

Eta-diagonal distributions and infinite divisibility for R-diagonals

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Received 29 August 2016; revised 13 February 2017; accepted 7 March 2017

Abstract. The class of R -diagonal $*$ -distributions is fairly well understood in free probability. In this class, we consider the concept of infinite divisibility with respect to the operation \boxplus of free additive convolution. We exploit the relation between free probability and the parallel (and simpler) world of Boolean probability. It is natural to introduce the concept of an η -diagonal distribution that is the Boolean counterpart of an R -diagonal distribution. We establish a number of properties of η -diagonal distributions, then we examine the canonical bijection relating η -diagonal distributions to infinitely divisible R -diagonal ones. The overall result is a parametrization of an arbitrary \boxplus -infinitely divisible R -diagonal distribution that can arise in a C^* -probability space by a pair of compactly supported Borel probability measures on $[0, \infty)$. Among the applications of this parametrization, we prove that the set of \boxplus -infinitely divisible R -diagonal distributions is closed under the operation \boxtimes of free multiplicative convolution.

Résumé. Dans le cadre des probabilités libres, la classe des $*$ -distributions R -diagonales est assez bien comprise. Dans cette classe, nous nous intéressons au concept d'infinie divisibilité par rapport à l'opération de convolution additive libre \boxplus . Nous exploitons la relation entre probabilités libres et le monde parallèle (et plus simple) des probabilités booléennes. Il est naturel d'introduire le concept de distributions η -diagonales, qui sont la contrepartie booléenne des distributions R -diagonales. Nous établissons un certain nombre de propriétés des distributions η -diagonales, avant d'examiner la bijection canonique reliant les distributions η -diagonales aux distributions R -diagonales indéfiniment divisibles. Le résultat principal est la paramétrisation par une paire de mesures boreliennes sur $[0, \infty)$ à support compact de toutes les lois R -diagonales \boxplus -indéfiniment divisibles pouvant apparaître dans un C^* -espace de probabilités. Parmi les applications de cette paramétrisation, nous montrons que l'ensemble des distributions R -diagonales \boxplus -indéfiniment divisibles est fermé sous l'opération \boxtimes de convolution multiplicative libre.

MSC: Primary 46L54; secondary 46L53; 60E07; 60E10

Keywords: R -diagonal distribution; η -diagonal distribution; Free additive convolution; Infinite divisibility

1. Introduction

Free additive convolution \boxplus is a binary operation on the set \mathcal{P} of Borel probability measures on \mathbb{R} , reflecting the addition operation for free selfadjoint elements in a noncommutative probability space. The properties of this operation parallel in many respects the ones of the usual convolution on \mathcal{P} , for instance in the treatment of infinite divisibility.

¹Supported in part by a grant from the National Science Foundation of the USA.

²Supported by a Discovery Grant from NSERC, Canada.

One way to approach \boxplus -infinite divisibility is to use a bijection constructed in [5] which relates free independence to another form of noncommutative independence, namely Boolean independence. In this paper we focus on probability measures with compact support, so we view this bijection as a map $\mathbb{B} : \mathcal{P}_c \rightarrow \mathcal{P}_c^{(\text{inf-div})}$, where \mathcal{P}_c is the set of probability measures with compact support on \mathbb{R} , while $\mathcal{P}_c^{(\text{inf-div})}$ consists of those measures $\mu \in \mathcal{P}_c$ which are \boxplus -infinitely divisible, that is, have the property that for every $n \in \mathbb{N}$, there exists $\mu_n \in \mathcal{P}_c$ satisfying

$$\underbrace{\mu_n \boxplus \cdots \boxplus \mu_n}_n = \mu. \tag{1.1}$$

The bijection \mathbb{B} connects the fundamental transforms of free and Boolean probability, the *R-transform* and respectively the *η -series*. For $\mu \in \mathcal{P}_c$, both of these transforms $R_\mu(z)$ and $\eta_\mu(z)$ are convergent power series. The bijection \mathbb{B} is described by the equation

$$R_{\mathbb{B}(\mu)} = \eta_\mu, \quad \mu \in \mathcal{P}_c. \tag{1.2}$$

More precisely, for every $\mu \in \mathcal{P}_c$ there exists a uniquely determined measure $\nu \in \mathcal{P}_c^{(\text{inf-div})}$ such that $R_\nu = \eta_\mu$, and one defines $\mathbb{B}(\mu) := \nu$.

At the level of compactly supported distributions, the bijection \mathbb{B} is precisely the parametrization of \boxplus -infinitely divisible distributions provided in [19]. This was extended in [5] to the space \mathcal{P} of all Borel probability measures on \mathbb{R} . In a different direction, the bijection \mathbb{B} was extended in [3] to the space of joint distributions for k -tuples of selfadjoint elements in a C^* -probability space. Our goal in this paper is to use a multivariate version of the bijection \mathbb{B} in order to study \boxplus -infinitely divisible R -diagonal distributions, a significant class of $*$ -distributions considered in free probability.

To explain our results, we introduce some notation. We let $\mathcal{D}_c(1, *)$ stand for the collection of all $*$ -distributions of (generally, not selfadjoint) elements in a (generally not tracial) C^* -probability space. There is a natural operation \boxplus on $\mathcal{D}_c(1, *)$ which corresponds to the addition $a + b$ of two variables a, b in the same space such that $\{a, a^*\}$ is free from $\{b, b^*\}$. Infinite divisibility in $\mathcal{D}_c(1, *)$ is defined as in (1.1), and we denote by $\mathcal{D}_c(1, *)^{(\text{inf-div})}$ the collection of \boxplus -infinitely divisible elements of $\mathcal{D}_c(1, *)$. The notions of R -transform and η -series also have natural extensions to the context of $*$ -distributions.

The results of [3], specialized to two selfadjoint variables, can be applied to $\mathcal{D}_c(1, *)$ after a simple change of coordinates. There is again a bijection $\mathbb{B}_{(1,*)} : \mathcal{D}_c(1, *) \rightarrow \mathcal{D}_c(1, *)^{(\text{inf-div})}$ defined by the requirement that

$$R_{\mathbb{B}_{(1,*)}(\mu)} = \eta_\mu, \quad \mu \in \mathcal{D}_c(1, *). \tag{1.3}$$

This is analogous to the condition (1.2) satisfied by the original bijection \mathbb{B} , but proving the existence of $\mathbb{B}_{(1,*)}$ is more than a trivial extension of the proof for \mathbb{B} , and requires a mixture of combinatorial and analytic methods.

We turn now to R -diagonal $*$ -distributions, which can be succinctly described as the distributions in $\mathcal{D}_c(1, *)$ that are invariant under multiplication by a free Haar unitary (see [15, Theorem 15.10, p. 244]). This class of $*$ -distributions has received quite a bit of attention in the free probability literature. In particular, elements with R -diagonal $*$ -distributions were among the first examples of non-normal elements in W^* -probability spaces for which the so-called ‘Brown spectral measure’ was calculated explicitly (in [10]), and for which the Brown measure techniques could be used to find invariant subspaces (in [17]). R -diagonal $*$ -distributions also appear in large N limit results for random matrices, in connection to the so-called ‘single ring theorem’ [9].

For our purposes, it is best to consider the original definition [14] of R -diagonal distributions which asks that the R -transform of the distribution be in some sense ‘diagonal’ [15, Definition 15.3, p. 241]. From this point of view, it is clear how to define the Boolean counterpart of R -diagonality: we simply say that a $*$ -distribution is η -diagonal if its η -series is diagonal. The map $\mathbb{B}_{(1,*)}$ defined by (1.3) will then give a bijection between the set of all η -diagonal distributions in $\mathcal{D}_c(1, *)$ and the set of R -diagonal distributions in $\mathcal{D}_c(1, *)$ which are \boxplus -infinitely divisible.

The above discussion shows that there is some interest in studying η -diagonal distributions. In this paper we point out a few general algebraic and combinatorial properties of such a distribution μ , which actually hold for μ in a larger, purely algebraic space $\mathcal{D}_{\text{alg}}(1, *)$. The property of a distribution $\mu \in \mathcal{D}_{\text{alg}}(1, *)$ of being η -diagonal has an elegant description phrased directly in terms of the $*$ -moments of μ . This result (Theorem 2.8) is reminiscent of

(but simpler than) the description [13, Theorem 1.2.1] of R -diagonal variables in terms of their $*$ -moments. The η -diagonal distributions also have other algebraic and combinatorial properties that are analogous to known properties of R -diagonal distributions. In particular, if a is an η -diagonal element in a $*$ -probability space (\mathcal{A}, φ) (which means, by definition, that a has η -diagonal $*$ -distribution with respect to φ) then it follows that aa^* and a^*a are Boolean independent elements of \mathcal{A} , and that the coefficients of the η -series of aa^* and a^*a are read from the so-called *determining sequences* for the $*$ -distribution of a . For details on the terms used above and for a discussion of why this is indeed analogous to known facts about R -diagonals, see Remark 3.4 below.

In the case in which the η -diagonal distribution μ is in $\mathcal{D}_c(1, *)$, we point out a natural parametrization for μ , given by a pair of compactly supported Borel probability measures on $[0, \infty)$. That is, we establish a canonical bijection

$$\{\mu \in \mathcal{D}_c(1, *) : \mu \text{ is } \eta\text{-diagonal}\} \ni \mu \leftrightarrow (\sigma_1, \sigma_2) \in \mathcal{P}_c^+ \times \mathcal{P}_c^+, \quad (1.4)$$

where $\mathcal{P}_c^+ := \{\sigma \in \mathcal{P}_c : \sigma([0, \infty)) = 1\}$. Without going into details, we mention that all the $*$ -distributions appearing in this paper are defined as linear functionals on the algebra $\mathbb{C}\langle Z, Z^* \rangle$ of complex polynomials in the non-commuting indeterminates Z and Z^* , and that the correspondence $\mu \leftrightarrow (\sigma_1, \sigma_2)$ from (1.4) amounts to the equalities

$$\mu((ZZ^*)^n) = \int_0^\infty t^n d\sigma_1(t) \quad \text{and} \quad \mu((Z^*Z)^n) = \int_0^\infty t^n d\sigma_2(t), \quad n \in \mathbb{N}. \quad (1.5)$$

In other words, the probability measures σ_1 and σ_2 which parametrize μ in (1.4) are simply the distributions of ZZ^* and of Z^*Z with respect to the functional μ . The relevant point here is that for any given $\sigma_1, \sigma_2 \in \mathcal{P}_c^+$ there exists a unique η -diagonal distribution $\mu \in \mathcal{D}_c(1, *)$ such that (1.5) holds; we will prove this point by producing an *operator model* for μ (an operator A on a Hilbert space \mathcal{K} , such that the required μ appears as $*$ -distribution of A with respect to a suitably chosen functional on $B(\mathcal{K})$).

When the bijection $\mathbb{B}_{(1,*)}$ is applied to $\{\mu \in \mathcal{D}_c(1, *) : \mu \text{ is } \eta\text{-diagonal}\}$ in (1.4), we obtain a bijection

$$\left\{ \nu \in \mathcal{D}_c(1, *) : \begin{array}{l} \nu \text{ is } R\text{-diagonal and} \\ \boxplus\text{-infinitely divisible} \end{array} \right\} \ni \nu \leftrightarrow (\sigma_1, \sigma_2) \in \mathcal{P}_c^+ \times \mathcal{P}_c^+. \quad (1.6)$$

Thus, we have a parametrization of a general \boxplus -infinitely divisible R -diagonal distribution by a pair of probability measures from \mathcal{P}_c^+ . Analogously to (1.5), it is possible to write explicitly the relation connecting σ_1, σ_2 to the distributions of the elements ZZ^* and Z^*Z in the noncommutative probability space $(\mathbb{C}\langle Z, Z^* \rangle, \nu)$. Theorem 6.4 below realizes this parametrization by providing precise formulas for the R -transforms of ZZ^* and of Z^*Z .

An important subclass of R -diagonal distributions consists of those which satisfy the KMS condition for some parameter $t \in (0, \infty)$. This is a generalization of the trace condition, where the latter corresponds to the special case $t = 1$ (see the review in Section 3 below). For an R -diagonal distribution ν which satisfies KMS with parameter t , one can process further the result of Theorem 6.4 in order to obtain explicit formulas for the distributions of ZZ^* and of Z^*Z , in terms of the probability measures σ_1 and σ_2 which parametrize ν . These formulas invoke some commonly used elements of free harmonic analysis on \mathcal{P}_c^+ , and are given in Proposition 6.8.

As an application of the parametrization from (1.6), we prove that the set of \boxplus -infinitely divisible R -diagonal $*$ -distributions is closed under the operation \boxtimes of free multiplicative convolution. This provides a non-selfadjoint counterpart to the line of studying closure properties of \boxplus -infinitely divisible distributions which was pursued in the selfadjoint framework in [1] (also, quite recently, in [8]).

In addition to the present **Introduction**, the paper contains 6 sections. Section 2 introduces η -diagonal $*$ -distributions and discusses some of their algebraic properties. Section 3 is devoted to a review of R -diagonal $*$ -distributions, with emphasis on facts that are needed in the present paper. In Section 4 we verify that the bijection $\mathbb{B}_{(1,*)}$ does indeed work on $\mathcal{D}_c(1, *)$ in the way described in (1.3). Section 5 presents the operator model for η -diagonals which is the main ingredient in establishing the bijection (1.4). Section 6 contains our results concerning the parametrization of infinitely divisible R -diagonal distributions, and a discussion of the KMS example. Finally, Section 7 discusses the application to free multiplicative convolution.

2. η -series and η -diagonal $*$ -distributions

Notation 2.1. (1) We denote by \mathcal{W}^+ the set $\bigsqcup_{n=1}^{\infty} \{1, *\}^n$ consisting of all non-empty words over the two-letter alphabet $\{1, *\}$. This is a semigroup (without unit) under the natural operation of concatenation. We denote by $|w|$ the number of letters in a word $w \in \mathcal{W}^+$.

(2) The algebra of complex polynomials in two non-commuting variables Z and Z^* is denoted, as usual, by $\mathbb{C}\langle Z, Z^* \rangle$. For every word $w = (\ell_1, \dots, \ell_n) \in \mathcal{W}^+$ we write

$$Z^w = Z^{\ell_1} \dots Z^{\ell_n} \in \mathbb{C}\langle Z, Z^* \rangle.$$

The set $\{1\} \cup \{Z^w : w \in \mathcal{W}^+\}$ is a basis of $\mathbb{C}\langle Z, Z^* \rangle$ as a complex vector space.

(3) An *algebraic $*$ -distribution* is a linear functional $\mu : \mathbb{C}\langle Z, Z^* \rangle \rightarrow \mathbb{C}$ such that $\mu(1) = 1$. (At this stage we do not require μ to have any additional properties.) The values of μ on monomials Z^w (with $w \in \mathcal{W}^+$) will be referred to as *$*$ -moments* of μ .

(4) The collection of all algebraic $*$ -distributions from (3) is denoted $\mathcal{D}_{\text{alg}}(1, *)$.

Notation 2.2 (Series and their coefficients). (1) The algebra of formal power series in two non-commuting indeterminates z and z^* is denoted, as usual, by $\mathbb{C}\langle\langle z, z^* \rangle\rangle$. The collection $\mathbb{C}_0\langle\langle z, z^* \rangle\rangle \subset \mathbb{C}\langle\langle z, z^* \rangle\rangle$ of power series with vanishing constant coefficient is a two-sided ideal in $\mathbb{C}\langle\langle z, z^* \rangle\rangle$. An arbitrary element $f \in \mathbb{C}_0\langle\langle z, z^* \rangle\rangle$ is of the form

$$f(z, z^*) = \sum_{n=1}^{\infty} \sum_{\ell_1, \dots, \ell_n \in \{1, *\}} \alpha_{(\ell_1, \dots, \ell_n)} z^{\ell_1} \dots z^{\ell_n} = \sum_{w \in \mathcal{W}^+} \alpha_w z^w, \quad (2.1)$$

where the coefficients α_w are complex numbers, and for $w = (\ell_1, \dots, \ell_n) \in \mathcal{W}^+$ we use the notation $z^w = z^{\ell_1} \dots z^{\ell_n}$.

(2) Given $w \in \mathcal{W}^+$, we denote by $\text{Cf}_w : \mathbb{C}_0\langle\langle z, z^* \rangle\rangle \rightarrow \mathbb{C}$ the linear functional which extracts the coefficient of z^w from a series $f \in \mathbb{C}_0\langle\langle z, z^* \rangle\rangle$. That is, if f is given by (2.1), we have $\text{Cf}_w(f) = \alpha_w$, $w \in \mathcal{W}^+$.

(3) It will be convenient to also have a dedicated notation for certain products made with the linear functionals Cf_w from the preceding paragraph. More precisely, let n be a positive integer and let $\pi = \{B_1, \dots, B_k\}$ be a *partition* of $\{1, \dots, n\}$; the latter term means that B_1, \dots, B_k (which are called the *blocks* of π) are non-empty sets with $B_i \cap B_j = \emptyset$ for $i \neq j$ and with $B_1 \cup \dots \cup B_k = \{1, \dots, n\}$. Given a word $w = (\ell_1, \dots, \ell_n) \in \mathcal{W}^+$ with $|w| = n$, we then define a functional (non-linear unless π consists of only one block) $\text{Cf}_{w;\pi} : \mathbb{C}_0\langle\langle z, z^* \rangle\rangle \rightarrow \mathbb{C}$, as follows. For every block $B_i = \{b_{i,1}, \dots, b_{i,m}\}$ where $1 \leq b_{i,1} < \dots < b_{i,m} \leq n$, we set

$$w|B_i = (\ell_1, \dots, \ell_n)|B_i := (\ell_{b_{i,1}}, \dots, \ell_{b_{i,m}}) \in \{1, *\}^m.$$

Then we define

$$\text{Cf}_{w;\pi}(f) := \prod_{i=1}^k \text{Cf}_{w|B_i}(f), \quad f \in \mathbb{C}_0\langle\langle z, z^* \rangle\rangle.$$

[Suppose, for instance, that $n = 5$, $\pi = \{\{1, 4, 5\}, \{2, 3\}\}$, and $w = (\ell_1, \dots, \ell_5)$. Then $\text{Cf}_{w;\pi}(f) = \text{Cf}_{(\ell_1, \ell_4, \ell_5)}(f) \cdot \text{Cf}_{(\ell_2, \ell_3)}(f)$, $f \in \mathbb{C}_0\langle\langle z, z^* \rangle\rangle$.]

