LIMITING BEHAVIOR OF 3-COLOR EXCITABLE MEDIA ON ARBITRARY GRAPHS

BY JANKO GRAVNER[*,1], HANBAEK LYU[†] AND DAVID SIVAKOFF[†,2]

University of California, Davis * *and Ohio State University*[†]

Fix a simple graph $G = (V, E)$ and choose a random initial 3-coloring of vertices drawn from a uniform product measure. The 3-color cycle cellular automaton is a process in which at each discrete time step in parallel, every vertex with color i advances to the successor color $(i + 1) \bmod 3$ if in contact with a neighbor with the successor color, and otherwise retains the same color. In the Greenberg–Hastings model, the same update rule applies only to color 0, while other two colors automatically advance. The limiting behavior of these processes has been studied mainly on the integer lattices. In this paper, we introduce a monotone comparison process defined on the universal covering space of the underlying graph, and characterize the limiting behavior of these processes on arbitrary connected graphs. In particular, we establish a phase transition on the Erdős–Rényi random graph. On infinite trees, we connect the rate of color change to the cloud speed of an associated tree-indexed walk. We give estimates of the cloud speed by generalizing known results to trees with leaves.

1. Introduction. An excitable medium is a network of coupled dynamic units whose states get excited upon a particular local event. It has the capacity to propagate waves of excitation, which often self-organize into spiral patterns. Examples of such systems in nature include neural networks, Belousov–Zhabotinsky reaction, as well as coupled oscillators such as fireflies and pacemaker cells. In a discrete setting, excitable media can be modeled using the framework of generalized cellular automaton (GCA). Given a simple connected graph $G = (V, E)$ and a fixed integer $\kappa \geq 2$, the microstate of the system at a given discrete time $t \geq 0$ is given by a κ-coloring of vertices $X_t : V \to \mathbb{Z}_\kappa = \mathbb{Z}/\kappa\mathbb{Z}$. A given initial coloring X_0 evolves in discrete time through iteration of a fixed deterministic transition map $\tau : X_t \mapsto X_{t+1}$, which depends only on local information at each time step. That is, for each $v \in V$, $X_{t+1}(v)$ is determined by X_t restricted on $N(v) \cup \{v\}$, where $N(v)$ is the set of neighbors of v in G. This generates a trajectory $(X_t)_{t \geq 0}$, and

its limiting behavior in relation to the topology of G and structure of τ is of our interest.

The Greenberg–Hastings Model (GHM) and cyclic cellular automaton (CCA) are two particular GCA models for excitable media [27] that have been studied extensively in the 1990s. GHM was introduced by Greenberg and Hastings [16] to capture phenomenological essence of neural networks in a discrete setting, whereas CCA was introduced by Bramson and Griffeath [5] as a discrete time analogue of the cyclic particle systems. In GHM, think of each vertex of a given graph as a κ-state neuron. An excited neuron (i.e., one in state 1) excites neighboring neurons at rest (i.e., in state 0) and then needs to wait for a refractory period of time (modeled by the remaining $\kappa - 2$ states) to become rested again. In CCA, each vertex of the graph is inhabited by one of κ different species in a cyclic food chain. Species of color i are eaten (and thus replaced) by species of color $(i + 1)$ mod κ in the neighborhood at each time step. More precisely, the transition maps $X_t \mapsto X_{t+1}$ for κ-color GHM and CCA are given below:

$$
\text{(GHM)} \qquad X_{t+1}(v) = \begin{cases} 1 \\ \qquad \text{if } X_t(v) = 0 \text{ and } \exists u \in N(v) \text{ s.t. } X_t(u) = 1, \\ 0 \\ \qquad \text{if } X_t(v) = 0 \text{ and } \nexists u \in N(v) \text{ s.t. } X_t(u) = 1, \\ X_t(v) + 1 \quad (\text{mod } \kappa) \\ \qquad \text{otherwise,} \end{cases}
$$

$$
\text{(CCA)} \qquad X_{t+1}(v) = \begin{cases} X_t(v) + 1 \quad (\text{mod } \kappa) \\ \qquad \text{if } \exists u \in N(v) \text{ s.t. } X_t(u) = X_t(v) + 1 \quad (\text{mod } \kappa), \\ X_t(v) \\ \qquad \text{otherwise.} \end{cases}
$$

We say a node x is *excited* at time t if: $X_{t+1}(x) = X_t(x) + 1 \bmod 3$ in the case of CCA; and if $X_t(x) = 0$ and $X_{t+1}(x) = 1$ in the case of GHM.

GHM and CCA are among the few models of discrete excitable media that have been studied rigorously in the probability literature, mostly on the integer lattice $\mathbb{Z}^d$ with randomly chosen initial κ-coloring X_0. For $d = 1$, Fisch [11] showed that CCA exhibits a phase transition "between" $\kappa = 4$ and 5; for $\kappa \in \{3, 4\}$ each vertex increments its color infinitely often with probability 1, whereas for $\kappa \geq 5$ the dynamics fixates. In a subsequent work [12], Fisch established clustering of the one-dimensional 3-color CCA by showing that the density of borders between different colors approaches 0 at the rate $t^{-1/2}$ as time t increases. The main technique was a connection to a random walk, which was adapted to the 3-color GHM on $\mathbb{Z}$ by Durrett and Steif [10], and to GHM on $\mathbb{Z}$ with arbitrary κ by Fisch and Gravner [13].

In higher dimensions, waves of excitation can feed back on themselves, resulting in self-sustained local wave generators, the stable periodic objects (SPOs).

Fisch, Gravner and Griffeath showed that the limiting behavior of CCA on $\mathbb{Z}^d$ for any $d \geq 2$ and $\kappa \geq 3$ is governed by the formation of SPOs [14]. The 3-color GHM on $\mathbb{Z}^d$ shares this behavior [10], and similar behavior of GHM and CCA for any $\kappa \geq 3$ on higher dimensions was studied by Fisch and Gravner [15], who introduced an additional parameter θ, the threshold number of excited neighbors required to excite a vertex.

A fundamental difficulty in understanding limiting behavior of excitable media models on general graphs is the complexity in generation of SPOs and interactions between them. On trees, however, these objects are topologically prohibited. This makes trees special, and a substantial portion of the paper addresses the behavior of CCA and GHM on them. Related models on trees arose in the context of distributed algorithms for digital clock synchronization [9]: a 3-color GCA model on finite trees studied by Herman and Ghosh [17]; odd $\kappa \geq 3$ models, which coincide with the 3-color CCA in case $\kappa = 3$, investigated by Boulinier, Petit and Villain [6]; and a GCA model for pulse-coupled inhibitory oscillators, called the firefly cellular automaton, whose behavior on finite trees was recently addressed by the second author [22].

In this paper, we revisit the 3-color GHM and CCA and characterize their limiting behavior on arbitrary graphs. Our main technique is the construction of a monotone comparison process on the universal covering space of the underlying graph. This comparison process itself is inspired by a famous consensus algorithm of Lamport [19]. Suppose the vertices of a connected graph are equipped with integer-valued opinions (not necessarily distinct), and at each step each node simultaneously adopts the maximum opinion among itself and its neighbors. Then in some finite time the entire graph reaches a consensus, which is the initial global maximum. The correspondence between the GHM and CCA dynamics and this comparison process is such that a vertex is excited in the original dynamics if and only if all vertices in its fiber increment their opinion by 1. We remark that a similar observation was made by Belitsky and Ferrari [2] for a one-dimensional ballistic annihilation system. A consequence of this comparison for the 3-color GHM and CCA on arbitrary graphs is that the SPOs are static and cannot arise spontaneously. After establishing the comparison process, we apply it to deterministic and random finite graphs and to infinite trees.

As we show in Section 3, our comparison dynamics on trees is also equivalent to an annihilating rule; although this equivalence is not directly useful for theoretical development, we include it for comparison between our model and related ones in the literature, and because it helps in developing intuition on the annihilating nature of the excitable dynamics.

2. Statements of results. Let $G = (V, E)$ be a connected (finite or infinite) graph and let $X_0 : V \to \mathbb{Z}_3$ be an initial 3-coloring, and let $(X_t)_{t \geq 0}$ be the resulting CCA or GHM dynamics. Define $\mathrm{ne}_t(x)$ to be the number of excitations $x \in V$

undergoes in the first t iterations,

$$(2.1) \qquad \mathrm{ne}_t(x) = \sum_{s=0}^{t-1} \mathbf{1}(x \text{ is excited at time } s), \qquad x \in V \text{ and } t \geq 1,$$

where $\mathbf{1}(A)$ denotes the indicator function of event A.

Let $\overline{E}$ be set of all ordered pairs of adjacent nodes, that is, $\overline{E} = \{(u, v) \in V^2 \mid uv \in E\}$. Let $X : V \to \mathbb{Z}_3$ be a 3-coloring on G. For the CCA, we define the associated antisymmetric 1-form $dX : \overline{E} \to \{-1, 0, 1\}$ as follows:

$$(2.2) \qquad dX(u, v) = X(v) - X(u),$$

where the subtraction is taken in $\mathbb{Z}_3$. Similarly, for the GHM dynamics, we define

$$(2.3) \qquad dX(u, v) = \begin{cases} 1 & \text{if } X(u) = 0 \text{ and } X(v) = 1, \\ -1 & \text{if } X(u) = 1 \text{ and } X(v) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The definition is such that a vertex $x \in V$ is excited at time t if and only if $dX_t(x, y) = 1$ for some $y \in N(x)$, in which case we say that y *excites* x at time t.

A *walk* $\vec{W}$ in G is a finite or infinite sequence of nodes $(v_i)_{i \geq 0}$ in G such that v_i is adjacent to v_{i+1} for all $i \geq 0$. We say that $\vec{W}$ is *nonbacktracking* if $v_i \neq v_{i+2}$ for all $i \geq 0$; *closed* if it consists of finitely many vertices and the first and last vertices coincide; and a *directed path* if all nodes in $\vec{W}$ are distinct. We say $\vec{W}$ is a *cycle* if it is a closed walk in which only the first and last vertices agree. For a finite walk $\vec{W} = (v_0, \ldots, v_m)$, we let $|\vec{W}| := m$ denote its length, and for an infinite walk $\vec{W}$ we let $|\vec{W}| = \infty$. For a walk $\vec{W} = (v_i)_{i \geq 0}$ and a 3-coloring $X : V \to \mathbb{Z}_3$, we define the *path integral* of X on $\vec{W}$ by

$$(2.4) \qquad \int_{\vec{W}} dX := \sum_{i \geq 0} dX(v_i, v_{i+1})$$

We say dX (with respect to CCA or GHM dynamics) is *conservative* if every contour integral of X over directed cycles is zero, that is,

$$(2.5) \qquad \oint_{\vec{C}} dX \equiv 0$$

for all closed directed cycles $\vec{C}$ in G. Note that any closed walk can be decomposed into a finite number of directed cycles. Hence if dX is conservative, then (2.5) holds for all closed walks $\vec{W}$.

One of our key tools is the following lemma.

LEMMA 1. *Let $G = (V, E)$ be an arbitrary connected graph and let $(X_t)_{t \geq 0}$ be a CCA or GHM trajectory of an initial 3-coloring $X_0 : V \to \mathbb{Z}_3$. Then for each $x \in V$, we have*

$$(2.6) \qquad \mathrm{ne}_t(x) = \max\left\{ \int_{\vec{W}} dX_0 : \vec{W} = (x, v_1, \ldots, v_m) \text{ and } m \leq t \right\}.$$

For each $x \in V$, we define the quantity

$$(2.7) \qquad \alpha(x) = \limsup_{t \to \infty} \frac{\mathrm{ne}_t(x)}{t}.$$

An immediate consequence from the above lemma is the following.

COROLLARY 2. *Let $G = (V, E)$ and $(X_t)_{t \geq 0}$ be as before. If $\sup_t \mathrm{ne}_t(x) < \infty$ holds for some $x \in V$, then it holds for all $x \in V$. Moreover, for each $x, y \in V$, we have $\alpha(x) = \alpha(y)$.*

We say that X_t *fixates* if $\mathrm{ne}_t(x)$ is bounded in time for some (and thus for all) $x \in V$, and that it *fluctuates* otherwise. We say X_t *synchronizes* if for every two vertices $x, y \in V$, there exists $N = N(x, y) \in \mathbb{N}$ such that $X_t(x) = X_t(y)$ for all $t \geq N$. It is not hard to see that fixation and synchronization are equivalent notions for 3-color CCA and GHM dynamics. In fact, this is true for the general κ-color GHM dynamics, but not for κ-color CCA dynamics with $\kappa \geq 4$ as there are many noninteracting pairs of colors. Furthermore, from now on we will denote the constant value of $\alpha(x)$ by α, and call it the *activity* of the dynamics $(X_t)_{t \geq 0}$. Finally, we say that X_t *synchronizes weakly* if $\alpha = 0$ and that it *oscillates* otherwise. In the case of finite graphs, synchronization and weak synchronization are equivalent notions, but this is not so for infinite graphs.

Our first result is a characterization of the limiting behavior of the 3-color CCA or GHM dynamics on finite graphs depending on whether the induced 1-form at time 0 is conservative.

THEOREM 3. *Let $G = (V, E)$ be a finite connected graph and let $(X_t)_{t \geq 0}$ be a CCA or GHM trajectory of an initial 3-coloring $X_0 : V \to \mathbb{Z}_3$. Then X_t synchronizes if and only if dX_0 is conservative. Furthermore, we have the following*:

  (i) *If dX_0 is conservative, then $X_t(x) = X_t(y)$ for all $x, y \in V$ and $t \geq D$, where D is the diameter of G.*
  (ii) *If dX_0 is not conservative, then for each node $x \in V$, we have*

$$(2.8) \qquad \alpha = \lim_{t \to \infty} \frac{\mathrm{ne}_t(x)}{t} = \max_{\vec{C}} \frac{1}{|\vec{C}|} \oint_{\vec{C}} dX_0,$$

where the maximum runs over all closed directed cycles $\vec{C}$ in G.

