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Mesoscopic central limit theorem for general β-ensembles

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Abstract. We prove that the linear statistics of eigenvalues of β-log gases satisfying the one-cut and off-critical assumption with a potential $V \in C^7(\mathbb{R})$ satisfy a central limit theorem at all mesoscopic scales $\alpha \in (0;1)$. We prove this for compactly supported test functions $f \in C^6(\mathbb{R})$ using loop equations at all orders along with rigidity estimates.

Résumé. Nous prouvons que les statistiques linéaires du β-gaz de Coulomb confiné par un potentiel $V \in C^7(\mathbb{R})$ et avec une mesure d'équilibre non critique à support connexe satisfont un théorème central limite à toutes les échelles mésoscopiques $\alpha \in (0;1)$. Nous prouvons ce résultat pour toute fonction test $f \in C^6(\mathbb{R})$ à support compact en utilisant les équations de boucles et des estimées de rigidité.

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1. Introduction

We consider a system of N particles on the real line distributed according to a density proportional to

$$\prod_{1\le i<j\le N} |\lambda_i - \lambda_j|^\beta e^{-N\sum V(\lambda_i)} \prod d\lambda_i,$$

where V is a continuous potential and $\beta > 0$. This system is called the β-log gas, or general β-ensemble. For classical values of $\beta \in \{1,2,4\}$, this distribution corresponds to the joint law of the eigenvalues of symmetric, hermitian or quaternionic random matrices with density proportional to $e^{-N \operatorname{Tr} V(M)}\, dM$ where N is the size of the random matrix M.

Recently, great progress has been made to understand the behavior of β-log gases. At the microscopic scale, the eigenvalues exhibit a universal behavior (see [3,4,8,9,22,25]) and the local statistics of the eigenvalues are described by the $Sine_\beta$ process in the bulk and the Stochastic Airy Operator at the edge (see [22,26] for definitions). In this article, we study the linear fluctuations of the eigenvalues of general β-ensembles at the mesoscopic scale; we prove that for $\alpha \in (0;1)$ fixed, f a smooth function (whose regularity and decay at infinity will be specified later), and E a fixed point in the bulk of the spectrum

$$\sum_{i=1}^N f\big(N^\alpha(\lambda_i - E)\big) - N \int f\big(N^\alpha(x-E)\big)\, d\mu_V(x)$$

converges towards a Gaussian random variable. At the macroscopic level (i.e when $\alpha = 0$), it is known that the eigenvalues satisfy a central limit theorem and the re-centered linear statistics of the eigenvalues converge towards a

Gaussian random variable. This was first proved in [16] for polynomial potentials satisfying the one-cut assumption. In [7], the authors derived a full expansion of the free energy in the one-cut regime from which they deduce the central limit theorem for analytic potentials. The multi-cut regime is more complicated and in this setting, the central limit theorem does not hold anymore for all test functions (see [6,24]). Similar results have also been obtained for the eigenvalues of Random Matrices from different ensembles (see [1,20,23]). Interest in mesoscopic linear statistics has surged in recent years. Results in this field of study were obtained in a variety of settings, for Gaussian random matrices [11,14], and for invariant ensembles [13,17]. In many cases the results were shown at all scales $\alpha \in (0;1)$, often with the use of distribution specific properties. In more general settings, the absence of such properties necessitates other approaches to obtain the limiting behavior at the mesoscopic regime. For example, an early paper studying mesoscopic statistics for Wigner Matrices was [12], here the regime studied was $\alpha \in (0;\frac{1}{8})$. This was pushed to $\alpha \in (0;\frac{1}{3})$ [19] using improved local law results, and recent work has pushed this to all scales [15]. The central limit theorem at all mesoscopic scales has also been obtained recently for the two dimensional β-log gas (or Coulomb gas) in [2,18].

Extending these results to one dimensional β-ensembles is a natural step. We also prove convergence at all mesoscopic scales. The proof of the main Theorem relies on the analysis of the loop equations (see Section 2.1) from which we can deduce a recurrence relationship between moments, and the rigidity results from [8,9] to control the linear statistics. Similar results have been obtained before in [10, Theorem 5.4]. There, the authors showed the mesoscopic CLT in the case of a quadratic potential, for small α (see Remark 5.5).

In Section 1, we introduce the model and recall some background results and Section 2 will be dedicated to the proof of Theorem 1.5.

1.1. *Definitions and background*

We consider the general β-matrix model. For a potential $V:\mathbb{R}\longrightarrow\mathbb{R}$ and $\beta>0$, we denote the measure on $\mathbb{R}^N$

$$\mathbb{P}_V^N(d\lambda_1,\dots,d\lambda_N) := \frac{1}{Z_V^N}\prod_{1\le i<j\le N}|\lambda_i-\lambda_j|^\beta e^{-N\sum V(\lambda_i)}\prod d\lambda_i, \tag{1.1}$$

with

$$Z_V^N = \int \prod_{1\le i<j\le N}|\lambda_i-\lambda_j|^\beta e^{-N\sum V(\lambda_i)}\prod d\lambda_i.$$

It is well known (see [21] for the Hölder case, and [16], Theorem 2.1 for the continuous case) that under $\mathbb{P}_V^N$ the empirical measure of the eigenvalues converge towards an equilibrium measure:

Theorem 1.1. *Assume that $V:\mathbb{R}\longrightarrow\mathbb{R}$ is continuous and that*

$$\liminf_{x\to\infty}\frac{V(x)}{\beta\log|x|}>1.$$

Then the energy defined by

$$E(\mu)=\iint\left(\frac{V(x_1)+V(x_2)}{2}-\frac{\beta}{2}\log|x_1-x_2|\right)d\mu(x_1)\,d\mu(x_2) \tag{1.2}$$

has a unique global minimum on the space $\mathcal{M}_1(\mathbb{R})$ of probability measures on $\mathbb{R}$.

Moreover, under $\mathbb{P}_V^N$ the normalized empirical measure $L_N=N^{-1}\sum_{i=1}^N\delta_{\lambda_i}$ converges almost surely and in expectation towards the unique probability measure μ_V which minimizes the energy.

The measure μ_V has compact support A and is uniquely determined by the existence of a constant C such that:

$$\beta\int\log|x-y|\,d\mu_V(y)-V(x)\le C,$$

with equality almost everywhere on the support.

1.2. *Results*

Hypothesis 1.2. *For what proceeds, we assume the following*

- V *is continuous and goes to infinity faster than* $\beta\log|x|$.
- *The support of* μ_V *is a connected interval* $A=[a;b]$ *and*

$$\frac{d\mu_V}{dx}=\rho_V(x)=S(x)\sqrt{(b-x)(x-a)}\quad \text{with } S>0 \text{ on } [a;b].$$

- *The function* $V(\cdot)-\beta\int\log|\cdot-y|\,d\mu_V(y)$ *achieves its minimum on the support only.*

Remark 1.3. The second and third assumptions are typically known as the one-cut and off-criticality assumptions. In the case where the support of the equilibrium measure is no longer connected, the macroscopic central limit theorem does not hold anymore in generality (see [6,24]). Whether the theorem holds for critical potentials is still an open question.

Remark 1.4. If the previous assumptions are fulfilled, and $V\in C^p(\mathbb{R})$ then $S\in C^{p-3}(\mathbb{R})$ (see [4], Lemma 3.2).

Theorem 1.5. *Let* $0<\alpha<1$, E *a point in the bulk* $(a;b)$, $V\in C^7(\mathbb{R})$ *and* $f\in C^6(\mathbb{R})$ *with compact support. Then, under* $\mathbb{P}_V^N$

$$\sum_{i=1}^N f\big(N^\alpha(\lambda_i-E)\big)-N\int f\big(N^\alpha(x-E)\big)\,d\mu_V(x)\xrightarrow{\mathcal{M}}\mathcal{N}\big(0,\sigma_f^2\big),$$

where the convergence holds in moments (and thus in distribution), and

$$\sigma_f^2=\frac{1}{2\beta\pi^2}\iint\left(\frac{f(x)-f(y)}{x-y}\right)^2 dx\,dy.$$

Note that, as in the macroscopic central limit theorem, the variance is universal in the potential with a multiplicative factor proportional to $1/\beta$. Interestingly and in contrast with the macroscopic scale, the limit is always centered.

The proof relies on an explicit computation of the moments of the linear statistics. We will use two tools: optimal rigidity for the eigenvalues of β-ensembles to provide a bound on the linear statistics (as in [8,9]) and the loop equations at all orders to derive a recurrence relationship between the moments.

2. Proof of Theorem 1.5

For what follows, set

$$L_N=\frac{1}{N}\sum_{i=1}^N\delta_{\lambda_i},\qquad M_N=\sum_{i=1}^N\delta_{\lambda_i}-N\mu_V,$$

and for a measure ν and an integrable function h set

$$\nu(h)=\int h\,d\nu\quad\text{and}\quad\tilde{\nu}(h)=\int h\,d\nu-\mathbb{E}_V^N\left(\int h\,d\nu\right),\tag{2.1}$$

when ν is random and where $\mathbb{E}_V^N$ is the expectation with respect to $\mathbb{P}_V^N$. Further f will be any function as in Theorem 1.5, and

$$f_N(x):=f\big(N^\alpha(x-E)\big).$$

Finally, for any function $g \in C^p(\mathbb{R})$, let

$$\|g\|_{C^p(\mathbb{R})} := \sum_{l=0}^{p} \sup_{x\in\mathbb{R}} \bigl|g^{(l)}(x)\bigr|,$$

when it exists.

2.1. *Loop equations*

To prove the convergence, we use the loop equations at all orders. Loop equations have been used previously to derive recurrence relationships between correlators and derive a full expansion of the free energy for β-ensembles in [6,7,24] (from which the authors also derive a macroscopic central limit theorem). The first loop equation was used to prove the central limit theorem at the macroscopic scale in [16] and used subsequently in [10]. Here, rather than using the first loop equation to control the Stieltjes transform as in [16] and [10], we rely on the analysis of the loop equations at all orders to compute directly the moments.

