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Generalized Dynkin games and doubly reflected BSDEs with jumps

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Abstract

We introduce a game problem which can be seen as a generalization of the classical Dynkin game problem to the case of a nonlinear expectation $\mathcal{E}^g$, induced by a Backward Stochastic Differential Equation (BSDE) with jumps with nonlinear driver g. Let ξ, ζ be two RCLL adapted processes with $\xi \le \zeta$. The criterium is given by

$$\mathcal{J}_{\tau,\sigma} = \mathcal{E}^g_{0,\tau\wedge\sigma}\left(\xi_\tau \mathbf{1}_{\{\tau\le\sigma\}} + \zeta_\sigma \mathbf{1}_{\{\sigma<\tau\}}\right),$$

where τ and σ are stopping times valued in $[0,T]$. Under Mokobodzki's condition, we establish the existence of a value function for this game, i.e. $\inf_\sigma \sup_\tau \mathcal{J}_{\tau,\sigma} = \sup_\tau \inf_\sigma \mathcal{J}_{\tau,\sigma}$. This value can be characterized via a doubly reflected BSDE. Using this characterization, we provide some new results on these equations, such as comparison theorems and a priori estimates. When ξ and ζ are left upper semicontinuous along stopping times, we prove the existence of a saddle point. We also study a generalized mixed game problem when the players have two actions: continuous control and stopping. We then study the generalized Dynkin game in a Markovian framework and its links with parabolic partial integro-differential variational inequalities with two obstacles.

Keywords: Dynkin game; mixed game problem; nonlinear expectation; g-evaluation; doubly reflected BSDEs; partial integro-differential variational inequalities; game option.
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1 Introduction

The classical Dynkin game has been widely studied: see e.g. Bismut [5], Alario-Nazaret et al. [1], Kobylanski et al. [31]. Let $\xi = (\xi_t), \zeta = (\zeta_t)$ be two right continuous left-limited (RCLL) adapted processes with $\xi \le \zeta$ and $\xi_T = \zeta_T$ a.s. The criterium is given, for all pair (τ, σ) of stopping times valued in $[0,T]$, by

$$J_{\tau,\sigma} := E\left(\xi_\tau \mathbf{1}_{\{\tau\le\sigma\}} + \zeta_\sigma \mathbf{1}_{\{\sigma<\tau\}}\right).$$

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Under Mokobodzki's condition, which states that there exist two supermartingales such that their difference is between ξ and ζ, the Dynkin game is fair, i.e. $\inf_\sigma \sup_\tau J_{\tau,\sigma} = \sup_\tau \inf_\sigma J_{\tau,\sigma}$. When the barriers ξ, ζ are left upper semicontinuous, and $\xi_t < \zeta_t$, $t < T$, there exists a saddle point.

Using a change of variable, these results can be generalized to the case of a criterium with an instantaneous reward process $g(t)$, of the form

$$J_{\tau,\sigma} := E\left(\int_0^{\tau\wedge\sigma} g(t)dt + \xi_\tau \mathbf{1}_{\{\tau\leq\sigma\}} + \zeta_\sigma \mathbf{1}_{\{\sigma<\tau\}}\right). \tag{1.1}$$

In the Brownian case and when ξ and ζ are continuous processes, Cvitanić and Karatzas have established links between these Dynkin games and doubly reflected Backward stochastic differential equations with driver process $g(t)$ and barriers ξ and ζ (see [9]).

In this paper, we introduce a new game problem, which generalizes the classical Dynkin game to the case of $\mathcal{E}^g$-expectations (or g-evaluations in the terminology of Peng [33]). Given a Lipschitz driver $g(t,y,z,k)$, a stopping time $\tau \leq T$ and a square integrable $\mathcal{F}_\tau$-measurable random variable η, the associated conditional $\mathcal{E}^g$-expectation process denoted by $(\mathcal{E}^g_{t,\tau}(\eta), 0 \leq t \leq \tau)$ is defined as the solution of the backward stochastic differential equation (BSDE) with driver g, terminal time τ and terminal condition η. We thus consider here a *generalized Dynkin game*, where the criterium is given, for each pair (τ,σ) of stopping times valued in $[0,T]$, by

$$\mathcal{J}_{\tau,\sigma} := \mathcal{E}^g_{0,\tau\wedge\sigma}\left(\xi_\tau \mathbf{1}_{\{\tau\leq\sigma\}} + \zeta_\sigma \mathbf{1}_{\{\sigma<\tau\}}\right),$$

with ξ, ζ two RCLL adapted processes satisfying $\xi \leq \zeta$.

When the driver g does not depend on the solution, that is, when it is given by a process $g(t)$, the criterium $\mathcal{J}_{\tau,\sigma}$ coincides with $J_{\tau,\sigma}$ given in (1.1). It is well-known that in this case, under Mokobodzki's condition, the value function for the Dynkin game problem can be characterized as the solution of the Doubly Reflected BSDE (DRBSDE) associated with driver process $g(t)$ and barriers ξ and ζ (see e.g. [9], Hamadène-Lepeltier [25], Lepeltier-Xu [32], Hamadène-Ouknine [27]). We generalize here this result to the case of a nonlinear driver $g(t,y,z,k)$ depending on the solution. More precisely, under Mokobodzki's condition, we prove that

$$\inf_\sigma \sup_\tau \mathcal{J}_{\tau,\sigma} = \sup_\tau \inf_\sigma \mathcal{J}_{\tau,\sigma},$$

and we characterize this common value function as the solution of the DRBSDE associated with driver g and barriers ξ and ζ. Moreover, when ξ and ζ are left-upper semicontinuous along stopping times, we show that there exist saddle points. Note that, contrary to the previous existence results given in the case of classical Dynkin games, we do not assume the strict separability of the barriers. We point out that the approach used in the classical case cannot be adapted to our case because of the nonlinearity of the driver. Using the characterization of the solution of a DRBSDE as the value function of a generalized Dynkin game, we prove some results on DRBSDEs, such as a comparison and a strict comparison theorem and a priori estimates which complete those given in the literature.

Moreover, we introduce a mixed game problem expressed in terms of $\mathcal{E}^g$-expectation/ g-evaluation, when the players have two possible actions: continuous control and stopping. The first (resp. second) player chooses a pair (u,τ) (resp. (v,σ)) of control and stopping time, and aims at maximizing (resp. minimizing) the criterium. This problem has been studied by [25] and [21] in the case of a classical expectation, that is when the criterium is given, for each quadruple (u,τ,v,σ) of controls and stopping times, by

$$E_{Q^{u,v}}\left[\int_0^{\tau\wedge\sigma} g(t,u_t,v_t)dt + \xi_\tau \mathbf{1}_{\{\tau\leq\sigma\}} + \zeta_\sigma \mathbf{1}_{\{\sigma<\tau\}}\right], \tag{1.2}$$

where $Q^{u,v}$ are a priori probability measures, and $g(t, u_t, v_t)$ represents the instantaneous reward process associated with controls u, v. In [25], Hamadène and Lepeltier (see also Hamadène [21]) have established some links between this mixed game problem and DRBSDEs. Here, we consider a *generalized mixed game* problem, where, for a given family of nonlinear drivers $g^{u,v}(t, y, z, k) := g(t, u_t, v_t, y, z, k)$ depending on the controls u, v, the criterium is defined by

$$\mathcal{E}^{u,v}_{0,\tau\wedge\sigma}\left(\xi_\tau \mathbf{1}_{\{\tau\leq\sigma\}} + \zeta_\sigma \mathbf{1}_{\{\sigma<\tau\}}\right), \tag{1.3}$$

evaluated under the nonlinear expectation $\mathcal{E}^{u,v} = \mathcal{E}^{g^{u,v}}$. Note that the criterium (1.3) corresponds to a criterium of the form (1.2) when the drivers $g^{u,v}$ are linear. We generalize the results of [25] and [21] to the case of nonlinear expectations. We provide some sufficient conditions which ensure the existence of a value function of our *generalized mixed game* problem, and show that the common value function can be characterized as the solution of a DRBSDE. Under additional regularity assumptions on ξ and ζ, we prove the existence of saddle points.

The paper is organized as follows. In Section 2 we introduce notation and definitions and provide some preliminary results. In Section 3 we consider a classical Dynkin game problem and study its links with a DRBSDE associated with a driver which does not depend on the solution. We provide an existence result for this game problem under relatively weak assumptions on ξ and ζ. Note that Section 3, although it contains new results, mainly situates our work and introduces the tools used in the sequel. In Section 4, we introduce the *generalized Dynkin game* with g-evaluation. We prove the existence of a value function for this game problem. We show that the common value function can be characterized as the solution of a nonlinear DRBSDE with jumps and RCLL barriers ξ and ζ. We then study in Section 5 a *generalized mixed game* problem when the players have two actions: continuous control and stopping. In Section 6, using the characterization of the solution of a DRBSDE as the value function of a *generalized Dynkin game*, we prove comparison theorems and a priori estimates for DRBSDEs. Finally in Section 7, we study the *generalized Dynkin game* in a Markovian framework and its links with parabolic partial integro-differential variational inequalities (PIDVI) with two obstacles. We prove that the value function of the generalized Dynkin game is a viscosity solution of a PIDVI. A uniqueness result is obtained under additional assumptions. Additional results and detailed proofs are given in the Appendix.

Motivating applications in mathematical finance As shown in El-Karoui-Quenez [18], in a market model with constraints such as taxes or large investor impact, the dynamics of the wealth process of an investor investing in this market can be written via a nonlinear driver g. In [18], a *nonlinear price system* (later called g-*evaluation*) is defined as follows: for each $S \in [0, T]$ and each $\eta \in L^2(\mathcal{F}_S)$, the initial (hedging) price of an European option with maturity S and payoff η is given by $\mathcal{E}^g_{0,S}(\eta)$.

In the case of an American option associated with a payoff process $\xi \in \mathcal{S}^2$, the buyer has the choice of the exercise (stopping) time τ. The g-*value of the American option*, defined as $\sup_\tau \mathcal{E}^g_{0,\tau}(\xi_\tau)$, is characterized as the initial value of the reflected BSDE associated with driver g and obstacle ξ (see [18] for a proof in a Brownian framework and Quenez-Sulem [36] for the generalization to the case with jumps and irregular payoff). In [14], we show that this price is the *super-hedging price* of the American option.

Game options are an extension of American options introduced by Kifer in the case of a perfect market model (see [29]). Beside the holder's ability to exercise at any (stopping) time τ, the issuer of the option also cancel the option at any (stopping) time σ and then pay the holder a cancellation fee ζ_σ. The cancellation fee ζ is greater than or equal to the exercise payoff ξ at all time. The difference $\zeta_\sigma - \xi_\sigma \geq 0$ is interpreted as a penalty that the seller pays to the buyer for the cancellation of the contract. Hence, the game option consists for the seller to select a cancellation time σ and for the buyer to choose an exercise time τ, so that the seller pays to the buyer at time $\tau \wedge \sigma$ the amount $I(\tau, \sigma) = \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\sigma < \tau}$. The main result of the present paper yields that, under Mokobodzki's condition, the g-*value* of the game option, defined as $\inf_\sigma \sup_\tau \mathcal{E}^g_{0,\tau\wedge\sigma}(I(\tau,\sigma))$, is equal to the *common value* of a *generalized* Dynkin game, and is characterized as the initial value of the solution of the doubly reflected BSDE associated with driver g and barriers ξ and ζ (see (2.2)).[1] This result, which is new even in the Brownian case, can be seen as the analogous, in the case of a game option, of the characterization of the g-*value* of an American option (i.e. $\sup_\tau \mathcal{E}^g_{0,\tau}(\xi_\tau)$), via a reflected BSDE. In [14], using this result, we show that the g-value of the game option is equal to the *super-hedging price*, that is the minimal initial wealth which allows the seller to be super-hedged.

2 Notation and definitions

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let W be a one-dimensional Brownian motion. Let $\mathbf{E} := \mathbf{R}^*$ and $\mathcal{B}(\mathbf{E})$ be its Borelian filtration. Suppose that it is equipped with a σ-finite positive measure ν and let $N(dt, de)$ be a Poisson random measure with compensator $\nu(de)dt$. Let $\tilde{N}(dt, de)$ be its compensated process. Let $I\!F = \{\mathcal{F}_t, t \geq 0\}$ be the completed natural filtration associated with W and N.

Notation 2.1. *Let $\mathcal{P}$ be the predictable σ-algebra on $\Omega \times [0,T]$. For each $T > 0$, we introduce the following sets:*

- *$L^2(\mathcal{F}_T)$: the set of random variables ξ which are $\mathcal{F}_T$-measurable and square integrable;*
- *$I\!H^2$: the set of real-valued predictable processes ϕ with $\|\phi\|^2_{I\!H^2} := E\left[\int_0^T \phi_t^2 dt\right] < \infty$;*
- *$\mathcal{S}^2$: the set of real-valued RCLL adapted processes ϕ with $\|\phi\|^2_{\mathcal{S}^2} := E(\sup_{0\leq t\leq T} |\phi_t|^2) < \infty$;*
- *$\mathcal{A}^2$ (resp. $\mathcal{A}^1$) : the set of real-valued non decreasing RCLL predictable processes A with $A_0 = 0$ and $E(A_T^2) < \infty$ (resp. $E(A_T) < \infty$).*
- *L^2_ν : the set of Borelian functions $\ell : \mathbf{E} \to \mathbf{R}$ such that $\int_{\mathbf{E}} |\ell(e)|^2 \nu(de) < +\infty$. The set L^2_ν is a Hilbert space equipped with the scalar product $\langle \ell, \ell' \rangle_\nu := \int_{\mathbf{E}} \ell(e)\ell'(e)\nu(de)$ for all $\ell, \ell' \in L^2_\nu \times L^2_\nu$, and the norm $\|\ell\|^2_\nu := \int_{\mathbf{E}} |\ell(e)|^2 \nu(de)$.*
- *$I\!H^2_\nu$: the set of all mappings $l : \Omega \times [0,T] \times \mathbf{E} \to \mathbf{R}$ that are $\mathcal{P} \otimes \mathcal{B}(\mathbf{E})/\mathcal{B}(\mathbf{R})$ measurable and satisfy $\|l\|^2_{I\!H^2_\nu} := E\left[\int_0^T \|l_t\|^2_\nu \, dt\right] < \infty$, where $l_t(\omega, e) = l(\omega, t, e)$ for all $(\omega, t, e) \in \Omega \times [0,T] \times \mathbf{E}$.*

Moreover, $\mathcal{T}_0$ is the set of stopping times τ such that $\tau \in [0,T]$ a.s. and for each S in $\mathcal{T}_0$, we denote by $\mathcal{T}_S$ the set of stopping times τ such that $S \leq \tau \leq T$ a.s.

Definition 2.2 (Driver, Lipschitz driver)**.** *A function g is said to be* a driver *if*

- *$g : \Omega \times [0,T] \times \mathbf{R}^2 \times L^2_\nu \to \mathbf{R}$ $(\omega, t, y, z, k(\cdot)) \mapsto g(\omega, t, y, z, k(\cdot))$ is $\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^2) \otimes \mathcal{B}(L^2_\nu)-$ measurable,*
- *$g(., 0, 0, 0) \in I\!H^2$.*

A driver g is called a Lipschitz driver *if moreover there exists a constant $C \geq 0$ such that $dP \otimes dt$-a.s., for each (y_1, z_1, k_1), (y_2, z_2, k_2),*

$$|g(\omega, t, y_1, z_1, k_1) - g(\omega, t, y_2, z_2, k_2)| \leq C(|y_1 - y_2| + |z_1 - z_2| + \|k_1 - k_2\|_\nu).$$

[1] Note that in the literature, this characterization has only been proven when g is linear with respect to y, z, k. The proof is based on a change of probability measure and an actualization procedure (see [21]). This approach cannot be adapted to the nonlinear case.

Recall that for each Lipschitz driver g, and each terminal condition $\xi \in L^2(\mathcal{F}_T)$, there exists a unique solution $(X^g, \pi^g, l^g) \in \mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu$ satisfying

$$-dX^g(t) = g(t, X^g_{t^-}, \pi^g(t), l^g(t)(\cdot))dt - \pi^g(t)dW_t - \int_{\mathbf{E}} l^g(t)(e)\tilde{N}(dt, de); \qquad X_T = \xi. \quad (2.1)$$

The solution is denoted by $(X^g_\cdot(\xi, T), \pi^g_\cdot(\xi, T), l^g_\cdot(\xi, T))$. The operator $X^g : (\xi, T) \mapsto X^g_\cdot(\xi, T)$, called *nonlinear pricing system* (associated with driver g), was first introduced in [18]. In [33], this operator X^g is called the *nonlinear evaluation* (associated with driver g) and is denoted by $\mathcal{E}^g$. In the sequel, we say that $\mathcal{E}^g_{\cdot,T}(\xi)$ is the $\mathcal{E}^g_{\cdot,T}$*-conditional expectation* process of ξ (or the g*-evaluation* of (ξ, T)). When there is no ambiguity on the driver, $\mathcal{E}^g$ will be simply denoted by $\mathcal{E}$. Recall that this notion can be extended to the case where T is replaced by a stopping time $\tau \in \mathcal{T}_0$ and ξ by a random variable $\eta \in L^2(\mathcal{F}_\tau)$.

We introduce a notion of mutually singular random measures associated with non decreasing RCLL predictable processes, which can be seen as a probabilistic version of a classical notion in analysis.

Definition 2.3. *Let $A = (A_t)_{0 \le t \le T}$ and $A' = (A'_t)_{0 \le t \le T}$ belonging to $\mathcal{A}^1$. We say that the random measures dA_t and dA'_t are* mutually singular *(in a probabilistic sense), and we write $dA_t \perp dA'_t$, if there exists $D \in \mathcal{P}$ such that:*

$$E[\int_0^T \mathbf{1}_{D^c} dA_t] = E[\int_0^T \mathbf{1}_D dA'_t] = 0,$$

which can also be written as $\int_0^T \mathbf{1}_{D^c_t} dA_t = \int_0^T \mathbf{1}_{D_t} dA'_t = 0$ a.s., where for each $t \in [0, T]$, D_t is the section at time t *of D, that is, $D_t := \{\omega \in \Omega\,,\, (\omega, t) \in D\}$.*[2]

We define now DRBSDEs with jumps, for which the solution is constrained to stay between two given RCLL processes called barriers $\xi \le \zeta$. Two nondecreasing processes A and A' are introduced in order to push the solution Y above ξ and below ζ in a minimal way. This *minimality property* of A and A' is ensured by the *Skorohod* conditions (condition (iii) below) together with the additional constraint $dA_t \perp dA'_t$ (condition (ii)).

Definition 2.4 (Doubly Reflected BSDEs with Jumps)**.** *Let $T > 0$ be a fixed terminal time and g be a Lipschitz driver. Let ξ and ζ be two adapted RCLL processes with $\zeta_T = \xi_T$ a.s., $\xi \in \mathcal{S}^2$, $\zeta \in \mathcal{S}^2$, $\xi_t \le \zeta_t$, $0 \le t \le T$ a.s.*

A process $(Y, Z, k(.), A, A')$ in $\mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu \times \mathcal{A}^2 \times \mathcal{A}^2$ is said to be a solution of the doubly reflected BSDE (DRBSDE) associated with driver g and barriers ξ, ζ if

$$-dY_t = g(t, Y_t, Z_t, k_t(\cdot))dt + dA_t - dA'_t - Z_t dW_t - \int_{\mathbf{E}} k_t(e)\tilde{N}(dt, de); \; Y_T = \xi_T, \quad (2.2)$$

with

(i) $\xi_t \le Y_t \le \zeta_t, \; 0 \le t \le T$ *a.s.,*

(ii) $dA_t \perp dA'_t$

(iii) $\int_0^T (Y_t - \xi_t)dA^c_t = 0$ *a.s. and* $\int_0^T (\zeta_t - Y_t)dA'^c_t = 0$ *a.s.*

$\Delta A^d_\tau = \Delta A^d_\tau \mathbf{1}_{\{Y_{\tau^-} = \xi_{\tau^-}\}}$ *and* $\Delta A'^d_\tau = \Delta A'^d_\tau \mathbf{1}_{\{Y_{\tau^-} = \zeta_{\tau^-}\}}$ *a.s. $\forall \tau \in \mathcal{T}_0$ predictable.*

Here A^c (resp A'^c) denotes the continuous part of A (resp A') and A^d (resp A'^d) its discontinuous part.

Remark 2.5. When A and A' are not required to be mutually singular, they can simultaneously increase on $\{\xi_{t^-} = \zeta_{t^-}\}$. The constraint $dA_t \perp dA'_t$ allows us to obtain the

[2] Note that if the random measures dA_t and dA'_t are mutually singular in the above probabilistic sense, then, for almost every ω, the deterministic measures on $[0, T]$ $dA_t(\omega)$ and $dA'_t(\omega)$ are mutually singular in the classical analysis sense. The converse is not straightforward. However, it holds by Proposition A.7.

uniqueness of the non decreasing RCLL processes A and A', without the usual strict separability condition $\xi < \zeta$ (see Theorem 3.5).