Definition and Remark 2.3 (Moment series, η -series). Fix $\mu \in \mathcal{D}_{\text{alg}}(1, *)$. (1) The *moment series* of μ is defined as $M_\mu := \sum_{w \in \mathcal{W}^+} \mu(Z^w) z^w \in \mathbb{C}_0\langle\langle z, z^* \rangle\rangle$.

(2) The *η -series* of μ is defined as

$$\eta_\mu := M_\mu(1 + M_\mu)^{-1} = (1 + M_\mu)^{-1} M_\mu \in \mathbb{C}_0\langle\langle z, z^* \rangle\rangle, \quad (2.2)$$

where all the algebraic operations are performed in the algebra $\mathbb{C}\langle\langle z, z^* \rangle\rangle$.

(3) It is immediate from (2.2) that the series M_μ can be retrieved from η_μ by the formula

$$M_\mu = \eta_\mu(1 - \eta_\mu)^{-1} = (1 - \eta_\mu)^{-1}\eta_\mu. \tag{2.3}$$

(4) The right-hand side of (2.3) can be written as a geometric series $\sum_{k=1}^\infty \eta_\mu^k$, which converges coefficientwise, in the sense that for every $w \in \mathcal{W}^+$ one has

$$\text{Cf}_w(M_\mu) = \sum_{k=1}^\infty \text{Cf}_w(\eta_\mu^k) \quad (\text{convergence in } \mathbb{C}).$$

The series on the right-hand side of the latter equation is convergent due to the obvious reason that, denoting $|w| =: n$, one has $\text{Cf}_w(\eta_\mu^k) = 0$ for all $k > n$. It is also worth recording that for $1 \leq k \leq n$ the coefficient $\text{Cf}_w(\eta_\mu^k)$ can be related (via a direct inspection of how the k -fold product of η_μ with itself is formed) to the generalized coefficients $\text{Cf}_{w;\pi}$ from Notation 2.2:

$$\text{Cf}_w(\eta_\mu^k) = \sum_{\substack{\pi \text{ interval partition} \\ \text{with } k \text{ blocks of } \{1, \dots, n\}}} \text{Cf}_{w;\pi}(\eta_\mu). \tag{2.4}$$

In (2.4) we used the customary term of *interval partition* for a partition π of $\{1, \dots, n\}$ such that every block B of π is of the form $\{a, a + 1, \dots, b\}$ for some $a \leq b$ in $\{1, \dots, n\}$. We will also use the customary notation $\text{Int}(n)$ for the set of all interval partitions of $\{1, \dots, n\}$. By summing over $1 \leq k \leq n$ in equation (2.4), one obtains a very useful explicit formula for the coefficients of M_μ in terms of those of η_μ , namely

$$\text{Cf}_w(M_\mu) = \sum_{\pi \in \text{Int}(n)} \text{Cf}_{w;\pi}(\eta_\mu), \quad w \in \mathcal{W}^+ \text{ with } |w| = n. \tag{2.5}$$

An analogous argument converts (2.2) into the formula

$$\text{Cf}_w(\eta_\mu) = \sum_{\pi \in \text{Int}(n)} (-1)^{1+|\pi|} \text{Cf}_{w;\pi}(M_\mu), \quad w \in \mathcal{W}^+ \text{ with } |w| = n, \tag{2.6}$$

where $|\pi|$ denotes the number of blocks of the partition π .

Remark 2.4. It is clear that the map $\mathcal{D}_{\text{alg}}(1, *) \ni \mu \mapsto M_\mu \in \mathbb{C}_0\langle\langle z, z^* \rangle\rangle$ is bijective. Equations (2.2) and (2.3) show that the map $\mathcal{D}_{\text{alg}}(1, *) \ni \mu \mapsto \eta_\mu \in \mathbb{C}_0\langle\langle z, z^* \rangle\rangle$ is a bijection as well. In other words, we can define a distribution $\mu \in \mathcal{D}_{\text{alg}}(1, *)$ by specifying its η -series.

Definition 2.5. A word $w \in \mathcal{W}^+$ is said to be *alternating* when it is of the form

$$w = \underbrace{(1, *, 1, *, \dots, 1, *)}_{2m} = (1, *)^m \quad \text{or} \quad w = \underbrace{(*, 1, *, 1, \dots, *, 1)}_{2m} = (*, 1)^m,$$

for some positive integer m . In the first case w is said to be *of type* $(1, *)$, and in the second case w is said to be *of type* $(*, 1)$. In these formulas, powers are taken relative to concatenation. Note in particular that alternating words have positive, even length.

Definition 2.6. (1) A distribution $\mu \in \mathcal{D}_{\text{alg}}(1, *)$ is said to be η -*diagonal* if $\text{Cf}_w(\eta_\mu) = 0$ for every word $w \in \mathcal{W}^+$ which is not alternating.

(2) If μ is η -diagonal, its η -series is thus of the form

$$\eta_\mu(z, z^*) = \sum_{n=1}^\infty \alpha_n (zz^*)^n + \sum_{n=1}^\infty \beta_n (z^*z)^n,$$

with $\alpha_n, \beta_n \in \mathbb{C}$ for $n \in \mathbb{N}$. The sequences $(\alpha_n)_{n=1}^\infty$ and $(\beta_n)_{n=1}^\infty$ will be called the *determining sequences* of the η -diagonal distribution μ .

The main goal of the present section is to reveal an equivalent characterization of η -diagonal distributions, which is phrased directly in terms of $*$ -moments. For this purpose, we require one more concept related to words in \mathcal{W}^+ .

Definition and Remark 2.7. (1) A word $w = (\ell_1, \dots, \ell_n) \in \mathcal{W}^+$ is said to be *mixed-alternating* if $n = 2m$ is even and if the letters of w are such that $\ell_{2k-1} \neq \ell_{2k}$, for $k = 1, 2, \dots, m$. Equivalently, w is mixed-alternating when it belongs to the concatenation subsemigroup of \mathcal{W}^+ generated by the words $(1, *)$ and $(*, 1)$.

(2) By grouping factors, it is easily seen that every mixed-alternating word w can be written in a unique way as a concatenation of alternating words such that consecutive words are of different types. Indeed, if we write such a word as

$$w = w_1 w_2 \cdots w_p, \tag{2.7}$$

where $p \geq 1$, each w_i is alternating, and w_i is not of the same type as w_{i+1} , $i = 1, 2, \dots, p - 1$, then the boundaries between the words w_1, \dots, w_p can be retrieved at the places where w has two consecutive identical letters.

[For example, $w = (1, *, 1, *, 1, *, *, 1, *, 1, 1, *, *, 1, *, 1)$ is mixed-alternating and its canonical factorization (2.7) is $w_1 w_2 w_3 w_4$ with $w_1 = (1, *)^3$, $w_2 = (*, 1)^2$, $w_3 = (1, *)$, $w_4 = (*, 1)^2$.]

Theorem 2.8. For every distribution $\mu \in \mathcal{D}_{\text{alg}}(1, *)$, the statements (a) and (b) are equivalent.

- (a) μ is η -diagonal.
- (b) μ satisfies the following conditions³:
 - (η DM1) Whenever $w \in \mathcal{W}^+$ is not mixed-alternating, it follows that $\mu(Z^w) = 0$.
 - (η DM2) Whenever $w = w_1 \cdots w_p \in \mathcal{W}^+$ is mixed-alternating and factored as in (2.7), it follows that $\mu(Z^w) = \mu(Z^{w_1}) \cdots \mu(Z^{w_p})$.

The proof of the implication (b) \Rightarrow (a) in the above theorem requires two auxilliary results.

Lemma 2.9. Suppose that a distribution $\mu \in \mathcal{D}_{\text{alg}}(1, *)$ satisfies the condition (η DM1). Then $\text{Cf}_w(\eta_\mu) = 0$ for every word $w \in \mathcal{W}^+$ that is not mixed-alternating.

Proof. Suppose that w is not mixed alternating and $|w| = n$. We prove that $\text{Cf}_w(\eta_\mu) = 0$ by showing that each term in the right-hand side of (2.6) vanishes. Indeed, let $\pi = \{J_1, \dots, J_m\} \in \text{Int}(n)$ be a partition where the intervals J_1, \dots, J_m are listed in increasing order. Observe that $w = (w|J_1) \cdots (w|J_m)$ (concatenation product). Since w is not mixed-alternating, there must exist an index $1 \leq k \leq m$ such that $w|J_k$ is not mixed-alternating. For this k , condition (η DM1) yields $\text{Cf}_{w|J_k}(M_\mu) = 0$. Therefore the term indexed by π in (2.6) vanishes as well because $\text{Cf}_{w|J_k}(M_\mu)$ is one of its factors. The lemma follows. □

Lemma 2.10. Let $\mu, \nu \in \mathcal{D}_{\text{alg}}(1, *)$ be such that

- (1) Both μ and ν satisfy conditions (η DM1) and (η DM2), and
- (2) $\text{Cf}_w(\eta_\mu) = \text{Cf}_w(\eta_\nu)$ for every alternating word $w \in \mathcal{W}^+$.

Then $\mu = \nu$.

Proof. If $w \in \mathcal{W}^+$ is not mixed-alternating then $\mu(Z^w) = \nu(Z^w) = 0$ because μ and ν satisfy (η DM1). Thus it suffices to verify that $\mu(Z^w) = \nu(Z^w)$ for mixed-alternating words w . In fact, it suffices to prove this equality when

³The acronym η DM is meant to suggest η -Diagonality-in-Moments.

w is alternating. Indeed, suppose for the moment that the equality has been proved for alternating words and let $w \in \mathcal{W}^+$ be a mixed-alternating word. Consider the canonical factorization $w = w_1 \cdots w_p$ indicated in (2.7). We have

$$\begin{aligned} \mu(Z^w) &= \mu(Z^{w_1}) \cdots \mu(Z^{w_p}) \quad (\text{by } (\eta\text{DM2}) \text{ for } \mu) \\ &= \nu(Z^{w_1}) \cdots \nu(Z^{w_p}) \quad (\text{by assumption on alternating moments}) \\ &= \nu(Z^w) \quad (\text{by } (\eta\text{DM2}) \text{ for } \nu). \end{aligned}$$

We conclude the proof by showing that $\mu(Z^w) = \nu(Z^w)$ for every alternating word w . By symmetry, it suffices to verify that $\mu((Z^*Z)^m) = \nu((Z^*Z)^m)$ for every $m \in \mathbb{N}$. Fix m and write $\mu((Z^*Z)^m)$ and $\nu((Z^*Z)^m)$ as sums indexed by $\text{Int}(2m)$, in the way indicated in (2.5). We show that for every $\pi \in \text{Int}(2m)$, the terms indexed by π in the two sums (for μ and for ν) are equal to each other. If π has a block B of odd cardinality, then the terms we are looking at are both equal to 0 because they include the factors $\text{Cf}_{w|_B}(\eta_\mu)$ and respectively $\text{Cf}_{w|_B}(\eta_\nu)$, and these factors are zero by Lemma 2.9. If all the blocks of π are even, we write $\pi = \{J_1, \dots, J_k\}$ where the intervals J_1, \dots, J_k are listed in increasing order, and where $|J_1| = 2d_1, \dots, |J_k| = 2d_k$ for some $d_1, \dots, d_k \in \mathbb{N}$. The terms indexed by π in the two sums we consider are then

$$\prod_{i=1}^k \text{Cf}_{(*,1)^{d_i}}(\eta_\mu) \quad \text{and respectively} \quad \prod_{i=1}^k \text{Cf}_{(*,1)^{d_i}}(\eta_\nu), \tag{2.8}$$

where we used the fact that $(*, 1)^m|_{J_i} = (*, 1)^{d_i}$, for $1 \leq i \leq k$. The two products in (2.8) are indeed equal by assumption (2) in the statement. \square

Proof of Theorem 2.8. Suppose first that μ is η -diagonal. We verify that it satisfies (ηDM1) and (ηDM2) . The hypothesis on μ says in particular that $\text{Cf}_w(\eta_\mu) = 0$ for every $w \in \mathcal{W}^+$ which is not mixed-alternating. To prove (ηDM1) , we must show that $\text{Cf}_w(M_\mu) = 0$ for every such word. This argument is carried precisely as in the proof of Lemma 2.9, with the roles of M_μ and η_μ being reversed and with (2.5) in place of (2.6). The reader will have no difficulty verifying the details.

In order to show that μ also satisfies (ηDM2) , fix a mixed-alternating word $w \in \mathcal{W}^+$, with canonical factorization $w = w_1 \cdots w_p$ as in (2.7). Set $n = |w| = |w_1| + \cdots + |w_p|$, and let ρ_0 be the partition of $\{1, \dots, n\}$ into intervals J_1, \dots, J_p (written in increasing order) with lengths $|J_1| = |w_1|, \dots, |J_p| = |w_p|$. Given a partition π of $\{1, \dots, n\}$, we write $\pi \leq \rho_0$ if every block B of π is contained in one of the blocks J_1, \dots, J_p of ρ_0 . (This relation is usually called the *reverse refinement order* on partitions.)

Next, we use (2.5) to express the coefficient $\text{Cf}_w(M_\mu) = \mu(Z^w)$ as a sum indexed by $\text{Int}(n)$. The special structure of the coefficients of η_μ implies that a partition $\pi \in \text{Int}(n)$ has a zero contribution to that sum unless $\pi \leq \rho_0$. It is immediate that the partitions $\pi \in \text{Int}(n)$ satisfying $\pi \leq \rho_0$ are in natural bijective correspondence to tuples of partitions (π_1, \dots, π_p) where $\pi_1 \in \text{Int}(J_1), \dots, \pi_p \in \text{Int}(J_p)$. This correspondence is such that for $\pi \leftrightarrow (\pi_1, \dots, \pi_p)$ we have

$$\text{Cf}_{w;\pi}(\eta_\mu) = \text{Cf}_{w_1;\pi_1}(\eta_\mu) \cdots \text{Cf}_{w_p;\pi_p}(\eta_\mu).$$

These observations lead to the formula

$$\mu(Z^w) = \sum_{\substack{\pi_1 \in \text{Int}(J_1), \dots, \\ \pi_p \in \text{Int}(J_p)}} \text{Cf}_{w_1;\pi_1}(\eta_\mu) \cdots \text{Cf}_{w_p;\pi_p}(\eta_\mu) = \prod_{i=1}^p \left(\sum_{\pi_i \in \text{Int}(J_i)} \text{Cf}_{w_i;\pi_i}(\eta_\mu) \right).$$

In the latter product, one more application of (2.5) identifies

$$\sum_{\pi_i \in \text{Int}(J_i)} \text{Cf}_{w_i;\pi_i}(\eta_\mu) = \mu(Z^{w_i}), \quad 1 \leq i \leq p,$$

thus implying the desired conclusion that $\mu(Z^w) = \prod_{i=1}^p \mu(Z^{w_i})$.

Conversely, assume now that μ satisfies conditions (ηDM1) and (ηDM2) . We show that it is η -diagonal by an indirect argument: we construct an η -diagonal distribution $\nu \in \mathcal{D}_{\text{alg}}(1, *)$ and then prove that $\mu = \nu$. The distribution ν is defined by specifying its η -series (see Remark 2.4), namely $\text{Cf}_w(\eta_\nu) = \text{Cf}_w(\eta_\mu)$ if w is alternating and $\text{Cf}_w(\eta_\nu) = 0$ otherwise. To prove that $\mu = \nu$ we show that μ and ν satisfy the hypothesis of Lemma 2.10. Indeed, both μ and ν satisfy (ηDM1) and (ηDM2) : μ does so by hypothesis, while ν does so because it is η -diagonal and by virtue of the implication $(a) \Rightarrow (b)$ proved above. On the other hand, if $w \in \mathcal{W}^+$ is an alternating word, the equality $\text{Cf}_w(\eta_\mu) = \text{Cf}_w(\eta_\nu)$ is true by the definition of ν . This concludes the proof of the theorem. \square

Remark 2.11. Let (\mathcal{A}, φ) be a noncommutative probability space (that is, \mathcal{A} is a unital algebra over \mathbb{C} , $\varphi : \mathcal{A} \rightarrow \mathbb{C}$ is a linear functional, and $\varphi(1) = 1$), and let $a_1, a_2 \in \mathcal{A}$. Recall [18] that a_1, a_2 are said to be *Boolean independent* provided that, given positive integers n, p_1, \dots, p_n and indices $i_1, \dots, i_n \in \{1, 2\}$ such that $i_k \neq i_{k+1}$ for $k = 1, \dots, n - 1$, the following identity is satisfied:

$$\varphi(a_{i_1}^{p_1} \cdots a_{i_n}^{p_n}) = \varphi(a_{i_1}^{p_1}) \cdots \varphi(a_{i_n}^{p_n}).$$

Now consider the noncommutative probability space $(\mathbb{C}\langle Z, Z^* \rangle, \mu)$, where $\mu \in \mathcal{D}_{\text{alg}}(1, *)$ is η -diagonal. Condition (ηDM2) of Theorem 2.8 can be restated as saying that ZZ^* and Z^*Z are Boolean independent in $(\mathbb{C}\langle Z, Z^* \rangle, \mu)$.

