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Triangulating stable laminations*

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Abstract

We study the asymptotic behaviour of random simply generated noncrossing planar trees in the space of compact subsets of the unit disk, equipped with the Hausdorff distance. Their distributional limits are obtained by triangulating at random the faces of stable laminations, which are random compact subsets of the unit disk made of non-intersecting chords and which are coded by stable Lévy processes. We also study other ways to “fill-in” the faces of stable laminations, which leads us to introduce the iteration of laminations and of trees.

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1 Introduction

We are interested in the structure of large random noncrossing trees. By definition, a noncrossing tree with n vertices is a tree drawn in the unit disk of the complex plane, having as vertices the n-th roots of unity and whose edges are straight line segments which do not cross. The enumeration problem for noncrossing trees was first proposed as Problem E3170 in the American Mathematical Monthly [20]. Dulucq & Penaud [15] established a bijection between noncrossing trees with n vertices and ternary trees with n internal vertices, thus showing that there are $\frac{1}{2n-1}\binom{3n-3}{n-1}$ noncrossing trees with n vertices in another way. Noy [36] pushed forward the enumerative study of noncrossing trees by counting them according to different statistics. Since then, various authors have studied combinatorial and algebraic properties of noncrossing trees [19, 12, 13, 37, 21]. See also [33] for motivations from linguistics and proof theory, where noncrossing trees are for instance connected to the number of different readings of an ambiguous sentence. Other families of noncrossing configurations have also attracted some attention [14, 19, 2, 9].

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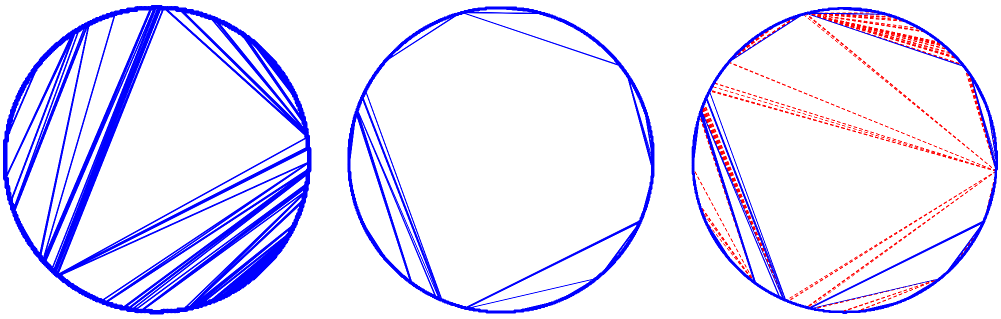

Figure 1: Simulations from left to right: the Brownian triangulation, an $\alpha = 1.1$ stable lamination, and the same lamination with its faces triangulated "uniformly" in dashed red.

In this work, we are interested in the properties of *random* noncrossing trees and we study in particular how the geometrical constraint of their planar embeddings influences their structure. Marckert & Panholzer [30] have shown that uniform random noncrossing trees on n vertices are almost conditioned Bienaymé–Galton–Watson trees, thus obtaining interesting results concerning the structure of noncrossing trees by using the theory of random plane trees. This was then used by Curien & Kortchemski [9] to establish limit theorems for large uniform random noncrossing trees as compact subsets of the unit disk. We shall generalise these results.

1.1 Noncrossing trees seen as subsets of the plane

Since noncrossing trees are given with a plane embedding, we naturally view them as (closed) subsets of the (closed) unit disk by considering each edge as a line segment. This idea goes back to Aldous [1], who showed that if P_n is the regular polygon spanned by the n-th roots of unity, then, as $n \to \infty$, a uniform random triangulation of P_n converges in distribution for the Hausdorff distance to a random compact subset of the unit disk $\mathbf{L}_2$ called the Brownian triangulation. This set is indeed a triangulation, as its complement in the unit disk is a disjoint union of triangles, and can be built from the Brownian excursion (see Sec. 3.1 below for details). Curien & Kortchemski [9] showed that the Brownian triangulation is the universal limit of various classes of uniform random noncrossing graphs built using the vertices of P_n, such as dissections (which are collections of noncrossing diagonals of P_n), noncrossing partitions or noncrossing trees.

Kortchemski [26] constructed a one parameter family $(\mathbf{L}_\alpha : \alpha \in (1, 2))$ of random compact subsets of the unit disk called stable laminations, which are the distributional limits of the more general model of Boltzmann-type random dissections chosen at random according to certain sequences of weights. They also appear as limits of large simply generated noncrossing partitions [27]. Stable laminations are coded by excursions of spectrally positive strictly stable Lévy processes, and unlike the Brownian triangulation, each face is surrounded by infinitely many chords; see Fig. 1 for a simulation and Sec. 3.2 below for details.

1.2 Simply generated noncrossing trees

In this work, we introduce and study the asymptotic behaviour of simply generated noncrossing trees in the space of compact subsets of the unit disk equipped with the Hausdorff distance. Given a sequence of nonnegative real numbers $(w(k) : k \geq 1)$, we define the weight of a noncrossing tree θ by

$$\Omega^w(\theta) = \prod_{u\in\theta} w(\deg u).$$

Next, for every integer $n \geq 1$, we denote by $\mathbb{NC}_n$ the set of noncrossing trees with n vertices and we set

$$Z_n^w = \sum_{\theta\in\mathbb{NC}_n} \Omega^w(\theta).$$

Finally, if $Z_n^w > 0$ (and we will always implicitly restrict our attention to those values of n for which it is the case), we define a probability measure on $\mathbb{NC}_n$ by

$$\mathbb{P}_n^w(\theta) = \frac{1}{Z_n^w}\Omega^w(\theta) \qquad \text{for all} \qquad \theta \in \mathbb{NC}_n. \tag{1.1}$$

A random noncrossing tree sampled according to $\mathbb{P}_n^w$ is called *simply generated*. We choose this terminology because of the similarity with the model of simply generated plane trees, introduced by Meir & Moon [34].

Note that if $w \equiv 1$, then $\mathbb{P}_n^w$ is the uniform distribution on $\mathbb{NC}_n$. More generally, if $\mathcal{A}$ is a subset of $\mathbb{N}$ and if $w(k) = \mathbb{1}_{k\in\mathcal{A}}$, then $\mathbb{P}_n^w$ is the uniform distribution on the set of all noncrossing trees with n vertices with all degrees belonging to $\mathcal{A}$ (provided this set is not empty).

Theorem 1.1. *Fix $\alpha \in (1,2]$. There exists a random compact subset of the unit disk, denoted by $\mathbf{L}_\alpha^U$, with Hausdorff dimension $1+\frac{1}{\alpha}$ such that the following holds. Let $(w(k) : k \geq 1)$ be a sequence of nonnegative real numbers such that there exists $b > 0$ satisfying*

$$\sum_{k=0}^{\infty}(k+1)(k-1)w(k+1)b^k = 0, \tag{1.2}$$

and, moreover, such that the probability measure

$$\mu(k) = \frac{(k+1)w(k+1)b^k}{\sum_{\ell=0}^{\infty}(\ell+1)w(\ell+1)b^\ell} \qquad (k \geq 0)$$

belongs to the domain of attraction of a stable law of index α. If Θ_n is a random noncrossing tree sampled according to $\mathbb{P}_n^w$, then the convergence in distribution

$$\Theta_n \quad \xrightarrow[n\to\infty]{(d)} \quad \mathbf{L}_\alpha^U$$

holds for the Hausdorff distance on the space of all compact subsets of the unit disk.

Recall that a probability distribution μ belongs to the domain of attraction of a stable law if either it has finite variance (in which case $\alpha = 2$), or there exists a slowly varying function $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mu([n,\infty)) = g(n)n^{-\alpha}$ for $n \geq 1$. See Remark 5.2 for a probabilistic interpretation of condition (1.2).

Let us give a rough description of $\mathbf{L}_\alpha^U$. In the case $\alpha = 2$, $\mathbf{L}_2^U = \mathbf{L}_2$ is simply Aldous' Brownian triangulation, whereas for $\alpha \in (1,2)$, $\mathbf{L}_\alpha^U$ is a triangulation that strictly contains the α-stable lamination $\mathbf{L}_\alpha$. Intuitively, $\mathbf{L}_\alpha^U$ is constructed from $\mathbf{L}_\alpha$ by "triangulating" each face of $\mathbf{L}_\alpha$ from a uniform random vertex, i.e. by joining this vertex to each other vertex of the face by a chord. We refer the reader to Fig. 1 for a simulation and to Sec. 3.3 for a precise definition. The random compact set $\mathbf{L}_\alpha^U$ is called the *uniform α-stable triangulation*. It is interesting to note that unlike the Brownian triangulation or stable laminations, $\mathbf{L}_\alpha^U$ is not simply coded by a function as we will see in Remark 3.3.

It is interesting to compare the Hausdorff dimension of $\mathbf{L}_\alpha^U$ with that of $\mathbf{L}_\alpha$ computed in [26], which is equal to $2-\frac{1}{\alpha}$. Since $1+\frac{1}{\alpha} > \frac{3}{2} > 2-\frac{1}{\alpha}$, any uniform stable triangulation is “fatter” than the Brownian triangulation and any stable lamination.

The main steps to prove Theorem 1.1 are the following. We first establish deterministic invariance principles in the space of compact subsets of the unit disk (Propositions 4.1 and 4.6) for noncrossing trees under conditions involving their *shape*, which is the plane tree structure that they carry (see Fig. 2 for an illustration). We then establish (Theorem 5.1) that the shape of Θ_n is a “modified” Bienaymé–Galton–Watson tree, where the root has a different offspring distribution, conditioned to have size n. This extends a result of Marckert & Panholzer [30] for the uniform distribution. Finally, we show that such trees fulfill the framework of our invariance principles with high probability.