We immediately obtain the following corollary.

COROLLARY 4. *Let $G = (V, E)$ be a finite graph. Then under either CCA or GHM dynamics, arbitrary 3-coloring $X_0 : V \to \mathbb{Z}_3$ on G synchronizes if and only if G is a tree.*

We next present a result on the CCA and GHM dynamics on general graphs $G = (V, E)$, starting from a random 3-coloring X_0 drawn from the uniform product measure on $\mathbb{Z}_3^V$, which generates a random 3-color CCA or GHM trajectory. See Figure 1 for an example. A *matching* in G is a set of edges $\{e_1, \ldots, e_k\} \subset E$ where two edges do not share a common vertex. For any graph H, we let $V(H)$ and $E(H)$ denote the vertex and edge sets of H.

THEOREM 5. *Consider a random 3-color CCA or GHM trajectory $(X_t)_{t \geq 0}$ on a graph $G = (V, E)$. If G contains a cycle, X_t oscillates with a positive probability. Furthermore, suppose G has a matching $\{e_1, \ldots, e_k\}$ and distinct cycles $C_1, \ldots, C_k$ (not necessarily vertex-disjoint) such that $e_i \in E(C_j)$ iff $i = j$ for all $1 \leq i, j \leq k$. Then*

$$(2.9) \qquad \mathbb{P}(X_t \text{ synchronizes weakly}) \leq (7/9)^k.$$

This bound is achieved when C_i's are vertex-disjoint triangles and assuming CCA dynamics. In particular, if $k = \infty$, that is, there is an infinite matching and an infinite sequence of cycles as above, then X_t oscillates almost surely.

An interesting application of the above theorem is the following phase transition of limiting behavior of X_t on the Erdős–Rényi random graph.

THEOREM 6. *Let $G = G(n, p)$ be the Erdős–Rényi random graph and let $(X_t)_{t \geq 0}$ be a random CCA or GHM trajectory, and let $\mathbb{P}$ denote the joint product probability measure for (G, X_0).*

   (i) *If $p = o(1/n)$, then X_t synchronizes on each component of G a.a.s.*
   (ii) *If $p = \lambda/n$ for any $0 < \lambda < 1$, then there exists a constant $C = C(\lambda) \in (0, 1)$ such that*

$$(2.10) \qquad \lim_{n \to \infty} \mathbb{P}(X_t \text{ synchronizes on each component of } G(n, p)) = C.$$

   (iii) *If $p = \lambda/n$ for any $\lambda > 1$, then there exists a constant $D = D(\lambda) > 0$ such that for all sufficiently large n,*

$$(2.11) \qquad \mathbb{P}(X_t \text{ oscillates on the largest component of } G(n, p)) \geq 1 - e^{-Dn}.$$

Explicit expressions for the constants C and D in the assertion are given in the proof.

We next address the limiting behavior on infinite trees. Let $\Gamma = (V, E)$ be an infinite, but locally finite, tree rooted at $0 \in V$. In the context of trees, X_0 will be a random 3-coloring on the vertices drawn from a general product measure $\mathbb{P} = \mathbb{P}_{p_0, p_1, p_2}$ with marginal distribution $\mathbb{P}(X_0(x) = i) = p_i$ for all $x \in V$ and $i \in \mathbb{Z}_3$. The reason we are considering this general situation is to highlight an intriguing symmetry between the three colors on infinite trees; by contrast, in The-
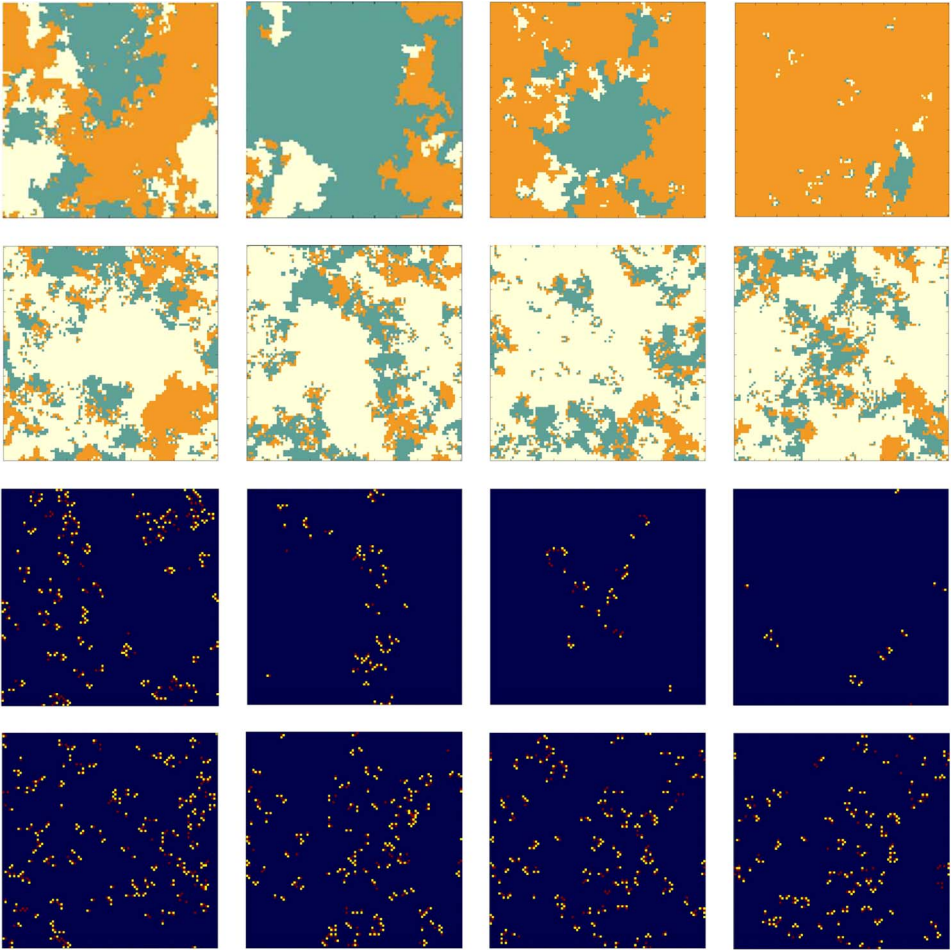
FIG. 1. (*Top row*) *Snapshots of* 3-*color CCA on a uniform spanning tree of a* 100 *by* 100 *torus, each* 100 *iterations from left to right.* (*Second row*) *Dynamics after* 12 *random edges are added to the spanning tree. Orange* = 0, *green* = 1 *and yellow* = 2. *Corresponding simulations for* 3-*color GHM are shown in the third and fourth rows. Dark blue* = 0, *yellow* = 1 *and red* = 2.

orems 5 and 6 the added generality provides little of interest. The random coloring X_0 induces random variables $dX_0(x, y)$ on each adjacent pair $(x, y) \in \overline{E}$, which we will refer to as *increments*. Note that the increments are bounded, have zero expectation and are identically distributed. Also note that they are in general not independent. For instance, no consecutive increments can be both 1 in case of GHM, for arbitrary marginal density. However, it is easy to see that in our case the increments are 1-*correlated*, by which we mean that two increments are independent if their underlying edges are vertex disjoint.

We append a random variable S_σ at each vertex $\sigma \in V$ by

$$(2.12) \qquad S_\sigma = \int_{\vec{P}_\sigma} dX_0,$$

where $\vec{P}_\sigma$ is the unique directed path from 0 to σ. The collection $\{S_\sigma\}_{\sigma \in V}$ is called a Γ-*indexed walk*. If we take Γ to be a Galton–Watson tree, for instance, then a Γ-indexed walk can be viewed as a branching random walk. In the beginning of Section 5, we will see that the activity α of the dynamics $(X_t)_{t \geq 0}$ coincides with the following quantity:

$$(2.13) \qquad v_c := \limsup_{n \to \infty} \frac{1}{n} \max_{|\sigma|=n} S_\sigma,$$

which is called the *cloud speed* of the associated Γ-indexed walk. Hence on infinite trees, understanding the CCA and GHM dynamics boils down to the study of the cloud speed of tree indexed walks. Hence we will be interested in the cloud speed of a Γ-indexed random walk, where Γ is arbitrary and increments are 1-correlated.

The value of v_c is closely related to the exponential growth rate of populations in Γ. Perhaps the simplest quantity to measure the growth is its *volume entropy* given by

$$(2.14) \qquad \mathrm{h}(\Gamma) = \limsup_{n \to \infty} \frac{1}{n} \log A_n,$$

where A_k is the number of vertices of Γ at level k. We remark that replacing A_k by the number B_k of vertices of Γ up to level k in the above definition does not change the value, which justifies its name. (This quantity is the *Minkowski dimension* of the boundary space of Γ; see, e.g., [3], which considers only trees without leaves.) Consider first $\Gamma = \mathbb{Z}$, in which case $\mathrm{h}(\Gamma) = 0$. The law of an iterated logarithm says that for random walks with centered i.i.d. increments, the maximum grows in the order of $\sqrt{n \log \log n}$ a.s., so $v_c = 0$ a.s. More generally, positive volume entropy is necessary for positive cloud speed; see Theorem 7(iii).

However, $\mathrm{h}(\Gamma) > 0$ is not sufficient to guarantee $v_c > 0$, which we illustrate in the following example. Let Γ^a be obtained from a single infinite ray γ (a directed path starting at the root) by attaching $2^{|\sigma|} - 1$ leaves to each vertex $\sigma \in \gamma$. Even though the kth level contains 2^k vertices so that $\mathrm{h}(\Gamma) = \log 2$, on this tree the cloud speed is zero, as there is no contribution of the increments on leaves to the cloud speed. Thus the volume entropy may not be a suitable notion to study the cloud speed. Nonetheless, under the assumption that Γ has no leaves and the increments of the Γ-indexed walk are centered, i.i.d., and satisfy a mild moment condition, Benjamini and Peres [3] showed that $v_c > 0$ if and only if $\mathrm{h}(\Gamma) > 0$. Furthermore, they obtained sharp upper and lower bounds on v_c in terms of the volume entropy and the large deviations rate of one-dimensional random walk with the same step distribution.

The example Γ^a illustrates that for trees with leaves we may need a refined quantity that measures the average number of infinite branches per vertex. Lyons [21] introduced such a quantity called the *branching number*. Given a tree Γ, a *cutset* Π is a finite set of vertices not including 0 such that every infinite path from 0 intersects Π and such that there is no pair $\sigma, \tau \in \Pi$ with $\sigma < \tau$. The *branching number* of Γ is defined by

$$(2.15) \qquad \mathrm{br}(\Gamma) = \inf\left\{\lambda > 0 \,\middle|\, \inf_{\Pi} \sum_{\sigma \in \Pi} \lambda^{-|\sigma|} = 0\right\}.$$

It follows from the definitions that $\log \mathrm{br}(\Gamma) \le \mathrm{h}(\Gamma)$, and it is known that equality holds when Γ is sufficiently regular, for example, for almost all trees generated by a Galton–Watson process [21]. Turning back to the example Γ^a, indeed we have $\log \mathrm{br}(\Gamma^a) = 0$ since there is only one infinite branch, which appears to support a connection with the fact that $v_c = 0$ on Γ^a. However, it turns out that the branching number does not decide when the cloud speed vanishes either, as there are trees with branching number 1 but large cloud speed. We will give such an example in Section 5 (Example 5.4).

This leads us to introduce another notion of dimension of Γ, with a better connection to the cloud speed in case Γ has leaves, which allows us to improve known lower and upper bounds. Given a tree Γ and for each $n \le m$, denote by $A_{n,m}$ the number of vertices at level n that have descendants at level m. For each $r > 1$, define the *r-volume entropy* of Γ by

$$(2.16) \qquad \mathrm{h}_r(\Gamma) = \limsup_{n \to \infty} \frac{1}{n} \log A_{n, \lfloor rn \rfloor},$$

and its *reduced volume entropy* by

$$(2.17) \qquad \overline{\mathrm{h}}(\Gamma) = \limsup_{r \searrow 1} \mathrm{h}_r(\Gamma).$$

In particular, if Γ has no leaves, then $A_n = A_{n, \lfloor rn \rfloor}$ for all $r > 1$ so $\mathrm{h}_r(\Gamma) = \mathrm{h}(\Gamma)$ for all $r > 1$. In general, we have $\log \mathrm{br}(\Gamma) \le \mathrm{h}_r(\Gamma) \le \overline{\mathrm{h}}(\Gamma) \le \mathrm{h}(\Gamma)$ for all $r > 1$.

In the statement of our main theorem to follow, recall that we consider a product measure with unequal densities p_0, p_1, p_2 of colors 0, 1, 2. In preparation, we introduce certain functions that determine the relevant large deviation rates. For each $t \ge 0$, let $x = x(t)$ be the largest positive root of the cubic equation

$$(2.18) \qquad x^3 - x^2 = p_0 p_1 p_2 \left(e^{3t} + e^{-3t} - 2\right),$$

and define $\Lambda_{\mathrm{CCA}}(t) = \log x(t)$. Note that $x(t)$ is strictly increasing on $[0, \infty)$, as the right-hand side of (2.18) is nonnegative and strictly increasing for $t \in [0, \infty)$, and the left-hand side strictly increases from 0 to ∞ for $x \in [1, \infty]$. Hence $\Lambda_{\mathrm{CCA}}(t) \ge 0$ is well-defined for $t \in [0, \infty)$, strictly increasing and continuous.

