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Convergence to equilibrium in the free Fokker–Planck equation with a double-well potential

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Abstract. We consider the one-dimensional free Fokker–Planck equation

$$\frac{\partial \mu_t}{\partial t} = \frac{\partial}{\partial x}\left[\mu_t \cdot \left(\frac{1}{2}V' - H\mu_t\right)\right],$$

where H denotes the Hilbert transform and V is a particular double-well quartic potential, namely $V(x) = \frac{1}{4}x^4 + \frac{c}{2}x^2$, with $c \geq -2$. We prove that the solution $(\mu_t)_{t\geq 0}$ of this PDE converges in Wasserstein distance of any order $p \geq 1$ to the equilibrium measure μ_V as t goes to infinity. This provides a first result of convergence for this equation in a non-convex setting. The proof involves free probability and complex analysis techniques.

Résumé. On considère l'équation de Fokker–Planck libre unidimensionnelle

$$\frac{\partial \mu_t}{\partial t} = \frac{\partial}{\partial x}\left[\mu_t \cdot \left(\frac{1}{2}V' - H\mu_t\right)\right],$$

où H désigne la transformée de Hilbert et V est un potentiel quartique à double puits particulier, à savoir $V(x) = \frac{1}{4}x^4 + \frac{c}{2}x^2$ avec $c \geq -2$. On démontre que la solution $(\mu_t)_{t\geq 0}$ de cette EDP converge pour une distance de Wasserstein d'ordre quelconque $p \geq 1$ vers la mesure d'équilibre μ_V quand t tend vers l'infini. Cela fournit un premier résultat de convergence pour cette équation dans un cadre non convexe. La démonstration fait intervenir les probabilités libres et l'analyse complexe.

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Keywords: Fokker–Planck equation; Granular media equation; Long-time behaviour; Double-well potential; Free probability; Equilibrium measure; Random matrices

1. Introduction

We consider the following *one-dimensional free Fokker–Planck equation*

$$\frac{\partial \mu_t}{\partial t} = \frac{\partial}{\partial x}\left[\mu_t \cdot \left(\frac{1}{2}V' - H\mu_t\right)\right]. \tag{1}$$

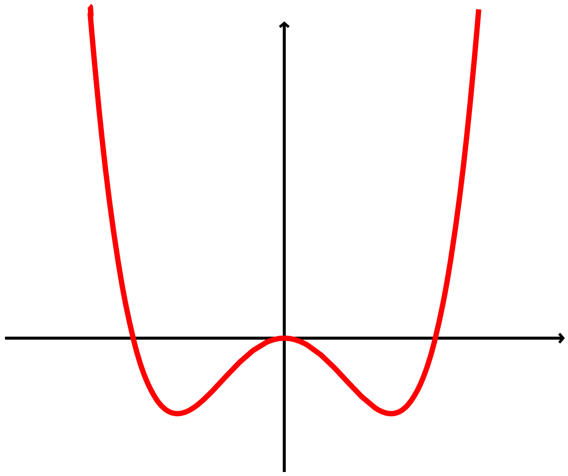

Fig. 1. Potential V defined by (3) with $-2 \le c < 0$.

In this equation, $(\mu_t)_{t\ge 0}$ denotes a family of probability measures on $\mathbb{R}$, $V:\mathbb{R}\to\mathbb{R}$ is a given potential, and H denotes the Hilbert transform, that is, for any probability measure μ on $\mathbb{R}$ and $x\in\mathbb{R}$,

$$H\mu(x) = ⨍_{\mathbb{R}} \frac{1}{x-y}\,\mathrm{d}\mu(y) := \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}\setminus[x-\varepsilon,x+\varepsilon]} \frac{1}{x-y}\,\mathrm{d}\mu(y),$$

where $⨍$ stands for the principal value of the integral. Partial differential equation (PDE) (1) must be understood in the sense of distributions, i.e. for any regular enough test function $\varphi:\mathbb{R}\to\mathbb{R}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \varphi(x)\,\mathrm{d}\mu_t(x) = -\frac{1}{2}\int V'(x)\varphi'(x)\,\mathrm{d}\mu_t(x) + \frac{1}{2}\iint \frac{\varphi'(x)-\varphi'(y)}{x-y}\,\mathrm{d}\mu_t(x)\,\mathrm{d}\mu_t(y).$$

Under this form, it is sometimes called the *McKean–Vlasov equation with logarithmic interaction*.

1.1. *Existence and uniqueness*

As far as we know, the problems of existence and uniqueness of the solution to the PDE (1) are not completely solved.

Existence was tackled by Biane and Speicher [8, Theorem 3.1]. Using the free stochastic calculus formalism (see [6,7] for an introduction), they proved that if, roughly speaking, V is locally Lipschitz and grows “nicely” at infinity then, for any initial condition X_0 whose distribution is compactly supported, the free stochastic differential equation (SDE)

$$\mathrm{d}X_t = \mathrm{d}S_t - \frac{1}{2}V'(X_t)\,\mathrm{d}t, \tag{2}$$

where S is a free Brownian motion, admits a unique solution $(X_t)_{t\ge 0}$ starting from X_0. As they also checked that the distribution of the solution $(X_t)_{t\ge 0}$ satisfies the PDE (1), this proves the existence of a solution for the latter.

As for uniqueness, using free transportation techniques, Li, Li, and Xie [25, Theorem 1.3] showed that the free Fokker–Planck equation (1) admits a unique solution starting from a compactly supported μ_0 as soon as V satisfies the same properties as in [8, Theorem 3.1] and V'' is uniformly bounded below.