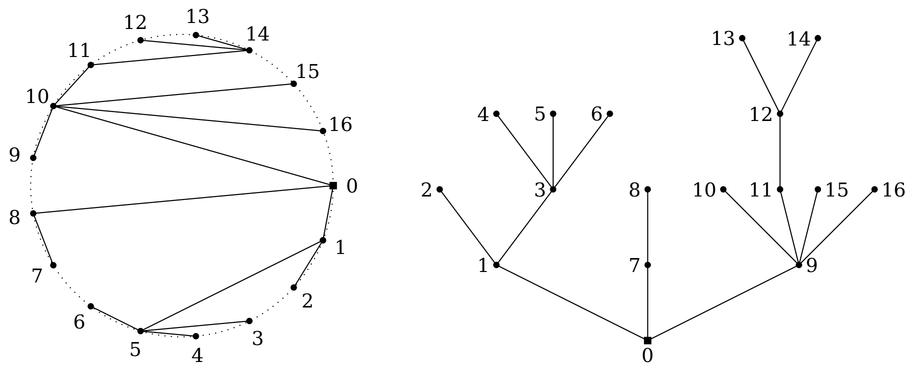

Figure 2: A non-crossing tree with its vertices labelled in clockwise-order and the associated plane tree, called its shape, with its vertices labelled in lexicographical order.

An interesting consequence of Theorem 1.1 is that the geometry of large simply generated noncrossing trees may be very different from that of large simply generated plane trees with the same weights, see Remark 5.5. Theorem 1.1 also has applications concerning the length of the longest chord of a noncrossing tree. By definition, the (angular) length of a chord $[\mathrm{e}^{-2i\pi s}, \mathrm{e}^{-2i\pi t}]$ with $0 \leq s \leq t \leq 1$ is $\min(t-s, 1-t+s)$. Denote by $\Lambda(\theta)$ the length of the longest chord of a noncrossing tree θ and by $\Lambda(\mathbf{L}_\alpha^U)$ the length of the longest chord of $\Lambda(\mathbf{L}_\alpha^U)$.

Corollary 1.2. *Under the assumptions of Theorem 1.1, we have*

$$\Lambda(\Theta_n) \quad \xrightarrow[n\to\infty]{(d)} \quad \Lambda(\mathbf{L}_\alpha^U).$$

This simply follows from Theorem 1.1 since the longest chord is a continuous functional for the Hausdorff distance on compact subsets of the unit disk obtained as the union of noncrossing chords. In the case $\alpha = 2$, it is known [1, 14] that the law of the longest chord of the Brownian triangulation is

$$\frac{1}{\pi} \frac{3x-1}{x^2(1-x)^2\sqrt{1-2x}} \mathbb{1}_{\frac{1}{3} \leq x \leq \frac{1}{2}} \, \mathrm{d}x. \tag{1.3}$$

It would be interesting to find an explicit formula for the length of the longest chord of the uniform α-stable triangulation for $\alpha \in (1,2)$. See [39, Proposition 4.3.] for the expression of the cumulative distribution function of the length of the longest chord in the α-stable lamination.

1.3 Degree-constrained noncrossing trees

For each integer $n \geq 1$ and each subset $\mathcal{A} \subset \mathbb{N}$, we denote by $\mathbb{NC}_n^{\mathcal{A}}$ the set of all noncrossing trees having n vertices and with degrees only belonging to $\mathcal{A}$.

Corollary 1.3. *For every $n \geq 1$ for which $\mathbb{NC}_n^{\mathcal{A}} \neq \varnothing$, sample a random noncrossing tree $\Theta_n^{\mathcal{A}}$ uniformly at random in $\mathbb{NC}_n^{\mathcal{A}}$. Then $\Theta_n^{\mathcal{A}}$ converges in distribution to the Brownian triangulation as $n \to \infty$.*

Indeed, this follows from Theorem 1.1 by taking $w(k) = \mathbb{1}_{k \in \mathcal{A}}$, as in this case μ admits finite small exponential moments (since $b < 1$, see the beginning of the proof of Theorem 1.4 below). Theorem 1.1 thus extends Theorem 3.1 in [9], which shows the convergence to the Brownian triangulation of large uniform noncrossing trees. Also, by Corollary 1.2, the length of the longest chord of $\Theta_n^{\mathcal{A}}$ converges in distribution to the random variable whose law is given by (1.3). It is remarkable that this limiting distribution does not depend on $\mathcal{A}$.

As an application of our techniques, we also establish the following enumerative result.

Theorem 1.4. *Assume that $\mathcal{A} \neq \{1,2\}$. Let $b > 0$ be such that $\sum_{k+1 \in \mathcal{A}} (k+1)(k-1)b^k = 0$ and define*

$$K_{\mathcal{A}} \quad := \quad \gcd(\mathcal{A}-1) \cdot \sqrt{\frac{\sum_{k+1 \in \mathcal{A}} (k+1)b^k}{2\pi \sum_{k+1 \in \mathcal{A}} (k+1)(k^2-1)b^k}} \cdot \left(\sum_{k \in \mathcal{A}} k b^k \right).$$

We have

$$\#\mathbb{NC}_n^{\mathcal{A}} \quad \underset{n \to \infty}{\sim} \quad K_{\mathcal{A}} \cdot \left(\sum_{k+1 \in \mathcal{A}} (k+1) b^{k-1} \right)^{n-1} \cdot n^{-3/2},$$

where the limit is taken along the subsequence of those values of n for which $\mathbb{NC}_n^{\mathcal{A}} \neq \varnothing$.

We give a simple proof of this by using the probabilistic structure of simply generated non-crossing trees. Observe that Theorem 1.4 is consistent with the fact that $\#\mathbb{NC}_n = \frac{1}{2n-1}\binom{3n-3}{n-1}$ since, for $\mathcal{A} = \mathbb{N}$, it reads $\#\mathbb{NC}_n \sim (9\sqrt{3\pi})^{-1} \cdot (27/4)^n \cdot n^{-3/2}$ as $n \to \infty$.

1.4 Iterating laminations

The random set $\mathbf{L}_\alpha^U$ is constructed from an α-stable lamination $\mathbf{L}_\alpha$ by triangulating independently each face of $\mathbf{L}_\alpha$. More generally, one can consider independent random β-laminations in each face of $\mathbf{L}_\alpha$ (see Fig. 3 for an illustration). We can also iterate this procedure: fix a sequence $(\alpha_k : k \geq 1)$ with values in $(1,2)$, let $\mathbf{L}^{(0)}$ be the unit circle and define next recursively for $n \geq 1$ random sets $\mathbf{L}^{(n)}$ by sampling independently an α_n-stable lamination in each face of $\mathbf{L}^{(n-1)}$. We give a formal definition of this procedure in Sec. 6, with several possible further directions of research concerning the study of $\mathbf{L}^{(n)}$.

The rest of this paper is organised as follows. In Section 2, we define discrete plane trees and their coding by a discrete paths, and we describe a bijection between noncrossing trees and plane trees. Next, in Section 3, we describe the continuous analogues which are the stable laminations of the disk and their triangulated versions, we also compute the Hausdorff dimension which appears in Theorem 1.1. In Section 4, we state and prove deterministic invariance principles for noncrossing trees and apply them to trees obtained by embedding in the disk a size-conditioned Bienaymé–Galton–Watson trees. In Section 5, we show that simply generated noncrossing trees are almost size-conditioned Bienaymé–Galton–Watson trees and prove Theorems 1.1 and 1.4. Finally, in Section 6 we give some extensions concerning the iteration of laminations.

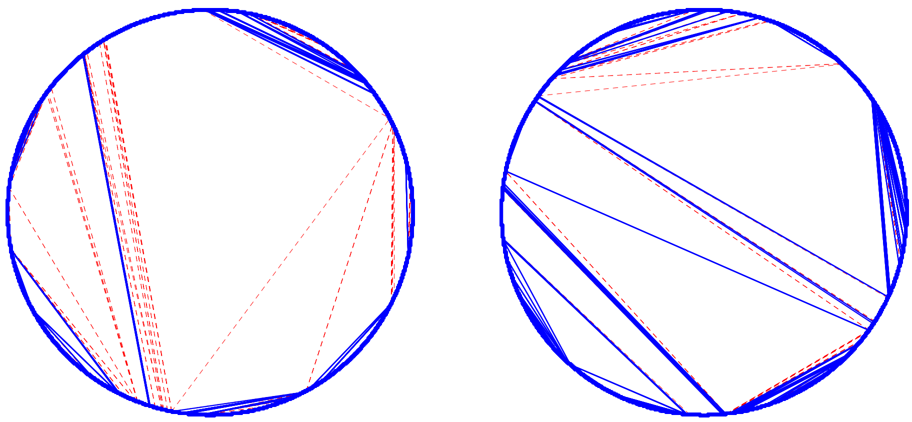

Figure 3: Simulations from left to right: $\beta = 1.4$ laminations iterated inside an $\alpha = 1.1$, and $\beta = 1.1$ laminations iterated inside an $\alpha = 1.4$. The chords of the β-stable laminations are in dashed red.

2 Coding plane trees and noncrossing trees

We start by explaining how we code plane trees and noncrossing trees. These codings are also useful to understand the intuition hiding behind the definitions of their continuous analogs.