Next, let $\Lambda^*_{\mathrm{CCA}}(u) = \sup_{t \geq 0}\{ut - \Lambda_{\mathrm{CCA}}(t)\}$ be its Legendre transform. Observe that $x(t) \sim (p_0 p_1 p_2)^{1/3} e^t$ as $t \to \infty$, so that

$$(2.19) \qquad \Lambda_{\mathrm{CCA}}(t) = t + \frac{1}{3}\log(p_0 p_1 p_2) + o(1).$$

Hence it follows that $\Lambda^*_{\mathrm{CCA}}(u)$ is finite, strictly increasing and continuous on $[0, 1]$, $\Lambda^*_{\mathrm{CCA}}(1) = -\frac{1}{3}\log(p_0 p_1 p_2)$, and $\Lambda^*_{\mathrm{CCA}} \equiv \infty$ on $(1, \infty)$. Moreover, it is easy to check that for $p_0 = p_1 = p_2 = 1/3$, we have the following closed-form expressions:

$$(2.20) \quad \Lambda_{\mathrm{CCA}}(t) = \log(1 + e^t + e^{-t}) - \log 3,$$

$$(2.21) \quad \Lambda^*_{\mathrm{CCA}}(u) = \begin{cases} u \log\left(\dfrac{u + \sqrt{4 - 3u^2}}{2(1 - u)}\right) \\ \qquad - \log\left(\dfrac{1 + \sqrt{4 - 3u^2}}{1 - u^2}\right) + \log 3 & \text{if } u \in [0, 1), \\ \log 3 & \text{if } u = 1, \\ \infty & \text{if } u \in (1, \infty). \end{cases}$$

Finally, define $\Lambda_{\mathrm{GHM}}(t) = \Lambda_{\mathrm{CCA}}(t/3)$ and let Λ^*_{GHM} be its Legendre transform, so that $\Lambda^*_{\mathrm{GHM}}(u) = \Lambda^*_{\mathrm{CCA}}(3u)$. We now state our main theorem.

THEOREM 7. *Let $\Gamma = (V, E)$ be an infinite rooted tree and $(X_t)_{t \geq 0}$ the random 3-color CCA or GHM trajectory on Γ, where X_0 is drawn from the product measure with marginal density $\mathbb{P}(X_0(\sigma) = i) = p_i$ for $i \in \mathbb{Z}_3$. Denote by Λ either Λ_{CCA} or Λ_{GHM}, depending on the dynamics, and let Λ^* be corresponding Legendre transform. Then we have the following*:

(i) *The process X_t synchronizes weakly if and only if $\overline{h}(\Gamma) = 0$.*

(ii) *The activity α is equal to the cloud speed v_c of the associated Γ-indexed random walk $\{S_\sigma\}_{\sigma \in V}$.*

(iii) *The cloud speed v_c satisfies the upper bound*

$$(2.22) \qquad \Lambda^*(v_c) \leq \overline{h}(\Gamma).$$

(iv) *Suppose $\log \mathrm{br}(\Gamma) = h(\Gamma)$ and let $B = 1$ for CCA and $B = 1/3$ for GHM. If $\mathrm{br}(\Gamma) \geq 1/\sqrt[3]{p_0 p_1 p_2}$, then $v_c = B$. Otherwise, $v_c < B$ and equality holds in (2.22), which determines v_c. In particular, v_c for GHM is a third of that for CCA, for arbitrary p_0, p_1, p_2.*

(v) *For each $r > 1$ such that $v_c/(1 - r^{-1}) < B$, where B is defined in (iv), we have*

$$(2.23) \qquad \frac{h_r(\Gamma)}{r - 1} \leq \Lambda^*\left(\frac{v_c}{1 - r^{-1}}\right).$$

In particular, if Γ *has no leaves, then*

(2.24) $$\Lambda\big(h(\Gamma)/v_c\big) \le h(\Gamma),$$

which is sharp in the sense that there exists a tree-indexed walk on some tree Γ^-
with cloud speed v_c^- *satisfying* $\Lambda(h(\Gamma)/(v_c^- - \varepsilon)) > h(\Gamma)$ *for each* $\varepsilon > 0$.

REMARK 8. Note that Theorem 7(iv) implies that, when $\log \mathrm{br}(\Gamma) = h(\Gamma)$, which for instance holds a.s. for Galton–Watson trees, the activity α for both CCA and GHM depends only on the product $p_0 p_1 p_2$; in particular, it is symmetric in the three densities. This is particularly surprising for GHM, in which the three colors play very different roles. For example, pick a small $\varepsilon > 0$ and compare two initial states: one with $p_0 = p_1 = (1 - \varepsilon)/2$ and $p_2 = \varepsilon$, and the other with $p_0 = p_2 = (1 - \varepsilon)/2$ and $p_1 = \varepsilon$. The two generate very different dynamics; nevertheless, the higher annihilation rate of the first one causes their activities to match.

This paper is organized as follows. In Section 3, we prove Lemma 1 by introducing a monotone comparison process defined on the universal covering space of the underlying graph. In Section 4, we give proofs for the main theorems concerning finite and random graphs, Theorems 3, 5 and 6. In Section 5, we extend Proposition 4.1 in Benjamini and Peres [3] and give sharp estimates of the cloud speed of a tree-index random walk when the underlying tree is arbitrary and the increments are 1-correlated. By the method developed in Section 3, this leads to the proof of Theorem 7.

3. Tournament expansion of 3-color GHM and CCA. We prove Lemma 1 in this section. One of the main complications in understanding the 3-color CCA and GHM dynamics is that the color space $\mathbb{Z}_3$ has a cyclic hierarchy so that the configuration space lacks monotonicity. To overcome this difficulty, we introduce a monotone comparison process and establish its relationship with our CCA and GHM dynamics.

Tournament process on a graph is a deterministic process in which initially all nodes have an integer rank, and in each step each node simultaneously adopts the maximum rank among itself and its neighbors. More precisely, given a connected graph $G = (V, E)$, a map $\mathrm{rk}_0 : V \to \mathbb{Z}$ is called a *ranking* on G. The transition map from time t to $t + 1$ is given by

(3.1) $$\mathrm{rk}_{t+1}(x) = \max\{\mathrm{rk}_t(y) \mid y \in N(x) \cup \{x\}\}.$$

Iteration of the above transition rule generates a discrete-time orbit $(\mathrm{rk}_t)_{t \ge 0}$ of rankings.

Observe that if G is finite, then for any initial ranking rk_0 on G there is a global maximum, which every node will eventually achieve. In general, locally maximum rank propagates with unit speed across the graph until it is overcome by waves from higher rankers. To make this observation precise, or each node $x \in V$ and radius $t \in \mathbb{N}$, define $M_t(x)$ to be the maximum initial rank in the t-ball centered at x:

$$(3.2) \qquad M_t(x) = \max\{\mathrm{rk}_0(y) \mid d(x, y) \le t\},$$

where $d(x, y)$ is the usual graph distance in G. The following proposition characterizes the dynamics in tournament processes.

PROPOSITION 3.1.   *Let $G = (V, E)$ be a connected graph and fix an initial ranking $\mathrm{rk}_0 : V \to \mathbb{Z}$. Then the local dynamic at any given node $x \in V$ is given by*

$$(3.3) \qquad \mathrm{rk}_t(x) = M_t(x).$$

PROOF.   Fix $x \in V$ and $t \ge 1$. By tracing back the origins of ranks, it is easy to see that if $\mathrm{rk}_t(x) = r > \mathrm{rk}_0(x)$, then necessarily there exists $y \in V$ with $d(x, y) \le t$ and $\mathrm{rk}_0(y) = r$. This yields $\mathrm{rk}_t(x) \le M_t(x)$. On the other hand, let $y \in V$ be the vertex that attains $M_t(x)$, that is, $d(x, y) \le t$ and $\mathrm{rk}_0(y) = M_t(x)$. Choose a shortest path P from x to y. Since P has length $\le t$, the rank $\mathrm{rk}_0(y)$ of y propagates along P and reaches x by time t. Since $\mathrm{rk}_s(x)$ is nondecreasing in s, this yields that $\mathrm{rk}_t(x) \ge M_t$. This shows the assertion.   □

Having introduced the tournament process, we now establish a comparison tournament process for CCA or GHM dynamics. Fix a graph $G = (V, E)$ and an initial 3-coloring $X_0 : V \to \mathbb{Z}_3$. Designate an arbitrary vertex x in G as its base point. If $xy \in E$ and $dX_0(x, y) = 1$, this means y excites x at time 0 and $X_1(x) = X_0(y)$. Hence when we construct an associated tournament process, it would be natural to give y a rank that is greater than that of x, for example, $\mathrm{rk}_0(y) = \mathrm{rk}_0(x) + dX_0(x, y)$. In general, if z is any node in G, we may try to define the rank of z by

$$(3.4) \qquad \mathrm{rk}_0(z) = \mathrm{rk}_0(x) + \int_{\vec{P}} dX_0,$$

where $\vec{P}$ is any walk from x to z with initial condition $\mathrm{rk}_0(x) = 0$. However, this gives a well-defined ranking on G if and only if dX_0 is conservative. In particular, rk_0 is well-defined when G is a tree. In that case, we define the *tournament expansion* of $(X_t)_{t \ge 0}$ to be the tournament process $(\mathrm{rk}_t)_{t \ge 0}$.

To make sense of the above definition on general graphs, we need to distinguish the endpoints of distinct walks from x. This encourages us to define the associated

tournament process on the universal covering space of G. The *universal covering space of G with base point x* is a tree $\mathcal{T}_x = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is the set of all finite nonbacktracking walks in G starting from x, and adjacency is given by one-step extension: $\{\vec{W}, \vec{W}'\} \in \mathcal{E}$ if and only if one of them can be obtained by adjoining a single vertex at the end of the other walk. We have the natural covering map $p : \mathcal{V} \to V$ which maps each finite walk $\vec{W}$ to its endpoint. We identify the length 0 walk from x to x with $x \in V$. Denote $p : \mathcal{T}_x \to G$. For each $y \in V$, $\tilde{y}$ denotes any node in $\mathcal{T}_x$ such that $p(\tilde{y}) = y$. For each 3-coloring X on G, we denote by $\tilde{X}$ the *lift* of X onto $\mathcal{T}_x$, defined by $\tilde{X}(\tilde{x}) = X(x)$ for all $\tilde{x} \in p^{-1}(x)$. A trivial but important observation is that any GCA dynamics of (G, X_0) naturally lifts to $(\mathcal{T}_x, \tilde{X}_0)$, due to the fact that the transition map is completely determined locally. Hence, in particular, we can lift any CCA or GHM dynamics on any simple graph G onto its universal covering space.

Now for a given 3-color CCA or GHM dynamics $(X_t)_{t \geq 0}$ on arbitrary connected graph $G = (V, E)$ with base point at $x \in V$, we define its *tournament expansion* by the tournament expansion of $(\tilde{X}_t)_{t \geq 0}$, the 3-color CCA or GHM dynamics lifted onto the universal covering space $\mathcal{T}_x$ with base point at $x \in \mathcal{V}$. The following lemma establishes that the two dynamical systems are compatible.

LEMMA 3.2. *Let $G = (V, E)$, $\mathcal{T}_x = (\mathcal{V}, \mathcal{E})$, $(X_t)_{t \geq 0}$, and $(\mathrm{rk}_t)_{t \geq 0}$ be as before. Then for all $t \geq 0$ and $\tilde{z} \in \mathcal{V}$, we have*

$$(3.5) \qquad \mathrm{rk}_{t+1}(\tilde{z}) - \mathrm{rk}_t(\tilde{z}) = \mathbf{1}(z \text{ is excited at time } t).$$

In particular, for all $t \geq 0$, we have

$$(3.6) \qquad \mathrm{rk}_t(x) = \mathrm{ne}_t(x).$$

Note that the key lemma (Lemma 1) follows from Lemma 3.2 in conjunction with Proposition 3.1. We prove Lemma 3.2 in the rest of this section.

In order to show Lemma 3.2, we define an auxiliary process $(\mathrm{Rk}_t)_{t \geq 0}$ on $\mathcal{T}_x$ as follow. For each $\tilde{z} \in \mathcal{V}$, let $\vec{P}(\tilde{z})$ be the unique directed path from x to $\tilde{z}$ in the universal cover $\mathcal{T}_x$. We define $\mathrm{Rk}_t : \mathcal{V} \to \mathbb{Z}$ first vertically by $\mathrm{Rk}_t(x) = \mathrm{ne}_t(x)$ for all $t \geq 0$ and then horizontally extend to all vertices $\tilde{z} \in \mathcal{V}$ by

$$(3.7) \qquad \mathrm{Rk}_t(\tilde{z}) = \mathrm{Rk}_t(x) + \int_{\vec{P}(\tilde{z})} d\tilde{X}_t$$

for each $t \geq 0$. We are going to show that this new process satisfies Lemma 3.2 and in fact equivalent to the original tournament expansion. We begin with the

following commutative diagram which summarizes our construction:

$$
\begin{array}{ccc}
\text{ranking } \mathrm{Rk}_t \text{ on } \mathcal{T}_x & \dashrightarrow & \text{ranking } \mathrm{Rk}_{t+1} \text{ on } \mathcal{T}_x \\
\uparrow & & \uparrow \\
\text{edge configuration } d\tilde{X}_t \text{ on } \mathcal{T}_x & \dashrightarrow & \text{edge configuration } d\tilde{X}_{t+1} \text{ on } \mathcal{T}_x \\
\uparrow & & \uparrow \\
\text{lift of 3-coloring } \tilde{X}_t \text{ on } \mathcal{T}_x & \longrightarrow & \text{lift of 3-coloring } \tilde{X}_{t+1} \text{ on } \mathcal{T}_x \\
\uparrow & & \uparrow \\
\text{3-coloring } X_t \text{ on } G & \longrightarrow & \text{3-coloring on } X_{t+1} \text{ on } G
\end{array}
$$

(3.8)

In the above diagram, the first two horizontal maps from bottom are the CCA (or GHM) transition map on G and $\mathcal{T}_x$, respectively, whereas the other two horizontal maps are to be described. In fact, the top horizontal map turns out to be the transition map for tournament processes. Since by definition $\mathrm{Rk}_0 = \mathrm{rk}_0$, this will show the aforementioned equivalence.