In this paper, we are interested in the free Fokker–Planck equation (1) for the particular potential (see Figure 1)

$$V(x) = \frac{1}{4}x^4 + \frac{c}{2}x^2, \quad c \ge -2. \tag{3}$$

Indeed, the quadratic potential $V(x) = \frac{x^2}{2}$ gives rise to the *free Ornstein–Uhlenbeck process* and is well understood; the quartic potential (3) is then the simplest example of a potential satisfying the assumptions of [8, Theorem 3.1] and

[25, Theorem 1.3]. Consequently, given any compactly supported initial condition, existence and uniqueness of the solution to Equation (1) are ensured and we can moreover identify this solution with the distribution of the solution $(X_t)_{t\geq 0}$ to the free SDE (2).

Note that the case when $c \geq 0$ is already covered in the existing literature, and the aim of this paper is to extend the study of the asymptotic behaviour of the solution $(\mu_t)_{t\geq 0}$ to the range $c \geq -2$ (see also [25, Conjecture 5.1]).

1.2. *Granular media equation*

We say that a family $(\mu_t)_{t\geq 0}$ of probability measures on $\mathbb{R}^d$ having densities $(\rho_t)_{t\geq 0}$ satisfies a *granular media equation* if we have in distribution

$$\frac{\partial \rho_t}{\partial t} = \nabla \cdot \big[\rho_t \nabla\big(\mathcal{U}'(\rho_t) + \mathcal{V} + \mathcal{W} * \rho_t\big)\big], \tag{4}$$

where $\mathcal{U} : \mathbb{R}_+ \to \mathbb{R}$ can be seen as an internal energy, $\mathcal{V} : \mathbb{R}^d \to \mathbb{R}$ as a confinement potential, $\mathcal{W} : \mathbb{R}^d \to \mathbb{R}$ as an interaction potential, and where the operation $*$ is the usual convolution in $\mathbb{R}^d$.

The free Fokker–Planck equation (1) corresponds to the particular case

$$d = 1, \qquad \mathcal{U}(s) = 0, \qquad \mathcal{V}(x) = \frac{1}{2}V(x), \quad \text{and} \quad \mathcal{W}(x) = -\log|x|.$$

Several classical partial differential equations arising from physics fall into the class of granular media equations (see [34, Chapter 9.1]), starting from the heat equation (for $\mathcal{U}(s) = s\log(s)$, $\mathcal{V} = 0$, $\mathcal{W} = 0$). Conditions are known to ensure that Equation (4) admits a unique solution, but we will not discuss this point here and rather focus on reviewing some existing results about the long-time behaviour of the solutions.

At least two main techniques can be identified. We focus on the *entropy dissipation method*, which is close to the techniques we will use in this work, but we will also briefly mention some results using an *approximation by a particle system*.

The first results using entropy dissipation are due to Benedetto et al. [3,4], who are interested in particular potentials arising from physics. In these works, to study the long-time behaviour of the solution, the authors consider an entropy functional naturally associated to (4) defined by

$$F(\mu) = \int_{\mathbb{R}^d} \mathcal{U}\big(\mu(x)\big)\,\mathrm{d}x + \int_{\mathbb{R}^d} \mathcal{V}(x)\,\mathrm{d}\mu(x) + \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathcal{W}(x-y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y),$$

as the sum of an internal energy, a potential energy, and an interaction energy associated to a given measure μ, and they show that this entropy is strictly decreasing along the trajectory $(\mu_t)_{t\geq 0}$. Under appropriate assumptions on $\mathcal{V}$ and $\mathcal{W}$, F admits a unique minimizer μ_∞, which is shown to be the limit of μ_t as $t \to +\infty$.

Combining this entropy dissipation method with optimal transport techniques, Carrillo et al. [14] establish convergence of μ_t in a more general setting, even leading in some cases to explicit rates of convergence. Note that Cattiaux et al. [15] and Bolley et al. [9–11] got various improvements of these results. Nevertheless, all these results require convexity, positivity, or smoothness assumptions on $\mathcal{V}$ and $\mathcal{W}$.

In a series of works, Tugaut [32,33] then tackled the problem of non-convex potentials in the case when $\mathcal{U}(s) = \sigma s\log(s)$ for a small $\sigma > 0$ but his results still require a smooth interaction $\mathcal{W}$.

An example of physically meaningful singular interaction is given by $\mathcal{W}(x) = -\log|x|$, which is out of reach of the previous methods as they are. This problem of the logarithmic interaction in dimension one has been recently tackled by Li, Li, and Xie [25], who could adapt Carrillo, McCann, and Villani's method to the free probability framework, at least in the case of a convex potential $\mathcal{V}$. We also mention the recent work by Carrillo et al. [13], in which a two-dimensional logarithmic interaction is considered, corresponding to the Keller–Segel model.

In view of these results, the study of the long-time behaviour of the solution of the granular media equation with a logarithmic interaction and a non-convex potential, such as (3), is of natural interest.

Another motivation to study the granular media equation with a logarithmic interaction is its link with a particle system which is well known in random matrix theory.

For any $N \geq 1$, we consider the system of stochastic differential equations (SDEs)

$$\forall i \in [\![1, N]\!], \quad \mathrm{d}X_i^N(t) = \sqrt{2}\,\mathrm{d}B_i(t) - \nabla\mathcal{V}\big(X_i^N(t)\big)\,\mathrm{d}t - \frac{1}{N}\sum_{j \neq i} \nabla\mathcal{W}\big(X_i^N(t) - X_j^N(t)\big)\,\mathrm{d}t, \tag{5}$$

where the B_i's are independent Brownian motions. The solution $(X_1^N(t), \ldots, X_N^N(t))_{t \geq 0}$ of (5) is a natural particle system that can be associated to PDE (4).

