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UNIQUENESS AND PROPAGATION OF CHAOS FOR THE BOLTZMANN EQUATION WITH MODERATELY SOFT POTENTIALS

BY LIPING XU

Université Pierre et Marie Curie (Paris VI)

We prove a strong/weak stability estimate for the 3D homogeneous Boltzmann equation with moderately soft potentials [$\gamma \in (-1, 0)$] using the Wasserstein distance with quadratic cost. This in particular implies the uniqueness in the class of all weak solutions, assuming only that the initial condition has a finite entropy and a finite moment of sufficiently high order. We also consider the Nanbu N-stochastic particle system, which approximates the weak solution. We use a probabilistic coupling method and give, under suitable assumptions on the initial condition, a rate of convergence of the empirical measure of the particle system to the solution of the Boltzmann equation for this singular interaction.

CONTENTS

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1. Introduction.

1.1. *The Boltzmann equation.* We consider a 3-dimensional spatially homogeneous Boltzmann equation, which depicts the density $f_t(v)$ of particles in a gas, moving with velocity $v \in \mathbb{R}^3$ at time $t \geq 0$. The density $f_t(v)$ solves

$$\partial_t f_t(v) = \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} d\sigma \, B(|v - v_*|, \theta)[f_t(v') f_t(v'_*) - f_t(v) f_t(v_*)], \tag{1.1}$$

where

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \qquad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma, \tag{1.2}$$

and θ is the *deviation angle* defined by $\cos\theta = \frac{v-v_*}{|v-v_*|} \cdot \sigma$. The *collision Kernel* $B(|v - v_*|, \theta) \geq 0$ depends on the type of interaction between particles. It only depends on $|v - v_*|$ and on the cosine of the deviation angle θ. Conservations of mass, momentum and kinetic energy hold for reasonable solutions and we may assume without loss of generality that $\int_{\mathbb{R}^3} f_t(v)\, dv = 1$ for all $t \geq 0$.

1.2. *Assumptions.* We will assume that there is a measurable function $\beta : (0, \pi] \to \mathbb{R}_+$ such that

$$\begin{cases} B(|v - v_*|, \theta) \sin\theta = |v - v_*|^\gamma \beta(\theta), \\ \exists 0 < c_0 < c_1, \forall \theta \in (0, \pi/2), \qquad c_0 \theta^{-1-\nu} \leq \beta(\theta) \leq c_1 \theta^{-1-\nu}, \\ \forall \theta \in [\pi/2, \pi], \qquad \beta(\theta) = 0, \end{cases} \tag{1.3}$$

for some $\nu \in (0, 1)$, and $\gamma \in (-1, 0)$ satisfying $\gamma + \nu > 0$.

The last assumption $\beta = 0$ on $[\pi/2, \pi]$ is not a restriction and can be obtained by symmetry as noted in the introduction of [2]. This assumption corresponds to a classical physical example, inverse power laws interactions: when particles collide by pairs due to a repulsive force proportional to $1/r^s$ for some $s > 2$, assumption (1.3) holds with $\gamma = (s - 5)/(s - 1)$ and $\nu = 2/(s - 1)$. Here, we will focus on the case of moderately soft potentials, that is, $s \in (3, 5)$.

1.3. *Some notation.* Let us denote by $\mathcal{P}(\mathbb{R}^3)$ the set of probability measures on $\mathbb{R}^3$ and by $\mathrm{Lip}(\mathbb{R}^3)$ the set of bounded globally Lipschitz functions $\phi : \mathbb{R}^3 \mapsto \mathbb{R}$. When $f \in \mathcal{P}(\mathbb{R}^3)$ has a density, we also denote this density by f. For $q > 0$, we set

$$\mathcal{P}_q(\mathbb{R}^3) = \{f \in \mathcal{P}(\mathbb{R}^3) : m_q(f) < \infty\} \qquad \text{with } m_q(f) := \int_{\mathbb{R}^3} |v|^q f(dv).$$

We now introduce, for $\theta \in (0, \pi/2)$ and $z \in [0, \infty)$,

$$H(\theta) = \int_\theta^{\pi/2} \beta(x)\, dx \quad \text{and} \quad G(z) = H^{-1}(z). \tag{1.4}$$

Under (1.3), it is clear that H is a continuous decreasing function valued in $[0, \infty)$, so it has an inverse function $G : [0, \infty) \mapsto (0, \pi/2)$ defined by $G(H(\theta)) = \theta$ and $H(G(z)) = z$. Furthermore, it is easy to verify that there exist some constants $0 < c_2 < c_3$ such that for all $z > 0$,

$$c_2(1+z)^{-1/\nu} \le G(z) \le c_3(1+z)^{-1/\nu}, \tag{1.5}$$

and we know from [9] that there exists a constant $c_4 > 0$ such that for all $x, y \in \mathbb{R}_+$,

$$\int_0^\infty \big(G(z/x) - G(z/y)\big)^2 \, dz \le c_4 \frac{(x-y)^2}{x+y}. \tag{1.6}$$

Let us now introduce the Wasserstein distance with quadratic cost on $\mathcal{P}_2(\mathbb{R}^3)$. For $g, \tilde{g} \in \mathcal{P}_2(\mathbb{R}^3)$, let $\mathcal{H}(g, \tilde{g})$ be the set of probability measures on $\mathbb{R}^3 \times \mathbb{R}^3$ with first marginal g and second marginal $\tilde{g}$. We then set

$$\mathcal{W}_2(g, \tilde{g}) = \inf\left\{\left(\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v - \tilde{v}|^2 R(dv, d\tilde{v})\right)^{1/2}, R \in \mathcal{H}(g, \tilde{g})\right\}.$$

Here, the infimum is actually a minimum, for more details on this distance, one can see [34], Chapter 2.

1.4. *Weak solutions*. We now introduce a suitable spherical parameterization of (1.2) as in [13]. For each $x \in \mathbb{R}^3 \setminus \{0\}$, we consider a vector $I(x) \in \mathbb{R}^3$ such that $|I(x)| = |x|$ and $I(x) \perp x$. We also set $J(x) = \frac{x}{|x|} \wedge I(x)$, where $\wedge$ is the vector product. Then the triplet $(\frac{x}{|x|}, \frac{I(x)}{|x|}, \frac{J(x)}{|x|})$ is an orthonormal basis of $\mathbb{R}^3$. Then for $x, v, v_* \in \mathbb{R}^3$, $\theta \in (0, \pi]$, $\varphi \in [0, 2\pi)$, we set

$$\begin{cases} \Gamma(x, \varphi) := (\cos\varphi) I(x) + (\sin\varphi) J(x), \\ v'(v, v_*, \theta, \varphi) := v - \dfrac{1-\cos\theta}{2}(v - v_*) + \dfrac{\sin\theta}{2}\Gamma(v - v_*, \varphi), \\ a(v, v_*, \theta, \varphi) := v'(v, v_*, \theta, \varphi) - v, \end{cases} \tag{1.7}$$

then we write $\sigma \in \mathbb{S}^2$ as $\sigma = \frac{v-v_*}{|v-v_*|}\cos\theta + \frac{I(v-v_*)}{|v-v_*|}\sin\theta\cos\varphi + \frac{J(v-v_*)}{|v-v_*|}\sin\theta\sin\varphi$, and observe at once that $\Gamma(x, \varphi)$ is orthogonal to x and has the same norm as x, from which it is easy to check that

$$\big|a(v, v_*, \theta, \varphi)\big| = \sqrt{\frac{1-\cos\theta}{2}}|v - v_*|. \tag{1.8}$$

Let us now give the definition of weak solutions to (1.1).

DEFINITION 1.1. Assume (1.3) is true for some $\nu \in (0, 1)$, $\gamma \in (-1, 0)$ with $\gamma + \nu > 0$. A measurable family of probability measures $(f_t)_{t \ge 0}$ is called a *weak solution* to (1.1) if it satisfies the following two conditions:

- For all $t \geq 0$,

$$\int_{\mathbb{R}^3} v f_t(dv) = \int_{\mathbb{R}^3} v f_0(dv) \quad \text{and} \quad \int_{\mathbb{R}^3} |v|^2 f_t(dv) = \int_{\mathbb{R}^3} |v|^2 f_0(dv) < \infty. \tag{1.9}$$

- For any bounded globally Lipschitz function $\phi \in \mathrm{Lip}(\mathbb{R}^3)$, any $t \in [0, T]$,

$$\begin{aligned} &\int_{\mathbb{R}^3} \phi(v) f_t(dv) \\ &\quad = \int_{\mathbb{R}^3} \phi(v) f_0(dv) + \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{A}\phi(v, v_*) f_s(dv_*) f_s(dv)\, ds, \end{aligned} \tag{1.10}$$

where

$$\mathcal{A}\phi(v, v_*) = |v - v_*|^\gamma \int_0^{\pi/2} \beta(\theta)\, d\theta \int_0^{2\pi} \big[\phi\big(v + a(v, v_*, \theta, \varphi)\big) - \phi(v)\big]\, d\varphi.$$

We observe that $|\mathcal{A}\phi(v, v_*)| \leq C_\phi |v - v_*|^{1+\gamma} \leq C_\phi(1 + |v - v_*|^2)$ from $|a(v, v_*, \theta, \varphi)| \leq C\theta|v - v_*|$ and $\int_0^{\pi/2} \theta\beta(\theta)\, d\theta < \infty$, (1.10) is thus well defined.

Let us now recall the well-posedness result of (1.1) in [15], Corollary 2.4 (more general existence results can be found in [33]).

THEOREM 1.2. *Assume* (1.3) *for some* $\gamma \in (-1, 0)$, $\nu \in (0, 1)$ *with* $\gamma + \nu > 0$. *Let* $q \geq 2$ *such that* $q > \gamma^2/(\gamma + \nu)$. *Let* $f_0 \in \mathcal{P}_q(\mathbb{R}^3)$ *with* $\int_{\mathbb{R}^3} f_0(v) |\log f_0(v)|\, dv < \infty$ *and let* $p \in (3/(3 + \gamma), p_0(\gamma, \nu, q))$, *where*

$$p_0(\gamma, \nu, q) = \frac{q - \gamma}{q(3 - \nu)/3 - \gamma} \in \big(3/(3 + \gamma), 3/(3 - \nu)\big). \tag{1.11}$$

Then (1.1) *has a unique weak solution* $f \in L^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{\mathrm{loc}}([0, \infty), L^p(\mathbb{R}^3))$.

The explicit value of $p_0(\gamma, \nu, q)$ are not properly stated in [15], Corollary 2.4. However, following its proof (see the end of Step 3), we see that $f \in L^1_{\mathrm{loc}}([0, \infty), L^p(\mathbb{R}^3))$ as soon as $1 < p < 3/(3 - \nu)$ and $-\gamma(p - 1)/(1 - p(3 - \nu)/3) < q$. This precisely rewrites as $p \in (1, p_0(\gamma, \nu, q))$.

1.5. *The particle system.* Let us now recall the Nanbu particle system introduced by [28]. It is the $(\mathbb{R}^3)^N$-valued Markov process with infinitesimal generator $\mathcal{L}_N$ defined as follows: for any bounded Lipschitz function $\phi : (\mathbb{R}^3)^N \mapsto \mathbb{R}$ and $\mathbf{v} = (v_1, \dots, v_N) \in (\mathbb{R}^3)^N$,

$$\begin{aligned} \mathcal{L}_N \phi(\mathbf{v}) = \frac{1}{N} \sum_{i \neq j} \int_{\mathbb{S}^2} &\big[\phi\big(\mathbf{v} + \big(v'(v_i, v_j, \sigma) - v_i\big)\mathbf{e}_i\big) - \phi(\mathbf{v})\big] \\ &\times B\big(|v_i - v_j|, \theta\big)\, d\sigma, \end{aligned}$$

where $v\mathbf{e}_i = (0, \dots, v, \dots, 0) \in (\mathbb{R}^3)^N$ with v at the ith place for $v \in \mathbb{R}^3$.

In other words, the system contains N particles with velocities $\mathbf{v} = (v_1, \dots, v_N)$. Each pair of particles [with velocities (v_i, v_j)], interact, for each $\sigma \in \mathbb{S}^2$, at rate $B(|v_i - v_j|, \theta)/N$. Then one changes the velocity v_i to $v'(v_i, v_j, \sigma)$ given by (1.2) but v_j remains unchanged, that is, only one particle is changed at each collision.

The fact that $\int_0^\pi \beta(\theta)\,d\theta = \infty$ (i.e., β is noncutoff) means that there are infinitely many jumps with a very small deviation angle. It is thus impossible to simulate it directly. For this reason, we will study a truncated version of Nanbu's particle system applying a cutoff procedure as [14], who were studying the Nanbu system for *hard potentials* and *Maxwell molecules*, and [4], who were dealing with the Kac system for *Maxwell molecules*. Our particle system with cutoff corresponds to the generator $\mathcal{L}_{N,K}$ defined, for any bounded Lipschitz function $\phi : (\mathbb{R}^3)^N \mapsto \mathbb{R}$ and $\mathbf{v} = (v_1, \dots, v_N) \in (\mathbb{R}^3)^N$, by

$$\begin{aligned} \mathcal{L}_{N,K}\phi(\mathbf{v}) = \frac{1}{N}\sum_{i\neq j}\int_{\mathbb{S}^2}&\big[\phi\big(\mathbf{v} + \big(v'(v_i, v_j, \sigma) - v_i\big)\mathbf{e}_i\big) - \phi(\mathbf{v})\big] \\ &\times B\big(|v_i - v_j|, \theta\big)\mathbf{1}_{\{\theta \geq G(K/|v_i - v_j|^\gamma)\}}\,d\sigma, \end{aligned} \tag{1.12}$$

with G defined by (1.4).

The generator $\mathcal{L}_{N,K}$ uniquely defines a strong Markov process with values in $(\mathbb{R}^3)^N$. This comes from the fact that the corresponding jump rate is finite and constant: for any configuration $\mathbf{v} = (v_1, \dots, v_N) \in (\mathbb{R}^3)^N$, it holds that $N^{-1}\sum_{i\neq j}\int_{\mathbb{S}^2} B(|v_i - v_j|, \theta)\mathbf{1}_{\{\theta \geq G(K/|v_i - v_j|^\gamma)\}}\,d\sigma = 2\pi(N-1)K$. Indeed, for any $z \in [0, \infty)$, we have $\int_{\mathbb{S}^2} B(x, \theta)\mathbf{1}_{\{\theta \geq G(K/x^\gamma)\}}\,d\sigma = 2\pi K$, which is easily checked recalling that $B(x, \theta) = x^\gamma \beta(\theta)$ and the definition of G.

1.6. *Main results.* Now, we give our uniqueness result for the Boltzmann equation.

THEOREM 1.3. *Assume* (1.3) *for some* $\gamma \in (-1, 0)$, $\nu \in (0, 1)$ *satisfying* $\gamma + \nu > 0$. *Let* $q \geq 2$ *such that* $q > \gamma^2/(\gamma + \nu)$. *Assume that* $f_0 \in \mathcal{P}_q(\mathbb{R}^3)$ *with a finite entropy, that is,* $\int_{\mathbb{R}^3} f_0(v)|\log f_0(v)|\,dv < \infty$. *Let* $p \in (3/(3+\gamma), p_0(\gamma, \nu, q))$, *recall* (1.11) *and* $(f_t)_{t\geq 0} \in L^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{\mathrm{loc}}([0, \infty), L^p(\mathbb{R}^3))$ *be the unique weak solution to* (1.1) *given by Theorem* 1.2. *Then for any other weak solution* $(\tilde{f}_t)_{t\geq 0} \in L^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3))$ *to* (1.1), *we have, for any* $t \geq 0$,

$$\mathcal{W}_2^2(f_t, \tilde{f}_t) \leq \mathcal{W}_2^2(f_0, \tilde{f}_0)\exp\left(C_{\gamma,p}\int_0^t \big(1 + \|f_s\|_{L^p}\big)\,ds\right).$$

In particular, we have uniqueness for (1.1) *when starting from* f_0 *in the space of all weak solutions in the sense of Definition* 1.1.

The novelty of Theorem 1.3 is that no regularity at all is assumed concerning $\tilde{f}$. In particular, we have uniqueness among all weak solutions, while in [15], uniqueness is proved only in the class of weak solutions lying in $L^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{\mathrm{loc}}([0, \infty), L^p(\mathbb{R}^3))$ for some $p > 3/(3+\gamma)$.

Next, we write the following conclusion concerning the particle system.

THEOREM 1.4. *Assume* (1.3) *for some* $\gamma \in (-1, 0)$, $\nu \in (0, 1)$ *with* $\gamma + \nu > 0$. *Let* $q > 6$ *such that* $q > \gamma^2/(\gamma + \nu)$ *and let* $f_0 \in \mathcal{P}_q(\mathbb{R}^3)$ *with a finite entropy. Let* $(f_t)_{t \geq 0}$ *be the unique weak solution to* (1.1) *given by Theorem* 1.2. *For each* $N \geq 1$, $K \in [1, \infty)$, *let* $(V_t^i)_{i=1,\dots,N}$ *be the Markov process with generator* $\mathcal{L}_{N,K}$ [*see* (1.12)] *starting from an i.i.d. family* $(V_0^i)_{i=1,\dots,N}$ *of* f_0*-distributed random variables. We denote the associated empirical measure by* $\mu_t^{N,K} = N^{-1} \sum_{i=1}^N \delta_{V_t^i}$. *Then for all* $T > 0$,

$$\sup_{[0,T]} \mathbb{E}[\mathcal{W}_2^2(\mu_t^{N,K}, f_t)] \leq C_{T,q}\big(N^{-(1-6/q)(2+2\gamma)/3} + K^{1-2/\nu} + N^{-1/2}\big).$$

We thus obtain a quantitative rate of chaos for the Nanbu's system with a singular interaction. To our knowledge, this is the first result in this direction. However, there is no doubt this rate is not the hoped optimal rate $N^{-1/2}$ like in the hard potential case [14].

1.7. *Known results*, *strategies and main difficulties*. Let us give a nonexhaustive overview of the known results on the well-posedness of (1.1) for different potentials. First, the global existence of weak solution for the Boltzmann equation concerning all potentials was concluded by Villani in [33], with rather few assumptions on the initial data (finite energy and entropy), using some compactness methods. However, the uniqueness results are less well understood. For *hard potentials* [$\gamma \in (0, 1)$] with *angular cutoff* [$\int_0^\pi \beta(\theta)\, d\theta < \infty$], there are some optimal results obtained by Mischler–Wennberg [27], where they gave the existence of a unique weak L^1 solution to (1.1) with the minimal assumption that $\int_{\mathbb{R}^3}(1 + |v|^2) f_0(v)\, dv < \infty$. This was extended to weak measure solutions by Lu–Mouhot [24]. For the difficult case *without* angular cutoff, the first uniqueness result was obtained by Tanaka [31] concerning *Maxwell molecules* ($\gamma = 0$). See also Toscani–Villani [32], who proved uniqueness for Maxwell molecules imposing that $\int_0^\pi \theta\beta(\theta)\, d\theta < \infty$ and that $\int_{\mathbb{R}^3}(1 + |v|^2) f_0(dv) < \infty$. Subsequently, Desvillettes–Mouhot [5] (relying on a weighted W_1^1 space) and Fournier–Mouhot [15] (using the Wasserstein distance $\mathcal{W}_1$) successively gave the uniqueness and stability for both *hard potentials* ($\gamma \in (0, 1]$) and *moderately soft potentials* [$\gamma \in (-1, 0)$ and $\nu \in (0, 1)$] under different assumptions on initial data. For *moderately soft potentials*, the result in [15] is much better since they use less assumptions on the initial condition than [5]. Finally, let us mention another work [9], where Fournier–Guérin proved a local (in time) uniqueness result with $f_0 \in L^p(\mathbb{R}^3)$ for some $p > 3/(3 + \gamma)$ for the *very soft potentials* [$\gamma \in (-3, 0)$ and $\nu \in (0, 2)$].

In this paper (Theorem 1.3), we obtain a better uniqueness result in the case of a collision kernel without angular cutoff when $\gamma \in (-1, 0)$ and $\nu \in (0, 1-\gamma)$, that is, the uniqueness holds in the class of all measure solutions in $L^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3))$.

This is very important when studying particle systems. For example, a convergence result without rate would be almost immediate from our uniqueness: the tightness of the empirical measure of the particle system is not very difficult, as well as the fact that any limit point is a weak solution to (1.1). Since such a weak solution is unique by Theorem 1.3, the convergence follows. Such a conclusion would be very difficult to obtain when using the uniqueness proved in [15], because one would need to check that any limit point of the empirical measure belongs to $L^1_{\mathrm{loc}}([0,\infty, L^p(\mathbb{R}^3))$ for some $p > 3/(3+\gamma)$, which seems very difficult.

In order to extend the uniqueness result for all measure solutions, extra difficulty is inevitable and the methods of [9, 15] will *not* work. However, Fournier–Hauray [11] provide some ideas to overcome this, in the simpler case of the Laudau equation for moderately soft potentials. Here, we follow these ideas, which rely on coupling methods. Consider two weak solutions f and $\tilde{f}$ in $L^\infty([0,\infty), \mathcal{P}_2(\mathbb{R}^3))$ to (1.1), with possibly two different initial conditions and assume that f is *strong*, in the sense that it belongs to $L^1_{\mathrm{loc}}([0,\infty), L^p(\mathbb{R}^3))$. First, we associate to the weak solution $\tilde{f}$ a *weak* solution $(X_t)_{t\geq 0}$ to some Poisson-driven SDE. This uses a smoothing procedure as in [6, 11], but the situation is consequently more complicated because we deal with jump processes. Next, we try to associate to the *strong* solution f a *strong* solution $(W_t)_{t\geq 0}$ to another SDE [driven by the same Poisson measure as $(X_t)_{t\geq 0}$], as [11] did. But we did not manage to do this properly and we had to use a truncation procedure which though complicates our computation. Then, roughly, we estimate $\mathcal{W}_2^2(f_t, \tilde{f}_t)$ by computing $\mathbb{E}[|X_t - W_t|^2]$ as precisely as possible.

The terminology *propagation of chaos*, which is equivalent to the convergence of the empirical measure of a particle system to the solution to a nonlinear equation, was first formulated by Kac [23]. He was studying the convergence of a *toy* particle system as a step to the rigorous derivation of the Boltzmann equation. Kac's particle system is similar to the one studied in the present paper, but each collision modifies the velocities of the two involved particles, while in Nanbu's system, only one of the two particles is deviated. Hence, Kac's system is physically more meaningful. Afterwards, McKean [25] and Grünbaum [18] extended Kac's ideas to study the chaos property for different models with bounded collision kernels. Sznitman [30] then showed the chaos property (for Kac's system without rate) for the *hard spheres* ($\gamma = 1$ and $\nu = 0$). Following Tanaka's probabilistic interpretation for the Boltzmann equation with *Maxwell molecules*, Graham–Méléard [17] were the first to give a rate of chaos for (1.1), concerning both Kac and Nanbu models, for Maxwell molecules with cutoff [$\gamma = 0$ and $\int_0^\pi \beta(\theta)\,d\theta < \infty$], using the total variation distance. Fontbona–Guérin–Méléard [7] first gave explicit rates for Nanbu type diffusive approximations of the Landau equation with *Maxwell molecules* by coupling arguments, using the $\mathcal{W}_2$ distance. Recently, some important progresses have been made. First, Mischler–Mouhot [26] obtained a uniform (in time) rate of convergence of Kac's particle system of order $N^{-\varepsilon}$ (for Maxwell molecules without cutoff) and $(\log N)^{-\varepsilon}$ (for hard spheres, i.e., $\gamma = 1$ and $\nu = 0$), with some small $\varepsilon > 0$, in $\mathcal{W}_1$ distance between the joint law of the particle system and $f_t^{\otimes N}$. This result, entirely relying on analytic methods, is noticeable, although the rates are clearly not sharp. Then Fournier–Mischler [14] proved the propagation of chaos at rate $N^{-1/4}$ for the Nanbu system and for hard potentials without cutoff ($\gamma \in [0, 1]$ and $\nu \in (0, 1)$) using the $\mathcal{W}_2$ distance. Finally, as mentioned in Section 1.5, Cortez–Fontbona [4] used two coupling techniques and the $\mathcal{W}_2$ distance for Kac's system and obtained a uniform in time estimate for the Boltzmann equation with *Maxwell molecules* ($\gamma = 0$) under some suitable moments assumptions on the initial datum. Let us mention that the time-uniformity uses the recent nice results of Rousset [29].

In this paper (Theorem 1.4), we obtain, to our knowledge, the first chaos result (with rate) for soft potentials (which are, of course, more difficult), but it is a bit unsatisfying: (1) we cannot study Kac's system (which is physically more reasonable than Nanbu's system) because it is not readily to exhibit a suitable coupling; (2) our consideration is merely for $\gamma \in (-1, 0)$, since some basic estimates in Section 2 do not hold any more if $\gamma \leq -1$; (3) our rate is not sharp. However, since the interaction is singular, it seems hopeless to get a perfect result.