The induced dynamics $(d\tilde{X}_0)_{t \geq 0}$ can be interpreted as a traditional comparison process called *embedded particle system*, which consists of branching and annihilating particles on the edges of $\mathcal{T}_x$. This allows us to view propagation of excitation (or information flow) as moving and branching edge particles. Without loss of generality, we consider a CCA (or GHM) trajectory $(X_t)_{t \geq 0}$ on a tree $T = (V, E)$ starting from an initial 3-coloring X_0. We view $\xi_0 := -dX_0$ as an initial *edge configuration* where on each adjacent pair $(u, v) \in \overline{E}$ with $dX_t(u, v) = -1$, we place a single particle heading to v. We denote a particle on the edge uv heading to v by $u \to v$. The orientation of particles is so that

(3.9)
$$\exists \text{ a particle } v \to u \text{ at time } 0 \iff dX_0(v, u) = -1$$
$$\iff v \text{ excites } u \text{ at time } 0.$$

We let the edge particles evolve in discrete time in parallel, by applying the following rules successively for each transition $\xi_t \mapsto \xi_{t+1}$:

1. (branching) If there is a particle $u \to v$ at time t, it branches into each incident edge vw without a particle $v \leftarrow w$ at time t, and becomes a *potential particle* $v \to w$ at time $t + 1/3$.

2. (annihilation) If an edge has at least two potential particles at time $t + 1/3$ with the opposite direction, then all potential particles on that edge get annihilated from the system at time $t + 2/3$.

3. (coalescence) All remaining potential particles on each edge at time $t + 2/3$, which have the same direction, coalesce into a single particle with the same direction at time $t + 1$. Then ξ_{t+1} counts the particles at time $t + 1$.
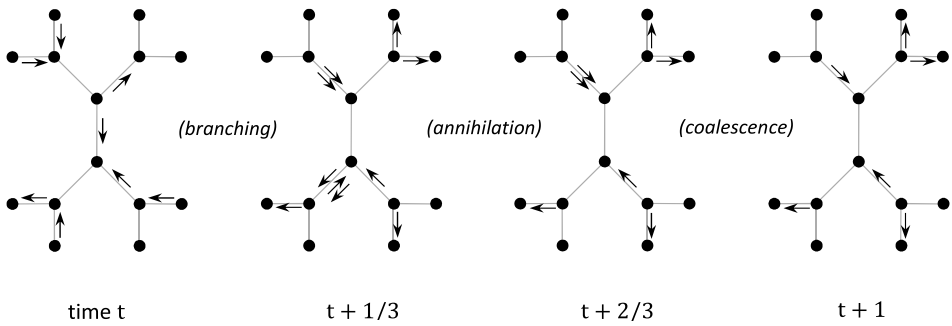
FIG. 2.   *Three-step edge particle evolution rule.*

We call this process $(\xi_t)_{t\geq 0}$ the *particle system expansion* of $(X_t)_{t\geq 0}$. See Figure 2 for an illustration. In the next proposition, we show that this particle evolution rule is compatible with the actual CCA and GHM dynamics.

PROPOSITION 3.3.   *Let $T = (V, E)$ be a tree, $X_0 : V \to \mathbb{Z}_3$ a 3-coloring, $(X_t)_{t\geq 0}$ the CCA (or GHM) trajectory starting from X_0, and $(\xi_t)_{t\geq 0}$ be its particle system expansion. Then for each $t \geq 0$, we have the following*:

(i) $dX_t = -\xi_t$.
(ii) $(dX_{t+1} - dX_t)(y, z) = \mathbf{1}(z$ *is excited at time* $t) - \mathbf{1}(y$ *is excited at time* $t)$.

PROOF.   First, we show (i) implies (ii). We show the following equivalent statement is implied from (i):

(3.10)
$$\xi_{t+1}(y, z) - \xi_t(y, z)$$
$$= \mathbf{1}(y \text{ is excited at time } t) - \mathbf{1}(z \text{ is excited at time } t).$$

Note that (i) says the relation (3.9) continues to hold for all times $t \geq 0$. Hence x is excited at time t iff it has an incoming particle at that time. Suppose first that $\xi_t(y, z) = 1$. By (i), $dX_t(y, z) = -1$, so z is excited at time t. So it suffices to show that the first terms in (3.10) agree. Indeed, since $\xi_t(y, z) = 1$, all other potential incoming particles to z at time t get annihilated. Thus $\xi_{t+1}(y, z) = 1$ iff y has an incoming particle at time t iff y is excited at time t. Since both sides of (3.10) are anti-symmetric, this deals with the opposite case $\xi_t(y, z) = -1$. Lastly, suppose $\xi_t(y, z) = 0$. Then $\xi_{t+1}(y, z) = 1$ if y has an incoming particle at time t but z does not, $\xi_{t+1}(y, z) = -1$ if z has an incoming particle at time t but y does not, and $\xi_{t+1}(y, z) = 0$ if y and z are both excited or both not at time t. Hence (i) implies (ii).

It remains to prove (i). We prove the assertion for GHM dynamics, and a similar argument applies for CCA. The assertion holds for $t = 0$ by definition. Suppose the assertion holds for $t \geq 1$. This means that at times $s \leq t$, particles always point

from color 1 to color 0. Now fix an edge $x_1 x_2 \in E$. First, suppose $X_t(x_1) = 1$ and $X_t(x_2) = 0$. Then $X_{t+1}(x_1) = 2$ and $X_{t+1}(x_2) = 1$, so $dX_{t+1}(x_1, x_2) = 0$. Hence we wish to show $\xi_{t+1}(x_1, x_2) = 0$. In this case, $dX_t(x_1, x_2) = -1$, and by the induction hypothesis, $\xi_t(x_1, x_2) = 1$. Since particles are always from color 1 to 0 at times $\leq t$, x_1 has no incoming particle at time t. Also, the particle $x_1 \rightarrow x_2$ at time t would annihilate all possible incoming particles to x_2 at time t. Thus $\xi_{t+1}(x_1, x_2) = 0$ as desired. Second, suppose that both $X_t(x_1)$ and $X_t(x_2)$ are from $\{1, 2\}$. Then $X_{t+1}(x_i) = X_t(x_i) + 1 \bmod 3$, so $X_{t+1}(x_i) \neq 1$ for $i = 1, 2$, and $dX_{t+1}(x_1, x_2) = 0$. Hence we also want to show $\xi_{t+1}(x_1, x_2) = 0$. But this is clear since both x_i's do not have incoming particles by the induction hypothesis.

Lastly, suppose $X_t(x_1) = X_t(x_2) = 0$. By the induction hypothesis, each x_i has an incoming particle at time t iff it has a neighbor of color 1 at time t. If neither of them are excited at time t, then they both keep color 0 at time $t + 1$, so $dX_{t+1}(x_1, x_2) = 0$, which agrees with the particle dynamics since there are no incoming particles in either side. If both of them are excited, then $X_{t+1}(x_1) = X_{t+1}(x_2) = 1$ so $dX_{t+1}(x_1, x_2) = 0$. Also $\xi_{t+1}(x_1, x_2) = 0$, as opposing particles come through both x_i's and annihilate, so there is no remaining particles on the edge $x_1 x_2$ at time $t + 1$. Otherwise, by symmetry we may assume that $X_{t+1}(x_1) = 1$ and $X_{t+1}(x_2) = 0$, so $dX_{t+1}(x_1, x_2) = -1$. In terms of particles, x_1 has an incoming particle at time t but x_2 does not, so there is a particle $x_1 \rightarrow x_2$ at time $t + 1$. This shows the assertion. $\square$

Next, we extend Proposition 3.3(ii) to the general case. This gives the desired property of the tournament expansion that a vertex increments its rank by 1 if and only if it gets excited in the original dynamics.

PROPOSITION 3.4. *Let $T = (V, E)$ be a tree and $(X_t)_{t \geq 0}$, $(dX_t)_{t \geq 0}$, and $(\mathrm{Rk}_t)_{t \geq 0}$ be as before. Fix any $y, z \in V$ and let $\vec{P}$ is the unique directed path from y to z in T. Then we have*

$$(3.11) \quad \int_{\vec{P}} dX_{t+1} - \int_{\vec{P}} dX_t = \mathbf{1}(z \text{ is excited at time } t) - \mathbf{1}(y \text{ is excited at time } t)$$

for all $t \geq 0$. Furthermore, for each $z \in V$ and $t \geq 0$, we have

$$(3.12) \quad \mathrm{Rk}_{t+1}(z) - \mathrm{Rk}_t(z) = \mathbf{1}(z \text{ is excited at time } t).$$

PROOF. The second part of the assertion follows immediately from the first part, definition of ranking of the base point x and the following relation coming from the definition:

$$(3.13) \quad \mathrm{Rk}_{t+1}(z) - \mathrm{Rk}_t(z) = \left[\mathrm{Rk}_{t+1}(x) - \mathrm{Rk}_t(x)\right] + \left[\int_{\vec{P}} dX_{t+1} - \int_{\vec{P}} dX_t\right].$$

To show the first part, label the vertices of $\vec{P}$ by $y = x_0, x_1, \ldots, x_k = z$ such that $x_i x_{i+1} \in E$. Note that Proposition 3.3(ii) gives

$$\sum_{i=0}^{k-1} (dX_{t+1} - dX_t)(x_i, x_{i+1})$$

$$= \sum_{i=0}^{k-1} \mathbf{1}(x_{i+1} \text{ is excited at time } t) - \mathbf{1}(x_i \text{ is excited at time } t)$$

$$= \mathbf{1}(z \text{ is excited at time } t) - \mathbf{1}(y \text{ is excited at time } t)$$

which is equivalent to (3.11). $\square$

As a remark, lifting dynamics into the universal covering space and applying (3.11) give that contour integrals of dX_t on arbitrary underlying graph G are time-invariant.

PROOF OF LEMMA 3.2.   The second assertion follows from the first since $\mathrm{rk}_0(x) = 0$. To show the first part, note that by Proposition 3.4 it suffices to show that

$$(3.14) \qquad\qquad \mathrm{Rk}_t = \mathrm{rk}_t \qquad \forall t \geq 0.$$

It holds for $t = 0$ by definition. Hence it suffices to show that each transition $\mathrm{Rk}_t \mapsto \mathrm{Rk}_{t+1}$ follows the transition map for tournament process. Indeed, by Proposition 3.4 and construction,

$$(3.15) \quad \mathrm{Rk}_{t+1}(\tilde{z}) - \mathrm{Rk}_t(\tilde{z}) = \mathbf{1}(\tilde{z} \text{ is excited at time } t)$$

$$(3.16) \qquad\qquad\qquad = \mathbf{1}(\exists \tilde{y} \in N(\tilde{z}) \text{ such that } d\tilde{X}_t(\tilde{y}, \tilde{z}) = -1)$$

$$(3.17) \qquad\qquad\qquad = \mathbf{1}(\exists y \in N(z) \text{ such that } \mathrm{Rk}_t(y) = \mathrm{Rk}_t(z) + 1)$$

for any $\tilde{z} \in \mathcal{V}$ and $t \geq 0$. This shows the assertion. $\square$

4. On finite graphs and the Erdős–Rényi random graph.   We begin with a lemma that provides a lower bound for the activity in terms of contour integrals with respect to dX_0.

LEMMA 4.1.   *Let $G = (V, E)$ be arbitrary connected graph (not necessarily finite) with a cycle $\vec{C}$. Suppose $X_0 : V \to \mathbb{Z}_3$ is an initial 3-coloring on G such that the contour integral of dX_0 on $\vec{C}$ is nonzero. Then we have*

$$(4.1) \qquad\qquad\qquad\qquad \alpha \geq \frac{1}{|\vec{C}|} \left| \oint_{\vec{C}} dX_0 \right|.$$

PROOF. Without loss of generality, we may assume that $\oint_{\vec{C}} dX_0 > 0$. Fix $x \in V$, and let $(\mathrm{rk}_t)_{t \geq 0}$ be the tournament expansion of $(X_t)_{t \geq 0}$ at base point $x \in V$. By definition, Lemma 3.2, and Proposition 3.1, we have

$$(4.2) \qquad \mathrm{ne}_t(x) = \mathrm{rk}_t(x) = M_t(x).$$

Let $\vec{P}$ be any finite directed path in G from x to the first vertex of $\vec{C}$, and let $\vec{W}_n$ be the walk formed by starting at x, traversing $\vec{P}$, and then concatenating n copies of $\vec{C}$. Let $p = |\vec{P}|$ and $c = |\vec{C}|$. Then we have

$$(4.3) \qquad M_{p+cn}(x) \geq \int_{\vec{W}_n} dX_0 = \int_{\vec{P}} dX_0 + n \int_{\vec{C}} dX_0$$

so by taking $n \to \infty$, we obtain

$$(4.4) \qquad \alpha = \limsup_{t \to \infty} \frac{M_t(x)}{t} \geq \frac{1}{|\vec{C}|} \oint_{\vec{C}} dX_0$$

as desired. $\square$

PROOF OF THEOREM 3. It suffices to show part (i) and (ii) of the assertion. First, we show part (i). Suppose $X_0 : V \to \mathbb{Z}_3$ is an initial 3-coloring such that dX_0 is conservative. Fix an arbitrary node $x \in V$. Let $(\mathrm{rk}_t)_{\geq 0}$ be the tournament expansion at x of the CCA (or GHM) trajectory $(X_t)_{t \geq 0}$. Let D be the diameter of G. By Lemma 1, the assertion follows from

$$(4.5) \qquad \|\mathrm{rk}_0\|_\infty \leq D.$$

Indeed, let $\vec{W}$ be any finite walk in G starting from x to some vertex, say z. Let $\vec{P}$ be a shortest directed path from x to z in G. Then $|\vec{P}| \leq D$, and since dX_0 is conservative we have

$$(4.6) \qquad |\mathrm{rk}_0(\tilde{z})| = \left| \int_{\vec{W}} dX_0 \right| = \left| \int_{\vec{P}} dX_0 \right| \leq |\vec{P}| \cdot \|dX_0\|_\infty \leq D$$

as desired.