Indeed, as the number of particles N goes to infinity, its empirical measure $(\frac{1}{N}\sum_{i=1}^N \delta_{X_i^N(t)})_{t\geq 0}$ converges to a solution of PDE (4), the Brownian term giving rise to an internal energy $\mathcal{U}(s) = s\log(s)$. If the Brownian term in (5) is multiplied by $\frac{1}{\sqrt{N}}$, it disappears in the limit and we get the solution of (4) with $\mathcal{U}(s) = 0$.

Using propagation of chaos for a modified approximating particle system, Malrieu [26] recovered some of the results of [14]. Bolley et al. [11] and Cattiaux et al. [15] also considered a particle system to prove convergence of the solution to the PDE they study.

In the case of a logarithmic interaction ($\mathcal{U}(s) = 0$, $\mathcal{V}(x) = 0$, and $\mathcal{W}(x) = -\log|x|$), the particle system (5) is the well-known *Dyson Brownian motion* introduced in [19] as the process of eigenvalues of Hermitian random matrices with Brownian entries. This process (or its variant when $\mathcal{V}$ is quadratic) has been much studied, among others in [16,17,20,29]; see also [2, Section 4.3]. For more general potentials $\mathcal{V}$, this particle system has been studied by Allez and Dumaz [1] in the cubic case and by Li et al. [25] in the convex case.

With this point of view, the reader who is familiar with random matrix theory will get an insight why a natural candidate for the long-time limit of the solution to the free Fokker–Planck equation (1) should be the equilibrium measure associated to potential V, that we now define.

1.3. *Main result of the paper*

Let D be a closed subset of $\mathbb{C}$ and $V : D \to \mathbb{C}$ be a polynomial such that

$$\lim_{|z|\to+\infty, z\in D} \operatorname{Re} V(z) - 2\log|z| = +\infty.$$

Then the functional

$$\Sigma_V : \mu \mapsto -\iint_{D^2} \log|z-t|\,\mathrm{d}\mu(z)\,\mathrm{d}\mu(t) + \int_D \operatorname{Re} V(z)\,\mathrm{d}\mu(z), \tag{6}$$

called *Voiculescu free entropy*, admits a unique minimizer among probability measures supported on D. This minimizer is called the *equilibrium measure* associated to V and D, and is denoted by μ_V. Note that when $D \subset \mathbb{R}$ and V is real-valued, we have

$$\Sigma_V(\mu) = -\iint_{\mathbb{R}^2} \log|x-y|\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) + \int_{\mathbb{R}} V(x)\,\mathrm{d}\mu(x).$$

We refer to [30] for a development on this topic for which the equilibrium measure is defined in a much more general setting.

For the quartic potential

$$V(x) = \frac{1}{4}x^4 + \frac{c}{2}x^2$$

and $D = \mathbb{R}$, the equilibrium measure is explicitly known (see [23, Example 3.2] for instance):

- when $c \geq -2$, its density is given by

$$\rho_V(x) = \frac{1}{\pi}\left(\frac{1}{2}x^2 + b_0\right)\sqrt{a^2 - x^2}\,\mathbf{1}_{[-a,a]}(x) \tag{7}$$

where

$$a^2 = \frac{2}{3}\big(\sqrt{c^2+12} - c\big), \qquad b_0 = \frac{1}{3}\left(c + \sqrt{\frac{c^2}{4}+3}\right),$$

- when $c < -2$, its density is given by

$$\rho_V(x) = \frac{1}{2\pi}|x|\sqrt{\big(x^2-a^2\big)\big(b^2-x^2\big)}\mathbf{1}_{[-b,-a]\cup[a,b]}(x), \tag{8}$$

where $a^2 = -2-c$, $b^2 = 2-c$.

In this paper, we focus on the case when $c \geq -2$, in which the equilibrium measure has connected support. Here is the main result of this paper.

For any real $p \geq 1$, if μ and ν are probability measures on $\mathbb{R}$ such that $|\cdot|^p$ is integrable for μ and ν, then the *Wasserstein distance* of order p between μ and ν is defined by

$$W_p(\mu,\nu) = \left(\inf \iint |x-y|^p \pi(\mathrm{d}x,\mathrm{d}y)\right)^{1/p},$$

where the infimum runs over all probability measures π on $\mathbb{R}\times\mathbb{R}$ with marginal distributions μ and ν.

Theorem 1.1. *Let $V(x) = \frac{1}{4}x^4 + \frac{c}{2}x^2$ with $c \geq -2$. Given any compactly supported probability measure μ_0 on $\mathbb{R}$, the solution $(\mu_t)_{t\geq 0}$ of the free Fokker–Planck equation (1) with initial condition μ_0 satisfies*

$$\lim_{t\to+\infty} W_p(\mu_t,\mu_V) = 0$$

for all $p \geq 1$, where μ_V is given by (7).

Let us mention that the convergence in W_p distance is equivalent to the weak convergence of measures together with the convergence of the moments of order p (see for instance [34, Theorem 7.12]).

The case when $c \geq 0$ was already covered by previous results of Li, Li, and Xie [25]. Indeed, in [25, Theorem 1.6 (i) and (ii)], they proved that

$$\lim_{t\to+\infty} W_2(\mu_t,\mu_V) = 0$$

as soon as V is convex (in the case when V is strictly convex, they even get that $t \mapsto W_2(\mu_t,\mu_V)$ exponentially decays to 0). They provide a proof for convergence in W_p, $p \leq 2$, that could be easily extended to any W_p, $p > 2$. We also refer to [21, Section 6.4] for some complements.

On the other hand, a result of [8, Section 7.1] implies that, if $c < 0$ and $|c|$ is large enough, then there exist initial conditions μ_0 for which the solution $(\mu_t)_{t\geq 0}$ does not converge towards the equilibrium measure μ_V.