In terms of the propagation of chaos with a singular interaction, there are only very few results. Hauray–Jabin [19] considered a deterministic system of particles interacting through a force of the type $1/|x|^\alpha$ with $\alpha < 1$, in dimension $d \geq 3$, and proved the mean field limit and the propagation of chaos to the Vlasov equation. Also, Fournier–Hauray–Mischler [12] proved the convergence of the vortex model to the 2D Navier–Stokes equation with a singular Biot–Savart kernel using some entropy dissipation technique. Following the method of [12], Godinho–Quiñinao [16] proved the propagation of chaos of some particle system to the 2D subcritical Keller–Segel equation. Recently, Fournier–Hauray [11] proved propagation of chaos for the Landau equation with a singular interaction [$\gamma \in (-2, 0)$] for the Nanbu diffusive particle system using the $\mathcal{W}_2$ distance. Actually, they gave a quantitative rate of chaos when $\gamma \in (-1, 0)$, while the convergence without rate was checked when $\gamma \in (-2, 0)$ by the entropy dissipation technique.

Roughly speaking, to prove our propagation of chaos result, we consider an approximate version of our stability principle, with a discrete L^p norm as in [11]. Here, we list the main difficulties: The trajectory of a typical particle related to the Boltzmann equation is a jump process so that all the continuity arguments used in [11] have to be changed. In particular, a detailed study of small and large jumps is required. Also, the solution to the Landau equation lies in $L^1_{\text{loc}}([0, \infty), L^2(\mathbb{R}^3))$, while the one of the Boltzmann equation lies in $L^1_{\text{loc}}([0, \infty), L^p(\mathbb{R}^3))$ for some p smaller than 2. This causes a few difficulties in Section 5, because working in L^p is slightly more complicated.

1.8. *Arrangement of the paper and final notation.* In Section 2, we give some basic estimates. In Section 3, we establish the strong/weak stability principle for (1.1). In Section 4, we construct the suitable coupling. In Section 5, we bound the L^p norm of a blob approximation of an empirical measure in terms of the L^p norm of the weak solution. Finally, in Section 6, we prove the convergence of the particle system.

In the sequel, C stands for a positive constant whose value may change from line to line. When necessary, we will indicate in subscript the parameters it depends on.

In the whole paper, we consider two probability spaces by Tanaka's idea for the probabilistic interpretation of the Boltzmann equation in Maxwell molecules case: the first space is the abstract space $(\Omega, \mathcal{F}, \mathbb{P})$ and the second is $([0,1], \mathcal{B}([0,1]), d\alpha)$. A stochastic process defined on the latter space is called an α-processes and we denote the expectation on $[0,1]$ by $\mathbb{E}_\alpha$ and the laws by $\mathcal{L}_\alpha$.

2. Preliminaries. Above all, let us recall that for $\gamma \in (-1,0)$, $p > 3/(3+\gamma)$ and $f \in \mathcal{P}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$, it holds that

$$\begin{aligned}
&\sup_{v\in\mathbb{R}^3} \int_{\mathbb{R}^3} |v-v_*|^\gamma f(dv_*) \\
&\quad \le \sup_{v\in\mathbb{R}^3} \int_{|v-v_*|\le 1} |v-v_*|^\gamma f(dv_*) \\
&\qquad + \sup_{v\in\mathbb{R}^3} \int_{|v-v_*|\ge 1} |v-v_*|^\gamma f(dv_*) \\
&\quad \le 1 + C_{\gamma,p}\|f\|_{L^p(\mathbb{R}^3)},
\end{aligned} \tag{2.1}$$

where $C_{\gamma,p} = \sup_{v\in\mathbb{R}^3}[\int_{|v-v_*|\le 1} |v - v_*|^{p\gamma/(p-1)}\, dv_*]^{(p-1)/p} = [\int_{|v_*|\le 1} |v_*|^{p\gamma/(p-1)}\, dv_*]^{(p-1)/p} < \infty$, since $p > 3/(3+\gamma)$ by assumption.

Let us now classically rewrite the collision operator by making disappear the velocity-dependence $|v-v_*|^\gamma$ in the *rate* using a substitution.

LEMMA 2.1. *We assume* (1.3) *and recall* (1.4) *and* (1.7). *For* $z \in [0,\infty)$, $\varphi \in [0, 2\pi)$, $v, v_* \in \mathbb{R}^3$ *and* $K \in [1,\infty)$, *we define*

$$\begin{aligned}
&c(v, v_*, z, \varphi) := a\bigl[v, v_*, G\bigl(z/|v-v_*|^\gamma\bigr), \varphi\bigr] \quad \textit{and} \\
&c_K(v, v_*, z, \varphi) := c(v, v_*, z, \varphi)\mathbf{1}_{\{z\le K\}}.
\end{aligned} \tag{2.2}$$

For any $\phi \in \mathrm{Lip}(\mathbb{R}^3)$, *any* $v, v_* \in \mathbb{R}$,

$$\mathcal{A}\phi(v, v_*) = \int_0^\infty dz \int_0^{2\pi} d\varphi\bigl[\phi\bigl(v + c(v, v_*, z, \varphi)\bigr) - \phi(v)\bigr]. \tag{2.3}$$

For any $N \geq 1$, $K \in [1, \infty)$, $\mathbf{v} = (v_1, \dots, v_N) \in (\mathbb{R}^3)^N$, *any bounded measurable* $\phi : (\mathbb{R}^3)^N \mapsto \mathbb{R}$,

$$\mathcal{L}_{N,K}\phi(\mathbf{v}) = \frac{1}{N} \sum_{i \neq j} \int_0^\infty dz \int_0^{2\pi} d\varphi \big[\phi\big(\mathbf{v} + c_K(v_i, v_j, z, \varphi)\mathbf{e}_i\big) - \phi(\mathbf{v})\big]. \tag{2.4}$$

This lemma is stated in [14], Lemma 2.2, when $\gamma \in [0, 1]$, but the proof does not use this fact: it actually holds true for any $\gamma \in \mathbb{R}$. Next, let us recall Lemma 2.3 in [14] which is an accurate version of Tanaka's trick in [31]. Here, we adopt the notation (1.7).

LEMMA 2.2. *There exists some measurable function* $\varphi_0 : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto [0, 2\pi)$ *such that for all* $X, Y \in \mathbb{R}^3$, *all* $\varphi \in [0, 2\pi)$,

$$\big|\Gamma(X, \varphi) - \Gamma\big(Y, \varphi + \varphi_0(X, Y)\big)\big| \leq |X - Y|.$$

The rest of the section is an adaption of Section 3 in [14], which assumes that $\gamma \in [0, 1]$, to the case where $\gamma \in (-1, 0)$. When compared with [9], what is new is that in the inequalities (2.5) and (2.6) below, only $|v - v_*|^\gamma$ appears (while in [9], there is $|v - v_*|^\gamma + |\tilde{v} - \tilde{v}_*|^\gamma$). This is very useful to get a strong/weak stability estimate: we will be able to use the regularity of only one of the two solutions to be compared. Let us mention that it seems impossible to extend our ideas to the more singular case where $\gamma \leq -1$.

LEMMA 2.3. *There is a constant* C *such that for any* $v, v_*, \tilde{v}, \tilde{v}_* \in \mathbb{R}^3$, *any* $K \geq 1$,

$$\begin{aligned}&\int_0^\infty \int_0^{2\pi} \big|c(v, v_*, z, \varphi) - c\big(\tilde{v}, \tilde{v}_*, z, \varphi + \varphi_0(v - v_*, \tilde{v} - \tilde{v}_*)\big)\big|^2 \, d\varphi \, dz \\ &\qquad \leq C\big(|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2\big)|v - v_*|^\gamma,\end{aligned} \tag{2.5}$$

$$\begin{aligned}&\int_0^\infty \int_0^{2\pi} \big(\big|v + c(v, v_*, z, \varphi) - \tilde{v} \\ &\qquad - c_K\big(\tilde{v}, \tilde{v}_*, z, \varphi + \varphi_0(v - v_*, \tilde{v} - \tilde{v}_*)\big)\big|^2 - |v - \tilde{v}|^2\big) \, d\varphi \, dz \\ &\qquad \leq C\big(|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2\big)|v - v_*|^\gamma \\ &\qquad\quad + C|v - v_*|^{2+2\gamma/\nu} K^{1-2/\nu},\end{aligned} \tag{2.6}$$

$$\begin{aligned}&\int_0^\infty \int_0^{2\pi} \big|c_K(v, v_*, z, \varphi)\big|^2 \, d\varphi \, dz \leq C|v - v_*|^{\gamma+2}, \\ &\int_0^\infty \left|\int_0^{2\pi} c_K(v, v_*, z, \varphi) \, d\varphi\right| dz \leq C|v - v_*|^{\gamma+1},\end{aligned} \tag{2.7}$$

$$\int_0^\infty \int_0^{2\pi} |c(v, v_*, z, \varphi)|^2 \, d\varphi \, dz \le C|v - v_*|^{\gamma+2},$$

(2.8)

$$\int_0^\infty \left| \int_0^{2\pi} c(v, v_*, z, \varphi) \, d\varphi \right| dz \le C|v - v_*|^{\gamma+1}.$$

PROOF. For $x > 0$, we set $\Phi_K(x) = \pi \int_0^K (1 - \cos G(z/x^\gamma)) \, dz$ and $\Psi_K(x) = \pi \int_K^\infty (1 - \cos G(z/x^\gamma)) \, dz$. We introduce the shortened notation $x = |v - v_*|$, $\tilde{x} = |\tilde{v} - \tilde{v}_*|$, $\varphi_0 = \varphi_0(v - v_*, \tilde{v} - \tilde{v}_*)$, $c = c(v, v_*, z, \varphi)$, $c_K = c_K(v, v_*, z, \varphi) = c\mathbf{1}_{\{z \le K\}}$, $\tilde{c} = c(\tilde{v}, \tilde{v}_*, z, \varphi + \varphi_0)$ and $\tilde{c}_K = c_K(\tilde{v}, \tilde{v}_*, z, \varphi + \varphi_0) = \tilde{c}\mathbf{1}_{\{z \le K\}}$.

Step 1. We first verify that $\Phi_K(x) \le Cx^\gamma$ and that $|\Phi_K(x) - \Phi_K(\tilde{x})| \le C|x^\gamma - \tilde{x}^\gamma|$. First, we immediately see that $\Phi_K(x) \le \pi \int_0^\infty G^2(z/x^\gamma) \, dz = x^\gamma \pi \int_0^\infty G^2(z) \, dz$ which implies the first point [recall (1.5)]. To check the second point, it suffices to verify that $F_K(x) = \int_0^K (1 - \cos G(z/x)) \, dz$ has a bounded derivative (uniformly in $K \ge 1$). But we have $F_K(x) = x \int_0^{K/x} (1 - \cos G(z)) \, dz$ so that

$$\begin{aligned} |F_K'(x)| &\le \int_0^\infty (1 - \cos G(z)) \, dz + x(K/x^2)(1 - \cos G(K/x)) \\ &\le C + (K/x)G^2(K/x), \end{aligned}$$

which is uniformly bounded by (1.5).

Step 2. Proceeding as in the proof of [14], Lemma 3.1, we see that $\int_0^\infty \int_0^{2\pi} |c_K|^2 \, d\varphi \, dz = x^2 \Phi_K(x)$, which is bounded by $Cx^{\gamma+2}$ by Step 1. Also, recalling (1.7) and (2.2), using that $\int_0^{2\pi} \Gamma(X, \varphi) \, d\varphi = 0$, we see that $\int_0^{2\pi} c_K \, d\varphi = -\pi(v - v_*)(1 - \cos G(z/x^\gamma))$, whence $\int_0^\infty |\int_0^{2\pi} c_K \, d\varphi| \, dz = x\Phi_K(x) \le Cx^{\gamma+1}$ by Step 1. All this proves (2.7), from which (2.8) follows by letting K increase to infinity.

Step 3. Let us denote by $I_K = \int_0^K \int_0^{2\pi} |c - \tilde{c}|^2 \, d\varphi \, dz$, by $J_K = \int_0^K \int_0^{2\pi} (|v + c - \tilde{v} - \tilde{c}|^2 - |v - \tilde{v}|^2) \, d\varphi \, dz$ and by $L_K = \int_K^\infty \int_0^{2\pi} (|v + c - \tilde{v}|^2 - |v - \tilde{v}|^2) \, d\varphi \, dz$. Proceeding exactly as in the proof of [14], Lemma 3.1, we see that $J_K \le A_1^K + A_2^K$ and $L_K \le A_3^K$, where

$$\begin{aligned} A_1^K &= 2x\tilde{x} \int_0^K \big(G(z/x^\gamma) - G(z/\tilde{x}^\gamma)\big)^2 \, dz, \\ A_2^K &= \big[|v - \tilde{v}| + |v_* - \tilde{v}_*|\big] \big|(v - v_*)\Phi_K(x) - (\tilde{v} - \tilde{v}_*)\Phi_K(\tilde{x})\big|, \\ A_3^K &= \big(x^2 + 2|v - \tilde{v}|x\big)\Psi_K(x). \end{aligned}$$

Also, $I_K = J_K - 2(v - \tilde{v}) \cdot \int_0^K \int_0^{2\pi} (c - \tilde{c}) \, d\varphi \, dz$ and, as seen in the proof of [14], Lemma 3.1, $\int_0^K \int_0^{2\pi} c \, d\varphi \, dz = -(v - v_*)\Phi_K(x)$, so that $I_K \le J_K + A_4^K$ with

$$A_4^K = 2|v - \tilde{v}| \big|(v - v_*)\Phi_K(x) - (\tilde{v} - \tilde{v}_*)\Phi_K(\tilde{x})\big|.$$

First, we immediately deduce from (1.6) that

$$A_1^K \le 2c_4 x\tilde{x}\frac{(x^\gamma - \tilde{x}^\gamma)^2}{x^\gamma + \tilde{x}^\gamma} \le 2c_4(x-\tilde{x})^2 \min\left(x^\gamma, \tilde{x}^\gamma\right)$$
$$\le C\left(|v-\tilde{v}|^2 + |v_* - \tilde{v}_*|^2\right)|v-v_*|^\gamma.$$

For the second inequality, we used that $|x^\gamma - \tilde{x}^\gamma| \le |x^{-1} - \tilde{x}^{-1}|(x \wedge \tilde{x})^{1+\gamma}$ [because $\gamma \in (-1, 0)$] so that

$$x\tilde{x}\frac{|x^\gamma - \tilde{x}^\gamma|^2}{x^\gamma + \tilde{x}^\gamma} \le (x\tilde{x})^{1+|\gamma|}\frac{|x^{-1} - \tilde{x}^{-1}|^2(x\wedge\tilde{x})^{2\gamma+2}}{x^{|\gamma|} + \tilde{x}^{|\gamma|}}$$
$$\le (x\tilde{x})^{|\gamma|-1}\frac{|x-\tilde{x}|^2(x\tilde{x})^{1+\gamma}}{x^{|\gamma|} + \tilde{x}^{|\gamma|}} = \frac{|x-\tilde{x}|^2}{x^{|\gamma|} + \tilde{x}^{|\gamma|}},$$

which is indeed bounded by $(x-\tilde{x})^2 \min(x^\gamma, \tilde{x}^\gamma)$.

We now verify that $A_2^K \le C(|v-\tilde{v}|^2 + |v_* - \tilde{v}_*|^2)|v-v_*|^\gamma$. By Step 1, for any $X, Y \in \mathbb{R}^3$,

$$\left|X\Phi_K\left(|X|\right) - Y\Phi_K\left(|Y|\right)\right| \le |Y|\left|\Phi_K\left(|X|\right) - \Phi_K\left(|Y|\right)\right| + |X-Y|\Phi_K\left(|X|\right)$$
$$\le C|Y|\left||X|^\gamma - |Y|^\gamma\right| + C|X-Y||X|^\gamma.$$

Since again $|x^\gamma - \tilde{x}^\gamma| \le |x^{-1} - \tilde{x}^{-1}|(x\wedge\tilde{x})^{1+\gamma}$, we conclude that $|X\Phi_K(|X|) - Y\Phi_K(|Y|)| \le C|X-Y||X|^\gamma$, whence

$$A_2^K \le C\left[|v-\tilde{v}| + |v_* - \tilde{v}_*|\right]\left|(v-v_*) - (\tilde{v} - \tilde{v}_*)\right| \min\{x^\gamma, \tilde{x}^\gamma\}$$

as desired.

We next observe that $A_4^K \le 2A_2^K$.

Finally, we see that $\Psi_K(x) \le C\int_K^\infty G^2(z/x^\gamma)\,dz \le C\int_K^\infty (z/x^\gamma)^{-2/\nu}\,dz = Cx^{2\gamma/\nu}K^{1-2/\nu}$ and that $\Psi_K(x) \le C\int_0^\infty G^2(z/x^\gamma)\,dz \le C\int_0^\infty (1+z/x^\gamma)^{-2/\nu}\,dz = Cx^\gamma$ according to (1.5), which imply $\Psi_K(x) \le C\min\{x^\gamma, x^{2\gamma/\nu}K^{1-2/\nu}\}$. Hence,

$$A_3^K = \left(x^2 + 2|v-\tilde{v}|x\right)\Psi_K(x) \le C|v-\tilde{v}|^2|v-v_*|^\gamma + C|v-v_*|^{2+2\gamma/\nu}K^{1-2/\nu},$$

because $2|v-\tilde{v}|x \le |v-\tilde{v}|^2 + x^2$ and $x^2\Psi_K(x) \le Cx^{2+2\gamma/\nu}K^{1-2/\nu}$.

The left-hand side of (2.6) is nothing but $J_K + L_K$, which is bounded by $A_1^K + A_2^K + A_3^K$: (2.6) is proved. Finally, the left-hand side of (2.5) equals $\lim_{K\to\infty} I_K$ and we know that $I_K \le A_1^K + A_2^K + A_4^K$, which is (uniformly in K) bounded by $(|v-\tilde{v}|^2 + |v_* - \tilde{v}_*|^2)|v-v_*|^\gamma$ as desired. □

3. Stability. In this section, our goal is to prove Theorem 1.3.

Let us first give the outline of the proof. Let $(f_t)_{t\ge0}$ be the *strong* solution to (1.1) and let $(\tilde{f}_t)_{t\ge0}$ be a weak solution. We first build $(X_t)_{t\ge0}$ with $\mathcal{L}(X_t) = \tilde{f}_t$ solving

$$X_t = X_0 + \int_0^t\int_0^1\int_0^\infty\int_0^{2\pi} c\left(X_{s-}, X_s^*(\alpha), z, \varphi\right)M(ds, d\alpha, dz, d\varphi),$$

where $(X_t^*)_{t\geq 0}$ is a measurable α-process with law $\tilde{f}_t$, and $M(ds, d\alpha, dz, d\varphi)$ is a Poisson measure. This process $(X_t)_{t\geq 0}$ can be interpreted as the velocity of a typical particle. Each time it has a jump, say at some time t, it means that the typical particle has collided with another particle, of which the velocity is independent and represented by X_t^*. Of course, X_t^* has to be $\tilde{f}_t$-distributed.

The existence of the process $(X_t)_{t\geq 0}$ is not easy and we only build a weak solution. The difficulty is mainly due to the singularity of the interaction, which cannot be compensated by some regularity of $\tilde{f}_t$, because $\tilde{f}_t$ is any weak solution. We thus use the strategy of [6] (which deals with continuous diffusion processes). We introduce $\tilde{f}_t^\varepsilon = \tilde{f}_t * \phi_\varepsilon$, where ϕ_ε is the centered Gaussian density with covariance matrix εI_3. We write the PDE satisfied by $\tilde{f}_t^\varepsilon$ and associate, for each $\varepsilon \in (0,1)$, a solution $(X_t^\varepsilon)_{t\geq 0}$ to some SDE. Since both the SDE and the PDE [with $\varepsilon \in (0,1)$ fixed] are well-posed (because the coefficients are regular enough, see Lemma 3.4), we conclude that $\mathcal{L}(X_t^\varepsilon) = \tilde{f}_t^\varepsilon$. Next, we prove that the family $\{(X_t^\varepsilon)_{t\geq 0}, \varepsilon \in (0,1)\}$ is tight using the Aldous criterion [1]. Finally, we consider a limit point $(X_t)_{t\geq 0}$, as $\varepsilon \to 0$, of $\{(X_t^\varepsilon)_{t\geq 0}, \varepsilon \in (0,1)\}$. Since $\mathcal{L}(X_t^\varepsilon) = \tilde{f}_t^\varepsilon$, we deduce that $\mathcal{L}(X_t) = \tilde{f}_t$ for each $t \geq 0$. Then we classically make use of martingale problems to show that $(X_t)_{t\geq 0}$ is indeed a solution of the desired SDE.

Next, we would like to associate to $(f_t)_{t\geq 0}$ a solution $(W_t)_{t\geq 0}$ to the SDE, driven by the same Poisson measure M, with f_t-distributed α-process $(W_t^*)_{t\geq 0}$ coupled with $(X_t^*)_{t\geq 0}$, that is,

$$\begin{aligned} W_t = W_0 + \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} c\big(&W_{s-}, W_s^*(\alpha), z, \varphi \\ &+ \varphi_0\big(X_{s-} - X_s^*(\alpha), W_{s-} - W_s^*(\alpha)\big)\big) M(ds, d\alpha, dz, d\varphi), \end{aligned}$$

where the f_t-distributed W_t^* is optimally coupled with X_t^* for each $t \geq 0$. Unfortunately, we cannot prove that such a process exists, because of the term $\varphi + \varphi_0(X_{s-} - X_s^*(\alpha), W_{s-} - W_s^*(\alpha))$. Such a problem was already encountered by Tanaka [31], and we more or less solve it as he did, by introducing, for all $K \geq 1$,

$$W_t^K = W_0 + \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} c_K\big(W_{s-}^K, W_s^*(\alpha), z, \varphi + \varphi_{s,\alpha,K}\big) M(ds, d\alpha, dz, d\varphi)$$

with $\varphi_{s,\alpha,K} = \varphi_0(X_{s-} - X_s^*(\alpha), W_{s-}^K - W_s^*(\alpha))$ as a coupling SDE. This equation of course has a unique strong solution $(W_t^K)_{t\geq 0}$, but the computation becomes more complicated.

Finally, we observe that

$$\mathcal{W}_2^2(f_t, \tilde{f}_t) \leq \limsup_{K\to\infty} \mathbb{E}\big[|W_t^K - X_t|^2\big],$$

because W_t^K goes in law to f_t for each $t \geq 0$.

Using the Itô formula, we find

$$\mathbb{E}\big[|W_t^K - X_t|^2\big] = \mathbb{E}\big[|W_0 - X_0|^2\big] + \mathbb{E}\bigg[\int_0^t \int_0^1 \Delta_s^K(\alpha)\, d\alpha\, ds\bigg],$$

where

$$\Delta_s^K(\alpha) := \int_0^\infty \int_0^{2\pi} \left(\left|W_{s-}^K - X_{s-} + c_{K,W}(s) - c_X(s)\right|^2 - \left|W_{s-}^K - X_{s-}\right|^2\right) d\varphi\, dz$$

with the shortened notation $c_{K,W}(s) := c_K(W_s^K, W_s^*(\alpha), z, \varphi + \varphi_{s,\alpha,K})$ and $c_X(s) := c(X_s, X_s^*(\alpha), z, \varphi)$. Then we deduce from Section 2 that

$$\begin{aligned}\Delta_s^K(\alpha) &\le C\left(\left|W_s^K - X_s\right|^2 + \left|W_s^*(\alpha) - X_s^*(\alpha)\right|^2\right)\left|W_s^K - W_s^*(\alpha)\right|^\gamma \\ &\quad + C\left|W_s^K - W_s^*(\alpha)\right|^{2+2\gamma/\nu} K^{1-2/\nu}.\end{aligned}$$

It is then not too hard to conclude, using technical computations, that

$$\limsup_{K\to\infty} \mathbb{E}\left[\left|W_t^K - X_t\right|^2\right] \le \mathcal{W}_2^2(f_0, \tilde{f}_0) \exp\left(C_{\gamma,p} \int_0^t \left(1 + \|f_s\|_{L^p}\right) ds\right),$$

which completes the proof.

We first state the following result, of which the proof lies at the end of the section.

PROPOSITION 3.1. *Assume* (1.3) *for some* $\gamma \in (-1, 0)$, $\nu \in (0, 1)$ *with* $\gamma + \nu > 0$. *Consider any weak solution* $(\tilde{f}_t)_{t\ge0} \in L^\infty([0,\infty), \mathcal{P}_2(\mathbb{R}^3))$ *to* (1.1). *Then there exists, on some probability space, a random variable* X_0 *with law* $\tilde{f}_0$, *independent of a Poisson measure* $M(ds, d\alpha, dz, d\varphi)$ *on* $[0,\infty) \times [0,1] \times [0,\infty) \times [0, 2\pi)$ *with intensity* $ds\, d\alpha\, dz\, d\varphi$, *a measurable family* $(X_t^*)_{t\ge0}$ *of* α*-random variables such that* $\mathcal{L}_\alpha(X_t^*) = \tilde{f}_t$ *and a càdlàg adapted process* $(X_t)_{t\ge0}$ *solving*

$$X_t = X_0 + \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} c\left(X_{s-}, X_s^*(\alpha), z, \varphi\right) M(ds, d\alpha, dz, d\varphi) \tag{3.1}$$

and such that for all $t \ge 0$, $\mathcal{L}(X_t) = \tilde{f}_t$.

We are unfortunately not able to say anything about uniqueness (in law) for this SDE, except if $\tilde{f}$ is a *strong* solution, and this is precisely the reason why things are complicated. We really need to use the ideas of [6] to produce, for $(\tilde{f}_t)_{t\ge0}$ given, a solution $(X_t)_{t\ge0}$ of which the time marginals are $(\tilde{f}_t)_{t\ge0}$.