We introduce the following definition.

Definition 2.6. *A progressively measurable process (ϕ_t) (resp. integrable) is said to be* left-upper semicontinuous (l.u.s.c.) along stopping times *(resp. along stopping times in expectation) if for all $\tau \in \mathcal{T}_0$ and for each non decreasing sequence of stopping times (τ_n) such that $\tau^n \uparrow \tau$ a.s.,*

$$\phi_\tau \geq \limsup_{n\to\infty} \phi_{\tau_n} \quad \textit{a.s.} \quad (\textit{resp. } E[\phi_\tau] \geq \limsup_{n\to\infty} E[\phi_{\tau_n}]). \tag{2.3}$$

Remark 2.7. When (ϕ_t) is left-limited, then (ϕ_t) is *left-upper semicontinuous (l.u.s.c.) along stopping times* if and only if for all predictable stopping time $\tau \in \mathcal{T}_0$, $\phi_\tau \geq \phi_{\tau_-}$ a.s.

3 Classical Dynkin games and links with doubly reflected BSDEs with a driver process

In this section, we suppose that the driver g does not depend on (y, z, k), that is $g(\omega, t, y, z, k) = g(\omega, t)$, where $g \in \mathbb{H}^2$. Let ξ and ζ be two adapted processes only supposed to be RCLL with $\zeta_T = \xi_T$ a.s., $\xi \in \mathcal{S}^2$, $\zeta \in \mathcal{S}^2$, $\xi_t \leq \zeta_t$, $0 \leq t \leq T$ a.s.

We prove below that the doubly reflected BSDE associated with the driver process $g(t)$ and the barriers ξ and ζ admits a unique solution $(Y, Z, k(\cdot), A, A')$, which is related to a classical Dynkin game problem. The results of this section complete previous works on classical Dynkin games and DRBSDEs (see e.g. [9, 22, 25, 7, 27, 26]). In particular, we provide an existence result of saddle points under weaker assumptions than those made in the previous literature.

For any $S \in \mathcal{T}_0$ and any stopping times $\tau, \sigma \in \mathcal{T}_S$, consider the gain (or payoff):

$$I_S(\tau, \sigma) = \int_S^{\sigma\wedge\tau} g(u)du + \xi_\tau \mathbf{1}_{\{\tau \leq \sigma\}} + \zeta_\sigma \mathbf{1}_{\{\sigma < \tau\}}. \tag{3.1}$$

For any $S \in \mathcal{T}_0$, the upper and lower value functions at time S are defined respectively by

$$\overline{V}(S) := \operatorname*{ess\,inf}_{\sigma \in \mathcal{T}_S} \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_S} E[I_S(\tau, \sigma)|\mathcal{F}_S] \tag{3.2}$$

$$\underline{V}(S) := \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_S} \operatorname*{ess\,inf}_{\sigma \in \mathcal{T}_S} E[I_S(\tau, \sigma)|\mathcal{F}_S]. \tag{3.3}$$

We clearly have the inequality $\underline{V}(S) \leq \overline{V}(S)$ a.s. By definition, we say that the Dynkin game is *fair* (or *there exists a value function*) at time S if $\overline{V}(S) = \underline{V}(S)$ a.s.

Definition 3.1 (S-saddle point)**.** *Let $S \in \mathcal{T}_0$. A pair $(\tau^*, \sigma^*) \in \mathcal{T}_S^2$ is called an* S-saddle point *if for each $(\tau, \sigma) \in \mathcal{T}_S^2$, we have*

$$E[I_S(\tau, \sigma^*)|\mathcal{F}_S] \leq E[I_S(\tau^*, \sigma^*)|\mathcal{F}_S] \leq E[I_S(\tau^*, \sigma)|\mathcal{F}_S] \ \textit{a.s.}$$

We introduce the following RCLL adapted processes (which depend on the process g):

$$\tilde{\xi}_t^g := \xi_t - E[\xi_T + \int_t^T g(s)ds|\mathcal{F}_t], \quad \tilde{\zeta}_t^g := \zeta_t - E[\zeta_T + \int_t^T g(s)ds|\mathcal{F}_t], \quad 0 \leq t \leq T. \tag{3.4}$$

Note that since $g \in \mathbb{H}^2$ and $\xi \in \mathcal{S}^2$, $\tilde{\xi}^g$ and $\tilde{\zeta}_t^g$ belong to $\mathcal{S}^2$. Moreover, we have $\tilde{\xi}_T^g = \tilde{\zeta}_T^g = 0$ a.s.

Definition 3.2. *An optional process ϕ valued in $[0, +\infty]$ is said to be a* strong supermartingale *if for any $\theta, \theta' \in \mathcal{T}_0$ such that $\theta \geq \theta'$ a.s., $E[\phi_\theta \,|\, \mathcal{F}_{\theta'}] \leq \phi_{\theta'}$ a.s.*

Set $J^{g,0}_{\cdot} = 0$ and $J'^{g,0}_{\cdot} = 0$. We define recursively for all $n \in \mathbb{N}$, the RCLL supermartingales $J^{g,n}_{\cdot}$ and $J'^{g,n}_{\cdot}$ satisfying for all $\theta \in \mathcal{T}_0$ the equalities [3]

$$J^{g,n+1}_\theta = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_\theta} E\left[J'^{g,n}_\tau + \tilde{\xi}^g_\tau | \mathcal{F}_\theta \right] \text{ a.s. and } J'^{g,n+1}_\theta = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_\theta} E\left[J^{g,n}_\tau - \tilde{\zeta}^g_\tau | \mathcal{F}_\theta \right] \text{ a.s.} \tag{3.5}$$

Lemma 3.3. *The sequences of processes $(J^{g,n}_{\cdot})_{n\in\mathbb{N}}$ and $(J'^{g,n}_{\cdot})_{n\in\mathbb{N}}$ are non decreasing.*

Moreover, the processes $J^g_{\cdot}$ and $J'^g_{\cdot}$ defined for all $t \in [0,T]$ by $J^g_t := \lim_{n\to+\infty} J^{g,n}_t$ and $J'^g_t := \lim_{n\to+\infty} J'^{g,n}_t$ are strong supermartingales valued in $[0,+\infty]$. They satisfy $J^g_T = J'^g_T = 0$ a.s. and for all $\theta \in \mathcal{T}_0$,

$$J^g_\theta = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_\theta} E\left[J'^g_\tau + \tilde{\xi}^g_\tau | \mathcal{F}_\theta \right] \text{ a.s. and } J'^g_\theta = \operatorname*{ess\,sup}_{\sigma \in \mathcal{T}_\theta} E\left[J^g_\sigma - \tilde{\zeta}^g_\sigma | \mathcal{F}_\theta \right] \text{ a.s.} \tag{3.6}$$

If $J^g_0 < +\infty$ and $J'^g_0 < +\infty$, then J^g and J'^g are RCLL supermartingales.

Proof. See Appendix. □

Using this lemma, we derive that if J^g and J'^g belong to S^2, then there exists a solution of the DRBSDE (2.2) associated with the driver $g(t)$.

Theorem 3.4. *Let ξ and ζ be two adapted RCLL processes in $\mathcal{S}^2$ with $\zeta_T = \xi_T$ a.s. and $\xi_t \le \zeta_t$, $0 \le t \le T$ a.s. Suppose that $J^g, J'^g \in \mathcal{S}^2$. Let $\overline{Y}$ be the RCLL adapted process defined by*

$$\overline{Y}_t := J^g_t - J'^g_t + E[\xi_T + \int_t^T g(s)ds | \mathcal{F}_t]; \; 0 \le t \le T. \tag{3.7}$$

There exist $(Z,k,A,A') \in I\!H^2 \times I\!H^2_\nu \times \mathcal{A}^2 \times \mathcal{A}^2$ such that $(\overline{Y}, Z, k, A, A')$ is a solution of DRBSDE (2.2) associated with the driver process $g(t)$.

Proof. As usual in the literature on DRBSDEs (see e.g. [9, 22, 7]), the proof is based on some results of Optimal Stopping Theory. By assumption, the processes J^g and J'^g are finite. Hence, the difference $J^g - J'^g$ and thus the process $\overline{Y}$ are well defined. By Lemma 3.3, we have $J^g_T = J'^g_T$ a.s. Hence, $\overline{Y}_T = \xi_T$ a.s. By (3.6), we have $J^g \ge J'^g + \tilde{\xi}^g$ and $J'^g \ge J^g - \tilde{\zeta}^g$. Using the definitions of $\tilde{\xi}^g$, $\tilde{\zeta}^g$ and $\overline{Y}$, we derive that $\xi \le \overline{Y} \le \zeta$.

Moreover, by the last assertion of Lemma 3.3 and the assumption $J^g, J'^g \in \mathcal{S}^2$, we obtain that J^g and J'^g are indistinguishable from RCLL supermartingales. We thus can apply the Doob-Meyer decomposition, and derive the existence of two square integrable martingales M and M' and two processes B and $B' \in \mathcal{A}^2$ such that:

$$dJ^g_t = dM_t - dB_t \quad ; \quad dJ'^g_t = dM'_t - dB'_t. \tag{3.8}$$

Set

$$\overline{M}_t := M_t - M'_t + E[\xi_T + \int_0^T g(s)ds | \mathcal{F}_t].$$

By (3.8), (3.7), we derive $d\overline{Y}_t = d\overline{M}_t - d\alpha_t - g(t)dt$, with $\alpha := B - B'$.

Now, by the martingale representation theorem, there exist $Z \in \mathbb{H}^2$ and $k \in \mathbb{H}^2_\nu$ such that $d\overline{M}_t = Z_t dW_t + \int_{\mathbf{E}} k_t(e)\tilde{N}(de,dt)$. Hence,

$$-d\overline{Y}_t = g(t)dt + d\alpha_t - Z_t dW_t - \int_{\mathbf{E}} k_t(e)\tilde{N}(dt,de).$$

By the optimal stopping theory (see [28, Appendix Sect. D] in the case of a continuous reward process, and [30, Proposition B.7, B.11] in the right-continuous case), the process B^c increases only when the value function J^g is equal to the corresponding

[3] Note that these processes exist by aggregation results (cf. [17]).

reward $J'^g + \tilde{\xi}^g$. Now, $\{ J_t^g = J_t'^g + \tilde{\xi}^g\} = \{\overline{Y_t} = \xi_t\}$. Hence, $\int_0^T (\overline{Y_t} - \xi_t) dB_t^c = 0$ a.s. Similarly the process B'^c satisfies $\int_0^T (\overline{Y_t} - \zeta_t) dB_t'^c = 0$ a.s. Moreover, thanks to a result from optimal stopping theory (cf. [17, Proposition 2.34] or [30]), for each predictable stopping time $\tau \in \mathcal{T}_0$ we have $\Delta B_\tau^d = \mathbf{1}_{J^g_{\tau^-} = J'^g_{\tau^-} + \tilde{\xi}^g_{\tau^-}} \Delta B_\tau^d = \mathbf{1}_{\overline{Y}_{\tau^-} = \xi_{\tau^-}} \Delta B_\tau^d$ a.s. and $\Delta B_\tau'^d = \mathbf{1}_{\overline{Y}_{\tau^-} = \zeta_{\tau^-}} \Delta B_\tau'^d$ a.s.

By the canonical decomposition of an RCLL predictable process with integrable total variation (see Proposition A.7), there exist $A, A' \in \mathcal{A}^2$ such that $\alpha = A - A'$ with $dA_t \perp dA'_t$. Also, $dA_t \ll dB_t$. Hence, since $\int_0^T \mathbf{1}_{\overline{Y}_{t^-} > \xi_{t^-}} dB_t = 0$ a.s., we get $\int_0^T \mathbf{1}_{\overline{Y}_{t^-} > \xi_{t^-}} dA_t = 0$ a.s. Similarly, we obtain $\int_0^T \mathbf{1}_{\overline{Y}_{t^-} < \zeta_{t^-}} dA'_t = 0$ a.s. The processes A and A' thus satisfy conditions (2.2)(iii) (with Y replaced by $\overline{Y}$). The process $(\overline{Y}, Z, k, A, A')$ is thus a solution of DRBSDE (2.2). □

From this result, we derive the following uniqueness and existence result for the DRBSDE associated with the driver process $g(t)$, as well as the characterization of the solution as the value function of the above Dynkin game problem.

Theorem 3.5. *Let ξ and ζ be two adapted RCLL processes in $\mathcal{S}^2$ with $\zeta_T = \xi_T$ a.s. and $\xi_t \leq \zeta_t$, $0 \leq t \leq T$ a.s. Suppose that J^g, $J'^g \in \mathcal{S}^2$. The doubly reflected BSDE* (2.2) *associated with driver process $g(t)$ admits a unique solution (Y, Z, k, A, A') in $\mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu \times (\mathcal{A}^2)^2$.*

For each $S \in \mathcal{T}_0$, Y_S is the common value function of the Dynkin game, that is

$$Y_S = \overline{V}(S) = \underline{V}(S) \quad \textit{a.s.} \tag{3.9}$$

Moreover, if the processes A, A' are continuous, then, for each $S \in \mathcal{T}_0$, the pair of stopping times (τ_s^*, σ_s^*) defined by

$$\sigma_S^* := \inf\{t \geq S,\ Y_t = \zeta_t\}; \quad \tau_S^* := \inf\{t \geq S,\ Y_t = \xi_t\} \tag{3.10}$$

is an S-saddle point for the Dynkin game problem associated with the gain I_S.

A short proof is given in the Appendix.

Remark 3.6. The condition $dA_t \perp dA'_t$ ensures that for each predictable stopping time $\tau \in \mathcal{T}_0$, $\Delta A_\tau^d \, \Delta A_\tau'^d = 0$ a.s. Now, since Y satisfies (2.2), we have $\Delta Y_\tau = \Delta A_\tau'^d - \Delta A_\tau^d$ a.s. We thus have $\Delta A_\tau^d = (\Delta Y_\tau)^-$ and $\Delta A_\tau'^d = (\Delta Y_\tau)^+$ a.s. for each predictable stopping time $\tau \in \mathcal{T}_0$.

We now provide a sufficient condition on ξ and ζ for the existence of saddle points. By the last assertion of Theorem 3.5, it is sufficient to give a condition which ensures the continuity of A and A'.

Theorem 3.7 (Existence of S-saddle points)**.** *Suppose that the assumptions of Theorem 3.5 are satisfied. Let $(Y, Z, k(.), A, A')$ be the solution of DRBSDE* (2.2)*. We have*

(i) If ξ (resp. $-\zeta$) is l.u.s.c. along stopping times, then the process A (resp. A') is continuous.

(ii) When ξ and $-\zeta$ are l.u.s.c. along stopping time, for each $S \in \mathcal{T}_0$, the pair of stopping times (τ_S^*, σ_S^*) defined by (3.10) *is an S-saddle point.*

Remark 3.8. For *(ii)*, the assumptions made on ξ and ζ are weaker than the ones made in the literature where it is supposed $\xi_t < \zeta_t, t < T$ a.s. (see e.g. [1, 9, 31]).

Proof. Suppose that ξ is l.u.s.c. along stopping times. Let $\tau \in \mathcal{T}_0$ be a predictable stopping time. Let us show $\Delta A_\tau = 0$ a.s.

Since $dA_t \perp dA'_t$, we have $\Delta A_\tau = (\Delta Y_\tau)^-$ a.s. (see Remark 3.6 above). Since A satisfies the Skorohod condition, we have a.s.

$$\Delta A_\tau = \mathbf{1}_{\{Y_{\tau^-} = \xi_{\tau^-}\}} (Y_{\tau^-} - Y_\tau)^+ = \mathbf{1}_{\{Y_{\tau^-} = \xi_{\tau^-}\}} (\xi_{\tau^-} - Y_\tau)^+ \leq \mathbf{1}_{\{Y_{\tau^-} = \xi_{\tau^-}\}} (\xi_\tau - Y_\tau)^+,$$

where the last inequality follows from the inequality $\xi_{\tau^-} \leq \xi_\tau$ a.s. (see Remark 2.7). Since $\xi \leq Y$, we derive that $\Delta A_\tau \leq 0$ a.s. Hence, $\Delta A_\tau = 0$ a.s., and this holds for each predictable stopping time τ. Consequently, A is continuous. Similarly, one can show that if $-\zeta$ is l.u.s.c. along stopping times, then A' is continuous, which completes the proof of *(i)*.

The assertion *(ii)* follows from *(i)* since, by the second assertion of Theorem 3.5, the continuity property of A and A' ensures the existence of saddle points. □

Definition 3.9 (Mokobodzki's condition)**.** *Let ξ and ζ be adapted RCLL processes in $\mathcal{S}^2$ with $\zeta_T = \xi_T$ a.s. and $\xi_t \leq \zeta_t$, $0 \leq t \leq T$ a.s. Mokobodzki's condition is said to be satisfied when there exist two nonnegative RCLL supermartingales H and H' in $\mathcal{S}^2$ such that:*

$$\xi_t \leq H_t - H'_t \leq \zeta_t \quad 0 \leq t \leq T \quad \text{a.s.} \tag{3.11}$$

Note that Mokobodzki's condition holds, for instance, when ξ or ζ is a semimartingale satisfying some integrability conditions (see Remark A.8 for details).

Proposition 3.10. *Let $g \in I\!H^2$. Let ξ and ζ be two adapted RCLL processes in $\mathcal{S}^2$ with $\zeta_T = \xi_T$ a.s. and $\xi_t \leq \zeta_t$, $0 \leq t \leq T$ a.s. The following assertions are equivalent:*

(i) $J^g \in \mathcal{S}^2$

(ii) $J^0 \in \mathcal{S}^2$

(iii) Mokobodzki's condition holds.

(iv) DRBSDE (2.2) *with driver process $g(t)$ has a solution.*

A short proof is given in the Appendix. Note that the equivalence between *(iii)* and *(iv)* is well-known.

4 Generalized Dynkin games and links with nonlinear doubly reflected BSDEs

In this section, we are given a Lipschitz driver g.

Theorem 4.1 (Existence and uniqueness for DRBSDEs)**.** *Suppose ξ and ζ are RCLL adapted process in $\mathcal{S}^2$ such that $\xi_T = \zeta_T$ a.s. and $\xi_t \leq \zeta_t$, $0 \leq t \leq T$ a.s. Suppose that $J^0 \in \mathcal{S}^2$ (or equivalently that Mokobodzki's condition is satisfied).*

Then, DRBSDE (2.2) *admits a unique solution $(Y, Z, k(.), A, A') \in \mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu \times (\mathcal{A}^2)^2$.*

If ξ (resp. $-\zeta$) is l.u.s.c. along stopping times, then the process A (resp. A') is continuous.

Remark 4.2. Note that the solution Y of the DRBSDE (2.2) coincides with the value function of the classical Dynkin game (3.2) and (3.3) with the gain:

$$I_S(\tau, \sigma) = \int_S^{\sigma \wedge \tau} g(u, Y_u, Z_u, k_u) du + \xi_\tau \mathbf{1}_{\{\tau \leq \sigma\}} + \zeta_\sigma \mathbf{1}_{\{\sigma < \tau\}}, \tag{4.1}$$

where Z, k are the associated processes with Y. However, this classical characterization of Y (see e.g. [9]) is not really exploitable because the instantaneous reward process $g(u, Y_u, Z_u, k_u)$ depends on the value function Y (and Z, K) of the associated Dynkin game.

The proof of the existence and uniqueness of the solution is based on classical contraction arguments and is given in the Appendix.

We now introduce a game problem, which can be seen as a *generalized Dynkin game* expressed in terms of $\mathcal{E}^g$-expectations.

In order to ensure that the $\mathcal{E}^g$-expectation is non decreasing, we make the following assumption.

Assumption 4.3. *Assume that $dP \otimes dt$-a.s for each $(y,z,k_1,k_2) \in \mathbb{R}^2 \times (L^2_\nu)^2$,*

$$g(t,y,z,k_1) - g(t,y,z,k_2) \geq \langle \gamma_t^{y,z,k_1,k_2}\,,\, k_1 - k_2 \rangle_\nu,$$

$$\textit{with} \quad \gamma : [0,T] \times \Omega \times \mathbb{R}^2 \times (L^2_\nu)^2 \to L^2_\nu\,;\, (\omega,t,y,z,k_1,k_2) \mapsto \gamma_t^{y,z,k_1,k_2}(\omega,.)$$

$\mathcal{P} \cdot \otimes \mathcal{B}(\mathbf{R}^2) \otimes \mathcal{B}((L^2_\nu)^2)$-measurable and satisfying the inequalities

$$\gamma_t^{y,z,k_1,k_2}(e) \geq -1 \quad \textit{and} \quad \|\gamma_t^{y,z,k_1,k_2}\|_\nu \leq K, \tag{4.2}$$

for each $(y,z,k_1,k_2) \in \mathbb{R}^2 \times (L^2_\nu)^2$, respectively $dP \otimes dt \otimes d\nu(e)$-a.s. and $dP \otimes dt$-a.s. (where K is a positive constant).