Proposition 2.1. *Let $h, h_1, h_2, \dots$ be a sequence of bounded functions in $C^1(\mathbb{R})$. Define*

$$F_1^N(h) := \frac{N\beta}{2} \iint \frac{h(x)-h(y)}{x-y}\, dL_N(x)\, dL_N(y) - N L_N\bigl(hV'\bigr) + \biggl(1-\frac{\beta}{2}\biggr) L_N\bigl(h'\bigr) \tag{2.2}$$

and for all $k \geq 1$

$$F_{k+1}^N(h, h_1, \dots, h_k) := F_k^N(h, h_1, \dots, h_{k-1}) \tilde{M}_N(h_k) + \Biggl(\prod_{l=1}^{k-1} \tilde{M}_N(h_l)\Biggr) L_N\bigl(hh_k'\bigr), \tag{2.3}$$

where the product is equal to 1 when $k = 1$ and $\tilde{M}_N$ was defined by the convention Eq. (2.1). Then we have for all $k \geq 1$

$$\mathbb{E}_V^N\bigl(F_k^N(h, h_1, \dots, h_{k-1})\bigr) = 0, \tag{2.4}$$

which is called the loop equation of order k.

Proof. The first loop equation (2.2) is derived by integration by parts (see also [16], Eq. (2.18) for a proof using a change of variables). More precisely, for a fixed index l, integration by parts with respect to λ_l yields the equality:

$$\mathbb{E}_V^N\bigl(h'(\lambda_l)\bigr) = -\mathbb{E}_V^N\Biggl(h(\lambda_l)\Biggl(\beta \sum_{\substack{1\leq i\leq N \\ i\neq l}} \frac{1}{\lambda_l - \lambda_i} - NV'(\lambda_l)\Biggr)\Biggr).$$

Summing over l we get by symmetry

$$\mathbb{E}_V^N\Biggl(\frac{\beta}{2} \sum_{l=1}^{N} \sum_{\substack{1\leq i\leq N \\ i\neq l}} \frac{h(\lambda_l)-h(\lambda_i)}{\lambda_l - \lambda_i} - N\sum_{l=1}^{N} V'(\lambda_l)h(\lambda_l) + \sum_{l=1}^{N} h'(\lambda_l)\Biggr) = 0.$$

Writing the sums in term of L_N and taking the diagonal terms to be equal to $h'(\lambda_l)$ gives Eq. (2.4) for $k = 1$.

To derive the loop equation at order $k+1$ from the one at order k, replace V by $V - \delta h_k$ and notice that for any functional F that is independent of δ,

$$\frac{\partial \mathbb{E}_{V-\delta h_k}^N(F)}{\partial \delta}\bigg|_{\delta=0} = N\mathbb{E}_V^N\bigl(F\tilde{M}_N(h_k)\bigr).$$

Also observe that the loop equation Eq. (2.4) is now

$$\mathbb{E}^N_{V-\delta h_k}\big(F^N_k(h,h_1,\dots,h_{k-1})\big)+\delta N\mathbb{E}^N_{V-\delta h_k}\left(\left(\prod_{l=1}^{k-1}\tilde{M}_N(h_l)\right)L_N\big(hh_k'\big)\right)=0,$$

by induction and the definitions given in eqns. (2.2) and (2.3). Differentiating both sides with respect to δ and setting $\delta=0$ yields the loop equation at order $k+1$. □

It will be easier to compute the moments of $M_N(f_N)$ by re-centering the first loop equation – that is, we wish to replace L_N by $L_N-\mu_V$. To that end, define the operator Ξ acting on smooth functions $h:\mathbb{R}\longrightarrow\mathbb{R}$ by

$$\Xi h(x):=\beta\int\frac{h(x)-h(y)}{x-y}\,d\mu_V(y)-V'(x)h(x).$$

This operator is central to our approach and had already been introduced in [4] to prove universality of general β-ensembles using transport methods. We now prove the equilibrium relation (2.5) to recenter L_N by μ_V. Consider for a smooth function $h:\mathbb{R}\longrightarrow\mathbb{R}$ and δ in a neighborhood of 0, $\mu_{V,\delta}=(x+\delta h(x))\#\mu_V$, where for a map T and measure μ, $T\#\mu$ refers to the push-forward measure of μ by T. Then by (1.2) we have $E(\mu_{V,\delta})\geq E(\mu_V)$, which writes

$$\begin{aligned}&\iint\bigg\{\frac{V(x_1+\delta h(x_1))+V(x_2+\delta h(x_2))}{2}\\&\qquad-\frac{\beta}{2}\log\big|x_1-x_2+\delta\big(h(x_1)-h(x_2)\big)\big|\bigg\}\,d\mu_V(x_1)\,d\mu_V(x_2)\\&\quad\geq\iint\left(\frac{V(x_1)+V(x_2)}{2}-\frac{\beta}{2}\log|x_1-x_2|\right)d\mu_V(x_1)\,d\mu_V(x_2).\end{aligned}$$

As δ approaches 0 we get

$$\delta\left(\int V'(x)h(x)\,d\mu_V(x)-\frac{\beta}{2}\iint\frac{h(x)-h(y)}{x-y}\,d\mu_V(x)\,d\mu_V(y)\right)+O\big(\delta^2\big)\geq 0.$$

So that

$$\frac{\beta}{2}\iint\frac{h(x)-h(y)}{x-y}\,d\mu_V(x)\,d\mu_V(y)=\int V'(x)h(x)\,d\mu_V(x),\tag{2.5}$$

and thus

$$\begin{aligned}&\frac{\beta}{2}\iint\frac{h(x)-h(y)}{x-y}\,dL_N(x)\,dL_N(y)-L_N\big(hV'\big)\\&\quad=\frac{1}{N}M_N(\Xi h)+\frac{\beta}{2N^2}\iint\frac{h(x)-h(y)}{x-y}\,dM_N(x)\,dM_N(y).\end{aligned}$$

Consequently, we can write

$$F^N_1(h)=M_N(\Xi h)+\left(1-\frac{\beta}{2}\right)L_N\big(h'\big)+\frac{1}{N}\left[\frac{\beta}{2}\iint\frac{h(x)-h(y)}{x-y}\,dM_N(x)\,dM_N(y)\right].\tag{2.6}$$

One of the key features of the operator Ξ is that, under the one-cut and non-critical assumptions, it is invertible (modulo constants) in the space of smooth functions. More precisely, we have the following Lemma (see [4, Lemma 3.2] for the proof):

Lemma 2.2 (Inversion of Ξ). *Assume that $V \in C^p(\mathbb{R})$ and satisfies Hypothesis 1.2. Let $[a;b]$ denote the support of μ_V and set*

$$\frac{d\mu_V}{dx} = S(x)\sqrt{(b-x)(x-a)} = S(x)\sigma(x),$$

where $S > 0$ on $[a;b]$.

Then for any $k \in C^r(\mathbb{R})$ there exists a unique constant c_k and $h \in C^{(r-2)\wedge(p-3)}(\mathbb{R})$ such that

$$\Xi(h) = k + c_k.$$

Moreover the inverse is given by the following formulas:

- $\forall x \in \operatorname{supp}(\mu_V)$

$$h(x) = -\frac{1}{\beta\pi^2 S(x)}\left(\int_a^b \frac{k(y)-k(x)}{\sigma(y)(y-x)}\,dy\right). \tag{2.7}$$

- $\forall x \notin \operatorname{supp}(\mu_V)$

$$h(x) = \frac{\beta\int \frac{h(y)}{x-y}\,d\mu_V(y) + k(x) + c_k}{\beta\int \frac{1}{x-y}\,d\mu_V(y) - V'(x)}. \tag{2.8}$$

And $c_k = -\beta\int \frac{h(y)}{a-y}\,d\mu_V(y) - k(a)$. Note that the definition (2.8) is proper since h has been defined on the support.

We shall denote this inverse by $\Xi^{-1}k$.

Remark 2.3. For f and V as in Theorem 1.5, $p = 7$ and $r = 6$ so $\Xi^{-1} f_N \in C^4(\mathbb{R})$.

Remark 2.4. The denominator $\beta\int \frac{1}{x-y}\,d\mu_V(y) - V'(x)$ is identically null on $\operatorname{supp}\mu_V$ and behaves like a square root at the edges. Since by the last point of Hypothesis 1.2 we can modify freely the potential outside any neighborhood of the support (see for instance the large deviation estimates Section 2.1 of [7]), we may assume that it doesn't vanish outside μ_V.

In order to bound the linear statistics we use the following lemma to bound $\Xi^{-1}(f_N)$ and its derivatives.

Lemma 2.5. *Let $\operatorname{supp} f \subset [-M, M]$ for some constant $M > 0$. For each $p \in \{0,1,2,3\}$, there is a constant $C > 0$ such that*

$$\left\|\Xi^{-1}(f_N)\right\|_{C^p(\mathbb{R})} \leq C N^{p\alpha}\log N. \tag{2.9}$$

Moreover, there is a constant C such that whenever $x \in \operatorname{supp}\mu_V$ and $N^\alpha|x-E| \geq M+1$

$$\left|\Xi^{-1}(f_N)^{(p)}(x)\right| \leq \frac{C}{N^\alpha(x-E)^{p+1}}, \tag{2.10}$$

and when $x \notin \operatorname{supp}\mu_V$

$$\left|\Xi^{-1}(f_N)^{(p)}(x)\right| \leq \frac{C\log N}{N^\alpha}. \tag{2.11}$$

Proof. We start by proving (2.9) on the support. For $x \in \operatorname{supp}\mu_V$ we use

$$\Xi^{-1}(f_N)(x) = -\frac{N^\alpha}{\beta\pi^2 S(x)}\int_a^b \frac{1}{\sigma(y)}\int_0^1 f'\big(N^\alpha t(x-E) + N^\alpha(1-t)(y-E)\big)\,dt\,dy$$

so that

$$\Xi^{-1}(f_N)^{(p)}(x) = -\frac{1}{\beta\pi^2}\sum_{l=0}^{p}\left\{\binom{p}{l}\left(\frac{1}{S}\right)^{(p-l)}(x) \times \int_a^b \frac{N^{(l+1)\alpha}}{\sigma(y)}\int_0^1 t^l f^{(l+1)}\big(N^\alpha t(x-E)+N^\alpha(1-t)(y-E)\big)\,dt\,dy\right\}.$$

Let $A(x)=\{(t,y)\in[0;1]\times[a;b], N^\alpha|t(x-E)+(1-t)(y-E)|\le M\}$. We have

$$\int_0^1 \mathbb{1}_{A(x)}(t,y)\,dt \le \frac{2M}{N^\alpha|x-y|}\wedge 1 \tag{2.12}$$

and thus

$$\int_a^b \frac{N^{(l+1)\alpha}}{\sigma(y)}\int_0^1 \big|f^{(l+1)}\big(N^\alpha t(x-E)+N^\alpha(1-t)(y-E)\big)\big|\,dt\,dy \le C\log N N^{l\alpha},$$

and this proves (2.9).