PROPOSITION 3.2. *Assume* (1.3) *for some* $\gamma \in (-1, 0)$, $\nu \in (0, 1)$ *with* $\gamma + \nu > 0$, *that* $f_0 \in \mathcal{P}_q(\mathbb{R}^3)$ *for some* $q \ge 2$ *such that* $q > \gamma^2/(\gamma + \nu)$ *and that* f_0 *has a finite entropy. Fix* $p \in (3/(3+\gamma), p_0(\gamma, \nu, q))$. *Let* $(f_t)_{t\ge0} \in L^\infty([0,\infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{\mathrm{loc}}([0,\infty), L^p(\mathbb{R}^3))$ *be the corresponding unique weak solution to* (1.1) *given by Theorem* 1.2. *Consider also the Poisson measure* M, *the process* $(X_t)_{t\ge0}$ *and the family* $(X_t^*)_{t\ge0}$ *built in Proposition* 3.1 [*associated to another weak solution* $(\tilde{f}_t)_{t\ge0} \in L^\infty([0,\infty), \mathcal{P}_2(\mathbb{R}^3))$]. *Let* $W_0 \sim f_0$ (*independent of* M) *be such that* $\mathbb{E}[|W_0 - X_0|^2] = \mathcal{W}_2^2(f_0, \tilde{f}_0)$ *and, for each* $t \ge 0$, *an* α*-random* *variable* W_t^* *such that* $\mathcal{L}_\alpha(W_t^*) = f_t$ *and* $\mathbb{E}_\alpha[|W_t^* - X_t^*|^2] = \mathcal{W}_2^2(f_t, \tilde{f}_t)$. *Then for* $K \geq 1$, *the equation*

(3.2)
$$W_t^K = W_0 + \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} c_K\big(W_{s-}^K, W_s^*(\alpha), z, \varphi + \varphi_{s,\alpha,K}\big) M(ds, d\alpha, dz, d\varphi),$$

with $\varphi_{s,\alpha,K} = \varphi_0(X_{s-} - X_s^*(\alpha), W_{s-}^K - W_s^*(\alpha))$, *has a unique solution. Moreover, setting* $f_t^K = \mathcal{L}(W_t^K)$ *for each* $t \geq 0$, *it holds that for all* $T > 0$,

$$\lim_{K\to\infty} \sup_{[0,T]} \mathcal{W}_2^2(f_t^K, f_t) = 0. \tag{3.3}$$

REMARK 3.3. As recalled in the previous section, the infimum in the definition of Wasserstein distance is actually a minimum. Since the strong solution $f_t \in \mathcal{P}_2(\mathbb{R}^3)$ has a density for all $t \geq 0$, there is a unique $R_t \in \mathcal{H}(f_t, \tilde{f}_t)$ such that $\mathcal{W}_2^2(f_t, \tilde{f}_t) = \int_{\mathbb{R}^3\times\mathbb{R}^3} |v - \tilde{v}|^2 R_t(dv, d\tilde{v})$ (see Villani [34], Theorem 2.12). We then know that $(t, \alpha) \mapsto (W_t^*(\alpha), X_t^*(\alpha))$ can be chosen measurable from Fontbona–Guérin–Méléard [7], Theorem 1.3.

PROOF. For any $K \geq 1$, the Poisson measure involved in (3.2) is actually finite (because $c_K = c\mathbf{1}_{\{z\leq K\}}$), so the existence and uniqueness for this equation is obvious. It only remains to prove (3.3), which has already been done in [9], Lemma 4.2, where the formulation of the equation is slightly different. But one easily checks that $(W_t^K)_{t\geq 0}$ is a (time-inhomogeneous) Markov process with the same generator as the one defined by [9], equation (4.1), because for all bounded measurable function $\phi : \mathbb{R}^3 \mapsto \mathbb{R}$ and all $t \geq 0$, a.s.,

$$\begin{aligned}
&\int_0^1 \int_0^\infty \int_0^{2\pi} \big[\phi\big(w + c_K\big(w, W_t^*(\alpha), z, \varphi \\
&\quad + \varphi_0\big(X_{t-} - X_t^*(\alpha), w - W_t^*(\alpha)\big)\big)\big) - \phi(w)\big] \, d\varphi \, dz \, d\alpha \\
&\qquad = \int_0^1 \int_0^\infty \int_0^{2\pi} \big[\phi\big(w + c_K(w, v, z, \varphi)\big) - \phi(w)\big] \, d\varphi \, dz f_t(dv)
\end{aligned}$$

by the 2π-periodicity of c_K (in φ) and since $\mathcal{L}_\alpha(W_t^*) = f_t$. □

Now, we use these coupled processes to conclude the following.

PROOF OF THEOREM 1.3. We consider a weak solution $(\tilde{f}_t)_{t\geq 0}$ to (1.1), with which we associate the objects M, $(X_t)_{t\geq 0}$, $(X_t^*)_{t\geq 0}$ as in Proposition 3.1. We then consider f_0 satisfying the assumptions of Theorem 1.2 and the corresponding unique weak solution $(f_t)_{t\geq 0}$ belonging to $L^\infty([0,\infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{\mathrm{loc}}([0,\infty), L^p(\mathbb{R}^3))$ [with $p \in (3/(3+\gamma), p_0(\gamma, \nu, q))$] and we consider $(W_t^K)_{t\geq 0}$, $(W_t^*)_{t\geq 0}$ built in Proposition 3.2 for any $K \geq 1$. We know that $\mathcal{W}_2^2(f_0, \tilde{f}_0) = \mathbb{E}[|W_0 - X_0|^2]$ and that $\mathcal{W}_2^2(f_t, \tilde{f}_t) = \mathbb{E}_\alpha[|W_t^* - X_t^*|^2]$ for all $t \geq 0$.

Using that $W_t^K \sim f_t^K$ and $X_t \sim \tilde{f}_t$ for each $t \geq 0$, we deduce from (3.3) that for all $t \geq 0$,

$$\mathcal{W}_2^2(f_t, \tilde{f}_t) \leq \limsup_{K\to\infty} \mathbb{E}\big[\big|W_t^K - X_t\big|^2\big] =: J_t. \tag{3.4}$$

Next, we focus on the time interval $[0, T]$ for any fixed $T > 0$, and split the proof into several steps.

Step 1. By the Itô formula we know that

$$\mathbb{E}\big[\big|W_t^K - X_t\big|^2\big] = \mathbb{E}\big[|W_0 - X_0|^2\big] + \mathbb{E}\bigg[\int_0^t \int_0^1 \Delta_s^K(\alpha)\, d\alpha\, ds\bigg],$$

where

$$\Delta_s^K(\alpha) := \int_0^\infty \int_0^{2\pi} \big(\big|W_s^K - X_s + c_{K,W}(s) - c_X(s)\big|^2 - \big|W_s^K - X_s\big|^2\big)\, d\varphi\, dz$$

with the shortened notation $c_{K,W}(s) := c_K(W_s^K, W_s^*(\alpha), z, \varphi + \varphi_{s,\alpha,K})$ and $c_X(s) := c(X_s, X_s^*(\alpha), z, \varphi)$. We then show that

$$\begin{aligned} \Delta_s^K(\alpha) &\leq C\big(\big|W_s^K - X_s\big|^2 + \big|W_s^*(\alpha) - X_s^*(\alpha)\big|^2\big)\big|W_s^K - W_s^*(\alpha)\big|^\gamma \\ &\quad + C\big|W_s^K - W_s^*(\alpha)\big|^{2+2\gamma/\nu} K^{1-2/\nu}, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \Delta_s^K(\alpha) &\leq C\big|W_s^K - W_s^*(\alpha)\big|^{\gamma+2} + C\big|X_s - X_s^*(\alpha)\big|^{\gamma+2} \\ &\quad + C\big|W_s^K - X_s\big|\big(\big|W_s^K - W_s^*(\alpha)\big|^{\gamma+1} + \big|X_s - X_s^*(\alpha)\big|^{\gamma+1}\big). \end{aligned} \tag{3.6}$$

First, Lemma 2.3 [inequality (2.6)] precisely tells us that (3.5) holds true. Next, we observe that

$$\begin{aligned} \Delta_s^K(\alpha) &\leq 2\int_0^\infty \int_0^{2\pi} \big(\big|c_{K,W}(s)\big|^2 + \big|c_X(s)\big|^2\big)\, d\varphi\, dz \\ &\quad + 2\big|W_s^K - X_s\big|\bigg|\int_0^\infty \int_0^{2\pi} \big(c_{K,W}(s) - c_X(s)\big)\, d\varphi\, dz\bigg|. \end{aligned}$$

Hence, using (2.7) and (2.8), the proof of (3.6) is concluded.

Step 2. Set $\kappa(\gamma) = \min((\gamma + 1)/|\gamma|, |\gamma|/2) > 0$. We verify that there exists a constant $C(T, f_0, \tilde{f}_0, f) > 0$ [depending on T, $m_2(f_0)$, $m_2(\tilde{f}_0)$, $\int_0^t \|f_s\|_{L^p}\, ds$], such that for all $\ell \geq 1$ (and all $K \geq 1$),

$$I_t^{i,\ell} \leq C(T, f_0, \tilde{f}_0, f)\ell^{-\kappa(\gamma)}, \qquad i = 1, 2, 3, 4,$$

where

$$I_t^{1,\ell} := \mathbb{E}\bigg[\int_0^t \int_0^1 \big|W_s^K - W_s^*(\alpha)\big|^{\gamma+2} \mathbf{1}_{\{|W_s^K - W_s^*(\alpha)|^\gamma \geq \ell\}}\, d\alpha\, ds\bigg],$$

$$I_t^{2,\ell} := \mathbb{E}\bigg[\int_0^t \int_0^1 \big|X_s - X_s^*(\alpha)\big|^{\gamma+2}\mathbf{1}_{\{|W_s^K - W_s^*(\alpha)|^\gamma \ge \ell\}}\, d\alpha\, ds\bigg],$$
$$I_t^{3,\ell} := \mathbb{E}\bigg[\int_0^t \int_0^1 \big|W_s^K - X_s\big|\big|W_s^K - W_s^*(\alpha)\big|^{\gamma+1}\mathbf{1}_{\{|W_s^K - W_s^*(\alpha)|^\gamma \ge \ell\}}\, d\alpha\, ds\bigg],$$
$$I_t^{4,\ell} := \mathbb{E}\bigg[\int_0^t \int_0^1 \big|W_s^K - X_s\big|\big|X_s - X_s^*(\alpha)\big|^{\gamma+1}\mathbf{1}_{\{|W_s^K - W_s^*(\alpha)|^\gamma \ge \ell\}}\, d\alpha\, ds\bigg].$$

Since $\gamma \in (-1, 0)$ and $\kappa(\gamma) \le (\gamma + 2)/|\gamma|$, we have

$$I_t^{1,\ell} \le \ell^{-(\gamma+2)/|\gamma|}T \le \ell^{-\kappa(\gamma)}T.$$

Similarly,

$$I_t^{3,\ell} \le \ell^{-(\gamma+1)/|\gamma|}\int_0^t \mathbb{E}\big[|W_s^K - X_s|\big]\, ds.$$

Using (1.9) for $(f_t)_{t\ge 0}$ and $(\tilde{f}_t)_{t\ge 0}$, (3.3), and that $m_2(f_s^K) \le 2m_2(f_s) + 2\mathcal{W}_2^2(f_s, f_s^K)$, we know that $\mathbb{E}[|W_s^K - X_s|] \le C(1 + m_2(f_s^K) + m_2(\tilde{f}_s)) \le C(T, f_0, \tilde{f}_0)$. Hence,

$$I_t^{3,\ell} \le C(T, f_0, \tilde{f}_0)\ell^{-\kappa(\gamma)}.$$

Since $\gamma + 2 \in (1, 2)$, it follows from the Hölder inequality that

$$\begin{aligned} I_t^{2,\ell} &\le \mathbb{E}\bigg[\bigg(\int_0^t \int_0^1 \big|X_s - X_s^*(\alpha)\big|^2\, d\alpha\, ds\bigg)^{\frac{\gamma+2}{2}} \\ &\qquad \times \bigg(\int_0^t \int_0^1 \mathbf{1}_{\{|W_s^K - W_s^*(\alpha)|^\gamma \ge \ell\}}\, d\alpha\, ds\bigg)^{\frac{|\gamma|}{2}}\bigg] \\ &\le C\mathbb{E}\bigg[\bigg(\int_0^t \big(|X_s|^2 + m_2(\tilde{f}_s)\big)\, ds\bigg)^{\frac{\gamma+2}{2}} \\ &\qquad \times \bigg(\int_0^t \int_0^1 \frac{|W_s^K - W_s^*(\alpha)|^\gamma}{\ell}\, d\alpha\, ds\bigg)^{\frac{|\gamma|}{2}}\bigg]. \end{aligned}$$

Since $\mathcal{L}_\alpha(W_s^*) = f_s$, we have $\int_0^1 |W_s^K - W_s^*(\alpha)|^\gamma\, d\alpha = \int_{\mathbb{R}^3} |W_s^K - v|^\gamma f_s(dv) \le 1 + C_{\gamma,p}\|f_s\|_{L^p}$ by (2.1), so that

$$\begin{aligned} I_t^{2,\ell} &\le \ell^{\gamma/2}\bigg(1 + \int_0^t \big(\mathbb{E}\big[|X_s|^2\big] + m_2(\tilde{f}_s)\big)\, ds\bigg)\bigg(\int_0^t \big(1 + C_{\gamma,p}\|f_s\|_{L^p}\big)\, ds\bigg)^{\frac{|\gamma|}{2}} \\ &\le \ell^{\gamma/2}\big(1 + 2m_2(\tilde{f}_0)T\big)\bigg(1 + \int_0^t \big(1 + C_{\gamma,p}\|f_s\|_{L^p}\big)\, ds\bigg) \\ &\le C(T, \tilde{f}_0, f)\ell^{-\kappa(\gamma)}. \end{aligned}$$

For $I_t^{4,\ell}$, we use the triple Hölder inequality to write

$$I_t^{4,\ell} \le \mathbb{E}\bigg[\int_0^t |W_s^K - X_s|^2\, ds\bigg]^{\frac{1}{2}} \mathbb{E}\bigg[\int_0^t \int_0^1 |X_s - X_s^*(\alpha)|^2\, d\alpha\, ds\bigg]^{\frac{1+\gamma}{2}} \times \mathbb{E}\bigg[\int_0^t \int_0^1 \mathbf{1}_{\{|W_s^K - W_s^*(\alpha)|^\gamma \ge \ell\}}\, d\alpha\, ds\bigg]^{\frac{|\gamma|}{2}}.$$

Thus, $I_t^{4,\ell} \le C(T, f_0, \tilde{f}_0, f)\ell^{-\kappa(\gamma)}$: use that $\mathbb{E}[|X_s|^2] = \mathbb{E}_\alpha[|X_s^*|^2] = m_2(\tilde{f}_0)$, that $m_2(f_s^K) \le 2m_2(f_s) + 2\mathcal{W}_2^2(f_s, f_s^K)$ as before and treat the last term of the product the same as we study $I_t^{2,\ell}$.

Step 3. According to Step 1, we now bound $\Delta_s^K(\alpha)$ by (3.5) when $|W_s^K - W_s^*(\alpha)|^\gamma \le \ell$ and by (3.6) when $|W_s^K - W_s^*(\alpha)|^\gamma \ge \ell$:

$$\begin{aligned}\mathbb{E}[|W_t^K - X_t|^2] \le \mathbb{E}[|W_0 - X_0|^2] &+ C\sum_{i=1}^4 I_t^{i,\ell} \\ &+ CK^{1-2/\nu}\mathbb{E}\bigg[\int_0^t \int_0^1 |W_s^K - W_s^*(\alpha)|^{2+2\gamma/\nu}\, d\alpha\, ds\bigg] \\ &+ C\mathbb{E}\bigg[\int_0^t \int_0^1 \big(|W_s^K - X_s|^2 \\ &+ |W_s^*(\alpha) - X_s^*(\alpha)|^2\big) \min\big(|W_s^K - W_s^*(\alpha)|^\gamma, \ell\big)\, d\alpha\, ds\bigg].\end{aligned}$$

It then follows from Step 2 that for all $\ell \ge 1$, all $K \ge 1$,

$$\begin{aligned}&\mathbb{E}[|W_t^K - X_t|^2] \\ &\quad \le \mathcal{W}_2^2(f_0, \tilde{f}_0) + C(T, f_0, \tilde{f}_0, f)\ell^{-\kappa(\gamma)} \\ &\qquad + CK^{1-2/\nu}\mathbb{E}\bigg[\int_0^t \int_0^1 |W_s^K - W_s^*(\alpha)|^{2+2\gamma/\nu}\, d\alpha\, ds\bigg] \\ &\qquad + C\mathbb{E}\bigg[\int_0^t \int_0^1 |W_s^K - X_s|^2 |W_s^K - W_s^*(\alpha)|^\gamma\, d\alpha\, ds\bigg] \\ &\qquad + C\mathbb{E}\bigg[\int_0^t \int_0^1 |W_s^*(\alpha) - X_s^*(\alpha)|^2 \\ &\qquad \times \min\big(|W_s^K - W_s^*(\alpha)|^\gamma, \ell\big)\, d\alpha\, ds\bigg].\end{aligned} \tag{3.7}$$

Since $\gamma + \nu > 0$, it holds that $2 + 2\gamma/\nu > 0$. As a consequence, like in Step 2,

$$\begin{aligned}\mathbb{E}\bigg[\int_0^t \int_0^1 |W_s^K - W_s^*(\alpha)|^{2+2\gamma/\nu}\, d\alpha\, ds\bigg] &\le C_T\big[1 + \mathbb{E}[|W_s^K|^2] + m_2(f_0)\big] \\ &\le C(T, f_0, \tilde{f}_0),\end{aligned}$$

which gives

$$\lim_{K\to\infty} K^{1-2/\nu}\mathbb{E}\left[\int_0^t\int_0^1 |W_s^K - W_s^*(\alpha)|^{2+2\gamma/\nu}\,d\alpha\,ds\right] = 0.$$

Moreover, we recall that a.s. $\int_0^1 |W_s^K - W_s^*(\alpha)|^\gamma\,d\alpha \le 1 + C_{\gamma,p}\|f_s\|_{L^p}$ as in Step 2, whence

$$\mathbb{E}\left[\int_0^t\int_0^1 |W_s^K - X_s|^2|W_s^K - W_s^*(\alpha)|^\gamma\,d\alpha\,ds\right]$$
$$\le \int_0^t \mathbb{E}[|W_s^K - X_s|^2](1 + C_{\gamma,p}\|f_s\|_{L^p})\,ds.$$

Letting $K\to\infty$, by dominated convergence, we find [recall (3.4)]

$$\limsup_K \mathbb{E}\left[\int_0^t\int_0^1 |W_s^K - X_s|^2|W_s^K - W_s^*(\alpha)|^\gamma\,d\alpha\,ds\right]$$
$$\le \int_0^t J_s(1 + C_{\gamma,p}\|f_s\|_{L^p})\,ds.$$

Next, it is obvious that for each $\ell\ge 1$ fixed, for all $s\in[0,T]$, all $\alpha\in[0,1]$, the function $v\mapsto \min(|v - W_s^*(\alpha)|^\gamma, \ell)$ is bounded and continuous. By (3.3), we conclude that $\lim_{K\to\infty}\mathbb{E}[\min(|W_s^K - W_s^*(\alpha)|^\gamma,\ell)] = \mathbb{E}[\min(|W_s - W_s^*(\alpha)|^\gamma,\ell)]$ and, by dominated convergence, that, still for $\ell\ge 1$ fixed,

$$\lim_{K\to\infty}\mathbb{E}\left[\int_0^t\int_0^1 |W_s^*(\alpha) - X_s^*(\alpha)|^2\min(|W_s^K - W_s^*(\alpha)|^\gamma,\ell)\,d\alpha\,ds\right]$$
$$= \int_0^t\int_0^1 |W_s^*(\alpha) - X_s^*(\alpha)|^2\mathbb{E}[\min(|W_s - W_s^*(\alpha)|^\gamma,\ell)]\,d\alpha\,ds.$$

But since $W_s\sim f_s$, we have, for each α fixed, $\mathbb{E}[\min(|W_s - W_s^*(\alpha)|^\gamma,\ell)]\le \int_{\mathbb{R}^3}|W_s^*(\alpha) - v|^\gamma f_s(dv)\le 1 + C_{\gamma,p}\|f_s\|_{L^p}$ by (2.1). Furthermore, we have $\int_0^1 |W_s^*(\alpha) - X_s^*(\alpha)|^2\,d\alpha = \mathbb{E}_\alpha[|W_s^* - X_s^*|^2] = \mathcal{W}_2^2(f_s,\tilde f_s)\le J_s$. All in all, we have checked that

$$\lim_{K\to\infty}\mathbb{E}\left[\int_0^t\int_0^1 |W_s^*(\alpha) - X_s^*(\alpha)|^2\min(|W_s^K - W_s^*(\alpha)|^\gamma,\ell)\,d\alpha\,ds\right]$$
$$\le C\int_0^t J_s(1 + \|f_s\|_{L^p})\,ds.$$

Gathering all the previous estimates to let $K\to\infty$ in (3.7): for each $\ell\ge 1$ fixed,

$$J_t \le \mathcal{W}_2^2(f_0,\tilde f_0) + C(T, f_0,\tilde f_0, f)\ell^{-\kappa(\gamma)}$$
$$+ C\int_0^t J_s(1 + \|f_s\|_{L^p})\,ds.$$

Letting now $\ell \to \infty$ and using the Grönwall lemma, we find

$$J_t \le \mathcal{W}_2^2(f_0, \tilde{f}_0) \exp\left(C_{\gamma,p} \int_0^t (1 + \|f_s\|_{L^p})\, ds\right).$$

Since $\mathcal{W}_2^2(f_t, \tilde{f}_t) \le J_t$, this completes the proof. $\square$

It remains to prove Proposition 3.1. We start with a technical result.

LEMMA 3.4. *Assume (1.3) for some $\gamma \in (-1, 0)$, some $\nu \in (0, 1)$ with $\gamma + \nu > 0$ and recall that the deviation function c was defined by (2.2). Consider $f \in \mathcal{P}_2(\mathbb{R}^3)$ and $\phi_\varepsilon(x) = (2\pi\varepsilon)^{-3/2} e^{-|x|^2/(2\varepsilon)}$. Set $f^\varepsilon(w) = (f * \phi_\varepsilon)(w)$:*

(i) *There exists a constant $C > 0$ such that for all $x \in \mathbb{R}^3$, all $\varepsilon \in (0, 1)$,*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_0^\infty\int_0^{2\pi} |c(v, v_*, z, \varphi)| \frac{\phi_\varepsilon(v - x)}{f^\varepsilon(x)}\, d\varphi\, dz f(dv) f(dv_*)$$
$$\le C\big(1 + \sqrt{m_2(f)} + |x|\big).$$

(ii) *For all $\varepsilon \in (0, 1)$, all $R > 0$, there is a constant $C_{R,\varepsilon} > 0$ [depending only on $m_2(f)$] such that for all $x, y \in B(0, R)$,*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_0^\infty\int_0^{2\pi} |c(v, v_*, z, \varphi)| \left|\frac{\phi_\varepsilon(v - x)}{f^\varepsilon(x)} - \frac{\phi_\varepsilon(v - y)}{f^\varepsilon(y)}\right| d\varphi\, dz f(dv) f(dv_*)$$
$$\le C_{R,\varepsilon}|x - y|.$$

PROOF. We start with (i) and set $I_\varepsilon(x) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_0^\infty\int_0^{2\pi} |c(v, v_*, z, \varphi)| \frac{\phi_\varepsilon(v-x)}{f^\varepsilon(x)}\, d\varphi\, dz f(dv) f(dv_*)$. Using (1.8) and (1.5), we see that $|c(v, v_*, z, \varphi)| \le G(z/|v - v_*|^\gamma)|v - v_*| \le C(1 + z/|v - v_*|^\gamma)^{-1/\nu}|v - v_*|$. Hence

$$\begin{aligned} I_\varepsilon(x) &\le C \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_0^\infty \big(1 + z/|v - v_*|^\gamma\big)^{-1/\nu} \\ &\qquad \times |v - v_*| \frac{\phi_\varepsilon(v - x)}{f^\varepsilon(x)}\, dz f(dv) f(dv_*) \\ &= C \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v - v_*|^{1+\gamma} \frac{\phi_\varepsilon(v - x)}{f^\varepsilon(x)} f(dv) f(dv_*). \end{aligned}$$

Using now that $|v - v_*|^{1+\gamma} \le 1 + |v| + |v_*|$, we find

$$\begin{aligned} I_\varepsilon(x) &\le C \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \big(1 + |v| + |v_*|\big) \frac{\phi_\varepsilon(v - x)}{f^\varepsilon(x)} f(dv) f(dv_*) \\ &\le C\left(1 + \sqrt{m_2(f)} + \frac{\int_{\mathbb{R}^3} |v| \phi_\varepsilon(v - x) f(dv)}{f^\varepsilon(x)}\right). \end{aligned}$$

To conclude the proof of (i), it remains to study $J_\varepsilon(x) = (f^\varepsilon(x))^{-1}\int_{\mathbb{R}^3}|v|\phi_\varepsilon(v-x)f(dv)$. We introduce $L := \sqrt{2m_2(f)}$, for which $f(B(0,L)) \geq 1/2$ [because $f(B(0,L)^c) \leq m_2(f)/L^2$]. Using that $\{v \in \mathbb{R}^3 : |v| \leq 2|x| + L\} \cup \{v \in \mathbb{R}^3 : |v-x| \geq |x| + L\} = \mathbb{R}^3$, we write

$$J_\varepsilon(x) = \frac{\int_{\mathbb{R}^3}|v|\phi_\varepsilon(v-x)f(dv)}{\int_{\mathbb{R}^3}\phi_\varepsilon(v-x)f(dv)} \leq 2|x| + L + \frac{\int_{|v-x|\geq|x|+L}|v|\phi_\varepsilon(v-x)f(dv)}{\int_{|v-x|\leq|x|+L}\phi_\varepsilon(v-x)f(dv)}.$$

Since ϕ_ε is radial and decreasing,

$$\int_{|v-x|\geq|x|+L}|v|\phi_\varepsilon(v-x)f(dv) \leq \phi_\varepsilon\big(|x|+L\big)\sqrt{m_2(f)}$$

and

$$\int_{|v-x|\leq|x|+L}\phi_\varepsilon(v-x)f(dv) \geq \phi_\varepsilon\big(|x|+L\big)f\big(B\big(x,|x|+L\big)\big) \geq \phi_\varepsilon\big(|x|+L\big)/2$$

owing to the fact that $B(0,L) \subset B(x,|x|+L)$. Hence, $J_\varepsilon(x) \leq 2|x| + L + 2\sqrt{m_2(f)} \leq 2|x| + 4\sqrt{m_2(f)}$ and this completes the proof of (i).