Assumption 4.3 is satisfied for example when g is $\mathcal{C}^1$ with respect to k with $\nabla_k g \geq -1$ and $|\nabla_k g| \leq \psi$, where $\psi \in L^2_\nu$. Assumption 4.3 is also satisfied when g is of the form $g(\omega,t,y,z,k) := \overline{g}(\omega,t,y,z,\int_{\mathbf{E}} k(e)\psi(e)\nu(de))$ where ψ is a nonnegative function in L^2_ν and $\overline{g} : \Omega \times [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ is Borelian and non-decreasing with respect to k, (see Proposition A.2 in the Appendix for details).

Assumption 4.3 ensures the non decreasing property of $\mathcal{E}^g$ by the comparison theorem for BSDEs with jumps (see [35, Theorem 4.2]). When in (4.2), $\gamma_t > -1$, the strict comparison theorem (see in [35, Theorem 4.4]) implies that $\mathcal{E}^g$ is strictly monotonous. For each $\tau, \sigma \in \mathcal{T}_0$, the *reward* (or *payoff*) at time $\tau \wedge \sigma$ is given by the random variable

$$I(\tau,\sigma) := \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\sigma < \tau}. \tag{4.3}$$

Note that $I(\tau,\sigma)$ is $\mathcal{F}_{\tau\wedge\sigma}$-measurable. Let $S \in \mathcal{T}_0$. For each $\tau \in \mathcal{T}_S$ and $\sigma \in \mathcal{T}_S$, the associated *criterium* is given by $\mathcal{E}^g_{S,\tau\wedge\sigma}(I(\tau,\sigma))$, the g-evaluation of the payoff $I(\tau,\sigma)$.

Recall that $\mathcal{E}^g_{\cdot,\tau\wedge\sigma}(I(\tau,\sigma)) = X^{\tau,\sigma}_\cdot$, where $(X^{\tau,\sigma}_\cdot, \pi^{\tau,\sigma}_\cdot, l^{\tau,\sigma}_\cdot)$ is the solution of the BSDE associated with driver g, terminal time $\tau \wedge \sigma$ and terminal condition $I(\tau,\sigma)$, that is

$$-dX^{\tau,\sigma}_s = g(s, X^{\tau,\sigma}_s, \pi^{\tau,\sigma}_s, l^{\tau,\sigma}_s)ds - \pi^{\tau,\sigma}_s dW_s - \int_{\mathbf{E}} l^{\tau,\sigma}_s(e)\tilde{N}(ds,de); \qquad X^{\tau,\sigma}_{\tau\wedge\sigma} = I(\tau,\sigma).$$

To simplify notation, $\mathcal{E}^g$ is denoted by $\mathcal{E}$ in the sequel.

For each stopping time $S \in \mathcal{T}_0$, the *upper* and *lower value functions* at time S are defined respectively by

$$\overline{V}(S) := \operatorname*{ess\,inf}_{\sigma\in\mathcal{T}_S} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_S} \mathcal{E}_{S,\tau\wedge\sigma}(I(\tau,\sigma)); \tag{4.4}$$

$$\underline{V}(S) := \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_S} \operatorname*{ess\,inf}_{\sigma\in\mathcal{T}_S} \mathcal{E}_{S,\tau\wedge\sigma}(I(\tau,\sigma)). \tag{4.5}$$

We clearly have the inequality $\underline{V}(S) \leq \overline{V}(S)$ a.s. By definition, we say that the game is *fair* (or *there exists a value function*) at time S if $\overline{V}(S) = \underline{V}(S)$ a.s.

We now give the definition of an S-saddle point for this game problem.

Definition 4.4. *Let $S \in \mathcal{T}_0$. A pair $(\tau^*,\sigma^*) \in \mathcal{T}_S^2$ is called an* S-saddle point *for the generalized Dynkin game if for each $(\tau,\sigma) \in \mathcal{T}_S^2$ we have*

$$\mathcal{E}_{S,\tau\wedge\sigma^*}(I(\tau,\sigma^*)) \leq \mathcal{E}_{S,\tau^*\wedge\sigma^*}(I(\tau^*,\sigma^*)) \leq \mathcal{E}_{S,\tau^*\wedge\sigma}(I(\tau^*,\sigma)) \quad \textit{a.s.}$$

We provide a sufficient condition for the existence of an S-saddle point and the characterization of the common value function as the solution of the DRBSDE. We first introduce the following definition.

Definition 4.5. *Let $Y \in \mathcal{S}^2$. The process Y is said to be a strong $\mathcal{E}^g$-supermartingale (resp $\mathcal{E}^g$-submartingale), if $\mathcal{E}^g_{\sigma,\tau}(Y_\tau) \leq Y_\sigma$ (resp. $\mathcal{E}^g_{\sigma,\tau}(Y_\tau) \geq Y_\sigma$) a.s. on $\sigma \leq \tau$, for all $\sigma, \tau \in \mathcal{T}_0$.*

Lemma 4.6. *Suppose that the driver g satisfies Assumption* (4.3). *Let ξ and ζ be RCLL adapted processes in $\mathcal{S}^2$ such that $\xi_T = \zeta_T$ a.s. and $\xi_t \le \zeta_t$, $0 \le t \le T$ a.s. Suppose that Mokobodzki's condition is satisfied. Let $(Y, Z, k(\cdot), A, A')$ be the solution of the DRBSDE* (2.2). *Let $S \in \mathcal{T}_0$. Let $(\hat\tau, \hat\sigma) \in \mathcal{T}_S^2$. Suppose that $(Y_t,\, S \le t \le \hat\tau)$ is a strong $\mathcal{E}$-submartingale and that $(Y_t,\, S \le t \le \hat\sigma)$ is a strong $\mathcal{E}$-supermartingale with $Y_{\hat\tau} = \xi_{\hat\tau}$ and $Y_{\hat\sigma} = \zeta_{\hat\sigma}$ a.s.*

The pair $(\hat\tau, \hat\sigma)$ is then an S-saddle point for the generalized Dynkin game (4.4)-(4.5) *and*

$$Y_S = \overline{V}(S) = \underline{V}(S) \text{ a.s.}$$

Remark 4.7. A well-known sufficient condition for a pair of stopping times $(\hat\tau, \hat\sigma)$ to be a saddle point for the classical Dynkin game (3.2)-(3.3) is that $\hat\tau$ (resp. $\hat\sigma$) is an optimal stopping time for the optimal stopping problem J_S^g (resp. $J_S'^{\,g}$) (see e.g. [1, Theorem 2.4] or [31, Proposition 3.1]). In the nonlinear case, we cannot have an analogous sufficient condition since there is no coupled optimal stopping problem associated with our generalized Dynkin game. Note also that in the classical linear case, the sufficient condition given in Lemma 4.6 is weaker than the one given in the literature.

Proof. Since the process $(Y_t,\, S \le t \le \hat\tau \wedge \hat\sigma)$ is a strong $\mathcal{E}$-martingale (see Definition 4.5) and since $Y_{\hat\tau} = \xi_{\hat\tau}$ and $Y_{\hat\sigma} = \zeta_{\hat\sigma}$ a.s., we have

$$Y_S = \mathcal{E}_{S,\hat\tau\wedge\hat\sigma}(Y_{\hat\tau\wedge\hat\sigma}) = \mathcal{E}_{S,\hat\tau\wedge\hat\sigma}(\xi_{\hat\tau}\mathbf{1}_{\hat\tau\le\hat\sigma} + \zeta_{\hat\sigma}\mathbf{1}_{\hat\sigma<\hat\tau}) = \mathcal{E}_{S,\hat\tau\wedge\hat\sigma}(I(\hat\tau,\hat\sigma)) \quad \text{a.s.}$$

Let $\tau \in \mathcal{T}_S$. We want to show that for each $\tau \in \mathcal{T}_S$

$$Y_S \;\ge\; \mathcal{E}_{S,\tau\wedge\hat\sigma}(I(\tau,\hat\sigma)) \quad \text{a.s.} \tag{4.6}$$

Since the process $(Y_t,\, S \le t \le \tau \wedge \hat\sigma)$ is a strong $\mathcal{E}$-supermartingale, we get

$$Y_S \;\ge\; \mathcal{E}_{S,\tau\wedge\hat\sigma}(Y_{\tau\wedge\hat\sigma}) \quad \text{a.s.} \tag{4.7}$$

Since $Y \ge \xi$ and $Y_{\hat\sigma} = \zeta_{\hat\sigma}$ a.s., we also have

$$Y_{\tau\wedge\hat\sigma} = Y_\tau\mathbf{1}_{\tau\le\hat\sigma} + Y_{\hat\sigma}\mathbf{1}_{\hat\sigma<\tau} \;\ge\; \xi_\tau\mathbf{1}_{\tau\le\hat\sigma} + \zeta_{\hat\sigma}\mathbf{1}_{\hat\sigma<\tau} = I(\tau,\hat\sigma) \quad \text{a.s.}$$

By inequality (4.7) and the monotonicity property of $\mathcal{E}$, we derive inequality (4.6).

Similarly, one can show that for each $\sigma \in \mathcal{T}_S$, we have:

$$Y_S \;\le\; \mathcal{E}_{S,\hat\tau\wedge\sigma}(I(\hat\tau,\sigma)) \quad \text{a.s.}$$

The pair $(\hat\tau, \hat\sigma)$ is thus an S-saddle point and $Y_S = \overline{V}(S) = \underline{V}(S)$ a.s. □

We now provide an existence result under an additional assumption.

Theorem 4.8 (Existence of S-saddle points)**.** *Suppose that g satisfies Assumption* (4.3). *Let ξ and ζ be RCLL adapted processes in $\mathcal{S}^2$ such that $\xi_T = \zeta_T$ a.s. and $\xi_t \le \zeta_t$, $0 \le t \le T$ a.s. Suppose that Mokobodzki's condition is satisfied.*

Let (Y, Z, k, A, A') be the solution of the DRBSDE (2.2). *Suppose that A, A' are continuous (which is the case if ξ and $-\zeta$ are l.u.s.c. along stopping times). For each $S \in \mathcal{T}_0$, let*

$$\tau_S^* := \inf\{t \ge S,\; Y_t = \xi_t\}; \quad \sigma_S^* := \inf\{t \ge S,\; Y_t = \zeta_t\}.$$

$$\overline{\tau}_S := \inf\{t \ge S,\; A_t > A_S\}; \quad \overline{\sigma}_S := \inf\{t \ge S,\; A'_t > A'_S\}.$$

Then, for each $S \in \mathcal{T}_0$, the pairs of stopping times (τ_S^*, σ_S^*) and $(\overline{\tau}_S, \overline{\sigma}_S)$ are S-saddle points for the generalized Dynkin game and $Y_S = \overline{V}(S) = \underline{V}(S)$ a.s.

Moreover, $Y_{\sigma_S^*} = \zeta_{\sigma_S^*}$, $Y_{\tau_S^*} = \xi_{\tau_S^*}$, $A_{\tau_S^*} = A_S$ and $A'_{\sigma_S^*} = A'_S$ a.s. The same properties hold for $\overline{\tau}_S, \overline{\sigma}_S$.

Remark 4.9. Note that $\sigma^*_S \leq \overline{\sigma}_S$ and $\tau^*_S \leq \overline{\tau}_S$ a.s. Moreover, by Proposition A.6 in the Appendix, $(Y_t,\, S \leq t \leq \overline{\tau}_S)$ is a strong $\mathcal{E}$-submartingale and $(Y_t,\, S \leq t \leq \overline{\sigma}_S)$ is a strong $\mathcal{E}$-supermartingale.

Proof. Let $S \in \mathcal{T}_0$. Since Y and ξ are right-continuous processes, we have $Y_{\sigma^*_S} = \zeta_{\sigma^*_S}$ and $Y_{\tau^*_S} = \xi_{\tau^*_S}$ a.s. By definition of τ^*_S, for almost every ω, we have $Y_t(\omega) > \xi_t(\omega)$ for each $t \in [S(\omega), \tau^*_S(\omega)[$. Hence, since Y is solution of the DRBSDE, the continuous process A is constant on $[S, \tau^*_S]$ a.s. because A is continuous. Hence, $A_{\tau^*_S} = A_S$ a.s. Similarly, $A'_{\sigma^*_S} = A'_S$ a.s. By Lemma 4.6, (τ^*_S, σ^*_S) is an S-saddle point and $Y_S = \overline{V}(S) = \underline{V}(S)$ a.s.

It remains to show that $(\overline{\tau}_S, \overline{\sigma}_S)$ is an S-saddle point. By definition of $\overline{\tau}_S$, $\overline{\sigma}_S$, we have $A_{\overline{\tau}_S} = A_S$ a.s. and $A'_{\overline{\sigma}_S} = A'_S$ a.s. because A and A' are continuous. Moreover, since the continuous process A increases only on $\{Y_t = \xi_t\}$, we have $Y_{\overline{\tau}_S} = \xi_{\overline{\tau}_S}$ a.s. Similarly, $Y_{\overline{\sigma}_S} = \zeta_{\overline{\sigma}_S}$ a.s. The result then follows from Lemma 4.6. □

In the case of irregular payoffs ξ and ζ, there does not generally exist a saddle point. However, we will now see that it is not necessary to have the existence of an S-saddle point to ensure the existence of a common value function and its characterization as the solution of a DRBSDE.

Theorem 4.10 (Existence and characterization of the value function)**.** *Suppose that g satisfies Assumption* (4.3)*. Let ξ and ζ be RCLL adapted processes in $\mathcal{S}^2$ such that $\xi_T = \zeta_T$ a.s. and $\xi_t \leq \zeta_t$, $0 \leq t \leq T$ a.s. Suppose that Mokobodzki's condition is satisfied. Let (Y, Z, k, A, A') be the solution of the DRBSDE* (2.2)*. Then, the generalized Dynkin game is* fair*, and for each stopping time $S \in \mathcal{T}_0$, we have*

$$Y_S = \overline{V}(S) = \underline{V}(S) \quad \textit{a.s.} \tag{4.8}$$

Proof. For each $S \in \mathcal{T}_0$ and for each $\varepsilon > 0$, let τ^ε_S and σ^ε_S be the stopping times defined by

$$\tau^\varepsilon_S := \inf\{t \geq S,\ Y_t \leq \xi_t + \varepsilon\} \quad \text{and} \quad \sigma^\varepsilon_S := \inf\{t \geq S,\ Y_t \geq \zeta_t - \varepsilon\}. \tag{4.9}$$

We first prove two lemmas.

Lemma 4.11.
- *We have*

$$Y_{\tau^\varepsilon_S} \ \leq\ \xi_{\tau^\varepsilon_S} + \varepsilon \quad \textit{a.s.} \tag{4.10}$$

$$Y_{\sigma^\varepsilon_S} \ \geq\ \zeta_{\sigma^\varepsilon_S} - \varepsilon \quad \textit{a.s.} \tag{4.11}$$

- *Moreover $A_{\tau^\varepsilon_S} = A_S$ a.s. and $A'_{\sigma^\varepsilon_S} = A'_S$ a.s.*

Remark 4.12. By the second point and Proposition $A.6$ in the Appendix, the process $(Y_t,\, S \leq t \leq \tau^\varepsilon_S)$ is a strong $\mathcal{E}$-submartingale and the process $(Y_t,\, S \leq t \leq \sigma^\varepsilon_S)$ is a strong $\mathcal{E}$-supermartingale.

Proof. The first point follows from the definitions of τ^ε_S and σ^ε_S and the right-continuity of ξ, ζ and Y. Let us show the second point. Note that $\tau^\varepsilon_S \in \mathcal{T}_S$ and $\sigma^\varepsilon_S \in \mathcal{T}_S$. Fix $\varepsilon > 0$. For a.e. ω, if $t \in [S(\omega), \tau^\varepsilon_S(\omega)[$, then $Y_t(\omega) > \xi_t(\omega) + \varepsilon$ and hence $Y_t(\omega) > \xi_t(\omega)$. It follows that almost surely, A^c is constant on $[S, \tau^\varepsilon_S]$ and A^d is constant on $[S, \tau^\varepsilon_S[$. Also, $Y_{(\tau^\varepsilon_S)^-} \geq \xi_{(\tau^\varepsilon_S)^-} + \varepsilon$ a.s. Since $\varepsilon > 0$, it follows that $Y_{(\tau^\varepsilon_S)^-} > \xi_{(\tau^\varepsilon_S)^-}$ a.s., which implies that $\Delta A^d_{\tau^\varepsilon_S} = 0$ a.s. Hence, almost surely, A is constant on $[S, \tau^\varepsilon_S]$. Similarly, A' is a.s. constant on $[S, \sigma^\varepsilon_S]$. □

Lemma 4.13. *Let $\varepsilon > 0$. For all $S \in \mathcal{T}_0$ and $(\tau, \sigma) \in \mathcal{T}^2_S$, we have*

$$\mathcal{E}_{S, \tau \wedge \sigma^\varepsilon_S}(I(\tau, \sigma^\varepsilon_S)) - K\varepsilon \ \leq\ Y_S \ \leq\ \mathcal{E}_{S, \tau^\varepsilon_S \wedge \sigma}(I(\tau^\varepsilon_S, \sigma)) + K\varepsilon \quad \textit{a.s.}, \tag{4.12}$$

where K is a positive constant which only depends on T and the Lipschitz constant C of g.

Proof. Let $\tau \in \mathcal{T}_S$. By Remark 4.9, the process $(Y_t,\ S \leq t \leq \sigma^\varepsilon_S)$ is a strong $\mathcal{E}$-supermartingale. Hence,

$$Y_S \ \geq\ \mathcal{E}_{S,\tau\wedge\sigma^\varepsilon_S}(Y_{\tau\wedge\sigma^\varepsilon_S}) \quad \text{a.s.} \tag{4.13}$$

Since $Y \geq \xi$ and $Y_{\sigma^\varepsilon_S} \geq \zeta_{\sigma^\varepsilon_S} - \varepsilon$ a.s. (see Lemma 4.11), we have:

$$Y_{\tau\wedge\sigma^\varepsilon_S} \ \geq\ \xi_\tau \mathbf{1}_{\tau\leq\sigma^\varepsilon_S} + (\zeta_{\sigma^\varepsilon_S} - \varepsilon)\mathbf{1}_{\sigma^\varepsilon_S<\tau} \ \geq\ I(\tau,\sigma^\varepsilon_S) - \varepsilon \quad \text{a.s.}$$

where the last inequality follows from the definition of $I(\tau,\sigma)$. Hence, using (4.13) and the monotonicity property of $\mathcal{E}^g$, we get

$$Y_S \ \geq\ \mathcal{E}_{S,\tau\wedge\sigma^\varepsilon_S}(I(\tau,\sigma^\varepsilon_S) - \varepsilon) \quad \text{a.s.} \tag{4.14}$$

Now, by a priori estimates on BSDEs (see [35, Proposition A.4]), we have

$$|\mathcal{E}_{S,\tau\wedge\sigma^\varepsilon_S}(I(\tau,\sigma^\varepsilon_S) - \varepsilon) - \mathcal{E}_{S,\tau\wedge\sigma^\varepsilon_S}(I(\tau,\sigma^\varepsilon_S))| \ \leq\ K\varepsilon \quad \text{a.s.}$$

It follows that

$$Y_S \ \geq\ \mathcal{E}_{S,\tau\wedge\sigma^\varepsilon_S}(I(\tau,\sigma^\varepsilon_S)) - K\varepsilon \quad \text{a.s.}$$

Similarly, one can show that $Y_S \ \leq\ \mathcal{E}_{S,\tau^\varepsilon_S\wedge\sigma}(I(\tau^\varepsilon_S,\sigma)) + K\varepsilon$. □

End of proof of Theorem 4.10. Using Lemma 4.13, we derive that for each $\varepsilon > 0$,

$$\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_S} \mathcal{E}_{S,\tau\wedge\sigma^\varepsilon_S}(I(\tau,\sigma^\varepsilon_S)) - K\varepsilon \ \leq\ Y_S \leq \operatorname*{ess\,inf}_{\sigma\in\mathcal{T}_S} \mathcal{E}_{S,\tau^\varepsilon_S\wedge\sigma}(I(\tau,\sigma^\varepsilon_S)) + K\varepsilon \ \text{ a.s.},$$

which implies

$$\overline{V}(S) - K\varepsilon \ \leq\ Y_S \ \leq\ \underline{V}(S) + K\varepsilon \quad \text{a.s.}$$

Since $\underline{V}(S) \leq \overline{V}(S)$ a.s., we get $\underline{V}(S) = Y_S = \overline{V}(S)$ a.s. The proof of Theorem 4.10 is thus complete. □

Remark 4.14. Inequality (4.12) shows that $(\tau^\varepsilon_S, \sigma^\varepsilon_S)$ defined by (4.9) is an ε'-saddle point at time S with $\varepsilon' = K\varepsilon$.