2.1 Plane trees

Definitions. We use Neveu's formalism [35] to define plane trees: let $\mathbb{N} = \{1, 2, \dots\}$ be the set of all positive integers, set $\mathbb{N}^0 = \{\varnothing\}$ and consider the set of labels $\mathbb{U} = \bigcup_{n \geq 0} \mathbb{N}^n$. For $u = (u_1, \dots, u_n) \in \mathbb{U}$, we denote by $|u| = n$ the length of u; if $n \geq 1$, we define $pr(u) = (u_1, \dots, u_{n-1})$ and for $i \geq 1$, we let $ui = (u_1, \dots, u_n, i)$; more generally, for $v = (v_1, \dots, v_m) \in \mathbb{U}$, we let $uv = (u_1, \dots, u_n, v_1, \dots, v_m) \in \mathbb{U}$ be the concatenation of u and v. We endow $\mathbb{U}$ with the lexicographical order: given $v, w \in \mathbb{U}$, let $z \in \mathbb{U}$ be their longest common prefix, that is $v = z(v_1, \dots, v_n)$, $w = z(w_1, \dots, w_m)$ and $v_1 \neq w_1$, then $v \prec w$ if $v_1 < w_1$.

A plane tree is a nonempty finite subset $\tau \subset \mathbb{U}$ such that (i) $\varnothing \in \tau$; (ii) if $u \in \tau$ with $|u| \geq 1$, then $pr(u) \in \tau$; (iii) if $u \in \tau$, then there exists an integer $k_u(\tau) \geq 0$ such that $ui \in \tau$ if and only if $1 \leq i \leq k_u(\tau)$. For $u, v \in \tau$, we let $[\![u, v]\!]$ be the vertices belonging to the shortest path from u to v.

We will view τ as a genealogical tree of a population, each vertex u being seen as an individual. The vertex $\varnothing$ is called the root of the tree and for every $u \in \tau$, $k_u(\tau)$ is the number of children of u (if $k_u(\tau) = 0$, then u is called a leaf, otherwise, u is called an internal vertex), $|u|$ is its generation, $pr(u)$ is its parent and more generally, the vertices $u, pr(u), pr \circ pr(u), \dots, pr^{|u|}(u) = \varnothing$ belonging to $[\![\varnothing, u]\!]$ are its ancestors. To simplify, we will sometimes write k_u instead of $k_u(\tau)$. We denote by $\mathbb{T}$ the set of all plane trees and by $\mathbb{T}_n$ the set of plane trees with n vertices for each integer $n \geq 1$.

Bienaymé–Galton–Watson trees. Let μ be a critical probability measure on $\mathbb{Z}_+$, by which we mean that $\mu(0) > 0$, $\mu(0) + \mu(1) < 1$ (to avoid trivial cases) and with expectation $\sum_{k=0}^{\infty} k\mu(k) = 1$. The law of a Bienaymé–Galton–Watson tree with offspring distribution μ is the unique probability measure BGW^μ on $\mathbb{T}$ such that for every $\tau \in \mathbb{T}$,

$$\mathrm{BGW}^\mu(\tau) = \prod_{u \in \tau} \mu(k_u).$$

For each integer $n \geq 1$, we denote by BGW_n^μ the law of a Bienaymé–Galton–Watson tree with offspring distribution μ conditioned to have n vertices; we shall always implicitly restrict ourselves to the values of n for which this conditioning makes sense.

Coding by the Łukasiewicz path. Fix a tree $\tau \in \mathbb{T}_n$ and let $\varnothing = u(0) \prec u(1) \prec \cdots \prec u(n-1)$ be its vertices, listed in lexicographical order. The Łukasiewicz path $\mathcal{W}(\tau) = (\mathcal{W}_j(\tau) : 0 \leq j \leq n)$ of τ is defined by $\mathcal{W}_0(\tau) = 0$ and for every $0 \leq j \leq n-1$,

$$\mathcal{W}_{j+1}(\tau) = \mathcal{W}_j(\tau) + k_{u(j)}(\tau) - 1.$$

One easily checks (see e.g. [28]) that $\mathcal{W}_j(\tau) \geq 0$ for every $0 \leq j \leq n-1$ but $\mathcal{W}_n(\tau) = -1$. Observe that $\mathcal{W}_{j+1}(\tau) - \mathcal{W}_j(\tau) \geq -1$ for every $0 \leq j \leq n-1$, with equality if and only if $u(j)$ is a leaf of τ. We shall think of such a path as the step function on $[0, n]$ given by $s \mapsto \mathcal{W}_{\lfloor s \rfloor}(\tau)$.

Scaling limits. Fix $\alpha \in (1, 2]$ and consider a strictly stable spectrally positive Lévy process of index α: X_α is a random process with paths in the set $\mathbb{D}([0,\infty), \mathbb{R})$ of càdlàg functions endowed with the Skorokhod J_1 topology (see e.g. Billingsley [4] for details) which has independent and stationary increments, no negative jump and such that $\mathbb{E}\left[\exp(-\lambda X_\alpha(t))\right] = \exp(t\lambda^\alpha)$ for every $t, \lambda > 0$. Using excursion theory, it is then possible to define X_α^{ex}, the normalised excursion of X_α, which is a random variable with values in $\mathbb{D}([0,1], \mathbb{R})$, such that $X_\alpha^{\mathrm{ex}}(0) = X_\alpha^{\mathrm{ex}}(1) = 0$ and, almost surely, $X_\alpha^{\mathrm{ex}}(t) > 0$ for every $t \in (0,1)$. We do not enter into details and refer to Bertoin [3] for background.

An important point is that X_α^{ex} is continuous for $\alpha = 2$, and indeed $X_2^{\mathrm{ex}}/\sqrt{2}$ is the standard Brownian excursion, whereas the set of discontinuities of X_α^{ex} is dense in $[0,1]$ for every $\alpha \in (1,2)$. We shall therefore treat the two cases separately.

Duquesne [16] (see also [25]) provides the following limit theorem which is the steppingstone of our convergence results. Fix $\alpha \in (1,2]$ and μ a critical probability measure on $\mathbb{Z}_+$ in the domain of attraction of a stable law of index α. For every $n \geq 1$ for which BGW_n^μ is well defined, sample $\mathcal{T}_n$ according to BGW_n^μ. Then there exists a sequence $(B_n)_{n \geq 1}$ of positive numbers satisfying $\lim_{n\to\infty} B_n = \infty$, such that the convergence in distribution

$$\left(\frac{1}{B_n}\mathcal{W}_{\lfloor ns \rfloor}(\mathcal{T}_n) : s \in [0,1]\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (X_\alpha^{\mathrm{ex}}(s) : s \in [0,1]) \tag{2.1}$$

holds in the space $\mathbb{D}([0,1], \mathbb{R})$. The sequence $(B_n)_{n\geq 1}$ is regularly varying with index $1/\alpha$, meaning that if $(u_n)_{n\geq 1}$ and $(v_n)_{n\geq 1}$ are two sequences of integers tending to ∞ and such that $u_n/v_n \to s > 0$, then $B_{u_n}/B_{v_n} \to s^{1/\alpha}$ as $n \to \infty$, and may be chosen to be increasing (see e.g. [24, Theorem 1.10], which also gives the dependence of B_n in terms of μ). When μ has finite positive variance σ^2, one can take $B_n = \sigma\sqrt{n/2}$.

2.2 Noncrossing trees

Let $\tau \in \mathbb{T}_n$ be a plane tree with vertices $\varnothing = u(0) \prec u(1) \prec \cdots \prec u(n-1)$ listed in lexicographical order. We set

$$\mathbb{C}(\tau) = \{(l_1, l_2, \ldots, l_{n-1}) : 0 \leq l_j \leq k_{u(j)}(\tau) \text{ for every } 1 \leq j \leq n-1\}$$

and

$$\mathbb{T}_n^{\mathsf{dec}} = \{(\tau, \mathbf{c}) : \tau \in \mathbb{T}_n \text{ and } \mathbf{c} \in \mathbb{C}(\tau)\}.$$

Elements of $\mathbb{T}_n^{\mathsf{dec}}$ are called decorated trees, and we can view l_j as the label carried by the vertex $u(j)$. Note that $\#\mathbb{C}(\tau) = \prod_{u\in\tau\setminus\{\varnothing\}}(k_u(\tau)+1)$ for every $\tau \in \mathbb{T}$.

If θ is a noncrossing tree, we let $S(\theta)$ be its shape, which is the plane tree associated with θ and rooted at the vertex corresponding to the complex number 1 (see Fig. 2 for an example). If θ is a noncrossing tree with n vertices and $\varnothing = u(0) \prec u(1) \prec \cdots \prec u(n-1)$ are the vertices of its shape listed in lexicographical order, for every $1 \leq i \leq n-1$, we let $L_i(\theta)$ be the number of children of $u(i)$ lying "on its left" (i.e. which are between the complex number 1 and $u(i)$ when we equip the circle with the clockwise order), and set

$$C(\theta) = (L_1(\theta), L_2(\theta), \ldots, L_{n-1}(\theta)) \quad \in \quad \mathbb{C}(S(\theta)).$$

The following result is a reformulation of the "left-right" coding of noncrossing trees in [37].

Proposition 2.1. *For every $n \geq 1$, the mapping*

$$\begin{array}{cccc} \Phi_n: & \mathbb{NC}_n & \longrightarrow & \mathbb{T}_n^{\mathsf{dec}} \\ & \theta & \longmapsto & (S(\theta), C(\theta)) \end{array}$$

is a bijection.

Proof. We describe the reverse map Φ_n^{-1}; this will also be useful later. Fix a decorated tree $(\tau, (l_1, l_2, \ldots, l_{n-1})) \in \mathbb{T}_n^{\mathsf{dec}}$. Let $\varnothing = u(0) \prec u(1) \prec \cdots \prec u(n-1)$ be the vertices of τ labelled in lexicographical order. To simplify notation, for every $u \in \tau$ with $u \neq \varnothing$, we set $n(u) = k$ if u is the k-th child of its parent and we let $l(u)$ be the label carried by u, that is $l(u) = l_j$ if $u = u(j)$. Then, for every $u \in \tau$, set

$$L(u) = \#\left\{v \in]\!]\varnothing, u]\!] : |v| \geq 2 \text{ and } n(v) \leq l(pr(v))\right\}, \qquad R(u) = |u| - L(u) - 1,$$

where we recall that $pr(v)$ is the parent of v. Intuitively speaking, $L(u)$ and $R(u)$ represent the number of vertices of $]\!]\varnothing, u]\!]$ that will be respectively folded to the left and to the right of u in the associated noncrossing tree $\Phi_n^*((\tau, (l_1, l_2, \ldots, l_{n-1})))$ which is defined as follows.