For point (ii), we set $\Delta_\varepsilon(x,y) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_0^\infty\int_0^{2\pi}|c(v,v_*,z,\varphi)||F_\varepsilon(x,v) - F_\varepsilon(y,v)|\,d\varphi\,dz f(dv)f(dv_*)$, where $F_\varepsilon(v,x) := (f^\varepsilon(x))^{-1}\phi_\varepsilon(v-x)$. Exactly as in point (i), we start with

$$\begin{aligned}\Delta_\varepsilon(x,y) &\leq C\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-v_*|^{1+\gamma}\big|F_\varepsilon(v,x) - F_\varepsilon(v,y)\big|f(dv)f(dv_*)\\ &\leq C\int_{\mathbb{R}^3}\big(1+\sqrt{m_2(f)}+|v|\big)\big|F_\varepsilon(v,x) - F_\varepsilon(v,y)\big|f(dv)\\ &\leq C|x-y|\int_{\mathbb{R}^3}\big(1+\sqrt{m_2(f)}+|v|\big)\Big(\sup_{a\in B(0,R)}\big|\nabla_x F_\varepsilon(v,a)\big|\Big)f(dv)\end{aligned}$$

for all $x, y \in B(0,R)$. But we have

$$\nabla_x F_\varepsilon(v,a) = \frac{1}{\varepsilon}\frac{\phi_\varepsilon(v-a)\int_{\mathbb{R}^3}(v-u)\phi_\varepsilon(u-a)f(du)}{(f^\varepsilon(a))^2}. \tag{3.8}$$

Indeed, recalling that $\phi_\varepsilon(x) = (2\pi\varepsilon)^{-3/2}e^{-|x|^2/(2\varepsilon)}$, we observe that

$$\nabla_x\phi_\varepsilon(v-x) = \frac{1}{\varepsilon}(v-x)\phi_\varepsilon(v-x)$$

and

$$\nabla_x f^\varepsilon(x) = \frac{1}{\varepsilon}\int_{\mathbb{R}^3}\phi_\varepsilon(u-x)(u-x)f(du).$$

Since $F_\varepsilon(v,a) := (f^\varepsilon(a))^{-1}\phi_\varepsilon(v-a)$, we have

$$\begin{aligned}
&\nabla_x F_\varepsilon(v,a)\\
&\quad= \frac{\nabla_x\phi_\varepsilon(v-a) f^\varepsilon(a) - \phi_\varepsilon(v-a)\nabla_x f^\varepsilon(a)}{(f^\varepsilon(a))^2}\\
&\quad= \frac{\phi_\varepsilon(v-a)}{\varepsilon}\frac{(v-a)f^\varepsilon(a) - \int_{\mathbb{R}^3}\phi_\varepsilon(u-a)(u-a)f(du)}{(f^\varepsilon(a))^2}\\
&\quad= \frac{\phi_\varepsilon(v-a)}{\varepsilon}\frac{\int_{\mathbb{R}^3}\phi_\varepsilon(u-a)(v-a)f(du) - \int_{\mathbb{R}^3}\phi_\varepsilon(u-a)(u-a)f(du)}{(f^\varepsilon(a))^2},
\end{aligned}$$

whence (3.8). Using now that $J_\varepsilon(a) = (f^\varepsilon(a))^{-1}\int_{\mathbb{R}^3}|u|\phi_\varepsilon(u-a)f(du) \le 2|a| + 4\sqrt{m_2(f)}$ as proved in (i),

$$\begin{aligned}
\left|\nabla_x F_\varepsilon(v,a)\right| &\le \frac{1}{\varepsilon}\frac{\phi_\varepsilon(v-a)}{f^\varepsilon(a)}\frac{\int_{\mathbb{R}^3}(|v|+|u|)\phi_\varepsilon(u-a)f(du)}{f^\varepsilon(a)}\\
&\le \frac{1}{\varepsilon}\frac{\phi_\varepsilon(v-a)}{f^\varepsilon(a)}\bigl(|v| + 2|a| + 4\sqrt{m_2(f)}\bigr).
\end{aligned}$$

But we know that $\phi_\varepsilon(x) \le (2\pi\varepsilon)^{-3/2}$ and that

$$\begin{aligned}
f^\varepsilon(a) &\ge \int_{|v-a|\le|a|+L}\phi_\varepsilon(v-a)f(dv) \ge \phi_\varepsilon\bigl(|a|+L\bigr)f\bigl(B\bigl(a,|a|+L\bigr)\bigr)\\
&\ge \phi_\varepsilon\bigl(|a|+L\bigr)/2
\end{aligned}$$

since $B(0,L)\subset B(a,|a|+L)$. Hence,

$$\sup_{a\in B(0,R)}\left|\nabla_x F_\varepsilon(v,a)\right| \le \frac{2}{\varepsilon}e^{(R+L)^2/(2\varepsilon)}\bigl(|v| + 2R + 4\sqrt{m_2(f)}\bigr).$$

Consequently, for all $x, y \in B(0,R)$,

$$\begin{aligned}
\Delta_\varepsilon(x,y) &\le \frac{2C}{\varepsilon}e^{(R+L)^2/(2\varepsilon)}\\
&\qquad\times|x-y|\int_{\mathbb{R}^3}\bigl(1+\sqrt{m_2(f)}+|v|\bigr)\bigl(|v|+2R+4\sqrt{m_2(f)}\bigr)f(dv)\\
&\le C_{R,\varepsilon}|x-y|,
\end{aligned}$$

where $C_{R,\varepsilon}$ depends only on R,ε and $m_2(f)$ [recall that $L := \sqrt{2m_2(f)}$]. □

Finally, we end the section with the following.

PROOF OF PROPOSITION 3.1. We consider any given weak solution $(\tilde{f}_t)_{t\ge0} \in L^\infty([0,\infty),\mathcal{P}_2(\mathbb{R}^3))$ to (1.1) and we write the proof in several steps.

Step 1. We introduce $\phi_\varepsilon(x) = (2\pi\varepsilon)^{-3/2} e^{-|x|^2/(2\varepsilon)}$ and $\tilde{f}_t^\varepsilon(w) = (\tilde{f}_t * \phi_\varepsilon)(w)$. For each $t \geq 0$, we see that $\tilde{f}_t^\varepsilon$ is a positive smooth function. We claim that for any $\psi \in \mathrm{Lip}(\mathbb{R}^3)$,

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^3} \psi(w) \tilde{f}_t^\varepsilon(dw) = \int_{\mathbb{R}^3} \tilde{f}_t^\varepsilon(dw) \tilde{\mathcal{A}}_{t,\varepsilon} \psi(w),$$

where

$$\begin{aligned} \tilde{\mathcal{A}}_{t,\varepsilon}\psi(w) = &\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_0^\infty\int_0^{2\pi} \big[\psi\big(w + c(v, v_*, z, \varphi)\big) - \psi(w)\big] \\ &\times \frac{\phi_\varepsilon(v-w)}{\tilde{f}_t^\varepsilon(w)}\, d\varphi\, dz\, \tilde{f}_t(dv_*)\, \tilde{f}_t(dv). \end{aligned} \tag{3.9}$$

Indeed, $\tilde{f}_t^\varepsilon(w) = \int_{\mathbb{R}^3} \phi_\varepsilon(v-w)\tilde{f}_t(dv)$ since $\phi_\varepsilon(x)$ is even. According to (1.10) and (2.3), we have

$$\begin{aligned} \frac{\partial}{\partial t}\tilde{f}_t^\varepsilon(w) = &\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_0^\infty\int_0^{2\pi} \big[\phi_\varepsilon\big(v - w \\ &+ c(v, v_*, z, \varphi)\big) - \phi_\varepsilon(v-w)\big]\, d\varphi\, dz\, \tilde{f}_t(dv_*)\, \tilde{f}_t(dv) \\ = &\int_{\mathbb{R}^3}\int_0^K\int_0^{2\pi}\bigg[\int_{\mathbb{R}^3}\phi_\varepsilon\big(v-w \\ &+ c(v, v_*, z, \varphi)\big)\tilde{f}_t(dv) - \tilde{f}_t^\varepsilon(w)\bigg]\, d\varphi\, dz\, \tilde{f}_t(dv_*) \\ &+ \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_K^\infty\int_0^{2\pi}\big[\phi_\varepsilon\big(v-w \\ &+ c(v, v_*, z, \varphi)\big) - \phi_\varepsilon(v-w)\big]\, d\varphi\, dz\, \tilde{f}_t(dv_*)\, \tilde{f}_t(dv) \end{aligned}$$

for any $K \geq 1$. We thus have, for any $\psi \in \mathrm{Lip}(\mathbb{R}^3)$,

$$\begin{aligned} &\frac{\partial}{\partial t}\int_{\mathbb{R}^3}\psi(w)\tilde{f}_t^\varepsilon(dw) \\ &\quad = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_0^K\int_0^{2\pi}\int_{\mathbb{R}^3}\phi_\varepsilon\big(v-w \\ &\quad\quad + c(v, v_*, z, \varphi)\big)\psi(w)\tilde{f}_t(dv)\, d\varphi\, dz\, \tilde{f}_t(dv_*)\, dw \\ &\quad\quad - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_0^K\int_0^{2\pi}\psi(w)\tilde{f}_t^\varepsilon(w)\, d\varphi\, dz\, \tilde{f}_t(dv_*)\, dw \\ &\quad\quad + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_K^\infty\int_0^{2\pi}\big[\phi_\varepsilon\big(v-w+c(v, v_*, z, \varphi)\big) \\ &\quad\quad - \phi_\varepsilon(v-w)\big]\psi(w)\, d\varphi\, dz\, \tilde{f}_t(dv_*)\, \tilde{f}_t(dv)\, dw. \end{aligned}$$

Using the change of variables $w - c(v, v_*, z, \varphi) \mapsto w$, we see that the first integral of the RHS equals

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_0^K\int_0^{2\pi}\int_{\mathbb{R}^3}\phi_\varepsilon(v-w)\psi\big(w+c(v,v_*,z,\varphi)\big)\tilde{f}_t(dv)\,d\varphi\,dz\,\tilde{f}_t(dv_*)\,dw.$$

Consequently,

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{\mathbb{R}^3}\psi(w)\tilde{f}_t^\varepsilon(dw)\\
&\quad=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_0^K\int_0^{2\pi}\bigg[\int_{\mathbb{R}^3}\psi\big(w+c(v,v_*,z,\varphi)\big)\frac{\phi_\varepsilon(v-w)}{\tilde{f}_t^\varepsilon(w)}\tilde{f}_t(dv)\\
&\qquad-\psi(w)\bigg]\tilde{f}_t^\varepsilon(w)\,d\varphi\,dz\,\tilde{f}_t(dv_*)\,dw\\
&\qquad+\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_K^\infty\int_0^{2\pi}\big[\phi_\varepsilon\big(v-w+c(v,v_*,z,\varphi)\big)\\
&\qquad-\phi_\varepsilon(v-w)\big]\psi(w)\,d\varphi\,dz\,\tilde{f}_t(dv_*)\,\tilde{f}_t(dv)\,dw\\
&\quad=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_0^K\int_0^{2\pi}\int_{\mathbb{R}^3}\big[\psi\big(w+c(v,v_*,z,\varphi)\big)\\
&\qquad-\psi(w)\big]\frac{\phi_\varepsilon(v-w)}{\tilde{f}_t^\varepsilon(w)}\tilde{f}_t(dv)\,d\varphi\,dz\,\tilde{f}_t(dv_*)\,\tilde{f}_t^\varepsilon(dw)\\
&\qquad+\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_K^\infty\int_0^{2\pi}\big[\phi_\varepsilon\big(v-w+c(v,v_*,z,\varphi)\big)\\
&\qquad-\phi_\varepsilon(v-w)\big]\psi(w)\,d\varphi\,dz\,\tilde{f}_t(dv_*)\,\tilde{f}_t(dv)\,dw.
\end{aligned}$$

Letting K increase to infinity, one easily ends the step.

Step 2. We set $F_{t,\varepsilon}(v,x) = (\tilde{f}_t^\varepsilon(x))^{-1}\phi_\varepsilon(v-x)$. For a given X_0^ε with law $\tilde{f}_0^\varepsilon$, and a given independent Poisson measure $N(ds, dv, dv_*, dz, d\varphi, du)$ on $[0,\infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \times [0,\infty) \times [0, 2\pi) \times [0,\infty)$ with intensity $ds\,\tilde{f}_s(dv)\,\tilde{f}_s(dv_*)\,dz\,d\varphi\,du$, there exists a pathwise unique solution to

$$\begin{aligned}
X_t^\varepsilon = X_0^\varepsilon + \int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_0^\infty\int_0^{2\pi}\int_0^\infty c(v,v_*,z,\varphi)\\
\times \mathbf{1}_{\{u\le F_{s,\varepsilon}(v,X_{s-}^\varepsilon)\}}N(ds,dv,dv_*,dz,d\varphi,du).
\end{aligned}\tag{3.10}$$

This classically follows from Lemma 3.4, which precisely tells us that the coefficients of this equation satisfy some *at most linear growth condition* [point (i)] and some *local Lipschitz condition* [point (ii)].

Step 3. We now prove that $\mathcal{L}(X_t^\varepsilon) = \tilde{f}_t^\varepsilon$ for each $t \geq 0$. We thus introduce $g_t^\varepsilon = \mathcal{L}(X_t^\varepsilon)$. By the Itô formula we see that for all $\psi \in \mathrm{Lip}(\mathbb{R}^3)$,

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{\mathbb{R}^3} \psi(w) g_t^\varepsilon(dw) \\
&\quad = \int_{\mathbb{R}^3} g_t^\varepsilon(dw) \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_0^\infty \int_0^{2\pi} \bigl(\psi\bigl(w + c(v, v_*, z, \varphi)\bigr) \\
&\qquad - \psi(w)\bigr) F_{t,\varepsilon}(v, w)\, d\varphi\, dz\, \tilde{f}_t(dv_*)\tilde{f}_t(dv) \\
&\quad = \int_{\mathbb{R}^3} g_t^\varepsilon(dw)\tilde{\mathcal{A}}_{t,\varepsilon}\psi(w).
\end{aligned}$$

Thus, $(\tilde{f}_t^\varepsilon)_{t\geq 0}$ and $(g_t^\varepsilon)_{t\geq 0}$ satisfy the same equation and we of course have $g_0^\varepsilon = \tilde{f}_0^\varepsilon$ by construction. The following uniqueness result allows us to conclude the step: for any $\mu_0 \in \mathcal{P}_2(\mathbb{R}^3)$, there exists at most one family $(\mu_t) \in L^\infty_{\mathrm{loc}}([0,\infty), \mathcal{P}_2(\mathbb{R}^3))$ such that for any $\psi \in \mathrm{Lip}(\mathbb{R}^3)$, any $t \geq 0$,

$$\int_{\mathbb{R}^3} \psi(w)\mu_t(dw) = \int_{\mathbb{R}^3} \psi(w)\mu_0(dw) + \int_0^t ds \int_{\mathbb{R}^3} \mu_s(dw)\tilde{\mathcal{A}}_{s,\varepsilon}\psi(w). \tag{3.11}$$

This must be classical (as well as Step 2 is), but we find no precise reference, and thus make use of martingale problems. A càdlàg adapted $\mathbb{R}^3$-valued process $(Z_t)_{t\geq 0}$ on some filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is said to solve the martingale problem $\mathrm{MP}(\tilde{\mathcal{A}}_{t,\varepsilon}, \mu_0, \mathrm{Lip}(\mathbb{R}^3))$ if $\mathbb{P} \circ Z_0 = \mu_0$ and if for all $\psi \in \mathrm{Lip}(\mathbb{R}^3)$, $(M_t^{\psi,\varepsilon})_{t\geq 0}$ is a $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$-martingale, where

$$M_t^{\psi,\varepsilon} = \psi(Z_t) - \int_0^t \tilde{\mathcal{A}}_{s,\varepsilon}\psi(Z_s)\, ds.$$

According to [3], Theorem 5.2 (see also [3], Remark 3.1, Theorem 5.1, and [20], Theorem B.1), it suffices to check the following points to conclude the uniqueness for (3.11):

(i) there exists a countable family $(\psi_k)_{k\geq 1} \subset \mathrm{Lip}(\mathbb{R}^3)$ such that for all $t \geq 0$, the closure (for the bounded pointwise convergence) of $\{(\psi_k, \tilde{\mathcal{A}}_{t,\varepsilon}\psi_k), k \geq 1\}$ contains $\{(\psi, \tilde{\mathcal{A}}_{t,\varepsilon}\psi), \psi \in \mathrm{Lip}(\mathbb{R}^3)\}$,

(ii) for each $w_0 \in \mathbb{R}^3$, there exists a solution to $\mathrm{MP}(\tilde{\mathcal{A}}_{t,\varepsilon}, \delta_{w_0}, \mathrm{Lip}(\mathbb{R}^3))$,

(iii) for each $w_0 \in \mathbb{R}^3$, uniqueness (in law) holds for $\mathrm{MP}(\tilde{\mathcal{A}}_{t,\varepsilon}, \delta_{w_0}, \mathrm{Lip}(\mathbb{R}^3))$.

The fact that (3.10) has a pathwise unique solution proved in Step 2 (there we can of course replace X_0^ε by any deterministic point $w_0 \in \mathbb{R}^3$) immediately implies (ii) and (iii). Point (i) is very easy (recall that $\varepsilon > 0$ is fixed here).

Step 4. In this step, we check that the family $((X_t^\varepsilon)_{t\geq 0})_{\varepsilon>0}$ is tight in $\mathbb{D}([0,\infty), \mathbb{R}^3)$. To do this, we use the Aldous criterion [1]; see also [22], p. 321, that is, it suffices to prove that for all $T > 0$,

$$\sup_{\varepsilon\in(0,1)} \mathbb{E}\Bigl[\sup_{[0,T]}|X_t^\varepsilon|\Bigr] < \infty, \qquad \lim_{\delta\to 0}\sup_{\varepsilon\in(0,1)}\sup_{S,S'\in\mathcal{S}_T(\delta)} \mathbb{E}\bigl[|X_{S'}^\varepsilon - X_S^\varepsilon|\bigr] = 0, \tag{3.12}$$

where $\mathcal{S}_T(\delta)$ is the set containing all pairs of stopping times (S, S') satisfying $0 \le S \le S' \le S + \delta \le T$.

First, since $X_t^\varepsilon \sim \tilde{f}_t^\varepsilon = \tilde{f}_t \star \phi_\varepsilon$, we have $\mathbb{E}[|X_t^\varepsilon|^2] \le 2(m_2(\tilde{f}_t) + 3\varepsilon) \le 2m_2(\tilde{f}_0) + 6$. Thus, for any $T > 0$, using Lemma 3.4(i),

$$\begin{aligned}
\mathbb{E}\Big[\sup_{[0,T]} |X_t^\varepsilon|\Big] &\le \mathbb{E}[|X_0^\varepsilon|] + \mathbb{E}\bigg[\int_0^T \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_0^\infty \int_0^{2\pi} |c(v, v_*, z, \varphi)| \\
&\qquad \times \frac{\phi_\varepsilon(v - X_s^\varepsilon)}{\tilde{f}_s^\varepsilon(X_s^\varepsilon)}\, d\varphi\, dz\, \tilde{f}_s(dv)\, \tilde{f}_s(dv_*)\, ds\bigg] \\
&\le \mathbb{E}[|X_0^\varepsilon|] + C\mathbb{E}\bigg[\int_0^T (1 + |X_s^\varepsilon|)\, ds\bigg] \le C_T.
\end{aligned}$$

Furthermore, for any $T > 0$, $\delta > 0$ and $(S, S') \in \mathcal{S}_T(\delta)$, using again Lemma 3.4(i),

$$\begin{aligned}
\mathbb{E}[|X_{S'}^\varepsilon - X_S^\varepsilon|] &\le \mathbb{E}\bigg[\int_S^{S+\delta} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_0^\infty \int_0^{2\pi} |c(v, v_*, z, \varphi)| \\
&\qquad \times \frac{\phi_\varepsilon(v - X_s^\varepsilon)}{\tilde{f}_s^\varepsilon(X_s^\varepsilon)}\, d\varphi\, dz\, \tilde{f}_s(dv)\, \tilde{f}_s(dv_*)\, ds\bigg] \\
&\le C\mathbb{E}\bigg[\int_S^{S+\delta} (1 + |X_s^\varepsilon|)\, ds\bigg] \\
&\le C\mathbb{E}\Big[\delta \sup_{[0,T]} (1 + |X_s^\varepsilon|)\Big] \\
&\le C_T \delta.
\end{aligned}$$

Hence, (3.12) holds true and this completes the step.

Step 5. We thus can find some $(X_t)_{t\ge0}$ which is the limit in law (for the Skorokhod topology) of a sequence $(X_t^{\varepsilon_n})_{t\ge0}$ with $\varepsilon_n \searrow 0$. Since $\mathcal{L}(X_t^{\varepsilon_n}) = \tilde{f}_t^{\varepsilon_n}$ by Step 3 and since $\tilde{f}_t^{\varepsilon_n} \to \tilde{f}_t$ by definition, we have $\mathcal{L}(X_t) = \tilde{f}_t$ for each $t \ge 0$. It only remains to show that $(X_t)_{t\ge0}$ is a (weak) solution to (3.1). Using the theory of martingale problems (see Jacod [21], Theorem 13.55), it classically suffices to prove that for any $\psi \in C_b^1(\mathbb{R}^3)$, the process $\psi(X_t) - \psi(X_0) - \int_0^t \mathcal{B}_s \psi(X_s)\, ds$ is a martingale, where

$$\mathcal{B}_t \psi(x) = \int_0^1 \int_0^\infty \int_0^{2\pi} \big(\psi\big(x + c\big(x, X_t^*(\alpha), z, \varphi\big)\big) - \psi(x)\big)\, d\varphi\, dz\, d\alpha.$$

But since $\mathcal{L}_\alpha(X_t^*) = \tilde{f}_t$, this rewrites [recall (2.3)]

$$\begin{aligned}
\mathcal{B}_t \psi(x) &= \int_{\mathbb{R}^3} \int_0^\infty \int_0^{2\pi} \big(\psi\big(x + c(x, v_*, z, \varphi)\big) - \psi(x)\big)\, d\varphi\, dz\, \tilde{f}_t(dv_*) \\
&= \int_{\mathbb{R}^3} \mathcal{A}\psi(x, v_*)\, \tilde{f}_t(dv_*).
\end{aligned}$$

We thus have to prove that for any $0 \le s_1 \le \cdots \le s_k \le s \le t \le T$, any $\psi_1, \dots, \psi_k \in C_b^1(\mathbb{R}^3)$, and any $\psi \in C_b^1(\mathbb{R}^3)$,

$$\mathbb{E}[\mathcal{F}(X)] = 0,$$

where $\mathcal{F} : \mathbb{D}([0,\infty), \mathbb{R}^3) \mapsto \mathbb{R}$ is defined by

$$\mathcal{F}(\lambda) = \left(\prod_{i=1}^k \psi_i(\lambda_{s_i})\right)\left(\psi(\lambda_t) - \psi(\lambda_s) - \int_s^t \mathcal{B}_r \psi(\lambda_r)\,dr\right).$$

We of course start from $\mathbb{E}[\mathcal{F}_{\varepsilon_n}(X^{\varepsilon_n})] = 0$, where, recalling (3.9),

$$\mathcal{F}_\varepsilon(\lambda) = \left(\prod_{i=1}^k \psi_i(\lambda_{s_i})\right)\left(\psi(\lambda_t) - \psi(\lambda_s) - \int_s^t \tilde{\mathcal{A}}_{r,\varepsilon} \psi(\lambda_r)\,dr\right).$$

We then write

$$\left|\mathbb{E}[\mathcal{F}(X)]\right| \le \left|\mathbb{E}[\mathcal{F}(X)] - \mathbb{E}[\mathcal{F}(X^{\varepsilon_n})]\right| + \left|\mathbb{E}[\mathcal{F}(X^{\varepsilon_n})] - \mathbb{E}[\mathcal{F}_{\varepsilon_n}(X^{\varepsilon_n})]\right|.$$

On the one hand, we know from [8], Lemma 3.3, that $(x, v_*) \mapsto \mathcal{A}\psi(x, v_*)$ is continuous on $\mathbb{R}^3 \times \mathbb{R}^3$ and bounded by $C|x - v_*|^{\gamma+1}$. We thus easily deduce that $\mathcal{F}$ is continuous at each $\lambda \in \mathbb{D}([0,\infty), \mathbb{R}^3)$ which does not jump at $s_1, \dots, s_k, s, t$ [this is a.s. the case of $X \in \mathbb{D}([0,\infty), \mathbb{R}^3)$ because it has no deterministic time jump by the Aldous criterion]. We also deduce that $|\mathcal{F}(\lambda)| \le C(1 + \int_0^t \int_{\mathbb{R}^3} |\lambda_r - v_*|^{\gamma+1} \tilde{f}_r(dv_*)\,dr)$. Using that $0 < \gamma + 1 < 1$, that $\sup_{\varepsilon \in (0,1)} \mathbb{E}[\sup_{[0,T]} |X_t^\varepsilon|] < \infty$ by Step 4 and recalling that X^{ε_n} goes in law to X, we easily conclude that $|\mathbb{E}[\mathcal{F}(X)] - \mathbb{E}[\mathcal{F}(X^{\varepsilon_n})]|$ tends to 0 as $n \to \infty$.