Remark 4.15. Note that contrary to the classical Dynkin game with payoff (4.1) (see Remark 4.2), the *generalized Dynkin game* is well-posed in the sense that the criterium does not depend on the value function. This new characterization of the solution Y of the nonlinear DRBSDE (2.2) in terms of the value function of the *generalized Dynkin game* is thus more interesting and exploitable than the one given in the literature. We will see in Section 6 that this result allows us to show a comparison theorem and a strict comparison theorem for DRBSDEs, as well as some new estimates with universal constants.

5 Generalized mixed game problems

We now introduce a game problem, which can be seen as a generalization of a mixed game problem studied in [4, 25, 21] to the case of $\mathcal{E}^g$-expectation/g-evaluation. The players have two actions: continuous control and stopping. Let $(g^{u,v}; (u,v) \in \mathcal{U} \times \mathcal{V})$ be a family of Lipschitz drivers satisfying Assumption (4.3). Let $S \in \mathcal{T}_0$. For each quadruple $(u,\tau,v,\sigma) \in \mathcal{U}\times\mathcal{T}_S\times\mathcal{V}\times\mathcal{T}_S$, the *criterium* at time S is given by $\mathcal{E}^{u,v}_{S,\tau\wedge\sigma}(I(\tau,\sigma))$, where $\mathcal{E}^{u,v}$ corresponds to the $g^{u,v}$-evaluation. The first (resp. second) player chooses a pair (u,τ) (resp. (v,σ)) of control and stopping time, and aims at maximizing (resp. minimizing) the criterium.

For each stopping time $S \in \mathcal{T}_0$, the *upper* and *lower value functions* at time S are defined respectively by

$$\overline{V}(S) := \operatorname*{ess\,inf}_{v\in\mathcal{V},\sigma\in\mathcal{T}_S}\ \operatorname*{ess\,sup}_{u\in\mathcal{U},\tau\in\mathcal{T}_S}\ \mathcal{E}^{u,v}_{S,\tau\wedge\sigma}(I(\tau,\sigma)); \tag{5.1}$$

$$\underline{V}(S) := \operatorname*{ess\,sup}_{u\in\mathcal{U},\tau\in\mathcal{T}_S} \operatorname*{ess\,inf}_{v\in\mathcal{V},\sigma\in\mathcal{T}_S} \mathcal{E}^{u,v}_{S,\tau\wedge\sigma}(I(\tau,\sigma)). \tag{5.2}$$

We say that the game is *fair* (or *there exists a value function*) at time S if $\overline{V}(S) = \underline{V}(S)$ a.s. We now introduce the definition of an S-saddle point for this game problem.

Definition 5.1. *Let $S \in \mathcal{T}_0$. A quadruple $(\overline{u},\overline{\tau},\overline{v},\overline{\sigma}) \in \mathcal{U}\times\mathcal{T}_S\times\mathcal{V}\times\mathcal{T}_S$ is called an* S-saddle point *for the generalized mixed game problem if for each $(u,\tau,v,\sigma) \in \mathcal{U}\times\mathcal{T}_S\times\mathcal{V}\times\mathcal{T}_S$ we have*

$$\mathcal{E}^{u,\overline{v}}_{S,\tau\wedge\overline{\sigma}}(I(\tau,\overline{\sigma})) \;\le\; \mathcal{E}^{\overline{u},\overline{v}}_{S,\overline{\tau}\wedge\overline{\sigma}}(I(\overline{\tau}\wedge\overline{\sigma})) \;\le\; \mathcal{E}^{\overline{u},v}_{S,\overline{\tau}\wedge\sigma}(I(\overline{\tau},\sigma)) \quad \textit{a.s.}$$

We prove below that when the obstacles are l.u.s.c. along stopping times, there exist saddle points for the above *generalized mixed game problem*.

Theorem 5.2. *Let $(g^{u,v};(u,v)\in\mathcal{U}\times\mathcal{V})$ be a family of Lipschitz drivers satisfying Assumptions* (4.3)*. Let ξ and ζ be RCLL adapted processes in $\mathcal{S}^2$ and l.u.s.c. along stopping times, such that $\xi_T=\zeta_T$ a.s. and $\xi_t\le\zeta_t$, $0\le t\le T$ a.s. Suppose that Mokobodzki's condition is satisfied and that there exist controls $\overline{u}\in\mathcal{U}$ and $\overline{v}\in\mathcal{V}$ such that for each $(u,v)\in\mathcal{U}\times\mathcal{V}$,*

$$g^{u,\overline{v}}(t,Y_t,Z_t,k_t) \;\le\; g^{\overline{u},\overline{v}}(t,Y_t,Z_t,k_t) \;\le\; g^{\overline{u},v}(t,Y_t,Z_t,k_t) \quad dt\otimes dP \text{ a.s. }, \tag{5.3}$$

where (Y,Z,k,A,A') is the solution of the DRBSDE (2.2) *associated with driver $g^{\overline{u},\overline{v}}$. Consider the stopping times*

$$\tau^*_S := \inf\{t\ge S : Y_t=\xi_t\} \quad ; \quad \sigma^*_S := \inf\{t\ge S : Y_t=\zeta_t\}.$$

The quadruple $(\overline{u},\tau^*_S,\overline{v},\sigma^*_S)$ is then an S-saddle point for the generalized mixed game *problem* (5.1)-(5.2)*, and we have $Y_S=\underline{V}(S)=\overline{V}(S)$ a.s.*

Proof. By the last assertion of Theoreom 4.8, the process $(Y_t,\ S\le t\le\tau^*_S\wedge\sigma^*_S)$ is a strong $\mathcal{E}^{\overline{u},\overline{v}}$-martingale and $Y_{\tau^*_S}=\xi_{\tau^*_S}$, $Y_{\sigma^*_S}=\zeta_{\sigma^*_S}$ a.s., which implies

$$Y_S=\mathcal{E}^{\overline{u},\overline{v}}_{S,\tau^*_S\wedge\sigma^*_S}(Y_{\tau^*_S\wedge\sigma^*_S})=\mathcal{E}^{\overline{u},\overline{v}}_{S,\tau^*_S\wedge\sigma^*_S}(\xi_{\tau^*_S}\mathbf{1}_{\tau^*_S\le\sigma^*_S}+\zeta_{\sigma^*_S}\mathbf{1}_{\sigma^*_S<\tau^*_S})=\mathcal{E}^{\overline{u},\overline{v}}_{S,\tau^*_S\wedge\sigma^*_S}(I(\tau^*_S,\sigma^*_S)) \quad \text{a.s.}$$

Let $\tau\in\mathcal{T}_S$. Since $Y\ge\xi$ and $Y_{\sigma^*_S}=\zeta_{\sigma^*_S}$ a.s., we have

$$Y_{\tau\wedge\sigma^*_S}=Y_\tau\mathbf{1}_{\tau\le\sigma^*_S}+Y_{\sigma^*_S}\mathbf{1}_{\sigma^*_S<\tau}\;\ge\;\xi_\tau\mathbf{1}_{\tau\le\sigma^*_S}+\zeta_{\sigma^*_S}\mathbf{1}_{\sigma^*_S<\tau}=I(\tau,\sigma^*_S) \quad \text{a.s.}$$

Moreover, by Theorem 4.8, $A'_{\sigma^*_S}=A'_S$ a.s., which implies that:

$$-dY_t=g^{\overline{u},\overline{v}}(t,Y_t,Z_t,k_t)dt+dA_t-Z_tdW_t-\int_{\mathbf{E}}k_t(e)\tilde{N}(dt,de); \quad S\le t\le\sigma^*_S, \quad dt\otimes dP \text{ a.s.}$$

Hence, $(Y_t)_{S\le t\le\tau\wedge\sigma^*_S}$ is the solution of the BSDE associated with generalized driver $g^{\overline{u},\overline{v}}(\cdot)dt+dA_t$ and terminal condition $Y_{\tau\wedge\sigma^*_S}$. By using Assumption (5.3), the inequality $Y_{\tau\wedge\sigma^*_S}\ge I(\tau,\sigma^*_S)$ and the comparison theorem for BSDEs with jumps, we obtain that for each $u\in\mathcal{U}$:

$$Y_S \;\ge\; \mathcal{E}^{u,\overline{v}}_{S,\tau\wedge\sigma^*_S}(I(\tau,\sigma^*_S)) \quad \text{a.s.}$$

Similarly, one can prove that for each $v\in\mathcal{V},\sigma\in\mathcal{T}_S$, we have:

$$Y_S \;\le\; \mathcal{E}^{\overline{u},v}_{S,\tau^*_S\wedge\sigma}(I(\tau^*_S,\sigma)) \quad \text{a.s.}$$

The quadruple $(\overline{u},\tau^*_S,\overline{v},\sigma^*_S)$ is thus an S-saddle point and $Y_S=\overline{V}(S)=\underline{V}(S)$ a.s. □

Under less restricted assumptions on the obstacles, we prove below that the above game problem is *fair*, and the common value function can be characterized as the solution of a DRBSDE.

Theorem 5.3 (Existence and characterization of the value function)**.** *Let $(g^{u,v};(u,v) \in \mathcal{U} \times \mathcal{V})$ be a family of drivers satisfying Assumptions* (4.3) *and uniformly Lipschitz with common Lipchitz constant C. Let ξ and ζ be RCLL adapted processes in $\mathcal{S}^2$ such that $\xi_T = \zeta_T$ a.s. and $\xi_t \leq \zeta_t$, $0 \leq t \leq T$ a.s. Suppose that Mokobodzki's condition is satisfied and that there exist controls $\overline{u} \in \mathcal{U}$ and $\overline{v} \in \mathcal{V}$ such that for each $u \in \mathcal{U}, v \in \mathcal{V}$:*

$$g^{u,\overline{v}}(t,Y_t,Z_t,k_t) \leq g^{\overline{u},\overline{v}}(t,Y_t,Z_t,k_t) \leq g^{\overline{u},v}(t,Y_t,Z_t,k_t), \quad dt \otimes dP \text{ a.s.} \tag{5.4}$$

where (Y,Z,k,A,A') is the solution of the DRBSDE (2.2) *associated with driver $g^{\overline{u},\overline{v}}$.*

Then, the generalized mixed game *problem* (5.1)-(5.2) *is* fair*, and for each stopping time $S \in \mathcal{T}_0$, we have*

$$Y_S = \overline{V}(S) = \underline{V}(S) \quad \text{a.s.}$$

Proof. For each $S \in \mathcal{T}_0$ and for each $\varepsilon > 0$, let τ_S^ε and σ_S^ε be the stopping times defined by

$$\tau_S^\varepsilon := \inf\{t \geq S,\ Y_t \leq \xi_t + \varepsilon\}; \quad \sigma_S^\varepsilon := \inf\{t \geq S,\ Y_t \geq \zeta_t - \varepsilon\}.$$

Let $\tau \in \mathcal{T}_S$. Since $Y \geq \xi$ and $Y_{\sigma_S^\varepsilon} \geq \zeta_{\sigma_S^\varepsilon} - \varepsilon$ a.s. (see Lemma 4.11), we have:

$$Y_{\tau \wedge \sigma_S^\varepsilon} \geq \xi_\tau \mathbf{1}_{\tau \leq \sigma_S^\varepsilon} + (\zeta_{\sigma_S^\varepsilon} - \varepsilon)\mathbf{1}_{\sigma_S^\varepsilon < \tau} \geq I(\tau, \sigma_S^\varepsilon) - \varepsilon \quad \text{a.s.}$$

By Lemma 4.11, $A'_{\sigma_S^\varepsilon} = A'_S$ a.s. which implies that:

$$-dY_t = g^{\overline{u},\overline{v}}(t,Y_t,Z_t,k_t)dt + dA_t - Z_t dW_t - \int_{\mathbf{E}} k_t(e)\tilde{N}(dt,de), \quad S \leq t \leq \sigma_S^\varepsilon, \quad dt \otimes dP \text{ a.s.}$$

Hence, $(Y_t)_{S \leq t \leq \tau \wedge \sigma^\varepsilon}$ is the solution of the BSDE associated with generalized driver $f(\cdot)dt + dA_t$ and terminal condition $Y_{\tau \wedge \sigma^\varepsilon}$. Using Assumption (5.4), the inequality $Y_{\tau \wedge \sigma^\varepsilon} \geq I(\tau, \sigma^\varepsilon) - \varepsilon$ and the comparison theorem for BSDEs with jumps, we obtain

$$Y_S \geq \mathcal{E}_S^{u,\overline{v}}(I(\tau,\sigma^\varepsilon) - \varepsilon) \geq \mathcal{E}_S^{u,\overline{v}}(I(\tau,\sigma^\varepsilon)) - K\varepsilon \quad \text{a.s. },$$

where the second inequality follows from a priori estimates for BSDEs with jumps.

Here, the constant K only depends on T and C, the common Lipschitz constant. Consequently, we get

$$Y_S \geq \operatorname*{ess\,inf}_{v \in \mathcal{V}, \sigma \in \mathcal{T}_S} \operatorname*{ess\,sup}_{u \in \mathcal{U}, \tau \in \mathcal{T}_S} \mathcal{E}_{S,\tau \wedge \sigma}^{u,v}(I(\tau,\sigma)) - K\varepsilon \quad \text{a.s.}$$

Similarly, one can prove that for each $\varepsilon > 0$,

$$Y_S \leq \operatorname*{ess\,sup}_{u \in \mathcal{U}, \tau \in \mathcal{T}_S} \operatorname*{ess\,inf}_{v \in \mathcal{V}, \sigma \in \mathcal{T}_S} \mathcal{E}_{S,\tau \wedge \sigma}^{u,v}(I(\tau,\sigma)) + K\varepsilon \quad \text{a.s.}$$

Hence, $\overline{V}(S) \leq \underline{V}(S)$ a.s. Since $\underline{V}(S) \leq \overline{V}(S)$ a.s., the equality follows. □

Remark 5.4. Theorem 5.3 still holds if $g^{\overline{u},\overline{v}}$ is replaced by any Lipschitz driver g which satisfies (5.4).

Application: the case of control processes Let U, V be compact Polish spaces. In the following, Ω is the canonical space defined in [13] (in Section 2). We are given a map $F : \Omega \times [0,T] \times U \times V \times \mathbf{R}^2 \times L_\nu^2 \to \mathbf{R}$, $(\omega,t,u,v,y,z,k) \mapsto F(\omega,t,u,v,y,z,k)$, supposed to be measurable with respect to $\mathcal{P} \otimes \mathcal{B}(U) \otimes \mathcal{B}(V) \otimes \mathcal{B}(\mathbf{R}^2) \otimes \mathcal{B}(L_\nu^2)$, continuous, concave with respect to u and convex with respect to v, and uniformly Lipchitz with respect to (y,z,k). Suppose that F is $\mathcal{C}^1$ with respect to k with $\nabla_k F \geq -1$, and that $F(\omega,t,u,v,0,0,0)$ is uniformly bounded. Let $\mathcal{U}$ (resp. $\mathcal{V}$) be the set of predictable processes valued in U (resp. V). For each $(u,v) \in \mathcal{U} \times \mathcal{V}$, let $g^{u,v}$ be the driver defined by

$$g^{u,v}(\omega,t,y,z,k) := F(\omega,t,u_t(\omega),v_t(\omega),y,z,k). \tag{5.5}$$

Let ξ and ζ be RCLL adapted processes in $\mathcal{S}^2$ such that $\xi_T = \zeta_T$ a.s. and $\xi_t \leq \zeta_t$, $0 \leq t \leq T$ a.s. Suppose that Mokobodzki's condition is satisfied. Let us consider the associated *generalized mixed game* problem. Define for each (t, ω, y, z, k) the map

$$g(t, \omega, y, z, k) = \sup_{u \in U} \inf_{v \in V} F(t, \omega, u, v, y, z, k). \tag{5.6}$$

Since U and V are Polish spaces, there exist some dense countable subsets $\overline{U}$ (resp. $\overline{V}$) of U (resp. V). Since F is continuous with respect to u, v, the *sup* (resp. *inf*) can be taken over $\overline{U}$ (resp. $\overline{V}$). Hence, g is a Lipschitz driver.

Let $(Y, Z, k, A, A') \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu \times (\mathcal{A}^2)^2$ be the solution of the DRBSDE associated with driver g and obstacles ξ and ζ. By classical convex analysis, for each (t, ω) there exist $(u^*, v^*) \in (U, V)$ such that

$$\begin{aligned} F(\omega, t, u, v^*, Y_{t^-}(\omega), Z_t(\omega), k_t(\omega)) &\leq F(\omega, t, u^*, v^*, Y_{t^-}(\omega), Z_t(\omega), k_t(\omega)) \\ &\leq F(\omega, t, u^*, v, Y_{t^-}(\omega), Z_t(\omega), k_t(\omega)),\ \forall (u, v) \in \overline{U} \times \overline{V}; \\ g(\omega, t, Y_{t^-}(\omega), Z_t(\omega), k_t(\omega))) &= F(\omega, t, u^*, v^*, Y_{t^-}(\omega), Z_t(\omega), k_t(\omega)) \end{aligned} \tag{5.7}$$

Let $(u, v) \in \overline{U} \times \overline{V}$. Since the processes Y_{t^-}, Z_t and k_t are predictable, the map $(\omega, t, u^*, v^*) \mapsto (\omega, t, u, v^*, Y_{t^-}(\omega), Z_t(\omega), k_t(\omega))$ is measurable with respect to the σ-algebras $\mathcal{P} \otimes \mathcal{B}(U) \otimes \mathcal{B}(V)$ and $\mathcal{P} \otimes \mathcal{B}(U) \otimes \mathcal{B}(V) \otimes \mathcal{B}(\mathbf{R}^2) \otimes \mathcal{B}(L^2_\nu)$. Using the measurability property of F, it follows by composition that the map $(\omega, t, u^*, v^*) \mapsto F(\omega, t, u, v^*, Y_{t^-}(\omega), Z_t(\omega), k_t(\omega))$ is $\mathcal{P} \otimes \mathcal{B}(U) \otimes \mathcal{B}(V)$-measurable, and similarly for the other maps which appear in (5.7). Hence, the set of all $(\omega, t, u^*, v^*) \in \Omega \times [0, T] \times U \times V$ satisfying conditions (5.7) belongs to $\mathcal{P} \otimes \mathcal{B}(U) \otimes \mathcal{B}(V)$. Since Ω is a Polish space for the Skorohod metric (see [13] sect. 2), by applying a measurable selection theorem (see e.g. Section 81 in the Appendix of Ch. III in [10]) and Lemma 1.2 in [8], we derive the existence of a pair of predictable process $(u^*, v^*) \in \mathcal{U} \times \mathcal{V}$ such that $dt \otimes dP$ a.s., for all $(u, v) \in \mathcal{U} \times \mathcal{V}$ we have:

$$F(t, u_t, v_t^*, Y_t, Z_t, k_t) \leq F(t, u_t^*, v_t^*, Y_t, Z_t, k_t) \leq F(t, u_t^*, v_t, Y_t, Z_t, k_t)$$

and $g(t, Y_t, Z_t, k_t) = F(t, u_t^*, v_t^*, Y_t, Z_t, k_t)$. Hence, Assumption (5.3) is satisfied. By applying Theorems 5.3 and 5.2, we derive the following result:

Proposition 5.5. *The generalized mixed game problem, associated with the controlled drivers $g^{u,v}$ given by* (5.5)*, is* fair. *Let Y be the solution of the DRBSDE associated with obstacles ξ, ζ and the driver g defined by* (5.6)*. For each stopping time $S \in \mathcal{T}_0$, we have $Y_S = \overline{V}(S) = \underline{V}(S)$ a.s. Suppose that ξ and ζ are l.u.s.c. along stopping times, and consider the stopping times*

$$\tau_S^* := \inf\{t \geq S : Y_t = \xi_t\} \quad ; \quad \sigma_S^* := \inf\{t \geq S : Y_t = \zeta_t\}.$$

The quadruple $(u^*, \tau_S^*, v^*, \sigma_S^*)$ is then an S-saddle point for this generalized mixed game *problem.*

We give now an example of application of the above proposition.