To show (ii), suppose that on some directed walk the contour integral of dX_0 does not vanish. We first note that $(X_t)_{t \geq 0}$ converges to a periodic limit cycle since G is finite and the dynamics is deterministic. Hence the limit of $E_t(x)/t$ as $t \to \infty$ exists. Recall that $\mathrm{rk}_t(x) = \mathrm{ne}_t(x) = M_t(x)$ by definition, Proposition 3.1, Lemma 3.2. Hence by Lemma 4.1, it suffices to show that

$$(4.7) \qquad \lim_{t \to \infty} \frac{M_t(x)}{t} \leq \sup_{\vec{C}} \frac{1}{|\vec{C}|} \oint_{\vec{C}} dX_0,$$

where the supremum runs over all directed cycles in G.

Choose $\vec{C}$ such that the right-hand side of (4.7) attains its supremum. Let $\vec{W}_t$ be any walk of length t in G starting from x. Since G is finite, $\vec{W}_t$ may have lots of self-intersections for $t \gg |V|$. Let $\vec{C}_1, \ldots, \vec{C}_{n_t}$ be the sequence of cycles arising in

$\vec{W}_t$, in chronological order as one traverses it. Note that the number of edges in $\vec{W}_t$ that are not used by $\vec{C}_i$'s is at most $|V|$. Hence by using the choice of $\vec{C}$, we get

$$(4.8) \qquad \int_{\vec{W}_t} dX_0 \leq |V| + \sum_{i=1}^{n_t} \left| \int_{\vec{C}_i} dX_0 \right|$$

$$(4.9) \qquad \leq |V| + \sum_{i=1}^{n_t} \frac{|\vec{C}_i|}{|\vec{C}|} \left| \int_{\vec{C}_i} dX_0 \right|$$

$$(4.10) \qquad \leq |V| + \frac{t}{|\vec{C}|} \int_{\vec{C}} dX_0.$$

Since $\vec{W}_t$ was arbitrary with length t, this yields

$$(4.11) \qquad M_t \leq |V| + \frac{t}{|\vec{C}|} \int_{\vec{C}} dX_0.$$

Thus after dividing both sides by t and letting $t \to \infty$, one has (4.7) as desired. $\qquad \square$

We now proceed to the proofs of Theorems 5 and 6. In both theorems, recall that the initial coloring is uniform.

PROOF OF THEOREM 5. Let $G = (V, E)$ be a connected graph, not necessarily finite, containing at least one cycle. It suffices to show the second assertion. Let $\{e_1, \ldots, e_k\} \subset E$ be a matching in G and let $C_1, \ldots, C_k$ be cycles in G such that $e_i \in E(C_j)$ iff $i = j$ for $1 \leq i, j \leq k$. Let $\vec{C}_i$ denote a directed cycle on C_i with any of the two orientations given. According to the claim, we have

$$(4.12) \qquad \mathbb{P}(X_t \text{ synchronizes weakly}) \leq \mathbb{P}\left(\oint_{\vec{C}_i} dX_0 = 0 \ \forall 1 \leq i \leq k \right).$$

Suppose all vertices but the $2k$ vertices used by the matching have been colored by X_0. Let x_i and y_i be the two endpoints of e_i for each $1 \leq i \leq k$. We are going to show that one can always assign colors on each pair (x_i, y_i) with at least probability $2/9$ in such a way that C_i is singular. This means that each C_i can be nonsingular with probability $\geq 2/9$, and this occurs independently for each i by assumption. Thus the right-hand side of (4.12) is at most $(7/9)^k$, as desired.

Fix $1 \leq i \leq k$, and recall the definition of dX_0 in CCA and GHM cases, which was given at the beginning of Section 2. Let x_i' and y_i' be the neighbors of x_i and y_i on C_i, which are distinct iff C_i has more than three vertices. Let $\vec{P}_i$ and $\vec{Q}_i$ be the two directed paths from x_i' to y_i' on C, where $\vec{P}_i$ is the one that contains x_i. Clearly, C is singular iff $\int_{\vec{P}_i} dX_0 = \int_{\vec{Q}_i} dX_0$. First, suppose CCA dynamics. By symmetry, we may assume that $(X_0(x_i'), X_0(y_i')) = (0, 0)$ or $(0, 1)$. Suppose $(X_0(x_i'), X_0(y_i')) = (0, 0)$. If $\int_{\vec{Q}_i} dX_0 = 0$, then we set $(X_0(x_i), X_0(y_i)) =$
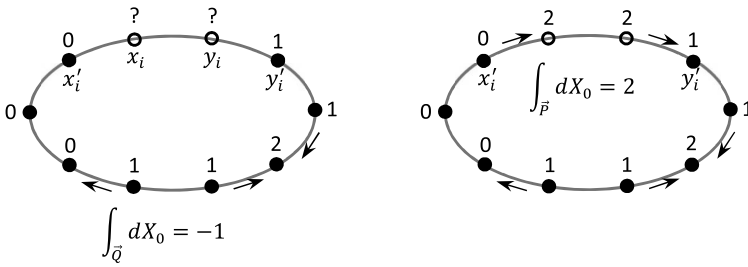
FIG. 3. *Controlling singularity of a cycle using two adjacent vertices in case of CCA dynamics.*

$(1, 2)$ or $(2, 1)$ so that $\int_{\vec{P}_i} dX_0 = \pm 2$. Otherwise, we set $X_0(x_i) = X_0(y_i)$ so that $\int_{\vec{P}_i} dX_0 = 0$. Second, suppose $(X_0(x_i'), X_0(y_i')) = (0, 1)$. In this case, $\int_{\vec{Q}_i} dX_0 \equiv 1 \bmod 3$, so it is enough to make $\int_{\vec{P}_i} dX_0$ to have the opposite sign. We can choose $(X_0(x_i), X_0(y_i)) = (2, 2)$ or $(0, 2)$ to make the integral over $\vec{P}_i$ equals to 2, or choose from $(0, 0)$ or $(1, 1)$ to make it -1. This takes care of the CCA case. See Figure 3 for examples.

Now we suppose GHM dynamics. Since there is no symmetry in colors as in CCA, we need to consider all six cases of $X_0(x_i') \leq X_0(y_i')$. We first suppose $\int_{\vec{Q}_i} dX_0 \neq 0$. Note that $\int_{\vec{P}_i} dX_0 = 0$ if $X_0(x_i) = X_0(y_i) = 2$ regardless of the colors of their neighbors. Hence it suffices to find one additional choices for which $\int_{\vec{P}_i} dX_0 = 0$ in each of the six cases. We list them as tuples $(X_0(x_i'), X_0(x_i), X_0(y_i), X_0(y_i'))$; $(0, 1, 1, 0)$, $(0, 2, 1, 1)$, $(0, 2, 1, 2)$, $(1, 1, 2, 1)$, $(1, 1, 2, 2)$ and $(2, 1, 1, 2)$. It remains to check the case when $\int_{\vec{Q}_i} dX_0 = 0$. In this case, we find two tuples for which the integral over $\vec{P}$ is nonzero for each of the six cases. Namely, $(0, 1, 2, 0)$, $(0, 2, 1, 0)$, $(0, 1, 0, 1)$, $(0, 1, 2, 1)$, $(0, 1, 1, 2)$, $(0, 1, 2, 2)$, $(1, 0, 2, 1)$, $(1, 2, 0, 1)$, $(1, 0, 2, 2)$, $(1, 1, 0, 2)$, $(2, 1, 0, 2)$ and $(2, 0, 1, 2)$. This shows the assertion. $\square$

Now we turn our attention to Erdős–Rényi random graphs. By Theorem 5, we know that the limiting behavior of X_t is closely related to the existence of cycles and a matching that separates a large number of cycles. We begin preparing for the proof of Corollary 6 with the following lemma.

LEMMA 4.2. *Let $G = G(m, \lambda/m)$ be the Erdős–Rényi random graph for some $\lambda > 0$. Let $M(G)$ be the size of the largest matching in G. Then for any $0 < \delta < 1$, there exists $m_\delta > 0$ such that*

$$(4.13) \qquad \mathbb{P}\left(M(G) \leq \frac{1}{3}(1 - \delta)\mu m \right) \leq e^{-\delta^2 m/24}$$

for all $m > m_\delta$, where $\mu = 1 - e^{-\lambda/3}$.

PROOF. Partition $V(G) = [m]$ into three sets A, B, and C of equal size (up to rounding) and in particular $|B| = \lceil m/3 \rceil$. We describe an algorithm that finds

a random matching in G of some size M_m. We will then show that the random variable M_m is large enough with sufficiently high probability to guarantee (4.13).

Label the vertices in A as $x_1, \ldots, x_{|A|}$. Initialize $A_0 = A$, $B_0 = B$, $C_0 = C$, and $\mathcal{M}_0 = \varnothing$. Define nested vertex sets A_i, B_i and C_i and matching $\mathcal{M}_i$ recursively as follows:

(i) If $x_i \in A_i$ it has a neighbor in B_i, say y_i, then set $\mathcal{M}_{i+1} = \mathcal{M}_i \cup \{x_i y_i\}$ and $A_{i+1} = A_i \setminus \{x_1\}$. Pick a vertex $z_i \in C_i$, and put $C_{i+1} = C_i \setminus \{z_i\}$ and $B_{i+1} = [B_i \setminus \{y_i\}] \cup \{z_i\}$.

(ii) If $x_i \in A_i$ has no neighbor in B_i, then put $A_{i+1} = A_i \setminus \{x_i\}$, $B_{i+1} = B_i$, $C_{i+1} = C_i$, and $\mathcal{M}_{i+1} = \mathcal{M}_i$.

This process terminates after $|A|$ steps with matching $\mathcal{M}_{|A|}$. Let $M_m = |\mathcal{M}_{|A|}|$.

Now define random variables $X_{m,i}$, $1 \le i \le |A|$, by

$$(4.14) \qquad X_{m,i} = \mathbf{1}(x_i \text{ has a neighbor in } B_i).$$

So $M_m = \sum_{i=1}^{|A|} X_{m,i}$. For each fixed $m \ge 1$, $X_{m,i}$ depends only on the size of B_i, which is constant and equals to $|B|$. Hence $X_{m,i}$'s are i.i.d. Bernoulli variables with $p_{m,i} = \mathbb{E}[X_{n,i}] = 1 - (1 - \lambda/m)^{|B|}$. Since $|B| = \lceil m/3 \rceil$, we have $p_{m,i} \to \mu := 1 - e^{-\lambda/3}$ as $m \to \infty$. Now for a fixed $\delta > 0$, there exists $m_\delta > 0$ such that for all $m > m_\delta$, we have

$$(4.15) \qquad \mathbb{P}\left[\frac{M_m}{m} \le \frac{1}{3}(1 - \delta)\mu \right] \le \mathbb{P}\left[\frac{M_m}{|A|} \le (1 - \delta/2) p_{m,i} \right]$$
$$\le e^{-\delta^2 |B|/8} \le e^{-\delta^2 m/24},$$

where the inequality in the middle is by Chernoff's bound for lower tail. This shows the assertion. $\square$

The second ingredient to the proof of Theorem 6 is a large deviations estimate for the size of largest component in $G(n, \lambda/n)$ for $\lambda > 1$, which we denote by $L(n, \lambda/n)$. It is well known (see, e.g., [25]) that in this regime, $L(n, \lambda/n)/n$ converges in probability as n tends to infinity, to the unique positive solution β_λ to the equation

$$(4.16) \qquad \beta_\lambda = 1 - e^{-\lambda \beta_\lambda}.$$

We are interested in the large deviations estimate of the probability that the largest component in $G(n, \lambda/n)$ contains a fraction x of vertices that is strictly less than the correct ratio β_λ. This is given by a reformulation of a more precise result by O'Connell (Lemma 3.2 in [23]). Namely, for each $0 < x < \beta_\lambda$, we have

$$(4.17) \qquad \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}[L(n, \lambda/n) < xn] = -A(x, \lambda),$$

where

$$(4.18) \quad A(x, \lambda) = -x \log(1 - e^{-\lambda x}) + x \log x + (1 - x) \log(1 - x) + \lambda x(1 - x).$$

We are now ready to give a proof of Theorem 6.