On the other hand, since $|\mathcal{F}(\lambda) - \mathcal{F}_\varepsilon(\lambda)| \le C|\int_s^t (\mathcal{B}_r\psi(\lambda_r) - \tilde{\mathcal{A}}_{r,\varepsilon}\psi(\lambda_r))\,dr|$ and $X_r^\varepsilon \sim \tilde{f}_r^\varepsilon$,

$$\begin{aligned}
&\left|\mathbb{E}[\mathcal{F}(X^{\varepsilon_n})] - \mathbb{E}[\mathcal{F}_{\varepsilon_n}(X^{\varepsilon_n})]\right| \\
&\quad \le C \int_s^t \mathbb{E}\biggl[\biggl|\int_{\mathbb{R}^3}\int_0^\infty\int_0^{2\pi}\int_{\mathbb{R}^3} \psi\bigl(X_r^{\varepsilon_n} + c(v, v_*, z, \varphi)\bigr) \\
&\qquad \times \left[\frac{\phi_{\varepsilon_n}(v - X_r^{\varepsilon_n})}{\tilde{f}_r^{\varepsilon_n}(X_r^{\varepsilon_n})}\tilde{f}_r(dv) - \boldsymbol{\delta}_{X_r^{\varepsilon_n}}(dv)\right] d\varphi\,dz\,\tilde{f}_r(dv_*)\biggr|\biggr]\,dr \\
&\quad = C \int_s^t \biggl|\int_{\mathbb{R}^3}\int_0^\infty\int_0^{2\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \psi\bigl(w + c(v, v_*, z, \varphi)\bigr) \\
&\qquad \times \bigl[\phi_{\varepsilon_n}(v - w)\tilde{f}_r(dv) - \tilde{f}_r^{\varepsilon_n}(w)\boldsymbol{\delta}_w(dv)\bigr]\,dw\,d\varphi\,dz\,\tilde{f}_r(dv_*)\biggr|\,dr.
\end{aligned}$$

But we can write $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \psi(w + c(v, v_*, z, \varphi))\tilde{f}_r^{\varepsilon_n}(w)\boldsymbol{\delta}_w(dv)\,dw = \int_{\mathbb{R}^3} \psi(w + c(w, v_*, z, \varphi))\tilde{f}_r^{\varepsilon_n}(w)\,dw = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \psi(w + c(w, v_*, z, \varphi))\phi_{\varepsilon_n}(v - w)\tilde{f}_r(dv)\,dw$, so that

$$
\begin{aligned}
&\big|\mathbb{E}[\mathcal{F}(X^{\varepsilon_n})]-\mathbb{E}[\mathcal{F}_{\varepsilon_n}(X^{\varepsilon_n})]\big| \\
&\quad\le C\int_s^t\bigg|\int_{\mathbb{R}^3}\int_0^\infty\int_0^{2\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\big[\psi\big(w+c(v,v_*,z,\varphi)\big) \\
&\qquad -\psi\big(w+c(w,v_*,z,\varphi)\big)\big]\phi_{\varepsilon_n}(v-w)\,\tilde f_r(dv)\,dw\,d\varphi\,dz\,\tilde f_r(dv_*)\bigg|\,dr \\
&\quad = C\int_s^t\bigg|\int_{\mathbb{R}^3}\int_0^\infty\int_0^{2\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\big[\psi\big(w+c\big(v,v_*,z,\varphi \\
&\qquad +\varphi_0(v-v_*,w-v_*)\big)\big) \\
&\qquad -\psi\big(w+c(w,v_*,z,\varphi)\big)\big]\phi_{\varepsilon_n}(v-w)\,\tilde f_r(dv)\,dw\,d\varphi\,dz\,\tilde f_r(dv_*)\bigg|\,dr.
\end{aligned}
$$

The last equality uses the 2π-periodicity of c. We now put

$$
\begin{aligned}
R_n(v,v_*,z,\varphi) := \int_{\mathbb{R}^3}&\big[\psi\big(w+c\big(v,v_*,z,\varphi+\varphi_0(v-v_*,w-v_*)\big)\big) \\
&-\psi\big(w+c(w,v_*,z,\varphi)\big)\big]\phi_{\varepsilon_n}(v-w)\,dw,
\end{aligned}
$$

and show the following two things:

(a) for all $v,v_*\in\mathbb{R}^3$, all $z\in[0,\infty)$ and $\varphi\in[0,2\pi)$, $\lim_{n\to\infty}R_n(v,v_*,z,\varphi)=0$;

(b) there is a constant $C>0$ such that for all $n\ge 1$, all $v,v_*\in\mathbb{R}^3$, all $z\in[0,\infty)$ and $\varphi\in[0,2\pi)$,

$$
\big|R_n(v,v_*,z,\varphi)\big|\le C\big(1+|v-v_*|\big)(1+z)^{-1/\nu},
$$

which belongs to $L^1([0,T]\times\mathbb{R}^3\times\mathbb{R}^3\times[0,\infty)\times[0,2\pi), dr\,\tilde f_r(dv_*)\,\tilde f_r(dv)\,dz\,d\varphi)$ because $(\tilde f_t)_{t\ge 0}\in L^\infty([0,T],\mathcal{P}_2(\mathbb{R}^3))$ by assumption.

By dominated convergence, we will deduce that $\lim_{n\to\infty}|\mathbb{E}[\mathcal{F}(X^{\varepsilon_n})]-\mathbb{E}[\mathcal{F}_{\varepsilon_n}(X^{\varepsilon_n})]|=0$ and this will conclude the proof.

We first study (a). Since $\psi\in C_b^1(\mathbb{R}^3)$, we immediately observe that

$$
\begin{aligned}
&\big|\psi\big(w+c\big(v,v_*,z,\varphi+\varphi_0(v-v_*,w-v_*)\big)\big) \\
&\quad -\psi\big(w+c(w,v_*,z,\varphi)\big)\big| \\
&\qquad \le C_\psi\big|c\big(v,v_*,z,\varphi+\varphi_0(v-v_*,w-v_*)\big)-c(w,v_*,z,\varphi)\big|.
\end{aligned}
\tag{3.13}
$$

Recalling that

$$
\begin{aligned}
c(v,v_*,z,\varphi) = &-\frac{1-\cos G(z/|v-v_*|^\gamma)}{2}(v-v_*) \\
&+\frac{\sin G(z/|v-v_*|^\gamma)}{2}\Gamma(v-v_*,\varphi),
\end{aligned}
$$

we have

$$
\begin{aligned}
&\big|c\big(v, v_*, z, \varphi+\varphi_0(v-v_*, w-v_*)\big)-c(w, v_*, z, \varphi)\big| \\
&\quad\le \frac{|\cos G(z/|v-v_*|^\gamma)-\cos G(z/|w-v_*|^\gamma)|}{2}|v-v_*| \\
&\qquad+\frac{|1-\cos G(z/|w-v_*|^\gamma)|}{2}|v-w| \\
&\qquad+\frac{|\sin G(z/|v-v_*|^\gamma)-\sin G(z/|w-v_*|^\gamma)|}{2}\big|\Gamma(v-v_*, \varphi+\varphi_0)\big| \\
&\qquad+\frac{|\sin G(z/|w-v_*|^\gamma)|}{2}\big|\Gamma(v-v_*, \varphi+\varphi_0)-\Gamma(w-v_*, \varphi)\big|.
\end{aligned}
$$

Using that $|\Gamma(v-v_*, \varphi+\varphi_0)|=|v-v_*|$ and Lemma 2.2, we obtain

$$
\begin{aligned}
&\big|c\big(v, v_*, z, \varphi+\varphi_0(v-v_*, w-v_*)\big)-c(w, v_*, z, \varphi)\big| \\
&\quad\le C\big|G\big(z/|v-v_*|^\gamma\big)-G\big(z/|w-v_*|^\gamma\big)\big||v-v_*|+C|v-w|.
\end{aligned}
$$

We deduce from (1.4) that $|G'(z)|=1/\beta(G(z))\le C$ by (1.3), whence

$$
\begin{aligned}
&\big|c\big(v, v_*, z, \varphi+\varphi_0(v-v_*, w-v_*)\big)-c(w, v_*, z, \varphi)\big| \\
&\quad\le Cz\big||v-v_*|^{|\gamma|}-|w-v_*|^{|\gamma|}\big||v-v_*|+C|v-w|.
\end{aligned}
$$

Using again the inequality $|x^\alpha-y^\alpha|\le|x-y|(x\vee y)^{\alpha-1}$ for $\alpha\in(0,1)$, and $x, y\ge 0$, we have

$$
\big||v-v_*|^{|\gamma|}-|w-v_*|^{|\gamma|}\big|\le|v-w||v-v_*|^{|\gamma|-1}.
$$

We thus get

$$
\begin{aligned}
&\big|c\big(v, v_*, z, \varphi+\varphi_0(v-v_*, w-v_*)\big)-c(w, v_*, z, \varphi)\big| \\
&\quad\le C\big(z|v-v_*|^{|\gamma|}+1\big)|v-w|.
\end{aligned}
$$

Consequently,

$$
R_n(v, v_*, z, \varphi)\le C_\psi\big(z|v-v_*|^{|\gamma|}+1\big)\int_{\mathbb{R}^3}|v-w|\phi_{\varepsilon_n}(v-w)\,dw,
$$

which clearly tends to 0 as $n\to\infty$. This completes the proof of (a).

For (b), start again from (3.13) to write

$$
\begin{aligned}
&\big|\psi\big(w+c\big(v, v_*, z, \varphi+\varphi_0(v-v_*, w-v_*)\big)\big)-\psi\big(w+c(w, v_*, z, \varphi)\big)\big| \\
&\quad\le\big|\psi\big(w+c(v, v_*, z, \varphi)\big)-\psi(w)\big|+\big|\psi(w)-\psi\big(w+c(w, v_*, z, \varphi)\big)\big| \\
&\quad\le C_\psi\big(\big|c(v, v_*, z, \varphi)\big|+\big|c(w, v_*, z, \varphi)\big|\big).
\end{aligned}
$$

Moreover, since $|c(v,v_*,z,\varphi)| \le G(z/|v-v_*|^\gamma)|v-v_*| \le C|v-v_*|(1+|v-v_*|^{|\gamma|}z)^{-1/\nu}$ by (1.8) and (1.5), we observe that

$$
\begin{aligned}
&R_n(v,v_*,z,\varphi)\\
&\qquad \le C|v-v_*|\bigl(1+|v-v_*|^{|\gamma|}z\bigr)^{-1/\nu}\\
&\qquad\quad + C\int_{\mathbb{R}^3}|w-v_*|\bigl(1+|w-v_*|^{|\gamma|}z\bigr)^{-1/\nu}\phi_{\varepsilon_n}(v-w)\,dw.
\end{aligned}
$$

Since $1+|v-v_*|^{|\gamma|}z \ge (1\wedge|v-v_*|^{|\gamma|})(1+z)$ for $z\in[0,\infty)$,

$$|v-v_*|\bigl(1+|v-v_*|^{|\gamma|}z\bigr)^{-1/\nu} \le |v-v_*|(1+z)^{-1/\nu}\bigl(1\wedge|v-v_*|^{|\gamma|}\bigr)^{-1/\nu}.$$

Using that $|\gamma|/\nu<1$, we deduce that

$$|v-v_*|\bigl(1+|v-v_*|^{|\gamma|}z\bigr)^{-1/\nu} \le \bigl(1+|v-v_*|\bigr)(1+z)^{-1/\nu}.$$

As a conclusion,

$$
\begin{aligned}
&R_n(v,v_*,z,\varphi)\\
&\qquad \le C\biggl(1+|v-v_*|+\int_{\mathbb{R}^3}|w-v_*|\phi_{\varepsilon_n}(v-w)\,dw\biggr)(1+z)^{-1/\nu},
\end{aligned}
$$

which is easily bounded [recall that $\varepsilon_n\in(0,1)$] by $C(1+|v|+|v_*|)(1+z)^{-1/\nu}$ as desired. $\square$

4. The coupling.

4.1. *Main ideas of the proof of Theorem* 1.4. The proof of Theorem 1.4 is very technical, so let us exhibit the main ideas. We consider the unique *strong* solution $(f_t)_{t\ge0}$ to (1.1) given in Theorem 1.2. We first couple $(W_t^1,\dots,W_t^N)_{t\ge0}$ [i.i.d. copies of $(W_t)_{t\ge0}$ solution to the SDE associated to $(f_t)_{t\ge0}$] and the Nanbu particle system $(V_t^1,\dots,V_t^N)_{t\ge0}$ in such a way that, roughly, as soon as possible, each time W_t^i has a jump $c(W_{t-}^i,W_t^*(\alpha),z,\varphi)$, V_t^i also has a jump $c_K(V_{t-}^i,V_t^j,z,\varphi)$ with V_t^j as close as possible to $W_t^*(\alpha)$. So, we construct a coupling between $W_t^*(\alpha)$ (with law f_t) and V_t^j (with law $\mu_t^{N,K}$) in Lemma 4.2 as Fournier–Mischler [14]; see also [7]. Unfortunately, there are many problems: we have to use in a complicated way the function φ_0 of Lemma 2.2, and to use an intermediate coupling between the empirical measure of the V_t^i's and the W_t^i's.

To get the convergence rate, we roughly apply the stability principle in Theorem 1.3, and find that $\mathcal{W}_2^2(\mu_t^{N,K},\mu_{\mathbf{W}_t}^N)$ should be bounded by (some natural error terms)$\times\exp(C_{\gamma,p}\int_0^t(1+\|\mu_{\mathbf{W}_s}^N\|_{L^p})\,ds)$, but it is not correct since the empirical measure does not have a finite L^p norm. We thus consider a regularized version (i.e., $\bar\mu_{\mathbf{W}_t}^N=\mu_{\mathbf{W}_t}^N*\psi_{\varepsilon_N}$), with a small parameter ε_N. Here, $\psi_\varepsilon=(3/(4\pi\varepsilon^3))\mathbf{1}_{\{|x|\le\varepsilon\}}$. This introduces some additional error terms, but we are able to bound, uniformly in N, the L^p-norm of $\bar{\mu}^N_{\mathbf{W}_t}$. This is difficult, but not surprising. Indeed, it is well known from statistics that, if $(X_1, \dots, X_N)$ are i.i.d. with density $g \in L^p$, then $\|\frac{1}{N}\sum_{i=1}^N \delta_{X_i} * \psi_{\varepsilon_N}\|_{L^p} \le 2\|g\|_{L^p}$ with high probability if ε_N is well chosen. So for each fixed $t \ge 0$, we apply such a principle, but we need to get something similar (locally) uniformly in time. For this, we use some continuity properties of the W^i_t's, and again this is complicated since they are only càdlàg.

Now we have all this in mind, we realize that we also need to take into account the regularization (by convolution with ψ_{ε_N}) when introducing the coupling between the V^i_t's and the W^i_t's.

4.2. *The coupling*. To get the convergence of the particle system, we construct a suitable coupling between the particle system with generator $\mathcal{L}_{N,K}$ defined by (2.4) and the realization of the weak solution to (1.1), following the ideas of [14].

LEMMA 4.1. *Assume* (1.3) *for some* $\gamma \in (-1, 0)$, $\nu \in (0, 1)$ *with* $\gamma + \nu \in (0, 1)$. *Let* $N \ge 1$ *be fixed. Let* $q \ge 2$ *such that* $q > \gamma^2/(\gamma + \nu)$. *Let* $f_0 \in \mathcal{P}_q(\mathbb{R}^3)$ *with a finite entropy and let* $(f_t)_{t\ge0} \in L^\infty([0,\infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{\text{loc}}([0,\infty), L^p(\mathbb{R}^3))$ [*with* $p \in (3/(3+\gamma), p_0(\gamma, \nu, q))$] *be the unique weak solution to* (1.1) *given by Theorem* 1.2. *Then there exists*, *on some probability space*, *a family of i.i.d. random variables* $(V^i_0)_{i=1,\dots,N}$ *with common law* f_0, *independent of a family of i.i.d. Poisson measures* $(M_i(ds, d\alpha, dz, d\varphi))_{i=1,\dots,N}$ *on* $[0,\infty) \times [0,1] \times [0,\infty) \times [0,2\pi)$, *with intensity* $ds\, d\alpha\, dz\, d\varphi$, *a measurable family* $(W^*_t)_{t\ge0}$ *of* α-*random variables with* α-*law* $(f_t)_{t\ge0}$ *and* N *i.i.d. càdlàg adapted processes* $(W^i_t)_{t\ge0}$ *solving*, *for each* $i = 1, \dots, N$,

$$W^i_t = V^i_0 + \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} c\big(W^i_{s-}, W^*_s(\alpha), z, \varphi\big) M_i(ds, d\alpha, dz, d\varphi). \tag{4.1}$$

Moreover, $W^i_t \sim f_t$ *for all* $t \ge 0$, *all* $i = 1, \dots, N$. *Also*, *for all* $T > 0$,

$$\mathbb{E}\Big[\sup_{[0,T]} |W^1_t|^q\Big] \le C_{T,q}. \tag{4.2}$$

PROOF. Except for the moment estimate (4.2), it suffices to apply Proposition 3.1. A simpler proof could be handled here because we deal with the *strong* solution $f \in L^\infty([0,\infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{\text{loc}}([0,\infty), L^p(\mathbb{R}^3))$. We now prove (4.2), which is more or less classical. We thus fix $q \ge 2$. It is clear that

$$\begin{aligned}
&\big||v + c(v, v_*, z, \varphi)|^q - |v|^q\big| \\
&\qquad \le C_q\big(|v|^{q-1} + |c(v, v_*, z, \varphi)|^{q-1}\big)|c(v, v_*, z, \varphi)|.
\end{aligned}$$

Due to (1.8) and (1.5), $|c(v, v_*, z, \varphi)| \le |v - v_*|$, $|c(v, v_*, z, \varphi)| \le (1 + z/|v - v_*|^\gamma)^{-1/\nu}|v - v_*|$, whence

$$\begin{aligned}
&\int_0^\infty \int_0^{2\pi} \big||v + c(v, v_*, z, \varphi)|^q - |v|^q\big|\, d\varphi\, dz \\
&\quad \le C_q \int_0^\infty \int_0^{2\pi} \big(1 + |v|^{q-1} + |v_*|^{q-1}\big) \\
&\qquad \times \big(1 + z/|v - v_*|^\gamma\big)^{-1/\nu}|v - v_*|\, d\varphi\, dz \\
&\quad = C_q\big(1 + |v|^{q-1} + |v_*|^{q-1}\big)|v - v_*|^{1+\gamma} \\
&\quad \le C_q\big(1 + |v|^q + |v_*|^q\big),
\end{aligned} \tag{4.3}$$

because $0 < 1 + \gamma < 1$. It then easily follows from the Itô formula and $\mathcal{L}_\alpha(W_t^*) = f_t = \mathcal{L}(W_t^1)$ that

$$\begin{aligned}
\mathbb{E}\Big[\sup_{[0,t]}|W_s^1|^q\Big] &\le \mathbb{E}\big[|V_0^1|^q\big] + C_q \int_0^t \int_0^1 \mathbb{E}\big[1 + |W_s^1|^q + |W_s^*(\alpha)|^q\big]\, d\alpha\, ds \\
&\le \mathbb{E}\big[|V_0^1|^q\big] + C_q \int_0^t \Big(1 + \mathbb{E}\Big[\sup_{[0,s]}|W_u^1|^q\Big]\Big)\, ds.
\end{aligned}$$

We thus conclude (4.2) by the Grönwall lemma. □

Next, let us recall [14], Lemma 4.3, below in order to construct our coupling.

LEMMA 4.2. *Consider $(f_t)_{t\ge0}$ and $(W_t^*)_{t\ge0}$ introduced in Lemma 4.1 and fix $N \ge 1$. For $\mathbf{v} = (v_1, v_2, \dots, v_N) \in (\mathbb{R}^3)^N$, we introduce the empirical measure $\mu_{\mathbf{v}}^N := N^{-1}\sum_{i=1}^N \delta_{v_i}$. Then for all $t \ge 0$, all $\mathbf{v} \in (\mathbb{R}^3)^N$ and all $\mathbf{w} \in (\mathbb{R}^3)_\bullet^N$, with $(\mathbb{R}^3)_\bullet^N := \{\mathbf{w} \in (\mathbb{R}^3)^N : w_i \ne w_j\ \forall i \ne j\}$, there are α-random variables $Z_t^*(\mathbf{w}, \alpha)$ and $V_t^*(\mathbf{v}, \mathbf{w}, \alpha)$ such that the α-law of $(Z_t^*(\mathbf{w}, \cdot), V_t^*(\mathbf{v}, \mathbf{w}, \cdot))$ is $N^{-1}\sum_{i=1}^N \delta_{(w_i, v_i)}$ and $\int_0^1 |W_t^*(\alpha) - Z_t^*(\mathbf{w}, \alpha)|^2\, d\alpha = \mathcal{W}_2^2(f_t, \mu_{\mathbf{w}}^N)$.*

REMARK 4.3. We know from [7] and the fact that f_t has a density for each $t \ge 0$ that the map $(t, \mathbf{v}, \mathbf{w}, \alpha) \mapsto (Z_t^*(\mathbf{w}, \alpha), V_t^*(\mathbf{v}, \mathbf{w}, \alpha))$ can be chosen measurable.

Observe that $\mathcal{L}_\alpha(Z_t^*(\mathbf{w}, \cdot)) = \mu_{\mathbf{w}}^N$ and $\mathcal{L}_\alpha(V_t^*(\mathbf{v}, \mathbf{w}, \cdot)) = \mu_{\mathbf{v}}^N$ for all fixed $t \ge 0$, $\mathbf{v} \in (\mathbb{R}^3)^N$ and $\mathbf{w} \in (\mathbb{R}^3)_\bullet^N$. No regularity of $Z_t^*(\mathbf{w}, \alpha)$ or $V_t^*(\mathbf{v}, \mathbf{w}, \alpha)$ is required in any of their variables.

Owing to technical reasons, we need to introduce some more notation.

NOTATION 4.4. We consider an α-random variable Y with uniform distribution on $B(0, 1)$ (independent of everything else) and, for $\varepsilon \in (0, 1)$, $t \ge 0$, $\alpha \in [0, 1]$, $\mathbf{v} \in (\mathbb{R}^3)^N$ and $\mathbf{w} \in (\mathbb{R}^3)_\bullet^N$, we set $W_t^{*,\varepsilon}(\alpha) = W_t^*(\alpha) + \varepsilon Y(\alpha)$ and $V_t^{*,\varepsilon}(\mathbf{v}, \mathbf{w}, \alpha) = V_t^*(\mathbf{v}, \mathbf{w}, \alpha) + \varepsilon Y(\alpha)$. It holds that $\mathcal{L}_\alpha(W_t^{*,\varepsilon}) = f_t * \psi_\varepsilon$ and $\mathcal{L}_\alpha(V_t^{*,\varepsilon}(\mathbf{v}, \mathbf{w}, \cdot)) = \mu_{\mathbf{v}}^N * \psi_\varepsilon$, where $\psi_\varepsilon(x) = (3/(4\pi\varepsilon^3))\mathbf{1}_{\{|x|\le\varepsilon\}}$.

At last, we built a suitable realisation for the particle system.