Example 5.6 (the classical linear case)**.** Consider the particular case when F takes the following form: $F(t, \omega, u, v, y, z, k) = \beta(t, \omega, u, v) z + < \gamma(t, \omega, u, v, \cdot), k >_\nu + c(t, \omega, u, v)$, with β, γ, c bounded. By classical results on linear BSDEs (see [35]), the criterium can be written

$$\mathcal{E}^{u,v}_{S, \tau \wedge \sigma}(I(\tau, \sigma)) = E_{Q^{u,v}} \left[\int_S^{\tau \wedge \sigma} c(t, u_t, v_t) dt + I(\tau, \sigma) | \mathcal{F}_S \right],$$

with $Q^{u,v}$ the probability measure which admits $Z_T^{u,v}$ as density with respect to P, where $(Z_t^{u,v})$ is the solution of the following SDE:

$$dZ_t^{u,v} = Z_t^{u,v} [\beta(t, u_t, v_t) dW_t + \int_{\mathbf{E}} \gamma(t, u_t, v_t, e) \tilde{N}(dt, de)]; \quad Z_0^{u,v} = 1.$$

The process $c(t, u_t, v_t)$ can be interpreted as an instantaneous reward associated with controls u, v. This linear model takes into account ambiguity on the model via the probability measures $Q^{u,v}$ as well as ambiguity on the instantaneous reward. This case corresponds to the "classical" mixed game problems studied in [4, 25, 21].

6 Comparison theorems for DRBSDEs with jumps and a priori estimates

Thanks to the characterization of the solution of the nonlinear DRBSDE as the value function of a generalized Dynkin game (Theorem 4.10), we now establish a comparison theorem and a strict comparison theorem for DRBSDEs, as well as some new estimates with universal constants.

6.1 Comparison theorems

Theorem 6.1 (Comparison theorem for DRBSDEs)**.** *Let ξ^1, ξ^2, ζ^1, ζ^2 be processes in $\mathcal{S}^2$ such that $\xi^i_T = \zeta^i_T$ a.s. and $\xi^i_t \leq \zeta^i_t$, $0 \leq t \leq T$ a.s. for $i = 1, 2$. Suppose that for $i = 1, 2$, ξ^i, ζ^i satisfies Mokobodzki's condition. Let $g^1$and g^2 be Lipschitz drivers satisfying Assumption (4.3). Let $(Y^i, Z^i, k^i, A^i, A'^i)$ be the solution of the DRBSDE associated with (ξ^i, ζ^i, g^i) , $i = 1, 2$. Suppose that*

(i) $\xi^2_t \leq \xi^1_t$ *and* $\zeta^2_t \leq \zeta^1_t$, $0 \leq t \leq T$ *a.s.*
(ii) $g^2(t, Y^2_t, Z^2_t, k^2_t) \leq g^1(t, Y^2_t, Z^2_t, k^2_t),\ 0 \leq t \leq T \quad dP \otimes dt - \text{a.s.}$

We then have:

$$Y^2_t \leq Y^1_t,\ 0 \leq t \leq T \quad \textit{a.s.}$$

Remark 6.2. A comparison theorem has been provided in [7] in the case of jumps under stronger assumptions, with a different proof based on Itô's calculus.

Proof. We give a short proof based on the characterization of solutions of DRBSDEs (Theorem 4.10) via *generalized* Dynkin games.

Step 1: Let us first suppose that condition (i) holds and that g^1 and g^2 satisfy: $g^2(t, y, z, k) \leq g^1(t, y, z, k)$ for all $(y, z, k) \in \mathbf{R}^2 \times L^2_\nu$ $dP \otimes dt$-a.s. (which is a stronger assumption than (ii)). Let $t \in [0, T]$. For each $\tau, \sigma \in \mathcal{T}_t$, let us denote by $\mathcal{E}^i_{\cdot,\tau\wedge\sigma}(I^i(\tau, \sigma))$ the unique solution of the BSDE associated with driver g^i, terminal time $\tau \wedge \sigma$ and terminal condition $I^i(\tau, \sigma) := \xi^i_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta^i_\sigma \mathbf{1}_{\sigma < \tau}$ for $i = 1, 2$. Since $g^2 \leq g^1$, and $I^2(\tau, \sigma) \leq I^1(\tau, \sigma)$, by the comparison theorem for BSDEs, the following inequality

$$\mathcal{E}^2_{t,\tau\wedge\sigma}(I^2(\tau, \sigma)) \ \leq\ \mathcal{E}^1_{t,\tau\wedge\sigma}(I^1(\tau, \sigma))\ a.s.$$

holds for each τ, σ in $\mathcal{T}_t$. Hence, by taking the essential supremum over τ in $\mathcal{T}_t$ and the essential infimum over σ in $\mathcal{T}_t$, and by using Theorem 4.10, we get

$$Y^2_t = \operatorname*{ess\,inf}_{\sigma\in\mathcal{T}_t} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathcal{E}^2_{t,\tau\wedge\sigma}(I^2(\tau, \sigma)) \leq \operatorname*{ess\,inf}_{\sigma\in\mathcal{T}_t} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathcal{E}^1_{t,\tau\wedge\sigma}(I^1(\tau, \sigma)) = Y^1_t\ a.s.$$

Step 2: Suppose that conditions (i) and (ii) hold. Let δg be the process defined by $\delta g_t := g^2(t, Y^2_t, Z^2_t, k^2_t) - g^1(t, Y^2_t, Z^2_t, k^2_t)$. Note that (Y^2, Z^2, k^2) is the solution the DRBSDE associated with barriers ξ^2, ζ^2 and driver $g^1(t, y, z, k) + \delta g_t$. Now, by (ii), we have $g^1(t, y, z, k) + \delta g_t \leq g^1(t, y, z, k)$ for all (y, z, k). By Step 1 applied to the driver g^1 and the driver $g^1(t, y, z, k) + \delta g_t$ (instead of g^2), we get $Y^2 \leq Y^1$. □

We now provide a strict comparison theorem for DRBSDEs. Note that no strict comparison theorem exists in the literature even in the Brownian case. The first assertion addresses the particular case when the non decreasing processes are continuous and the second one deals with the general case.

Theorem 6.3 (Strict comparison for DRBSDEs.). *Suppose that the assumptions of Theorem 6.1 hold and that the driver g^1 satisfies Assumption 4.3 with $\gamma_t > -1$ in* (4.2)*. Let S in $\mathcal{T}_0$ and suppose that $Y^1_S = Y^2_S$ a.s.*

1. *Suppose that A^i, A'^i, $i = 1,2$ are continuous. For $i = 1,2$, let $\overline{\tau}_i = \overline{\tau}_{i,S} := \inf\{s \geq S;\ A^i_s > A^i_S\}$ and $\overline{\sigma}_i = \overline{\sigma}_{i,S} := \inf\{s \geq S;\ A'^i_s > A'^i_S\}$. Then*
$$Y^1_t = Y^2_t,\ \ S \leq t \leq \overline{\tau}_1 \wedge \overline{\tau}_2 \wedge \overline{\sigma}_1 \wedge \overline{\sigma}_2 \ \text{ a.s.}$$
and
$$g^2(t, Y^2_t, Z^2_t, k^2_t) \ = \ g^1(t, Y^2_t, Z^2_t, k^2_t) \quad S \leq t \leq \overline{\tau}_1 \wedge \overline{\tau}_2 \wedge \overline{\sigma}_1 \wedge \overline{\sigma}_2,\ dP \otimes dt - a.s. \quad (6.1)$$
2. *Consider the case when A^i, A'^i, $i = 1,2$ are not necessarily continuous. For $i = 1,2$, define for each $\varepsilon > 0$,*
$$\tau_i^\varepsilon := \inf\{t \geq S,\ Y^i_t \leq \xi^i_t + \varepsilon\}\ ;\ \ \sigma_i^\varepsilon := \inf\{t \geq S,\ Y^i_t \geq \zeta^i_t - \varepsilon\}.$$
Setting $\tilde{\tau}_i := \lim_{\varepsilon \downarrow 0} \uparrow \tau_i^\varepsilon$ and $\tilde{\sigma}_i := \lim_{\varepsilon \downarrow 0} \uparrow \sigma_i^\varepsilon$, we have
$$Y^1_t = Y^2_t,\ \ S \leq t < \tilde{\tau}_1 \wedge \tilde{\tau}_2 \wedge \tilde{\sigma}_1 \wedge \tilde{\sigma}_2. \quad \text{a.s.} \qquad (6.2)$$
Moreover, equality (6.1) *holds on $[S, \tilde{\tau}_1 \wedge \tilde{\tau}_2 \wedge \tilde{\sigma}_1 \wedge \tilde{\sigma}_2]$.*

Proof. We adopt the same notation as in the proof of the comparison theorem.

1. Suppose first that A^i, A'^i, $i = 1,2$ are continuous. By Theorem 4.8, for $i = 1,2$, $(\overline{\tau}_i, \overline{\sigma}_i)$ is a saddle point for the game problem associated with $g = g^i$, $\xi = \xi^i$ and $\zeta = \zeta^i$. By Remark 4.9, $(Y^i_t, S \leq t \leq \overline{\tau}_i \wedge \overline{\sigma}_i)$ is an $\mathcal{E}^i$ martingale. Hence we have
$$Y^i_t = \mathcal{E}^i_{t, \overline{\tau}_i \wedge \overline{\sigma}_i}(I(\overline{\tau}_i, \overline{\sigma}_i)),\ \ S \leq t \leq \overline{\tau}_i \wedge \overline{\sigma}_i \ \text{ a.s.}$$
Setting $\overline{\theta} = \overline{\tau}_1 \wedge \overline{\tau}_2 \wedge \overline{\sigma}_1 \wedge \overline{\sigma}_2$, we thus have
$$Y^i_t = \mathcal{E}^i_{t, \overline{\theta}}(Y^i_{\overline{\theta}}),\ \ S \leq t \leq \overline{\theta} \ \text{ a.s. for } i = 1,2.$$
By hypothesis, $Y^1_S = Y^2_S$ a.s. Now, we apply the strict comparison theorem for non reflected BSDEs with jumps (see [35, Th 4.4]) for terminal time $\overline{\theta}$. Hence, we get $Y^1_t = Y^2_t,\ \ S \leq t \leq \overline{\theta}$ a.s., as well as equality (6.1), which provides the desired result.

2. Consider now the general case.

Let $\varepsilon > 0$. By Remark 4.12, $(Y^i_t, S \leq t \leq \tau_i^\varepsilon \wedge \sigma_i^\varepsilon)$ is an $\mathcal{E}^i$ martingale. Hence we have
$$Y^i_t = \mathcal{E}^i_{t, \tau_i^\varepsilon \wedge \sigma_i^\varepsilon}(I(\tau_i^\varepsilon, \sigma_i^\varepsilon)),\ \ S \leq t \leq \tau_i^\varepsilon \wedge \sigma_i^\varepsilon \ \text{ a.s.}$$
By the same arguments as above with τ_1^*,τ_2^* and σ_1^*,σ_2^* replaced by τ_1^ε,τ_2^ε and σ_1^ε,σ_2^ε respectively, we derive $Y^1_t = Y^2_t,\ \ S \leq t \leq \tau_1^\varepsilon \wedge \tau_2^\varepsilon \wedge \sigma_1^\varepsilon \wedge \sigma_2^\varepsilon$ a.s., and equality (6.1) holds on $[S, \tau_1^\varepsilon \wedge \tau_2^\varepsilon \wedge \sigma_1^\varepsilon \wedge \sigma_2^\varepsilon]$, $dt \otimes dP$-a.s. By letting ε tend to 0, we obtain the desired result. □

We now give an application of the comparison theorem to a control game problem for DRBSDEs.

Proposition 6.4 (Control game problem for DRBSDEs). *Suppose that the assumptions of Theorem* 5.3 *hold. For each $(u, v) \in \mathcal{U} \times \mathcal{V}$, let $Y^{u,v}$ be the solution of the DRBSDE* (2.2) *associated with driver $g^{u,v}$. Then, for each $S \in \mathcal{T}_0$, $Y^{u,\overline{v}}_S \leq Y^{\overline{u},\overline{v}}_S \leq Y^{\overline{u},v}_S$ a.s.*

Proof. By using Assumption (5.3) and by applying the comparison theorem for DRBSDEs (Theorem 6.1), we get that for each $u \in \mathcal{U}$, $Y^{u,\overline{v}}_S \leq Y^{\overline{u},\overline{v}}_S$ a.s. Similarly, for all $v \in \mathcal{V}$, we have $Y^{\overline{u},\overline{v}}_S \leq Y^{\overline{u},v}_S$ a.s. □

Remark 6.5. We point out that the above control game problem for DRBSDEs is different from the *generalized mixed game* problem studied in Section 5. However, from the above proposition, it follows that, under Assumption (5.3), the value functions of these two game problems coincide.

6.2 A priori estimates with universal constants

Using Theorem 4.10, we now prove the following estimates on the spread of the solutions of two DRBSDEs, where the constants are universal (i.e. they only depend on the terminal time T and the common Lipschitz constant C).

Proposition 6.6 (A priori estimates for DBBSDEs). *Let $\xi^1, \xi^2, \zeta^1, \zeta^2 \in \mathcal{S}^2$ such that $\xi^i_T = \zeta^i_T$ a.s. and $\xi^i_t \leq \zeta^i_t$, $0 \leq t \leq T$ a.s. Suppose that for $i = 1, 2$, ξ^i and ζ^i satisfy Mokobodzki's condition. Let g^1, g^2 be Lipschitz drivers satisfying Assumption 4.3 with common Lipschitz constant $C > 0$. For $i = 1, 2$, let Y^i be the solution of the DRBSDE associated with driver g^i and barriers ξ^i, ζ^i.*

Let $\overline{Y} := Y^1 - Y^2$, $\overline{\xi} := \xi^1 - \xi^2$, $\overline{\zeta} = \zeta^1 - \zeta^2$. Let $\eta, \beta > 0$ with $\beta \geq \frac{3}{\eta} + 2C$ and $\eta \leq \frac{1}{C^2}$. Let $\delta g_s = g^2(t, Y^2_s, Z^2_s, k^2_s) - g^1(t, Y^2_s, Z^2_s, k^2_s)$. For each t, we have

$$\overline{Y}_t^2 \leq e^{\beta(T-t)} E[\sup_{s \geq t} \overline{\xi_s}^2 + \sup_{s \geq t} \overline{\zeta_s}^2 | \mathcal{F}_t] + \eta E[\int_t^T e^{\beta(s-t)} (\delta g_s)^2 ds | \mathcal{F}_t] \quad \text{a.s.} \tag{6.3}$$

Proof. The proof is divided into two steps.

Step 1: For $i = 1, 2$ and for each $\tau, \sigma \in \mathcal{T}_0$, let $(X^{i,\tau,\sigma}, \pi^{i,\tau,\sigma}, l^{i,\tau,\sigma})$ be the solution of the BSDE associated with driver g^i, terminal time $\tau \wedge \sigma$ and terminal condition $I^i(\tau, \sigma)$, where $I^i(\tau, \sigma) = \xi^i_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta^i_\sigma \mathbf{1}_{\sigma < \tau}$. Set $\overline{X}^{\tau,\sigma} := X^{1,\tau,\sigma} - X^{2,\tau,\sigma}$ and $\overline{I}^{\tau,\sigma} := I^1(\tau, \sigma) - I^2(\tau, \sigma) = \overline{\xi}_\tau \mathbf{1}_{\tau \leq \sigma} + \overline{\zeta}_\sigma \mathbf{1}_{\sigma < \tau}$. By an estimate on BSDEs (see Proposition $A.4$ in [36]), we have a.s.:

$$(\overline{X}_t^{\tau,\sigma})^2 \leq e^{\beta(T-t)} E[\overline{I}(\tau, \sigma)^2 \mid \mathcal{F}_t] + \eta E[\int_t^T e^{\beta(s-t)} [(g^1 - g^2)(s, X^{2,\tau,\sigma}_s, \pi^{2,\tau,\sigma}_s, l^{2,\tau,\sigma}_s)]^2 ds \mid \mathcal{F}_t] \tag{6.4}$$

from which we derive that

$$(\overline{X}_t^{\tau,\sigma})^2 \leq e^{\beta(T-t)} E[\sup_{s \geq t} \overline{\xi}_s^2 + \sup_{s \geq t} \overline{\zeta}_s^2 | \mathcal{F}_t] + \eta E[\int_t^T e^{\beta(s-t)} \overline{g}_s^2 ds | \mathcal{F}_t] \quad a.s., \tag{6.5}$$

where $\overline{g}_s := \sup_{y,z,k} |g^1(s, y, z, k) - g^2(s, y, z, k)|$. Now, by using inequality (4.12), we obtain that for each $\varepsilon > 0$ and for all stopping times τ, σ,

$$Y^1_t - Y^2_t \leq X^{1,\tau_1^\epsilon,\sigma}_t - X^{2,\tau,\sigma_2^\epsilon}_t + 2K\epsilon.$$

Applying this inequality to $\tau = \tau_1^\epsilon, \sigma = \sigma_2^\epsilon$ we get

$$Y^1_t - Y^2_t \leq X^{1,\tau_1^\epsilon,\sigma_2^\epsilon}_t - X^{2,\tau_1^\epsilon,\sigma_2^\epsilon}_t + 2K\epsilon \leq |X^{1,\tau_1^\epsilon,\sigma_2^\epsilon}_t - X^{2,\tau_1^\epsilon,\sigma_2^\epsilon}_t| + 2K\epsilon. \tag{6.6}$$

By (6.5) and (6.6), we have:

$$Y^1_t - Y^2_t \leq \sqrt{e^{\beta(T-t)} E[\sup_{s \geq t} \overline{\xi_s}^2 + \sup_{s \geq t} \overline{\zeta_s}^2 | \mathcal{F}_t] + \eta E[\int_t^T e^{\beta(s-t)} \overline{g}_s^2 ds | \mathcal{F}_t]} + 2K\epsilon.$$

By symmetry, the last inequality is also verified by $Y^2_t - Y^1_t$. We thus derive that

$$\overline{Y}_t^2 \leq e^{\beta(T-t)} E[\sup_{s \geq t} \overline{\xi_s}^2 + \sup_{s \geq t} \overline{\zeta_s}^2 | \mathcal{F}_t] + \eta E[\int_t^T e^{\beta(s-t)} \overline{g}_s^2 ds | \mathcal{F}_t] \; a.s. \tag{6.7}$$

This result holds for all Lipschitz drivers g^1 and g^2 satisfying Assumption 4.3.

Step 2: Note that (Y^2, Z^2, k^2) is the solution the DRBSDE associated with barriers ξ^2, ζ^2 and driver $g^1(t, y, z, k) + \delta g_t$. By applying the result of Step 1 to the driver $g^1(t, y, z, k)$ and the driver $g^1(t, y, z, k) + \delta g_t$ (instead of g^2), we get the desired result. □

Remark 6.7. The arguments of the above proof are different from those used in the literature. Based on Theorem 4.10, they allow us to obtain universal constants, which is not the case for the a priori estimates on DRBSDEs given in the literature (for details see Remark A.5 in the Appendix). This new estimate with universal constants is useful to study the Markovian case (see the next section), in particular to obtain the continuity of the value function. Moreover, this estimate is a powerful tool which allows us to study a mixed generalized Dynkin game problem (see [15]), in particular to obtain a weak dynamic programming principle under mild assumptions and a classical one in the regular case.

We also state the following estimate on the common value function Y of our generalized Dynkin game problem (4.4)-(4.5) (or equivalently the solution of the DRBSDE associated with driver g).

Proposition 6.8. *Let g be a driver satisfying Assumption* (4.3). *Let ξ and ζ be RCLL adapted processes in $\mathcal{S}^2$ such that $\xi_T = \zeta_T$ a.s. and $\xi_t \leq \zeta_t$, $0 \leq t \leq T$ a.s. Suppose that Mokobodzki's condition is satisfied.*

Let (Y, Z, k, A, A') be the solution of the DRBSDE (2.2). *For each $t \in [0, T]$, we have:*

$$Y_t^2 \leq e^{\beta(T-t)} E[\sup_{s \geq t} {\xi_s}^2 + \sup_{s \geq t} {\zeta_s}^2 | \mathcal{F}_t] + \eta E[\int_t^T e^{\beta(s-t)} g(s,0,0,0)^2 ds | \mathcal{F}_t] \ a.s. \tag{6.8}$$

Proof. Let $X^{\tau,\sigma}$ be the solution of the BSDE associated with driver g, terminal time $\tau \wedge \sigma$ and terminal condition $I(\tau, \sigma)$. By applying inequality (6.4) with $g^1 = g$, $\xi_1 = \xi$, $\zeta_1 = \zeta$, $g^2 = 0$, $\xi^2 = 0$ and $\zeta^2 = 0$, we get:

$$(X_t^{\tau,\sigma})^2 \leq e^{\beta(T-t)} E[I(\tau,\sigma)^2 | \mathcal{F}_t] + \eta E[\int_t^T e^{\beta(s-t)} (g(s,0,0,0))^2 | \mathcal{F}_t]. \tag{6.9}$$

By using the same procedure as in the proof of Proposition 6.6, the result follows. □

7 Relation with partial integro-differential variational inequalities (PIDVI)

We consider now the Markovian case, and we study the links between Markovian generalized Dynkin games (or equivalently DRBSDEs) and obstacle problems.