We now proceed with the proof of (2.10). First, let $x\in\operatorname{supp}\mu_V$ such that $N^\alpha|x-E|\ge M+1$. The inversion formula (2.7) writes

$$\begin{aligned}\Xi^{-1}(f_N)(x) &= -\frac{1}{\beta\pi^2 S(x)}\int_a^b \frac{f(N^\alpha(y-E))}{\sigma(y)(y-x)}\,dy\\ &= -\frac{1}{\beta\pi^2 S(x)}\int_{-M}^{M}\frac{f(u)}{\sigma(E+\frac{u}{N^\alpha})(u-N^\alpha(x-E))}\,du,\end{aligned} \tag{2.13}$$

and we can conclude in this setting by differentiating under the integral. Moreover we see that $\Xi^{-1}(f_N)$ is in fact of class C^5 on $\operatorname{supp}\mu_V$ and similar bounds holds for $p\in\{4,5\}$.

We now prove the bounds for $x\notin\operatorname{supp}\mu_V$. Let ψ_N be an arbitrary extension of $\Xi^{-1}(f_N)|_{\operatorname{supp}\mu_V}$ in $C^5(\mathbb{R})$, bounded by C/N^α outside the support (and its five first derivatives as well). This is possible by what we just proved and a Taylor expansion. Using (2.8) we notice that

$$\begin{aligned}\Xi^{-1}(f_N)(x) &= \frac{\beta\int\frac{\psi_N(y)}{x-y}\,d\mu_V(y)+c_{f_N}}{\beta\int\frac{1}{x-y}\,d\mu_V(y)-V'(x)}\\ &= \frac{-\beta\int\frac{\psi_N(x)-\psi_N(y)}{x-y}\,d\mu_V(y)+\beta\psi_N(x)\int\frac{d\mu_V(y)}{x-y}+c_{f_N}}{\beta\int\frac{1}{x-y}\,d\mu_V(y)-V'(x)}\\ &= \psi_N(x)-\frac{\Xi(\psi_N)(x)-c_{f_N}}{\beta\int\frac{1}{x-y}\,d\mu_V(y)-V'(x)}.\end{aligned} \tag{2.14}$$

Since f has compact support we may write $\Xi(\psi_N)-c_{f_N}=\Xi(\psi_N)-c_{f_N}-f_N$ on $[a;a+\epsilon]$ and $[b-\epsilon;b]$ for ϵ small enough. Furthermore this quantity vanishes identically on these intervals by definition of ψ_N. In particular, $\Xi(\psi_N)-c_{f_N}$ and its four first derivatives vanish at the edges. By definition, and using the previous bounds we also get that

$$\begin{aligned}|c_{f_N}| &= \left|\beta\int\frac{\psi_N(y)}{a-y}\,d\mu_V(y)\right|\\ &\le C\log N\int_{|y-E|\le 2M/N^\alpha}\frac{d\mu_V(y)}{y-a}+\frac{C}{N^\alpha}\int_{|y-E|\ge 2M/N^\alpha}\frac{d\mu_V(y)}{(y-a)|y-E|}\\ &\le C\frac{\log N}{N^\alpha}.\end{aligned}$$

On the other hand, for $p \in [\![0;4]\!]$ and $x \notin \operatorname{supp} \mu_V$,

$$\Xi(\psi_N)^{(p)}(x) = \beta p! \int \frac{\psi_N(y) - \psi_N(x) - \cdots - \psi_N^{(p)}(x)(y-x)^p/p!}{(y-x)^{p+1}} \, d\mu_V(y) \\ - \big(V'\psi_N\big)^{(p)}(x).$$

By doing a similar splitting, and bounding the fifth derivative of ψ_N uniformly away from E, we obtain the same bound $C \log N/N^\alpha$ on $\Xi(\psi_N)^{(p)}$ outside the support. By Remark 2.4 and (2.14), we conclude that we can bound the C^3 norm of $\Xi^{-1}(f_N)$ by $C \log N/N^\alpha$ outside the support. □

2.2. *Sketch of the proof*

We have developed the tools we need to prove Theorem 1.5. In order to motivate the technical estimates in the following section, we now sketch the proof by computing the first moments. The full proof of the theorem will be given in Section 2.4. Consider a function f satisfying the hypothesis of Theorem 1.5. Applying (2.6) to $\Xi^{-1}(f_N)$ yields

$$F_1^N\big(\Xi^{-1}(f_N)\big) = M_N(f_N) + \left(1 - \frac{\beta}{2}\right) L_N\big(\big(\Xi^{-1} f_N\big)'\big) \\ + \frac{1}{N}\left[\frac{\beta}{2} \iint \frac{\Xi^{-1} f_N(x) - \Xi^{-1} f_N(y)}{x-y} \, dM_N(x)\, dM_N(y)\right].$$

If the central limit theorem holds, we expect terms of the type $M_N(h)$ where h is fixed to be almost of constant order, and this an easy consequence of the rigidity estimates from [8] (stated as Theorem 2.6 below). Due to the dependency in N of f_N (and its inverse under Ξ), a little care must be taken for these estimates to yield a bound on the last term in the right handside, and this is precisely the point of Lemma 2.9. Similarly, we have

$$L_N\big(\big(\Xi^{-1} f_N\big)'\big) = \mu_V\big(\big(\Xi^{-1} f_N\big)'\big) + \frac{1}{N} M_N\big(\big(\Xi^{-1} f_N\big)'\big),$$

and Lemma 2.8 shows the term in the right handside is a small error term. Thus admitting the results of the next section, we would get with high probability and for ε_N small

$$F_1^N\big(\Xi^{-1}(f_N)\big) = M_N(f_N) + \left(1 - \frac{\beta}{2}\right) \mu_V\big(\big(\Xi^{-1} f_N\big)'\big) + \varepsilon_N.$$

By the first loop equation from Proposition 2.1, the expectation of F_1^N is zero and this shows that the first moment

$$\mathbb{E}_V^N\big(M_N(f_N)\big) = -\left(1 - \frac{\beta}{2}\right) \mu_V\big(\big(\Xi^{-1} f_N\big)'\big) + o(1).$$

The term on the right handside is deterministic and is shown to decrease towards zero in Lemma 2.10. Thus the first moment converges to 0.

In order to exhibit all the terms we will need to control, we proceed with the computation of the second moment. By definition

$$F_2^N\big(\Xi^{-1}(f_N), f_N\big) = F_1^N\big(\Xi^{-1}(f_N)\big) \tilde{M}_N(f_N) + L_N\big(\Xi^{-1}(f_N) f_N{}'\big),$$

which we can write (with now an ε_N incorporating the deterministic mean converging to zero)

$$F_2^N\big(\Xi^{-1}(f_N), f_N\big) = M_N(f_N) \tilde{M}_N(f_N) + \varepsilon_N \tilde{M}_N(f_N) + L_N\big(\Xi^{-1}(f_N) f_N'\big).$$

Lemma 2.8 ensures that $\varepsilon_N \tilde{M}_N(f_N)$ remains small, and that the term in the right handside of the decomposition

$$L_N\big(\Xi^{-1}(f_N)f_N'\big) = \mu_V\big(\Xi^{-1}(f_N)f_N'\big) + \frac{1}{N}M_N\big(\Xi^{-1}(f_N)f_N'\big),$$

is also controlled. Consequently, using the second loop equation we see that

$$\mathbb{E}_V^N\big(M_N(f_N)^2\big) = -\mu_V\big(\Xi^{-1}(f_N)f_N'\big) + o(1). \tag{2.15}$$

The limit of the term appearing on the right handside is then computed in Lemma 2.10, equation (2.33). The following moments are computed similarly (see Section 2.4).

In the following section, we establish all the bounds we need for the proof of Theorem 1.5. The previous steps will then be made rigorous in the last section.

2.3. *Control of the linear statistics*

We now make use of the strong rigidity estimates proved in [8] (Theorem 2.4) to control the linear statistics. We recall the result here

Theorem 2.6. *Let γ_i the quantile defined by*

$$\int_a^{\gamma_i} d\mu_V(x) = \frac{i}{N}. \tag{2.16}$$

Then, under Hypothesis 1.2 and for all $\xi > 0$ there exists constants $c > 0$ such that for N large enough

$$\mathbb{P}_V^N\big(|\lambda_i - \gamma_i| \geq N^{-2/3+\xi}\hat{\imath}^{-1/3}\big) \leq e^{-N^c},$$

where $\hat{\imath} = i \wedge (N+1-i)$.

We will use the following lemma quite heavily in what proceeds.

Lemma 2.7. *Let γ_i and $\hat{\imath}$ be as in Theorem 2.6. Let λ_i, $i \in [\![1, N]\!]$, be a configuration of points such that $|\lambda_i - \gamma_i| \leq N^{-2/3+\xi}\hat{\imath}^{-1/3}$ for $0 < \xi < (1-\alpha) \wedge \frac{2}{3}$, and let $M > 1$ be a constant. Define the pairwise disjoint sets*:

$$J_1 := \big\{i \in [\![1; N]\!], \big|N^\alpha(\gamma_i - E)\big| \leq 2M\big\}, \tag{2.17}$$

$$J_2 := \left\{i \in J_1^c, \big|(\gamma_i - E)\big| \leq \frac{1}{2}(E-a) \wedge (b-E)\right\}, \tag{2.18}$$

$$J_3 := J_1^c \cap J_2^c. \tag{2.19}$$

The following statements hold:

(a) *For all $i \in J_1 \cup J_2$, $\hat{\imath} \geq CN$, for some $C > 0$ that depend only on μ_V. For all such i, $|\gamma_i - \gamma_{i+1}| \leq \frac{C}{N}$ for a constant $C > 0$.*

(b) *Uniformly in all $i \in J_1^c = J_2 \cup J_3$, $x \in [\gamma_i, \gamma_{i+1}]$ and all $t \in [0; 1]$,*

$$\big|N^\alpha t(\lambda_i - x) + N^\alpha(x - E)\big| > M + 1, \tag{2.20}$$

for N large enough.