PROOF OF THEOREM 6. We use standard facts about Erdős–Rényi random graph model, which may be found in many references including [18]. If $p = o(1/n)$, then we know that all components of G are trees a.a.s., so (i) follows from Corollary 4. To show (ii), suppose $p = \lambda/n$ for $\lambda \in (0, 1)$. In this subcritical regime, every component in $G(n, p)$ is either a tree or contains a single cycle a.a.s. (see, e.g., Corollary 5.8 in Bollobás [4]). Hence with Corollary 4, we have

$$(4.19) \quad \mathbb{P}(X_t \text{ synchronizes on each component of } G(n, p))$$

$$(4.20) \quad \begin{aligned} &= \mathbb{P}(X_t \text{ synchronizes on all unicyclic components of } G(n, p)) \\ &\quad + o(1). \end{aligned}$$

Now let V_{ns} denote the total number of unicyclic components in $G(n, p)$ whose cycle is of length s for $s \geq 3$. Let $\tau(x) = \sum_{s \geq 1} s^{s-1} x^s / s!$ be the exponential moment generating function of the sequence $\tau(k) = k^{k-1}$, $k \in \mathbb{N}$, which equals to the number of rooted trees on vertex set $[k]$. Pittel showed (see page 63 in [24]) that for each fixed $k \geq 3$, $V_{n3}, \ldots, V_{nk}$ are asymtotically independent Poisson random variables with means $\nu_3, \ldots, \nu_k$, where $\nu_s = \tau^s(\lambda e^{-\lambda})/2s$ for each $3 \leq s \leq k$. Hence by Theorem 3, the assertion holds with

$$(4.21) \quad C(\lambda) = \prod_{k=3}^{\infty} \prod_{l=1}^{\infty} \frac{(\nu_k)^l e^{-\nu_k}}{l!} \left[\mathbb{P}\left(\int_{C_k} dX_0 = 0 \right) \right]^l,$$

where C_k is a cycle of length k. It is not apparent, however, from the formula above, that the constant $C(\lambda)$ is bounded away from both 0 and 1 for any $\lambda \in (0, 1)$. To see this, we use Theorem 1 in Pittel [24], which says

$$c(\lambda) := \lim_{n \to \infty} \mathbb{P}(G(n, p) \text{ has no cycle}) = (1 - \lambda)^{1/2} e^{\lambda/2 + \lambda^2/4} \in (0, 1).$$

Thus by Theorems 3 and 5, we have

$$0 < c(\lambda) \leq C(\lambda) \leq 1 - \frac{2}{9}(1 - c(\lambda)) < 1.$$

Next, suppose $p = \lambda/n$ for $\lambda > 1$. We first introduce some notation. Let H be the largest component of $G = G(n, \lambda/n)$. Choose a spanning tree T of H, and let $S \subset V(H)$ such that no two vertices in S have an edge in T (i.e., a stable set in T). Since T is bipartite, we may choose S so that $|S| \geq \lceil |V(H)|/2 \rceil$. Finally, let $\mathbb{M} = M(H[S])$ be the size of largest matching in the induced subgraph $H[S] \subset H$ on the vertex set S. See Figure 4 for illustration.
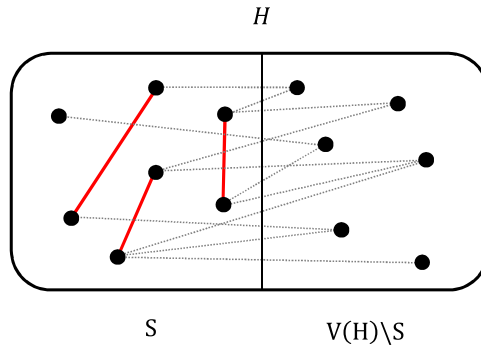
H

FIG. 4.   *Largest component $H \subset G$, a spanning tree T of H (dotted edges), a stable set $S \subset V(T)$, and a matching in $H[S]$ (red edges).*

Define parameters $\lambda' = (\lambda + 1)/2 \in (1, \lambda)$, $\beta = \beta(\lambda)$ the unique solution of $\beta = 1 - e^{-\lambda'\beta}$, and $c = c(\lambda) := \frac{\beta}{24}(1 - e^{-(\lambda-1)\beta/24})$. Now define following events:

$$A_n = \{X_t \text{ synchronizes on the largest component of } G\},$$

$$B_n = \{\text{The largest component } H \subseteq G \text{ has size } < \beta n/2\},$$

$$C_n = B_n^c \cap \{\texttt{M} < cn\},$$

$$D_n = B_n^c \cap \{\texttt{M} \ge cn\} \cap \{X_t \text{ synchronizes on } H\}.$$

Then by partitioning, we may write

(4.22) $$\mathbb{P}(A_n) \le \mathbb{P}(B_n) + \mathbb{P}(C_n) + \mathbb{P}(D_n).$$

Using (4.17), the first term in the right-hand side of (4.22) can be bounded by

$$\mathbb{P}(B_n) \le \mathbb{P}\big[L(n, a/n) < \beta n/2\big] \le e^{-A(\beta/2, a)n}$$

for large n. To bound the third term, observe that any matching $\{e_1, \ldots, e_k\}$ in the induced subgraph $H[S] \subset H$ satisfies the hypothesis in Theorem 5. Indeed, for each i, let C_i be the unique cycle which uses the edge e_i and the unique path between its two endpoints in T (see Figure 4). Then for any $j \ne i$, $e_j \notin E(C_i)$ since otherwise $e_i \in E(T)$, which contradicts the construction of S. Since on the event D_n we have a matching in $G[S]$ of size at least cn, we have

$$\mathbb{P}(D_n) \le (7/9)^{cn}.$$

Finally, let $a = \lambda'/n$ and $b = (\lambda - \lambda')/n$ and choose two independent random graphs $G(n, a)$ and $G(n, b)$. Their union has the law of $G(n, a+b-ab)$, which we may view as a subgraph of $G(n, \lambda/n)$ by a standard coupling. Denote by $G(S, b)$ the induced subgraph of $G(n, b)$ on the vertex set S. Note that

$$G\big(\lceil xn/2 \rceil, b\big) \subseteq G(S, b) \subseteq G(S, \lambda/n) \qquad \text{on the event } B_n^c$$

after some possible relabeling of the vertices. Hence by Lemma 4.2 with $\delta = 1/2$ and $\mu = \lim_{n \to \infty} 1 - e^{-\lceil \beta n/4 \rceil b/3} = 1 - e^{-(\lambda - 1)\beta/24}$, we have

$$\mathbb{P}(C_n) \leq \mathbb{P}\big[M\big(G\big(\lceil xn/2 \rceil, b\big)\big) \text{ has no matching of size} \geq cn\big]$$
$$\leq e^{-\lceil \beta n/4 \rceil/96} \leq e^{-\beta n/384}$$

for all sufficiently large n.

Combining the above estimates, we have $\mathbb{P}(A_n) \leq \exp[-Dn]$ for all sufficiently large n where $D = D(\lambda)$ is given by

$$D(\lambda) = \frac{1}{384} \min\{192A(\beta/2, (1+\lambda)/2), \beta, 4\beta(1 - e^{-(\lambda-1)\beta/24})\log(9/7)\}.$$
$$\square$$

5. On infinite trees.

In this section, we prove Theorem 7. Throughout this section, $\Gamma = (V, E)$ will denote an infinite but locally finite tree rooted at a vertex $0 \in V$. None of our discussion depends on the choice of root. Let X_0 be a random 3-coloring on Γ drawn from product measure $\mathbb{P}$ with marginal density $\mathbb{P}(X_0(\sigma) = i) = p_i$ for $i \in \mathbb{Z}_3$. Recall the definition of the Γ-indexed walk associated to the CCA or GHM dynamics $(X_t)_{t \geq 0}$. Theorem 7 gives quantitative estimates on the activity α. The first step is to establish part (i) of the assertion, which states that

$$\alpha = v_c \qquad \text{a.s..}$$

According to Lemma 1, we have

$$(5.1) \qquad \qquad \text{ne}_t(0) = \max_{|\sigma| \leq t} S_\sigma \qquad \text{a.s.}$$

for all $t \geq 0$. This implies

$$\alpha = \limsup_{t \to \infty} \frac{1}{t} \max_{|\sigma| \leq t} S_\sigma \qquad \text{a.s.}$$

Hence part (i) of Theorem 7 follows from (5.1) and the following observation.

PROPOSITION 5.1. *Almost surely, we have*

$$v_c = \limsup_{t \to \infty} \frac{1}{t} \max_{|\sigma|=t} S_\sigma = \limsup_{t \to \infty} \frac{1}{t} \max_{|\sigma| \leq t} S_\sigma = \alpha.$$

PROOF.   It suffices to show that

$$(5.2) \qquad \qquad \limsup_{t \to \infty} \frac{1}{t} \max_{|\sigma| \leq t} S_\sigma \leq v_c.$$

We may assume that $v_c < \infty$. Fix any $\varepsilon > 0$. Then there exists $T = T(\varepsilon) > 0$ such that

$$\max_{|\sigma|=t} S_\sigma \leq (v_c + \varepsilon)t \qquad \forall t > T.$$

Let $M = \max_{|\sigma| \leq T} S_\sigma$, and choose large $T' > 0$ so that $T' > T$ and $M < (v_c + \varepsilon)T'$. Then for all $t > T'$, we have

$$
\max_{|\sigma| \leq t} S_\sigma \leq \max_{|\sigma| \leq T'} S_\sigma + \max_{T' < |\sigma| \leq t} S_\sigma
$$
$$
\leq M + (v_c + \varepsilon)t,
$$

which implies (5.2) as desired. □

While Theorem 7(ii)–(v) gives an estimation on the cloud speed of the associated tree indexed walk, here we prove the assertion for general Γ-indexed walks with 1-correlated increments. Namely, let Y_σ be a real-valued random variable attached to the edge (σ^-, σ), and assume throughout this section that they are identically distributed, centered and that the moment generating function $\mathbb{E}[e^{tY_\sigma}] < \infty$ for some $t > 0$. The Γ-index random walk $\{S_\sigma\}_{\sigma \in V}$ with increments $\{Y_\sigma\}_{\sigma \in V}$ is defined similarly:

$$
S_\sigma = \sum_{0 \leq \tau \leq \sigma} Y_\sigma \qquad \forall \sigma \in V.
$$

Benjamini and Peres [3] obtained sharp upper and lower bounds on v_c of a Γ-indexed walk for the case when Γ with minimum degree ≥ 2 and the increments are i.i.d. In the present work, we generalize their result twofold: general infinite trees without degree constraint, and 1-correlated increments. Our arguments, which is largely based on Benjamini and Peres' original proof, easily carries over to any finite range correlation.

The analysis of cloud speed is essentially based on the large deviations principle for the random walk obtained by restricting the Γ-indexed walk on a single ray. Let γ be an infinite ray in Γ starting from 0, and let $0 = \sigma_0, \sigma_1, \sigma_2, \ldots$ be the successive vertices on it. By the 1-correlation, $\{Y_{\sigma_i}\}_{i \geq 0}$ is a Markov chain, or more generally, a functional $g : \mathfrak{X} \to \mathbb{R}$ of some underlying Markov chain $\{\mathcal{X}_i\}_i$ on state space $\mathfrak{X}$. In our special case of CCA and GHM increments, we may take $\mathfrak{X} = (\mathbb{Z}_3)^2$, $\mathcal{X}_i = (X_0(\sigma_i), X_0(\sigma_{i+1}))$, and g determined by

$$
Y_{\sigma_i} = dX_0(\sigma_i, \sigma_{i+1}) = g\big(X_0(\sigma_i), X_0(\sigma_{i+1})\big).
$$

Furthermore, since X_0 is drawn from the uniform product measure $\mathbb{P}$ on $(\mathbb{Z}_3)^V$, the product measure is the unique invariant measure for our chain and transition probabilities are given by

$$
\pi\big[(a, b), (c, d)\big] = \begin{cases} 1/3 & \text{if } b = c, \\ 0 & \text{otherwise.} \end{cases}
$$

Denote $\tilde{S}_n = \sum_{i=0}^{n-1} Y_{\sigma_i}$ with $\tilde{S}_0 = 0$. The large deviations principle for the measures associated to $\tilde{S}_n/n$'s deals with the probability that this sample average deviates from a typical behavior. When Y_σ's are i.i.d., then this typical behavior is

dictated by the strong law of large numbers, whereas ergodicity takes place in the Markovian context.

To make a statement in the most convenient form for our purpose, let $\Lambda(t) = \log \mathbb{E}[e^{tY_\sigma}]$ be the logarithmic moment generating function of Y_σ and Λ^* be its *Legendre transform*

$$\Lambda^*(u) = \sup_{t \in \mathbb{R}}[ut - \Lambda(t)].$$

Note that this function may take ∞. When Y_σ's are independent and satisfy the moment condition we gave at the beginning of this section, Cramér's theorem on large deviations asserts that

$$(5.3) \qquad \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}[\tilde{S}_n \geq nu] = -\Lambda^*(u).$$

This formula is still valid when $\Lambda^*(u)$ takes the value ∞.

In order to extend this relation to Markovian setting, we may assume that the following (sufficient but not necessary) conditions are satisfied:

(a) The state space $\mathfrak{X}$ is finite.
(b) The Markov chain $\{\mathcal{X}_i\}_{i \geq 0}$ has a unique stationary distribution.
(c) For each $t \in \mathbb{R}$, the principal eigenvalue $\lambda_\pi(tg)$ of a matrix π_{tg} is positive, where π_{tg} is the exponentially weighted transition matrix defined by

$$\pi_{tg}(x, y) = \pi(x, y)e^{tg(y)}.$$

Then (5.3) holds for the sums $\tilde{S}_n = \sum_{i=0}^{n-1} g(\mathcal{X}_i)$ where Λ is given by

$$\Lambda(t) := \log \lambda_\pi(tg).$$

We may refer to this relationship as Cramér's theorem for Markov chains. For references, see [8, 26], and in particular Theorem 3.1.2 in [7].

We are now ready to state and prove a generalization of Proposition 4.1 in Benjamini and Peres [3], from which the proof of Theorem 7 will then follow.

PROPOSITION 5.2. *Let $\Gamma = (V, E)$ be an infinite tree, and suppose the increments $\{Y_\sigma\}_{\sigma \in V}$ restricted on a ray starting from 0 can be realized as a functional g of some Markov chain $\{\mathcal{X}_i\}_{i \geq 0}$, which satisfies the conditions* (a)–(c) *above. Then we have the following*:

(i) $v_c > 0$ *if and only if* $\overline{\mathrm{h}}(\Gamma) > 0$.
(ii) v_c *satisfies the following upper bound*

$$(5.4) \qquad \Lambda^*(v_c) \leq \overline{\mathrm{h}}(\Gamma).$$

(iii) *If* $\log \mathrm{br}(\Gamma) = \mathrm{h}(\Gamma)$, *then either* $v_c = B$ *or equality in* (5.4) *is achieved, where* $B := \sup\{v : \Lambda^*(v) < \infty\}$.