LEMMA 4.5. *Consider all the objects introduced in Lemmas 4.1–4.2 and Notation 4.4. Set* $\mathbf{W}_s = (W_s^1, \dots, W_s^N)$, *which a.s. belongs to* $(\mathbb{R}^3)^N_\bullet$ *(because* f_s *has a density for all* $s \geq 0$*). Fix* $K \geq 1$ *and* $\varepsilon \in (0, 1)$*. There is a unique strong solution* $(\mathbf{V}_t)_{t\geq 0} = (V_t^1, \dots, V_t^N)_{t\geq 0}$ *to*

(4.4)
$$V_t^i = V_0^i + \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} c_K\bigl(V_{s-}^i, V_s^*(\mathbf{V}_{s-}, \mathbf{W}_{s-}, \alpha), z, \varphi + \varphi_{i,\alpha,s}\bigr) M_i(ds, d\alpha, dz, d\varphi), \qquad i = 1, \dots, N,$$

where $\varphi_{i,\alpha,s} := \varphi^1_{i,\alpha,s} + \varphi^2_{i,\alpha,s} + \varphi^3_{i,\alpha,s}$ *with*

$$\begin{aligned}
\varphi^1_{i,\alpha,s} &= \varphi_0\bigl(W_{s-}^i - W_s^*(\alpha), W_{s-}^i - W_s^{*,\varepsilon}(\alpha)\bigr),\\
\varphi^2_{i,\alpha,s} &= \varphi_0\bigl(W_{s-}^i - W_s^{*,\varepsilon}(\alpha), V_{s-}^i - V_s^{*,\varepsilon}(\mathbf{V}_{s-}, \mathbf{W}_{s-}, \alpha)\bigr),\\
\varphi^3_{i,\alpha,s} &= \varphi_0\bigl(V_{s-}^i - V_s^{*,\varepsilon}(\mathbf{V}_{s-}, \mathbf{W}_{s-}, \alpha), V_{s-}^i - V_s^*(\mathbf{V}_{s-}, \mathbf{W}_{s-}, \alpha)\bigr).
\end{aligned}$$

Moreover, $(\mathbf{V}_t)_{t\geq 0}$ *is a Markov process with generator* $\mathcal{L}_{N,K}$*. If* $f_0 \in \mathcal{P}_q(\mathbb{R}^3)$ *for some* $q \geq 2$*, then* $\mathbb{E}[\sup_{[0,T]} |V_t^1|^q] \leq C_{T,q}$ *[this last constant not depending on* N*,* K *nor* $\varepsilon \in (0, 1)$*].*

PROOF. Since $c_K = c\mathbf{1}_{\{z\leq K\}}$, the Poisson measures involved in (4.4) are finite. Hence, the existence and uniqueness results hold for (4.4). Next, we check that $(\mathbf{V}_t)_{t\geq 0}$ is a Markov process with generator $\mathcal{L}_{N,K}$: for all bounded measurable function $\phi : (\mathbb{R}^3)^N \mapsto \mathbb{R}$, all $t \geq 0$, a.s.,

$$\begin{aligned}
&\sum_{i=1}^N \int_0^1 \int_0^\infty \int_0^{2\pi} \bigl[\phi\bigl(\mathbf{v} + c_K\bigl(v_i, V_t^*(\mathbf{v}, \mathbf{w}, \alpha), z, \varphi + \varphi_{i,\alpha,t}\bigr)\mathbf{e}_i\bigr) - \phi(\mathbf{v})\bigr]\, d\varphi\, dz\, d\alpha\\
&\quad = \sum_{i=1}^N \int_0^1 \int_0^\infty \int_0^{2\pi} \bigl[\phi\bigl(\mathbf{v} + c_K\bigl(v_i, V_t^*(\mathbf{v}, \mathbf{w}, \alpha), z, \varphi\bigr)\mathbf{e}_i\bigr) - \phi(\mathbf{v})\bigr]\, d\varphi\, dz\, d\alpha\\
&\quad = \sum_{i=1}^N N^{-1} \sum_{j=1}^N \int_0^\infty \int_0^{2\pi} \bigl[\phi\bigl(\mathbf{v} + c_K(v_i, v_j, z, \varphi)\mathbf{e}_i\bigr) - \phi(\mathbf{v})\bigr]\, d\varphi\, dz\\
&\quad = N^{-1} \sum_{i\neq j} \int_0^\infty \int_0^{2\pi} \bigl[\phi\bigl(\mathbf{v} + c_K(v_i, v_j, z, \varphi)\mathbf{e}_i\bigr) - \phi(\mathbf{v})\bigr]\, d\varphi\, dz.
\end{aligned}$$

This is nothing but $\mathcal{L}_{N,K}\phi(\mathbf{v})$, recall Lemma 2.1. We used the 2π-periodicity of c_K in φ for the first equality, that $\mathcal{L}_\alpha(V_t^*(\mathbf{v}, \mathbf{w}, \cdot)) = \mu^N_{\mathbf{v}}$ for the second one, and that $c_K(v_i, v_i, z, \varphi) = 0$ for the last one.

Finally, we verify that $\sup_{[0,T]}\mathbb{E}[|V_t^1|^q] \le C_{T,q}$ if $f_0 \in \mathcal{P}_q(\mathbb{R}^3)$ for some $q \ge 2$: it immediately follows from the Itô formula, (4.3) and exchangeability that

$$\begin{aligned}\mathbb{E}[|V_t^1|^q] &\le \mathbb{E}[|V_0^1|^q] + C_q \int_0^t \int_0^1 \mathbb{E}[1 + |V_s^1|^q + |V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha)|^q]\, d\alpha\, ds \\ &\le \mathbb{E}[|V_0^1|^q] + C_q N^{-1} \sum_{i=1}^N \int_0^t \mathbb{E}[1 + |V_s^1|^q + |V_s^i|^q]\, ds \\ &\le \mathbb{E}[|V_0^1|^q] + C_q \int_0^t \mathbb{E}[1 + |V_s^1|^q]\, ds.\end{aligned}$$

The Grönwall lemma allows us to complete the proof. □

REMARK 4.6. The exchangeability holds for the family $\{(W_t^i, V_t^i)_{t\ge 0}, i = 1, \dots, N\}$. Indeed, the family $\{(W_t^i)_{t\ge 0}, i = 1, \dots, N\}$ is i.i.d. by construction, so that the exchangeability follows from the symmetry and pathwise uniqueness for (4.4).

5. Bound in L^p of a blob approximation of an empirical measure.

An empirical measure cannot be in some L^p space with $p > 1$, so we will consider a blob approximation, inspired by Proposition 5.5 in [11] and [19]. But we deal with a jump process, so we need to overcome a few additional difficulties.

First, the following fact can be checked as Lemma 5.3 in [11] (the norm and the step of the subdivision are different, but this obviously changes nothing).

LEMMA 5.1. *Let $p \in (1, 2)$ and $(f_t)_{t\ge 0} \in L^\infty([0,\infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{\mathrm{loc}}([0,\infty), L^p(\mathbb{R}^3))$ such that $m_2(f_t) = m_2(f_0)$ for all $t \ge 0$:*

(i) *There is a constant $M_p > 0$, such that for all $t \ge 0$, $\|f_t\|_{L^p} \ge M_p$.*

(ii) *For any $T > 0$, we can find a subdivision $(t_\ell^N)_{\ell=0}^{K_N+1}$ satisfying $0 = t_0^N < t_1^N < \cdots < t_{K_N}^N \le T \le t_{K_N+1}^N$, such that $\sup_{\ell=0,\dots,K_N}(t_{\ell+1}^N - t_\ell^N) \le N^{-2}$ with $K_N \le 2TN^2$ and*

$$\int_0^T h_N(t)\, dt \le 2 \int_0^T \|f_t\|_{L^p}\, dt,$$

with $h_N(t) = \sum_{\ell=1}^{K_N+1} \|f_{t_\ell^N}\|_{L^p} \mathbf{1}_{\{t\in(t_{\ell-1}^N, t_\ell^N]\}}$.

The goal of the section is to prove the following crucial fact.

PROPOSITION 5.2. *Assume (1.3) for some $\gamma \in (-1, 0)$, $\nu \in (0, 1)$ with $\gamma + \nu > 0$. Let $q \ge 2$ such that $q > \gamma^2/(\gamma + \nu)$ and let $p \in (3/(3+\gamma), p_0(\gamma, \nu, q)) \subset (1, 3/2)$. Consider $f_0 \in \mathcal{P}_q(\mathbb{R}^3)$ with a finite entropy and $(f_t)_{t\ge 0} \in L^\infty([0,\infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{\mathrm{loc}}([0,\infty), L^p(\mathbb{R}^3))$ the corresponding unique solution to (1.1) given* *by Theorem 1.2. Consider* $(W_t^i)_{i=1,\dots,N,t\geq 0}$ *the solution to* (4.1) *and set* $\mu^N_{\mathbf{W}_t} = N^{-1}\sum_1^N \delta_{W_t^i}$. *Fix* $\delta \in (0,1)$, *set* $\varepsilon_N = N^{-(1-\delta)/3}$ *and define* $\bar{\mu}^N_{\mathbf{W}_t} = \mu^N_{\mathbf{W}_t} * \psi_{\varepsilon_N}$, *where* ψ_ε *was defined in Notation 4.4. Finally, fix* $T > 0$ *and consider* h_N *built in Lemma 5.1. We have*

$$\mathbb{P}\big(\forall t \in [0,T], \|\bar{\mu}^N_{\mathbf{W}_t}\|_{L^p} \leq 13{,}500\big(1 + h_N(t)\big)\big) \geq 1 - C_{T,q,\delta} N^{1-\delta q/3}.$$

Throughout the section, we fix $N \geq 1$, $\delta \in (0,1)$, and $\varepsilon_N = N^{-(1-\delta)/3}$ and adopt the assumptions and notation of Proposition 5.2. We also put $r = p/(p-1)$.

In order to extend Proposition 5.5 in [11], it is necessary to study some properties of the paths of the processes defined by (4.1). Following Lemma 3.11 in [35], we introduce, for each $i = 1,\dots,N$,

$$\widetilde{W}_t^i = V_0^i + \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} c\big(W_{s-}^i, W_s^*(\alpha), z, \varphi\big) \times \mathbf{1}_{\{|c(W_{s-}^i, W_s^*(\alpha), z,\varphi)| \leq N^{-1/3}\}} M_i(ds, d\alpha, dz, d\varphi). \tag{5.1}$$

LEMMA 5.3. *For all* $T > 0$,

$$\mathbb{P}\Big[\sup_{[0,T]} |W_t^1| \leq N^{\delta/3}, \sup_{s,t\in[0,T],|s-t|\leq N^{-2}} \big|\widetilde{W}_t^1 - \widetilde{W}_s^1\big| \geq \varepsilon_N\Big] \leq C_T N^2 e^{-N^{\delta/3}}.$$

PROOF. Let us denote by $\tilde{p}$ the probability we want to bound.

Step 1. We introduce

$$Z_t^1 = \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} G\big(z/|W_{s-}^1 - W_s^*(\alpha)|^\gamma\big)\big|W_{s-}^1 - W_s^*(\alpha)\big| \times \mathbf{1}_{\{G(z/|W_{s-}^1 - W_s^*(\alpha)|^\gamma)|W_{s-}^1 - W_s^*(\alpha)|/4 \leq N^{-1/3}\}} M_1(ds, d\alpha, dz, d\varphi).$$

It is clear that Z_t^1 is almost surely increasing in t, and that a.s., for all $s,t \in [0,T]$,

$$\big|\widetilde{W}_t^1 - \widetilde{W}_s^1\big| \leq \big|Z_t^1 - Z_s^1\big|, \tag{5.2}$$

since for any $v, v_* \in \mathbb{R}^3$ [recall (1.8)]

$$G\big(z/|v-v_*|^\gamma\big)|v-v_*|/4 \leq \big|c(v,v_*,z,\varphi)\big| \leq G\big(z/|v-v_*|^\gamma\big)|v-v_*|.$$

We now consider the stopping time $\tau_N = \inf\{t \geq 0 : |W_t^1| > N^{\delta/3}\}$ and deduce from (5.2) and the Markov inequality that

$$\begin{aligned}\tilde{p} &\leq \mathbb{P}\Big[\sup_{[0,T]} |W_t^1| \leq N^{\delta/3}, \sup_{s,t\in[0,T],|s-t|\leq N^{-2}} \big|Z_t^1 - Z_s^1\big| \geq \varepsilon_N\Big] \\ &\leq \mathbb{P}\Big[\sup_{s,t\in[0,T],|s-t|\leq N^{-2}} \big|Z_{t\wedge\tau_N}^1 - Z_{s\wedge\tau_N}^1\big| \geq \varepsilon_N\Big].\end{aligned}$$

Since $[0,T]\subset\bigcup_{k=0}^{\lfloor N^2T\rfloor}[k/N^2,(k+1)/N^2)$ and Z_t^N is almost surely increasing in t, we deduce that on $\{\sup_{s,t\in[0,T],|s-t|\le N^{-2}}|Z^1_{t\wedge\tau_N}-Z^1_{s\wedge\tau_N}|\ge\varepsilon_N\}$, there exists $k\in\{0,1,\dots,\lfloor N^2T\rfloor\}$ for which there holds $(Z^1_{((k+1)N^{-2})\wedge\tau_N}-Z^1_{(kN^{-2})\wedge\tau_N})\ge\varepsilon_N/3$. Hence,

$$\begin{aligned}\tilde p&\le\sum_{k=0}^{\lfloor N^2T\rfloor}\mathbb{P}\Big[\big(Z^1_{((k+1)N^{-2})\wedge\tau_N}-Z^1_{(kN^{-2})\wedge\tau_N}\big)\ge\frac{\varepsilon_N}{3}\Big]\\&\le\sum_{k=0}^{\lfloor N^2T\rfloor}e^{-N^{\delta/3}}\mathbb{E}\big[\exp\{3N^{1/3}(Z^1_{((k+1)N^{-2})\wedge\tau_N}-Z^1_{(kN^{-2})\wedge\tau_N})\}\big]\\&=:\sum_{k=0}^{\lfloor N^2T\rfloor}e^{-N^{\delta/3}}I_k.\end{aligned}$$

Step 2. We now prove that I_k is (uniformly) bounded, which will complete the proof. We put

$$J_k(t)=:\mathbb{E}\big[\exp\{3N^{1/3}(Z^1_{(t+kN^{-2})\wedge\tau_N}-Z^1_{(kN^{-2})\wedge\tau_N})\}\big].$$

It is obvious that $I_k=J_k(N^{-2})$. Applying the Itô formula, we find

$$\begin{aligned}J_k(t)=1+2\pi\mathbb{E}\Big[&\int_{(kN^{-2})\wedge\tau_N}^{(t+kN^{-2})\wedge\tau_N}\int_0^1\int_0^\infty\exp\{3N^{1/3}(Z^1_s-Z^1_{(kN^{-2})\wedge\tau_N})\}\\&\times\big(e^{3N^{1/3}G(z/|W^1_s-W^*_s(\alpha)|^\gamma)|W^1_s-W^*_s(\alpha)|}-1\big)\\&\times\mathbf{1}_{\{G(z/|W^1_s-W^*_s(\alpha)|^\gamma)|W^1_s-W^*_s(\alpha)|/4\le N^{-1/3}\}}\,dz\,d\alpha\,ds\Big].\end{aligned}$$

Since $3N^{1/3}G(z/|W^1_s-W^*_s(\alpha)|^\gamma)|W^1_s-W^*_s(\alpha)|\le 12$ (thanks to the indicator function), we have

$$\begin{aligned}&e^{3N^{1/3}G(z/|W^1_s-W^*_s(\alpha)|^\gamma)|W^1_s-W^*_s(\alpha)|}-1\\&\qquad\le CN^{1/3}G\big(z/\big|W^1_s-W^*_s(\alpha)\big|^\gamma\big)\big|W^1_s-W^*_s(\alpha)\big|\end{aligned}$$

for a positive constant C. Then using (1.5), we see that

$$\mathbf{1}_{\{G(z/|W^1_s-W^*_s(\alpha)|^\gamma)|W^1_s-W^*_s(\alpha)|/4\le N^{-1/3}\}}\le\mathbf{1}_{\{z\ge CN^{\nu/3}|W^1_s-W^*_s(\alpha)|^{\gamma+\nu}-|W^1_s-W^*_s(\alpha)|^\gamma\}}.$$

Hence,

$$\begin{aligned}J_k(t)\le 1+CN^{1/3}\mathbb{E}\Big[&\int_{(kN^{-2})\wedge\tau_N}^{(t+kN^{-2})\wedge\tau_N}\int_0^1\int_0^\infty\exp\{3N^{1/3}(Z^1_s-Z^1_{(kN^{-2})\wedge\tau_N})\}\\&\times\big(1+z/\big|W^1_s-W^*_s(\alpha)\big|^\gamma\big)^{-1/\nu}\big|W^1_s-W^*_s(\alpha)\big|\\&\times\mathbf{1}_{\{z\ge CN^{\nu/3}|W^1_s-W^*_s(\alpha)|^{\gamma+\nu}-|W^1_s-W^*_s(\alpha)|^\gamma\}}\,dz\,d\alpha\,ds\Big].\end{aligned}$$

But, we have

$$\begin{aligned}
&\left|W_s^1 - W_s^*(\alpha)\right| \int_0^\infty \left(1 + z/\left|W_s^1 - W_s^*(\alpha)\right|^\gamma\right)^{-1/\nu} \\
&\quad \times \mathbf{1}_{\{z \geq CN^{\nu/3}|W_s^1 - W_s^*(\alpha)|^{\gamma+\nu} - |W_s^1 - W_s^*(\alpha)|^\gamma\}}\, dz \\
&\quad = CN^{-(1-\nu)/3}\left|W_s^1 - W_s^*(\alpha)\right|^{\nu+\gamma} \\
&\quad \leq CN^{-(1-\nu)/3}\left(1 + \left|W_s^1\right|^2 + \left|W_s^*(\alpha)\right|^2\right)
\end{aligned}$$

since $\gamma + \nu \in (0, 1)$. Using now that $\int_0^1 |W_s^*(\alpha)|^2\, d\alpha = m_2(f_0)$ and that $|W_s^1| \leq N^{\delta/3}$ for all $s \leq \tau_N$, we conclude that

$$\begin{aligned}
J_k(t) &\leq 1 + CN^{\nu/3}\left(1 + m_2(f_0) + N^{2\delta/3}\right) \int_0^t J_k(s)\, ds \\
&\leq 1 + CN^{(\nu+2\delta)/3} \int_0^t J_k(s)\, ds.
\end{aligned}$$

It follows from the Grönwall lemma that $J_k(t) \leq \exp\left(CN^{(\nu+2\delta)/3} t\right)$, and thus that $I_k = J_k(N^{-2})$ is uniformly bounded, because $(\nu + 2\delta)/3 < 2$ [recall that $\nu \in (0, 1)$ and $\delta \in (0, 1)$]. □

Next, we study the *large* jumps of $(W_t^1)_{t\geq 0}$.

LEMMA 5.4. *There exists $C > 0$ such that for any $\ell \in \{1, \ldots, K_N + 1\}$,*

$$\mathbb{P}\left[\exists t \in \left(t_{\ell-1}^N, t_\ell^N\right] : \left|\Delta W_t^1\right| > N^{-1/3}\right] \leq CN^{-(2-\nu/3)}.$$

PROOF. Let us fix ℓ and set $A = \{\exists t \in (t_{\ell-1}^N, t_\ell^N] : |\Delta W_t^1| > N^{-1/3}\}$. After noting that

$$A = \left\{\int_{t_{\ell-1}^N}^{t_\ell^N} \int_0^1 \int_0^\infty \int_0^{2\pi} \mathbf{1}_{\{|c(W_{s-}^i, W_s^*(\alpha), z, \varphi)| > N^{-1/3}\}} M_1(ds, d\alpha, dz, d\varphi) \geq 1\right\},$$

we have

$$\mathbb{P}(A) \leq \mathbb{E}\left[\int_{t_{\ell-1}^N}^{t_\ell^N} \int_0^1 \int_0^\infty \int_0^{2\pi} \mathbf{1}_{\{|c(W_{s-}^1, W_s^*(\alpha), z, \varphi)| > N^{-1/3}\}} M_1(ds, d\alpha, dz, d\varphi)\right]$$

by the Markov inequality. But (1.8) and (1.5) tell us that $|c(v, v_*, z, \varphi)| \leq C(1 + z/|v - v_*|^\gamma)^{-1/\nu}|v - v_*|$. Hence,

$$\begin{aligned}
\mathbb{P}(A) &\leq 2\pi\mathbb{E}\left[\int_{t_{\ell-1}^N}^{t_\ell^N} \int_0^1 \int_0^\infty \mathbf{1}_{\{C(1+z/|W_s^1 - W_s^*(\alpha)|^\gamma)^{-1/\nu}|W_s^1 - W_s^*(\alpha)| > N^{-1/3}\}}\, dz\, d\alpha\, ds\right] \\
&\leq 2\pi\mathbb{E}\left[\int_{t_{\ell-1}^N}^{t_\ell^N} \int_0^1 \int_0^\infty \mathbf{1}_{\{z < CN^{\nu/3}|W_s^1 - W_s^*(\alpha)|^{\gamma+\nu}\}}\, dz\, d\alpha\, ds\right]
\end{aligned}$$

$$= CN^{\nu/3}\mathbb{E}\bigg[\int_{t_{\ell-1}^N}^{t_\ell^N}\int_0^1 \big|W_s^1 - W_s^*(\alpha)\big|^{\gamma+\nu}\,d\alpha\,ds\bigg].$$

Finally, using that $|W_s^1 - W_s^*(\alpha)|^{\gamma+\nu} \le 1 + |W_s^1|^2 + |W_s^*(\alpha)|^2$ and that $\int_0^1 |W_s^*(\alpha)|^2\,d\alpha = \mathbb{E}[|W_s^1|^2] < \infty$, we conclude that $\mathbb{P}(A) \le CN^{\nu/3}(t_{\ell+1}^N - t_\ell^N) \le CN^{\nu/3-2}$ as desired. $\square$

LEMMA 5.5. *For $\ell = 1, \dots, K_N + 1$, we introduce*

$$I_\ell = \big\{i \in \{1,\dots,N\} : \exists t \in \big(t_{\ell-1}^N, t_\ell^N\big] \text{ such that } \big|\Delta W_t^i\big| > N^{-1/3}\big\}, \tag{5.3}$$

and the event

$$\begin{aligned}\Omega_{T,N}^1 = \Big\{&\forall i \in \{1,\dots,N\}, \sup_{[0,T]}\big|W_t^i\big| \le N^{\delta/3} \text{ and} \\ &\sup_{s,t\in[0,T],|s-t|\le N^{-2}}\big|\widetilde{W}_t^i - \widetilde{W}_s^i\big| \le \varepsilon_N\Big\} \\ &\cap \big\{\forall \ell = 1,\dots,K_N+1, \#(I_\ell) \le N\varepsilon_N^{3/r}\big\}.\end{aligned}$$

Then we have

$$\mathbb{P}\big[\Omega_{T,N}^1\big] \ge 1 - C_{T,q,\delta}N^{1-q\delta/3}.$$

PROOF. We write $\Omega_{T,N}^1 = \Omega_{T,N}^{1,1} \cap \Omega_{T,N}^{1,2}$, where

$$\begin{aligned}\Omega_{T,N}^{1,1} &:= \Big\{\forall i \in \{1,\dots,N\}, \sup_{[0,T]}\big|W_t^i\big| \le N^{\delta/3} \\ &\qquad \text{and} \sup_{s,t\in[0,T],|s-t|\le N^{-2}}\big|\widetilde{W}_t^i - \widetilde{W}_s^i\big| \le \varepsilon_N\Big\}, \\ \Omega_{T,N}^{1,2} &:= \big\{\forall \ell = 1,\dots,K_N+1, \#(I_\ell) \le N\varepsilon_N^{3/r}\big\}.\end{aligned}$$

Step 1. Here, we estimate $\mathbb{P}[(\Omega_{T,N}^{1,1})^c]$. Using the Markov inequality, (4.2) and Lemma 5.3, we get

$$\begin{aligned}\mathbb{P}\big[(\Omega_{T,N}^{1,1})^c\big] &\le N\mathbb{P}\Big[\Big\{\sup_{[0,T]}\big|W_t^1\big| \le N^{\delta/3} \text{ and } \sup_{|s-t|\le N^{-2}}\big|\widetilde{W}_t^1 - \widetilde{W}_s^1\big| \le \varepsilon_N\Big\}^c\Big] \\ &= N\mathbb{P}\Big[\sup_{[0,T]}\big|W_t^1\big| \ge N^{\delta/3}\Big] \\ &\quad + N\mathbb{P}\Big[\sup_{[0,T]}\big|W_t^1\big| \le N^{\delta/3} \text{ and } \sup_{|s-t|\le N^{-2}}\big|\widetilde{W}_t^1 - \widetilde{W}_s^1\big| \ge \varepsilon_N\Big] \\ &\le N\mathbb{E}\Big[\sup_{[0,T]}\big|W_t^1\big|^q\Big]N^{-q\delta/3} + C_T N^3 e^{-N^{\delta/3}} \le C_{T,q}N^{1-q\delta/3}.\end{aligned}$$

Step 2. We now prove that $\mathbb{P}[(\Omega^{1,2}_{T,N})^c] \le C_T \exp(-N^\delta)$. For any fixed $\ell \in \{1, \dots, K_N+1\}$, we introduce $A^\ell_N = \{\exists t \in (t^N_{\ell-1}, t^N_\ell] : |\Delta W^1_t| > N^{-1/3}\}$. Then we observe that $\#(I_\ell)$ follows a Binomial distribution with parameters N and $\mathbb{P}(A^\ell_N)$. Using again the Markov inequality, we observe that

$$\begin{aligned}\mathbb{P}\big[(\Omega^{1,2}_{T,N})^c\big] &\le \sum_{\ell=1}^{K_N+1} \mathbb{P}\big[\#(I_\ell) \ge N\varepsilon_N^{3/r}\big]\\ &\le \sum_{\ell=1}^{K_N+1} \mathbb{E}\big[\exp(\#(I_\ell))\big]\exp\big(-N\varepsilon_N^{3/r}\big).\end{aligned}$$

But

$$\begin{aligned}\mathbb{E}\big[\exp\big(\#(I_\ell)\big)\big] &= \exp\big(N\log\big(1+(e-1)\mathbb{P}\big(A^\ell_N\big)\big)\big)\\ &\le \exp\big(N(e-1)\mathbb{P}\big(A^\ell_N\big)\big).\end{aligned}$$

Hence,

$$\mathbb{P}\big[(\Omega^{1,2}_{T,N})^c\big] \le \sum_{\ell=1}^{K_N+1} \exp\big(N(e-1)\mathbb{P}\big(A^\ell_N\big)\big)\exp\big(-N\varepsilon_N^{3/r}\big).$$

We know from Lemma 5.4 that $\mathbb{P}(A^\ell_N) \le CN^{-(2-\nu/3)}$, hence $N\mathbb{P}(A^\ell_N) \le CN^{-1+\nu/3} \le C$. We thus deduce that

$$\begin{aligned}\mathbb{P}\big[(\Omega^{1,2}_{T,N})^c\big] &\le C(K_N+1)\exp\big(-N\varepsilon_N^{3/r}\big)\\ &\le C\big(2TN^2+1\big)\exp\big(-N\varepsilon_N^{3/r}\big)\\ &\le C_T\exp\big(-N^\delta\big),\end{aligned}$$

since $N\varepsilon_N^{3/r} = N^{1/p+\delta/r}$ and since $1/p+\delta/r > \delta$. This completes the proof. $\square$

We now give the following.