First map $\varnothing$ to the complex number 1. Then, for every $1 \leq p \leq n-1$, let k_p be the number of children of $u(p)$. If $k_p = 0$, map $u(p)$ to $\mathrm{e}^{-2\mathrm{i}\pi\cdot(p-R(u(p)))/n}$. Otherwise, for $1 \leq i \leq k_p$, let T_i be the size of the subtree grafted on the i-th child of $u(p)$ (so that T_i is the number of its non strict descendants) with the convention $T_0 = 0$. Then map $u(p)$ to $\mathrm{e}^{-2\mathrm{i}\pi\cdot(p-R(p)+T_1+T_2+\cdots+T_{l_p})/n}$. It is then a simple matter to check that $\Phi_n \circ \Phi_n^*$ and $\Phi_n^* \circ \Phi_n$ are the identity, which completes the proof. $\square$

In Section 4, we give sufficient conditions on a sequence $(\tau_n^{\mathrm{dec}})_{n\geq 1}$ of decorated trees which ensure that the associated noncrossing trees $(\Phi_n^{-1}(\tau_n^{\mathrm{dec}}))_{n\geq 1}$ converge to triangulated laminations, which form a family of compact subsets of the unit disk which we now define.

3 Triangulations, laminations and triangulated laminations

We denote by $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ the closed unit disk. A geodesic lamination of $\overline{\mathbb{D}}$ is a closed subset of $\overline{\mathbb{D}}$ which can be written as the union of a collection of noncrossing chords. In the sequel, by lamination we will always mean geodesic lamination of $\overline{\mathbb{D}}$. A lamination is said to be maximal when it is maximal for the inclusion relation among laminations. We call faces of a lamination the connected components of its complement in $\overline{\mathbb{D}}$; note that the faces of a maximal lamination are open triangles whose vertices belong to $\mathbb{S}^1$, a maximal lamination is also called a triangulation.

3.1 Triangulations coded by continuous functions

Let $f : [0,1] \to \mathbb{R}_+$ be a continuous function with $f(0) = f(1) = 0$ and such that the following assumption (H_f) holds:

$$\text{The local minima of } f \text{ are distinct.} \tag{H_f}$$

This means that if $0 \leq a < b < c < d \leq 1$ are such that the infimum of f over $]a,b[$ is achieved at a point of $]a,b[$, and that over $]c,d[$ is achieved at a point of $]c,d[$ as well, then $\min_{]a,b[} f \neq \min_{]c,d[} f$. We define an equivalence relation on $[0,1]$ by setting $s \sim^f t$ whenever $f(s) = f(t) = \min_{[s\wedge t, s\vee t]} f$. We then define a subset of $\overline{\mathbb{D}}$ by

$$\mathbf{L}(f) \quad := \quad \bigcup_{s \sim^f t} \left[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}\right].$$

Using the fact that f is continuous and its local minima are distinct, one can prove (see e.g. [29, Proposition 2.1]) that $\mathbf{L}(f)$ is a maximal lamination; we say that $\mathbf{L}(f)$ is the *triangulation coded by* f.

Now let $\mathbb{e} = X_2^{\mathrm{ex}}$ be $\sqrt{2}$ times the standard Brownian excursion. Since $\mathbb{e}$ has almost surely distinct local minima, the lamination $\mathbf{L}(\mathbb{e})$ is maximal, it is called the Brownian triangulation and we also denote it by $\mathbf{L}_2$. This set has been introduced first by Aldous [1].

3.2 Laminations coded by càdlàg functions

Recall that $\mathbb{D}([0,1],\mathbb{R})$ denotes the space of real-valued càdlàg functions on $[0,1]$ equipped with the Skorokhod J_1 topology. If $X \in \mathbb{D}([0,1],\mathbb{R})$ and $t \in [0,1]$, we set $\Delta X(t) = X(t) - X(t-)$ with the convention $X(0-) = X(0)$. We fix a function $Z \in \mathbb{D}([0,1],\mathbb{R})$ satisfying the following properties:

(H0) $Z(0) = Z(1) = 0$ and for every $t \in (0,1)$, $Z(t) > 0$ and $\Delta Z(t) \geq 0$.

(H1) For every $0 \leq s < t \leq 1$, there exists at most one value $r \in (s,t)$ such that $Z(r) = \inf_{[s,t]} Z$.

(H2) For every $t \in (0,1)$ such that $\Delta Z(t) > 0$, we have $\inf_{[t,t+\varepsilon]} Z < Z(t)$ for every $0 < \varepsilon \leq 1-t$;

(H3) For every $t \in (0,1)$ such that $\Delta Z(t) > 0$, we have $\inf_{[t-\varepsilon,t]} Z < Z(t-)$ for every $0 < \varepsilon \leq t$;

(H4) For every $t \in (0,1)$ such that Z attains a local minimum at t (which implies $\Delta Z(t) = 0$), if $s = \sup\{u \in [0,t] : Z(u) < Z(t)\}$, then $\Delta Z(s) > 0$ and $Z(s-) < Z(t) < Z(s)$.

We recall the construction in [26] of a lamination $L(Z)$ from Z. To this end, we define a relation (not equivalence relation in general) on $[0,1]$ as follows: for every $0 \leq s < t \leq 1$, set

$$s \simeq^Z t \qquad \text{if} \qquad t = \inf\{u > s : Z(u) \leq Z(s-)\};$$

then for $0 \leq t < s \leq 1$, we set $s \simeq^Z t$ if $t \simeq^Z s$, and we agree that $s \simeq^Z s$ for every $s \in [0,1]$. We finally define a subset of $\overline{\mathbb{D}}$ by

$$L(Z) \quad := \quad \bigcup_{s \simeq^Z t} \left[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}\right]. \tag{3.1}$$

Using the above properties, it is proved in [26, Proposition 2.9] that $L(Z)$ is a lamination, called the *lamination coded by* Z.

Recall that X_α^{ex} denotes the normalised excursion of a spectrally positive strictly stable Lévy process for $\alpha \in (1,2]$. For every $\alpha \in (1,2)$, X_α^{ex} fulfils the above properties with probability one ([26, Proposition 2.10]), we can therefore set

$$\mathbf{L}_\alpha \quad := \quad L(X_\alpha^{\mathrm{ex}}),$$

which is called the stable lamination of index α.

We recall from [26, Proposition 3.10] the description of the faces of $L(Z)$ (this reference actually only covers the case where $Z = X_\alpha^{\mathrm{ex}}$, but the arguments carry out in this setting as well), which are the connected components of the complement of $L(Z)$ in $\overline{\mathbb{D}}$. The faces of $L(Z)$ are in one-to-one correspondence with the jump times of Z (observe that the latter set is countable since Z is càdlàg). For every $s, t \in (0,1)$, let $\mathbb{H}(s,t)$ be the open half-plane bounded by the line containing $\mathrm{e}^{-2\mathrm{i}\pi s}$ and $\mathrm{e}^{-2\mathrm{i}\pi t}$, which does not contain the complex number 1. Then for every jump time s of Z, letting $t = \inf\{u > s : Z(u) = Z(s-)\}$, the face V_s of $L(Z)$ associated with s is the unique one contained in $\mathbb{H}(s,t)$ whose boundary contains the chord $[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}]$. Moreover, the "boundary" of the face V_s which belongs to $\mathbb{S}^1$ is given by

$$B_s \quad := \quad \overline{V_s} \cap \mathbb{S}^1 = \left\{ r \in [s,t] : Z(r) = \inf_{[s,r]} Z \right\}, \tag{3.2}$$

where we identify the interval $[0,1)$ with the circle $\mathbb{S}^1$ via the mapping $t \mapsto \mathrm{e}^{-2\mathrm{i}\pi t}$ to ease notation.

3.3 Triangulated laminations

We next define triangulations which are, informally, obtained from $L(Z)$ by "triangulating" all its faces, i.e. for each face of $L(Z)$ we choose a special vertex on its boundary on $\mathbb{S}^1$ and join it to all the other vertices of this face by chords.

Fix $Z \in \mathbb{D}([0,1],\mathbb{R})$ satisfying (H0), ..., (H4). Let $J(Z) = \{u \in [0,1] : \Delta Z(u) > 0\}$ be the set of all jump times of Z, and let $\boldsymbol{\ell} = (\ell_u; u \in J(Z))$ be a sequence of nonnegative real numbers indexed by these jump times such that $0 \le \ell_u \le 1$ for every $u \in J(Z)$. By convention, we shall always assume that $\ell_u = 0$ if $u \notin J(Z)$. The sequence $\boldsymbol{\ell}$ will be called a *jumps labelling*. For every $u \in J(Z)$, set

$$p_u(\boldsymbol{\ell}) = \inf\{r \ge u : Z_r = Z_u - \Delta Z(u) \cdot \ell_u\} \tag{3.3}$$

and

$$C_u(\boldsymbol{\ell}) = \bigcup_{r \in B_u} \left[\mathrm{e}^{-2\mathrm{i}\pi p_u(\boldsymbol{\ell})}, \mathrm{e}^{-2\mathrm{i}\pi r} \right],$$

where we recall that B_u is defined by (3.2). Note that $p_u(\boldsymbol{\ell}) \in B_u$ for every $u \in J(Z)$. Finally define

$$L(Z, \boldsymbol{\ell}) \quad := \quad L(Z) \cup \bigcup_{s \in J(Z)} C_s(\boldsymbol{\ell}). \tag{3.4}$$

Intuitively speaking, $L(Z, \boldsymbol{\ell})$ is obtained from $L(Z)$ by triangulating each face as follows: inside every face V_s of $L(Z)$ indexed by a jump time s, choose a special vertex on its boundary B_s indexed by $p_s(\boldsymbol{\ell})$, and draw chords from this special vertex to all the other points of B_s. The point is that the latter set is uncountable, so some care is needed to define the special vertex, hence the purpose of the jumps labelling $\boldsymbol{\ell}$. Roughly speaking, $x \in [0,1] \mapsto \inf\{u \ge s : Z_u = Z_s - \Delta Z(s) \cdot x\} \in B_s$ plays the role of the inverse of the local time of vertices of B_s (that is a measurement of the evolution of "number" of vertices of B_s as one goes around $\mathbb{S}^1$) and allows to identify $[0,1]$ with B_s.