Remark and Notation 2.12. Another relevant fact concerning an η -diagonal $*$ -distribution μ concerns the individual η -series of ZZ^* and Z^*Z in the noncommutative probability space $(\mathbb{C}\langle Z, Z^* \rangle, \mu)$. If a is an element in a noncommutative probability space (\mathcal{A}, φ) , then its moment series and η -series $M_a, \eta_a \in \mathbb{C}[[z]]$ are defined as in Definition 2.3 but using moments in place of $*$ -moments: first we set $M_a(z) = \sum_{n=1}^\infty \varphi(a^n)z^n$, and then define

$$\eta_a(z) = M_a(z)/(1 + M_a(z)) \in \mathbb{C}[[z]].$$

The coefficients of M_a and η_a are related to each other via summations over interval partitions which are analogous to those shown in (2.5), (2.6) (and are derived the same way, by starting from the algebraic relations satisfied by the series themselves). We explicitly record here the analogue of (2.6):

$$\text{Cf}_n(\eta_a) = \sum_{\rho \in \text{Int}(n)} (-1)^{1+|\rho|} \prod_{B \in \rho} \varphi(a^{|B|}), \quad n \in \mathbb{N}, \tag{2.9}$$

where (by analogy with Notation 2.2(2)) we use the notation $\text{Cf}_n : \mathbb{C}[[z]] \rightarrow \mathbb{C}$ for the linear map that extracts the n th coefficient of a series in $\mathbb{C}[[z]]$.

When applied to the elements ZZ^* and Z^*Z from the framework of Theorem 2.8, these observations yield the following result.

Proposition 2.13. *Suppose that $\mu \in \mathcal{D}_{\text{alg}}(1, *)$ is an η -diagonal distribution, and let $(\alpha_n)_{n=1}^\infty, (\beta_n)_{n=1}^\infty$ be its determining sequences (as introduced in Definition 2.6(2)). Then in the noncommutative probability space $(\mathbb{C}\langle Z, Z^* \rangle, \mu)$, the elements ZZ^* and Z^*Z have η -series given by*

$$\eta_{ZZ^*}(z) = \sum_{n=1}^\infty \alpha_n z^n \quad \text{and} \quad \eta_{Z^*Z}(z) = \sum_{n=1}^\infty \beta_n z^n.$$

Proof. By symmetry, it suffices to prove the first formula. Equation (2.9) yields

$$\text{Cf}_n(\eta_{ZZ^*}) = \sum_{\rho \in \text{Int}(n)} (-1)^{1+|\rho|} \prod_{B \in \rho} \mu((ZZ^*)^{|B|}), \quad n \in \mathbb{N}. \tag{2.10}$$

For the remainder of the proof, we fix $n \in \mathbb{N}$ and verify that the right-hand side of (2.10) is equal to α_n .

For every partition $\rho = \{J_1, \dots, J_k\} \in \text{Int}(n)$, with intervals J_1, \dots, J_k written in increasing order, we define a *doubled* partition $\widehat{\rho} = \{\widehat{J}_1, \dots, \widehat{J}_k\} \in \text{Int}(2n)$. This is the interval partition uniquely determined by the requirement

that $\widehat{J}_1, \dots, \widehat{J}_k$ come in increasing order and satisfy $|\widehat{J}_i| = 2|J_i|$ for $1 \leq i \leq k$. With this notation, it is easily seen that the right-hand side of (2.10) can be written as

$$\sum_{\rho \in \text{Int}(n)} (-1)^{1+|\widehat{\rho}|} \text{Cf}_{(1,*)^n; \widehat{\rho}}(M_\mu).$$

This, however, is the same as

$$\sum_{\pi \in \text{Int}(2n)} (-1)^{1+|\pi|} \text{Cf}_{(1,*)^n; \pi}(M_\mu).$$

Indeed, due to the special structure of the $*$ -moments of μ described in Theorem 2.8, all the terms in the latter sum, corresponding to partitions $\pi \in \text{Int}(2n)$ which are not of the form $\widehat{\rho}$, are equal to 0. We conclude that

$$\text{Cf}_n(\eta_{ZZ^*}) = \sum_{\pi \in \text{Int}(2n)} (-1)^{1+|\pi|} \text{Cf}_{(1,*)^n; \pi}(M_\mu) = \text{Cf}_{(1,*)^n}(\eta_\mu) = \alpha_n,$$

where (2.6) is used in the second equality. □

3. R -transforms and R -diagonal $*$ -distributions

The discussion in Section 2 is better put into perspective when one compares it to the parallel (more elaborate) free probability framework. In the free probability framework, instead of η -series one works with R -transforms, and one has the concept of what it means for a $*$ -distribution $\mu \in \mathcal{D}_{\text{alg}}(1, *)$ to be R -diagonal. The class of R -diagonal $*$ -distributions is in fact rather well-studied in the free probability literature. Here we review some of their basic properties, mostly following [15, Lecture 15], and with emphasis on the facts we need in the present work.

Remark 3.1. On a combinatorial level, switching to the world of free probability comes to using non-crossing partitions instead of interval partitions. We recall that a *crossing* of a partition π of $\{1, \dots, n\}$ consists of integers $1 \leq a < b < c < d \leq n$ such that the set $\{a, c\}$ is contained in a block of π and $\{b, d\}$ is contained in a different block of π . A partition is *non-crossing* if it has no crossings. We denote by $NC(n)$ the collection of all non-crossing partitions of $\{1, \dots, n\}$.

Similarly to the lattice $\text{Int}(n)$, the set $NC(n)$ is partially ordered by reverse refinement. The minimal and maximal elements with respect to this partial order are denoted by 0_n (the partition of $\{1, \dots, n\}$ into n singleton blocks) and respectively 1_n (the partition of $\{1, \dots, n\}$ into one block).

We record a notation and an elementary observation needed in the final part of this section. For every $n \in \mathbb{N}$, we denote by $NCE(2n)$ the collection of all the partitions $\pi \in NC(2n)$ with the property that every block of π has even cardinality. Observe that if $\pi \in NCE(2n)$ and if $V = \{k_1 < k_2 < \dots < k_{2m}\}$ is a block of π , then the numbers k_1, k_2, \dots, k_{2m} have alternating parities. Indeed, for every $i = 1, \dots, 2m - 1$, the set $\{k_i + 1, k_i + 2, \dots, k_{i+1} - 1\}$ is a union of blocks of π , and hence has even cardinality, which implies that k_{i+1} is of opposite parity from k_i .

For a discussion of other elementary facts concerning $NC(n)$, we refer to [15, Lecture 9].

Remark 3.2 (Review of R -transforms). (1) The R -transform of a $*$ -distribution $\mu \in \mathcal{D}_{\text{alg}}(1, *)$ is the series $R_\mu \in \mathbb{C}_0\langle\langle z, z^* \rangle\rangle$, whose coefficients are uniquely determined by the requirement that they relate to the $*$ -moments of μ by the formula

$$\text{Cf}_w(M_\mu) = \sum_{\pi \in NC(n)} \text{Cf}_{w; \pi}(R_\mu), \quad w \in \mathcal{W}^+ \text{ and } n = |w|. \tag{3.1}$$

Equation (3.1) is the free probabilistic counterpart of (2.5). It is often referred to as the *moment-cumulant formula* for free cumulants (see [15, Lecture 11] for an explanation of this terminology).

One can also define the series R_μ by an equation involving the series M_μ and R_μ themselves (rather than their coefficients, as in (3.1)). More precisely, R_μ is the unique series in $\mathbb{C}_0\langle\langle z, z^* \rangle\rangle$ that satisfies the functional equation

$$R_\mu(z(1 + M_\mu(z, z^*)), z^*(1 + M_\mu(z, z^*))) = M_\mu(z, z^*) \tag{3.2}$$

(see [15, Corollary 16.16]). This is the free probabilistic analogue of (2.2), but now we only have an implicit functional equation rather than an explicit formula describing the series R_μ .

An easy inductive argument in the moment-cumulant formula (3.1) shows that one can recover M_μ from R_μ and that (as in Remark 2.4) we have a bijection

$$\mathcal{D}_{\text{alg}}(1, *) \ni \mu \mapsto R_\mu \in \mathbb{C}_0\langle\langle z, z^* \rangle\rangle.$$

In other words, one can define a distribution $\mu \in \mathcal{D}_{\text{alg}}(1, *)$ by specifying its R -transform.

(2) Consider the framework of Remark 2.11, where we discussed the moment series and η -series $M_a, \eta_a \in \mathbb{C}\langle\langle z \rangle\rangle$ associated to an element a in a noncommutative probability space (\mathcal{A}, φ) . In that framework one also has an R -transform associated with the element $a \in \mathcal{A}$. This is the series $R_a \in \mathbb{C}\langle\langle z \rangle\rangle$ that relates to M_a by

$$\text{Cf}_n(M_a) = \sum_{\pi \in \text{NC}(n)} \text{Cf}_{n;\pi}(R_a), \quad n \in \mathbb{N} \tag{3.3}$$

and

$$R_a(z(1 + M_a(z))) = M_a(z). \tag{3.4}$$

These formulas are analogous to (3.1) or (3.2), respectively. For a detailed discussion of the algebraic aspects of R -transforms (covering both the series R_μ in part (1) of this remark and the series R_a in part (2)), see [15, Lecture 16].

Definition 3.3. (1) A $*$ -distribution $\mu \in \mathcal{D}_{\text{alg}}(1, *)$ is said to be R -diagonal when $\text{Cf}_w(R_\mu) = 0$ for every word $w \in \mathcal{W}^+$ that is not alternating.

(2) If μ is R -diagonal, its R -transform is thus of the form

$$R_\mu(z, z^*) = \sum_{n=1}^{\infty} \alpha_n (zz^*)^n + \sum_{n=1}^{\infty} \beta_n (z^*z)^n, \tag{3.5}$$

with $\alpha_n, \beta_n \in \mathbb{C}$ for $n \in \mathbb{N}$. The sequences $(\alpha_n)_{n=1}^{\infty}$ and $(\beta_n)_{n=1}^{\infty}$ are called the *determining sequences* of μ .

Remark 3.4. The concept of an η -diagonal $*$ -distribution from Section 2 obviously parallels the one of an R -diagonal $*$ -distribution, with the η -series in place of the R -transform. The basic properties of η -diagonal distributions proved in Section 2 are the counterparts of known facts concerning R -diagonal distributions, as noted below.

(1) Remark 2.11 is the Boolean counterpart of [15, Corollary 15.11, p. 244]: if μ is R -diagonal, then Z^*Z and ZZ^* are freely independent in the noncommutative probability space $(\mathbb{C}\langle Z, Z^* \rangle, \mu)$.

(2) Theorem 2.8 is the counterpart of [13, Theorem 1.2.1] which describes R -diagonal distributions in terms of their $*$ -moments.

(3) Proposition 2.13 is analogous to [15, Proposition 15.6, p. 241] which gives a precise formula, first found in [11], for the coefficients of the one-variable R -transforms R_{ZZ^*} and R_{Z^*Z} in terms of the determining sequences $(\alpha_n)_{n=1}^{\infty}$ and $(\beta_n)_{n=1}^{\infty}$ from (3.5). This formula is more elaborate than the relation found in Proposition 2.13 for η -diagonal elements. It states that

$$\begin{cases} \text{Cf}_n(R_{ZZ^*}) = \sum_{\pi = \{V_1, \dots, V_k\} \in \text{NC}(n)} \alpha_{|V_1|} \beta_{|V_2|} \cdots \beta_{|V_k|} & \text{and} \\ \text{Cf}_n(R_{Z^*Z}) = \sum_{\pi = \{V_1, \dots, V_k\} \in \text{NC}(n)} \beta_{|V_1|} \alpha_{|V_2|} \cdots \alpha_{|V_k|}, \end{cases} \tag{3.6}$$

where the blocks of π are arranged so that $1 \in V_1$. For example, the first three coefficients of R_{ZZ^*} are $\alpha_1, \alpha_2 + \alpha_1\beta_1$ and $\alpha_3 + 2\alpha_2\beta_1 + \alpha_1\beta_2 + \alpha_1\beta_1^2$.

Equations (3.6) lead to the following observation: an easy induction on n (where one singles out the terms indexed by the partition $1_n \in \text{NC}(n)$ on the right-hand sides of the equations) shows that the determining sequences $(\alpha_n)_{n=1}^{\infty}$ and $(\beta_n)_{n=1}^{\infty}$ can be retrieved from the coefficients of R_{ZZ^*} and R_{Z^*Z} . Hence the R -diagonal $*$ -distribution $\mu \in \mathcal{D}_{\text{alg}}(1, *)$ is completely determined by these R -transforms.

In Section 6 we require a reformulation of (3.6) in terms of operations with series rather than individual coefficients. This reformulation is given in the next proposition. The formulas (3.7) bear a striking resemblance to the functional equation (3.4) of the R -transform. In fact, (3.7) collapse to $R_{ZZ^*}(z) = R_{Z^*Z}(z) = M_a(z)$ in the special case $\alpha_n = \beta_n$, $n \in \mathbb{N}$, in which case we can take $a = b$.

Proposition 3.5. *Let $\mu \in \mathcal{D}_{\text{alg}}(1, *)$ be an R -diagonal $*$ -distribution with determining sequences $(\alpha_n)_{n=1}^\infty$ and $(\beta_n)_{n=1}^\infty$. Suppose we are given some elements a and b in a noncommutative probability space (\mathcal{A}, φ) such that*

$$R_a(z) = \sum_{n=1}^\infty \alpha_n z^n \quad \text{and} \quad R_b(z) = \sum_{n=1}^\infty \beta_n z^n.$$

Then, in the noncommutative probability space $(\mathbb{C}\langle Z, Z^* \rangle, \mu)$, we have

$$R_{ZZ^*}(z) = R_a(z(1 + M_b(z))) \quad \text{and} \quad R_{Z^*Z}(z) = R_b(z(1 + M_a(z))). \tag{3.7}$$

Proof. The argument is analogous to the proof of (3.2) (see, for instance, the proof of [15, Theorem 16.15]). For the reader's convenience, we describe the basic idea.

By symmetry, it suffices to prove the first equality in (3.7). We show that the coefficients of z^n in the series $R_{ZZ^*}(z)$ and $R_a(z(1 + M_b(z)))$ are equal to each other for every $n \in \mathbb{N}$. The formal series expansion

$$R_a(z(1 + M_b(z))) = \sum_{m=1}^\infty \alpha_m (z(1 + M_b(z)))^m$$

yields

$$\text{Cf}_n(R_a(z(1 + M_b(z)))) = \sum_{m=1}^n \alpha_m \text{Cf}_{n-m}((1 + M_b)^m), \quad n \in \mathbb{N}.$$

Recall that the coefficients of $1 + M_b$ are moments of b , and expand $(1 + M_b)^m$, to obtain

$$\text{Cf}_n[R_a(z(1 + M_b)(z))] = \sum_{m=1}^n \sum_{\substack{k_1, \dots, k_m \geq 0 \\ \text{with } k_1 + \dots + k_m = n-m}} \alpha_m \varphi(b^{k_1}) \cdots \varphi(b^{k_m}). \tag{3.8}$$

On the other hand, (3.6) yields

$$\text{Cf}_n(R_{ZZ^*}) = \sum_{m=1}^n \sum_{\substack{S \subseteq \{1, \dots, n\} \\ |S|=m \text{ and } 1 \in S}} \sum_{\substack{\pi = \{V_1, \dots, V_p\} \in \text{NC}(n) \\ \text{with } V_1 = S}} \alpha_m \beta_{|V_2|} \cdots \beta_{|V_p|}, \quad n \in \mathbb{N}.$$

For a fixed set $S = \{s_1, \dots, s_m\} \subseteq \{1, \dots, n\}$ with $1 = s_1 < s_2 < \dots < s_m \leq n$, the collection of non-crossing partitions $\{\pi \in \text{NC}(n) : S \text{ is a block of } \pi\}$ is naturally identified with the Cartesian product

$$\text{NC}(s_2 - s_1 - 1) \times \cdots \times \text{NC}(s_m - s_{m-1} - 1) \times \text{NC}(n - s_m),$$

in a way that converts the sum

$$\sum_{\substack{\pi = \{V_1, \dots, V_p\} \in \text{NC}(n) \\ \text{with } V_1 = S}} \beta_{|V_2|} \cdots \beta_{|V_p|}$$

into the product

$$\prod_{\ell=1}^m \left(\sum_{\pi_\ell \in NC(s_{\ell+1}-s_\ell-1)} \prod_{V \in \pi_\ell} \beta_{|V|} \right), \tag{3.9}$$

where we set $s_{m+1} = n + 1$. An application of the moment-cumulant formula (3.3) shows that $\prod_{V \in \pi_\ell} \beta_{|V|} = \varphi(b^{s_{\ell+1}-s_\ell-1})$. (See the proof of [15, Theorem 16.15] for more details.) We conclude that

$$\text{Cf}_n(R_{ZZ^*}) = \sum_{m=1}^n \sum_{\substack{S \subseteq \{1, \dots, n\} \\ |S|=m \text{ and } 1 \in S}} \alpha_m \prod_{\ell=1}^m \varphi(b^{s_{\ell+1}-s_\ell-1}), \quad n \in \mathbb{N}. \tag{3.10}$$

Finally, observe that for every fixed $m \in \{1, \dots, n\}$ there is a natural bijection between tuples $(k_1, \dots, k_m) \in (\mathbb{N} \cup \{0\})^m$ with $k_1 + \dots + k_m = n - m$ (on the one hand) and subsets $1 \in S \subseteq \{1, \dots, n\}$ with $|S| = m$ (on the other), given by the formula

$$(k_1, \dots, k_m) \mapsto S = \{1, k_1 + 2, k_1 + k_2 + 3, \dots, k_1 + \dots + k_{m-1} + m\}.$$

The inner sums on the right-hand sides of (3.8) and (3.10) are identified term by term via this bijection, and this concludes the proof. □

In the remainder of this section we discuss the R -diagonal $*$ -distributions that satisfy the KMS condition. This is a special case of the class of $*$ -distribution studied in [16] (see, for instance [16, Remark 2.10]). The best known example of a KMS R -diagonal distribution is the one where, in the framework of the next definition, one sets $\alpha_1 = \lambda$, $\beta_1 = 1$ and $\alpha_n = \beta_n = 0$ for $n \geq 2$; this is called the λ -circular distribution, and is studied in detail in [16, Section 4].

Definition 3.6. Let $\mu \in \mathcal{D}_{\text{alg}}(1, *)$ be an R -diagonal $*$ -distribution with determining sequences $(\alpha_n)_{n=1}^\infty$ and $(\beta_n)_{n=1}^\infty$, and let t be a positive real number. We say that μ satisfies the *KMS condition* with parameter t if

$$\alpha_n = t\beta_n, \quad n \in \mathbb{N}. \tag{3.11}$$

The following result shows that the KMS condition is a generalization of the trace property, where the latter property occurs for the value $t = 1$ of the parameter.