(c) *The cardinality of J_1 is of order $CN^{1-\alpha}$, where again, $C > 0$ depends only on μ_V in a neighborhood of E.*

Proof. The first part of statement (a) holds by the observation that for $i \in J_1 \cup J_2$, γ_i is in the bulk, so

$$0 < c \leq \int_a^{\gamma_i} d\mu_V(x) = \frac{i}{N} \leq C < 1$$

for constants $C, c > 0$ depending only on μ_V. For the second part of statement (a), the density of μ_V is bounded below uniformly in $i \in J_1 \cup J_2$, so

$$c|\gamma_i - \gamma_{i+1}| \leq \int_{\gamma_i}^{\gamma_{i+1}} d\mu_V(x) = \frac{1}{N}.$$

Statement (b) can be seen as follows: let $i \in J_2$ and consider first $x = \gamma_i$. On this set $\hat{\imath} \geq CN$ by (a), so uniformly in such i, $N^\alpha|\lambda_i - \gamma_i| \leq CN^{\alpha-1+\xi}$, which goes to zero, while $N^\alpha|\gamma_i - E| > 2M$. On the other hand, for $i \in J_3$, we have $N^\alpha|\gamma_i - E| > \frac{1}{2}N^\alpha(E-a) \wedge (b-E)$, which goes to infinity faster than $N^\alpha|\lambda_i - \gamma_i| \leq N^{\alpha-\frac{2}{3}+\xi}$, by our choice of ξ. When we substitute γ_i by x, the same argument holds because $N^\alpha|x-\gamma_i| \leq N^\alpha|\gamma_i - \gamma_{i+1}|$, which is of order $N^{\alpha-1}$ on J_2 (as we showed in statement (a)) and bounded by $CN^{\alpha-\frac{2}{3}}$ on J_3.

Statement (c) follows by the observation that on the set $x \in [a,b]$ such that $|x-E| \leq \frac{2M}{N^\alpha}$ the density of μ_V is bounded uniformly above and below, so

$$\frac{c}{N^\alpha} \leq \int_{|x-E|\leq \frac{2M}{N^\alpha}} d\mu_V(x) = \sum_{i\in J_1}\int_{\gamma_i}^{\gamma_{i+1}} d\mu_V(x) + O\left(\frac{1}{N}\right) \leq \frac{C}{N^\alpha},$$

giving the required result. □

The rigidity of eigenvalues, Theorem 2.6, along with the previous Lemma leads to the following estimates

Lemma 2.8. *For all* $0 < \xi < (1-\alpha) \wedge \frac{2}{3}$ *there exists constants* $C, c > 0$ *such that for* N *large enough we have the concentration bounds*

$$\mathbb{P}_V^N\left(\left|M_N(f_N)\right| \geq CN^\xi \|f\|_{C^1(\mathbb{R})}\right) \leq e^{-N^c}, \tag{2.21}$$

$$\mathbb{P}_V^N\left(\left|M_N\left(\Xi^{-1}(f_N)'\right)\right| \geq CN^{\alpha+\xi} \|f\|_{C^1(\mathbb{R})}\right) \leq e^{-N^c}, \tag{2.22}$$

$$\mathbb{P}_V^N\left(\left|M_N\left(\Xi^{-1}(f_N)f_N'\right)\right| \geq CN^{\alpha+\xi} \|f\|_{C^1(\mathbb{R})}\right) \leq e^{-N^c}. \tag{2.23}$$

Proof. Let $M > 1$ such that supp $f \subset [-M, M]$ and fix $0 < \xi < (1-\alpha) \wedge \frac{2}{3}$. For the remainder of the proof, we may assume that we are on the event $\Omega := \{\forall i, |\lambda_i - \gamma_i| \leq N^{-2/3+\xi}\hat{\imath}^{-1/3}\}$. This follows from the fact that, for example,

$$\begin{aligned}
&\mathbb{P}_N^V\left(\left|M_N(f_N)\right| \geq CN^\xi \|f\|_{C^1(\mathbb{R})}\right) \\
&\quad\leq \mathbb{P}_N^V\left(\left\{\left|M_N(f_N)\right| \geq CN^\xi \|f\|_{C^1(\mathbb{R})}\right\} \cap \Omega\right) + \mathbb{P}_N^V\left(\Omega^c\right),
\end{aligned}$$

and by Theorem 2.6, we may bound $\mathbb{P}_N^V(\Omega^c)$ by e^{-N^c} for some constant $c > 0$, and N large enough. On Ω, as the λ_i satisfy the conditions of Lemma 2.7 we will utilize the sets J_1, J_2, and J_3 as defined there.

We begin by controlling (2.21). We have that

$$\begin{aligned}
\left|M_N(f_N)\right| &= \left|\sum_{i=1}^N f\left(N^\alpha(\lambda_i - E)\right) - N\mu_V(f_N)\right| \\
&= \left|\sum_{i=1}^N f\left(N^\alpha(\lambda_i - E)\right) - N\int_{\gamma_i}^{\gamma_{i+1}} f\left(N^\alpha(x-E)\right) d\mu_V(x)\right| \\
&\leq N\sum_{i=1}^N \int_{\gamma_i}^{\gamma_{i+1}} \left|f\left(N^\alpha(\lambda_i - E)\right) - f\left(N^\alpha(x-E)\right)\right| d\mu_V(x) \\
&\leq N^{1+\alpha}\sum_{i\in J_1}\int_{\gamma_i}^{\gamma_{i+1}}\int_0^1 |\lambda_i - x|\left|f'\left(N^\alpha t(\lambda_i - E) + N^\alpha(1-t)(x-E)\right)\right| dt\, d\mu_V(x),
\end{aligned} \tag{2.24}$$

where we used Eq. (2.20). Using Lemma 2.7 item ((c)) and the definition of Ω we obtain

$$\big|M_N(f_N)\big| \leq N^{1+\alpha}|J_1|N^{\xi-1}\|f\|_{C^1(\mathbb{R})}\int_{\gamma_i}^{\gamma_{i+1}} d\mu_V(x) \leq CN^{\xi}\|f\|_{C^1(\mathbb{R})}.$$

This proves (2.21). We now proceed with the proof of (2.22).

$$\begin{aligned}\big|M_N\big(\Xi^{-1}(f_N)'\big)\big| &= \left|\sum_{i=1}^N\left(\Xi^{-1}(f_N)'(\lambda_i) - N\int_{\gamma_i}^{\gamma_{i+1}}\Xi^{-1}(f_N)'(x)\,d\mu_V(x)\right)\right| \\ &\leq N\sum_{i=1}^N\int_{\gamma_i}^{\gamma_{i+1}}\big|\Xi^{-1}(f_N)'(\lambda_i) - \Xi^{-1}(f_N)'(x)\big|\,d\mu_V(x) \\ &\leq N\sum_{i=1}^N\int_{\gamma_i}^{\gamma_{i+1}}\int_0^1|\lambda_i - x|\big|\Xi^{-1}(f_N)^{(2)}\big(t(\lambda_i - x)+x\big)\big|\,dt\,d\mu_V(x).\end{aligned}$$

Recall from the proof of Lemma 2.7 that uniformly in $i\in J_2$ and $x\in[\gamma_i,\gamma_{i+1}]$, $|\gamma_i - E|\geq\frac{2M}{N^\alpha}$ while $|x-\gamma_i|\leq\frac{C}{N}$. For what follows, as $|\lambda_i - x|\leq CN^{-1+\xi}$ for N large enough we can replace $|t(\lambda_i - x)+(\gamma_i - E)|$ by $|\gamma_i - E|$ uniformly in $t\in[0;1]$. Likewise, uniformly in $i\in J_3$, $x\in[\gamma_i,\gamma_{i+1}]$ and $t\in[0;1]$ we can bound below $|t(\lambda_i - x)+(x-E)|$ by a constant.

For $i\in J_2$, by the observations in the previous paragraph, along with Lemma 2.7(b), Lemma 2.5 (2.10) and Lemma 2.7(a),

$$\begin{aligned}&N\sum_{i\in J_2}\int_{\gamma_i}^{\gamma_{i+1}}\int_0^1|\lambda_i - x|\big|\Xi^{-1}(f_N)^{(2)}\big(t(\lambda_i - x)+x\big)\big|\,dt\,d\mu_V(x) \\ &\quad\leq N\sum_{i\in J_2}\int_{\gamma_i}^{\gamma_{i+1}}\int_0^1\frac{C|\lambda_i - x|}{N^\alpha(|t(\lambda_i - x)+x-E|^3)}\,dt\,d\mu_V(x)\leq\sum_{i\in J_2}\frac{CN^{\xi-1-\alpha}}{(\gamma_i - E)^3}.\end{aligned}$$

The same reasoning for $i\in J_3$ using also (2.11) yields

$$\begin{aligned}&N\sum_{i\in J_3}\int_{\gamma_i}^{\gamma_{i+1}}\int_0^1|\lambda_i - x|\big|\Xi^{-1}(f_N)^{(2)}\big(t(\lambda_i - x)+x\big)\big|\,dt\,d\mu_V(x) \\ &\quad\leq\log N\sum_{i\in J_3}CN^{\xi-\alpha-\frac{2}{3}}\hat{\imath}^{-\frac{1}{3}}.\end{aligned}$$

For $i\in J_1$, by Lemma 2.5 Eq. (2.9) and Lemma 2.7(a),

$$\begin{aligned}&N\sum_{i\in J_1}\int_{\gamma_i}^{\gamma_{i+1}}\int_0^1|\lambda_i - x|\big|\Xi^{-1}(f_N)^{(2)}\big(t(\lambda_i - x)+x\big)\big|\,dt\,d\mu_V(x) \\ &\quad\leq N\sum_{i\in J_1}\int_{\gamma_i}^{\gamma_{i+1}}CN^{2\alpha}\log N|\lambda_i - x|\,d\mu_V(x)\leq\sum_{i\in J_1}CN^{2\alpha+\xi-1}\log N.\end{aligned}$$

It follows that

$$\begin{aligned}\big|M_N\big(\Xi^{-1}(f_N)'\big)\big| &\leq\sum_{i\in J_1}CN^{2\alpha+\xi-1}\log N+\sum_{i\in J_2}\frac{CN^{\xi-1-\alpha}}{(\gamma_i - E)^3}+\log N\sum_{i\in J_3}CN^{\xi-\alpha-\frac{2}{3}}\hat{\imath}^{-\frac{1}{3}} \\ &\leq CN^{\alpha+\xi}\log N+CN^{\xi+\alpha}\leq CN^{\alpha+\xi}\log N,\end{aligned}$$

where we have used $|J_1| \le CN^{1-\alpha}$ and the following estimates:

$$\sum_{i\in J_2} \frac{N^{\xi-\alpha-1}}{(\gamma_i - E)^3} \le CN^{\xi-\alpha}\left(\int_a^{E-\frac{2M}{N^\alpha}} \frac{dx}{(x-E)^3} + \int_{E+\frac{2M}{N^\alpha}}^b \frac{dx}{(x-E)^3}\right) \le CN^{\xi+\alpha},$$

$$CN^{\xi-\alpha-\frac{2}{3}} \sum_{i\in J_3} \hat{\imath}^{-\frac{1}{3}} \le CN^{\xi-\alpha} \times \frac{1}{N}\sum_{i=1}^N \left(\frac{i}{N}\right)^{-\frac{1}{3}} \le CN^{\xi-\alpha}.$$

This proves (2.22). The bound (2.23) is obtained in a similar way and we omit the details. □

For convenience we introduce the following notation: for a sequence of random variable $(X_N)_{N\in\mathbb{N}}$ we write $X_N = \omega(1)$ if there exists constants c, C and $\delta > 0$ such that the bound $|X_N| \le \frac{C}{N^\delta}$ holds with probability greater than $1 - e^{-N^c}$.