(iv) *For each $r > 1$ such that $v_c/(1 - r^{-1}) < B$, where B is defined in* (iii), *we have*

(5.5)
$$\frac{h_r(\Gamma)}{r - 1} \leq \Lambda^*\left(\frac{v_c}{1 - r^{-1}}\right).$$

In particular, if Γ has no leaves, then

(5.6)
$$\Lambda\big(h(\Gamma)/v_c\big) \leq h(\Gamma),$$

which is sharp in the sense that there exists a tree-indexed walk on some tree Γ^- on which its cloud speed v_c^- satisfies $\Lambda(h(\Gamma)/(v_c^- - \varepsilon)) > h(\Gamma)$ for each $\varepsilon > 0$.

PROOF. (i) follows from (ii) and (iv).

(ii) By Cramér's theorem, $v_c \leq B$. We first show that $v_c < B$ implies $\Lambda^*(v_c) \leq h(\Gamma)$, and improve the later result. Suppose $0 \leq v_c < B$. Let $\tilde{S}_n = \sum_{i=0}^{n-1} Y_{\sigma_i}$ be the one-dimensional random walk as before. Observe that

$$\sum_{n \geq 1} \sum_{|\sigma| = n} \mathbb{P}(S_\sigma \geq nv) = \sum_{n \geq 1} A_n \mathbb{P}(\tilde{S}_n \geq nv)$$

$$= \sum_{n \geq 1} \exp\big[\log A_n \mathbb{P}(\tilde{S}_n \geq nv)\big]$$

$$= \sum_{n \geq 1} \exp\left[n\left(\frac{1}{n}\log A_n + \frac{1}{n}\log \mathbb{P}(\tilde{S}_n \geq nv)\right)\right].$$

Thus, if $h(\Gamma) < \Lambda^*(v)$, Cramér's theorem for Markov chains yields that the expression in the parenthesis above is negative and bounded away from 0 for large n. Thus, the left-hand side is summable, so by the Borel–Cantelli lemma we get $v_c < v$. This shows $\Lambda^*(v_c) \leq h(\Gamma)$.

To improve the upper bound, suppose $v_c < B$. Fix $\varepsilon > 0$ and we construct a subtree Γ_ε of Γ as follows. At each level $n = 1, 2, \ldots$ successively, delete all vertices σ in Γ if they have no descendant at level $\geq (1 + \varepsilon)n$. Call this operation ε-pruning. Let $v_c(\varepsilon)$ be the cloud speed of the Γ-indexed walk restricted on Γ_ε. Since we can recover the original tree from Γ_ε by attaching deleted parts at levels $n = 1, 2, \ldots$ successively and since they all have depth at most εn, we get

$$v_c - \varepsilon B \leq v_c(\varepsilon) \leq v_c.$$

Moreover, since Λ^* is nondecreasing and according to the first part, we know

$$\Lambda^*(v_c - \varepsilon B) \leq h(\Gamma_\varepsilon).$$

Then by continuity of Λ^* at $v = v_c$, it suffices to show that

(5.7)
$$\limsup_{\varepsilon \to 0} h(\Gamma_\varepsilon) \leq \overline{h}(\Gamma).$$

To this end, recall that for any $n \leq m$, $A_{n,m}$ denotes the number of vertices in Γ at level n which have descendants at level m. Let $A_n(\varepsilon)$ denote the number of

vertices in Γ_ε at level n. Note that $A_n(\varepsilon)$ equals to the number of vertices in Γ at level n which survives the ε-pruning up to level $n - 1$ *and* with descendants at level $\lfloor (1 + \varepsilon)n \rfloor$. This yields

$$(5.8) \qquad A_n(\varepsilon) \leq A_{n, \lfloor (1+\varepsilon)n \rfloor}$$

for all $n \geq 1$, which easily implies $h(\Gamma_\varepsilon) \leq h_{1+\varepsilon}(\Gamma)$. This gives (5.7), as desired.

(iii) Here, we follow Lyons' argument in [21] with a minor modification. Suppose $\log \mathrm{br}(\Gamma) = \overline{h}(\Gamma) = h(\Gamma)$. Fix arbitrary $v > 0$ such that $\Lambda^*(v) < \log \mathrm{br}(\Gamma) < \infty$. We will show that $v \leq v_c$. Then the assertion follows since Λ^* is strictly increasing on $[0, B]$. Choose $0 < \varepsilon < h(\Gamma) - \Lambda^*(v)$. Then by Cramér's theorem for Markov chains, we may choose a large $k \geq 1$ for which

$$(5.9) \qquad \mathbb{P}\big[\tilde{S}_{k-1} \geq (k-1)v + B\big] > e^{-(\Lambda^*(v)+\varepsilon)k} > e^{-h(\Gamma)k} = (\mathrm{br}\,\Gamma)^{-k}.$$

For this choice of k, define a tree $\Gamma^k = (V_k, E_k)$ from Γ by $V_k = \{\sigma \in V : k \mid |\sigma|\}$ and $(\sigma \to \sigma') \in E_k$ iff the unique path between σ and σ' in Γ, which we denote by $\gamma(\sigma, \sigma')$, has length k. In words, Γ^k describes how vertices of Γ at levels multiple of k are interconnected. It is easy to check that $\mathrm{br}\,\Gamma^k = (\mathrm{br}\,\Gamma)^k$. We are going to define a bond percolation process on Γ^k, by deleting each edge $(\sigma \to \sigma')$ from Γ^k if, roughly speaking, the partial sum on the corresponding path in Γ grows slowly. More precisely, form a random subgraph $\Gamma^k(\omega)$ of Γ^k by deleting each edge $(\sigma \to \sigma')$ unless

$$(5.10) \qquad \sum_{\sigma^+ < \tau < \sigma'} Y_\tau \geq (k-1)v + B,$$

where σ^+ denotes the unique descendant of σ on the path $\gamma(\sigma, \sigma')$. Note that the partial sum above is over the path $\gamma(\sigma^+, \sigma') = \gamma(\sigma, \sigma') - \sigma$, and these paths are vertex disjoint in Γ. Hence by the 1-correlation, each edge in Γ^k is selected independently. This defines a quasi-Bernoulli percolation process on Γ^k, which was introduced in [20]. There it was shown that percolation occurs a.s. if $q_k \mathrm{br}(\Gamma^k) > 1$, where q_k is the survival probability of each edge. Since this condition is satisfied by (5.9) and $\mathrm{br}(\Gamma^k) = [\mathrm{br}(\Gamma)]^k$ in our case, $\Gamma^k(\omega)$ contains an infinite ray γ a.s. This gives an infinite ray γ_0 in Γ emanating from 0 such that all but finitely many vertices are in γ. Then $\limsup_{\sigma \in \gamma_0} S_\sigma / |\sigma| \geq v$, so this shows $v_c \geq v$ as desired.

(iv) We follow the argument in Benjanini and Peres [3] with a minor modification. Suppose $h_r(\Gamma) > 0$ for some $r > 1$. For brevity, we shall omit roundings in this proof. Fix $\varepsilon > 0$, and choose a strictly increasing $f : \mathbb{N} \to \mathbb{N}$ such that

$$(5.11) \qquad \lim_{n \to \infty} \frac{1}{f(n)} \log A_{f(n), rf(n)} > h_r(\Gamma) - \varepsilon.$$

For each $v > 0$, define the following event:

$$(5.12) \qquad \Omega_n(v) = \Big\{ \max_{|\sigma| = rf(n)} S_\sigma \leq rf(n)v \Big\}.$$

For each vertex σ at level n, which has a descendant, say σ', at level $rf(n)$, let $\gamma(\sigma, \sigma')$ be the unique path between the two vertices. Note that there are at least $A_{f(n),rf(n)}$ of such paths, and they are mutually vertex disjoint. Denote by M_k the random variable

$$(5.13) \qquad M_k = \sum_{|\sigma|=k} \mathbf{1}(S_\sigma \geq -k\varepsilon \text{ and } \sigma \text{ has a descendant at level } rk).$$

By conditioning on the values of S_σ for $|\sigma| = f(n)$, we have

$$(5.14) \qquad \begin{aligned} \mathbb{P}[\Omega_n(v)] &\leq \mathbb{P}[M_{f(n)} \leq A_{f(n),rf(n)}/2] \\ &\quad + \mathbb{P}[\tilde{S}_{(r-1)f(n)} \leq rf(n)v + f(n)\varepsilon]^{A_{f(n),rf(n)}/2}. \end{aligned}$$

The first term in the right-hand side should tend to 0 as $k \to \infty$, since it is unlikely that for half of $\sigma's$ at level $f(n)$ which has descendant at level $rf(n)$ we have $S_\sigma \leq -f(n)\varepsilon$, which is well below the correct mean 0. Since

$$(5.15) \qquad \lim_{n\to\infty} \frac{\tilde{S}_n}{n} = 0 \qquad \text{a.s.,}$$

we have

$$(5.16) \qquad \frac{\mathbb{E}[M_k]}{A_{k,rk}} = \mathbb{P}(\tilde{S}_k \geq -k\varepsilon) \to 1 \qquad \text{as } k \to \infty.$$

But since $M_k \leq A_{k,rk}$, this implies that

$$(5.17) \qquad \mathbb{P}(M_k \leq A_{k,rk}/2) \to 0 \qquad \text{as } k \to \infty.$$

Now we may assume $v_c < B(1 - r^{-1})$ and pick $v \in (v_c, B]$. The definition of cloud speed enforces that $\lim_{n\to\infty} \mathbb{P}(\Omega_n(v)) = 1$. Hence combining with (5.14) and (5.17), we get

$$(5.18) \qquad \mathbb{P}[\tilde{S}_{(r-1)f(n)} \leq f(n)(v+\varepsilon)]^{A_{f(n),rf(n)}} \to 1 \qquad \text{as } n \to \infty.$$

Taking logarithm and and using the fact that $\log(1-x) \leq -x$ for $|x| \ll 0$, this yields

$$(5.19) \qquad A_{f(n),rf(n)}\mathbb{P}[\tilde{S}_{(r-1)f(n)} \geq f(n)(v+\varepsilon)] \to 0.$$

By Cramér's theorem for Markov chains, if $(r-1)f(n)$ is sufficiently large, then

$$(5.20) \qquad \mathbb{P}[\tilde{S}_{(r-1)f(n)} \geq f(n)(v+\varepsilon)] \geq \exp[-(r-1)f(n)(\Lambda^*(u)+\varepsilon)],$$

where

$$(5.21) \qquad u = \frac{v+\varepsilon}{1-r^{-1}}.$$

For large enough n, the terms in (5.19) is less than 1. At the same time, by (5.11), we have $\log A_{f(n),rf(n)} \geq f(n)(\mathrm{h}(\Gamma) - \varepsilon)$ for n large. Thus (5.19) and (5.20) gives

$$(5.22) \qquad (\mathrm{h}_r(\Gamma) - \varepsilon) \leq (r-1)(\Lambda^*(u) + \varepsilon).$$

Then by the continuity of Λ^* at $v_c < B$, letting $v \searrow v_c$ and $\varepsilon \searrow 0$ establishes the first assertion.

To show the second part, we may further assume that Γ has no leaves. Then we have $h_r(\Gamma) = h(\Gamma)$ for all $r > 1$. We may write (5.22) as

$$\tag{5.23} \frac{h(\Gamma) - \varepsilon}{v + \varepsilon} u - \Lambda^*(u) \leq h(\Gamma),$$

where u is given by (5.21). Since this is valued for all $r > 1$, the above inequality is valid for all $u > v + \varepsilon$. Since it is also valid for $0 < u \leq v + \varepsilon$ trivially, we may take supremum of left hand side of (5.23) over all $u > 0$, which makes it the Legendre transform of Λ^*. But since Λ is convex, we have $\Lambda^{**} = \Lambda$ by the Fenchel–Moreu theorem [1]. This gives

$$\tag{5.24} \Lambda\left(\frac{h(\Gamma) - \varepsilon}{v + \varepsilon}\right) \leq h(\Gamma),$$

and by left continuity of Λ, letting $v \searrow v_c$ and $\varepsilon \searrow 0$ establishes second assertion.

Lastly, to show the lower bound is sharp, consider a tree $\Gamma^- = (V, E)$ where every vertex has only one descendant except one special vertex σ_j at every level $j!$ for each $j \geq 1$, which has $e^{dj!}$ descendants. This is called the "exploding tree" in [3], and in the reference it is shown that the cloud speed on this tree satisfies the assertion. This completes the proof. □

PROOF OF THEOREM 7.   Part (i) follows from Proposition 5.1. By Proposition 5.2, it remains to verify that Λ in the cases of CCA and GHM agree with Λ_{CCA} and Λ_{GHM}, as stated in Section 2. The corresponding exponentially weighted transition matrices π_{tg}^{CCA} and π_{tg}^{GHM} are

$$\tag{5.25}
\begin{aligned}
\pi_{tg}^{\text{CCA}} &= \begin{bmatrix} p_0 B_1 & p_1 e^{-t} B_1 & p_2 e^t B_1 \\ p_0 e^t B_2 & p_1 B_2 & p_2 e^{-t} B_2 \\ p_0 e^{-t} B_3 & p_1 e^t B_3 & p_2 B_3 \end{bmatrix}, \\
\pi_{tg}^{\text{GHM}} &= \begin{bmatrix} p_0 B_1 & p_1 e^{-t} B_1 & p_2 B_1 \\ p_0 e^t B_2 & p_1 B_2 & p_2 B_2 \\ p_0 B_3 & p_1 B_3 & p_2 B_3 \end{bmatrix},
\end{aligned}
$$

where B_j is the 3×3 matrix with all zeros but ones in the jth column, for $1 \leq j \leq 3$. Hence $\Lambda(t)$ is the logarithm of the largest positive root $x = x(t)$ in the corresponding characteristic polynomial. Elementary computations show that the resulting equations are as follows:

$$\tag{5.26} \text{(CCA)} \quad x^3 - x^2 = p_0 p_1 p_2 (e^{3t} + e^{-3t} - 2),$$

$$\tag{5.27} \text{(GHM)} \quad x^3 - x^2 = p_0 p_1 p_2 (e^t + e^{-t} - 2).$$

This shows that Λ for CCA and GHM agree with Λ_{CCA} and Λ_{GHM}, and that $\Lambda_{\text{GHM}}(t) = \Lambda_{\text{CCA}}(t/3)$. □

EXAMPLE 5.3 (d-ary tree).   Fix an integer $d \geq 2$, and let $\Gamma = (V, E)$ be the d-ary tree. Suppose uniform color density for X_0. By part (iv) of Theorem 7 and using the closed-form expression for Λ^*_{CCA}, we compute the activity α of CCA dynamics on Γ as

(5.28) $$\alpha = 0.86824163\ldots$$

for $d = 2$ and $\alpha \equiv 1$ for $d \geq 3$. In case of GHM dynamics, part (iv) of Theorem 7 gives

(5.29) $$\alpha = 0.28941386\ldots$$

for $d = 2$ and $\alpha \equiv 1/3$ for $d \geq 3$.