The rest of the paper is organized as follows. Some tools, such as properties of $(\mu_t)_{t\geq 0}$ viewed as the law of a free diffusion and the description of critical measures via complex analysis techniques, are introduced in Section 2, and Section 3 uses these tools to prove Theorem 1.1. Section 4 is the final section of this paper, in which we present some perspectives for future work.

2. Free probability and complex analysis tools

2.1. *Some properties of the solution of the free Fokker–Planck equation*

As we explained in the introduction of the paper, the solution of the free Fokker–Planck equation (1) can be interpreted as the distribution of the solution to the free SDE

$$\mathrm{d}X_t = \mathrm{d}S_t - \frac{1}{2}V'(X_t)\,\mathrm{d}t,$$

where S is a free Brownian motion. As a consequence, it inherits some properties of free diffusions with regular drift, studied by Biane and Speicher [8], the most important of which are the following.

Proposition 2.1 (See [8, Theorems 3.1 and 5.2]). *Let V be a $\mathcal{C}^1$ potential such that V' is locally Lipschitz, and such that there exist $a<0$ and $b>0$ such that, for all $x \in \mathbb{R}$,*

$$-xV'(x) \leq ax^2 + b. \tag{9}$$

Let $(\mu_t)_{t\geq 0}$ be the solution of the free Fokker–Planck equation (1) starting from a compactly supported μ_0.

(i) *There exists $M>0$ such that, for every $t>0$,*

$$\operatorname{supp}(\mu_t) \subset [-M, M]. \tag{10}$$

(ii) *There exist $K_1, K_2 > 0$ depending only on V such that, for every $t>0$, the density ρ_t of μ_t satisfies*

$$\|\rho_t\|_\infty \leq \frac{K_1}{\sqrt{t}} + K_2, \qquad \left\|D^{1/2}\rho_t\right\|_2 \leq \frac{K_1}{t} + K_2, \tag{11}$$

where $D^{1/2}$ is the fractional derivative of order $1/2$.

(iii) *The family $\{\rho_t\}_{t\geq 1}$ lives in a subset $\mathcal{A}$ of $L^2([-M, M])$ which is compact for the topology induced by $\|\cdot\|_2$.*

In the statement of Point (ii), the notion of half-derivative appears. It can be defined in several ways; we will just mention that for $u \in L^2$, the derivative of order $1/2$ of u is the inverse Fourier transform of $\xi \mapsto (1+\xi^2)^{1/4}\hat{u}(\xi)$, where $\hat{u}$ is the Fourier transform of u (see [18, Chapter 4] for instance).

We notice that the potential (3) satisfies the assumptions of Proposition 2.1. Nevertheless, we include here a proof of Point (iii), since it will play a key role in extending the entropy dissipation method to this singular interaction, as we will explain at the beginning of Section 3.

Proof of Proposition 2.1(iii). By Proposition 2.1(i)–(ii), there exist $M, K_1, K_2 > 0$ such that, for every $t>0$, (10) and (11) hold. For every $t>0$, we denote by $\mathcal{A}_t$ the set of probability density functions f with support in $[-M, M]$ which satisfy $\|f\|_\infty \leq \frac{K_1}{\sqrt{t}} + K_2$ and $\|D^{1/2}f\|_2 \leq \frac{K_1}{t} + K_2$. Note that, for $t>0$, $\mathcal{A}_t$ contains all the ρ_{t+s}'s, $s \geq 0$, where ρ_{t+s} denotes the density of the measure μ_{t+s} as in Point (ii).

Furthermore, for every $t>0$, $\mathcal{A}_t$ is a subset of the Sobolev space $H^{1/2}([-M, M])$, defined as the set of L^2-probability density functions whose derivative of order $1/2$ belongs to L^2. Because the injection of $H^{1/2}([-M, M])$ in $L^p([-M, M])$ is compact for every $p \in [1, \infty)$ (see [18, Theorem 4.54] for instance) and $\mathcal{A}_t$ is bounded in $H^{1/2}([-M, M])$, we can deduce that the set $\mathcal{A}_t$ is relatively compact in $L^2([-M, M])$. Hence, we can choose for $\mathcal{A}$ the closure of $\mathcal{A}_1$ in $L^2([-M, M])$. □

Furthermore, even if we consider here a singular interaction in granular media equation (4), thanks to the bounds stated in Proposition 2.1, we have an entropy dissipation formula as in [14].

Proposition 2.2 (See [8, Proposition 6.1]). *Under the assumptions and notations of Proposition 2.1, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Sigma_V(\mu_t) = -2\int\left|\frac{1}{2}V' - H\mu_t\right|^2 \mathrm{d}\mu_t. \tag{12}$$

As this formula suggests, probability measures μ supported in $\mathbb{R}$ satisfying the *Euler–Lagrange equation*

$$H\mu = \frac{1}{2}V' \quad \mu\text{-a.e.} \tag{13}$$

for a real potential V are the candidates for the long-time limit of μ_t. We will call these measures *stationary measures* because they are exactly the stationary solutions of Equation (1) in the sense of PDEs, i.e. they are solutions of Equation (1) that are constant in time.

2.2. *Critical measures and their identification*

In addition to the equilibrium measure and stationary measures that we encounter in our problem, we introduce the notion of critical measure, as defined by Martínez-Finkelshtein and Rakhmanov [27].