Proposition 3.1. *Under the assumptions (H0), ..., (H4), for every jumps labelling $\boldsymbol{\ell}$, the set $L(Z, \boldsymbol{\ell})$ is a triangulation of $\overline{\mathbb{D}}$.*

Proof. First note that the chords defining $L(Z, \boldsymbol{\ell})$ in (3.4) are noncrossing: there exists no 4-tuple $0 \le s < s' < t < t' \le 1$ such that both chords $[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}]$ and $[\mathrm{e}^{-2\mathrm{i}\pi s'}, \mathrm{e}^{-2\mathrm{i}\pi t'}]$ belong to $L(Z, \boldsymbol{\ell})$. Indeed, suppose there exists such a 4-tuple. Clearly, we cannot have $[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}] \subset C_u(\boldsymbol{\ell})$ and $[\mathrm{e}^{-2\mathrm{i}\pi s'}, \mathrm{e}^{-2\mathrm{i}\pi t'}] \subset C_u(\boldsymbol{\ell})$ for any $u \in J(Z)$ and neither do we have $[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}] \subset L(Z)$ and $[\mathrm{e}^{-2\mathrm{i}\pi s'}, \mathrm{e}^{-2\mathrm{i}\pi t'}] \subset L(Z)$ since $L(Z)$ is a lamination.

Assume next that $[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}] \subset C_u(\boldsymbol{\ell})$ for a certain $u \in J(Z)$ and $[\mathrm{e}^{-2\mathrm{i}\pi s'}, \mathrm{e}^{-2\mathrm{i}\pi t'}] \subset L(Z)$; then $s, t \in B_u$ so $u \le s < s' < t < t'$ and $Z(t) = \inf_{[u,t]} Z$. It follows that $Z(t) \le Z(s'-)$ which contradicts $t' = \inf\{r > s' : Z(r) \le Z(s'-)\}$. The case $[\mathrm{e}^{-2\mathrm{i}\pi s'}, \mathrm{e}^{-2\mathrm{i}\pi t'}] \subset C_u(\boldsymbol{\ell})$ for a certain $u \in J(Z)$ and $[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}] \subset L(Z)$ yields a similar contradiction.

The last case to consider is $[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}] \subset C_u(\boldsymbol{\ell})$ for a certain $u \in J(Z)$ and $[\mathrm{e}^{-2\mathrm{i}\pi s'}, \mathrm{e}^{-2\mathrm{i}\pi t'}] \subset C_{u'}(\boldsymbol{\ell})$ for a certain $u' \in J(Z)$ with $u' \neq u$. Let $v = \inf\{r > u : Z(r) = Z(u-)\}$ and $v' = \inf\{r > u' : Z(r) = Z(u'-)\}$; then $u \le s < t \le v$ and $u' \le s' < t' \le v'$. If $u' < u$, then $u' < u \le s < s' < t$; with the same reasoning as above, we conclude that $\Delta Z(u) = \Delta Z(s') = 0$ and $Z(u) = Z(s') = Z(t) = \inf_{[u',t]} Z$ which contradicts (H1). Similarly, if $u' > u$, then $u < u' \le s' < t < t' \le v' < v$ and we conclude that $\Delta Z(u') = \Delta Z(t) = 0$ and $Z(u') = Z(t) = Z(t') = \inf_{[u,t']} Z$.

Next, we need to show that $L(Z, \boldsymbol{\ell})$ is closed. Consider a sequence of points of the plane $(x_n)_{n \ge 1}$ on $L(Z, \boldsymbol{\ell})$ which converges as $n \to \infty$ to $x \in \overline{\mathbb{D}}$. Let us show that $x \in L(Z, \boldsymbol{\ell})$. If $x \in \overline{\mathbb{D}} \setminus L(Z)$, then there exists a face V of the latter such that $x \in V$ and, moreover, $x_n \in V$ for every n large enough. Note that if u is the jump time of Z associated with V, then $V \cap L(Z, \boldsymbol{\ell})$ is the union of the open chords $\bigcup_{t \in B_u}]\mathrm{e}^{-2\mathrm{i}\pi p_u(\boldsymbol{\ell})}, \mathrm{e}^{-2\mathrm{i}\pi t}[$. Thus, for every n large enough, x_n belongs to a chord $[\mathrm{e}^{-2\mathrm{i}\pi p_u(\boldsymbol{\ell})}, \mathrm{e}^{-2\mathrm{i}\pi t_n}]$, where $t_n \in B_u$. Since B_u is compact, t_n converges along a subsequence to a certain $t \in B_u$ and we conclude that $x \in [\mathrm{e}^{-2\mathrm{i}\pi p_u(\boldsymbol{\ell})}, \mathrm{e}^{-2\mathrm{i}\pi t}]$.

Finally, we show that $L(Z, \boldsymbol{\ell})$ is a maximal lamination. We argue by contradiction that for every $a, b \in \mathbb{S}^1$ with $a \neq b$, the open chord $]a, b[= [a, b] \setminus \{a, b\}$ must intersect $L(Z, \boldsymbol{\ell})$, otherwise $L(Z, \boldsymbol{\ell}) \cup [a, b]$ would be a bigger lamination. Fix $0 \le s < t \le 1$ and suppose that $]\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}[\cap L(Z, \boldsymbol{\ell}) = \varnothing$. Then $]\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}[$ belongs to a face V_u for a certain $u \in J(Z)$. As a consequence $s, t \in B_u$, so that, setting $v = \inf\{r > u : Z(r) = Z(u-)\}$, we have $s, t \in [u, v]$, $Z(s) = \inf_{[u,s]} Z$ and $Z(t) = \inf_{[u,t]} Z$. We claim that $Z(s) \neq Z(t)$ and so $Z(s) > Z(t)$. Indeed suppose $Z(s) = Z(t)$ and observe that Z is continuous at s by (H3); either $Z(r) > Z(s)$ for every $r \in (s, t)$ and then $[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}] \subset L(Z)$, or there exists $r \in (s, t)$ such that $Z(s) = Z(r) = Z(t)$, which contradicts (H1). Let $x = \inf\{r > u : Z(r) \le (Z(s) + Z(t))/2\}$, then $x \in (s, t) \cap B_u$. Finally, note that $(s, t) \neq (u, v)$ so, similarly, there exists $y \in B_u \cap ((u, s) \cup (t, v))$. Since $p_u(\boldsymbol{\ell}) \in B_u \setminus \{s, t\}$, we conclude that one of the open chords $]\mathrm{e}^{-2\mathrm{i}\pi p_u(\boldsymbol{\ell})}, \mathrm{e}^{-2\mathrm{i}\pi x}[$ or $]\mathrm{e}^{-2\mathrm{i}\pi p_u(\boldsymbol{\ell})}, \mathrm{e}^{-2\mathrm{i}\pi y}[$ belonging to $L(Z, \boldsymbol{\ell})$ intersects $]\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}[$. $\square$

As a consequence, note that $C_u(\boldsymbol{\ell})$ is compact for every $u \in L(Z)$.

Remark 3.2. *For $\alpha \in (1, 2)$, the triangulation $\widehat{\mathbf{L}}_\alpha$ introduced in [31] is a particular case of a triangulated lamination. More precisely, we have $\widehat{\mathbf{L}}_\alpha = L(X_\alpha^{\mathrm{ex}}, \boldsymbol{\ell})$ with $\ell_s = 0$ for every s. In other words, $\widehat{\mathbf{L}}_\alpha$ is obtained from the stable lamination $\mathbf{L}_\alpha$ by drawing chords from the "leftmost" vertex of a face to all the other vertices of this face.*

An interesting example of a triangulated lamination is the so-called uniform α-stable triangulation, which is defined as follows. For $\alpha \in (1, 2)$, conditionally given X_α^{ex}, let $\boldsymbol{\ell}^U = (\ell_s)_{s \in J(X_\alpha^{\mathrm{ex}})}$ be a sequence of i.i.d. uniform random variables on $[0, 1]$. The uniform stable triangulation $\mathbf{L}_\alpha^U$ is then defined to be

$$\mathbf{L}_\alpha^U \quad := \quad L(X_\alpha^{\mathrm{ex}}, \boldsymbol{\ell}^U).$$

We will see that this set is the distributional limit of certain simply generated noncrossing trees as well as large critical Bienaymé–Galton–Watson trees in the domain of attraction of a stable law of index α which are uniformly embedded in a noncrossing way.