Proposition 3.7. Let $\mu \in \mathcal{D}_{\text{alg}}(1, *)$ be an R -diagonal $*$ -distribution, let t be a positive real number, and suppose that μ satisfies the KMS condition with parameter t . Denote by

$$U_t : \mathbb{C}\langle Z, Z^* \rangle \rightarrow \mathbb{C}\langle Z, Z^* \rangle$$

the unique unital algebra homomorphism such that

$$U_t(Z) = tZ \quad \text{and} \quad U_t(Z^*) = \frac{1}{t}Z^*.$$

Then

$$\mu(PQ) = \mu(QU_t(P)), \quad P, Q \in \mathbb{C}\langle Z, Z^* \rangle. \tag{3.12}$$

Proof. Both sides of (3.12) are bilinear in P and Q , so it suffices to check the equation when both P and Q are monomials. Using the notation $Z^\emptyset = 1$, we must show that

$$\mu(Z^v Z^w) = \mu(Z^w U_t(Z^v)), \quad v, w \in \mathcal{W}^+ \cup \{\emptyset\}. \tag{3.13}$$

Equivalently, we must show that the set

$$\mathcal{S} = \{v \in \mathcal{W}^+ \cup \{\emptyset\} : \mu(Z^v Z^w) = \mu(Z^w U_t(Z^v)), w \in \mathcal{W}^+ \cup \{\emptyset\}\}$$

is equal to $\mathcal{W}^+ \cup \{\emptyset\}$. The set \mathcal{S} is clearly closed under concatenation and contains \emptyset . Therefore, it suffices to show that $\{1, *\} \subset \mathcal{S}$. In other words it suffices to prove that

$$\mu(ZZ^w) = t\mu(Z^w Z) \quad \text{and} \quad \mu(Z^*Z^w) = \frac{1}{t}\mu(Z^w Z^*), \quad w \in \mathcal{W}^+ \cup \{\emptyset\}. \quad (3.14)$$

We only prove the first equality in (3.14); the verification of the second one is analogous. The case $w = \emptyset$ follows from the fact (incorporated in the definition of an R -diagonal distribution) that $\mu(Z) = 0$. For the remainder part of the proof we fix a word $w = (\ell_1, \dots, \ell_n) \in \mathcal{W}^+$, for which we prove that $\mu(ZZ^w) = t\mu(Z^w Z)$. Observe that

$$\mu(ZZ_w) = \text{Cf}_{w_1}(M_\mu), \quad \mu(Z_w Z) = \text{Cf}_{w_2}(M_\mu),$$

where $w_1 := (1, \ell_1, \dots, \ell_n)$ and $w_2 := (\ell_1, \dots, \ell_n, 1)$. It is convenient to view w_1 and w_2 as functions from $\{1, \dots, n+1\}$ to $\{1, *\}$ and to record the fact that

$$w_2 = w_1 \circ \gamma_{n+1}, \quad (3.15)$$

where γ_{n+1} is the cyclic permutation $1 \mapsto 2 \mapsto \dots \mapsto n+1 \mapsto 1$ of $\{1, \dots, n+1\}$.

For every partition $\pi = \{V_1, \dots, V_k\} \in \text{NC}(n+1)$ we denote by $\gamma_{n+1}^{-1}(\pi)$ the partition (still in $\text{NC}(n+1)$) whose blocks are $\gamma_{n+1}^{-1}(V_1), \dots, \gamma_{n+1}^{-1}(V_k)$. The desired conclusion $\text{Cf}_{w_1}(M_\mu) = t\text{Cf}_{w_2}(M_\mu)$ is obtained from

$$\text{Cf}_{w_1;\pi}(R_\mu) = t\text{Cf}_{w_2;\gamma_{n+1}^{-1}(\pi)}(R_\mu), \quad \pi \in \text{NC}(n+1), \quad (3.16)$$

using the moment-cumulant formula. Indeed, sum both sides of (3.16) over $\pi \in \text{NC}(n+1)$ and invoke (3.1) applied to the words w_1 and w_2 . The sums thus obtained are precisely $\text{Cf}_{w_1}(M_\mu)$ on the left side and $t\text{Cf}_{w_2}(M_\mu)$ on the right.

Thus, it remains to prove (3.16). Fix a partition $\pi = \{V_1, \dots, V_k\} \in \text{NC}(n+1)$ such that $1 \in V_1$. Then $\gamma_{n+1}^{-1}(\pi) = \{W_1, \dots, W_k\}$, where $W_j = \gamma_{n+1}^{-1}(V_j)$, $1 \leq j \leq k$, and $n+1 \in W_1$. It follows from (3.15) that $w_1|_{V_j} = w_2|_{W_j} \in \mathcal{W}^+$ if $2 \leq j \leq k$, and thus

$$\text{Cf}_{w_1|_{V_j}}(R_\mu) = \text{Cf}_{w_2|_{W_j}}(R_\mu), \quad 2 \leq j \leq k \quad (3.17)$$

For the remaining block, we show that

$$\text{Cf}_{w_1|_{V_1}}(R_\mu) = t \cdot \text{Cf}_{w_2|_{W_1}}(R_\mu). \quad (3.18)$$

Indeed, suppose that $V_1 = \{j_1, \dots, j_m\}$ with $1 = j_1 < j_2 < \dots < j_m$, and therefore $W_1 = \{j_2 - 1, \dots, j_m - 1, n+1\}$. Both sides of (3.18) are 0 if m is odd or if m is even but $w_1|_{V_1}$ is not an alternating word. If m is even and $w_1|_{V_1}$ is alternating, then we find that $w_1|_{V_1} = \{1, *\}^{m/2}$ and $w_2|_{W_1} = \{*, 1\}^{m/2}$, which implies that $\text{Cf}_{w_1|_{V_1}}(R_\mu) = \alpha_{m/2}$ and $\text{Cf}_{w_2|_{W_1}}(R_\mu) = \beta_{m/2}$. In this case, (3.18) follows from the KMS hypothesis.

Finally, for the partition π fixed in the preceding paragraph we write:

$$\begin{aligned} \text{Cf}_{w_1;\pi}(R_\mu) &= \text{Cf}_{w_1|_{V_1}}(R_\mu) \prod_{j=2}^k \text{Cf}_{w_1|_{V_j}}(R_\mu) \\ &= t\text{Cf}_{w_2|_{W_1}}(R_\mu) \prod_{j=2}^k \text{Cf}_{w_2|_{W_j}}(R_\mu) \quad (\text{by (3.17) and (3.18)}) \\ &= t\text{Cf}_{w_2;\gamma_{n+1}^{-1}(\pi)}(R_\mu), \end{aligned}$$

thus concluding the proof of (3.16). \square

Remark 3.8. The converse of Proposition 3.7 is also true. More precisely, every R -diagonal distribution $\mu \in \mathcal{D}_{\text{alg}}(1, *)$ that satisfies (3.12) for some $t \in (0, +\infty)$ must also satisfy the KMS condition for the same value of t . To see this, let $(\alpha_n)_{n=1}^\infty$ and $(\beta_n)_{n=1}^\infty$ be the determining sequences of μ . Equation (3.12) yields, in particular, the identity

$$\mu((ZZ^*)^n) = t\mu((Z^*Z)^n), \quad n \in \mathbb{N}. \tag{3.19}$$

This identity implies $\alpha_n = t\beta_n$, $n \in \mathbb{N}$, by induction on n . For the induction step one invokes the moment-cumulant formula in order to expand both sides of (3.19) as sums over $NC(2n)$; then the action of the cyclic permutation γ_{2n}^{-1} on $NC(2n)$ can be used in the same way as it was done in the proof of Proposition 3.7.

4. The framework of $\mathcal{D}_c(1, *)$ and $\mathcal{D}_c(k)$, BBP bijections

We now introduce the analytic framework which is of interest for the present paper.

Definition 4.1. (1) Let (\mathcal{A}, φ) be a C^* -probability space (which means that \mathcal{A} is a unital C^* -algebra, $\varphi : \mathcal{A} \rightarrow \mathbb{C}$ is a positive linear functional, and $\varphi(1) = 1$), and let $a \in \mathcal{A}$. The **-distribution of a* is the functional $\mu \in \mathcal{D}_{\text{alg}}(1, *)$ determined by the requirement that

$$\mu(Z^w) = \varphi(a^w), \quad w \in \mathcal{W}^+.$$

(2) We denote by $\mathcal{D}_c(1, *)$ the set of all elements of $\mathcal{D}_{\text{alg}}(1, *)$ that are equal to the *-distribution of some element in a C^* -probability space.

(3) Free additive (respectively, multiplicative) convolution is a binary operation on $\mathcal{D}_c(1, *)$ denoted by \boxplus (respectively, \boxtimes). This operation is uniquely determined by the following property: given elements a, a' in a C^* -probability space (\mathcal{A}, φ) such that $\{a, a^*\}$ is free from $\{a', (a')^*\}$, the *-distribution of $a + a'$ (respectively, aa') is the free additive (respectively, multiplicative) convolution of the *-distributions of a and a' . See [15, Lectures 5 and 7] for more details.

(4) An element $\mu \in \mathcal{D}_c(1, *)$ is said to be *\boxplus -infinitely divisible* if for every $n \in \mathbb{N}$ there exists $\mu_n \in \mathcal{D}_c(1, *)$ such that

$$\mu = \underbrace{\mu_n \boxplus \cdots \boxplus \mu_n}_{n \text{ times}}.$$

The set of all \boxplus -infinitely divisible distributions in $\mathcal{D}_c(1, *)$ is denoted by $\mathcal{D}_c^{(\text{inf-div})}(1, *)$.

The main result of this section is the following theorem. The series R_ν and η_μ appearing in the statement of the theorem are as defined in Sections 2 and 3.

Theorem 4.2 (BBP bijection on $\mathcal{D}_c(1, *)$). *There exists a bijection*

$$\mathbb{B}_{(1,*)} : \mathcal{D}_c(1, *) \rightarrow \mathcal{D}_c^{(\text{inf-div})}(1, *),$$

determined by the requirement that

$$R_{\mathbb{B}_{(1,*)}(\mu)} = \eta_\mu, \quad \mu \in \mathcal{D}_c(1, *). \tag{4.1}$$

More precisely, for every $\mu \in \mathcal{D}_c(1, *)$ there exists a unique *-distribution $\nu \in \mathcal{D}_c^{(\text{inf-div})}(1, *)$ such that $R_\nu = \eta_\mu$, and we define $\mathbb{B}_{(1,*)}(\mu) := \nu$.

Definition and Remark 4.3 (Framework of $\mathcal{D}_c(k)$). We reduce Theorem 4.2 to an analogous theorem proved in [3] for the space, denoted by $\mathcal{D}_c(2)$, of joint distributions of pairs of selfadjoint elements in a C^* -probability space. The passage from $\mathcal{D}_c(1, *)$ to $\mathcal{D}_c(2)$ is natural, and essentially amounts to the change of variables

$$(a, a^*) \mapsto \left(\frac{a + a^*}{2}, \frac{a - a^*}{2i} \right),$$

for a in a C^* -probability space (\mathcal{A}, φ) . In order to clarify this idea, we review briefly the framework of $\mathcal{D}_c(k)$. Fix $k \in \mathbb{N}$.

(1) We denote by $\mathbb{C}\langle X_1, \dots, X_k \rangle$ the algebra of polynomials in the non-commuting indeterminates X_1, \dots, X_k .

(2) Let (\mathcal{A}, φ) be a C^* -probability space and let $b_1, \dots, b_k \in \mathcal{A}$ be selfadjoint. The *joint distribution* of b_1, \dots, b_k is the linear functional $\lambda : \mathbb{C}\langle X_1, \dots, X_k \rangle \rightarrow \mathbb{C}$ which is determined by the requirement that $\lambda(1) = 1$ and

$$\lambda(X_{i_1} \cdots X_{i_n}) = \varphi(b_{i_1} \cdots b_{i_n}), \quad n \in \mathbb{N}, i_1, \dots, i_n \in \{1, \dots, k\}.$$

(3) We denote by $\mathcal{D}_c(k)$ the set of all linear functionals $\lambda : \mathbb{C}\langle X_1, \dots, X_k \rangle \rightarrow \mathbb{C}$ that can arise as joint distributions of k -tuples of selfadjoint elements in some C^* -probability space.

(4) Free additive convolution is a binary operation on $\mathcal{D}_c(k)$ denoted⁴ by \boxplus . This operation is uniquely determined by the following property: given selfadjoint elements a_1, \dots, a_k and b_1, \dots, b_k in a C^* -probability space (\mathcal{A}, φ) such that $\{a_1, \dots, a_k\}$ is free from $\{b_1, \dots, b_k\}$, the joint distribution of $a_1 + b_1, \dots, a_k + b_k$ is the free additive convolution of the joint distributions of a_1, \dots, a_k and b_1, \dots, b_k . The concept of \boxplus -infinite divisibility in $\mathcal{D}_c(k)$ is introduced as in Definition 4.1(4). The set of \boxplus -infinitely divisible distributions in $\mathcal{D}_c(k)$ is denoted by $\mathcal{D}_c^{(\text{inf-div})}(k)$.

(5) We denote by $\mathbb{C}_0\langle\langle x_1, \dots, x_k \rangle\rangle$ the space of those formal power series with complex coefficients in k non-commuting indeterminates x_1, \dots, x_k whose constant term is equal to 0. We denote by $\text{Cf}_{(i_1, \dots, i_n)}(f)$ the coefficient of $x_{i_1} \cdots x_{i_n}$ in a series $f \in \mathbb{C}_0\langle\langle x_1, \dots, x_k \rangle\rangle$. Every joint distribution $\lambda \in \mathcal{D}_c(k)$ has a moment series M_λ , an R -transform R_λ and an η -series η_λ . These are elements of $\mathbb{C}_0\langle\langle x_1, \dots, x_k \rangle\rangle$, and their definitions are analogous to Definitions 2.3 and 3.2. A detailed description of these power series and of the relations between their coefficients can be found in [3, pp. 14–17].

The proof of Theorem 4.2 will be reduced to the following result from [3] (see also [5] for the case $k = 1$).

Theorem 4.4 (BBP bijection on $\mathcal{D}_c(k)$). *Let k be a positive integer. There exists a bijection $\mathbb{B}_k : \mathcal{D}_c(k) \rightarrow \mathcal{D}_c^{(\text{inf-div})}(k)$, determined by the requirement that*

$$R_{\mathbb{B}_k(\lambda)} = \eta_\lambda, \quad \lambda \in \mathcal{D}_c(k). \tag{4.2}$$

More precisely, for every $\lambda \in \mathcal{D}_c(k)$ there exists a unique $v \in \mathcal{D}_c^{(\text{inf-div})}(k)$ such that $R_v = \eta_\lambda$, and we define $\mathbb{B}_k(\lambda) := v$.

Remark 4.5. We only need Theorem 4.4 for $k = 1$ and $k = 2$. When $k = 1$, the space $\mathcal{D}_c(1)$ is naturally identified with the space \mathcal{P}_c of compactly supported Borel probability measures on \mathbb{R} . Indeed, given $b = b^*$ in a C^* -probability space (\mathcal{A}, φ) , Definition 4.3(2) produces a linear functional $\lambda : \mathbb{C}\langle x_1 \rangle \rightarrow \mathbb{C}$ which becomes, via the Riesz representation theorem, a Borel probability measure supported on the spectrum of b . The original BBP bijection from [5] was defined on \mathcal{P}_c (and on the larger set \mathcal{P} of all Borel probability measures on \mathbb{R}). In Section 6, we will simply talk about $\mathbb{B}(\sigma)$ for $\sigma \in \mathcal{P}_c$. In other words if λ denotes the functional in $\mathcal{D}_c(1)$ corresponding to σ , then $\mathbb{B}(\sigma) \in \mathcal{P}_c$ denotes the probability measure corresponding to $\mathbb{B}_1(\lambda)$.

The following result creates bijections C and D that we use in conjunction with the case $k = 2$ of Theorem 4.4. (The letters C and D are meant to suggest *complexification* and *decomplexification*.) The proof is immediate, and therefore omitted.

Proposition 4.6. *There exists a bijection $D : \mathcal{D}_c(1, *) \rightarrow \mathcal{D}_c(2)$ defined as follows. Given $\mu \in \mathcal{D}_c(1, *)$ that is the $*$ -distribution of an element a in a C^* -probability space (\mathcal{A}, φ) , $D(\mu)$ is the joint distribution of the pair*

$$\left(\frac{a + a^*}{2}, \frac{a - a^*}{2i} \right).$$

*The inverse of D is the bijection $C : \mathcal{D}_c(2) \rightarrow \mathcal{D}_c(1, *)$ defined as follows. Given $\lambda \in \mathcal{D}_c(2)$ that is the joint distribution of a pair b_1, b_2 of selfadjoint elements in a C^* -probability space (\mathcal{A}, φ) , $C(\lambda)$ is the $*$ -distribution of $b_1 + ib_2$.*

⁴It is customary to always denote free additive convolution by “ \boxplus ”. The setting in which the symbol \boxplus is used should be clear, in each case, from the context.