A probability measure μ on $\mathbb{C}$ such that $\Sigma_V(\mu) < +\infty$ is called a *critical measure* associated to V if, for every $h : \mathbb{C} \to \mathbb{C}$ regular enough, the quantity

$$D_h \Sigma_V(\mu) = \lim_{s\to 0} \frac{\Sigma_V(\mu^{h,s}) - \Sigma_V(\mu)}{s}$$

is zero, where $\mu^{h,s}$ is the push-forward measure of μ by the deformation of identity $z \mapsto z + sh(z)$, $s \in \mathbb{C}$.

The reason why we consider critical measures is that, by [27, Lemma 3.7], we have

$$D_h \Sigma_V(\mu) = \mathrm{Re}\left(\int V'(x)h(x)\,\mathrm{d}\mu(x) - \iint \frac{h(x)-h(y)}{x-y}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\right),$$

hence for a probability measure μ supported on $\mathbb{R}$, the previous condition is equivalent to the Euler–Lagrange equation (13). As a result, critical measures supported on $\mathbb{R}$ are exactly stationary measures, and we will be able to use some tools developed to identify critical measures in order to identify stationary measures.

Note that in general, several critical measures may exist while there is only one equilibrium measure. This is the case for a potential satisfying the conditions given in [8, Section 7.1] for instance. A key point in the proof of Theorem 1.1 will be to show that for the quartic potential (3), there is no other critical measure than the equilibrium measure.

The following statement gives the most important properties of critical measures supported in $\mathbb{R}$ we will use in the sequel. The key point is that the Stieltjes transform of a critical measure μ, defined on $\mathbb{C} \setminus \mathbb{R}$ by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-x}\,\mathrm{d}\mu(x),$$

satisfies an algebraic equation.

Proposition 2.3 (See [22,24]). *Let V be a polynomial and μ be a critical measure supported on $\mathbb{R}$.*

(i) *There exists a polynomial R of degree $2\deg(V) - 2$ such that*

$$R(z) = \left(\frac{1}{2}V'(z) - G_\mu(z)\right)^2 \tag{14}$$

almost everywhere for Lebesgue measure on $\mathbb{C}$. Moreover, we have

$$R(z) = \frac{1}{4}V'(z)^2 - \int_{\mathbb{R}} \frac{V'(x)-V'(z)}{x-z}\,\mathrm{d}\mu(x). \tag{15}$$

(ii) *Every non-real root of R has even multiplicity.*
(iii) *The support of μ is a finite union of intervals connecting zeros of R.*

Point (i) combines Proposition 3.7 and Formula (3.31) from [24]. Point (ii) is an easy consequence of analyticity of Stieltjes transform, see [22, Lemma 2.6]. At last, Point (iii) comes from [24, Proposition 3.9].

Let us remark that a critical measure μ is completely determined by the associated polynomial R. Hence, finding critical measures boils down to determining all possible polynomials satisfying Equations (14) and (15). For the quartic potential and other polynomials with few monomials, it is possible to do so (see [22,27] for examples). However, in the quartic case, we will only use R in order to show that a critical measure has connected support. Indeed, as soon as this is the case, we can just recover μ by solving a singular integral equation, as we will do in the next subsection.

Proposition 2.4. *For the potential $V(x)=\frac{1}{4}x^4+\frac{c}{2}x^2$ with $c\geq -2$, every critical measure supported in $\mathbb{R}$ has connected support.*

Proof. By (15), the polynomial R defined in (14) is given by

$$R(z)=\frac{1}{4}z^6+\frac{c}{2}z^4+\frac{1}{4}\left(c^2-4\right)z^2-z\int x\,\mathrm{d}\mu(x)-\int x^2\,\mathrm{d}\mu(x)-c.$$

We can not find the roots of this polynomial because the two first moments of μ are unknown. However, we will be able to count its real roots applying Descartes' rule of signs.

Lemma 2.5 (Descartes' rule of signs, see [35]). *Let*

$$P(X)=a_nX^n+\cdots+a_1X+a_0$$

be a polynomial with real coefficients. We denote by p, resp. q, the number of sign changes in the sequence $(a_n,\dots,a_1,a_0)$, resp. $((-1)^na_n,\dots,-a_1,a_0)$, in which we have removed the zeros. Then, the number of positive, resp. negative, roots of P is at most p, resp. q, and has the same parity as p, resp. q.

If we distinguish all the possible cases, it easily follows that the polynomial R admits 0, 2, or 4 non-zero real roots, whatever the value of $c\geq -2$ is and whatever the signs of the quantities $\int x\,\mathrm{d}\mu(x)$ and $\int x^2\,\mathrm{d}\mu(x)+c$ are.

In addition to this, every non-real root of R has even multiplicity by Proposition 2.3(ii). Since R admits 6 roots, it follows that the multiplicity of 0 is necessarily even.

- If 0 is not a root of R, then R admits at most 4 real roots, thus at least two conjugate non-real roots. But, by Proposition 2.3(ii), every non-real root is at least a double root, thus R has in fact at most two real roots. By Proposition 2.3(iii), μ has connected support in this case.
- If 0 is a root of R, then it is at least a double root. Thus R is explicit and we have $R(z)=\frac{1}{4}z^2(z^2+c+2)(z^2+c-2)$. This is impossible for $c>-2$ by Proposition 2.3(ii). For $c=-2$, this leads to $R(z)=\frac{1}{4}z^4(z-2)(z+2)$, thus by Proposition 2.3(iii), the support of μ is $[-2,0]$, $[0,2]$, or $[-2,2]$.

In both cases, we have shown that μ has connected support. □

2.3. *Singular integral equations*

Euler–Lagrange equations are singular integral equations that can be solved once we know the support of the solution, or at least its number of connected components, thanks to the following result. For a slightly different approach, see [31].