Let $b : \mathbf{R} \to \mathbf{R}$, $\sigma : \mathbf{R} \to \mathbf{R}$ be continuous mappings, globally Lipschitz and $\beta : \mathbf{R} \times \mathbf{E} \to \mathbf{R}$ a measurable function such that for some nonnegative real C, and for all $e \in \mathbf{E}$

$$|\beta(x,e)| \leq C\varphi(e), \qquad |\beta(x,e) - \beta(x',e)| \leq C|x - x'|\varphi(e), \ \ x, x' \in \mathbf{R},$$

where φ is a bounded map belonging to L^2_ν. For each $(t,x) \in [0,T] \times \mathbf{R}$, let $(X_s^{t,x}, t \leq s \leq T)$ be the unique $\mathbf{R}$-valued solution of the SDE with jumps:

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x}) dr + \int_t^s \sigma(X_r^{t,x}) dW_r + \int_t^s \int_{\mathbf{E}} \beta(X_{r^-}^{t,x}, e) \tilde{N}(dr, de),$$

and set $X_s^{t,x} = x$ for $s \leq t$. We consider the DRBSDE associated with obstacles $\xi^{t,x}$, $\zeta^{t,x}$ of the following form:

$$\xi_s^{t,x} := h_1(s, X_s^{t,x}), \ \zeta_s^{t,x} := h_2(s, X_s^{t,x}), \ s < T; \ \ \xi_T^{t,x} = \zeta_T^{t,x} := g(X_T^{t,x}).$$

We suppose that $g \in \mathcal{C}(\mathbf{R})$, $h_1, h_2 : [0,T] \times \mathbf{R} \to \mathbf{R}$ are continuous with respect to t and Lipschitz continuous with respect to x, uniformly in t and that g, h_1, h_2 have at most polynomial growth with respect to x. Moreover, the obstacles $\xi^{t,x}$ and $\zeta^{t,x}$ are supposed to satisfy Mokobodzki's condition, which holds for example when h_1 or h_2 is $\mathcal{C}^{1,2}$ (see [15] for details).

We consider two functions γ and f satisfying Assumption 2.1 in [12]. More precisely, let $\gamma : \mathbf{R} \times \mathbf{E} \to \mathbf{R}$ be a $\mathcal{B}(\mathbf{R}) \otimes \mathcal{K}$-measurable map, such that $|\gamma(x,e) - \gamma(x',e)| < C|x - x'|\varphi(e)$ and $-1 \leq \gamma(x,e) \leq C\varphi(e)$ for all $x, x' \in \mathbf{R}, e \in \mathbf{E}$, and let $f : [0,T] \times \mathbf{R}^3 \times L^2_\nu \to \mathbf{R}$ be a map, continuous in t uniformly with respect to x, y, z, k, uniformly Lipschitz with respect to x, y, z, k uniformly in t, such that $f(t,x,0,0,0)$ has at most polynomial growth with respect to x, and such that for all t, x, y, z, k_1, k_2, $f(t,x,y,z,k_1) - f(t,x,y,z,k_2) \geq <\gamma(x,\cdot), k_1 - k_2>_\nu$. The driver is defined by $f(s, X_s^{t,x}(\omega), y, z, k)$.

By Theorem 4.1, for each initial conditions $(t,x) \in [0,T] \times \mathbf{R}$, there exists an unique square integrable solution $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x}, A_s^{t,x}, A_s'^{t,x})_{t \leq s \leq T}$ of the associated DRBSDE. Note that $Y_t^{t,x}$ is constant.[4] By Theorem 4.10, $Y^{t,x}$ is the *value function* process of the associated generalized Dynkin game. We define:

$$u(t,x) := Y_t^{t,x}, \quad t \in [0,T],\ x \in \mathbf{R} \tag{7.1}$$

which is a deterministic function. Thanks to the a priori estimates for DBBSDEs with universal constants (see Propositions 6.6 and 6.8) and the same arguments as those used in the proofs of Lemma 3.1 and Lemma 3.2 in [12], we derive that u is continuous in (t,x) and has at most polynomial growth at infinity. It follows that the process $Y_s^{t,x} = u(s, X_s^{t,x})$ admits only totally inaccessible jumps. Hence, the processes $A^{t,x}$, $A'^{t,x}$ are continuous.

A solution of the obstacle problem is a function $u : [0,T] \times \mathbf{R} \to \mathbf{R}$ which satisfies the equality $u(T,x) = g(x)$ and

$$\begin{cases} h_1(t,x) \leq u(t,x) \leq h_2(t,x), \\ \text{if } u(t,x) < h_2(t,x) \text{ then } \mathcal{H}u \geq 0, \\ \text{if } h_1(t,x) < u(t,x) \text{ then } \mathcal{H}u \leq 0, \end{cases} \tag{7.2}$$

where $L := A + K$ and

- $A\phi(x) := \dfrac{1}{2}\sigma^2(x)\dfrac{\partial^2 \phi}{\partial x^2}(x) + b(x)\dfrac{\partial \phi}{\partial x}(x)$,
- $B\phi(t,x)(\cdot) := \phi(t, x + \beta(x,\cdot)) - \phi(t,x)$,
- $K\phi(x) := \int_{\mathbf{E}} \left(\phi(x + \beta(x,e)) - \phi(x) - \dfrac{\partial \phi}{\partial x}(x)\beta(x,e) \right) \nu(de)$,
- $\mathcal{H}\phi(t,x) := -\dfrac{\partial \phi}{\partial t}(t,x) - L\phi(t,x) - f(t,x,\phi(t,x), (\sigma\dfrac{\partial \phi}{\partial x})(t,x), B\phi(t,x))$.

Definition 7.1. • A continuous function u is said to be a *viscosity subsolution* of (7.2) if $u(T,x) \leq g(x), x \in \mathbf{R}$, and if for any point $(t_0,x_0) \in [0,T) \times \mathbf{R}$, we have $h_1(t_0,x_0) \leq u(t_0,x_0) \leq h_2(t_0,x_0)$ and, for any $\phi \in C^{1,2}([0,T] \times \mathbf{R})$ such that $\phi(t_0,x_0) = u(t_0,x_0)$ and $\phi - u$ attains its minimum at (t_0,x_0), if $u(t_0,x_0) > h_1(t_0,x_0)$, then $(\mathcal{H}\phi)(t_0,x_0) \leq 0$.

• A continuous function u is said to be a *viscosity supersolution* of (7.2) if $u(T,x) \geq g(x), x \in \mathbf{R}$, and if for any point $(t_0,x_0) \in [0,T) \times \mathbf{R}$, we have $h_1(t_0,x_0) \leq u(t_0,x_0) \leq h_2(t_0,x_0)$ and, for any $\phi \in C^{1,2}([0,T] \times \mathbf{R})$ such that $\phi(t_0,x_0) = u(t_0,x_0)$ and $\phi - u$ attains its maximum at (t_0,x_0), if $u(t_0,x_0) < h_2(t_0,x_0)$ then $(\mathcal{H}\phi)(t_0,x_0) \geq 0$.

Theorem 7.2. *The function u defined by* (7.1) *is a viscosity solution (i.e. both a viscosity sub- and supersolution) of the obstacle problem* (7.2)*.*

Proof. We propose a short direct proof, contrary to the previous literature on doubly reflected BSDEs where an approach by penalization is used (see [6]). We prove that u is a viscosity supersolution of (7.2), the proof in the case of subsolution being similar.

[4] Indeed, the DRBSDE can be solved with respect to the t-translated Brownian motion $W^t := (W_s - W_t)_{s \geq t}$ and the t-translated Poisson random measure $N^t := N(]t,s],\cdot)_{s \geq t}$. The solution is thus adapted to the filtration $\mathcal{F}_s^t$, $t \leq s \leq T$ equal to the completion of $\sigma(W_r^t, N_r^t, t \leq r \leq s)$, $t \leq s \leq T$. Hence, $Y_t^{t,x}$ is measurable with respect to $\mathcal{F}_t^t = \mathcal{F}_0$, which implies that it is constant up to a P-null set.

Let $(t_0, x_0) \in (0,T) \times \mathbb{R}$ and $\phi \in C^{1,2}([0,T] \times \mathbb{R})$ be such that $\phi(t_0,x_0) = u(t_0,x_0)$ and $\phi(t,x) \leq u(t,x)$, $\forall (t,x) \in [0,T] \times \mathbb{R}$. Suppose that $u(t_0,x_0) < h_2(t_0,x_0)$ and that

$$-\frac{\partial}{\partial t}\phi(t_0,x_0) - L\phi(t_0,x_0) - g\left(t_0, x_0, \phi(t_0,x_0), (\sigma\frac{\partial \phi}{\partial x})(t_0,x_0), B\phi(t_0,x_0)\right) < 0.$$

By continuity, we can suppose that there exists $\epsilon > 0$ and $\eta_\epsilon > 0$ such that: $\forall (t,x)$ such that $t_0 \leq t \leq t_0 + \eta_\epsilon < T$ and $|x - x_0| \leq \eta_\epsilon$, we have: $u(t,x) \leq h_2(t,x) - \epsilon$ and

$$-\frac{\partial}{\partial t}\phi(t,x) - L\phi(t,x) - g\left(t, x, \phi(t,x), (\sigma\frac{\partial \phi}{\partial x})(t,x), B\phi(t,x)\right) \leq -\epsilon. \tag{7.3}$$

Let θ be the stopping time defined as follows:

$$\theta := (t_0 + \eta_\epsilon) \wedge \inf\{s \geq t_0 / |X_s^{t_0,x_0} - x_0| > \eta_\epsilon\}.$$

By this definition, we have

$$u(s, X_s^{t_0,x_0}) \leq h_2(s, X_s^{t_0,x_0}) - \epsilon < h_2(s, X_s^{t_0,x_0}), t_0 \leq s < \theta \text{ a.s.}$$

Hence, the process $(Y_s^{t_0,x_0} = u(s, X_s^{t_0,x_0}), s \in [t_0, \theta[)$ stays strictly below the upper barrier. It follows that the continuous process $A_s'^{t_0,x_0}$ is constant on $[t, \theta]$. The process $(Y_s^{t_0,x_0}, s \in [t_0, \theta])$ is thus the solution of the classical BSDE associated with terminal condition $Y_\theta^{t_0,x_0} = u(\theta, X_\theta^{t_0,x_0})$ and the generalized driver

$$g(s, X_s^{t_0,x_0}, y, z, q)ds + dA_s^{t_0,x_0}.$$

Our aim now is to use the comparison theorem. We apply as above Itô's lemma to $\phi(s, X_s^{t_0,x_0})$ and we get that $\left(\phi(s, X_s^{t_0,x_0}), (\sigma\frac{\partial \phi}{\partial x})(s, X_s^{t_0,x_0}), \Phi(s, X_{s^-}^{t_0,x_0}, \cdot); s \in [t_0, \theta]\right)$ is the solution of the BSDE associated to the terminal value $\phi(\theta, X_\theta^{t_0,x_0})$ and driver $-\psi(s, X_s^{t_0,x_0})$, where $\psi(s,x) := \frac{\partial \phi}{\partial s}(s,x) + L\phi(s,x)$. By assumption (7.3) and the definition of the stopping time, we have :

$$\begin{aligned}
&-\psi(s, X_s^{t_0,x_0})ds \leq (g(s, X_s^{t_0,x_0}, \phi(s, X_s^{t_0,x_0}),\\
&\qquad (\sigma\frac{\partial \phi}{\partial x})(s, X_s^{t_0,x_0}), B\phi(s, X_s^{t_0,x_0})))ds + dA_s^{t_0,x_0} - \epsilon\, ds, \quad \forall s \in [t_0, \theta].
\end{aligned}$$

The above inequality gives a relation between the drivers of the two BSDEs. Moreover, $\phi(\theta, X_\theta^{t_0,x_0}) \leq u(\theta, X_\theta^{t_0,x_0}) = Y_\theta^{t_0,x_0}$. By applying the extended comparison theorem for BSDEs with jumps given in [12, Proposition A.3] we get:

$$\phi(t_0, x_0) = \phi(t_0, X_{t_0}^{t_0,x_0}) < Y_{t_0}^{t_0,x_0} = u(t_0, x_0),$$

which provides a contradiction. □

Uniqueness of the viscosity solution In the sequel, we suppose that the function φ is defined by $\varphi(e) := 1 \wedge |e|$ and belongs to L_ν^2. We also suppose that g, h_1 and h_2 are bounded, and that Assumption 4.1 in [12] holds. More precisely, we assume:
(i) $\quad f(s, X_s^{t,x}(\omega), y, z, k) := \overline{f}\left(s, X_s^{t,x}(\omega), y, z, \int_{\mathbf{R}^*} k(e)\gamma(X_s^{t,x}(\omega), e)\nu(de)\right)\mathbf{1}_{s\geq t}$,
where $\overline{f} : [0,T] \times \mathbf{R}^4 \to \mathbf{R}$ is a map which is continuous with respect to t uniformly in x, y, z, k, continuous with respect x uniformly in y, z, k, uniformly Lipschitz with respect to y, z, k and the map $\overline{f}(t, x, 0, 0, 0)$ is uniformly bounded.
The map $k \mapsto \overline{f}(t, x, y, z, k)$ is also non-decreasing, for all $t \in [0,T]$, $x, y, z \in \mathbf{R}$.
(ii) For all $R > 0$, there exists a continuous function $m_R : \mathbf{R}_+ \to \mathbf{R}_+$ with $m_R(0) = 0$ and

$|\overline{f}(t,x,y,z,k) - \overline{f}(t,x',y,z,k)| \le m_R(|x-x'|(1+|z|))$, for all $t \in [0,T]$, $|x|, |x'| \le R, |y| \le R,\ z, k \in \mathbf{R}$.
(iii) $\quad |\gamma(x,e) - \gamma(y,e)| \le C|x-y|(1 \wedge e^2)$; $0 \le \gamma(x,e) \le C(1 \wedge |e|)$, $x, y \in \mathbf{R}, e \in \mathbf{R}^*$.
(iv) $\quad \overline{f}(t,x,y,z,l) - \overline{f}(t,x,y,z,l) \ge r(u-v)$, $u \ge v$, $t \in [0,T]$, $x,u,v,p,l \in \mathbf{R}$, where $r > 0$.

To simplify notation, in the sequel, $\overline{f}$ is denoted by f.

The operator B has now the following form: $B\phi(x) := \int_{\mathbf{R}^*}(\phi(x+\beta(x,e)) - \phi(x))\gamma(x,e)\nu(de)$.

Theorem 7.3 (Comparison principle)**.** *Suppose that Assumptions (i) to (iv) hold. If U is a bounded viscosity subsolution and V is a bounded viscosity supersolution of the obstacle problem* (7.2)*, then $U(t,x) \le V(t,x)$, for all $(t,x) \in [0,T] \times \mathbf{R}$.*

Proof. The proof is similar to the proof given in [12] (in the case of one barrier). For the convenience of the reader, we give a sketch of proof, where we draw attention to some points which differ from the proof in [12]. Set

$$\psi^{\epsilon,\eta}(t,s,x,y) := U(t,x) - V(s,y) - \frac{|x-y|^2}{\epsilon^2} - \frac{|t-s|^2}{\epsilon^2} - \eta^2(|x|^2 + |y|^2),$$

where ϵ, η are small parameters devoted to tend to 0. Let $M^{\epsilon,\eta}$ be a maximum of $\psi^{\epsilon,\eta}(t,s,x,y)$. This maximum is reached at some point $(t^{\epsilon,\eta}, s^{\epsilon,\eta}, x^{\epsilon,\eta}, y^{\epsilon,\eta})$. We define:

$$\Psi_1(t,x) := V(s^{\epsilon,\eta}, y^{\epsilon,\eta}) + \frac{|x-y^{\epsilon,\eta}|^2}{\epsilon^2} + \frac{|t-s^{\epsilon,\eta}|^2}{\epsilon^2} + \eta^2(|x|^2 + |y^{\epsilon,\eta}|^2);$$

$$\Psi_2(s,y) := U(t^{\epsilon,\eta}, x^{\epsilon,\eta}) - \frac{|x^{\epsilon,\eta}-y|^2}{\epsilon^2} - \frac{|t^{\epsilon,\eta}-s|^2}{\epsilon^2} - \eta^2(|x^{\epsilon,\eta}|^2 + |y|^2).$$

As $(t,x) \to (U - \Psi_1)(t,x)$ reaches its maximum at $(t^{\epsilon,\eta}, x^{\epsilon,\eta})$ and U is a subsolution, we have the two following cases:

• $t^{\epsilon,\eta} = T$ and then $U(t^{\epsilon,\eta}, x^{\epsilon,\eta}) \le g(x^{\epsilon,\eta})$,

• $t^{\epsilon,\eta} \neq T$, $h_1(t^{\epsilon,\eta}, x^{\epsilon,\eta}) \le U(t^{\epsilon,\eta}, x^{\epsilon,\eta}) \le h_2(t^{\epsilon,\eta}, x^{\epsilon,\eta})$ and, if $U(t^{\epsilon,\eta}, x^{\epsilon,\eta}) > h_1(t^{\epsilon,\eta}, x^{\epsilon,\eta})$, we then have:

$$-\frac{\partial \Psi_1}{\partial t}(t^{\epsilon,\eta}, x^{\epsilon,\eta}) - L\Psi_1(t^{\epsilon,\eta}, x^{\epsilon,\eta}) \\ - f\left(t^{\epsilon,\eta}, x^{\epsilon,\eta}, U(t^{\epsilon,\eta}, x^{\epsilon,\eta}), (\sigma\frac{\partial \Psi_1}{\partial x})(t^{\epsilon,\eta}, x^{\epsilon,\eta}), B\Psi_1(t^{\epsilon,\eta}, x^{\epsilon,\eta})\right) \le 0. \quad (7.4)$$

As $(s,y) \to (\Psi_2 - V)(s,y)$ reaches its maximum at $(s^{\epsilon,\eta}, y^{\epsilon,\eta})$ and V is a supersolution, we have the two following cases:

- $s^{\epsilon,\eta} = T$ and $V(s^{\epsilon,\eta}, y^{\epsilon,\eta}) \ge g(y^{\epsilon,\eta})$,
- $s^{\epsilon,\eta} \neq T$, $h_1(s^{\epsilon,\eta}, y^{\epsilon,\eta}) \le V(s^{\epsilon,\eta}, y^{\epsilon,\eta}) \le h_2(s^{\epsilon,\eta}, y^{\epsilon,\eta})$ and, if $V(s^{\epsilon,\eta}, y^{\epsilon,\eta}) < h_2(s^{\epsilon,\eta}, y^{\epsilon,\eta})$ then

$$-\frac{\partial \Psi_2}{\partial t}(s^{\epsilon,\eta}, y^{\epsilon,\eta}) - L\Psi_2(s^{\epsilon,\eta}, y^{\epsilon,\eta}) \\ - f(s^{\epsilon,\eta}, y^{\epsilon,\eta}, V(s^{\epsilon,\eta}, y^{\epsilon,\eta}), (\sigma\frac{\partial \Psi_2}{\partial x})(s^{\epsilon,\eta}, y^{\epsilon,\eta})), B\Psi_2(s^{\epsilon,\eta}, y^{\epsilon,\eta}) \ge 0.$$

As in [12], we have: $|x^{\epsilon,\eta} - y^{\epsilon,\eta}| + |t^{\epsilon,\eta} - s^{\epsilon,\eta}| \le C\epsilon$, $|x^{\epsilon,\eta}| \le \dfrac{C}{\eta}$ and $|y^{\epsilon,\eta}| \le \dfrac{C}{\eta}$.

Extracting a subsequence if necessary, we may suppose that for each η the sequences $(t^{\epsilon,\eta})_\epsilon$ and $(s^{\epsilon,\eta})_\epsilon$ converge to a common limit t^η, and the sequences $(x^{\epsilon,\eta})_\epsilon$ and $(y^{\epsilon,\eta})_\epsilon$ converge to a common limit x^η. Here, we have to consider four cases.