Definition and Remark 4.7. In addition to the transformations C and D , the proof of Theorem 4.2 requires the corresponding change of variables for power series. Denote by

$$t_{1,1} = t_{1,*} = \frac{1}{2} \quad \text{and} \quad t_{2,1} = \frac{1}{2i}, \quad t_{2,*} = -\frac{1}{2i}, \tag{4.3}$$

the coefficients of the linear transformation $b_1 = (a + a^*)/2, b_2 = (a - a^*)/2i$. This transformation can now be written more compactly as

$$b_i = \sum_{\ell \in \{1,*\}} t_{i,\ell} a^\ell, \quad \text{for } i = 1, 2. \tag{4.4}$$

Using the coefficients $t_{i,\ell}$ we define a map $\tilde{D} : \mathbb{C}_0\langle\langle z, z^* \rangle\rangle \rightarrow \mathbb{C}_0\langle\langle x_1, x_2 \rangle\rangle$ as follows: given a series $f \in \mathbb{C}_0\langle\langle z, z^* \rangle\rangle$, the coefficients of the series $g = \tilde{D}(f) \in \mathbb{C}_0\langle\langle x_1, x_2 \rangle\rangle$ are given by

$$\text{Cf}_{(i_1, \dots, i_n)}(g) := \sum_{\ell_1, \dots, \ell_n \in \{1,*\}} t_{i_1, \ell_1} \cdots t_{i_n, \ell_n} \text{Cf}_{(\ell_1, \dots, \ell_n)}(f), \quad n \in \mathbb{N}, i_1, \dots, i_n \in \{1, 2\}. \tag{4.5}$$

The map \tilde{D} is clearly linear and bijective. Its inverse $\tilde{C} : \mathbb{C}_0\langle\langle x_1, x_2 \rangle\rangle \rightarrow \mathbb{C}_0\langle\langle z, z^* \rangle\rangle$ is defined by a formula analogous to (4.5), but with $[t_{i,\ell}]$ replaced by the inverse matrix

$$t'_{1,1} = 1, \quad t'_{1,2} = i \quad \text{and} \quad t'_{*,1} = 1, \quad t'_{*,2} = -i.$$

Lemma 4.8. For every $*$ -distribution $\mu \in \mathcal{D}_c(1, *)$, we have

$$M_{D(\mu)} = \tilde{D}(M_\mu), \quad R_{D(\mu)} = \tilde{D}(R_\mu) \quad \text{and} \quad \eta_{D(\mu)} = \tilde{D}(\eta_\mu). \tag{4.6}$$

Proof. Suppose that $\mu \in \mathcal{D}_c(1, *)$ is the $*$ -distribution of an element a in a C^* -probability space (\mathcal{A}, φ) , and set $b_i = \sum_{\ell \in \{1,*\}} t_{i,\ell} a^\ell$ for $i = 1, 2$. By definition, the joint distribution of b_1, b_2 is $D(\mu) \in \mathcal{D}_c(2)$. To verify the first identity in (4.6), fix $n \in \mathbb{N}$ and $i_1, \dots, i_n \in \{1, 2\}$, and calculate directly

$$\begin{aligned} \text{Cf}_{(i_1, \dots, i_n)}(M_{D(\mu)}) &= \varphi(b_{i_1} \cdots b_{i_n}) \\ &= \varphi((t_{i_1,1} a^1 + t_{i_1,*} a^*) \cdots (t_{i_n,1} a^1 + t_{i_n,*} a^*)) \\ &= \sum_{\ell_1, \dots, \ell_n \in \{1,*\}} t_{i_1, \ell_1} \cdots t_{i_n, \ell_n} \varphi(a^{\ell_1} \cdots a^{\ell_n}) \\ &= \sum_{\ell_1, \dots, \ell_n \in \{1,*\}} t_{i_1, \ell_1} \cdots t_{i_n, \ell_n} \text{Cf}_{(\ell_1, \dots, \ell_n)}(M_\mu) \\ &= \text{Cf}_{(i_1, \dots, i_n)} \tilde{D}(M_\mu). \end{aligned}$$

The second equality in (4.6) follows from a similar multilinearity argument, using the fact that the C^* -probability space (\mathcal{A}, φ) carries a family of multilinear functionals $(\kappa_n : \mathcal{A}^n \rightarrow \mathbb{C})_{n=1}^\infty$, called free cumulant functionals, such that

$$\begin{aligned} \text{Cf}_{(i_1, \dots, i_n)}(R_{D(\mu)}) &= \kappa_n(b_{i_1}, \dots, b_{i_n}), \quad n \in \mathbb{N}, i_1, \dots, i_n \in \{1, 2\} \quad \text{and} \\ \text{Cf}_{(\ell_1, \dots, \ell_n)}(R_\mu) &= \kappa_n(a^{\ell_1}, \dots, a^{\ell_n}), \quad n \in \mathbb{N}, \ell_1, \dots, \ell_n \in \{1, *\}. \end{aligned}$$

(This multilinearity argument is precisely the one used to describe the behavior of the R -transform under linear transformations [15, Proposition 16.12].)

The third equality (4.6) follows from a similar multilinearity argument, using the Boolean cumulant functionals $(\beta_n : \mathcal{A}^n \rightarrow \mathbb{C})_{n=1}^\infty$ (for a discussion of Boolean cumulants see, for instance, [12, Section 4.6]). \square

Lemma 4.9. Let $D : \mathcal{D}_c(1, *) \rightarrow \mathcal{D}_c(2)$ be the bijection defined in Proposition 4.6. Then:

- (1) $D(\mu \boxplus \mu') = D(\mu) \boxplus D(\mu')$ for every $\mu, \mu' \in \mathcal{D}_c(1, *)$.
- (2) $D(\mathcal{D}_c^{(\text{inf-div})}(1, *)) = \mathcal{D}_c^{(\text{inf-div})}(2)$.

Proof. (1) Since every $\lambda \in \mathcal{D}_c(2)$ is uniquely determined by its R -transform, it suffices to verify that $D(\mu \boxplus \mu')$ and $D(\mu) \boxplus D(\mu')$ have the same R -transform. Indeed,

$$\begin{aligned} R_{D(\mu \boxplus \mu')} &= \tilde{D}(R_{\mu \boxplus \mu'}) \quad (\text{by Lemma 4.8}) \\ &= \tilde{D}(R_\mu) + \tilde{D}(R_{\mu'}) \quad (\text{since } R_{\mu \boxplus \mu'} = R_\mu + R_{\mu'} \text{ and } \tilde{D} \text{ is linear}) \\ &= R_{D(\mu)} + R_{D(\mu')} \quad (\text{by Lemma 4.8}) \\ &= R_{D(\mu) \boxplus D(\mu')}. \end{aligned}$$

Part (2) follows immediately from (1) and from the definition of \boxplus -infinite divisibility. □

Proof of Theorem 4.2. We define the required bijection $\mathbb{B}_{(1,*)}$ so that the diagram

$$\begin{array}{ccc} \mathcal{D}_c(1, *) & \xrightarrow{\mathbb{B}_{(1,*)}} & \mathcal{D}_c^{(\text{inf-div})}(1, *) \\ D \downarrow & & \downarrow D \\ \mathcal{D}_c(2) & \xrightarrow{\mathbb{B}_2} & \mathcal{D}_c^{(\text{inf-div})}(2) \end{array}$$

is commutative, where D is defined in Proposition 4.6 and \mathbb{B}_2 is provided by Theorem 4.4 for $k = 2$. More precisely, let $D_0 : \mathcal{D}_c^{(\text{inf-div})}(1, *) \rightarrow \mathcal{D}_c^{(\text{inf-div})}(2)$ be the restriction of D ; this is a bijection by Lemma 4.9(2). Then define

$$\mathbb{B}_{(1,*)} := D_0^{-1} \circ \mathbb{B}_2 \circ D.$$

Pick an arbitrary $\mu \in \mathcal{D}_c(1, *)$, denote $\mathbb{B}_{(1,*)}(\mu) = \nu$. We prove that $R_\nu = \eta_\mu$. Since \tilde{D} is injective, it suffices to verify that $\tilde{D}(R_\nu) = \tilde{D}(\eta_\mu)$. Indeed, the definition of $\mathbb{B}_{(1,*)}$ implies $\mathbb{B}_2(D(\mu)) = D(\nu)$, and the definition of \mathbb{B}_2 yields $R_{D(\nu)} = \eta_{D(\mu)}$. Thus

$$\tilde{D}(R_\nu) = R_{D(\nu)} = \eta_{D(\mu)} = \tilde{D}(\eta_\mu),$$

as required. □

5. Parametrization of η -diagonal distributions in $\mathcal{D}_c(1, *)$

We show that an η -diagonal $*$ -distribution $\mu \in \mathcal{D}_c(1, *)$ is naturally parametrized by the pair of compactly supported probability measures on $[0, \infty)$ that arise as the distributions of ZZ^* and of Z^*Z in the noncommutative probability space $(\mathbb{C}\langle Z, Z^* \rangle, \mu)$.

Definition and Remark 5.1. Suppose that $\mu \in \mathcal{D}_c(1, *)$ is the $*$ -distribution of an element a in a C^* -probability space (\mathcal{A}, φ) . Basic considerations on positive elements in a C^* -probability space (see, for instance, [15, Propositions 3.13 and 3.6]) show the existence of compactly supported Borel probability measures σ_1, σ_2 on $[0, \infty)$ such that

$$\varphi((aa^*)^n) = \int_0^\infty t^n d\sigma_1(t) \quad \text{and} \quad \varphi((a^*a)^n) = \int_0^\infty t^n d\sigma_2(t), \quad n \in \mathbb{N}.$$

Thus σ_1 and σ_2 satisfy

$$\int_0^\infty t^n d\sigma_1(t) = \mu((ZZ^*)^n), \quad n \in \mathbb{N} \tag{5.1}$$

and

$$\int_0^\infty t^n d\sigma_2(t) = \mu((Z^*Z)^n), \quad n \in \mathbb{N}. \tag{5.2}$$

Moreover, σ_1 and σ_2 are uniquely determined by (5.1) and (5.2) since a compactly supported probability measure on \mathbb{R} is determined by its moments. We refer to σ_1 and σ_2 as the *distributions* of ZZ^* and Z^*Z , respectively, in the $*$ -probability space $(\mathbb{C}\langle Z, Z^* \rangle, \mu)$.

The following theorem provides the parametrization announced in the title of the section.

Theorem 5.2. *Let \mathcal{P}_c^+ denote the set of all compactly supported Borel probability measures on $[0, \infty)$. There is bijective map*

$$\Phi : \mathcal{P}_c^+ \times \mathcal{P}_c^+ \rightarrow \{ \mu \in \mathcal{D}_c(1, *) : \mu \text{ is } \eta\text{-diagonal} \}$$

described as follows: given $\sigma_1, \sigma_2 \in \mathcal{P}_c^+$, $\Phi(\sigma_1, \sigma_2)$ is the unique η -diagonal $$ -distribution $\mu \in \mathcal{D}_c(1, *)$ such that the distributions of ZZ^* and Z^*Z in $(\mathbb{C}\langle Z, Z^* \rangle, \mu)$ are equal to σ_1 and σ_2 , respectively.*

The point of Theorem 5.2 is that the map Φ is defined on all of $\mathcal{P}_c^+ \times \mathcal{P}_c^+$. In other words, for every $\sigma_1, \sigma_2 \in \mathcal{P}_c^+$ there exists an η -diagonal $*$ -distribution $\mu \in \mathcal{D}_c(1, *)$ such that (5.1) and (5.2) hold. We prove this by producing an *operator model* for μ : starting from σ_1 and σ_2 we construct explicitly an operator A on a Hilbert space \mathcal{K} such that the $*$ -distribution of A with respect to a suitably chosen functional on $B(\mathcal{K})$ is the required η -diagonal distribution. The bulk of this section is devoted to the description of the operator model. At the end, we complete the proof of Theorem 5.2. The construction of the operator model is described in the next remark.

Remark and Notation 5.3 (Description of the operator model). Fix $\sigma_1, \sigma_2 \in \mathcal{P}_c^+$ which we take as the input for our construction of an η -diagonal operator. In the description of the construction, it is convenient to use the *symmetric square roots* of σ_1 and σ_2 . These are the symmetric compactly supported Borel probability measures $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ on \mathbb{R} with moments given by the formula

$$\int_{-\infty}^\infty t^n d\tilde{\sigma}_j(t) = \begin{cases} 0, & n \text{ odd,} \\ \int_0^\infty t^{n/2} d\sigma_j(t), & n \text{ even,} \end{cases}$$

for $j = 1, 2$. Our construction of an η -diagonal operator proceeds in three steps.

Step 1. We construct a Hilbert space \mathcal{H} , a selfadjoint operator $X \in B(\mathcal{H})$, and vectors $\xi_1, \xi_2 \in \mathcal{H}$ with the following properties:

- (1a) $\|\xi_1\| = \|\xi_2\| = 1$,
- (1b) $\langle \xi_1, \xi_2 \rangle = 0$,
- (1c) $\langle X^k \xi_1, \xi_2 \rangle = \langle X^k \xi_2, \xi_1 \rangle = 0$ for $k \in \mathbb{N}$,
- (1d) $\langle X^{2k-1} \xi_j, \xi_j \rangle = 0$ and $\langle X^{2k} \xi_j, \xi_j \rangle = \int_0^\infty t^k d\sigma_j(t)$ for $k \in \mathbb{N}$ and $j = 1, 2$.

In other words, property (1d) says that X has distribution $\tilde{\sigma}_1$ with respect to the vector state defined by ξ_1 , and distribution $\tilde{\sigma}_2$ with respect to the one defined by ξ_2 , on the operator algebra $B(\mathcal{H})$. For the actual construction of X consider, for $j = 1, 2$, Hilbert spaces \mathcal{M}_j , selfadjoint operators $T_j \in B(\mathcal{M}_j)$, and unit vectors $\eta_j \in \mathcal{M}_j$, such that the distribution of T_j with respect to the vector state η_j is $\tilde{\sigma}_j$. Then set $\mathcal{H} = \mathcal{M}_1 \oplus \mathcal{M}_2$, $X = T_1 \oplus T_2$, $\xi_1 = \eta_1 \oplus 0$, and $\xi_2 = 0 \oplus \eta_2$. Properties (1a)–(1d) are then easily verified. (In subsequent steps we only use the properties (1a)–(1d). The precise description of \mathcal{H} , X , ξ_1 , and ξ_2 is not necessary.)

Step 2. Define a rank-one partial isometry $Y \in B(\mathcal{H})$ by setting $Y(\zeta) = \langle \zeta, \xi_1 \rangle \xi_2$ for $\zeta \in \mathcal{H}$. We have $Y\xi_1 = \xi_2$, $Y^*\zeta = \langle \zeta, \xi_2 \rangle \xi_1$ for $\zeta \in \mathcal{H}$, and

$$YY^*\zeta = \langle \zeta, \xi_2 \rangle \xi_2, \quad Y^*Y\zeta = \langle \zeta, \xi_1 \rangle \xi_1. \tag{5.3}$$

Thus YY^* and Y^*Y are the orthogonal projections onto the 1-dimensional spaces generated by ξ_2 and ξ_1 , respectively.

Step 3. Consider the Hilbert space $\mathcal{K} = \mathcal{H} \otimes \mathcal{H}$ and the unit vector $\xi = \xi_1 \otimes \xi_2 \in \mathcal{K}$, then consider the C^* -probability space $(B(\mathcal{K}), \varphi_\xi)$, where $\varphi_\xi(T) = \langle T\xi, \xi \rangle$ for $T \in B(\mathcal{K})$. Let $V \in B(\mathcal{H} \otimes \mathcal{H})$ be the flip operator determined by the requirement that by $V(\zeta \otimes \zeta') = \zeta' \otimes \zeta$, $\zeta, \zeta' \in \mathcal{H}$. Note that V is a symmetry (that is, it is self-adjoint and $V^2 = I$). Finally, define $A = V(Y \otimes X)$.

This concludes the construction of the variable A in $(B(\mathcal{K}), \varphi_\xi)$.

We now take on the proof that the operator A constructed above has the desired η -diagonal distribution with respect to the functional φ_ξ . We start by recording some easily verified identities satisfied by A , the proof of which is left to the reader.

Lemma 5.4. *Consider the framework of Remark 5.3. We have*

$$AA^* = X^2 \otimes YY^*, \quad A^*A = Y^*Y \otimes X^2 \quad (5.4)$$

and

$$A^2 = XY \otimes YX, \quad (A^*)^2 = Y^*X \otimes XY^*. \quad (5.5)$$

The following lemma establishes the distributions of AA^* and A^*A along with a few non-alternating $*$ -moments of A .

Lemma 5.5. *Let A and ξ be as above, then for any integer $k \geq 0$ we have*

- (1) $\varphi_\xi((AA^*)^k) = \int_0^\infty t^k d\sigma_1(t)$,
- (2) $\varphi_\xi(A(AA^*)^k) = 0$,
- (3) $\varphi_\xi(A^*(AA^*)^k) = 0$,
- (4) $\varphi_\xi((A^*A)^k) = \int_0^\infty t^k d\sigma_2(t)$,
- (5) $\varphi_\xi(A(A^*A)^k) = 0$,
- (6) $\varphi_\xi(A^*(A^*A)^k) = 0$.

Proof. We verify only the first three equations. The proof of (4)–(6) is similar. We have

$$(AA^*)^k = X^{2k} \otimes (YY^*)^k = X^{2k} \otimes (YY^*),$$

so

$$\varphi_\xi((AA^*)^k) = \langle (X^{2k} \otimes (YY^*))\xi, \xi \rangle = \langle X^{2k}\xi_1, \xi_1 \rangle,$$

and (1) follows from property (1d) in Step 1 of the construction of A . To prove (2), we calculate

$$A(AA^*)^k = V(Y \otimes X)(X^{2k} \otimes YY^*),$$

thus

$$\begin{aligned} \varphi_\xi(A(AA^*)^k) &= \langle (Y \otimes X)(X^{2k} \otimes YY^*)\xi_1 \otimes \xi_2, V\xi_1 \otimes \xi_2 \rangle \\ &= \langle (Y \otimes X)(X^{2k}\xi_1 \otimes \xi_2), \xi_2 \otimes \xi_1 \rangle \\ &= \langle YX^{2k}\xi_1, \xi_2 \rangle \langle X\xi_2, \xi_1 \rangle = 0, \end{aligned}$$

because $\langle X\xi_2, \xi_1 \rangle = 0$, thereby concluding the proof of (2). Similarly,

$$A^*(AA^*)^k = (X \otimes Y^*)V(X^{2k} \otimes YY^*),$$

so

$$\begin{aligned} \varphi_\xi(A^*(AA^*)^k) &= \langle (X \otimes Y^*)V(X^{2k} \otimes YY^*)\xi, \xi \rangle = \langle X \otimes Y^*(\xi_2 \otimes X^{2k}\xi_1), \xi_1 \otimes \xi_2 \rangle \\ &= \langle X\xi_2, \xi_1 \rangle \langle Y^*X^{2k}\xi_1, \xi_2 \rangle = 0 \end{aligned}$$

by property (1c) in Step 1 of the construction of A . □

The next lemma gives some properties of the operator A that are useful in verifying its η -diagonality.