Theorem 2.6 (See [28, Section 88]). *Let L be a finite union of intervals $\bigcup_{j=1}^p[a_{2j-1},a_{2j}]$ and let f be a given Hölder continuous function on L. The singular integral equation*

$$\forall x\in L,\quad \fint_L\frac{\varphi(t)}{t-x}\,\mathrm{d}t=f(x)$$

admits a Hölder continuous, bounded solution φ if and only if f satisfies the following p conditions:

$$\forall k\in [\![0,p-1]\!],\quad \int_L\frac{t^kf(t)}{\prod_{j=1}^{2p}\sqrt{|t-a_j|}}\,\mathrm{d}t=0.$$

In this case, the solution is unique and it is given by

$$\forall x\in L,\quad \varphi(x)=-\frac{1}{\pi^2}\prod_{j=1}^{2p}\sqrt{|x-a_j|}\fint_L\frac{f(t)}{(t-x)\prod_{j=1}^{2p}\sqrt{|t-a_j|}}\,\mathrm{d}t.$$

Applying this theorem to the quartic potential (3), we get the following.

Proposition 2.7. *For the potential $V(x) = \frac{1}{4}x^4 + \frac{c}{2}x^2$ with $c \geq -2$, the only stationary probability measure with bounded density and connected support is the equilibrium measure μ_V, which is defined by* (7).

Proof. Let μ be a stationary probability measure with bounded density, denoted by ρ, and with connected support, denoted by $[a,b]$. By Theorem 2.6 applied to $f(x) = -\frac{1}{2}V'(x)$ and $p = 1$, the existence of μ is ensured by the condition

$$\int_a^b \frac{t^3 + ct}{\sqrt{(t-a)(b-t)}}\,\mathrm{d}t = 0. \tag{16}$$

An elementary computation leads to

$$\int_a^b \frac{t^3 + ct}{\sqrt{(t-a)(b-t)}}\,\mathrm{d}t = \frac{\pi}{16}\left(5b^3 + 3ab^2 + 3a^2b + 5a^3\right) + c\frac{\pi}{2}(a+b),$$

thus condition (16) reads

$$(a+b)\left(5b^2 - 2ab + 5a^2 + 8c\right) = 0. \tag{17}$$

Moreover, by Theorem 2.6 again, the density of μ is given by

$$\begin{aligned}\rho(x) &= \frac{\sqrt{(x-a)(b-x)}}{2\pi^2} \fint_a^b \frac{t^3 + ct}{(t-x)\sqrt{(t-a)(b-t)}}\,\mathrm{d}t \\ &= \frac{1}{2\pi}\sqrt{(x-a)(b-x)}\left(x^2 + \frac{a+b}{2}x + \frac{3}{8}b^2 + \frac{1}{4}ab + \frac{3}{8}a^2 + c\right).\end{aligned} \tag{18}$$

This result has been obtained by standard integral computations. By integrating this expression between a and b, since ρ is a probability density function, we get a new constraint on a and b:

$$\frac{(b-a)^2}{256}\left(15a^2 + 18ab + 15b^2 + 16c\right) = 1. \tag{19}$$

The two equations (17) and (19) allow us to determine a and b. First, Equation (17) gives three families of possible solutions:

$$a = -b, \qquad a = \frac{1}{5}b + \frac{2}{5}\sqrt{-10c - 6b^2}, \qquad a = \frac{1}{5}b - \frac{2}{5}\sqrt{-10c - 6b^2}.$$

Equation (19) then eliminates some cases. Note first that, if c is non-negative, only the first case would be possible, and that the same situation occurs when c is negative but $b^2 > -\frac{5}{3}c$.

- Case 1: $a = -b$.

In this case, Equation (19) gives

$$b = \sqrt{\frac{2}{3}\left(\sqrt{c^2 + 12} - c\right)},$$

so the density given by (18) becomes

$$\rho(x) = \frac{1}{2\pi}\sqrt{b^2 - x^2}\left(x^2 + \frac{2}{3}c + \frac{1}{3}\sqrt{c^2 + 12}\right).$$

This is exactly the equilibrium measure of V for $c \geq -2$, see (7).

- Case 2: $a = \frac{1}{5}b + \frac{2}{5}\sqrt{-10c - 6b^2}$.

Equation (19) now implies that

$$45b^8 + 156cb^6 + (182c^2 - 552)b^4 + (76c^3 - 880c)b^2 + 5c^4 - 200c^2 + 2000 = 0.$$

We will show this is not possible under the conditions $-2 \le c \le 0$ and $0 \le b^2 \le -\frac{5}{3}c$. Indeed, we can study the polynomial function

$$f : (x, c) \mapsto 45x^4 + 156cx^3 + (182c^2 - 552)x^2 + (76c^3 - 880c)x + 5c^4 - 200c^2 + 2000$$

on the compact set

$$K = \left\{(x, c) \in \mathbb{R}^2 \,\middle|\, -2 \le c \le 0, 0 \le x \le -\frac{5}{3}c\right\}.$$

The resolution of $\frac{\partial f}{\partial x}(x, c) = \frac{\partial f}{\partial c}(x, c) = 0$ shows that the only critical point of f in K is $(0, 0)$. Consequently, f attains its minimum on the boundary of K. The study of the three functions

$$c \mapsto f(0, c) = 5(c^2 - 20)^2,$$
$$x \mapsto f(x, -2) = 45x^4 - 312x^3 + 176x^2 + 1152x + 1280,$$

and

$$c \mapsto f\left(-\frac{5}{3}c, c\right) = \frac{80}{9}(c^2 - 15)^2$$

allows us to conclude that the minimum of f on K is attained at $(\frac{10}{3}, -2)$ and is equal to $\frac{9680}{9}$. Consequently, f does not vanish on K and Case 2 does not lead to a suitable solution μ.