Lemma 2.9. *We have*

$$\frac{1}{N}\iint \frac{\Xi^{-1}(f_N)(x) - \Xi^{-1}(f_N)(y)}{x-y}\, dM_N(x)\, dM_N(y) = \omega(1). \tag{2.25}$$

Proof. The proof will be similar to the proof of Lemma 2.8. As in Lemma 2.8 we may restrict our attention to the event $\Omega = \{\forall i : |\lambda_i - \gamma_i| \le N^{-\frac{2}{3}+\xi}\hat{\imath}^{-\frac{1}{3}}\}$ by applying Theorem 2.6. Further, we use again the sets J_1, J_2 and J_3 defined in Lemma 2.7.

The general idea will be that we can use the uniform bounds (2.9) for particles close to the bulk point E (corresponding to the indices in J_1), and control the number of such particles. In the intermediary regime we will use the bounds (2.11) or the explicit formula (2.13). On the other hand, for the particles far away from E (corresponding to J_3) we can use the uniform decay of $\Xi^{-1} f_N$ and its derivative by (2.10) and (2.11).

Define for $j \in \{1, 2, 3\}$:

$$M_N^{(j)} = \sum_{i\in J_j} (\delta_{\lambda_i} - N\mathbb{1}_{[\gamma_i,\gamma_{i+1}]}\mu_V)$$

so that $M_N = M_N^{(1)} + M_N^{(2)} + M_N^{(3)}$. We can write

$$\begin{aligned}&\iint \frac{\Xi^{-1}(f_N)(x) - \Xi^{-1}(f_N)(y)}{x-y}\, dM_N(x)\, dM_N(y)\\ &\quad= \sum_{1\le j_1,j_2\le 3} \iint \frac{\Xi^{-1}(f_N)(x) - \Xi^{-1}(f_N)(y)}{x-y}\, dM_N^{(j_1)}(x)\, dM_N^{(j_2)}(y).\end{aligned}$$

Integrating repeatedly for each (j_1, j_2), and using that $N\mu_V([\gamma_i, \gamma_{i+1}]) = 1$ for all indices i yields:

$$\begin{aligned}&\iint \frac{\Xi^{-1}(f_N)(x) - \Xi^{-1}(f_N)(y)}{x-y}\, dM_N^{(j_1)}(x)\, dM_N^{(j_2)}(y)\\ &\quad= N^2 \sum_{\substack{i_1\in J_{j_1}\\ i_2\in J_{j_2}}} \int_{\gamma_{i_1}}^{\gamma_{i_1+1}} d\mu_V(x_1) \int_{\gamma_{i_2}}^{\gamma_{i_2+1}} d\mu_V(x_2) \int_T du\, dv\, dt \big\{(\lambda_{i_1} - x_1)(\lambda_{i_2} - x_2)t(1-t)\\ &\qquad\times \Xi^{-1}(f_N)^{(3)}\big(tv(\lambda_{i_1} - x_1) + ut(x_2 - \lambda_{i_2}) + u(\lambda_{i_2} - x_2) + t(x_1 - x_2) + x_2\big)\big\},\end{aligned} \tag{2.26}$$

where $T = [0; 1]^3$. We will bound (2.26) for each pair (j_1, j_2).

For $(j_1, j_2) = (1, 1)$. Recall by Lemma 2.7(c) that $|J_1| \le CN^{1-\alpha}$, and from the proof of Lemma 2.7, uniformly in $i \in J_1$ $|\lambda_i - x| \le CN^{\xi-1}$ whenever $x \in [\gamma_i, \gamma_{i+1}]$. We use (2.26), Lemma 2.5 Eq. (2.9) to obtain the upper bound

$$\iint \frac{\Xi^{-1}(f_N)(x) - \Xi^{-1}(f_N)(y)}{x-y}\, dM_N^{(1)}(x)\, dM_N^{(1)}(y)$$
$$\le N^2 \sum_{\substack{i_1 \in J_1 \\ i_2 \in J_1}} \int_{\gamma_{i_1}}^{\gamma_{i_1+1}} \int_{\gamma_{i_2}}^{\gamma_{i_2+1}} N^{3\alpha} \log N |\lambda_{i_1} - x_1||\lambda_{i_2} - x_2|\, d\mu_V(x_1)\, d\mu_V(x_2) \le CN^{2\xi+\alpha} \log N,$$

which is $\omega(1)$ when divided by N.

For $(j_1, j_2) = (3, 3)$. we do as in the previous case. Using (2.11) instead and the fact that uniformly uniformly in $i \in J_3$, $|\lambda_i - x| \le CN^{-\frac{2}{3}+\xi} \hat{\imath}^{-\frac{1}{3}}$,

$$\iint \frac{\Xi^{-1}(f_N)(x) - \Xi^{-1}(f_N)(y)}{x-y}\, dM_N^{(3)}(x)\, dM_N^{(3)}(y)$$
$$\le N^2 \sum_{\substack{i_1 \in J_3 \\ i_2 \in J_3}} \int_{\gamma_{i_1}}^{\gamma_{i_1+1}} \int_{\gamma_{i_2}}^{\gamma_{i_2+1}} \frac{\log N}{N^\alpha} |\lambda_{i_1} - x_1||\lambda_{i_2} - x_2|\, d\mu_V(x_1)\, d\mu_V(x_2) \le CN^{2\xi-\alpha} \log N,$$

which is $\omega(1)$ when divided by N.

For $(j_1, j_2) = (2, 2)$. We remark that the strategy is not as straightforward as the case $i \in J_2$ in the proof of Lemma 2.8 Eq. (2.22). This is because the term $t(x_1 - x_2) + x_2$ appearing as an argument in (2.26) may enter a neighborhood of E depending on the indices $i_1, i_2 \in J_2$ and we may not use the bound Lemma 2.5 Eq. (2.10) uniformly in $i_1, i_2 \in J_2$. Some care is needed also because M_N is a signed measure so $|M_N(g)|$ need not be bounded by $M_N(|g|)$.

It will be convenient to use directly Eq. (2.13) from the proof of Lemma 2.5 (this can be done as J_2 corresponds to indices i such that γ_i is located outside the support of f). We can write for $x, y \in \{z \in \operatorname{supp} \mu_V, N^\alpha |z - E| > M + 1\}$

$$\begin{aligned} &\frac{\Xi^{-1}(f_N)(x) - \Xi^{-1}(f_N)(y)}{x-y} \\ &\quad = \frac{1}{\beta\pi^2} \int_{-M}^{M} \frac{f(u)}{\sigma(E + \frac{u}{N^\alpha})(x-y)} \left(\frac{1}{S(y)(u - N^\alpha(y-E))} - \frac{1}{S(x)(u - N^\alpha(x-E))} \right) du \\ &\quad = \frac{1}{\beta\pi^2} \int_{-M}^{M} \frac{f(u)}{\sigma(E + \frac{u}{N^\alpha})} \left\{ \frac{S(x) - S(y)}{(x-y)} \frac{1}{S(x)S(y)(u - N^\alpha(y-E))} \right. \\ &\qquad \left. + \frac{N^\alpha}{S(x)(u - N^\alpha(x-E))(u - N^\alpha(y-E))} \right\} du. \end{aligned} \tag{2.27}$$

When we integrate the term on the third line of (2.27) against $M_N^{(2)} \otimes M_N^{(2)}$, we obtain

$$\int_{-M}^{M} \frac{f(u)}{\sigma(E + \frac{u}{N^\alpha})} \left\{ \int M_N^{(2)} \left(\int_0^1 \frac{S'(t(\cdot - y) + y)}{S(\cdot)S(y)}\, dt \right) \frac{1}{(u - N^\alpha(y-E))}\, dM_N^{(2)}(y) \right\} du. \tag{2.28}$$

Define the function

$$g(y) := M_N^{(2)} \left(\int_0^1 \frac{S'(t(\cdot - y) + y)}{S(\cdot)S(y)}\, dt \right).$$

First, $g(y)$ is bounded for any $y \in [a;b]$:

$$\begin{aligned}
&\left| M_N^{(2)}\left(\int_0^1 \frac{S'(t(\cdot - y) + y)}{S(\cdot)S(y)}\, dt \right) \right| \\
&\quad = \left| \frac{N}{S(y)} \sum_{i \in J_2} \int_{\gamma_i}^{\gamma_{i+1}} \int_0^1 \left(\frac{S'(t(\lambda_i - y) + y)}{S(\lambda_i)} - \frac{S'(t(x - y) + y)}{S(x)} \right) dt\, d\mu_V(x) \right| \\
&\quad \le \left| \frac{N}{S(y)} \sum_{i \in J_2} \int_{\gamma_i}^{\gamma_{i+1}} \int_0^1 \frac{S'(t(\lambda_i - y) + y) - S'(t(x - y) + y)}{S(\lambda_i)}\, dt\, d\mu_V(x) \right| \\
&\qquad + \left| \frac{N}{S(y)} \sum_{i \in J_2} \int_{\gamma_i}^{\gamma_{i+1}} \int_0^1 \frac{S(x) - S(\lambda_i)}{S(x)S(\lambda_i)} S'\big(t(x - y) + y\big)\, dt\, d\mu_V(x) \right| \le C N^{\xi},
\end{aligned}$$

where in the final line we used S and S' are smooth on $[a;b]$ (and therefore uniformly Lipschitz), $S > 0$ in a neighborhood of $[a;b]$, further $|x - \lambda_i| \le C N^{\xi - 1}$, and $|J_2| \le C N$. Moreover, $g(y)$ is uniformly Lipschitz in $[a;b]$ with constant $C N^{\xi}$, since:

$$\begin{aligned}
&M_N^{(2)}\left(\int_0^1 \frac{S'(t(\cdot - y) + y)}{S(\cdot)S(y)} - \frac{S'(t(\cdot - z) + z)}{S(\cdot)S(z)}\, dt \right) \\
&\quad = (z - y) M_N^{(2)}\left(\int_0^1 \int_0^1 \frac{t S''(ut(z - y) + t(\cdot - z) + y)}{S(\cdot)S(y)}\, dt\, du \right) \\
&\qquad + \frac{S(z) - S(y)}{S(z)S(y)} M_N^{(2)}\left(\int_0^1 \frac{S'(t(\cdot - z) + z)}{S(\cdot)}\, dt \right)
\end{aligned}$$

and both terms appearing in $M_N^{(2)}$ above are of the same form as g so they are bounded by $C N^{\xi}$. Returning to (2.28), we may bound

$$\begin{aligned}
&\left| M_N^{(2)}\left(\frac{g(y)}{u - N^{\alpha}(y - E)} \right) \right| \\
&\quad = \left| N \sum_{i \in J_2} \int_{\gamma_i}^{\gamma_{i+1}} \frac{g(\lambda_i) - g(x)}{(u - N^{\alpha}(\lambda_i - E))} + \frac{N^{\alpha}(\lambda_i - x) g(x)}{(u - N^{\alpha}(\lambda_i - E))(u - N^{\alpha}(x - E))}\, d\mu_V(x) \right| \\
&\quad \le \int_{[a;b] \cap \{|x - E| \ge \frac{2M}{N^{\alpha}}\}} \frac{C N^{2\xi}}{|u - N^{\alpha}(x - E)|} + \frac{C N^{2\xi + \alpha}}{(u - N^{\alpha}(x - E))^2}\, dx \\
&\quad \le C N^{2\xi - \alpha} \log N + C N^{2\xi},
\end{aligned}$$

uniformly in $u \in [-M;M]$. Thus (2.28) is bounded by $C N^{2\xi}$ as f is bounded.

The remaining term in (2.27) is

$$N^{\alpha} \int_{-M}^{M} \frac{f(u)}{\sigma(E + \frac{u}{N^{\alpha}})} M_N^{(2)}\left(\frac{1}{S(\cdot)(u - N^{\alpha}(\cdot - E))} \right) M_N^{(2)}\left(\frac{1}{u - N^{\alpha}(\cdot - E)} \right) du. \tag{2.29}$$

Repeating our argument in the previous paragraph gives:

$$\begin{aligned}
&\left| M_N^{(2)}\left(\frac{1}{S(\cdot)(u - N^{\alpha}(\cdot - E))} \right) \right| \le C N^{\xi - \alpha} \log N + C N^{\xi}, \\
&\left| M_N^{(2)}\left(\frac{1}{u - N^{\alpha}(\cdot - E)} \right) \right| \le C N^{\xi},
\end{aligned}$$

where in the first inequality we use $1/S$ is uniformly bounded and uniformly Lipschitz on $[a; b]$. Inserting the bounds into (2.29) gives an upper bound of $CN^{2\xi+\alpha}$, as f is bounded.

Altogether

$$\left| \frac{\Xi^{-1}(f_N)(x) - \Xi^{-1}(f_N)(y)}{x-y} \, dM_N^{(2)}(x) \, dM_N^{(2)}(y) \right| \leq CN^{2\xi+\alpha},$$

which is $\omega(1)$ when divided by N.

For $(j_1, j_2) = (1, 2)$. By the bounds $|\lambda_{i_j} - \gamma_{i_j}| \leq CN^{\xi-1}$, $|\gamma_{i_j} - x_j| \leq \frac{C}{N}$ for $x_j \in [\gamma_{i_j}; \gamma_{i_j+1}]$, whenever

$$N^\alpha \left| tv(\lambda_{i_1} - x_1) + ut(x_2 - \lambda_{i_2}) + t(x_1 - x_2) + u(\lambda_{i_2} - x_2) + x_2 - E \right| \geq M + 1, \tag{2.30}$$

we have

$$N^\alpha \left(\left| t(\gamma_{j_1} - \gamma_{j_2}) + (\gamma_{j_2} - E) \right| + CN^{\xi-1} \right) \geq M + 1,$$

and

$$\begin{aligned} &\frac{1}{|tv(\lambda_{i_1} - x_1) + ut(x_2 - \lambda_{i_2}) + t(x_1 - x_2) + u(\lambda_{i_2} - x_2) + x_2 - E|} \\ &\quad \leq \frac{C}{|t(\gamma_{i_1} - \gamma_{i_2}) + (\gamma_{i_2} - E)|}, \end{aligned}$$

where the constant C only depends on M. Therefore, whenever (2.30) is satisfied, applying Lemma 2.5 Eq. (2.10) yields

$$\begin{aligned} &\left| \Xi^{-1}(f_N)^{(3)} \left(tv(\lambda_{i_1} - x_1) + ut(x_2 - \lambda_{i_2}) + t(x_1 - x_2) + u(\lambda_{i_2} - x_2) + x_2 \right) \right| \\ &\quad \leq \frac{C}{N^\alpha (t(\gamma_{i_1} - \gamma_{i_2}) + \gamma_{i_2} - E)^4}. \end{aligned} \tag{2.31}$$

Now fix $t \in (0, 1)$ and define the sets

$$\begin{aligned} K_t^1 &:= \left\{ j \in J_2, t\left(E - \frac{2M}{N^\alpha} - \gamma_j \right) + \gamma_j - E \geq \frac{2M}{N^\alpha} \right\}, \\ K_t^2 &:= \left\{ j \in J_2, t\left(E + \frac{2M}{N^\alpha} - \gamma_j \right) + \gamma_j - E \leq -\frac{2M}{N^\alpha} \right\}, \\ K_t &:= K_t^1 \cup K_t^2. \end{aligned}$$

By construction, if $i_2 \in K_t^1$ then

$$\left| t(\gamma_{i_1} - \gamma_{i_2}) + (\gamma_{i_2} - E) \right| \geq \frac{2M}{N^\alpha}$$

uniformly in $i_1 \in J_1$. Thus for such $i_2 \in K_t^1$, (2.30) is satisfied for N sufficiently large. The same statement holds for K_t^2.

We now proceed to bound (2.26) for $j_1 = 1$ and $j_2 = 2$ by splitting J_2 into the regions K_t^1, K_t^2 and $J_2 \backslash K_t$. We start with K_t^1 (the argument for K_t^2 is identical). Our observations from the previous paragraph along with (2.3) gives:

$$\begin{aligned}
&\int_T du\,dv\,dt \biggl| N^2 \sum_{\substack{i_1\in J_1\\ i_2\in K_t^1}} \int_{\gamma_{i_1}}^{\gamma_{i_1+1}} d\mu_V(x_1) \int_{\gamma_{i_2}}^{\gamma_{i_2+1}} d\mu_V(x_2) \bigl\{ (\lambda_{i_1}-x_1)(\lambda_{i_2}-x_2)t(1-t) \\
&\quad \times \Xi^{-1}(f_N)^{(3)}\bigl(tv(\lambda_{i_1}-x_1)+ut(x_2-\lambda_{i_2})+u(\lambda_{i_2}-x_2)+t(x_1-x_2)+x_2\bigr)\bigr\}\biggr| \\
&\quad \le \int_0^1 \sum_{\substack{i_1\in J_1\\ i_2\in K_t^1}} \frac{CN^{2\xi-2-\alpha}t(1-t)}{(t(\gamma_{i_1}-\gamma_{i_2})+(\gamma_{i_2}-E))^4}\,dt \le \int_0^1 \sum_{i_2\in K_t^1} \frac{CN^{2\xi-1-2\alpha}t(1-t)}{((1-t)(\gamma_{i_2}-E)-\frac{t2M}{N^\alpha})^4}\,dt,
\end{aligned}$$

where in the final line we used $|J_1| \le CN^{1-\alpha}$ from Lemma 2.7(c). Next, note that

$$\begin{aligned}
&\frac{1}{N}\sum_{i_2\in K_t^1} \frac{1}{((1-t)(\gamma_{i_2}-E)-\frac{t2M}{N^\alpha})^4} \\
&\quad \le C\int_{E+\frac{2M}{N^\alpha}(\frac{1+t}{1-t})}^{E+\frac{1}{2}(E-a)\wedge(b-E)} \frac{dx}{((1-t)(x-E)-\frac{t2M}{N^\alpha})^4} \le \frac{CN^{3\alpha}}{1-t},
\end{aligned}$$

since, by definition of K_t^1, $\gamma_{i_2} \ge E + \frac{2M}{N^\alpha}(\frac{1+t}{1-t})$. We conclude,

$$\int_0^1 \sum_{i_2\in K_t^1} \frac{CN^{2\xi-1-2\alpha}t(1-t)}{((1-t)(\gamma_{i_2}-E)-\frac{t2M}{N^\alpha})^4}\,dt \le CN^{2\xi+\alpha}.$$

We continue with $J_2\backslash K_t$. By the same argument as in Lemma 2.7(c) $|J_2\backslash K_t| \le \frac{CN^{1-\alpha}}{1-t}$ where the constant C does not depend on t, we use this in addition with Lemma 2.5 Eq. (2.10), $|J_1| \le CN^{1-\alpha}$, and $|\lambda_{i_j} - x_j| \le CN^{\xi-1}$ to obtain the bound

$$\begin{aligned}
&\int_T du\,dv\,dt \biggl| N^2 \sum_{\substack{i_1\in J_1\\ i_2\in J_2\backslash K_t}} \int_{\gamma_{i_1}}^{\gamma_{i_1+1}} d\mu_V(x_1) \int_{\gamma_{i_2}}^{\gamma_{i_2+1}} d\mu_V(x_2) \bigl\{ (\lambda_{i_1}-x_1)(\lambda_{i_2}-x_2)t(1-t) \\
&\quad \times \Xi^{-1}(f_N)^{(3)}\bigl(tv(\lambda_{i_1}-x_1)+ut(x_2-\lambda_{i_2})+u(\lambda_{i_2}-x_2)+t(x_1-x_2)+x_2\bigr)\bigr\}\biggr| \\
&\quad \le C\int_0^1 N^{3\alpha}\log N \times N^{2\xi-2} \times N^{2-2\alpha}t\,dt \le CN^{\alpha+2\xi}\log N.
\end{aligned}$$

Combining the bounds we have obtained gives

$$\left| \iint \frac{\Xi^{-1}f_N(x)-\Xi^{-1}f_N(y)}{x-y}\,dM_N^{(1)}(x)\,dM_N^{(2)}(y)\right| \le CN^{\alpha+2\xi}\log N,$$

which is $\omega(1)$ when divided by N for ξ small enough.