1st case: there exists a subsequence of (t^η) such that $t^\eta = T$ for all η (of this subsequence)

2nd case: there exists a subsequence of (t^η) such that $t^\eta \neq T$ and for all η belonging to this subsequence, there exist a subsequence of $(x^{\epsilon,\eta})_\epsilon$ and a subsequence of $(t^{\epsilon,\eta})_\epsilon$, such that $U(t^{\epsilon,\eta}, x^{\epsilon,\eta}) - h_1(t^{\epsilon,\eta}, x^{\epsilon,\eta}) = 0$.
3rd case: there exists a subsequence such that $t^\eta \neq T$, and for all η belonging to this subsequence, there exist a subsequence of $(y^{\epsilon,\eta})_\epsilon$ and a subsequence of $(s^{\epsilon,\eta})_\epsilon$, such that $V(s^{\epsilon,\eta}, y^{\epsilon,\eta}) - h_2(s^{\epsilon,\eta}, y^{\epsilon,\eta}) = 0$.
Last case: we are left with the case when, for a subsequence of η we have $t^\eta \neq T$, and for all η belonging to this subsequence, there exists a subsequence of $(x^{\epsilon,\eta})_\epsilon$, $(y^{\epsilon,\eta})_\epsilon$, $(t^{\epsilon,\eta})_\epsilon$ and $(s^{\epsilon,\eta})_\epsilon$ such that

$$U(t^{\epsilon,\eta}, x^{\epsilon,\eta}) - h_1(t^{\epsilon,\eta}, x^{\epsilon,\eta}) > 0; \quad h_2(s^{\epsilon,\eta}, y^{\epsilon,\eta}) - V(s^{\epsilon,\eta}, y^{\epsilon,\eta}) > 0.$$

We are thus in the case when the solution if strictly between the barriers, that is when there is no reflection. We can then use the same arguments as in the case of one barrier when there is no reflection. For convenience of the reader, we recall below the main arguments. We argue by contradiction by assuming that $M > 0$. We set

$$\varphi(t,s,x,y) := \frac{|x-y|^2}{\epsilon^2} + \frac{|t-s|^2}{\epsilon^2} + \eta^2(|x|^2 + |y|^2). \tag{7.5}$$

We know that he maximum of the function $\psi_{\epsilon,\eta} := U(t,x) - V(s,y) - \varphi(t,s,x,y)$ is reached at the point $(t^{\epsilon,\eta}, s^{\epsilon,\eta}, x^{\epsilon,\eta}, y^{\epsilon,\eta})$. We can thus apply a generalized Jensen-Ishii's lemma [5] (see [3]), which leads to the desired result, by using the same arguments as in [12] (see Theorem 4.1, last case).

Note that the first, second and fourth case are identical to the three cases considered for reflected BSDEs (see [12]). The third one, which didn't appear in the case of reflected BSDEs, can be treated similarly to the second one. □

Corollary 7.4. *We derive that under Assumptions (i) to (iv), there exists an unique solution of the obstacle problem* (7.2) *in the class of bounded continuous functions.*

Conclusion and perspectives We have introduced a game problem which can be seen as a generalization of the classical Dynkin game problem to the case when the linear expectation in the performance is replaced by a nonlinear $\mathcal{E}^g$-expectation/g-evaluation. Our main result is the characterization of the value of this game problem as the solution of a nonlinear DRBSDE. An interesting application concerns the pricing of a game option associated with payoffs ξ and ζ in a market with imperfections. In [14], we prove that under a regularity assumption on the upper barrier ζ, which corresponds to the payoff at cancellation time, the value of the generalized Dynkin game is equal to the super-hedging price of the game option, defined as the minimal initial amount which allows the seller to be hedged.

In the Markovian case, our results provide a new probabilistic interpretation of semi linear PDEs with two barriers in terms of game problems. A Markovian approach is also studied in [15] in a more general setup allowing for an additional control.

A Appendix

Remark A.1. Note that L^2_ν is a separable Hilbert space. Indeed, by a result of Measure Theory (see e.g. Proposition 3.4.5 of Cohn's book on Measure Theory [8]), given a measurable space $(Y, \mathcal{B}, \mu)$, if μ is σ-finite and $\mathcal{B}$ is countably generated, then $L^2(Y, \mathcal{B}, \mu)$

[5] Jensen-Ishii's lemma (also called Ishii's lemma) is the main tool to prove the comparison theorem for viscosity solutions of HJB equations (without integro-differential operators). The extension to HJB equations with integro-differential operators is due to Barles and Imbert (see [3]).

is separable. Applying this property to $Y = \mathbf{E}$ (where $\mathbf{E} = \mathbb{R}^*$), $\mathcal{B} = \mathcal{B}(\mathbf{E})$ and $\mu = \nu$, since $\mathcal{B}(\mathbf{E})$ is countably generated, it follows that $L^2_\nu = L^2(\mathbf{E}, \mathcal{B}(\mathbf{E}), \nu)$ is separable.

Using this remark, we now prove an analysis result which provides some sufficient conditions ensuring that a given driver satisfies the technical Assumption 4.3, which is essential in the case of jumps. These conditions are more tractable than Assumption 4.3.

Proposition A.2. *Let $(X, \mathcal{A})$ be a measurable space. Let $f : (X \times L^2_\nu, \mathcal{A} \otimes \mathcal{B}(L^2_\nu) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}));\ (\alpha, k) \to f(\alpha, k)$. Suppose that f satisfies one of the three following conditions:*

1. f is of class $\mathcal{C}^1$ with respect to k such that for all $(\alpha, k) \in X \times L^2_\nu$,

$$\|\nabla_k f(\alpha, k)\|_\nu \leq C \quad \text{and} \quad \nabla_k f(\alpha, k)(e) \geq -1 \quad d\nu(e) - \text{a.s.} \tag{A.1}$$

where C is a positive constant.

2. f is convex (resp. concave) with respect to k and Gâteaux-differentiable with respect to k such that the Gâteaux-gradiant $\nabla^g_k f(\alpha, k)$, which is also the sub- (resp. super-) differential with respect to k, satisfies (A.1)*.*

3. f of the form $f(\alpha, k) := \overline{f}(\alpha, \int_{\mathbf{E}} k(e)\psi(e)\nu(de))$, where ψ is a nonnegative function in L^2_ν and $\overline{f} : X \times \mathbb{R} \to \mathbb{R}$ is a measurable map, supposed to be non-decreasing with respect to its second variable and Lipschitz continuous with Lipschitz constant denoted by C.

Then, there exists a measurable map $\gamma : \big(X \times (L^2_\nu)^2, \mathcal{A} \otimes \mathcal{B}((L^2_\nu)^2)\big) \to (L^2_\nu, \mathcal{B}(L^2_\nu));$ $(\alpha, k_1, k_2) \mapsto \gamma(\alpha, k_1, k_2)$ such that $\|\gamma(.)\|_\nu \leq C$, where $C > 0$; $\gamma(.)(e) \geq -1 \quad \nu(de) - \text{a.s.}$ and

$$f(\alpha, k_2) - f(\alpha, k_1) \geq < \gamma(\alpha, k_1, k_2)\, , \, k_2 - k_1 >_\nu, \quad \forall (\alpha, k_1, k_2) \in X \times (L^2_\nu)^2.$$

Proof. 1. Since L^2_ν is a separable Hilbert space, it admits a countable orthonormal basis $\{e^i, i \in \mathbb{N}\}$. Let $(\alpha, k) \in X \times L^2_\nu$. Since f is differentiable at k, for each h in V we have: $f(\alpha, k+h) = f(\alpha, k) + < \nabla_k f(\alpha, k), h >_\nu + ||h||_\nu \varepsilon(||h||_\nu)$, where $\lim_{x \to 0} \varepsilon(x) = 0$. By taking $h = te_i$, $t \in \mathbb{R}, i \in \mathbb{N}$ we obtain that

$$< \nabla_k f(\alpha, k), e_i >_\nu = \lim_{t \to 0} \frac{f(\alpha, k + te_i) - f(\alpha, k)}{t}. \tag{A.2}$$

Hence, the map δ_i defined for each $(\alpha, k) \in X \times V$ by $\delta_i(\alpha, k) :=< \nabla_k f(\alpha, k), e_i >$ is $\mathcal{A} \otimes \mathcal{B}(L^2_\nu)$ -measurable. We thus obtain that $\nabla_k f(.,.) : (X \times L^2_\nu, \mathcal{A} \otimes \mathcal{B}(L^2_\nu) \to (L^2_\nu, \mathcal{B}(L^2_\nu));\ (\alpha, k) \mapsto \nabla_k f(\alpha, k) = \sum_{i \in \mathbb{N}} \delta_i(\alpha, k) e_i$ is measurable.

Now, for each $(\alpha, k_1, k_2) \in X \times (L^2_\nu)^2$, the map $t \mapsto f(\alpha, k_1 + t(k_2 - k_1))$ is $\mathcal{C}^1$. Hence, by the mean value theorem, we have that

$$\begin{aligned} f(\alpha, k_2) - f(\alpha, k_1) &= \int_0^1 < \nabla_k f(\alpha, k_1 + t(k_2 - k_1))\, , \, k_2 - k_1 >_\nu dt \\ &= \int_0^1 \sum_{i \in \mathbb{N}} \nabla^i_k f(\alpha, k_1 + t(k_2 - k_1))\ (k^i_2 - k^i_1) dt. \end{aligned}$$

where for each $l \in L^2_\nu$, we have denoted its coordinates in the basis $(e^i)_{i \in \mathbb{N}}$ by $(l^i)_{i \in \mathbb{N}}$.

Now, by (A.1), $||\nabla_k f(.)||_\nu$ is uniformly bounded. Using this property and Fubini's theorem, one can show that

$$f(\alpha, k_2) - f(\alpha, k_1) = < \gamma(\alpha, k_1, k_2)\, , \, k_2 - k_1 >_\nu,$$

where $\gamma(\alpha, k_1, k_2) := \int_0^1 \nabla_k f(\alpha, k_1 + t(k_2 - k_1)) dt$. Here, for each continuous map $F : [0,1] \to L^2_\nu;\ t \mapsto F(t)$, the integral $\int_0^1 F(t) dt$ is defined as $\int_0^1 F(t) dt := \sum_{i \in \mathbb{N}} (\int_0^1 F^i(t) dt) e_i$. The desired result follows.

2. Suppose f is convex. By [16, Proposition 5.4], since f is convex and Gâteaux-differentiable, f is sub-differentiable. By [16, Proposition 5.3], the Gâteaux-gradient $\nabla_k^g f(\alpha,k)$ coincides with the sub-differential at k. Hence, for each k,h in L^2_ν, we have: $f(\alpha,k+h) \geq \ f(\alpha,k) + \ < \nabla_k^g f(\alpha,k), h >_\nu$. By definition of the Gâteaux-gradiant (see Definition 5.2. in [16]), we have $< \nabla_k^g f(\alpha,k), e_i >_\nu = \lim_{t\to 0} \dfrac{f(\alpha,k+te_i)-f(\alpha,k)}{t}$, for each $i \in \mathbb{N}$. Setting $\gamma(\alpha,k_1,k_2) := \nabla_k^g f(\alpha,k_1)$, the result follows.

Suppose f is concave. By applying the previous property to the convex map $-f$ and with (k_2,k_1) instead of (k_1,k_2), we get $-f(\alpha,k_1)+f(\alpha,k_2) \geq \ < -\nabla_k^g f(\alpha,k_2)\,,\, k_1-k_2 >_\nu$, for each $(\alpha,k_1,k_2) \in X \times (L^2_\nu)^2$. Setting $\gamma(\alpha,k_1,k_2) := \nabla_k^g f(\alpha,k_2)$, the result follows.

3. Setting $\gamma(\alpha,k_1,k_2) := C\psi(e)\mathbf{1}_{\{\int_{\mathbf{E}}(k_2(e)-k_1(e))\psi(e)\nu(de)\,\leq\, 0\}}$, the result follows. □

Sketch of proof of Lemma 3.3. Here we omit the exponent g in $J^{g,n}$ and $J'^{g,n}$ for sake of simplicity. By classical results of optimal control theory, for each n, $J^n_\cdot$ and $J'^n_\cdot$ are RCLL supermartingales belonging to $\mathcal{S}^2$. We also have

$$J^{n+1}_\cdot := \bar{\mathcal{R}}(J'^n_\cdot + \tilde{\xi}^g_\cdot) \quad ; \quad J'^{n+1}_\cdot := \bar{\mathcal{R}}(J^n_\cdot - \tilde{\zeta}^g_\cdot), \tag{A.3}$$

where $\bar{\mathcal{R}}$ is the classical Snell envelop operator. It can be shown recursively, using the equalities $\tilde{\xi}^g_T = \tilde{\zeta}^g_T = 0$ a.s., that $J^n_T = J'^n_T = 0$ a.s. for each n. Hence, $J^n_\cdot$ and $J'^n_\cdot$ are non negative since they are supermartingales. Let us now prove that $(J^n_\cdot)$ and $(J'^n_\cdot)$ are non decreasing sequences of processes. The arguments are classical and are given for the convenience of the reader. We have $J^1_\cdot \geq 0 = J^0_\cdot$ and $J'^1_\cdot \geq 0 = J'^0_\cdot$. Suppose that $J^n_\cdot \geq J^{n-1}_\cdot$ and $J'^n_\cdot \geq J'^{n-1}_\cdot$. Since the operator $\bar{\mathcal{R}}$ is non decreasing, it follows that $\bar{\mathcal{R}}(J'^n_\cdot + \tilde{\xi}^g_\cdot) \geq \bar{\mathcal{R}}(J'^{n-1}_\cdot + \tilde{\xi}^g_\cdot)$ and $\bar{\mathcal{R}}(J^n_\cdot - \tilde{\zeta}^g_\cdot) \geq \bar{\mathcal{R}}(J^{n-1}_\cdot - \tilde{\zeta}^g_\cdot)$. Hence, $J^{n+1}_\cdot \geq J^n_\cdot$ and $J'^{n+1}_\cdot \geq J'^n_\cdot$, which gives the desired result.

The processes $J^g_\cdot := \lim \uparrow J^n_\cdot$ and $J'^g_\cdot := \lim \uparrow J'^n_\cdot$ are optional and valued in $[0,+\infty]$. Since for each n, $J^n_T = J'^n_T = 0$ a.s. we have $J^g_T = J'^g_T = 0$ a.s. Using the monotone convergence theorem, one can show that $J^g_\cdot$ and $J'^g_\cdot$ are strong supermartingales valued in $[0,+\infty]$.

We now show that the processes $J^g_\cdot$ and $J'^g_\cdot$ satisfy equalities (3.6). In the following, we use the Snell envelope operator $\mathcal{R}$ which acts on admissible families of random variables (r.v.). The reader is referred to [31, Section 1.1] for the definition of an admissible family of r.v. indexed by stopping times, as well as the definition of a supermartingale family. Recall that for each admissible family $\phi = (\phi(\theta), \theta \in \mathcal{T}_0)$ valued in $\mathbf{R} \cup \{+\infty\}$ with $E\,[\operatorname{ess\,sup}_{\theta\in\mathcal{T}} \phi(\theta)^-] < +\infty$, $\mathcal{R}(\phi)$ is defined as the smallest supermartingale family greater than ϕ. By some results of optimal stopping (see [31, Section 1.1]), we have

$$\mathcal{R}(\phi)(\theta) = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_\theta} E[\phi(\tau)\,|\,\mathcal{F}_\theta] \quad \text{a.s.} \tag{A.4}$$

for each stopping time θ. In the following, for each optional process $\phi_\cdot = (\phi_t)_{0\leq t\leq T}$ valued in $\mathbf{R} \cup \{+\infty\}$, we denote by $\phi = (\phi(\theta), \theta \in \mathcal{T}_0)$ its associated family of r.v. defined for each $\theta \in \mathcal{T}_0$ by $\phi(\theta) := \phi_\theta$. If $\phi_\cdot \in \mathcal{S}^2$, we then have

$$\mathcal{R}(\phi)(\theta) = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_\theta} E[\phi_\tau\,|\,\mathcal{F}_\theta] = \bar{\mathcal{R}}(\phi_\cdot)_\theta \quad \text{a.s.} \tag{A.5}$$

for each stopping time θ. This property and equalities (A.3) lead to the following equalities written in terms of families and the operator $\mathcal{R}$:

$$J^{n+1} = \mathcal{R}(J'^n + \tilde{\xi}^g) \quad ; \quad J'^{n+1} = \mathcal{R}(J^n - \tilde{\zeta}^g). \tag{A.6}$$

Since $J^g_\cdot = \lim \uparrow J^n_\cdot$, we derive that for each $\theta \in \mathcal{T}_0$, we have $J^g_\theta = \lim \uparrow J^n_\theta$ a.s. Hence, $J^g = \lim \uparrow J^n$. Similarly, we have $J'^g = \lim \uparrow J'^n$. Note that the supermartingale property of the families J^g and J'^g corresponds to the strong supermartingale property of the optional processes $J^g_\cdot$ and $J'^g_\cdot$. This formulation in terms of admissible families allows us to let n tend to ∞ in (A.6) even if the limits J^g and J'^g are not necessarily finite. We use below some arguments used in the proof of Theorem 2.3 in [31]. As $\mathcal{R}$ is nondecreasing, for each $n \in \mathbb{N}$, we have $J^{n+1} = \mathcal{R}(J'^n + \tilde{\xi}^g) \leq \mathcal{R}(J'^g + \tilde{\xi}^g)$. By letting n tend to $+\infty$, we get that $J^g \leq \mathcal{R}(J'^g + \tilde{\xi}^g)$. Now, for each $n \in \mathbb{N}$, $J^{n+1} \geq J'^n + \tilde{\xi}^g$. By letting n tend to $+\infty$, we derive that $J^g \geq J'^g + \tilde{\xi}^g$. Since J^g is a supermartingale family, it follows that $J^g \geq \mathcal{R}(J'^g + \tilde{\xi}^g)$. Since $J^g \leq \mathcal{R}(J'^g + \tilde{\xi}^g)$, we get $J^g = \mathcal{R}(J'^g + \tilde{\xi}^g)$. Similarly, $J'^g = \mathcal{R}(J^g - \tilde{\zeta}^g)$, which, by (A.5), leads to $J^g_\cdot = \bar{\mathcal{R}}(J'^g_\cdot + \tilde{\xi}^g_\cdot)$ and $J'^g_\cdot = \bar{\mathcal{R}}(J^g_\cdot - \tilde{\zeta}^g_\cdot)$, which gives the equalities (3.6).

Moreover, if $J_0^g < +\infty$ and $J_0'^g < +\infty$, by [11, Theorem 18, Chapter VI], $J^g_\cdot$ and $J'^g_\cdot$ are indistinguishable from nonnegative RCLL supermartingales, as the non decreasing limits of nonnegative RCLL supermartingales. □

Remark A.3. By [31, Proposition 5.1], we derive that (J_t^g) and $(J_t'^g)$ are the smallest *strong supermartingales* valued in $[0, +\infty]$ satisfying $J_t^g \geq J_t'^g + \tilde{\xi}_t^g$ and $J_t'^g \geq J_t^g - \tilde{\zeta}_t^g$, $0 \leq t \leq T$ a.s.

Remark A.4. The proof of Theorem 3.4 together with Lemma 3.3 ensures that $B = A$. Indeed, set $H_t := E[A_T - A_t|\mathcal{F}_t]$ (resp. $H'_t := E[A'_T - A'_t|\mathcal{F}_t]$). Since $dA_t << dB_t$ (resp. $dA'_t << dB'_t$), we have $H_t \leq J_t^g = E[B_T - B_t|\mathcal{F}_t]$ (resp. $H'_t \leq {}^gJ'_t = E[B'_T - B'_t|\mathcal{F}_t]$). Moreover, $H - H' = J^g - J'^g$. Hence, we have $H \geq H' + \tilde{\xi}^g$ and $H' \geq H - \tilde{\zeta}^g$. By the minimality property of J^g, J'^g, we derive that $J^g = H$ (resp. $J'^g = H'$).

Proof of Theorem 3.5. Theorem 3.4 gives the existence. Let (Y, Z, k, A, A') be a solution of the DRBSDE associated with driver process $g(t)$ and obstacles (ξ, ζ). Let us prove that it is unique. We first show the uniqueness of Y. For each $S \in \mathcal{T}_0$ and for each $\varepsilon > 0$, let

$$\tau_S^\varepsilon := \inf\{t \geq S,\ Y_t \leq \xi_t + \varepsilon\} \quad \sigma_S^\varepsilon := \inf\{t \geq S,\ Y_t \geq \zeta_t - \varepsilon\}. \tag{A.7}$$

Note that σ_S^ε and $\tau_S^\varepsilon \in \mathcal{T}_S$. Fix $\varepsilon > 0$. By the same arguments as in the proof of Lemma 4.11, the function $t \mapsto A_t$ is constant a.s. on $[S, \tau_S^\varepsilon]$ and $Y_{\tau_S^\varepsilon} \leq \xi_{\tau_S^\varepsilon} + \varepsilon$ a.s. Similarly, A' is constant on $[S, \sigma_S^\varepsilon]$ and $Y_{\sigma_S^\varepsilon} \geq \zeta_{\sigma_S^\varepsilon} - \varepsilon$ a.s.