- Case 3: $a = \frac{1}{5}b - \frac{2}{5}\sqrt{-10c - 6b^2}$.

Very similar computations lead to the fact that the same function f must vanish on the same compact K, and thus to the same conclusion.

Finally, the only stationary probability measure with bounded density and connected support is indeed the equilibrium measure μ_V. $\square$

Remark. The previous calculations also show that there does not exist a stationary probability measure with bounded density and connected support when $-\sqrt{15} < c < -2$ because, in this situation, Case 1 of the proof leads to a density taking negative values, and Cases 2 and 3 still lead to unsuitable solutions.

In addition to this, the same technique allows us to prove that, when $c < -2$, the only symmetric stationary probability measure having a bounded density and a support with two connected components is the equilibrium measure. We recall that, by Proposition 2.4, there does not exist such a symmetric stationary measure when $c \ge -2$.

Finally, we mention that, for $c = -\sqrt{15}$, there exist two stationary measures with bounded density and connected support. The first one is given by the density

$$x \mapsto \frac{1}{2\pi}\sqrt{\left(x - \frac{1}{\sqrt[4]{15}}\right)\left(\frac{5}{\sqrt[4]{15}} - x\right)}\left(x + \frac{4}{\sqrt[4]{15}}\right)\left(x - \frac{1}{\sqrt[4]{15}}\right)$$

on the interval $[\frac{1}{\sqrt[4]{15}}; \frac{5}{\sqrt[4]{15}}]$ and the second one is its symmetrical measure with respect to the origin. We refer to [21, Chapter 7] for the detailed computations.

3. Proof of Theorem 1.1

We are now able to prove Theorem 1.1. The ideas are as follows. First, thanks to properties of free diffusions stated in Section 2.1, we find an accumulation point of $(\mu_t)_{t\geq 0}$ which is a stationary measure with a bounded density. By Propositions 2.4 and 2.7, this accumulation point is necessarily the equilibrium measure μ_V. We then prove that all accumulation points have the same entropy, using again the estimates of Proposition 2.1. Since μ_V is the unique minimizer of Σ_V, this proves it is the only accumulation point. A compactness argument allows us to prove the convergence of $(\mu_t)_{t\geq 0}$ towards μ_V.

We emphasize our proof depends on the special potential (3) only through Propositions 2.4 and 2.7; the other arguments given below are valid for every potential V satisfying the assumptions of Proposition 2.1.

From now, for every $t \geq 0$, we denote by ρ_t the density of μ_t.

Proof of Theorem 1.1. By (12), the function $t \mapsto \Sigma_V(\mu_t)$ is decreasing on $[0, +\infty)$. As it is also bounded below (by $\Sigma_V(\mu_V)$), this function admits a finite limit as t goes to infinity. Therefore, by (12) again, there exists a sequence $(t_k)_{k\in\mathbb{N}}$ such that $t_k \to \infty$ and $\frac{\mathrm{d}}{\mathrm{d}t}\Sigma_V(\mu_{t_k}) \to 0$ when $k \to \infty$.

By Proposition 2.1(iii), extracting a further subsequence if necessary, we can assume that the densities ρ_{t_k} converge in the L^2-topology to a limit ρ. As the ρ_{t_k}'s are defined on the compact set $[-M, M]$, L^2-convergence implies that $\int \rho_{t_k}(x)\,\mathrm{d}x$ converges to $\int \rho(x)\,\mathrm{d}x$, hence the limit ρ is a probability density function defined on $[-M, M]$. We denote by μ the probability measure associated to ρ. By Scheffé's lemma, μ_{t_k} also converges in distribution towards μ.

We will now prove that μ is a stationary probability measure with a bounded density. First, as the densities ρ_{t_k} converge in $L^2([-M, M])$, extracting a further subsequence if necessary, we can assume that they converge almost everywhere on $[-M, M]$. Thus, we have $\|\rho\|_\infty \leq K_2$.

Furthermore, for all $k \in \mathbb{N}$, we can decompose

$$\begin{aligned}
&\left|\int\left|H\mu_{t_k} - \frac{1}{2}V'\right|^2 \mathrm{d}\mu_{t_k} - \int\left|H\mu - \frac{1}{2}V'\right|^2 \mathrm{d}\mu\right| \\
&\quad\leq \left|\int\left|H\mu_{t_k} - \frac{1}{2}V'\right|^2 \mathrm{d}\mu_{t_k} - \int\left|H\mu - \frac{1}{2}V'\right|^2 \mathrm{d}\mu_{t_k}\right| \\
&\qquad + \left|\int\left|H\mu - \frac{1}{2}V'\right|^2 \mathrm{d}\mu_{t_k} - \int\left|H\mu - \frac{1}{2}V'\right|^2 \mathrm{d}\mu\right|,
\end{aligned} \tag{20}$$

where the integrals are taken over $[-M, M]$.