For $j_1 = 1$ *or* 2 *and* $j_2 = 3$. The proof is similar and we omit the details. □

Using Lemma 2.8 we also prove the following bounds:

Lemma 2.10. *The following estimates hold*:

$$L_N\bigl(\Xi^{-1}(f_N)'\bigr) = \omega(1), \tag{2.32}$$

$$L_N\bigl(\Xi^{-1}(f_N)f_N'\bigr) + \sigma_f^2 = \omega(1). \tag{2.33}$$

Proof. For both (2.32) and (2.33), we use

$$L_N\big(\Xi^{-1}(f_N)'\big)=\frac{M_N(\Xi^{-1}(f_N)')}{N}+\mu_V\big(\Xi^{-1}(f_N)'\big),$$
$$L_N\big(\Xi^{-1}(f_N)f_N'\big)=\frac{M_N(\Xi^{-1}(f_N)f_N')}{N}+\mu_V\big(\Xi^{-1}(f_N)f_N'\big).$$

Lemma 2.8 implies that the first term in both equations are $\omega(1)$ so (2.32) and (2.33) simplify to deterministic statements about the speed of convergence of the integrals against μ_V above.

To show (2.32), integration by parts yields:

$$\int\big(\Xi^{-1}f_N\big)'(x)\,d\mu_V(x)=-\int_a^b\big(\Xi^{-1}f_N\big)(x)\big(S'(x)\sigma(x)+S(x)\sigma'(x)\big)\,dx.$$

Inserting the formula for $\Xi^{-1}f_N$ we obtain

$$\left|\int\big(\Xi^{-1}f_N\big)'(x)\,d\mu_V(x)\right|\le\frac{1}{\beta\pi^2}\int_a^b\int_a^b\left|\frac{f_N(x)-f_N(y)}{y-x}\right|\left(\left|\frac{S'(x)\sigma(x)}{S(x)\sigma(y)}\right|+\left|\frac{\sigma'(x)}{\sigma(y)}\right|\right)dx\,dy.$$

Recall that S is bounded below on $[a,b]$, S' is bounded above on $[a,b]$, further, up to a constant, $\frac{\sigma'(x)}{\sigma(y)}$ can be bounded above by $(\sigma(x)\sigma(y))^{-1}$. We define the sets

$$A_N:=\big[N^\alpha(a-E);N^\alpha(b-E)\big],$$
$$B_N:=\left[\frac{1}{2}N^\alpha(a-E);\frac{1}{2}N^\alpha(b-E)\right].$$

By the observations above, and the change of variable $u=N^\alpha(x-E)$ and $v=N^\alpha(y-E)$ we get

$$\left|\int\big(\Xi^{-1}f_N\big)'(x)\,d\mu_V(x)\right|$$
$$\le\frac{C}{N^\alpha}\iint_{A_N^2}\left|\frac{f(u)-f(v)}{u-v}\right|\left(\frac{\sigma(E+\frac{u}{N^\alpha})}{\sigma(E+\frac{v}{N^\alpha})}+\frac{1}{\sigma(E+\frac{u}{N^\alpha})\sigma(E+\frac{v}{N^\alpha})}\right)du\,dv. \tag{2.34}$$

For large enough N, on the set $(u,v)\in(A_N\backslash B_N)^2$, the function $|f(u)-f(v)|$ is always zero, thus the integral on the right above can be divided into integrals over the sets:

$$(A_N\times A_N)\cap(A_N\backslash B_N\times A_N\backslash B_N)^c=B_N\times B_N\cup B_N\times(A_N\backslash B_N)\cup(A_N\backslash B_N)\times B_N. \tag{2.35}$$

We bound the integral in (2.3) over each set in (2.35). We begin with the first set in (2.35). For $(u,v)\in B_N\times B_N$, $\sigma(E+\frac{u}{N^\alpha})$ and $\sigma(E+\frac{v}{N^\alpha})$ are uniformly bounded above and below. Therefore, the integral in (2.3) can be bounded in this region by

$$\iint_{B_N^2}\left|\frac{f(u)-f(v)}{u-v}\right|du\,dv$$
$$=\iint_{[-M;M]^2}\left|\frac{f(u)-f(v)}{u-v}\right|du\,dv+2\int_{-M}^M\int_{B_N\cap\{|u|\ge M\}}\left|\frac{f(v)}{u-v}\right|du\,dv,$$

the integral over $[-M;M]^2$ exists by the differentiability of f, while:

$$\int_{-M}^M\int_{B_N\cap\{|u|\ge M\}}\left|\frac{f(v)}{u-v}\right|du\,dv\le C\int_{-M}^M\big|f(v)\big|\log\big[N|v+M||v-M|\big]\,dv\le C\log N,$$

for N large enough.

For the second set in (2.3), observe that for $(u,v)\in B_N\times(A_N\backslash B_N)$, $f(v)$ is 0 for N sufficiently large, and $\sigma(E+\frac{u}{N^\alpha})$ is bounded uniformly above and below while $f(u)$ is 0 outside $[-M;M]$. This implies that the integral in (2.3) can be bounded in this region by

$$\int_{A_N\backslash B_N}\int_{-M}^{M}\left|\frac{f(u)}{u-v}\right|\left(\frac{\sigma(E+\frac{u}{N^\alpha})}{\sigma(E+\frac{v}{N^\alpha})}+\frac{1}{\sigma(E+\frac{u}{N^\alpha})\sigma(E+\frac{v}{N^\alpha})}\right)du\,dv$$
$$\leq\frac{C\|f\|_{C(\mathbb{R})}}{N^\alpha}\int_{A_N\backslash B_N}\frac{1}{\sigma(E+\frac{v}{N^\alpha})}dv\leq C,$$

where in the final line we used $|u-v|\geq cN^\alpha$ for $u\in[-M;M]$, $v\in A_N\backslash B_N$.

We can do similarly for the third set in (2.3) and putting together these bounds on the right hand side of (2.3) gives

$$\left|\int\left(\Xi^{-1}f_N\right)'(x)\,d\mu_V(x)\right|\leq\frac{C\log N}{N^\alpha},$$

which is $\omega(1)$ as claimed.

We continue with (2.33). Recall that we reduced this problem to computing the limit of $\mu_V(\Xi^{-1}(f_N)f_N')$. Using the inversion formula we see that

$$\int\Xi^{-1}f_N(x)f_N'(x)\,d\mu_V(x)=-\frac{1}{\beta\pi^2}\int_a^b\int_a^b\frac{\sigma(x)f_N'(x)(f_N(x)-f_N(y))}{\sigma(y)(x-y)}\,dx\,dy.$$

Observe that

$$\frac{1}{2}\partial_x\left(f_N(x)-f_N(y)\right)^2=f_N'(x)\left(f_N(x)-f_N(y)\right),$$
$$\partial_x\left(\frac{\sigma(x)}{x-y}\right)=\frac{-\frac{1}{2}(a+b)(x+y)+ab+xy}{\sigma(x)(x-y)^2}.$$

Therefore, integration by parts yields

$$\int\Xi^{-1}f_N(x)f_N'(x)\,d\mu_V(x)=-\frac{1}{2\beta\pi^2}\int_a^b\int_a^b\frac{\sigma(x)\partial_x(f_N(x)-f_N(y))^2}{\sigma(y)(x-y)}\,dx\,dy$$
$$=\frac{1}{2\beta\pi^2}\int_a^b\int_a^b\left(\frac{f_N(x)-f_N(y)}{x-y}\right)^2\left(\frac{ab+xy-\frac{1}{2}(a+b)(x+y)}{\sigma(x)\sigma(y)}\right)dx\,dy.$$

By changing variables again to $(u,v)=(N^\alpha(x-E),N^\alpha(y-E))$ and observing that

$$ab+xy-\frac{1}{2}(a+b)(x+y)=-\sigma(E)^2+\frac{u+v}{N^\alpha}\left(\frac{a+b}{2}+E\right)+\frac{uv}{N^{2\alpha}},$$

we obtain

$$\int\Xi^{-1}f_N(x)f_N'(x)\,d\mu_V(x)$$
$$=-\frac{1}{2\beta\pi^2}\iint_{A_N^2}\left(\frac{f(u)-f(v)}{u-v}\right)^2\left(\frac{\sigma(E)^2-\frac{u+v}{N^\alpha}(\frac{a+b}{2}+E)-\frac{uv}{N^{2\alpha}}}{\sigma(E+\frac{u}{N^\alpha})\sigma(E+\frac{v}{N^\alpha})}\right)du\,dv. \tag{2.36}$$

As before, $(f(u)-f(v))^2$ is zero for all $(u,v)\in(A_N\backslash B_N)^2$ for large enough N, therefore we split the above integral into the regions defined in (2.35).

Notice that uniformly in $u \in B_N$

$$\frac{1}{\sigma(E+\frac{u}{N^\alpha})} = \frac{1}{\sigma(E)} + O\left(\frac{|u|}{N^\alpha}\right),$$

and further notice $(u+v)/N^\alpha$ and $uv/N^{2\alpha}$ are bounded uniformly by constants in the entire region $A_N \times A_N$.

Consequently the integral (2.3) over the region $B_N \times B_N$ is:

$$\begin{aligned}
&\iint_{B_N^2} \left(\frac{f(u)-f(v)}{u-v}\right)^2 \left(\frac{\sigma(E)^2 - \frac{u+v}{N^\alpha}(\frac{a+b}{2}+E) - \frac{uv}{N^{2\alpha}}}{\sigma(E+\frac{u}{N^\alpha})\sigma(E+\frac{v}{N^\alpha})}\right) du\,dv \\
&\quad = \iint_{B_N^2} \left(\frac{f(u)-f(v)}{u-v}\right)^2 du\,dv + O\left(\frac{1}{N^\alpha}\iint_{B_N^2}\left(\frac{f(u)-f(v)}{u-v}\right)^2 \big(|u|+|v|\big)\,du\,dv\right).
\end{aligned} \tag{2.37}$$

The first term of (2.3) is equal to,

$$\frac{1}{2\beta\pi^2}\iint \left(\frac{f(u)-f(v)}{u-v}\right)^2 du\,dv + O\left(\frac{1}{N^\alpha}\right)$$

while the second term in (2.3) can be written as

$$\begin{aligned}
\iint_{B_N^2}\left(\frac{f(u)-f(v)}{u-v}\right)^2 \big(|u|+|v|\big)\,du\,dv &= \iint_{[-M;M]^2}\left(\frac{f(u)-f(v)}{u-v}\right)^2\big(|u|+|v|\big)\,du\,dv \\
&\quad + 2\int_{-M}^{M}\int_{B_N\cap\{|u|\ge M\}}\left(\frac{f(v)}{u-v}\right)^2\big(|u|+|v|\big)\,du\,dv,
\end{aligned}$$

the integral over $[-M;M]^2$ is finite by differentiability of f while the second is bounded by

$$\begin{aligned}
&\int_{-M}^{M}\int_{B_N\cap\{|u|\ge M\}} \big|f(v)\big|^2\left(\frac{1}{|u-v|} + \frac{2|v|}{|u-v|^2}\right) du\,dv \\
&\quad \le C\int_{-M}^{M}\big|f(v)\big|^2\left(\frac{1}{|v-M|} + \frac{1}{|M+v|} + \log\big[N|v-M||v+M|\big]\right) \le C\log N
\end{aligned}$$

since $\operatorname{supp} f \subset [-M, M]$.