Lemma 5.6. *Let A and ξ be as above, then for any integer $k \geq 0$ we have*

- (1) $A^2(AA^*)^k\xi = 0$,
- (2) $(A^*)^2(AA^*)^k\xi = 0$,
- (3) $(A^*A)(AA^*)^k\xi = \int_0^\infty t^k d\sigma_1(t) \cdot A^*A\xi$,
- (4) $A^2(A^*A)^k\xi = 0$,
- (5) $(A^*)^2(A^*A)^k\xi = 0$,
- (6) $(AA^*)(A^*A)^k\xi = \int_0^\infty t^k d\sigma_2(t) \cdot AA^*\xi$.

Proof. As in the previous proof we only verify (1)–(3). We have

$$(A)^2(AA^*)^k = (XY \otimes YX)(X^{2k} \otimes YY^*) = XYX^{2k} \otimes YXY Y^*,$$

and using the fact that $YY^*\xi_2 = \xi_2$ we see that $YXY Y^*\xi_2 = \langle X\xi_2, \xi_1 \rangle \xi_2 = 0$ by property (1c). Similarly,

$$(A^*)^2(AA^*)^k = (Y^*X \otimes XY^*)(X^{2k} \otimes YY^*) = Y^*X^{2k+1} \otimes XY^*,$$

and (2) follows because $Y^*X^{2k+1}\xi_1 = \langle X^{2k+1}\xi_1, \xi_2 \rangle \xi_1 = 0$ by (1c). Finally, (5.4) yields

$$(A^*A)(AA^*)^k = (Y^*Y \otimes X^2)(X^{2k} \otimes YY^*) = Y^*YX^{2k} \otimes X^2YY^*.$$

Observe that $Y^*YX^{2k}\xi_1 = \langle X^{2k}\xi_1, \xi_1 \rangle \xi_1 = \int_0^\infty t^k d\sigma_1(t) \cdot \xi_1$ by (1d), while $YY^*\xi_2 = \xi_2$. Therefore

$$(A^*A)(AA^*)^k = \int_0^\infty t^k d\sigma_1(t) \cdot (\xi_1 \otimes X^2\xi_2) = \int_0^\infty t^k d\sigma_1(t) \cdot A^*A\xi,$$

thus proving (3). □

Corollary 5.7. *Let $W = W_1 W_2 \cdots W_d$ be a mixed-alternating word in A and A^* , factored as in (2.7) with $d \geq 2$. Then*

$$W\xi = \varphi_\xi(W_2)\varphi_\xi(W_3)\cdots\varphi_\xi(W_d)W_1\xi.$$

Proof. Parts (3) and (6) of the preceding lemma yield the conclusion when $d = 2$. The general case follows easily by induction on d . □

Proposition 5.8. *Let σ_1 and σ_2 be probability measures in \mathcal{P}_c^+ , and let the operator A in $(\mathcal{B}(\mathcal{K}), \varphi_\xi)$ be constructed as in Remark 5.3. Then the $*$ -distribution of A is η -diagonal. Moreover, the distributions of AA^* and A^*A are σ_1 and σ_2 , respectively.*

Proof. The second assertion follows from parts (1) and (4) of Lemma 5.5. It remains to prove that the distribution of A is η -diagonal, and to do this we verify the conditions in Theorem 2.8. Let $W = W_1 W_2 \cdots W_d$ be a mixed-alternating word in A and A^* , factored as in (2.7). Corollary 5.7 yields

$$\varphi_\xi(W) = \varphi_\xi(W_2)\varphi_\xi(W_3)\cdots\varphi_\xi(W_d)\langle W_1\xi, \xi \rangle = \prod_{j=1}^d \varphi_\xi(W_j),$$

thus verifying condition (η DM2). Finally we verify condition (η DM1). Suppose that V is a word in A and A^* that is not mixed-alternating, and choose a mixed-alternating word W of maximum length with the property that V can be written as $V = UW$ for some non-empty word U . Also, write $W = W_1W_2 \cdots W_d$ as in (2.7). We have

$$\varphi_\xi(V) = \langle V\xi, \xi \rangle = \prod_{j=2}^d \varphi_\xi(W_j) \langle UW_1\xi, \xi \rangle$$

by Corollary 5.7. If $|U| = 1$, the equality $\varphi_\xi(V) = 0$ follows from Lemma 5.5. If $|U| \geq 2$, then U is of the form $U'AA$ or $U'A^*A^*$ for some (possibly empty) word U' . In this case, $\varphi_\xi(V) = 0$ by Lemma 5.6. \square

We conclude the discussion of the parametrization announced at the beginning of the section.

Proof of Theorem 5.2. We first note that the map Φ is well-defined. Indeed, let $\sigma_1, \sigma_2 \in \mathcal{P}_c^+$ be given. The existence of an η -diagonal $*$ -distribution $\mu \in \mathcal{D}_c(1, *)$ which fulfils the conditions (5.2) is ensured by Proposition 5.8. The uniqueness of μ follows from the fact that an η -diagonal $*$ -distribution is completely determined by its alternating $*$ -moments, as we saw in Theorem 2.8.

The surjectivity of Φ is immediate from its definition: every η -diagonal $*$ -distribution $\mu \in \mathcal{D}_c(1, *)$ can be written as $\Phi(\sigma_1, \sigma_2)$, where $\sigma_1, \sigma_2 \in \mathcal{P}_c^+$ are the distributions of ZZ^* and respectively Z^*Z in $(\mathbb{C}\langle Z, Z^* \rangle, \mu)$, in the sense discussed in Definition 5.1.

Finally, the injectivity of Φ is immediate as well. Indeed, if $\Phi(\sigma_1, \sigma_2) = \mu$, then the moments of σ_1 and σ_2 can be retrieved as alternating moments of μ , and compactly supported probability measures on \mathbb{R} are determined by their moments. \square

6. Parametrization of infinitely divisible R-diagonal distributions

In this section we use the BBP method to characterize \boxplus -infinitely divisible R -diagonal distributions. The parametrization mentioned in the title of the section arises naturally, in the way indicated in the following remark.

Remark and Notation 6.1. Let $\mathcal{R}_c^{(\text{inf-div})}$ denote the set of all the R -diagonal distributions in $\mathcal{D}_c(1, *)$ that are \boxplus -infinitely divisible. It is immediate that the bijection $\mathbb{B}_{(1,*)}$ from Theorem 4.2 induces a bijection (still denoted $\mathbb{B}_{(1,*)}$)

$$\mathbb{B}_{(1,*)} : \{ \mu \in \mathcal{D}_c(1, *) : \mu \text{ is } \eta\text{-diagonal} \} \rightarrow \mathcal{R}_c^{(\text{inf-div})}. \tag{6.1}$$

On the other hand, Theorem 5.2 provides a natural bijection

$$\Phi : \mathcal{P}_c^+ \times \mathcal{P}_c^+ \rightarrow \{ \mu \in \mathcal{D}_c(1, *) : \mu \text{ is } \eta\text{-diagonal} \}. \tag{6.2}$$

The map

$$\Psi := \mathbb{B}_{(1,*)} \circ \Phi : \mathcal{P}_c^+ \times \mathcal{P}_c^+ \rightarrow \mathcal{R}_c^{(\text{inf-div})} \tag{6.3}$$

is therefore a bijection as well. We refer to Ψ as the *BBP parametrization of $\mathcal{R}_c^{(\text{inf-div})}$* . Every choice of parameters $\sigma_1, \sigma_2 \in \mathcal{P}_c^+$ yields a distribution $\nu = \Psi(\sigma_1, \sigma_2) \in \mathcal{R}_c^{(\text{inf-div})}$, and every $\nu \in \mathcal{R}_c^{(\text{inf-div})}$ arises from a unique pair σ_1, σ_2 .

We emphasize that the bijection Ψ works in a really straightforward way: the coefficients of the η -series of σ_1 and σ_2 give the determining sequences of $\nu = \Psi(\sigma_1, \sigma_2)$. It is actually worth recording a direct consequence of this fact, as follows.

Notation 6.2. We denote by \mathcal{E}_c^+ the collection of those series $f \in \mathbb{C}[[z]]$ with the property that $f = \eta_\sigma$ for some $\sigma \in \mathcal{P}_c^+$ (where σ is, a fortiori, uniquely determined).

Proposition 6.3. *Let $\nu \in \mathcal{D}_c(1, *)$ be an R -diagonal distribution, and let $(\alpha_n)_{n=1}^\infty$ and $(\beta_n)_{n=1}^\infty$ be its determining sequences. Set $f(z) = \sum_{n=1}^\infty \alpha_n z^n$ and $g(z) = \sum_{n=1}^\infty \beta_n z^n$. Then ν is \boxplus -infinitely divisible if and only if both f and g belong to \mathcal{E}_c^+ .*

Proof. If $\nu \in \mathcal{R}_c^{(\text{inf-div})}$, then $\nu = \Psi(\sigma_1, \sigma_2)$ for some $\sigma_1, \sigma_2 \in \mathcal{P}_c^+$, hence $f = \eta_{\sigma_1}$ and $g = \eta_{\sigma_2}$, and so $f, g \in \mathcal{E}_c^+$. Conversely, suppose that $f, g \in \mathcal{E}_c^+$, so $f = \eta_{\sigma_1}$ and $g = \eta_{\sigma_2}$ for some $\sigma_1, \sigma_2 \in \mathcal{P}_c^+$. Then the distribution $\tilde{\nu} := \Psi(\sigma_1, \sigma_2)$ belongs to $\mathcal{R}_c^{(\text{inf-div})}$, and the definition of Ψ shows that $\tilde{\nu}$ has the same determining sequences as ν . This forces $\nu = \tilde{\nu}$, hence $\nu \in \mathcal{R}_c^{(\text{inf-div})}$. \square

The criterion provided by Proposition 6.3 is useful because one can (following the work in [2]) characterize the series from \mathcal{E}_c^+ in terms of the associated analytic functions. We will follow up on this in the application presented in Section 7.

Since we are dealing with free probabilistic structures, it is natural to ask what is the description of the BBP parametrization Ψ in terms of R -transforms. Recall (Remark 3.4) that an R -diagonal $*$ -distribution $\nu \in \mathcal{D}_c(1, *)$ is uniquely determined by the R -transforms $R_{ZZ^*}, R_{Z^*Z} \in \mathbb{C}[[z]]$. The following result thus provides an alternative characterization of $\Psi(\sigma_1, \sigma_2)$.

Theorem 6.4. *Let $\sigma_1, \sigma_2 \in \mathcal{P}_c^+$ and set $\nu = \Psi(\sigma_1, \sigma_2)$. Then the R -transforms of ZZ^* and of Z^*Z in the $*$ -probability space $(\mathbb{C}\langle Z, Z^* \rangle, \nu)$ are described as follows:*

$$R_{ZZ^*}(z) = R_{\mathbb{B}(\sigma_1)}(z(1 + M_{\mathbb{B}(\sigma_2)}(z))), \quad R_{Z^*Z}(z) = R_{\mathbb{B}(\sigma_2)}(z(1 + M_{\mathbb{B}(\sigma_1)}(z))), \tag{6.4}$$

where $\mathbb{B}(\sigma_1)$ and $\mathbb{B}(\sigma_2)$ indicate the original BBP bijection (as discussed in Remark 4.5).

Proof. We set $\mu := \Phi(\sigma_1, \sigma_2)$, so ν is R -diagonal, μ is η -diagonal, and $\mathbb{B}_{(1,*)}(\mu) = \nu$. Thus

$$R_\nu(z, z^*) = \eta_\mu(z, z^*) = \sum_{n=1}^\infty \alpha_n (zz^*)^n + \sum_{n=1}^\infty \beta_n (z^*z)^n,$$

where $(\alpha_n)_{n=1}^\infty$ and $(\beta_n)_{n=1}^\infty$ are the (common) determining sequences for μ and for ν . By the definition of the bijection Φ in Theorem 5.2, σ_1 has the same moments as the element ZZ^* in the noncommutative probability space $(\mathbb{C}\langle Z, Z^* \rangle, \mu)$. This implies that $\eta_{\sigma_1} = \eta_{ZZ^*}$, and then Proposition 2.13 gives us the formula $\eta_{\sigma_1}(z) = \sum_{n=1}^\infty \alpha_n z^n$. In a similar way we find that $\eta_{\sigma_2}(z) = \sum_{n=1}^\infty \beta_n z^n$.

Consider now the probability measures $\mathbb{B}(\sigma_1), \mathbb{B}(\sigma_2) \in \mathcal{P}_c$. The definition of \mathbb{B} implies

$$R_{\mathbb{B}(\sigma_1)}(z) = \eta_{\sigma_1}(z) = \sum_{n=1}^\infty \alpha_n z^n, \quad R_{\mathbb{B}(\sigma_2)}(z) = \eta_{\sigma_2}(z) = \sum_{n=1}^\infty \beta_n z^n.$$

But then Proposition 3.5 applies to the R -diagonal $*$ -distribution ν and yields (6.4). \square

As a consequence of Theorem 6.4, we obtain a natural connection between the notions of \boxplus -infinite divisibility in $\mathcal{D}_c(1, *)$ and in \mathcal{P}_c . This is stated in the next corollary. The converse of the corollary fails even in the tracial framework (see Remark 6.9 below).

Corollary 6.5. *Let $\nu \in \mathcal{D}_c(1, *)$ be R -diagonal and let $\tau_1, \tau_2 \in \mathcal{P}_c^+$ be the distributions of ZZ^* and Z^*Z in the $*$ -probability space $(\mathbb{C}\langle Z, Z^* \rangle, \nu)$ (as discussed in Definition 5.1). If ν is \boxplus -infinitely divisible in $\mathcal{D}_c(1, *)$, then τ_1 and τ_2 are \boxplus -infinitely divisible in \mathcal{P}_c .*

Proof. By symmetry, it suffices to show that τ_1 is \boxplus -infinitely divisible. According to [19, Theorem 4.3], a compactly supported Borel probability measure on \mathbb{R} is \boxplus -infinitely divisible if and only if its R -transform can be extended to an analytic self-map of the upper half-plane \mathbb{C}^+ . Suppose that $\nu = \Phi(\sigma_1, \sigma_2)$, where $\sigma_1, \sigma_2 \in \mathcal{P}_c^+$. The R -transform R_{τ_1} , which is the same as R_{ZZ^*} , is given by the first equation (6.4). By [6, Proposition 6.1], the moment series of the probability measure $\mathbb{B}(\sigma_2)$ can be extended analytically to \mathbb{C}^+ and this extension satisfies

$$z \in \mathbb{C}^+ \quad \Rightarrow \quad z(1 + M_{\mathbb{B}(\sigma_2)}(z)) \in \mathbb{C}^+.$$

Finally, since $\mathbb{B}(\sigma_1)$ is \boxplus -infinitely divisible, [19, Theorem 4.3] assures us that $R_{\mathbb{B}(\sigma_1)}$ extends analytically to a self-map of \mathbb{C}^+ . We conclude that for every $z \in \mathbb{C}^+$, $R_{\mathbb{B}(\sigma_1)}$ is defined at $z(1 + M_{\mathbb{B}(\sigma_2)}(z))$, and that

$$z \mapsto R_{\mathbb{B}(\sigma_1)}(z(1 + M_{\mathbb{B}(\sigma_2)}(z)))$$

is an analytic self-map on \mathbb{C}^+ , as required. \square

In the remainder of this section, we discuss the KMS example. In this special case one can process further the formulas from Theorem 6.4 and arrive at explicit formulas (stated in Proposition 6.8) for the distributions of ZZ^* and of Z^*Z in terms of the probability measures σ_1, σ_2 that parametrize ν . These formulas call on some commonly used operations from the free harmonic analysis of \mathcal{P}_c^+ , that are reviewed in the following remark.

Remark 6.6 (Some elements of free harmonic analysis on \mathcal{P}_c^+). (1) Measures $\sigma \in \mathcal{P}_c$ have *free additive convolution powers* with real exponent $t \in [1, \infty)$. More precisely, for every $\sigma \in \mathcal{P}_c$ and $t \in [1, \infty)$, there exists a unique measure $\tau \in \mathcal{P}_c$ such that $R_\tau = tR_\sigma$ (see [15, pp. 228–231]). This measure τ is denoted $\sigma^{\boxplus t}$. When t is an integer, $\sigma^{\boxplus t}$ is simply the t -fold convolution $\sigma \boxplus \cdots \boxplus \sigma$. The argument in [15, pp. 228–231] also shows that $\sigma^{\boxplus t} \in \mathcal{P}_c^+$ for all $t \in [1, \infty)$ if $\sigma \in \mathcal{P}_c^+$.

The analogous result for Boolean convolution provides for every $\sigma \in \mathcal{P}_c$ and $t \in (0, \infty)$ a *Boolean convolution power* $\sigma^{\uplus t} \in \mathcal{P}_c$ such that $\eta_{\sigma^{\uplus t}} = t\eta_\sigma$ (see [18, Theorem 3.6]). As in the free case, $\sigma^{\uplus t} \in \mathcal{P}_c^+$ for every $t \in (0, \infty)$ if $\sigma \in \mathcal{P}_c^+$ (see, for instance, the operator model constructed in [3, Proposition 4.8]).

(2) The original BBP bijection $\mathbb{B} : \mathcal{P}_c \rightarrow \mathcal{P}_c^{(\text{inf-div})}$ (Remark 4.5) can be expressed using convolution powers, by the formula

$$\mathbb{B}(\sigma) = (\sigma^{\boxplus 2})^{\uplus 1/2}, \quad \sigma \in \mathcal{P}_c,$$

which was proved in [4, Theorem 1.2]). The facts reviewed in (1) above imply that $\mathbb{B}(\sigma) \in \mathcal{P}_c^+$ for every $\sigma \in \mathcal{P}_c^+$.