Remark 3.3. *If $f : [0,1] \to \mathbb{R}_+$ is a continuous function such that $f(0) = f(1) = 0$ but which does not fulfill (H_f), one can still adapt the construction of $\mathbf{L}(f)$ in Section 3.1 to define a (non-maximal) lamination from f, see Curien & Le Gall [11, Proposition 2.5]. As shown in [26], the stable laminations $\mathbf{L}_\alpha$ can be coded in this sense by H_α^{ex}, the normalised excursion of the so-called height process associated with X_α^{ex}. In the same way, the sets $L(X_\alpha^{\mathrm{ex}}, \boldsymbol{\ell})$ could also be defined from H_α^{ex} (although in a different sense than that of Curien & Le Gall since it would involve $\boldsymbol{\ell}$). Nonetheless, H_α^{ex} is a more complicated object than X_α^{ex}, the definition of p_u and the invariance principles of Section 4 would be more technical and may even require more assumptions (see Remark 4.4 below).*

Conversely, if L is a maximal lamination, by adapting the argument of [29, Proposition 2.2] and using [17, Corollary 1.2], we believe that there exists a continuous function $f : [0,1] \to \mathbb{R}_+$ with $f(0) = f(1) = 0$ satisfying (H_f) such that $L = \mathbf{L}(f)$. However, if L is the lamination

$$L = [1, \mathrm{e}^{-\mathrm{i}\pi/2}] \cup [\mathrm{e}^{-\mathrm{i}\pi/2}, -1] \cup [-1, \mathrm{e}^{\mathrm{i}\pi/2}] \cup [\mathrm{e}^{\mathrm{i}\pi/2}, 1] \cup [-1, 1],$$

there does not exist a continuous function $f : [0,1] \to \mathbb{R}_+$ with $f(0) = f(1) = 0$ such that $L = \mathbf{L}(f)$ in the sense of Curien & Le Gall, and there does not exist a càdlàg function $Z \in \mathbb{D}([0,1], \mathbb{R})$ satisfying (H0), ..., (H4) such that $L = L(Z)$ either. In the same way, $L(X_\alpha^{\mathrm{ex}}, \boldsymbol{\ell})$ cannot be coded by a continuous nor a càdlàg function in this manner for $\alpha \in (1,2)$.

3.4 The Hausdorff dimension of triangulated stable laminations

If L is a lamination, we denote by $A(L) \subset \mathbb{S}^1$ the set of all end-points of its chords. We denote by $\dim(K)$ the Hausdorff dimension of a subset K of $\mathbb{C}$, and refer to Mattila [32] for background. Recall that X_α^{ex} is the normalised excursion of the α-stable Lévy process.

Theorem 3.4. *For every $\alpha \in (1,2)$ and for every jumps labelling $\boldsymbol{\ell}$, almost surely,*

$$\dim(A(L(X_\alpha^{\mathrm{ex}}, \boldsymbol{\ell}))) = \frac{1}{\alpha} \qquad \textit{and} \qquad \dim(L(X_\alpha^{\mathrm{ex}}, \boldsymbol{\ell})) = 1 + \frac{1}{\alpha}. \tag{3.5}$$

These results should be compared with [26, Theorem 5.1], where these dimensions are calculated for stable laminations:

$$\dim(A(L(X_\alpha^{\mathrm{ex}}))) = 1 - \frac{1}{\alpha} \qquad \text{and} \qquad \dim(L(X_\alpha^{\mathrm{ex}})) = 2 - \frac{1}{\alpha}. \tag{3.6}$$

We mention that (3.6) also holds for $\alpha = 2$ by results of Aldous [1] and Le Gall & Paulin [29] when $L(X_2^{\mathrm{ex}})$ is taken to be the Brownian triangulation. Also, Theorem 3.4 is established in [31] in the particular case where $\ell_s = 0$ for every s. The general case only requires mild modifications, but we give a full proof for completeness.

Remark 3.5. *We see that the dimensions of the sets in* (3.5) *and* (3.6) *have the same limit as $\alpha \uparrow 2$. Indeed, the stable lamination and actually any triangulated stable lamination converges to the Brownian triangulation in this limit. On the other hand, we also see that*

$$(\dim(L(X_\alpha^{\mathrm{ex}})), \dim(L(X_\alpha^{\mathrm{ex}}, \boldsymbol{\ell}))) \underset{\alpha \downarrow 1}{\longrightarrow} (1,2).$$

Let us give an intuitive explanation of this fact. Informally, as $\alpha \downarrow 1$, the process X_α^{ex} converges towards the deterministic function $f : [0,1] \to \mathbb{R}$ defined by $f(0) = 0$ and $f(x) = 1 - x$ for every $x \in (0,1]$ (f is not càdlàg, but we refer to [10, Theorem 3.6] for a precise statement and proof). If we try then to define $L(f)$ and $L(f, \boldsymbol{\ell})$ mimicking (3.1) *and* (3.4)*, we obtain $L(f) = \mathbb{S}^1$ and $L(f, \boldsymbol{\ell}) = \overline{\mathbb{D}}$.*

Proof of Theorem 3.4. Fix a face V of $L(X_\alpha^{\mathrm{ex}}, \boldsymbol{\ell})$ and let s be the jump-time of X_α^{ex} associated with V. Notice from (3.4) that all the chords of $L(X_\alpha^{\mathrm{ex}}, \boldsymbol{\ell})$ which lie in $\overline{V}$ either belong to the boundary ∂V or are of the form $[\mathrm{e}^{-2\mathrm{i}\pi p_s(\boldsymbol{\ell})}, \mathrm{e}^{-2\mathrm{i}\pi r}]$ for $r \in \overline{V} \cap \mathbb{S}^1$. To simplify notation, denote by L_V the lamination $L(X_\alpha^{\mathrm{ex}}, \boldsymbol{\ell}) \cap \overline{V}$ and by A_V the set of all its end-points, so that

$$A_V = \overline{V} \cap \mathbb{S}^1.$$

and $\dim(A_V) = 1/\alpha$ by [26, Theorem 5.1]. As a consequence, since $A(L(X_\alpha^{\mathrm{ex}}, \boldsymbol{\ell})) = \bigcup_V A_V$, where the union runs over the countable set of faces of $L(X_\alpha^{\mathrm{ex}})$, we have

$$\dim(A(L(X_\alpha^{\mathrm{ex}}, \boldsymbol{\ell}))) = \sup_{V \text{ face of } L(X_\alpha^{\mathrm{ex}})} \dim(A_V) = \dim(\overline{V} \cap \mathbb{S}^1) = \frac{1}{\alpha}.$$

Similarly, we have

$$\dim(L(X_\alpha^{\mathrm{ex}}, \boldsymbol{\ell})) = \sup_{V \text{ face of } L(X_\alpha^{\mathrm{ex}})} \dim(L_V)$$

so it only remains to show that for any given face V of $L(X_\alpha^{\mathrm{ex}})$, we have

$$\dim(L_V) = 1 + \dim(A_V) = 1 + \frac{1}{\alpha}. \tag{3.7}$$

If s is the jump time associated with V, it is actually sufficient to establish (3.7) with L_V replaced by the compact set $C_s(\boldsymbol{\ell})$, which is the union of the chords $[\mathrm{e}^{-2\mathrm{i}\pi p_s(\boldsymbol{\ell})}, z]$ for $z \in A_V$. Indeed as we remarked previously, $L_V \setminus C_s(\boldsymbol{\ell}) \subset L(X_\alpha^{\mathrm{ex}})$ which, by (3.6), has Hausdorff dimension $2 - \frac{1}{\alpha} < 1 + \frac{1}{\alpha}$ for every $\alpha \in (1,2)$. We adapt the argument of Le Gall & Paulin [29, Proposition 2.3] to show that $\dim(C_s(\boldsymbol{\ell})) = 1 + \dim(A_V)$.

We first show that $\dim(C_s(\boldsymbol{\ell})) \geq 1 + \dim(A_V)$. Fix $0 < \gamma < \dim(A_V)$; thanks to Frostman's lemma [32, Theorem 8.8], there exists a non-trivial finite Borel measure ν supported on A_V such that $\nu(B(x,r)) \leq r^\gamma$ for every $x \in \mathbb{C}$ and every $r > 0$, where $B(x,r)$ is the Euclidean ball centred at x and of radius r. Next, for every $x \in A_V$, denote by λ_x the one-dimensional Lebesgue measure on the chord joining x to $\mathrm{e}^{-2\mathrm{i}\pi p_s(\boldsymbol{\ell})}$. We define a finite Borel measure Λ on $\mathbb{C}$, supported on $C_s(\boldsymbol{\ell})$, by setting for every Borel set B

$$\Lambda(B) = \int \nu(\mathrm{d}x)\lambda_x(B).$$

Fix $0 < R < 1$ such that $\Lambda(B(0,R)) > 0$; let $z_0 \in B(0,R) \cap C_s(\boldsymbol{\ell})$ and then $x_0 \in A_V$ such that the chord $[x_0, \mathrm{e}^{-2\mathrm{i}\pi p_s(\boldsymbol{\ell})}]$ contains z_0. Fix $\varepsilon \in (0,1]$; every $x \in A_V$ such that the chord $[x, \mathrm{e}^{-2\mathrm{i}\pi p_s(\boldsymbol{\ell})}]$ intersects the ball $B(z_0, \varepsilon)$ must satisfy $|x - x_0| \leq C\varepsilon$, where the constant C only depends on R. We conclude that

$$\Lambda(B(z_0, \varepsilon)) = \int_{|x - x_0| \leq C\varepsilon} \nu(\mathrm{d}x)\lambda_x(B(z_0, \varepsilon)) \leq C'\varepsilon^{1+\gamma},$$

where the constant C' does not depend on ε nor z_0. Appealing again to Frostman's lemma, we obtain $\dim(C_s(\boldsymbol{\ell})) \geq 1 + \gamma$, whence, as $\gamma < \dim(A_V)$ is arbitrary, $\dim(C_s(\boldsymbol{\ell})) \geq 1 + \dim(A_V)$.