PROOF OF PROPOSITION 5.2. Consider the partition $\mathscr{P}_N$ of $\mathbb{R}^3$ in cubes with side length ε_N and its subset $\mathscr{P}^\delta_N$ consisting of cubes that have nonempty intersection with $B(0, N^{\delta/3})$. Then we observe that $\#(\mathscr{P}^\delta_N) \le (2(N^{\delta/3}+\varepsilon_N)\varepsilon_N^{-1})^3 \le 64N^\delta\varepsilon_N^{-3} = 64N$. We split the proof into several steps.

Step 1. For $(x_1, \dots, x_N) \in (B(0, N^{\delta/3}))^N$ and $(y_1, \dots, y_N) \in (B(0, N^{\delta/3}))^N$, we set

$$I = \big\{i \in \{1, \dots, N\} : |x_i - y_i| > \varepsilon_N\big\},$$

and denote the empirical measure of $\mathbf{y} = (y_1, \dots, y_N) \in (\mathbb{R}^3)^N$ by $\mu_{\mathbf{y}}^N = N^{-1} \sum_{i=1}^N \delta_{y_i}$. The goal of this step is to show that

$$\begin{aligned}&\|\mu_{\mathbf{y}}^N * \psi_{\varepsilon_N}\|_{L^p}\\&\quad \le \left(\frac{3}{4\pi}\right)^{1/r} \frac{\#(I)}{N\varepsilon_N^{3/r}}\\&\qquad + 3375\Big(N^{-p}\varepsilon_N^{-3(p-1)} \sum_{D\in\mathscr{P}_N^\delta} \big(\#\{i \in \{1,\dots,N\} : x_i \in D\}\big)^p\Big)^{1/p}.\end{aligned}$$

Indeed, recalling that $\psi_\varepsilon(x) = (3/(4\pi\varepsilon^3))\mathbf{1}_{\{|x|\le\varepsilon\}}$, we observe that

$$\begin{aligned}&\mu_{\mathbf{y}}^N * \psi_{\varepsilon_N}(v)\\&\quad = \frac{1}{N}\sum_{i=1}^N \psi_{\varepsilon_N}(v-y_i)\mathbf{1}_{\{|x_i-y_i|>\varepsilon_N\}} + N^{-1}\sum_{i=1}^N \psi_{\varepsilon_N}(v-y_i)\mathbf{1}_{\{|x_i-y_i|\le\varepsilon_N\}}\\&\quad = \frac{1}{N}\sum_{i\in I}\psi_{\varepsilon_N}(v-y_i)\\&\qquad + \frac{3}{4\pi N\varepsilon_N^3}\#\big\{i\in\{1,\dots,N\} : y_i \in B(v,\varepsilon_N), |y_i - x_i| \le \varepsilon_N\big\}\\&\quad \le \frac{1}{N}\sum_{i\in I}\psi_{\varepsilon_N}(v-y_i) + \frac{3}{4\pi N\varepsilon_N^3}\#\big\{i\in\{1,\dots,N\} : x_i \in B(v,2\varepsilon_N)\big\}.\end{aligned}$$

Hence,

$$\begin{aligned}&\mu_{\mathbf{y}}^N * \psi_{\varepsilon_N}(v)\\&\quad \le \frac{1}{N}\sum_{i\in I}\psi_{\varepsilon_N}(v-y_i)\\&\qquad + \frac{3}{4\pi N\varepsilon_N^3}\sum_{D\in\mathscr{P}_N^\delta}\#\big\{i\in\{1,\dots,N\} : x_i\in D\big\}\mathbf{1}_{\{D\cap B(v,2\varepsilon_N)\neq\varnothing\}}.\end{aligned}$$

We then deduce that

$$\begin{aligned}&\|\mu_{\mathbf{y}}^N * \psi_{\varepsilon_N}\|_{L^p}\\&\quad \le \frac{1}{N}\Big\|\sum_{i\in I}\psi_{\varepsilon_N}(\cdot - y_i)\Big\|_{L^p}\\&\qquad + \frac{3}{4\pi N\varepsilon_N^3}\Big\|\sum_{D\in\mathscr{P}_N^\delta}\#\big\{i\in\{1,\dots,N\} : x_i\in D\big\}\mathbf{1}_{\{D\cap B(\cdot,2\varepsilon_N)\neq\varnothing\}}\Big\|_{L^p}.\end{aligned}$$

Since $\|\psi_{\varepsilon_N}(\cdot - y_i)\|_{L^p} = (\frac{3}{4\pi})^{1/r}\varepsilon_N^{-3/r}$, we have

$$\frac{1}{N}\Big\|\sum_{i\in I}\psi_{\varepsilon_N}(\cdot - y_i)\Big\|_{L^p} \le \frac{1}{N}\sum_{i\in I}\|\psi_{\varepsilon_N}(\cdot - y_i)\|_{L^p} \le \left(\frac{3}{4\pi}\right)^{1/r}\frac{\#(I)}{N\varepsilon_N^{3/r}}.$$

On the other hand, let

$$A := \Big\|\sum_{D\in\mathscr{P}_N^\delta}\#\{i\in\{1,\dots,N\}: x_i\in D\}\mathbf{1}_{\{D\cap B(\cdot,2\varepsilon_N)\neq\varnothing\}}\Big\|_{L^p},$$

then

$$\begin{aligned}
A^p &= \int_{\mathbb{R}^3}\Big(\sum_{D\in\mathscr{P}_N^\delta}\#\{i: x_i\in D\}\mathbf{1}_{\{D\cap B(v,2\varepsilon_N)\neq\varnothing\}}\Big)^p\,dv\\
&= \int_{\mathbb{R}^3}\Big(\sum_{D,D'\in\mathscr{P}_N^\delta}\#\{i: x_i\in D\}\#\{i: x_i\in D'\}\\
&\qquad\times\mathbf{1}_{\{D\cap B(v,2\varepsilon_N)\neq\varnothing, D'\cap B(v,2\varepsilon_N)\neq\varnothing\}}\Big)^{p/2}\,dv\\
&\le \int_{\mathbb{R}^3}\sum_{D,D'\in\mathscr{P}_N^\delta}\big(\#\{i: x_i\in D\}\big)^{p/2}\big(\#\{i: x_i\in D'\}\big)^{p/2}\\
&\qquad\times\mathbf{1}_{\{D\cap B(v,2\varepsilon_N)\neq\varnothing, D'\cap B(v,2\varepsilon_N)\neq\varnothing\}}\,dv
\end{aligned}$$

because $p\in(1,2)$. From $x^2+y^2\ge 2xy$ and a symmetry argument, we see that

$$A^p \le \sum_{D\in\mathscr{P}_N^\delta}\big(\#\{i: x_i\in D\}\big)^p\int_{\mathbb{R}^3}\mathbf{1}_{\{D\cap B(v,2\varepsilon_N)\neq\varnothing\}}\sum_{D'\in\mathscr{P}_N^\delta}\mathbf{1}_{\{D'\cap B(v,2\varepsilon_N)\neq\varnothing\}}\,dv.$$

But, for each $v\in\mathbb{R}^3$, $\sum_{D'\in\mathscr{P}_N^\delta}\mathbf{1}_{\{D'\cap B(v,2\varepsilon_N)\neq\varnothing\}} = \#\{D'\in\mathscr{P}_N^\delta: D'\cap B(v,2\varepsilon_N)\neq\varnothing\}\le 5^3$. And for each $D\in\mathscr{P}_N^\delta$, $\{v\in\mathbb{R}^3: D\cap B(v,2\varepsilon_N)\neq\varnothing\}$ is included by a ball of radius $3\varepsilon_N$. Therefore, $\int_{\mathbb{R}^3}\mathbf{1}_{\{D\cap B(v,2\varepsilon_N)\neq\varnothing\}}\,dv\le 4\pi(3\varepsilon_N)^3/3$. Hence,

$$A^p\le\frac{5^3 4\pi(3\varepsilon_N)^3}{3}\sum_{D\in\mathscr{P}_N^\delta}\big(\#\{i: x_i\in D\}\big)^p.$$

Consequently,

$$\begin{aligned}
&\|\mu_{\mathbf{y}}^N*\psi_{\varepsilon_N}(v)\|_{L^p}\\
&\qquad\le\left(\frac{3}{4\pi}\right)^{1/r}\frac{\#(I)}{N\varepsilon_N^{3/r}}+\frac{3}{4\pi N\varepsilon_N^3}A\\
&\qquad\le\left(\frac{3}{4\pi}\right)^{1/r}\frac{\#(I)}{N\varepsilon_N^{3/r}}
\end{aligned}$$

$$+\left(\frac{3}{4\pi}\right)^{1/r}(15)^{3/p}\left(N^{-p}\varepsilon_N^{-3(p-1)}\sum_{D\in\mathscr{P}_N^\delta}\big(\#\{i:x_i\in D\}\big)^p\right)^{1/p}.$$

Since $(15)^{3/p}\leq 15^3=3375$, this ends the step.

Step 2. In this step, we extend the proof of [11], Step 3–Proposition 5.5, to show that there are some constants $C>0$ and $c>0$ (depending on δ and M_p of Lemma 5.1) such that for all fixed $t\in[0,T+1]$,

$$\mathbb{P}\big[\big(\Omega_{t,N}^2\big)^c\big]\leq C\exp\big(-cN^{\delta/r}\big),$$

where

$$\Omega_{t,N}^2=\left\{N^{-p}\varepsilon_N^{-3(p-1)}\sum_{D\in\mathscr{P}_N^\delta}\big(\#\{i\in\{1,\dots,N\}:W_t^i\in D\}\big)^p\leq 2^{p+1}\|f_t\|_{L^p}^p\right\}.$$

To this end, we introduce, for $D\in\mathscr{P}_N^\delta$, $A_D=\#\{i:W_t^i\in D\}$. Then $A_D\sim B(N,f_t(D))$ and we have

$$\mathbb{P}(A_D\geq x)\leq\exp(-x/8)\qquad\text{for all } x\geq 2Nf_t(D).\tag{5.4}$$

Indeed, $\mathbb{P}(A_D\geq x)\leq e^{-x}\mathbb{E}[\exp(A_D)]=e^{-x}\exp[N\log(1+f_t(D)(e-1))]\leq e^{-x}\exp[N(e-1)f_t(D)]$. If $x\geq 2Nf_t(D)$, we thus have

$$\mathbb{P}(A_D\geq x)\leq\exp\big[-x+x(e-1)/2\big]\leq\exp(-x/8).$$

Next, it follows from the Hölder inequality that

$$\|f_t\|_{L^p}^p\geq\sum_{D\in\mathscr{P}_N^\delta}\int_D|f_t(v)|^p\,dv\geq\varepsilon_N^{-3p/r}\sum_{D\in\mathscr{P}_N^\delta}\big(f_t(D)\big)^p.$$

On the other hand, we observe from $\#(\mathscr{P}_N^\delta)\leq 64N^\delta\varepsilon_N^{-3}$ that

$$\|f_t\|_{L^p}^p\geq 64^{-1}N^{-\delta}\varepsilon_N^3\sum_{D\in\mathscr{P}_N^\delta}\|f_t\|_{L^p}^p.$$

Using the two previous inequalities, we find that

$$2^{p+1}\|f_t\|_{L^p}^p\geq\sum_{D\in\mathscr{P}_N^\delta}\big(2^p\varepsilon_N^{-3p/r}\big(f_t(D)\big)^p+2^p64^{-1}N^{-\delta}\varepsilon_N^3\|f_t\|_{L^p}^p\big).$$

Consequently, on $(\Omega_{t,N}^2)^c$, we have

$$\begin{aligned}\sum_{D\in\mathscr{P}_N^\delta}A_D^p&>N^p\varepsilon_N^{3(p-1)}2^{p+1}\|f_t\|_{L^p}^p\\&\geq N^p\varepsilon_N^{3(p-1)}\sum_{D\in\mathscr{P}_N^\delta}\big(2^p\varepsilon_N^{-3p/r}\big(f_t(D)\big)^p+2^p64^{-1}N^{-\delta}\varepsilon_N^3\|f_t\|_{L^p}^p\big),\end{aligned}$$

so that there is at least one $D \in \mathscr{P}_N^\delta$ with $A_D^p \geq N^p \varepsilon_N^{3(p-1)}[2^p \varepsilon_N^{-3p/r}(f_t(D))^p + 2^p 64^{-1} N^{-\delta} \varepsilon_N^3 \|f_t\|_{L^p}^p]$. Hence,

$$\mathbb{P}\big[(\Omega_{t,N}^2)^c\big] \leq \sum_{D \in \mathscr{P}_N^\delta} \mathbb{P}\big(A_D \geq N\varepsilon_N^{3/r}\big[2^p \varepsilon_N^{-3p/r}(f_t(D))^p + 2^p 64^{-1} N^{-\delta} \varepsilon_N^3 \|f_t\|_{L^p}^p\big]^{1/p}\big).$$

But we can apply (5.4), because $x_N := N\varepsilon_N^{3/r}[2^p \varepsilon_N^{-3p/r}(f_t(D))^p + 2^p 64^{-1} N^{-\delta} \times \varepsilon_N^3 \|f_t\|_{L^p}^p]^{1/p}$ enjoys the property that $x_N \geq N\varepsilon_N^{3/r}[2^p \varepsilon_N^{-3p/r}(f_t(D))^p]^{1/p} = 2Nf_t(D)$:

$$\mathbb{P}\big[(\Omega_{t,N}^2)^c\big] \leq \sum_{D \in \mathscr{P}_N^\delta} \exp(-x_N/8).$$

Using that $x_N \geq N\varepsilon_N^{3/r}(2^p 64^{-1} N^{-\delta} \varepsilon_N^3 \|f_t\|_{L^p}^p)^{1/p} = cN^{\delta/r}\|f_t\|_{L^p}$, that $\#(\mathscr{P}_N^\delta) \leq 64N$ and that $\|f_t\|_{L^p} \geq M_p$, we deduce that

$$\begin{aligned}\mathbb{P}\big[(\Omega_{t,N}^2)^c\big] &\leq \sum_{D \in \mathscr{P}_N^\delta} \exp\big(-cN^{\delta/r}\|f_t\|_{L^p}/8\big)\\ &\leq 64N \exp\big(-cM_p N^{\delta/r}/8\big)\\ &\leq C\exp\big(-cM_p N^{\delta/r}/10\big).\end{aligned}$$

This ends the step.

Step 3. We finally consider the event

$$\Omega_{T,N} = \Omega_{T,N}^1 \cap \left(\bigcap_{\ell=1}^{K_N+1} \Omega_{t_\ell^N,N}^2\right),$$

where $\Omega_{T,N}^1$ is defined in Lemma 5.5, and the sequence $(t_\ell^N)_{\ell=0}^{K_N+1}$ satisfying $0 = t_0^N < t_1^N < \cdots < t_{K_N}^N \leq T \leq T_{K_N+1}^N$, with $K_N \leq 2TN^2$ and $\sup_{i=0,\dots,K_N}(t_{\ell+1}^N - t_\ell^N) \leq N^{-2}$ is built in Lemma 5.1. We also recall that $h_N(t) = \sum_{\ell=1}^{K_N+1}\|f_{t_\ell^N}\|_{L^p} \times \mathbf{1}_{\{t \in (t_{\ell-1}^N, t_\ell^N]\}}$.

According to Lemma 5.5 and Step 2, we see that

$$\begin{aligned}\mathbb{P}\big[\Omega_{T,N}^c\big] &\leq \mathbb{P}\big[(\Omega_{T,N}^1)^c\big] + \sum_{\ell=1}^{K_N+1} \mathbb{P}\big[(\Omega_{t_\ell^N,N}^2)^c\big]\\ &\leq C_{T,q,\delta} N^{1-q\delta/3} + C(K_N+1)\exp\big(-cN^{\delta/r}\big)\\ &\leq C_{T,q,\delta} N^{1-q\delta/3}.\end{aligned}$$

Finally, we show that on $\Omega_{T,N}$, for all $t \in [0,T]$, $\|\bar{\mu}_{\mathbf{W}_t}^N\|_{L^p} \leq 13{,}500(1 + h_N(t))$. Recall that $\widetilde{W}_t^i$ is defined by (5.1) and that I_ℓ is given by (5.3), we have:

(i) for all $i = 1, \dots, N$, and for all $t \in [0, T+1]$, $W_t^i \in B(0, N^{\delta/3})$ (according to $\Omega^1_{T,N}$);

(ii) for all $\ell = 1, \dots, K_N + 1$, all $t \in (t^N_{\ell-1}, t^N_\ell]$, all $i \in \{1, \dots, N\} \setminus I_\ell$, $|W_t^i - W^i_{t^N_\ell}| = |\widetilde{W}^i_t - \widetilde{W}^i_{t^N_\ell}| \le \varepsilon_N$, and $\#(I_\ell) \le N\varepsilon_N^{3/r}$ (by definition of $\widetilde{W}^i$ and I_ℓ and thanks to $\Omega^1_{T,N}$);

(iii) For all $\ell = 1, \dots, K_N + 1$, $N^{-p}\varepsilon_N^{-3(p-1)} \sum_{D \in \mathscr{P}^\delta_N} (\#\{i \in \{1, \dots, N\} : W^i_{t^N_\ell} \in D\})^p \le 2^{p+1} \| f_{t^N_\ell} \|^p_{L^p}$ (according to $\bigcap_{\ell=1}^{K_N+1} \Omega^2_{t^N_\ell, N}$).

Using Step 1 with $\bar{\mu}^N_{\mathbf{W}_t} = \mu^N_{\mathbf{W}_t} * \psi_{\varepsilon_N}$, we deduce that on $\Omega_{T,N}$, for all $t \in [0, T]$, choosing ℓ such that $t \in (t^N_{\ell-1}, t^N_\ell]$, we have

$$
\begin{aligned}
\| \bar{\mu}^N_{\mathbf{W}_t} \|_{L^p} \le \left(\frac{3}{4\pi} \right)^{1/r} \frac{\#(I_\ell)}{N\varepsilon_N^{3/r}} \\
&+ 3375 \bigg(N^{-p} \varepsilon_N^{-3(p-1)} \\
&\times \sum_{D \in \mathscr{P}^\delta_N} \left(\#\{ i \in \{1, \dots, N\} : W^i_{t^N_\ell} \in D \} \right)^p \bigg)^{1/p} \\
&\le 1 + 3375.2^{(p+1)/p} \| f_{t^N_\ell} \|_{L^p} \\
&= 1 + 3375.2^{(p+1)/p} h_N(t).
\end{aligned}
$$

This completes the proof, since $3375.2^{(p+1)/p} \le 3375.4 = 13{,}500$. $\square$

6. Estimate of the Wasserstein distance. This last section is devoted to the proof of Theorem 1.4. In the whole section, we assume (1.3) for some $\gamma \in (-1, 0)$, $\nu \in (0, 1)$ with $\gamma + \nu > 0$. We consider $q > 6$ such that $q > \gamma^2/(\gamma + \nu)$, $f_0 \in \mathcal{P}_q(\mathbb{R}^3)$ with a finite entropy, and $(f_t)_{t \ge 0}$ the unique weak solution to (1.1) given by Theorem 1.2. We fix $p \in (3/(3+\gamma), p_0(\gamma, \nu, q))$ and know that $(f_t)_{t \ge 0} \in L^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{\mathrm{loc}}([0, \infty), L^p(\mathbb{R}^3))$.

We fix $N \ge 1$, $K \ge 1$ and put $\varepsilon_N = N^{-(1-\delta)/3}$ with $\delta = 6/q$. Consider $(V^i_t)_{t \ge 0}$ for $i = 1, \dots, N$, defined by (4.4) with the choice $\varepsilon = \varepsilon_N$. We know by Lemma 4.5 that $(V^i_t)_{i=1,\dots,N, t \ge 0}$ is a Markov process with generator $\mathcal{L}_{N,K}$ [see (1.12)], starting from $(V^i_0)_{i=1,\dots,N}$, which is an i.i.d. family of f_0-distributed random variables. We set $\mu^N_{\mathbf{V}_t} = N^{-1} \sum_1^N \delta_{V^i_t}$. So the goal of the section is to prove that

$$
\begin{aligned}
\sup_{[0,T]} & \mathbb{E}[\mathcal{W}_2^2(\mu^N_{\mathbf{V}_t}, f_t)] \\
&\le C_{T,q} \big(N^{-(1-6/q)(2+2\gamma)/3} + K^{1-2/\nu} + N^{-1/2} \big).
\end{aligned} \tag{6.1}
$$

We consider $(W_t^i)_{t\geq 0}$, for $i=1,\dots,N$ defined by (4.1) and recall that for all $t\geq 0$, the family $(W_t^i)_{i=1,\dots,N}$ is i.i.d. and f_t-distributed.

First, we introduce the following shortened notation:

$$
\begin{aligned}
c_W(s) &:= c\big(W_s^1, W_s^*(\alpha), z, \varphi\big),\\
c_W^N(s) &:= c\big(W_s^1, W_s^{*,\varepsilon_N}(\alpha), z, \varphi+\varphi_{1,\alpha,s}^1\big),\\
c_V^N(s) &:= c\big(V_s^1, V_s^{*,\varepsilon_N}(\mathbf{V}_s,\mathbf{W}_s,\alpha), z, \varphi+\varphi_{1,\alpha,s}^1+\varphi_{1,\alpha,s}^2\big),\\
c_{K,V}^N(s) &:= c_K\big(V_s^1, V_s^{*,\varepsilon_N}(\mathbf{V}_s,\mathbf{W}_s,\alpha), z, \varphi+\varphi_{1,\alpha,s}^1+\varphi_{1,\alpha,s}^2\big),\\
c_{K,V}(s) &:= c_K\big(V_s^1, V_s^{*}(\mathbf{V}_s,\mathbf{W}_s,\alpha), z, \varphi+\varphi_{1,\alpha,s}\big),
\end{aligned}
$$

with the notation of Section 4. Let us now prove an intermediate result.