(3) *Free multiplicative convolution* \boxtimes is another binary operation defined on the set \mathcal{P}_c^+ . This operation corresponds to the product of free random variables. Quite remarkably, $\mathbb{B}|\mathcal{P}_c^+$ was shown in [4, Remark 3.9] to be a homomorphism for \boxtimes , that is,

$$\mathbb{B}(\sigma \boxtimes \sigma') = \mathbb{B}(\sigma) \boxtimes \mathbb{B}(\sigma'), \quad \sigma, \sigma' \in \mathcal{P}_c^+.$$

(4) The free counterpart of the standard Poisson distribution is the *Marchenko–Pastur distribution* Π_1 (also known as the *free Poisson distribution*). This distribution is supported on the interval $[0, 4]$ and it is Lebesgue absolutely continuous with density

$$d\Pi_1(t)/dt = \frac{1}{2\pi} \sqrt{(4-t)/t}, \quad 0 < t \leq 4.$$

Its R -transform is

$$R_{\Pi_1}(z) = z/(1-z),$$

and a simple calculation using the definition of \mathbb{B} shows that

$$\Pi_1 = \mathbb{B}\left(\frac{1}{2}(\delta_0 + \delta_2)\right). \tag{6.5}$$

A useful property of Π_1 is that it converts moment series into R -transforms via the formula

$$R_{\sigma \boxtimes \Pi_1} = M_\sigma, \quad \sigma \in \mathcal{P}_c^+. \tag{6.6}$$

See, for instance, [15, Propositions 17.2 and 17.4].

Remark 6.7. Let $\sigma \in \mathcal{P}_c^+$ and let $t > 0$ be a real number. The determining sequences $(\alpha_n)_{n=1}^\infty$ and $(\beta_n)_{n=1}^\infty$ of the infinitely divisible R -diagonal $*$ -distribution $\nu := \Psi(\sigma^{\boxplus t}, \sigma) \in \mathcal{R}_c^{(\text{inf-div})}$ satisfy

$$\sum_{n=1}^\infty \beta_n z^n = \eta_\sigma(z), \quad \sum_{n=1}^\infty \alpha_n z^n = \eta_{\sigma^{\boxplus t}}(z) = t \eta_\sigma(z).$$

Thus ν satisfies the KMS condition with parameter t : $\alpha_n = t\beta_n$, $n \in \mathbb{N}$ (Definition 3.6).

Proposition 6.8. *With the notation of the preceding remark, let $\tau_1, \tau_2 \in \mathcal{P}_c^+$ be the distributions of ZZ^* and of Z^*Z in the noncommutative probability space $(\mathbb{C}\langle Z, Z^* \rangle, \nu)$ (as discussed in Definition 5.1). Then*

$$\tau_1 = (\mathbb{B}(\sigma) \boxtimes \Pi_1)^{\boxplus t} \quad \text{and} \quad \tau_2 = (\mathbb{B}(\sigma)^{\boxplus t} \boxtimes \Pi_1)^{\boxplus 1/t}. \tag{6.7}$$

Proof. The two formulas in (6.7) have similar proofs. We only verify the first one. Since τ_1 is the distribution of ZZ^* , we have $R_{\tau_1} = R_{ZZ^*}$, and Theorem 6.4 yields

$$R_{\tau_1}(z) = R_{\mathbb{B}(\sigma_1)}(z(1 + M_{\mathbb{B}(\sigma_2)}(z))), \tag{6.8}$$

where $\sigma_1 = \sigma^{\boxplus t}$ and $\sigma_2 = \sigma$. The relation $\sigma_1 = \sigma^{\boxplus t}$ and (1.2) imply $\mathbb{B}(\sigma_1) = \mathbb{B}(\sigma)^{\boxplus t}$, and hence $R_{\mathbb{B}(\sigma_1)} = t R_{\mathbb{B}(\sigma)}$. The equality (6.8) can be continued as follows:

$$\begin{aligned} R_{\tau_1}(z) &= t \cdot R_{\mathbb{B}(\sigma)}(z(1 + M_{\mathbb{B}(\sigma)}(z))) \\ &= t \cdot M_{\mathbb{B}(\sigma)}(z) \quad (\text{by (3.4)}) \\ &= t \cdot R_{\mathbb{B}(\sigma) \boxtimes \Pi_1}(z) \quad (\text{by (6.6)}) \\ &= R_{(\mathbb{B}(\sigma) \boxtimes \Pi_1)^{\boxplus t}}(z). \end{aligned}$$

Thus the probability measures τ_1 and $(\mathbb{B}(\sigma) \boxtimes \Pi_1)^{\boxplus t}$ are equal because they have the same R -transform. □

Remark 6.9 (Tracial case). In the special case when $t = 1$, the preceding proposition reduces to

$$\tau_1 = \tau_2 = \mathbb{B}(\sigma) \boxtimes \Pi_1. \tag{6.9}$$

Using (6.5) and invoking the multiplicativity of \mathbb{B} (Remark 6.6(3)), we can rewrite (6.9) as

$$\tau_1 = \tau_2 = \mathbb{B}\left(\sigma \boxtimes \frac{1}{2}(\delta_0 + \delta_2)\right). \tag{6.10}$$

This confirms the fact (Corollary 6.5) that τ_1 and τ_2 are \boxplus -infinitely divisible in \mathcal{P}_c .

We conclude with an argument showing that the converse of Corollary 6.5 does not hold. Choose a distribution $\tilde{\sigma} \in \mathcal{P}_c^+$ that cannot be written as $\sigma \boxtimes \frac{1}{2}(\delta_0 + \delta_2)$ for any $\sigma \in \mathcal{P}_c^+$. (For instance, $\tilde{\sigma} = \frac{1}{3}(\delta_0 + \delta_1 + \delta_2)$ is such a distribution.) Let $\nu \in \mathcal{D}_c(1, *)$ be the tracial R -diagonal $*$ -distribution defined by the requirement that the common distribution of ZZ^* and Z^*Z in $(\mathbb{C}\langle Z, Z^* \rangle, \nu)$ is equal to $\mathbb{B}(\tilde{\sigma})$ (see [15, Proposition 15.13] for an argument that ν exists). The distributions of ZZ^* and Z^*Z are \boxplus -infinitely divisible in \mathcal{P}_c , by construction. We show that ν is not \boxplus -infinitely divisible in $\mathcal{D}_c(1, *)$. Suppose, to get a contradiction, that ν is \boxplus -infinitely divisible. Then $\nu = \Psi(\sigma, \sigma)$ for some $\sigma \in \mathcal{P}_c^+$. Since $\tau_1 = \tau_2 = \mathbb{B}(\tilde{\sigma})$, (6.10) yields $\mathbb{B}(\tilde{\sigma}) = \mathbb{B}(\sigma \boxtimes \frac{1}{2}(\delta_0 + \delta_2))$, and thus $\tilde{\sigma} = \sigma \boxtimes \frac{1}{2}(\delta_0 + \delta_2)$ because \mathbb{B} is injective, contrary to the choice of $\tilde{\sigma}$.

Example 6.10 (λ -circular distribution). Let $\lambda > 0$ be a parameter. If in the setting of Remark 6.7 and Proposition 6.8 we take $\sigma = \delta_1$ (Dirac mass at 1) and $t = \lambda$, then the resulting $*$ -distribution $\nu \in \mathcal{R}_c^{(\text{inf-div})}$ is the λ -circular distribution mentioned right before Definition 3.6. Indeed, it is immediate that in this case the series $\sum_{n=1}^\infty \beta_n z^n$ and $\sum_{n=1}^\infty \alpha_n z^n$

from Remark 6.7 are reduced to $\eta_{\delta_1}(z) = z$ and respectively to $t\eta_{\delta_1}(z) = \lambda z$; hence we have $\alpha_1 = \lambda, \beta_1 = 1$ and $\alpha_n = \beta_n = 0$ for all $n \geq 2$, as required in the definition of the λ -circular distribution.

In this example, the formulas indicated in Proposition 6.8 for the distributions of ZZ^* and of Z^*Z give free Poisson distributions. In order to make this precise, we need to review another bit of notation: for any two parameters $p, q > 0$ one has a *free Poisson distribution of rate p and jump size q* , which we will denote as $\Pi_{p,q}$, and which appears in the free analogue of the Poisson limit theorem (see e.g. Proposition 12.11 in [15]). The Marchenko–Pastur distribution reviewed in Remark 6.6(4) corresponds to $p = q = 1$ (so “ Π_1 ” from there becomes “ $\Pi_{1;1}$ ”). For general $p, q > 0$, the formula given in Remark 6.6(4) for the R -transform of Π_1 extends to

$$R_{\Pi_{p,q}}(z) = \frac{pqz}{1 - qz}.$$

Returning to the example of the λ -circular distribution, an immediate processing of the formulas (6.4) from Theorem 6.4 gives us that the R -transforms of ZZ^* and of Z^*Z in the noncommutative probability space $(\mathbb{C}\langle Z, Z^* \rangle, \nu)$ are

$$R_{ZZ^*}(z) = \frac{\lambda z}{1 - z}, \quad R_{Z^*Z}(z) = \frac{z}{1 - \lambda z}.$$

For our example, this shows that the distributions τ_1 and τ_2 appearing in (6.7) (Proposition 6.8) are free Poisson distributions:

$$\tau_1 = \Pi_{\lambda;1} \quad \text{and} \quad \tau_2 = \Pi_{1/\lambda;\lambda}.$$

7. Stability of $\mathcal{R}_c^{(\text{inf-div})}$ under free multiplicative convolution

Remark 7.1. In this section we consider the operation \boxtimes on $\mathcal{D}_c(1, *)$, which follows the *multiplication* of $*$ -free random variables (cf. Definition 4.1(3)). One has the remarkable fact that whenever $\mu, \mu' \in \mathcal{D}_c(1, *)$ and at least one of μ, μ' is R -diagonal, it follows that $\mu \boxtimes \mu'$ is R -diagonal as well (see [15, Proposition 15.8]). If we make the additional assumption that both μ and μ' are R -diagonal, then we have explicit formulas for the determining sequences of $\mu \boxtimes \mu'$ in terms of the determining sequences of μ and of μ' . To be precise, denote the determining sequences of μ by $(\alpha_n)_{n=1}^\infty, (\beta_n)_{n=1}^\infty$, and those of μ' by $(\alpha'_n)_{n=1}^\infty, (\beta'_n)_{n=1}^\infty$. Then the determining sequences $(\widehat{\alpha}_n)_{n=1}^\infty, (\widehat{\beta}_n)_{n=1}^\infty$ of $\mu \boxtimes \mu'$ are given by:

$$\left\{ \begin{array}{l} \widehat{\alpha}_n = \sum_{\substack{\pi \sqcup \rho \in NC(2n) \\ \pi = \{V_1, \dots, V_p\} \in NC(1, 3, \dots, 2n-1) \\ \text{with } 1 \in V_1, \text{ and} \\ \rho = \{W_1, \dots, W_r\} \in NC(2, 4, \dots, 2n)}} \alpha_{|V_1|} \beta_{|V_2|} \cdots \beta_{|V_p|} \alpha'_{|W_1|} \cdots \alpha'_{|W_r|}, \\ \widehat{\beta}_n = \sum_{\substack{\pi \sqcup \rho \in NC(2n) \\ \pi = \{V_1, \dots, V_p\} \in NC(1, 3, \dots, 2n-1) \\ \text{with } 1 \in V_1, \text{ and} \\ \rho = \{W_1, \dots, W_r\} \in NC(2, 4, \dots, 2n)}} \beta'_{|V_1|} \alpha'_{|V_2|} \cdots \alpha'_{|V_p|} \beta_{|W_1|} \cdots \beta_{|W_r|}. \end{array} \right. \quad (7.1)$$

The formulas (7.1) were proved in [11, Proposition 3.9]. They can also be rephrased in terms of equations for power series, as shown in the next proposition. The formulas (7.2) in the proposition have appeared before (but only as a conjecture, without proof), in [13, Section 5.3]. For the reader’s convenience, we include the proof of how (7.2) is derived from (7.1).

Proposition 7.2. *With the notation of Remark 7.1, suppose that we have elements a, b, a', b' in a noncommutative probability space (\mathcal{A}, φ) such that $\{a, b\}$ is free from $\{a', b'\}$ and such that*

$$R_a(z) = \sum_{n=1}^\infty \alpha_n z^n, \quad R_b(z) = \sum_{n=1}^\infty \beta_n z^n,$$

$$R_{a'}(z) = \sum_{n=1}^{\infty} \alpha'_n z^n, \quad R_{b'}(z) = \sum_{n=1}^{\infty} \beta'_n z^n.$$

Assume moreover that $\beta_1 \neq 0 \neq \alpha'_1$, so the series R_b and $R_{a'}$ have inverses $R_b^{(-1)}$ and $R_{a'}^{(-1)}$ relative to composition. Then:

$$\begin{cases} \sum_{n=1}^{\infty} \widehat{\alpha}_n z^n = (R_a \circ R_b^{(-1)} \circ M_{ba'})(z), \\ \sum_{n=1}^{\infty} \widehat{\beta}_n z^n = (R_{b'} \circ R_{a'}^{(-1)} \circ M_{a'b})(z). \end{cases} \tag{7.2}$$

Proof. The second equation in (7.2) follows from the first one if we substitute b', a', b for a, b, a' , respectively. To prove the first equation, we fix an $n \in \mathbb{N}$ and we suitably structure the formula for $\widehat{\alpha}_n$ provided in (7.1). Let us also momentarily fix an $m \leq n$ and a set $V_1 = \{2i_1 - 1, \dots, 2i_m - 1\}$, where $1 = i_1 < \dots < i_m \leq n$. Denote $n_k = i_{k+1} - i_k$, $k = 1, \dots, m$, where $i_{m+1} = n + 1$. Note that $n_1 + \dots + n_m = n$ and that V_1 can be recovered from n_1, \dots, n_m . Use the moment-cumulant formula as in the proof of Proposition 3.5 (using the cumulant functionals and the fact that the mixed cumulants of a' and b vanish on account of freeness) to obtain

$$\sum_{\substack{\pi \sqcup \rho \in NC(2n) \\ \pi = \{V_1, \dots, V_p\} \in NC(1, 3, \dots, 2n-1) \\ \rho = \{W_1, \dots, W_r\} \in NC(2, 4, \dots, 2n)}} \alpha_{|V_1|} \beta_{|V_2|} \cdots \beta_{|V_p|} \alpha'_{|W_1|} \cdots \alpha'_{|W_r|} = \alpha_m \prod_{k=1}^m \varphi(a'(ba')^{n_k-1}).$$

Letting V_1 vary, (7.1) yields, for the $n \in \mathbb{N}$ that we had fixed:

$$\widehat{\alpha}_n = \sum_{m=1}^n \alpha_m \sum_{n_1 + \dots + n_m = n} \prod_{k=1}^m \varphi(a'(ba')^{n_k-1}).$$

We now let n vary in \mathbb{N} , and get that

$$\sum_{n=1}^{\infty} \widehat{\alpha}_n z^n = \sum_{n=1}^{\infty} \alpha_m (g(z))^m = R_a(g(z)), \tag{7.3}$$

where

$$g(z) = \sum_{n=1}^{\infty} \varphi(a'(ba')^{n-1}) z^n \in \mathbb{C}\llbracket z \rrbracket. \tag{7.4}$$

It remains to show that the series g introduced in (7.4) is equal to $R_b^{(-1)} \circ M_{ba'}$ or, equivalently, that one has $R_b \circ g = M_{ba'}$. To see this, apply again the moment-cumulant formula (using, as in [15, Theorem 14.4] the fact that b is free from a') to obtain

$$\varphi((ba')^n) = \sum_{\substack{\pi \sqcup \rho \in NC(2n) \\ \pi = \{V_1, \dots, V_p\} \in NC(1, 3, \dots, 2n-1) \\ \rho = \{W_1, \dots, W_r\} \in NC(2, 4, \dots, 2n)}} \beta_{|V_1|} \cdots \beta_{|V_p|} \alpha'_{|W_1|} \cdots \alpha'_{|W_r|}. \tag{7.5}$$

In (7.5) we can list the blocks of π such that $1 \in V_1$. A similar argument to the one used above to structure the formula for $\widehat{\alpha}_n$ shows now that

$$\varphi((ba')^n) = \sum_{m=1}^n \beta_m \sum_{n_1 + \dots + n_m = n} \prod_{k=1}^m \varphi(a'(ba')^{n_k-1}), \quad n \in \mathbb{N},$$

and this implies the desired relation $M_{ba'} = R_b \circ g$. □

Remark 7.3. In the notation of Remark 7.1, consider the case (disjoint from the one covered by the preceding proposition) when $\beta_n = 0$ for every $n \in \mathbb{N}$. In anticipation of the proof of Theorem 7.8 below, it is useful to record that in this case we have

$$\widehat{\alpha}_n = \alpha_n (\alpha_1')^n, \quad \forall n \in \mathbb{N}.$$

Indeed, in this case the only way to get a non-zero term in the sum on the right-hand side of the first equation (7.1) is by requiring that $\pi = \{V_1\}$ with $V_1 = \{1, 3, \dots, 2n - 1\}$. But then the condition that $\pi \sqcup \rho$ is non-crossing forces ρ to be the partition of $\{2, 4, \dots, 2n\}$ into n singleton blocks. Hence the sum giving $\widehat{\alpha}_n$ in (7.1) is reduced to just one term, and yields the formula displayed above.

The next corollary presents a reformulation of (7.2) which has the advantage that it introduces in discussion two power series F and \widetilde{F} , related with the subordination results of [7].

Corollary 7.4. *In the framework of Proposition 7.2, we have*

$$\begin{cases} \sum_{n=1}^{\infty} \widehat{\alpha}_n z^n = R_a(F(z)(1 + M_b(F(z)))) \\ \sum_{n=1}^{\infty} \widehat{\beta}_n z^n = R_{b'}(\widetilde{F}(z)(1 + M_{a'}(\widetilde{F}(z)))) \end{cases} \tag{7.6}$$

where $F = M_b^{(-1)} \circ M_{ba'}$ and $\widetilde{F} = M_{a'}^{(-1)} \circ M_{a'b}$.