It remains to show the converse inequality. We denote respectively by $\underline{\dim}_M(K)$ and $\overline{\dim}_M(K)$ the lower and upper Minkowski dimensions of a subset K of $\mathbb{C}$ (see e.g. Mattila

[32, Chapter 5]); recall that for every $K \subset \overline{\mathbb{D}}$, we have $\dim(K) \leq \underline{\dim}_M(K) \leq \overline{\dim}_M(K)$. Observe from the proof of Theorem 5.1 in [26] (in particular, Proposition 5.3 there) that we have $\dim(A_V) = \overline{\dim}_M(A_V)$. Fix $\beta > \dim(A_V) = \underline{\dim}_M(A_V)$; there exists a sequence $(\varepsilon_k; k \geq 1)$ which decreases to 0 such that for every $k \geq 1$, there exists a positive integer $M(\varepsilon_k) \leq \varepsilon_k^{-\beta}$ and $M(\varepsilon_k)$ disjoint subarcs of $\mathbb{S}^1$ with length less than ε_k and which cover A_V. It follows that the two-dimensional Lebesgue measure of the ε_k-enlargement of $C_s(\boldsymbol{\ell})$ is bounded above by $C\varepsilon_k^{1-\beta}$, where the constant C does not depend on k. We conclude from [32, page 79] that $\dim(C_s(\boldsymbol{\ell})) \leq \overline{\dim}_M(C_s(\boldsymbol{\ell})) \leq 1+\beta$ for every $\beta > \dim(A_V)$, which completes the proof. □

4 Invariance principles for triangulated laminations

In this section, we establish invariance principles for different classes of noncrossing trees which converge to triangulated stable laminations. As an application, we obtain limit theorems for large discrete random trees embedded in a noncrossing way.

4.1 The continuous case

If τ is a plane tree, we let $\mathsf{H}(\tau) = \max_{u\in\tau}|u|$ be its height. Recall that $\mathcal{W}(\tau)$ is its Łukasiewicz path.

Proposition 4.1. *Let $f : [0,1] \to \mathbb{R}_+$ be a continuous function satisfying (H_f) and such that $f(0) = f(1) = 0$. For every $n \geq 1$, let θ_n be a noncrossing tree with n vertices and let τ_n be its shape. Assume that, as $n \to \infty$,*

(i) $\mathsf{H}(\tau_n)/n \to 0$*;*

(ii) There exists a sequence $B_n \to \infty$ such that $\mathcal{W}(\tau_n)/B_n \to f$ for the uniform topology.

Then the convergence $\theta_n \to \mathbf{L}(f)$ holds for the Hausdorff topology.

In other words, as soon as the Łukasiewicz path of the shape of a sequence of noncrossing trees converges to a continuous function having distinct local minima, the limit of the noncrossing trees is a triangulation that only depends on their shapes and not on their embeddings, provided that their height is negligible compared to their total size. Notice that Assumption (i) is crucial, as it is simple to construct a sequence of noncrossing trees satisfying (ii) but which does not converge for the Hausdorff topology.

As a direct application of Proposition 4.1, we obtain a limit theorem for Bienaymé–Galton–Watson trees conditioned to be large, embedded in the disk, as well as noncrossing trees with a prescribed degree sequence.

Corollary 4.2. *Let μ be critical offspring distribution with finite variance. For every $n \geq 1$, let Θ_n be a random noncrossing tree with n vertices such that its shape has the law BGW_n^μ. Then Θ_n converges in distribution to the Brownian triangulation as $n \to \infty$.*

This result simply follows from Proposition 4.1 by applying Skorokhod's representation theorem and combining (2.1) with the well-known fact that $\mathsf{H}(S(\Theta_n))/\sqrt{n}$ converges in distribution to a positive random variable as $n \to \infty$.

Corollary 4.3. *For every $n \geq 1$, fix a sequence $d(n) = (d_i(n); i \geq 0)$ of nonnegative integers such that $\Delta(n) \coloneqq \max\{i \geq 0 : d_i(n) > 0\} < \infty$ and sample then a noncrossing tree Θ_n uniformly at random in the set of all noncrossing trees with shape having $d_i(n)$ vertices with i children for every $i \geq 0$. Set $|d(n)| \coloneqq \sum_{i\geq 0} d_i(n) = 1 + \sum_{i\geq 0} i d_i(n)$ the total number of vertices of Θ_n and then*

$$p_i(n) \coloneqq \frac{d_i(n)}{|d(n)|} \quad (i \geq 0), \qquad \textit{and} \qquad \sigma^2(n) \coloneqq \sum_{i\geq 0} i^2 p_i(n) - 1.$$

If, as $n \to \infty$, we have $|d(n)| \to \infty$, $\Delta(n) = o(|d(n)|^{1/2})$, $p_i(n) \to p_i$ for each $i \geq 0$ where $\sum_{i\geq 0} p_i = \sum_{i\geq 0} ip_i = 1$ and $\sigma^2(n) \to \sigma_p$ where $\sigma_p \coloneqq \sum_{i\geq 0} i^2 p_i - 1 \in (0,\infty)$, then Θ_n converges in distribution to the Brownian triangulation as $n \to \infty$.

As previously, this follows from Proposition 4.1 and Skorokhod's representation theorem since, under these assumptions, Broutin & Marckert [5, Theorem 3] have shown the convergence in distribution as $n \to \infty$ of the rescaled Łukasiewicz path of $S(\Theta_n)$ towards the Brownian excursion, and that of $\mathsf{H}(S(\Theta_n))/\sqrt{|d(n)|}$ to a positive random variable.

Remark 4.4. *In [9, Sec. 3.2], a similar result to Proposition 4.1 is established using the contour function with the additional assumptions that the leaves of τ_n are "uniformly distributed" and that the local minima of f are dense. An important point is that we do not require the local minima of f to be dense in Proposition 4.1, which in particular allows triangulations with nonempty interior. We lift these restrictions by using the Łukasiewicz path instead of the contour function. Another advantage of this approach is that invariance principles are usually simpler to establish for the Łukasiewicz path than for the contour function. Furthermore, the fact that the leaves of τ_n are "uniformly distributed" does not necessarily follow from a functional invariance principle.*

We start with a preliminary observation which will be crucial in the proof of Proposition 4.1: roughly speaking, if the height of a plane tree is small compared to its size, then in any possible embedding of this plane tree as a noncrossing tree, the position of every vertex having a small number of descendants is known, up to a small error. In addition, if a vertex is such that only one of the subtrees grafted on its children is large, then it can only have two possible locations in the noncrossing embedding, up to a small error.

Lemma 4.5. *Let θ be a noncrossing tree with shape τ having n vertices. Denote by $\varnothing = u_0 \prec u_1 \prec \cdots \prec u_{n-1}$ the vertices of τ labelled in lexicographical order. Fix $\eta, \varepsilon \in (0,1)$. Let $0 \leq k \leq n-1$ and denote by S_k the number of (strict) descendants of u_k. Assume that $\mathsf{H}(\tau)/n \leq \varepsilon$.*

(i) Assume that $S_k \leq \eta n$. Then

$$\left| \mathrm{e}^{-2\mathrm{i}\pi k/n} - u_k \right| \leq 7(\varepsilon + \eta),$$

where we identify u_k with its associated complex number in the noncrossing tree θ.

(ii) Let M_k be the size of the largest subtree grafted on a child of u_k. Assume that $S_k - M_k \leq \eta n$. Then

$$\min\left(\left| \mathrm{e}^{-2\mathrm{i}\pi k/n} - u_k \right|, \left| \mathrm{e}^{-2\mathrm{i}\pi (k+S_k)/n} - u_k \right| \right) \leq 7(\varepsilon + \eta).$$

Proof. Let $P_k \in \{0, 1, \ldots, n-1\}$ be such that the vertex u_k is the complex number $\exp(-2\mathrm{i}\pi P_k/n)$ in θ_n. Then

$$|k - P_k| \leq \mathsf{H}(\tau) + S_k.$$

This readily follows from the description of the bijection Φ_n^{-1} given in the proof of Proposition 2.1: the error $\mathsf{H}(\tau)$ corresponds to the vertices belonging to $\rrbracket \varnothing, u_k \llbracket$ which may be folded to the right of u_k in θ_n, and the error S_k corresponds to all the vertices after u_k (in the lexicographical order) which may be folded to the left of u_k. Assertion (i) then follows from the fact that $|\mathrm{e}^{-2\mathrm{i}\pi s} - \mathrm{e}^{-2\mathrm{i}\pi t}| \leq 2\pi|s-t|$ for $s, t \in [0,1]$.

For (ii), let $\widetilde{u}$ be a child of u_k having M_k descendants (including itself). Then either $\widetilde{u}$ is folded to the right of u_k in θ, in which case all these M_k descendants are also folded to the right of u_k in θ, so that $|k - P_k| \leq \mathsf{H}(\tau) + S_k - M_k$, or $\widetilde{u}$ is folded to the left of u_k in θ, in which case all these M_k descendants are also folded to the left of u_k in θ, so that $|k + M_k - P_k| \leq \mathsf{H}(\tau) + S_k - M_k$ (the errors $S_k - M_k$ come from the descendants of u_k which are not descendants of $\widetilde{u}$ and which may be folded to the left of u_k). This completes the proof. $\square$

We next prove Proposition 4.1.

Proof of Proposition 4.1. Since $\overline{\mathbb{D}}$ is compact, the space of compact subsets of $\overline{\mathbb{D}}$ equipped with the Hausdorff distance is compact as well. Furthermore the space of laminations is closed, therefore the sequence $(\theta_n)_{n\geq 1}$ converges along a subsequence towards a lamination L of $\overline{\mathbb{D}}$ and we aim at showing that $L = \mathbf{L}(f)$. Since $\mathbf{L}(f)$ is maximal, it suffices to check that $\mathbf{L}(f) \subset L$. To simplify notation, we assume that $\theta_n \to L$ without extracting a subsequence.