Let $\tau \in \mathcal{T}_S$. Since A' is constant on $[S, \sigma_S^\varepsilon]$, the process $(Y_t + \int_0^t g(s)ds,\ S \leq t \leq \tau \wedge \sigma_S^\varepsilon)$ is a supermartingale. Hence

$$Y_S \geq E[Y_{\tau \wedge \sigma_S^\varepsilon} + \int_S^{\tau \wedge \sigma_S^\varepsilon} g(s)ds \mid \mathcal{F}_S] \quad \text{a.s.}$$

We also have that $Y_{\tau \wedge \sigma_S^\varepsilon} = Y_\tau \mathbf{1}_{\tau \leq \sigma_S^\varepsilon} + Y_{\sigma_S^\varepsilon} \mathbf{1}_{\sigma_S^\varepsilon < \tau} \geq \xi_\tau \mathbf{1}_{\tau \leq \sigma_S^\varepsilon} + (\zeta_{\sigma_S^\varepsilon} - \varepsilon)\mathbf{1}_{\sigma_S^\varepsilon < \tau}$ a.s. We get $Y_S \geq E[I_S(\tau, \sigma_S^\varepsilon) \mid \mathcal{F}_S] - \varepsilon$ a.s. Similarly, one can show that for each $\sigma \in \mathcal{T}_S$, $Y_S \leq E[I_S(\tau_S^\varepsilon, \sigma) \mid \mathcal{F}_S] + \varepsilon$ a.s. It follows that for each $\varepsilon > 0$,

$$\operatorname*{ess\,sup}_{\tau \in \mathcal{T}_S} E[I_S(\tau, \sigma_S^\varepsilon) \mid \mathcal{F}_S] - \varepsilon \ \leq\ Y_S \ \leq\ \operatorname*{ess\,inf}_{\sigma \in \mathcal{T}_S} E[I_S(\tau_S^\varepsilon, \sigma) \mid \mathcal{F}_S] + \varepsilon \ \text{ a.s.},$$

which implies $\overline{V}(S) - \varepsilon \leq Y_S \leq \underline{V}(S) + \varepsilon$ a.s. Since $\underline{V}(S) \leq \overline{V}(S)$ a.s. we get $\underline{V}(S) = Y_S = \overline{V}(S)$ a.s. This equality holds of each stopping time $S \in \mathcal{T}_0$, which implies the uniqueness of Y. It remains to show the uniqueness of (Z, k, A, A'). By the uniqueness of the decomposition of the semimartingale $Y_t + \int_0^t g(s)ds$, there exists an unique square integrable martingale M and an unique square integrable finite variation RCLL adapted process α with $\alpha_0 = 0$ such that $dY_t + g(t)dt = dM_t - d\alpha_t$. The martingale representation theorem applied to M ensures the uniqueness of the pair $(Z, k) \in \mathbb{H}^2 \times \mathbb{H}_\nu^2$.

The uniqueness of the processes A, A' follows from the uniqueness of the canonical decomposition of an RCLL process with integrable variation (see Proposition A.7).

Suppose that A and A' are continuous. Since Y and ξ are right-continuous, we have $Y_{\sigma_S^*} = \zeta_{\sigma_S^*}$ and $Y_{\tau_S^*} = \xi_{\tau_S^*}$ a.s. By definition of τ_S^*, on $[S, \tau_S^*[$, we have $Y_t > \xi_t$

a.s. Since $(Y, Z, k(.), A, A')$ is the solution of the DRBSDE, A is constant on $[S, \tau_S^*[$ a.s. and even on $[S, \tau_S^*]$ because A is continuous. Similarly, A' is constant on $[S, \sigma_S^*]$ a.s. The process $(Y_t + \int_0^t g(s)ds, S \leq t \leq \tau_S^* \wedge \sigma_S^*)$ is thus a martingale. Hence, we have $Y_S = E[I_S(\tau_S^*, \sigma_S^*) \mid \mathcal{F}_S]$ a.s. By similar arguments as above, one can show that for each $\tau, \sigma \in \mathcal{T}_S$, $E[I_S(\tau, \sigma_S^*) \mid \mathcal{F}_S] \leq Y_S$ and $Y_S \leq E[I_S(\tau_S^*, \sigma) \mid \mathcal{F}_S]$ a.s., which yields that (τ_S^*, σ_S^*) is an S-saddle point. □

Proof of Proposition 3.10. Since $J^g \geq J'^g + \tilde{\xi}^g$ and $J'^g \geq J^g - \tilde{\zeta}^g$, $J^g \in \mathcal{S}^2 \Leftrightarrow J'^g \in \mathcal{S}^2$. Using the minimality property of J and J' given in Remark A.3, one can show that $J^g \in \mathcal{S}^2$ if and only if there exist two non-negative supermartingales $H^g, H'^g \in \mathcal{S}^2$ such that

$$\tilde{\xi}_t^g \leq H_t^g - H_t'^g \leq \tilde{\zeta}_t^g \quad 0 \leq t \leq T \quad \text{a.s.} \tag{A.8}$$

Since this equivalence holds for all $g \in I\!H^2$, in particular when $g = 0$, we get *(ii)* $\Leftrightarrow$ *(iii)*.

To prove *(i)* $\Leftrightarrow$ *(ii)*, it is sufficient to show that (3.11) is equivalent to (A.8). Suppose that (3.11) is satisfied. By setting

$$\begin{cases} H_t^g := H_t + E[\xi_T^- | \mathcal{F}_t] + E[\int_t^T g^-(s)ds | \mathcal{F}_t], , 0 \leq t \leq T \\ H_t'^g := H_t' + E[\xi_T^+ | \mathcal{F}_t] + E[\int_t^T g^+(s)ds | \mathcal{F}_t], 0 \leq t \leq T, \end{cases}$$

(A.8) holds. Similarly, (A.8) implies (3.11). We have that *(i)* implies *(iv)*. It remains to prove that *(iv)* implies *(i)*. Let (Y, Z, k, A, A') be the solution of the DRBSDE (2.2) associated with driver process $g(t)$. Let $H_t^g := E[A_T - A_t | \mathcal{F}_t]$ and $H_t^{'g} := E[A_T' - A_t' | \mathcal{F}_t]$. We have $H_t^g - H_t'^g = Y_t - E[\int_t^T g(s)ds | \mathcal{F}_t]$. Since $\xi \leq Y \leq \zeta$, condition (A.8) holds. □

Proof of Theorem 4.1. For $\beta > 0$, $\phi \in I\!H^2$, and $l \in I\!H_\nu^2$, we introduce the norms $\|\phi\|_\beta^2 := E[\int_0^T e^{\beta s} \phi_s^2 ds]$, and $\|l\|_{\nu,\beta}^2 := E[\int_0^T e^{\beta s} \|l_s\|_\nu^2 \, ds]$.

Let $I\!H_{\beta,\nu}^2$ (below simply denoted by $I\!H_\beta^2$) the space $I\!H^2 \times I\!H^2 \times I\!H_\nu^2$ equipped with the norm $\|Y, Z, k(\cdot)\|_\beta^2 := \|Y\|_\beta^2 + \|Z\|_\beta^2 + \|k\|_{\nu,\beta}^2$.

We define a mapping Φ from $I\!H_\beta^2$ into itself as follows. Given $(U, V, l) \in I\!H_\beta^2$, by Theorem 3.5 there exists a unique process $(Y, Z, k) = \Phi(U, V, l)$ solution of the DRBSDE associated with driver process $g(s) = g(s, U_s, V_s, l_s)$. Note that $(Y, Z, k) \in I\!H_\beta^2$. Let A, A' be the associated non decreasing processes. Let us show that Φ is a contraction and hence admits a unique fixed point (Y, Z, k) in $I\!H_\beta^2$, which corresponds to the unique solution of DRBSDE (2.2). The associated finite variation process is then uniquely determined in terms of (Y, Z, k) and the pair (A, A') corresponds to the unique canonical decomposition of this finite variation process. Let (U^2, V^2, l^2) be another element of $I\!H_\beta^2$ and define $(Y^2, Z^2, k^2) = \Phi(U^2, V^2, l^2)$. Let A^2, A'^2 be the associated non decreasing processes. Set $\overline{U} = U - U^2$, $\overline{V} = V - V^2$, $\bar{l} = l - l^2$ and, $\overline{Y} = Y - Y^2$, $\overline{Z} = Z - Z^2$, $\overline{k} = k - k^2$. By Itô's formula, for any $\beta > 0$, we have

$$\begin{aligned} &\overline{Y}_0^2 + E \int_0^T e^{\beta s} [\beta \overline{Y}_s^2 + \overline{Z}_s^2 + \|\overline{k}_s^2\|] ds + E \sum_{0 < s \leq T} e^{\beta s} (\Delta A_s - \Delta A_s^2 - \Delta A_s' + \Delta A_s'^2)^2 \\ &= 2E \int_0^T e^{\beta s} \overline{Y}_s [g(s, U_s, V_s, l_s) - g(s, U_s^2, V_s^2, l_s^2)] \, ds \\ &+ 2E[\int_0^T e^{\beta s} \overline{Y}_{s^-} \, dA_s - \int_0^T e^{\beta s} \overline{Y}_{s^-} \, dA_s^2] \\ &- 2E[\int_0^T e^{\beta s} \overline{Y}_{s^-} \, dA_s' - \int_0^T e^{\beta s} \overline{Y}_{s^-} \, dA_s'^2]. \end{aligned} \tag{A.9}$$

Now, we have a.s. $\overline{Y}_s dA_s^c = (Y_s - \xi_s) dA_s^c - (Y_s^2 - \xi_s) dA_s^c = -(Y_s^2 - \xi_s) dA_s^c \leq 0$, and by symmetry, $\overline{Y}_s dA_s^{2c} \geq 0$ a.s. Also, we have a.s.

$$\overline{Y}_{s^-}\Delta A^d_s = (Y_{s^-} - \xi_{s^-})\Delta A^d_s - (Y^2_{s^-} - \xi_{s^-})\Delta A^d_s = -(Y^2_{s^-} - \xi_{s^-})\Delta A^d_s \leq 0$$

and $\overline{Y}_{s^-}\Delta A^{2d}_s \geq 0$ a.s. Similarly, we have $\overline{Y}_s dA'^c_s = -(Y^2_s - \zeta_s)dA'^c_s \geq 0$ and $\overline{Y}_{s^-}\Delta A'^d_s = -(Y^2_{s^-} - \zeta_{s^-})\Delta A'^d_s \geq 0$ a.s. By symmetry, $\overline{Y}_s dA'^{2c}_s \leq 0$ and $\overline{Y}_{s^-}\Delta A'^{2d}_s \leq 0$ a.s.

Consequently, the second and the third term of (A.9) are non positive. By using the Lipschitz property of g and the inequality $2Cyu \leq 2C^2y^2 + \frac{1}{2}u^2$, we get

$$\beta\|\overline{Y}\|^2_\beta + \|\overline{Z}\|^2_\beta + \|\overline{k}\|^2_{\nu,\beta} \leq 6C^2\|\overline{Y}\|^2_\beta + \frac{1}{2}(\|\overline{U}\|^2_\beta + \|\overline{V}\|^2_\beta + \|\bar{l}\|^2_{\nu,\beta}).$$

Choosing $\beta = 6C^2 + 1$, we deduce $\|(\overline{Y}, \overline{Z}, \overline{k})\|^2_\beta \leq \frac{1}{2}\|(\overline{U}, \overline{V}, \bar{l})\|^2_\beta$.

The last assertion of the theorem follows from Theorem 3.7 *(i)* and Remark 4.2. □

Remark A.5. By similar arguments as above (see [7, proof of Theorem 3.2]), one can show the following estimate, which is expressed in terms of the associated increasing processes. More precisely, under the assumptions of Proposition 6.6, we have

$$\|\overline{Y}\|^2_{\mathcal{S}^2} \leq K\left(E[\bar{\xi}^2_T] + \|\overline{g}^2\|_2 + \|A^1_T + A^2_T\|_{L^2}\|\bar{\xi}\|_{S^2} + \|A'^1_T + A'^2_T\|_{L^2}\|\bar{\zeta}\|_{S^2}\right). \tag{A.10}$$

In [7], in the particular case when for each $i = 1, 2$, the lower barrier ξ^i is of the form $\xi^i = M^i + B^i$, where M^i is a square integrable martingale and B^i is a square integrable RCLL predictable non decreasing process with $B^i_0 = 0$, the authors derive from (A.10) the following estimate: $\|\overline{Y}\|^2_{\mathcal{S}^2} \leq K\left(E[\bar{\xi}^2_T] + \|\overline{g}^2\|_2 + \phi(\|\bar{\xi}\|_{S^2} + \|\bar{\zeta}\|_{S^2})\right)$, where the constant $\phi > 0$ is not necessarily universal, depending in particular on $\|\xi^i\|_{\mathcal{S}^2}$, $\|\zeta^i\|_{\mathcal{S}^2}$, $\|g^i(s,0,0,0)\|_{I\!H^2}$ and B^i, for $i = 1, 2$ (see [7, estimate (14) of Theorem 3.2]).

We now easily show an $\mathcal{E}^g$-Doob-Meyer decomposition of $\mathcal{E}^g$-supermartingales, which generalizes the results given in [37] under stronger assumptions. Moreover, our proof gives an alternative short proof of this result.

Proposition A.6. *Suppose that g satisfies Assumption* (4.3).

- *Let A be a non decreasing (resp non increasing) RCLL predictable process in $\mathcal{S}^2$ with $A_0 = 0$. Let $(Y, Z, k) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu$ following the dynamics:*

$$-dY_s = g(s, Y_s, Z_s, k_s)ds + dA_s - Z_s dW_s - \int_{\mathbf{E}} k_s(e)\tilde{N}(ds, de). \tag{A.11}$$

  Then the process (Y_t) a strong $\mathcal{E}^g$-supermartingale (resp $\mathcal{E}^g$-submartingale).
- *($\mathcal{E}^g$-Doob-Meyer decomposition) Let (Y_t) be a strong $\mathcal{E}^g$-supermartingale (resp. $\mathcal{E}^g$-submartingale). Then, there exists a non decreasing (resp non increasing) RCLL predictable process A in $\mathcal{S}^2$ with $A_0 = 0$ and $(Z, k) \in I\!H^2 \times I\!H^2_\nu$ such that* (A.11) *holds.*

Proof. Suppose A is non decreasing. Let $(X^\tau, \pi^\tau, l^\tau)$ be the solution of the BSDE associated with driver g, terminal time τ, and terminal condition Y_τ. Since g satisfies Assumption 4.3 and since $g(s,y,z,k)ds + dA_s \geq g(s,y,z,k)ds$, the comparison theorem for BSDEs (see [35, Theorem 4.2]) gives that $Y_\sigma \geq X^\tau_\sigma = \mathcal{E}^g_{\sigma,\tau}(Y_\tau)$ a.s. on $\{\sigma \leq \tau\}$. The case when A is non-increasing can be shown similarly.

Let us show the second assertion. Fix $S \in \mathcal{T}_0$. Since (Y_t) is a strong $\mathcal{E}^g$-supermartingale, we derive that for all $\tau \in \mathcal{T}_S$, we have $Y_S \geq \mathcal{E}^g_{S,\tau}(Y_\tau)$ a.s. We get $Y_S \geq \operatorname{ess\,sup}_{\tau \in \mathcal{T}_S} \mathcal{E}^g_{S,\tau}(Y_\tau)$ a.s. Now, by definition of the essential supremum, $Y_S \leq \operatorname{ess\,sup}_{\tau \in \mathcal{T}_S} \mathcal{E}^g_{S,\tau}(Y_\tau)$ a.s. because $S \in \mathcal{T}_S$. Hence, $Y_S = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_S} \mathcal{E}^g_{S,\tau}(Y_\tau)$ a.s. By [36, Theorem 3.3], the process (Y_t) coincides with the solution of the reflected BSDE associated with the RCLL obstacle (Y_t). The result follows. □

We now show a result on RCLL adapted processes with integrable total variation, which can be seen as a probabilistic version of the well-known existence and uniqueness result of the canonical decomposition of a function of bounded variation on $[0, T]$.

Proposition A.7. *Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a completed right-continuous filtration $(\mathcal{F}_t)_{0\leq t\leq T}$. Let $\alpha = (\alpha_t)_{0\leq t\leq T}$ be a RCLL predictable process with integrable total variation, that is, $E(|\alpha|_T) < \infty$, where $|\alpha|_T$ is the total variation at time T. There exists an unique pair $(A, A') \in (\mathcal{A}^1)^2$ such that $\alpha = A - A'$ with $dA_t \perp dA'_t$. This decomposition is called the* canonical decomposition *of the process α. If $E(|\alpha|_T^2) < \infty$, then A_T and $A'_T \in L^2$.*

Moreover, if $(B, B') \in (\mathcal{A}^1)^2$ satisfies $\alpha = B - B'$, then $dA_t << dB_t$ in the (probabilistic) sense, that is, for each $K \in \mathcal{P}$ with $\int_0^T \mathbf{1}_K dB_t = 0$ a.s., then $\int_0^T \mathbf{1}_K dA_t = 0$ a.s.

An analogous result holds in the optional case (that is replacing predictable *by* optional *and $\mathcal{P}$ by the optional σ-algebra in the above properties).*

Proof. By classical results, the process α can be written as $\alpha = B - B'$ with $B, B' \in \mathcal{A}^1$. Let $C_t := B_t + B'_t$. This process belongs to $\mathcal{A}^1$. For almost every ω, the measures $dB_.(\omega)$ and $dB'_.(\omega)$ on $[0,T]$ are absolutely continuous with respect to $dC_.(\omega)$. By using the Radon-Nikodym Theorem for predictable RCLL non decreasing processes (see [11, Theorem 67, Chap. VI]), there exist nonnegative predictable processes H and H' such that for each $t \in [0,T]$, $B_t = \int_0^t H_s dC_s$ and $B'_t = \int_0^t H'_s dC_s$ a.s. Let A and A' be the processes defined by

$$A_t := \int_0^t (H_s - H'_s)^+ dC_s \quad \text{and} \quad A'_t := \int_0^t (H_s - H'_s)^- dC_s.$$

They belong to $\mathcal{A}^1$. Now, the set $D := \{(t,\omega)\,,\, H_t(\omega) - H'_t(\omega) \geq 0\}$ belongs to $\mathcal{P}$. We have $\int_0^T \mathbf{1}_{D_t^c} dA_t = \int_0^T \mathbf{1}_{\{H_t - H'_t < 0\}} (H_t - H'_t)^+ dC_t = 0$ a.s. Similarly $\int_0^T \mathbf{1}_{D_t} dA'_t = 0$ a.s., which implies that $dA_t \perp dA'_t$. It remains to show the uniqueness of this decomposition. Since $dA_t \perp dA'_t$, it follows that, for almost every ω, the deterministic measures $dA_t(\omega)$ and $dA'_t(\omega)$ are mutually singular in the classical analysis sense. Hence, for almost every ω, the non decreasing maps $A(\omega)$ and $A'(\omega)$ correspond to the unique canonical decomposition of the RCLL bounded variational map $\alpha.(\omega)$ by a well-known analysis result. This implies the uniqueness of A, A'. Note that since $|\alpha|_T = A_T + A'_T$, if $|\alpha|_T \in L^2$, then A_T and A'_T belong to L^2. Moreover, since $(H_t - H'_t)^+ \leq H_t$, the last assertion holds. □

Remark A.8. Let ξ be an adapted RCLL process in $\mathcal{S}^2$. Suppose that ξ is a semimartingale of the form $\xi_t := M_t + \alpha_t$, where M a square integrable martingale and α is an RCLL adapted process with $\alpha_0 = 0$ and with square integrable total variation, that is $E(|\alpha|_T^2) < \infty$. Let us show that ξ can be written as the difference of two non negative square integrable supermartingales. By the above Proposition A.7, there exists an unique pair (A, A') of square integrable non decreasing RCLL adapted processes with $A_0 = A'_0 = 0$, and such that $\alpha = A' - A$ with $dA_t \perp dA'_t$. The processes H and H' defined by $H_t := E[\xi_T^+ + A_T - A_t|\mathcal{F}_t]$ and $H'_t := E[\xi_T^- + A'_T - A'_t|\mathcal{F}_t]$ are non negative RCLL supermartingales belonging to $\mathcal{S}^2$. Moreover, we have $\xi_t = E[\xi_T + \alpha_T - \alpha_t|\mathcal{F}_t] = H_t - H'_t$, which gives the desired result.

From this property, we derive that if ζ is an adapted RCLL process in $\mathcal{S}^2$ with $\zeta_T = \xi_T$ and $\xi \leq \zeta$, Mokobodzki's condition (3.11) then holds.

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