In the region $(u, v) \in B_N \times (A_N \backslash B_N)$, $\sigma(E+\frac{u}{N^\alpha})$ is bounded above and below while, for N large enough $f(v) = 0$, thus the integral over $B_N \times (A_N\backslash B_N)$ is bounded above by

$$\begin{aligned}
&\int_{A_N\backslash B_N}\int_{B_N}\left(\frac{f(u)-f(v)}{u-v}\right)^2\left(\frac{1}{\sigma(E+\frac{u}{N^\alpha})\sigma(E+\frac{v}{N^\alpha})}\right) du\,dv \\
&\quad \le \int_{A_N\backslash B_N}\int_{-M}^{M}\left(\frac{f(u)}{u-v}\right)^2 \frac{1}{\sigma(E+\frac{v}{N^\alpha})}\,du\,dv \le \frac{C}{N^{2\alpha}}\int_{A_N\backslash B_N}\frac{1}{\sigma(E+\frac{v}{N^\alpha})}\,dv \le \frac{C}{N^\alpha},
\end{aligned}$$

where in the second line we used $|u-v| \ge cN^\alpha$ for $u \in [-M; M]$ and $v \in A_N\backslash B$. By symmetry of the integrand in (2.3) this argument extends to the region $(u, v) \in (A_N\backslash B_N) \times B_N$.

Altogether, our bounds show

$$\int \Xi^{-1} f_N(x) f_N'(x)\,d\mu_V(x) = -\frac{1}{2\beta\pi^2}\iint\left(\frac{f(x)-f(y)}{x-y}\right)^2 dx\,dy + O\left(\frac{\log N}{N^\alpha}\right),$$

which shows (2.33). □

2.4. *Proof of Theorem* 1.5

We proceed with the proof of Theorem 1.5. As we did in the sketch of the proof, (2.6) applied to $h = \Xi^{-1}(f_N)$ yields

$$F_1^N\big(\Xi^{-1}(f_N)\big) = M_N(f_N) + \left(1 - \frac{\beta}{2}\right) L_N\big(\big(\Xi^{-1} f_N\big)'\big) + \frac{1}{N}\left[\frac{\beta}{2} \iint \frac{\Xi^{-1} f_N(x) - \Xi^{-1} f_N(y)}{x - y} \, dM_N(x) \, dM_N(y)\right].$$

Combining Lemma 2.9 Eq. 2.25 and Lemma 2.10 Eq. (2.32) we can bound the two terms on the right hand side to get

$$F_1^N\big(\Xi^{-1}(f_N)\big) = M_N(f_N) + \omega(1). \tag{2.38}$$

We consider an event A_1 of probability higher than $1 - e^{-N^c}$ on which

$$\big|F_1^N\big(\Xi^{-1}(f_N)\big) - M_N(f_N)\big| \le \frac{C}{N^\delta}, \tag{2.39}$$

for some positive constants c, C and δ. Using the first loop equation from Proposition 2.1, and the trivial deterministic bounds

$$M_N(f_N) = O\big(N\|f\|_\infty\big), \qquad F_1^N\big(\Xi^{-1}(f_N)\big) = O\big(N\big(\|f\|_\infty + \big\|\Xi^{-1}(f_N)\big\|_{C^1(\mathbb{R})}\big)\big) = O\big(N^3\big),$$

we obtain

$$\begin{aligned} 0 &= \mathbb{E}_V^N\big(F_1^N\big(\Xi^{-1}(f_N)\big)\big) = \mathbb{E}_V^N\big(F_1^N\big(\Xi^{-1}(f_N)\big)\mathbb{1}_{A_1}\big) + \mathbb{E}_V^N\big(F_1^N\big(\Xi^{-1}(f_N)\big)\mathbb{1}_{A_1^c}\big) \\ &= \mathbb{E}_V^N\big(M_N(f_N)\mathbb{1}_{A_1}\big) + o(1) + O\big(N^3 \mathbb{P}_V^N\big(A_1^c\big)\big) \\ &= \mathbb{E}_V^N\big(M_N(f_N)\big) + o(1), \end{aligned} \tag{2.40}$$

and thus

$$\mathbb{E}_V^N\big(M_N(f_N)\big) = o(1). \tag{2.41}$$

We now show recursively that

$$F_k^N\big(\Xi^{-1}(f_N), f_N, \dots, f_N\big) = \tilde{M}_N(f_N)^k - (k-1)\sigma_f^2 \tilde{M}_N(f_N)^{k-2} + \omega(1). \tag{2.42}$$

Here, the set on which the bound holds might vary from one k to another but each bound has probability greater than $1 - e^{-N^{c_k}}$ for each fixed k.

The bound holds for $k = 1$, by (2.38). Now, assume this holds for $k \ge 1$. On a set of probability greater than $1 - e^{-N^{c_{k+1}}}$ we have by the induction hypothesis, Lemma 2.8 Eq. (2.21) and Lemma 2.10 Eq. (2.33), for some $\delta > 0$ and a constant C

$$\begin{aligned} &\big|F_k^N\big(\Xi^{-1}(f_N), f_N, \dots, f_N\big) - \tilde{M}_N(f_N)^k + (k-1)\sigma_f^2 \tilde{M}_N(f_N)^{k-2}\big| \le \frac{C}{N^\delta}, \\ &\big|L_N\big(\Xi^{-1}(f_N) f_N'\big) + \sigma_f^2\big| \le \frac{C}{N^\delta}, \\ &\big|M_N(f_N)\big| \le N^{\delta/2k}. \end{aligned}$$

On this set, using the definition of F_{k+1}^N from Proposition 2.1,

$$F_{k+1}^N\big(\Xi^{-1}(f_N), f_N, \dots, f_N\big) = F_k^N\big(\Xi^{-1}(f_N), f_N, \dots, f_N\big)\tilde{M}_N(f_N) + \tilde{M}_N(f_N)^{k-1} L_N\big(\Xi^{-1}(f_N) f_N'\big)$$

$$= \left(\tilde{M}_N(f_N)^k - (k-1)\sigma_f^2 \tilde{M}_N(f_N)^{k-2} + O\left(\frac{1}{N^\delta}\right)\right)\tilde{M}_N(f_N)$$
$$+ \tilde{M}_N(f_N)^{k-1}\left(-\sigma_f^2 + O\left(\frac{1}{N^\delta}\right)\right)$$
$$= \tilde{M}_N(f_N)^{k+1} - k\sigma_f^2 \tilde{M}_N(f_N)^{k-1} + O\left(\frac{1}{N^{\delta/2}}\right)$$

and this proves the induction. Using the fact that F_k is bounded polynomially and deterministically as before, we see that for any $k \geq 1$

$$\mathbb{E}_V^N\left(M_N(f_N)^{k+1}\right) = \sigma_f^2 k \mathbb{E}_V^N\left(M_N(f_N)^{k-1}\right) + o(1). \tag{2.43}$$

Coupled with (2.41), the computation of the moments is then straightforward and we obtain for all $k \in \mathbb{N}$

$$\begin{aligned} &\mathbb{E}_V^N\left(M_N(f_N)^{2k}\right) = \sigma_f^{2k}\frac{(2k)!}{2^k k!} + o(1), \\ &\mathbb{E}_V^N\left(M_N(f_N)^{2k+1}\right) = o(1). \end{aligned} \tag{2.44}$$

This concludes the proof of Theorem 1.5.

2.5. *A few remarks*

The result of Theorem 1.5 naturally extends to the joint law of the fluctuations of finite families. More precisely, for any fixed k, if $f^1, \dots, f^k$ satisfy the hypothesis of the theorem then $(M_N(f_N^1), \dots, M_N(f_N^k))$ converges in distribution towards a centered Gaussian vector with covariance matrix

$$\Sigma_{i,j} = \frac{1}{2\beta\pi^2}\iint \left(\frac{f^i(x) - f^i(x)}{x-y}\right)\left(\frac{f^j(x) - f^j(x)}{x-y}\right) dx\, dy.$$

We would also like to point out that a similar proof should also yield the macroscopic central limit Theorem already shown in [6,16,24] (one-cut and off-critical cases) with appropriate decay conditions on f. Indeed, in the macroscopic case we get uniform bounds on $\Xi^{-1} f$ and its derivatives instead of the bounds obtained in Lemma 2.5. The major issue when dealing with the multicut and critical cases is that the operator Ξ is not invertible (as an operator acting on smooth functions). When dealing with functions that lie in the image of Ξ and with additional regularity assumptions, one can show using transport methods similar to [18] that the central limit Theorem do hold at the macroscopic scale. This is the object of a future work [5].

Another interesting direction would be to study the fluctuations at the edge (i.e, $E = a$ or b). We expect the same result to hold with covariance matrix(if for instance $E = a$) equal to

$$\Sigma_{i,j} = \frac{1}{2\beta\pi^2}\int_0^\infty \int_0^\infty \left(\frac{f^i(x) - f^i(y)}{x-y}\right)\left(\frac{f^j(x) - f^j(y)}{x-y}\right)\frac{x+y}{\sqrt{xy}}\, dx\, dy.$$

Additional technical estimates as in Lemma 2.5 would be needed to reproduce the proof in the edge case. These estimates are not straightforward because of the singular behaviour of $\Xi^{-1}(f_N)$ at the edges and this is the subject of a future work. However the covariance (in the case $k = 1$) would still be given by taking the limit of $-\mu_V(\Xi^{-1}(f_N) f_N')$ as in Lemma 2.10, which yields the above formula.

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