Let us show why the first term in the right-hand side goes to 0 as $k \to +\infty$. Denoting by K a uniform bound on the densities and using the Cauchy–Schwarz inequality, we have for all $k \in \mathbb{N}^*$,

$$\begin{aligned}
&\left|\int\left|H\mu_{t_k} - \frac{1}{2}V'\right|^2 \mathrm{d}\mu_{t_k} - \int\left|H\mu - \frac{1}{2}V'\right|^2 \mathrm{d}\mu_{t_k}\right| \\
&\quad\leq K\left|\int\left|H\mu_{t_k}(x) - H\mu(x)\right|\left(\left|H\mu_{t_k}(x)\right| + \left|H\mu(x)\right| + \left|V'(x)\right|\right)\mathrm{d}x\right| \\
&\quad\leq K\|H\mu_{t_k} - H\mu\|_2 \times \left(\|H\mu_{t_k}\|_2 + \|H\mu\|_2 + \left\|V'\right\|_2\right)
\end{aligned}$$

and by the continuity of the Hilbert transform from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$, $H\mu_{t_k}$ converges to $H\mu$ in L^2. On the other hand, it follows from similar arguments and from $\rho \in L^4$ that the second term in the right-hand side of (3) also tends to 0 when $k \to +\infty$. By (12), we finally have

$$0 = \lim_{k\to+\infty} \frac{\mathrm{d}}{\mathrm{d}t}\Sigma_V(\mu_{t_k}) = \lim_{k\to+\infty} -2\int\left|H\mu_{t_k} - \frac{1}{2}V'\right|^2 \mathrm{d}\mu_{t_k} = -2\int\left|H\mu - \frac{1}{2}V'\right|^2 \mathrm{d}\mu.$$

The limit measure μ is thus a stationary probability measure.

As μ is also a critical measure supported in $\mathbb{R}$, when V is the quartic potential (3), by Proposition 2.4, μ has connected support and, by Proposition 2.7, μ is the equilibrium measure μ_V.

We will now prove that two accumulation points for $(\rho_t)_{t\geq 0}$ in the L^2-topology must have the same entropy.

Indeed, let $(\rho_{s_k})_{k\in\mathbb{N}}$ be a convergent subsequence from $(\rho_t)_{t\geq 0}$ in the L^2-topology. We denote by ρ its limit. Then ρ is a probability density function supported in $[-M, M]$ and it is bounded by $K_1 + K_2$, as these properties hold for ρ_t with $t \geq 1$. We denote by μ the associated probability measure.

By the Cauchy–Schwarz inequality, we have

$$\left| \int V(x)\,\mathrm{d}\mu_{s_k}(x) - \int V(x)\,\mathrm{d}\mu(x) \right| \leq \|\rho_{s_k} - \rho\|_2 \|V\|_2$$

and

$$\begin{aligned}
&\left| \iint \log|x-y| \rho_{s_k}(x)\rho_{s_k}(y)\,\mathrm{d}x\,\mathrm{d}y - \iint \log|x-y|\rho(x)\rho(y)\,\mathrm{d}x\,\mathrm{d}y \right| \\
&\quad\leq \left| \iint \log|x-y|\rho_{s_k}(x)\big(\rho_{s_k}(y) - \rho(y)\big)\,\mathrm{d}x\,\mathrm{d}y \right| \\
&\qquad + \left| \iint \log|x-y|\rho(y)\big(\rho_{s_k}(x) - \rho(x)\big)\,\mathrm{d}x\,\mathrm{d}y \right| \\
&\quad\leq 2(K_1 + K_2)\cdot\sqrt{2M}\|\rho_{s_k} - \rho\|_2 \left(\iint \log^2|x-y|\,\mathrm{d}x\,\mathrm{d}y \right)^{1/2}
\end{aligned}$$

for k large enough. Therefore, we get

$$\lim_{k\to+\infty} \Sigma_V(\mu_{s_k}) = \Sigma_V(\mu).$$

This proves the "continuity" of entropy along a solution.

Since the function $t \mapsto \Sigma_V(\mu_t)$ is decreasing, we conclude that two accumulation points lead to the same entropy.

This allows us to complete the proof. Indeed, since we proved that the density ρ_V of μ_V is an accumulation point of $(\rho_t)_{t\geq 0}$ in the L^2-topology, since μ_V is the unique minimizer of free entropy Σ_V, and since all accumulation points have the same entropy, the only possible accumulation point in the L^2-topology is ρ_V. But, by Proposition 2.1(iii), the ρ_t's, $t \geq 1$, are contained in a compact set $\mathcal{A}$ for this topology, so ρ_t converges towards ρ_V in the L^2-topology. As we explained at the beginning of this proof, this implies that μ_t converges in distribution towards μ_V. Since weak convergence and W_p-convergence, $p \in [1, +\infty)$, coincide for distributions on a given compact set, the conclusion of Theorem 1.1 follows. □

4. Perspectives

Many natural questions follow this work.

- *The case* $c < -2$. Our result uses the fact that, when $c \geq -2$, we have only one critical measure that can be an accumulation point for $(\mu_t)_{t\geq 0}$ (Propositions 2.4 and 2.7). When $c < -2$, can we describe the critical measures that are candidates to be accumulation points? For instance, there is no critical measure with bounded density and connected support when $-\sqrt{15} < c < -2$. In this case, is the equilibrium measure μ_V the only suitable critical measure? Is the convergence of the solution of (1) towards μ_V possible in this case?

  Besides, the value $c = -\sqrt{15}$ appears as the value under which the existence of unilateral critical measures for the quartic potential becomes possible. This threshold also appears in [5] in a slightly different context. Are the measures described in [5] the only critical measures?

  Finally, when c is very negative, can we describe the basins of attraction associated to each possible limit for the solution of the free Fokker–Planck equation?

- *Other confining potentials*. We only used the special form of the quartic potential in order to get Propositions 2.4 and 2.7. Do our methods apply in other cases? For instance, can we change the potential V, take a higher degree, or consider higher dimensions?
- *Non-confining potentials*. Several works deal with non-confining potentials. For instance, [1] studied a cubic potential, and [12] considered the quartic potential $V(x) = \frac{1}{2}x^2 + \frac{g}{4}x^4$ with $g < 0$. For these potentials, once the problems of definitions are solved, we can tackle the problem of long-time behaviour. Can we prove a convergence result for the cubic potential or for the quartic potential with $-\frac{1}{12} < g < 0$, as Biane and Speicher conjectured for the latter?

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