LEMMA 6.1. *There is $C>0$ such that a.s.,*

$$
\begin{aligned}
&I_0^N(s)+I_1^N(s)+I_2^N(s)+I_3^N(s)\\
&\quad\leq C\varepsilon_N^{2+2\gamma}+C|W_s^1-V_s^1|^2\\
&\qquad+CK^{1-2/\nu}\int_0^1\big|W_s^1-W_s^{*,\varepsilon_N}(\alpha)\big|^{2+2\gamma/\nu}\,d\alpha\\
&\qquad+C\int_0^1\big(|W_s^1-V_s^1|^2\\
&\qquad+\big|W_s^*(\alpha)-V_s^*(\mathbf{V}_s,\mathbf{W}_s,\alpha)\big|^2\big)\big|W_s^1-W_s^{*,\varepsilon_N}(\alpha)\big|^\gamma\,d\alpha,
\end{aligned}
$$

where

$$
\begin{aligned}
I_0^N(s) &:= \int_0^1\int_0^\infty\int_0^{2\pi}\big(2\big(W_s^1-V_s^1\big)\cdot\big(c_W^N(s)-c_{K,V}^N(s)\big)\\
&\qquad+\big|c_W^N(s)-c_{K,V}^N(s)\big|^2\big)\,d\varphi\,dz\,d\alpha,\\
I_1^N(s) &:= \int_0^1\int_0^\infty\int_0^{2\pi}2\big(W_s^1-V_s^1\big)\\
&\qquad\cdot\big(c_W(s)-c_W^N(s)+c_{K,V}^N(s)-c_{K,V}(s)\big)\,d\varphi\,dz\,d\alpha,\\
I_2^N(s) &:= \int_0^1\int_0^\infty\int_0^{2\pi}\big|c_W(s)-c_W^N(s)+c_{K,V}^N(s)-c_{K,V}(s)\big|^2\,d\varphi\,dz\,d\alpha,\\
I_3^N(s) &:= \int_0^1\int_0^\infty\int_0^{2\pi}2\big(c_W^N(s)-c_{K,V}^N(s)\big)\cdot\big(c_W(s)-c_W^N(s)\\
&\qquad+c_{K,V}^N(s)-c_{K,V}(s)\big)\,d\varphi\,dz\,d\alpha.
\end{aligned}
$$

PROOF. First recall that $|W_s^{*,\varepsilon_N}(\alpha)-V_s^{*,\varepsilon_N}(\mathbf{V}_s,\mathbf{W}_s,\alpha)|^2=|W_s^*(\alpha)-V_s^*(\mathbf{V}_s,\mathbf{W}_s,\alpha)|^2$; see Notation 4.4. It thus follows from (2.6) [with $v=W_s^1$, $v_*=W_s^{*,\varepsilon_N}(\alpha)$, $\tilde v=V_s^1$ and $\tilde v_*=V_s^{*,\varepsilon_N}(\mathbf{V}_s,\mathbf{W}_s,\alpha)$] that

$$\begin{aligned}I_0^N(s)\le C\int_0^1\bigl(&|W_s^1-V_s^1|^2\\&+|W_s^*(\alpha)-V_s^*(\mathbf{V}_s,\mathbf{W}_s,\alpha)|^2\bigr)|W_s^1-W_s^{*,\varepsilon_N}(\alpha)|^\gamma\,d\alpha\\&+CK^{1-2/\nu}\int_0^1|W_s^1-W_s^{*,\varepsilon_N}(\alpha)|^{2+2\gamma/\nu}\,d\alpha.\end{aligned}$$

Next, we study $I_1^N(s)$. As seen in the proof of Lemma 2.3,

$$\int_0^\infty\int_0^{2\pi}c(v,v_*,z,\varphi)\,d\varphi\,dz=-(v-v_*)\Phi(|v-v_*|)$$

and

$$\int_0^\infty\int_0^{2\pi}c_K(v,v_*,z,\varphi)\,d\varphi\,dz=-(v-v_*)\Phi_K(|v-v_*|),$$

where $\Phi(x)=\pi\int_0^\infty(1-\cos G(z/x^\gamma))\,dz$ and $\Phi_K(x)=\pi\int_0^K(1-\cos G(z/x^\gamma))\,dz$. Then

$$\begin{aligned}I_1^N(s)=2(W_s^1-V_s^1)\cdot\int_0^1\bigl[&-(W_s^1-W_s^*(\alpha))\Phi(|W_s^1-W_s^*(\alpha)|)\\&+(W_s^1-W_s^{*,\varepsilon_N}(\alpha))\Phi(|W_s^1-W_s^{*,\varepsilon_N}(\alpha)|)\\&-(V_s^1-V_s^{*,\varepsilon_N}(\mathbf{V}_s,\mathbf{W}_s,\alpha))\Phi_K(|V_s^1-V_s^{*,\varepsilon_N}(\mathbf{V}_s,\mathbf{W}_s,\alpha)|)\\&+(V_s^1-V_s^*(\mathbf{V}_s,\mathbf{W}_s,\alpha))\Phi_K(|V_s^1-V_s^*(\mathbf{V}_s,\mathbf{W}_s,\alpha)|)\bigr]\,d\alpha.\end{aligned}$$

But we have checked that $|X\Phi_K(|X|)-Y\Phi_K(|Y|)|\le C|X-Y||X|^\gamma$ for any $X,Y\in\mathbb{R}^3$ in the proof of Lemma 2.3, and it of course also holds true that $|X\Phi(|X|)-Y\Phi(|Y|)|\le C|X-Y||X|^\gamma$. Thus,

$$\begin{aligned}I_1^N(s)&\le C|W_s^1-V_s^1|\int_0^1\bigl[|W_s^*(\alpha)-W_s^{*,\varepsilon_N}(\alpha)||W_s^1-W_s^{*,\varepsilon_N}(\alpha)|^\gamma\\&\qquad+|V_s^{*,\varepsilon_N}(\mathbf{V}_s,\mathbf{W}_s,\alpha)-V_s^*(\mathbf{V}_s,\mathbf{W}_s,\alpha)|\\&\qquad\times|V_s^1-V_s^{*,\varepsilon_N}(\mathbf{V}_s,\mathbf{W}_s,\alpha)|^\gamma\bigr]\,d\alpha\\&=C|W_s^1-V_s^1|\int_0^1|\varepsilon_NY(\alpha)|\bigl[|W_s^1-W_s^*(\alpha)-\varepsilon_NY(\alpha)|^\gamma\\&\qquad+|V_s^1-V_s^*(\mathbf{V}_s,\mathbf{W}_s,\alpha)-\varepsilon_NY(\alpha)|^\gamma\bigr]\,d\alpha\\&\le C|W_s^1-V_s^1|^2\end{aligned}$$

$$+ C\varepsilon_N^2 \int_0^1 |Y(\alpha)|^2 \big[|W_s^1 - W_s^*(\alpha) - \varepsilon_N Y(\alpha)|^{2\gamma} + |V_s^1 - V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha) - \varepsilon_N Y(\alpha)|^{2\gamma} \big] \, d\alpha.$$

But Y is independent of $(W_s^*, V_s^*(\mathbf{V}_s, \mathbf{W}_s, \cdot))$ and it holds that $\sup_{x\in\mathbb{R}^3} \int_0^1 |x - \varepsilon_N Y(\alpha)|^{2\gamma} |Y(\alpha)|^2 \, d\alpha \leq \int_0^1 |\varepsilon_N Y(\alpha)|^{2\gamma} |Y(\alpha)|^2 \, d\alpha = C\varepsilon_N^{2\gamma}$ [recall that $\gamma \in (-1, 0)$ and that Y is uniformly distributed on $B(0, 1)$], so that finally,

$$I_1^N(s) \leq C |W_s^1 - V_s^1|^2 + C\varepsilon_N^{2+2\gamma}.$$

For $I_2^N(s)$, we first write $I_2^N(s) \leq A + B$, where

$$A = 2 \int_0^1 \int_0^\infty \int_0^{2\pi} |c_W(s) - c_W^N(s)|^2 \, d\varphi \, dz \, d\alpha$$

and

$$B = 2 \int_0^1 \int_0^\infty \int_0^{2\pi} |c_{K,V}^N(s) - c_{K,V}(s)|^2 \, d\varphi \, dz \, d\alpha.$$

We first apply (2.5) with $v = W_s^1$, $v_* = W_s^{*,\varepsilon_N}(\alpha)$, $\tilde{v} = W_s^1$ and $\tilde{v}_* = W_s^*(\alpha)$:

$$\begin{aligned} A &\leq C \int_0^1 |W_s^*(\alpha) - W_s^{*,\varepsilon_N}(\alpha)|^2 |W_s^1 - W_s^{*,\varepsilon_N}(\alpha)|^\gamma \, d\alpha \\ &= C\varepsilon_N^2 \int_0^1 |Y(\alpha)|^2 |W_s^1 - W_s^*(\alpha) - \varepsilon_N Y(\alpha)|^\gamma \, d\alpha. \end{aligned}$$

Using that $\sup_{x\in\mathbb{R}^3} \int_0^1 |x - \varepsilon_N Y(\alpha)|^\gamma |Y(\alpha)|^2 \, d\alpha \leq \int_0^1 |\varepsilon_N Y(\alpha)|^\gamma |Y(\alpha)|^2 \, d\alpha = C\varepsilon_N^\gamma$ and arguing as in the study of $I_1^N(s)$, we conclude that $A \leq C\varepsilon_N^{2+\gamma} \leq C\varepsilon_N^{2+2\gamma}$. The other term B is treated in the same way [observe that (2.5) obviously also holds when replacing c by $c_K = c\mathbf{1}_{\{z\leq K\}}$].

We finally treat $I_3^N(s)$. It is obvious that

$$I_3^N(s) \leq \int_0^1 \int_0^\infty \int_0^{2\pi} |c_W^N(s) - c_{K,V}^N(s)|^2 \, d\varphi \, dz \, d\alpha + I_2^N(s).$$

But

$$\begin{aligned} &\int_0^\infty \int_0^{2\pi} |c_W^N(s) - c_{K,V}^N(s)|^2 \, d\varphi \, dz \\ &\quad = \int_0^K \int_0^{2\pi} |c_W^N(s) - c_V^N(s)|^2 \, d\varphi \, dz + \int_K^\infty \int_0^{2\pi} |c_W^N(s)|^2 \, d\varphi \, dz. \end{aligned}$$

Applying first (2.5) with $v = W_s^1$, $v_* = W_s^{*,\varepsilon_N}(\alpha)$, $\tilde{v} = V_s^1$ and $\tilde{v}_* = V_s^{*,\varepsilon_N}(\mathbf{V}_s, \mathbf{W}_s, \alpha)$, we find that

$$
\begin{aligned}
&\int_0^K \int_0^{2\pi} \left|c_W^N(s) - c_V^N(s)\right|^2 d\varphi\, dz \\
&\quad \le C\left(\left|W_s^1 - V_s^1\right|^2 + \left|W_s^{*,\varepsilon_N}(\alpha) - V_s^{*,\varepsilon_N}(\mathbf{V}_s, \mathbf{W}_s, \alpha)\right|^2\right) \\
&\qquad \times \left|W_s^1 - W_s^{*,\varepsilon_N}(\alpha)\right|^\gamma \\
&\quad = C\left(\left|W_s^1 - V_s^1\right|^2 + \left|W_s^*(\alpha) - V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha)\right|^2\right)\left|W_s^1 - W_s^{*,\varepsilon_N}(\alpha)\right|^\gamma.
\end{aligned}
$$

Moreover, as seen in the proof of Lemma 2.3, $\int_K^\infty \int_0^{2\pi} |c_W^N(s)|^2 \, d\varphi\, dz = |W_s^1 - W_s^{*,\varepsilon_N}(\alpha)|^2 \Psi_K(|W_s^1 - W_s^{*,\varepsilon_N}(\alpha)|)$, where $\Psi_K(x) = \Phi(x) - \Phi_K(x) \le C \int_K^\infty G^2(z/x^\gamma)\, dz \le C x^{2\gamma/\nu} K^{1-2/\nu}$. Hence,

$$
\int_K^\infty \int_0^{2\pi} \left|c_W^N(s)\right|^2 d\varphi\, dz \le C\left|W_s^1 - W_s^{*,\varepsilon_N}(\alpha)\right|^{2+2\gamma/\nu} K^{1-2/\nu}.
$$

All this shows that

$$
\begin{aligned}
I_3^N(s) \le I_2^N(s) + C \int_0^1 &\left(\left|W_s^1 - V_s^1\right|^2\right. \\
&\left. + \left|W_s^*(\alpha) - V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha)\right|^2\right)\left|W_s^1 - W_s^{*,\varepsilon_N}(\alpha)\right|^\gamma d\alpha \\
&+ C K^{1-2/\nu} \int_0^1 \left|W_s^1 - W_s^{*,\varepsilon_N}(\alpha)\right|^{2+2\gamma/\nu} d\alpha
\end{aligned}
$$

and this completes the proof. $\square$

To prove our main result, we need the following estimate which can be found in [10], Theorem 1.

LEMMA 6.2. *Fix $A > 0$ and $q > 4$. There is a constant $C_{A,q}$ such that for all $f \in \mathcal{P}_q(\mathbb{R}^3)$ verifying $\int_{\mathbb{R}^3} |v|^q f(dv) \le A$, all i.i.d. family $(X_i)_{i=1,\dots,N}$ of f-distributed random variables*,

$$
\mathbb{E}\left[\mathcal{W}_2^2\left(f, N^{-1} \sum_{i=1}^N \delta_{X_i}\right)\right] \le C_{A,q} N^{-1/2}.
$$

PROPOSITION 6.3. *Fix $T > 0$ and recall that h_N was defined in Lemma 5.1. Consider the stopping time*

$$
\sigma_N = \inf\left\{t \ge 0 : \|\bar{\mu}_{\mathbf{W}_t}^N\|_{L^p} \ge 13{,}500\left(1 + h_N(t)\right)\right\},
$$

where $\bar{\mu}_{\mathbf{W}_t}^N = \mu_{\mathbf{W}_t}^N * \psi_{\varepsilon_N}$ with $\psi_{\varepsilon_N}(x) = (3/(4\pi\varepsilon_N^3))\mathbf{1}_{\{|x| \le \varepsilon_N\}}$ and $\mu_{\mathbf{W}_t}^N = N^{-1} \times \sum_1^N \delta_{W_t^i}$. We have for all $T > 0$,

$$
\sup_{[0,T]} \mathbb{E}\left[\left|W_{t\wedge\sigma_N}^1 - V_{t\wedge\sigma_N}^1\right|^2\right] \le C_T\left(\varepsilon_N^{2+2\gamma} + K^{1-2/\nu} + N^{-1/2}\right).
$$

PROOF. We fix $T > 0$ and set $u_t^N = \mathbb{E}[|W_{t\wedge\sigma_N}^1 - V_{t\wedge\sigma_N}^1|^2]$ for all $t \in [0, T]$. By the Itô formula we have

$$
\begin{aligned}
u_t^N &= \mathbb{E}\Big[\int_0^{t\wedge\sigma_N}\int_0^1\int_0^\infty\int_0^{2\pi}\big(\big|W_s^1 - V_s^1 + c_W(s) - c_{K,V}(s)\big|^2 \\
&\qquad - |W_s^1 - V_s^1|^2\big)\,d\varphi\,dz\,d\alpha\Big] \\
&= \mathbb{E}\Big[\int_0^{t\wedge\sigma_N}\int_0^1\int_0^\infty\int_0^{2\pi}\big(2(W_s^1 - V_s^1)\cdot\big(c_W(s) - c_{K,V}(s)\big) \\
&\qquad + \big|c_W(s) - c_{K,V}(s)\big|^2\big)\,d\varphi\,dz\,d\alpha\Big] \\
&= \mathbb{E}\Big[\int_0^{t\wedge\sigma_N}\big(I_0^N(s) + I_1^N(s) + I_2^N(s) + I_3^N(s)\big)\,ds\Big],
\end{aligned}
$$

where $I_i^N(s)$ was introduced in Lemma 6.1 for $i = 0, 1, 2, 3$. We know from Lemma 6.1 that

$$
u_t^N \le Ct\varepsilon_N^{2+2\gamma} + C\int_0^t u_s^N\,ds + C\big(J_1^N(t) + J_2^N(t) + J_3^N(t)\big),
$$

where

$$
\begin{aligned}
J_1^N(t) &= \mathbb{E}\Big[\int_0^{t\wedge\sigma_N}\int_0^1 |W_s^1 - V_s^1|^2 |W_s^1 - W_s^{*,\varepsilon_N}(\alpha)|^\gamma\,d\alpha\,ds\Big], \\
J_2^N(t) &= \mathbb{E}\Big[\int_0^{t\wedge\sigma_N}\int_0^1 \big|W_s^*(\alpha) - V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha)\big|^2 \\
&\qquad \times \big|W_s^1 - W_s^{*,\varepsilon_N}(\alpha)\big|^\gamma\,d\alpha\,ds\Big], \\
J_3^N(t) &= K^{1-2/\nu}\mathbb{E}\Big[\int_0^{t\wedge\sigma_N}\int_0^1 \big|W_s^1 - W_s^{*,\varepsilon_N}(\alpha)\big|^{2+2\gamma/\nu}\,d\alpha\,ds\Big].
\end{aligned}
$$

First, we have

$$
J_3^N(t) \le CK^{1-2/\nu}t.
$$

Indeed, it suffices to use that $|W_s^1 - W_s^{*,\varepsilon_N}(\alpha)|^{2+2\gamma/\nu} \le C(1 + |W_s^1|^2 + |W_s^{*,\varepsilon_N}(\alpha)|^2)$ [because $2 + 2\gamma/\nu \in (0, 2)$], that $|W_s^{*,\varepsilon_N}(\alpha)|^2 \le 2 + 2|W_s^*(\alpha)|^2$ [because $\varepsilon_N \in (0, 1)$ and Y takes its values in $B(0, 1)$] and finally that $\mathbb{E}[|W_s^1|^2] = \int_0^1 |W_s^*(\alpha)|^2\,d\alpha = m_2(f_0)$.

Next, $\mathcal{L}_\alpha(W_s^{*,\varepsilon_N}) = f_s * \psi_{\varepsilon_N}$, so that $\int_0^1 |W_s^1 - W_s^{*,\varepsilon_N}(\alpha)|^\gamma\,d\alpha \le 1 + C_{\gamma,p}\|f_s * \psi_{\varepsilon_N}\|_{L^p}$ by (2.1) [recall that $p > 3/(3+\gamma)$ is fixed since the beginning of the section]. Of course, $\|f_s * \psi_{\varepsilon_N}\|_{L^p} \le \|f_s\|_{L^p}$, and we conclude that

$$
J_1^N(t) \le C_{\gamma,p}\int_0^t (1 + \|f_s\|_{L^p})u_s^N\,ds.
$$

On the other hand, using the exchangeability and that $W_s^{*,\varepsilon_N}(\alpha) = W_s^*(\alpha) + \varepsilon_N Y(\alpha)$, with $Y(\alpha)$ independent of $W_s^*(\alpha)$ and $V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha)$ introduced in Notation 4.4, we have

$$
\begin{aligned}
J_2^N(t) &= \mathbb{E}\bigg[\int_0^{t\wedge\sigma_N}\int_0^1 \big|W_s^*(\alpha) - V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha)\big|^2 N^{-1} \\
&\qquad \times \sum_{i=1}^N \big|W_s^i - \varepsilon_N Y(\alpha) - W_s^*(\alpha)\big|^\gamma \, d\alpha \, ds\bigg] \\
&= \mathbb{E}\bigg[\int_0^{t\wedge\sigma_N}\int_0^1 \big|W_s^*(\alpha) - V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha)\big|^2 \\
&\qquad \times \bigg(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |w - x - W_s^*(\alpha)|^\gamma \psi_{\varepsilon_N}(x) \mu_{\mathbf{W}_s}^N(dw)\, dx\bigg) d\alpha \, ds\bigg] \\
&= \mathbb{E}\bigg[\int_0^{t\wedge\sigma_N}\int_0^1 \big|W_s^*(\alpha) - V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha)\big|^2 \\
&\qquad \times \bigg(\int_{\mathbb{R}^3} |w - W_s^*(\alpha)|^\gamma \bar{\mu}_{\mathbf{W}_s}^N(dw)\bigg) d\alpha \, ds\bigg].
\end{aligned}
$$

But $\int_{\mathbb{R}^3} |W_s^*(\alpha) - w|^\gamma \bar{\mu}_{\mathbf{W}_s}^N(dw) \le C_{\gamma,p}(1 + \|\bar{\mu}_{\mathbf{W}_s}^N\|_{L^p})$ by (2.1), so that

$$
\begin{aligned}
J_2^N(t) \le C_{\gamma,p}\mathbb{E}\bigg[\int_0^{t\wedge\sigma_N}\int_0^1 &\big(1 + \|\bar{\mu}_{\mathbf{W}_s}^N\|_{L^p}\big) \\
&\times \big|W_s^*(\alpha) - V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha)\big|^2 \, d\alpha \, ds\bigg].
\end{aligned}
$$

We now deduce from Lemma 4.2 that

$$
\begin{aligned}
\int_0^1 &\big|W_s^*(\alpha) - V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha)\big|^2 \, d\alpha \\
&\le 2\int_0^1 \big(\big|W_s^*(\alpha) - Z_s^*(\mathbf{W}_s, \alpha)\big|^2 \\
&\qquad + \big|Z_s^*(\mathbf{W}_s, \alpha) - V_s^*(\mathbf{V}_s, \mathbf{W}_s, \alpha)\big|^2\big) \, d\alpha \\
&= 2\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s}^N) + 2\frac{1}{N}\sum_{i=1}^N |W_s^i - V_s^i|^2.
\end{aligned}
$$

Using the exchangeability and that $\|\bar{\mu}_{\mathbf{W}_s}^N\|_{L^p} \le 13{,}500(1 + h_N(s))$ for all $s \le \tau_N$, it holds that

$$
J_2^N(t) \le C\int_0^t (1 + h_N(s))\mathbb{E}[\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s}^N)] \, ds + C\int_0^t (1 + h_N(s)) u_s^N \, ds.
$$

We thus have checked that

$$u_t^N \le Ct(\varepsilon_N^{2+2\gamma} + K^{1-2/\nu}) + C\int_0^t (1+h_N(s))\mathbb{E}[\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s}^N)]\,ds$$
$$+ C\int_0^t (1+\|f_s\|_{L^p} + h_N(s))u_s^N\,ds.$$

But for each $t \ge 0$, the family $(W_t^i)_{i=1,\dots,N}$ is i.i.d. and f_t-distributed. Furthermore, $\sup_{[0,T]} \mathbb{E}[|W_t^1|^q] < \infty$ ($q > 6$) by (4.2). Hence, Lemma 6.2 tells us that

$$\sup_{[0,T]} \mathbb{E}[\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s}^N)] \le C_T N^{-1/2}. \tag{6.2}$$

Using the Grönwall lemma, we deduce that

$$\sup_{[0,T]} u_t^N \le C_T\left(\varepsilon_N^{2+2\gamma} + K^{1-2/\nu} + N^{-1/2}\int_0^T (1+h_N(s))\,ds\right)$$
$$\times \exp\left(C\int_0^T (1+\|f_s\|_{L^p} + h_N(s))\,ds\right).$$

But $\int_0^T h_N(s)\,ds \le 2\int_0^T \|f_s\|_{L^p}\,ds$ by Lemma 5.1(ii). And we know that $f \in L^1_{\text{loc}}([0,\infty), L^p(\mathbb{R}^3))$. We thus conclude that

$$\sup_{[0,T]} u_t^N \le C_T(\varepsilon_N^{2+2\gamma} + K^{1-2/\nu} + N^{-1/2})$$

as desired. $\square$

PROOF OF THEOREM 1.4. As explained at the beginning of the section, we only have to prove (6.1). Recall that $\sigma_N = \inf\{t \ge 0 : \|\bar{\mu}_{\mathbf{W}_t}^N\|_{L^p} \ge 13{,}500(1 + h_N(t))\}$, that $q > 6$ and that $\delta = 6/q$. It is clear that $\mathbb{P}[\sigma_N \le T] \le C_{T,q,\delta} N^{1-q\delta/3} = C_{T,q} N^{-1}$ from Proposition 5.2. Then for $t \in [0,T]$, we write

$$\sup_{[0,T]} \mathbb{E}[\mathcal{W}_2^2(\mu_{\mathbf{V}_t}^N, f_t)] \le 2\sup_{[0,T]} \mathbb{E}[\mathcal{W}_2^2(\mu_{\mathbf{V}_t}^N, \mu_{\mathbf{W}_t}^N) + \mathcal{W}_2^2(\mu_{\mathbf{W}_t}^N, f_t)]$$
$$\le 2\sup_{[0,T]} \mathbb{E}[\mathcal{W}_2^2(\mu_{\mathbf{V}_t}^N, \mu_{\mathbf{W}_t}^N)] + C_T N^{-1/2}$$

by (6.2). But, by exchangeability, we have

$$\mathbb{E}[\mathcal{W}_2^2(\mu_{\mathbf{V}_t}^N, \mu_{\mathbf{W}_t}^N)] \le \mathbb{E}\left[N^{-1}\sum_{i=1}^N |W_t^i - V_t^i|^2\right]$$
$$= \mathbb{E}[|W_t^1 - V_t^1|^2].$$

Moreover,

$$\begin{aligned}\mathbb{E}\big[|W_t^1 - V_t^1|^2\big] &\le \mathbb{E}\big[|W_{t\wedge\sigma_N}^1 - V_{t\wedge\sigma_N}^1|^2\big] + \mathbb{E}\big[|W_t^1 - V_t^1|^2 \mathbf{1}_{\{\sigma_N \le T\}}\big] \\ &\le C_T\big(\varepsilon_N^{2+2\gamma} + K^{1-2/\nu} + N^{-1/2}\big) \\ &\quad + C\mathbb{E}\big[|W_t^1|^4 + |V_t^1|^4\big]^{1/2}\big(\mathbb{P}(\sigma_N \le T)\big)^{1/2},\end{aligned}$$

by Proposition 6.3, and the Cauchy–Schwarz inequality. Noting that $\mathbb{E}[|W_t^1|^4] \le C_T$ by (4.2), and that $\mathbb{E}[|V_t^1|^4] \le C_T\mathbb{E}[|V_0^1|^4]$ by Lemma 4.5, we deduce that

$$\mathbb{E}\big[|W_t^1 - V_t^1|^2\big] \le C_{T,q}\big(\varepsilon_N^{2+2\gamma} + K^{1-2/\nu} + N^{-1/2}\big).$$

All in all, we have proved that

$$\sup_{[0,T]} \mathbb{E}\big[\mathcal{W}_2^2\big(\mu_{\mathbf{V}_t}^N, f_t\big)\big] \le C_{T,q}\big(\varepsilon_N^{2+2\gamma} + K^{1-2/\nu} + N^{-1/2}\big).$$

This is precisely (6.1), since $\varepsilon_N^{2+2\gamma} = N^{-(1-6/q)(2+2\gamma)/3}$, with $\varepsilon_N = N^{-(1-\delta)/3}$ and $\delta = 6/q$. $\square$

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LABORATOIRE DE PROBABILITÉS
ET MODÈLES ALÉATOIRES (CNRS, UMR 7599)
UNIVERSITÉ PIERRE-ET-MARIE CURIE
CASE 188, 4 PLACE JUSSIEU
F-75252 PARIS CEDEX 5
FRANCE
E-MAIL: liping.xu@upmc.fr
URL: www.lpma-paris.fr/~xu