EXAMPLE 5.4 (Geometric tree).   Fix an integer $d \geq 2$, and let $\Gamma^d = (V, E)$ be obtained by a single infinite ray γ emanating from root 0, by attaching a d-ary tree of depth 4^k to each vertex $\sigma \in \gamma$ at level $|\sigma| = 4^k$. Consider CCA dynamics on Γ, so that $v_c \leq 1$. Since Γ^d has a single infinite branch, one has $\text{br}(\Gamma^d) = 1$. It is also easy to check that $\text{h}(\Gamma) = \frac{\log d}{2}$. Moreover, for any $r \in [1, 2]$, Γ^d has a r-cut f defined by $f(k) = (2/r)4^k$. This gives $\text{h}_r(\Gamma) \geq (1 - r/2) \log d$, so Proposition 5.2(iii) gives

(5.30) $$\frac{1 - r/2}{r - 1} \log d \leq \Lambda^*\left(\frac{v_c}{1 - r^{-1}}\right).$$

Thus v_c attains the maximum value 1 for large d, despite $\text{br}(\Gamma) = 1$. Also note that the optimal value of r for the above lower bound is strictly between 1 and 2. This makes sense because if $r = 1$, we have the largest r-entropy which matches the volume entropy, but there is not enough room for the maximum to grow. On the other hand, $r = 2$ gives the longest time for the maximum to grow, but the corresponding r-entropy is the lowest. In fact, in this example it is possible to calculate v_c explicitly. By the second part of Proposition 5.2(ii), we know that the cloud speed v'_c on a d-ary tree is given by $\Lambda^*(v'_c) = \log d$. Observe that on Γ^d, for each interval $[4^k, 4^{k+1}]$ of levels, the maximum grows linearly with speed v'_c only for the first third and stays the same for the later two thirds. Thus $v_c = v'_c/3$.

6. Concluding remarks and open problems.

6.1. *CCA with more than three colors.*   Our key tool in characterizing the limiting behavior of 3-color CCA is the tournament expansion, which enables us to unfold the dynamics of cyclically arranged colors into a monotone dynamics of linearly ordered ranks. An interesting and natural question is whether a similar technique might work for $\kappa \geq 4$. In this case, the particle representation is more complex [11], and existence of a suitable tournament expansion remains unclear. Proving (or disproving) the following conjecture would be a key step in understanding these dynamics.
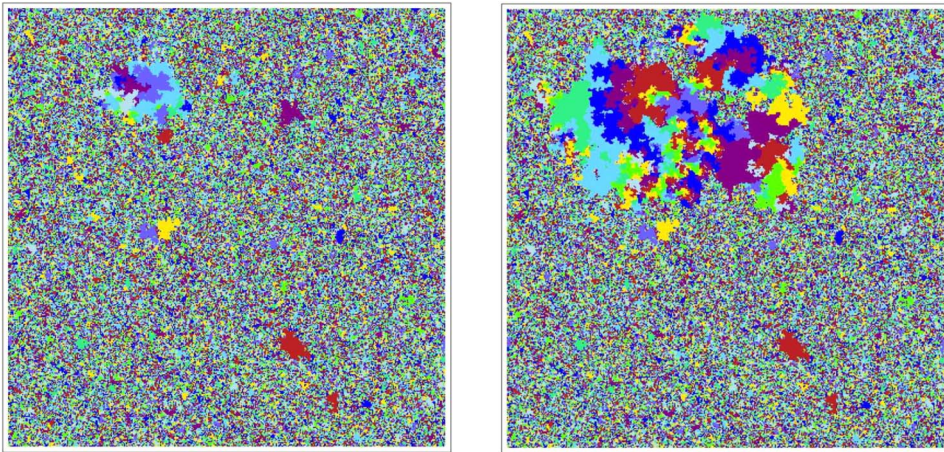
FIG. 5. *Two snapshots of 9-color CCA dynamics on a uniform spanning tree of a 400×400 torus, at times about 3000 and 17,000. By the former time, almost the entire square fixates, except for a single droplet that continues to grow slowly until it takes over the available space.*

CONJECTURE 6.1. *Fix any $\kappa \geq 4$. Let $\Gamma = (V, E)$ be a tree with $\mathrm{br}(\Gamma) > 1$, and let X_0 be a random κ-coloring of V drawn from the uniform product measure. Then, almost surely, every vertex oscillates.*

Methods from the present paper yield some partial results. For example, it is not hard to prove the above conjecture for $\kappa = 4$ and regular binary tree Γ. For very large κ, we expect that the activity is caused by large connected sets in which all edges have color differences 0 or ± 1. Once large enough, these regular droplets grow without bounds with high probability. This nucleation scenario seems to hold even for some trees with branching number 1 (see Figure 5), but due to correlations in the growth environment, techniques for proving its validity remain elusive.

6.2. *Activity for CCA and GHM on arbitary trees.* In Theorem 7(iv), we have seen that the activity for GHM is exactly a third of that for CCA for a given random 3-coloring X_0 with arbitrary color density, provided the tree is regular enough that $\log \mathrm{br}(\Gamma) = \mathrm{h}(\Gamma)$. In fact, it is not hard to see that an inequality holds for an arbitrary infinite tree Γ in the special case of uniform initialization. Namely, let α_{CCA} and α_{GHM} be the activities of CCA and GHM dynamics on Γ, starting from a random 3-coloring X_0 with equal densities. Then we claim that

$$(6.1) \qquad \alpha_{\mathrm{CCA}} \leq 3\alpha_{\mathrm{GHM}}.$$

Indeed, let $X_0^{(i)}$ for $i \in \mathbb{Z}_3$ be the random 3-coloring on Γ obtained by color shifts from X_0, that is, $X_0^{(i)}(x) = X_0(x) + i \bmod 3$. Since $d_{\mathrm{CCA}} X_0 = \sum_{i=0}^{2} d_{\mathrm{GHM}} X_0^{(i)}$, taking path integrals and maxima on both sides gives

$$(6.2) \qquad \alpha_{\mathrm{CCA}} \leq \alpha_{\mathrm{GHM}}^{(0)} + \alpha_{\mathrm{GHM}}^{(1)} + \alpha_{\mathrm{GHM}}^{(2)},$$

where $\alpha_{\mathrm{GHM}}^{(i)}$ is the activity of GHM with initial coloring $X_0^{(i)}$ for $i \in \mathbb{Z}_3$. As the initial color densities are equal, $X_0^{(i)}$'s are identically distributed so that $\alpha_{\mathrm{GHM}}^{(i)}$'s coincide and, therefore, (6.2) implies (6.1).

The message of Theorem 7(iv) is that, provided $\log \mathrm{br}(\Gamma) = \mathrm{h}(\Gamma)$, the correlation between the three GHMs in the above paragraph does not increase the cloud speed for the CCA, resulting in equality in (6.1). To make sense of this result, observe from the percolation arguments in Section 5 that, on such regular trees, the cloud speed is essentially attained on a single ray. By contrast, on irregular trees the cloud speed might be determined by disconnected chunks. We thus pose the following question.

QUESTION 6.2. Let $\Gamma = (V, E)$ be an arbitrary infinite tree and let X_0 be a random 3-coloring of V drawn from the uniform product measure. Is it true that

$$(6.3) \qquad\qquad\qquad \alpha_{\mathrm{CCA}} = 3\alpha_{\mathrm{GHM}}?$$

One may attempt to resolve this issue by generalizing the characterization of the cloud speed v_c from Theorem 7(iv) to arbitrary trees. However, the cloud speed on irregular trees may not be characterized solely by a dimensionality of the underlying tree and step size distribution: there are two trees Γ_1, Γ_2 and two step size distributions F_1, F_2 such that $v_c(\Gamma_1, F_1) > v_c(\Gamma_2, F_2)$ but $v_c(\Gamma_1, F_2) < v_c(\Gamma_2, F_1)$ (see Remark in [3] following Proposition 4.3). Hence characterizing the cloud speed on irregular trees, which may be of an independent interest, seems to require novel techniques.

REFERENCES

[1] ARNOL'D, V. I. (2013). *Mathematical Methods of Classical Mechanics. Graduate Texts in Mathematics* **60**. Springer, New York. MR0690288

[2] BELITSKY, V. and FERRARI, P. A. (1995). Ballistic annihilation and deterministic surface growth. *J. Stat. Phys.* **80** 517–543. MR1342240

[3] BENJAMINI, I. and PERES, Y. (1994). Tree-indexed random walks on groups and first passage percolation. *Probab. Theory Related Fields* **98** 91–112. MR1254826

[4] BOLLOBÁS, B. (1998). Random graphs. In *Modern Graph Theory* 215–252. Springer, Berlin.

[5] BRAMSON, M. and GRIFFEATH, D. (1989). Flux and fixation in cyclic particle systems. *Ann. Probab.* **17** 26–45. MR0972768

[6] DATTA, A. K., PETIT, F. and GUERRAOUI, R. (2016). Toward a time-optimal odd phase clock unison in trees. In *Stabilization, Safety, and Security of Distributed Systems* 137–151. Springer, Cham.

[7] DEMBO, A. and ZEITOUNI, O. (2009). *Large Deviations Techniques and Applications. Applications of Mathematics* (*New York*) **38**. Springer, New York. MR1619036

[8] DEUSCHEL, J.-D. and STROOCK, D. W. (1989). *Large Deviations. Pure and Applied Mathematics* **137**. Academic Press, Boston, MA. MR0997938

[9] DOLEV, S. (2000). *Self-Stabilization*. MIT Press, Cambridge, MA.

[10] DURRETT, R. and STEIF, J. E. (1991). Some rigorous results for the Greenberg-Hastings model. *J. Theoret. Probab.* **4** 669–690. MR1132132

[11] FISCH, R. (1990). The one-dimensional cyclic cellular automaton: A system with deterministic dynamics that emulates an interacting particle system with stochastic dynamics. *J. Theoret. Probab.* **3** 311–338. MR1046336

[12] FISCH, R. (1992). Clustering in the one-dimensional three-color cyclic cellular automaton. *Ann. Probab.* **20** 1528–1548. MR1175276

[13] FISCH, R. and GRAVNER, J. (1995). One-dimensional deterministic Greenberg-Hastings models. *Complex Systems* **9** 329–348. MR1434149

[14] FISCH, R., GRAVNER, J. and GRIFFEATH, D. (1991). Cyclic cellular automata in two dimensions. In *Spatial Stochastic Processes*. *Progress in Probability* **19** 171–185. Birkhäuser, Boston, MA. MR1144096

[15] FISCH, R., GRAVNER, J. and GRIFFEATH, D. (1991). Threshold-range scaling of excitable cellular automata. *Stat. Comput.* **1** 23–39.

[16] GREENBERG, J. M. and HASTINGS, S. P. (1978). Spatial patterns for discrete models of diffusion in excitable media. *SIAM J. Appl. Math.* **34** 515–523. MR0484504

[17] HERMAN, T. and GHOSH, S. (1995). Stabilizing phase-clocks. *Inform. Process. Lett.* **54** 259–265.

[18] JANSON, S., ŁUCZAK, T. and RUCINSKI, A. (2011). *Random Graphs. Wiley-Interscience Series in Discrete Mathematics and Optimization* **45**. Wiley-Interscience, New York. MR1782847

[19] LAMPORT, L. (1978). Time, clocks, and the ordering of events in a distributed system. *Commun. ACM* **21** 558–565.

[20] LYONS, R. (1989). The Ising model and percolation on trees and tree-like graphs. *Comm. Math. Phys.* **125** 337–353. MR1016874

[21] LYONS, R. (1990). Random walks and percolation on trees. *Ann. Probab.* **18** 931–958. MR1062053

[22] LYU, H. (2015). Synchronization of finite-state pulse-coupled oscillators. *Phys. D* **303** 28–38. MR3349515

[23] O'CONNELL, N. (1998). Some large deviation results for sparse random graphs. *Probab. Theory Related Fields* **110** 277–285. MR1616567

[24] PITTEL, B. (1988). A random graph with a subcritical number of edges. *Trans. Amer. Math. Soc.* **309** 51–75. MR0957061

[25] SPENCER, J. (1993). Nine lectures on random graphs. In *École D'Été de Probabilités de Saint-Flour XXI—1991*. *Lecture Notes in Math.* **1541** 293–347. Springer, Berlin. MR1242577

[26] VARADHAN, S. R. S. (2008). Large deviations. *Ann. Probab.* **36** 397–419. MR2393987

[27] WIENER, N. and ROSENBLUETH, A. (1946). The mathematical formulation of the problem of conduction of impulses in a network of connected excitable elements, specifically in cardiac muscle. *Arch. Inst. Cardiol. Méx.* **16** 205–265. MR0025140

J. GRAVNER
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA, DAVIS
DAVIS, CALIFORNIA 95616
USA
E-MAIL: gravner@math.ucdavis.edu

H. LYU
D. SIVAKOFF
DEPARTMENT OF MATHEMATICS
OHIO STATE UNIVERSITY
COLUMBUS, OHIO 43210
USA
E-MAIL: colourgraph@gmail.com
dsivakoff@stat.osu.edu