Proof. By symmetry, it suffices to prove the first of the two equations. Using (7.2), we see that we must verify the identity

$$(R_b^{(-1)} \circ M_{ba'})(z) = F(z)(1 + M_b(F(z))). \tag{7.7}$$

Recalling the assumption that $\varphi(b) \neq 0$, the functional equation $M_b(z) = R_b(z(1 + M_b(z)))$ can be rewritten as

$$R_b^{(-1)}(w) = (1 + w)M_b^{(-1)}(w) \tag{7.8}$$

(see [15, Remark 16.18]). Substitute $M_{ba'}$ for w in (7.8) to find that

$$(R_b^{(-1)} \circ M_{ba'})(z) = (1 + M_{ba'}(z)) \cdot (M_b^{(-1)} \circ M_{ba'})(z) = (1 + M_{ba'}(z)) \cdot F(z).$$

Finally, the definition of F implies that $M_{ba'}(z) = M_b(F(z))$, and using this equality in the right-hand side of the preceding equality yields (7.7). □

The following lemma is an immediate consequence of the definition of Ψ (Remark 6.1).

Lemma 7.5. *Consider a distribution $\nu \in \mathcal{R}_c^{(\text{inf-div})}$, and let $(\alpha_n)_{n=1}^{\infty}$ and $(\beta_n)_{n=1}^{\infty}$ be its determining sequences. There exist positive elements a, b in a C^* -probability space (\mathcal{A}, φ) such that*

$$R_a(z) = \sum_{n=1}^{\infty} \alpha_n z^n \quad \text{and} \quad R_b(z) = \sum_{n=1}^{\infty} \beta_n z^n.$$

Moreover, the distributions of a and b are \boxplus -infinitely divisible.

Proof. Write $\nu = \Psi(\sigma_1, \sigma_2)$, with $\sigma_1, \sigma_2 \in \mathcal{P}_c^+$. Then

$$\sum_{n=1}^{\infty} \alpha_n z^n = \eta_{\sigma_1}(z) = R_{\mathbb{B}(\sigma_1)}(z),$$

where $\mathbb{B}(\sigma_1)$ is a \boxplus -infinitely divisible distribution in \mathcal{P}_c^+ (cf. Remark 6.6(2)). Thus taking a to be a positive element with distribution $\mathbb{B}(\sigma_1)$ in some C^* -probability space will fulfill the required conditions. The argument for b is similar. \square

In reference to the set of power series \mathcal{E}_c^+ introduced in Notation 6.2, we record a result which follows easily from [2, Proposition 2.2].

Proposition 7.6. *A series $f \in \mathbb{C}[[z]]$ belongs to the set \mathcal{E}_c^+ if and only if it satisfies the following three conditions:*

- (i) f has real coefficients;
- (ii) f has positive convergence radius;
- (iii) f can be extended to an analytic map (still denoted f) of \mathbb{C}^+ into $\overline{\mathbb{C}^+}$ such that $f(0) = 0$ and $\text{Arg}(z) \leq \text{Arg}(f(z))$ for $z \in \mathbb{C}^+$.

Corollary 7.7. *Suppose that the series $f(z) = \sum_{n=1}^\infty \alpha_n z^n \in \mathcal{E}_c^+$ is not identically zero. Then $\alpha_1 > 0$.*

Proof. Let n be the smallest integer such that $\alpha_n \neq 0$ and suppose, to get a contradiction, that either $n > 1$ or $n = 1$ and $\alpha_n < 0$. Choose $\gamma \in \mathbb{C}^+$ such that $|\gamma| = 1$ and $\Im(\gamma^n \alpha_n) < 0$. We have $\lim_{r \downarrow 0} (f(r\gamma)/r^n) = \gamma^n \alpha_n$, and therefore $\Im f(r\gamma) < 0$ for sufficiently small r , contrary to Proposition 7.6(iii). \square

We are now ready for the main result of this section.

Theorem 7.8. *For every $\nu, \nu' \in \mathcal{R}_c^{(\text{inf-div})}$ we have $\nu \boxtimes \nu' \in \mathcal{R}_c^{(\text{inf-div})}$.*

Proof. Let $(\alpha_n)_{n=1}^\infty$ and $(\beta_n)_{n=1}^\infty$ (respectively $(\alpha'_n)_{n=1}^\infty$ and $(\beta'_n)_{n=1}^\infty$) denote the determining sequences of ν (respectively, ν'). Two applications of Lemma 7.5, combined with a free product construction, allow us to construct a C^* -probability space (\mathcal{A}, φ) and positive elements $a, b, a', b' \in \mathcal{A}$ such that

- (a) $R_a(z) = \sum_{n=1}^\infty \alpha_n z^n$ and $R_b(z) = \sum_{n=1}^\infty \beta_n z^n$,
- (b) $R_{a'}(z) = \sum_{n=1}^\infty \alpha'_n z^n$ and $R_{b'}(z) = \sum_{n=1}^\infty \beta'_n z^n$, and
- (c) $\{a, b\}$ is free from $\{a', b'\}$.

We know from Remark 7.1 that $\nu \boxtimes \nu'$ is an R -diagonal distribution in $\mathcal{D}_c(1, *)$. Let $(\widehat{\alpha}_n)_{n=1}^\infty$ and $(\widehat{\beta}_n)_{n=1}^\infty$ denote the determining sequences of $\nu \boxtimes \nu'$, and set

$$\widehat{f}(z) := \sum_{n=1}^\infty \widehat{\alpha}_n z^n, \quad \widehat{g}(z) := \sum_{n=1}^\infty \widehat{\beta}_n z^n.$$

By Proposition 6.3, we have to prove that $\widehat{f}, \widehat{g} \in \mathcal{E}_c^+$. By symmetry, it suffices to show that $\widehat{f} \in \mathcal{E}_c^+$, and this is done by verifying that \widehat{f} satisfies conditions (i)–(iii) of Proposition 7.6. We dispose first of the simple case in which $\beta_1 = 0$. Corollary 7.7 yields $\beta_n = 0$ for all $n \in \mathbb{N}$, and Remark 7.3 implies that $\widehat{f}(z) = \sum_{n=1}^\infty \alpha_n (\alpha'_1 z)^n$. The desired conclusion follows because $\alpha'_1 \geq 0$ and the series $\sum_{n=1}^\infty \alpha_n z^n$ belongs to \mathcal{E}_c^+ . Similarly, if $\alpha'_1 = 0$, Corollary 7.7 yields $\alpha'_n = 0$ for all $n \in \mathbb{N}$, and then (7.1) implies that $\widehat{f} = 0$.

It remains to show that $\widehat{f} \in \mathcal{E}_c^+$ when $\beta_1 \neq 0 \neq \alpha'_1$. In this case, Corollary 7.4 shows that $\widehat{f} = R_a(F(z)(1 + M_b(F(z))))$, where $F = M_b^{(-1)} \circ M_{ba'}$. In other words, \widehat{f} is the composition of the three power series $R_a(z)$, $z(1 + M_b(z))$, and $F(z)$. We know that $R_a \in \mathcal{E}_c^+$. It was proved in [7] that $F \in \mathcal{E}_c^+$. The series $z(1 + M_b(z))$ also belongs to \mathcal{E}_c^+ by [6, Proposition 6.1]. Proposition 7.6 shows that the set \mathcal{E}_c^+ is closed under composition. Therefore $\widehat{f} \in \mathcal{E}_c^+$, thus concluding the proof. \square

Corollary 7.9. *Suppose that $\nu \in \mathcal{R}_c^{(\text{inf-div})}$ is the $*$ -distribution of an element a in some C^* -probability space. Then the $*$ -distribution of a^n belongs to $\mathcal{R}_c^{(\text{inf-div})}$ for every $n \in \mathbb{N}$.*

Proof. This follows from Theorem 7.8 and the known fact [11, Proposition 3.11] that the distribution of a^n is equal to $\nu^{\boxtimes n}$. □

We conclude the section by looking again at the KMS example, and by describing explicitly the BBP parametrization for the powers of a λ -circular element.

Remark 7.10. (1) Suppose that $\nu, \nu' \in \mathcal{D}_c(1, *)$ are R -diagonal and satisfy the KMS condition with parameters $t, t' \in (0, \infty)$, respectively. Consider the $*$ -distribution $\nu \boxtimes \nu'$, which is R -diagonal as well (see Remark 7.1). We claim that $\nu \boxtimes \nu'$ also satisfies the KMS condition, with parameter tt' . Using the same notations for determining sequences as in Remark 7.1, this claim amounts to the fact that $\widehat{\alpha}_n = (tt')\widehat{\beta}_n$ for every $n \in \mathbb{N}$. In order to prove this, we replace $\alpha_{|V_1|}$ and $\alpha'_{|W_1|}$ by $t\beta_{|V_1|}$ and $t'\beta'_{|W_1|}$, respectively, in the first formula (7.1) to obtain

$$\widehat{\alpha}_n = (tt') \sum_{\substack{\pi \sqcup \rho \in NC(2n) \\ \pi = \{V_1, \dots, V_k\} \in NC(1, 3, \dots, 2n-1) \\ \rho = \{W_1, \dots, W_l\} \in NC(2, 4, \dots, 2n) \\ 1 \in V_1, 2 \in W_1}} \beta'_{|W_1|} \alpha'_{|W_2|} \cdots \alpha'_{|W_l|} \beta_{|V_1|} \beta_{|V_2|} \cdots \beta_{|V_k|}.$$

To see that the last sum equals $\widehat{\beta}_n$, we observe that pairs (π, ρ) as above are in a bijective correspondence with pairs $(\widetilde{\pi}, \widetilde{\rho})$ such that $\widetilde{\pi} \sqcup \widetilde{\rho} \in NC(2n)$ and $\widetilde{\pi} = \{\widetilde{W}_1, \dots, \widetilde{W}_l\} \in NC(1, 3, \dots, 2n-1)$ and $\widetilde{\rho} = \{\widetilde{V}_1, \dots, \widetilde{V}_k\} \in NC(2, 4, \dots, 2n)$. Indeed, $\widetilde{\rho}$ and $\widetilde{\pi}$ are obtained as $\widetilde{\pi} \sqcup \widetilde{\rho} = \gamma_{2n}^{-1}(\pi \sqcup \rho)$, where we use the permutation γ_{2n} from the proof of Proposition 3.7. Thus, the sum above is equal to

$$\sum_{\substack{\pi' \sqcup \rho' \in NC(2n) \\ \widetilde{\pi} = \{\widetilde{W}_1, \dots, \widetilde{W}_l\} \in NC(1, 3, \dots, 2n-1) \\ \widetilde{\rho} = \{\widetilde{V}_1, \dots, \widetilde{V}_k\} \in NC(2, 4, \dots, 2n) \\ 1 \in \widetilde{W}_1, 2 \in \widetilde{V}_1}} \beta'_{|\widetilde{W}_1|} \alpha'_{|\widetilde{W}_2|} \cdots \alpha'_{|\widetilde{W}_l|} \beta_{|\widetilde{V}_1|} \cdots \beta_{|\widetilde{V}_k|},$$

and this equals $\widehat{\beta}_n$ by the second formula (7.1).

(2) Now fix a real number $\lambda > 0$ and consider the λ -circular distribution $\nu = \Psi(\delta_\lambda, \delta_1)$, as in Example 6.10. If a is an element in some $*$ -probability space such that the $*$ -distribution of a is equal to ν , then we will say that a is a λ -circular element. Such elements do of course exist, for instance we can just take $a = Z$ in the $*$ -probability space $(\mathbb{C}\langle Z, Z^* \rangle, \nu)$. If a is a λ -circular element, then Theorem 7.8 and Corollary 7.9 tell us that every power a^k has $*$ -distribution $\nu^{\boxtimes k} \in \mathcal{R}_c^{(\text{inf-div})}$. Moreover, part (1) of the present remark assures us that $\nu^{\boxtimes k}$ satisfies the KMS condition with parameter λ^k . Hence for every $k \in \mathbb{N}$ we have a BBP parametrization of the form

$$\nu^{\boxtimes k} = \Psi(\sigma_k^{\uplus \lambda^k}, \sigma_k),$$

for some probability measure $\sigma_k \in \mathcal{P}_c^+$. For $k = 1$, we know from Example 6.10 that σ_1 is the Dirac mass δ_1 . The next proposition gives a way of describing σ_k for $k \geq 2$.

Proposition 7.11. *Let λ and $(\sigma_k)_{k=1}^\infty$ be as above, and consider on the other hand the probability measures with finite support $(\tau_k)_{k=1}^\infty$ defined by*

$$\tau_k := \frac{\lambda^k}{1 + \lambda^k} \delta_0 + \frac{1}{1 + \lambda^k} \delta_{1+\lambda^k}, \quad k \in \mathbb{N}.$$

Then one has

$$\sigma_k = \tau_1 \boxtimes \cdots \boxtimes \tau_{k-1}, \quad k \geq 2. \tag{7.9}$$

Proof. As in Remark 7.10(2), we use the notation ν for the λ -circular distribution. We fix a $k \in \mathbb{N}$ and invoke Proposition 7.2 in the special case in which the $*$ -distributions μ, μ' considered there are $\nu^{\boxtimes k}$ and ν , respectively. The power series

$$\sum_{n=1}^{\infty} \beta_n z^n, \quad \sum_{n=1}^{\infty} \beta'_n z^n, \quad \sum_{n=1}^{\infty} \widehat{\beta}_n z^n \tag{7.10}$$

from Proposition 7.2 are equal in this case to the η -series of the probability measures σ_k, σ_1 and σ_{k+1} , respectively. (For instance the equality $\sum_{n=1}^{\infty} \beta_n z^n = \eta_{\sigma_k}(z)$ follows from the comments at the end of Remark 6.1 and the fact that $\nu^{\boxtimes k} = \Psi(\sigma_k^{\uplus \lambda^k}, \sigma_k)$.) Note that, since $\sigma_1 = \delta_1$, for the second power series in (7.10) we actually have $\sum_{n=1}^{\infty} \beta'_n z^n = z$.

The notation of Proposition 7.2 also include some non-commutating random variables a, b, a', b' , where b is such that

$$R_b(z) = \sum_{n=1}^{\infty} \beta_n z^n = \eta_{\sigma_k}(z) = R_{\mathbb{B}(\sigma_k)}(z). \tag{7.11}$$

From (7.11) we infer that the distribution of b is $\mathbb{B}(\sigma_k)$. Similar reasoning, based on the formulas $R_{b'}(z) = z$ and $R_{a'}(z) = \lambda z$, leads to the fact that a' and b' have distributions δ_λ and δ_1 , respectively. As a consequence, we may assume without loss of generality that $a' = \lambda$ and $b' = 1$ in their noncommutative probability space.

We are interested in the second relation (7.2) from Proposition 7.2. Due to the very simple form of $R_{a'}$ and $R_{b'}$, this equation simplifies to

$$\sum_{n=1}^{\infty} \widehat{\beta}_n z^n = \frac{1}{\lambda} M_{\lambda b}(z). \tag{7.12}$$

The same argument as used in (7.11) shows that the left-hand side of (7.12) is equal to $R_{\mathbb{B}(\sigma_{k+1})}(z)$. On the right-hand side of (7.12) we perform the obvious transformation $M_{\lambda b}(z) = M_b(\lambda z) = M_{\mathbb{B}(\sigma_k)}(\lambda z)$, and this leads us to a direct connection between σ_k and σ_{k+1} :

$$R_{\mathbb{B}(\sigma_{k+1})}(z) = \frac{1}{\lambda} M_{\mathbb{B}(\sigma_k)}(\lambda z). \tag{7.13}$$

In order to make use of (7.13), it is convenient to resort to another well-known transform of free probability, the S -transform. For a probability measure $\sigma \in \mathcal{P}_c$ with non-vanishing mean, one defines the S -transform of σ as the power series

$$S_\sigma(z) = \frac{1}{z} R_\sigma^{(-1)}(z) = \frac{z+1}{z} M_\sigma^{(-1)}(z)$$

(see, for instance, [15, Definition 18.15 and Remark 18.16 on p. 294]). Some straightforward processing of equation (7.13) (multiply both sides by λ , take inverses under composition, and write the resulting series in terms of the suitable S -transforms) then leads to the formula

$$S_{\mathbb{B}(\sigma_{k+1})}(z) = \frac{1}{1 + \lambda z} S_{\mathbb{B}(\sigma_k)}(\lambda z). \tag{7.14}$$

The formula (7.14) was obtained for a fixed (but arbitrary) $k \in \mathbb{N}$. We now let k vary and use a straightforward induction argument, with base case $S_{\mathbb{B}(\sigma_1)}(z) = S_{\delta_1}(z) = 1$, in order to infer that

$$S_{\mathbb{B}(\sigma_k)}(z) = \prod_{j=1}^{k-1} \frac{1}{1 + \lambda^j z}, \quad \forall k \in \mathbb{N}. \tag{7.15}$$

It remains to make the connection to the τ_k indicated in the statement of the proposition. For every $j \in \mathbb{N}$, an elementary calculation shows that $\mathbb{B}(\tau_j)$ is the free Poisson distribution $\Pi_{1/\lambda^j; \lambda^j}$, where the notation “ $\Pi_{p;q}$ ” is as

in Example 6.10. Another elementary calculation shows that the S -transform of $\Pi_{1/\lambda^j; \lambda^j}$ is $1/(1 + \lambda^j z)$. Thus the right-hand side of (7.15) can be written as $S_{\mathbb{B}(\tau_1)}(z)S_{\mathbb{B}(\tau_2)}(z) \cdots S_{\mathbb{B}(\tau_{k-1})}(z)$.

Now, the S -transform is multiplicative with respect to the operation \boxtimes ([15, Corollary 18.17]). Since \mathbb{B} is multiplicative as well (Remark 6.6(3)), the observations made in the preceding paragraph lead to the formula

$$S_{\mathbb{B}(\sigma_k)} = S_{\mathbb{B}(\tau_1 \boxtimes \cdots \boxtimes \tau_{k-1})}, \quad k \geq 2.$$

The required equation (7.9) follows from here, since \mathbb{B} is injective and since a compactly supported probability measure with non-vanishing mean is uniquely determined by its S -transform. \square

Acknowledgement

The authors would like to thank the anonymous referee for several insightful comments on this work, and for suggesting some interesting new problems which can be considered in its continuation.

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