Fix $0 < s < t < 1$ such that $s \sim^f t$ and let us show that $[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}] \subset L$. To this end, we fix $\varepsilon \in (0, (t-s)/10)$ and show that $[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}] \subset \theta_n^{(49\varepsilon)}$ for every n sufficiently large, where $X^{(\varepsilon)}$ is the ε-enlargement of a closed subset $X \subset \overline{\mathbb{D}}$. Observe from (H_f) that either $f(s) = f(t) < f(r)$ for every $r \in (s,t)$, or there exists a unique $r \in (s,t)$ such that $f(s) = f(t) = f(r)$ and neither s nor t are times of a local minimum. We may restrict our attention to the first case since, in the second one, there exists $s' \in (s-\varepsilon, s)$ and $t' \in (t, t+\varepsilon)$ such that $f(s') = f(t') < f(r)$ for every $r \in (s', t')$. We assume in the sequel that $f(s) = f(t) < f(r)$ for every $r \in (s,t)$ and that n is sufficiently large so that $\mathsf{H}(\tau_n)/n \leq \varepsilon$.

The strategy of the proof is to define, for every $n \geq 1$ large enough, two integers $r_n < l_n$ such that the geodesic path in θ_n going from the r_n-th vertex of τ_n to its l_n-th is close to $[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}]$ for the Hausdorff distance. A number of intermediate times is needed.

We start with some preliminary observations. Let $\mathcal{W}^{(n)}$ be the Łukasiewicz path of τ_n and denote by $\varnothing = u_0^{(n)} \prec u_1^{(n)} \prec \cdots \prec u_{n-1}^{(n)}$ the vertices of τ_n labelled in lexicographical order. It is well known that $u_i^{(n)}$ is an ancestor of $u_j^{(n)}$ if and only if $i \leq j$ and $\mathcal{W}_i^{(n)} = \min_{[i,j]} \mathcal{W}^{(n)}$ (see e.g. [28, Proposition 1.5]). As a consequence, for every $0 \leq k \leq n-1$, if $S_k^{(n)}$ denotes the number of (strict) descendants of $u_k^{(n)}$, we have

$$|u_k^{(n)}| = \# \left\{ 0 \leq j \leq k-1 : \mathcal{W}_j^{(n)} = \min_{[j,k]} \mathcal{W}^{(n)} \right\},$$

and

$$S_k^{(n)} = \min \left\{ j \geq k : \mathcal{W}_j^{(n)} < \mathcal{W}_k^{(n)} \right\} - k - 1.$$

Since $f(r) > f(s) = f(t)$ for every $r \in (s,t)$, there exists $z \in (s, s+\varepsilon)$ such that $\inf\{u > z : f(u) \leq f(z)\} \in (t-\varepsilon, t)$. As a consequence, setting $\eta = (z-s)/10$, for every n sufficiently large, there exists $z_n \in \{1, \dots, n-1\}$ such that

$$z - \eta \leq n^{-1} z_n \leq z + \eta \qquad \text{and} \qquad t - 2\varepsilon < n^{-1} \min \left\{ i > z_n : \mathcal{W}_i^{(n)} \leq \mathcal{W}_{z_n}^{(n)} \right\} < t. \tag{4.1}$$

Similarly, since $f(s) < \inf_{[z-4\eta, z+2\eta]} f$, we can find $y_n \in \{1, \dots, n-1\}$ such that

$$\begin{cases} s \leq n^{-1} y_n \leq z - 4\eta, \\ t - 2\varepsilon < n^{-1} \min \left\{ i > y_n : \mathcal{W}_i^{(n)} \leq \mathcal{W}_{y_n}^{(n)} \right\}, \\ n^{-1} \min \left\{ i > y_n : \mathcal{W}_i^{(n)} < \mathcal{W}_{y_n}^{(n)} \right\} < t. \end{cases} \tag{4.2}$$

We claim that for every n sufficiently large there exists integers $r_n^0 < j_n^0 \leq z_n$ such that

$$z - 3\eta < n^{-1} r_n^0, \qquad z - 2\eta < n^{-1} j_n^0, \qquad \text{and} \qquad \mathcal{W}^{(n)}_{r_n^0} > \mathcal{W}^{(n)}_{j_n^0}. \tag{4.3}$$

Indeed, if this were not the case, for every $j \in ((z-2\eta)n, z_n)$, we would have $\mathcal{W}^{(n)}_r \leq \mathcal{W}^{(n)}_j$ for every $r \in ((z-3\eta)n, j)$, yielding $\mathcal{W}^{(n)}_r = \min_{[r,z_n]} \mathcal{W}^{(n)}$ for every $(z-3\eta)n < r < (z-2\eta)n$, which would imply that $|u^{(n)}_{z_n}| \geq \eta n$ and contradict Assumption (i).

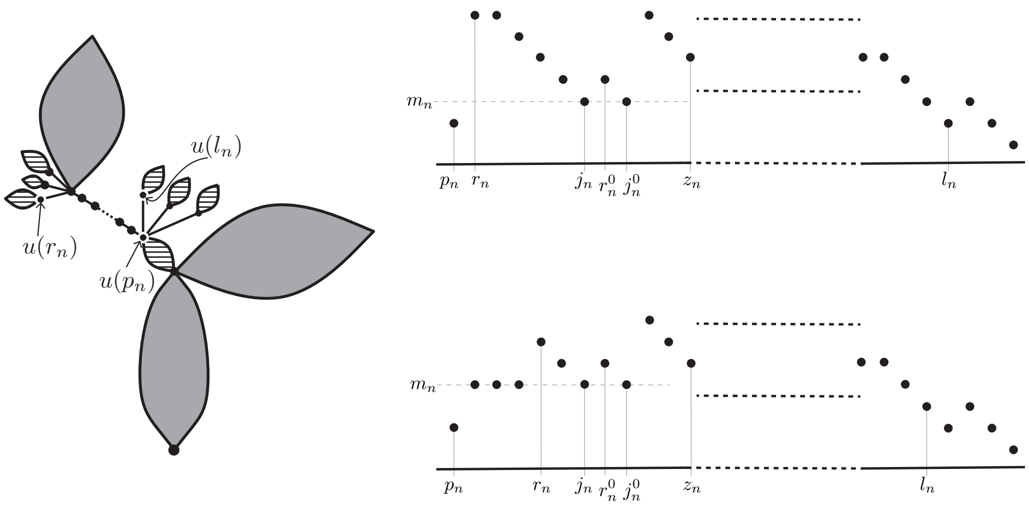

Figure 4: Illustration of the proof. On the left, the sizes of the dashed subtrees are small compared to the size of the three grey subtrees. On the top right is illustrated the case where $\mathcal{W}^{(n)}_{p_n+1} > m_n$ (so that $r_n = p_n + 1$), and on the bottom right is illustrated the case where $\mathcal{W}^{(n)}_{p_n+1} = m_n$ (so that $r_n > p_n + 1$).

Choose $r_n^0 < j_n^0 \leq z_n \in \{1, \ldots, n-1\}$ such that (4.3) holds. Set

$$m_n = \min_{[r_n^0, z_n]} \mathcal{W}^{(n)}, \quad p_n = \max\left\{ i < r_n^0 : \mathcal{W}^{(n)}_i < m_n \right\}, \quad r_n = \min\left\{ i > p_n ; \mathcal{W}^{(n)}_i > m_n \right\},$$

as well as

$$j_n = \min\left\{ i > r_n : \mathcal{W}^{(n)}_i = m_n \right\}, \quad l_n = \min\left\{ i > p_n : \ \mathcal{W}^{(n)}_i < m_n \right\},$$

so that

$$y_n \leq p_n < r_n < j_n \leq z_n < l_n \qquad \text{and} \qquad \mathcal{W}^{(n)}_i = m_n \quad \text{for every} \quad p_n < i < r_n.$$

For the first inequality, note that $p_n < y_n$ would imply $\mathcal{W}^{(n)}_{y_n} \geq m_n$ and so $\min\{i > y_n : \mathcal{W}^{(n)}_i \leq \mathcal{W}^{(n)}_{y_n}\} \leq z_n$ which, by (4.1), contradicts (4.2). In addition, for every n sufficiently large,

$$s \leq n^{-1} p_n, \qquad n^{-1} j_n < s + 2\varepsilon, \qquad n^{-1} j_n - n^{-1} p_n \leq 2\varepsilon, \qquad t - 2\varepsilon < n^{-1} l_n < t.$$

The first inequality follows from the fact that $p_n \geq y_n$, the second one from the fact that $n^{-1} j_n \leq n^{-1} z_n \leq z + \eta \leq s + 2\varepsilon$, the third one from the first two, and the last one from (4.1) and (4.2). Observe that $\mathcal{W}^{(n)}_{y_n} < \mathcal{W}^{(n)}_{p_n} < m_n$; we also have,

$$n^{-1} \min\left\{ i > p_n : \mathcal{W}^{(n)}_i < \mathcal{W}^{(n)}_{p_n} \right\} < t \tag{4.4}$$

by (4.2).

Note that either $\mathcal{W}^{(n)}_{p_n+1} > m_n$, in which case $r_n = p_n + 1$ and $u^{(n)}_{r_n}$ is the first child of $u^{(n)}_{p_n}$, or $\mathcal{W}^{(n)}_{p_n+1} = m_n$ and so $u^{(n)}_{p_n+1}, \dots, u^{(n)}_{r_n-2}$ all have one child, and $u^{(n)}_{r_n}$ is the first child of $u^{(n)}_{r_n-1}$.

This implies (see Fig. 4 for an illustration) that:

(a) $u^{(n)}_{l_n}$ is a child of $u^{(n)}_{p_n}$, since $\mathcal{W}^{(n)}_{p_n} \le \mathcal{W}^{(n)}_{l_n} < \mathcal{W}^{(n)}_i$ for every $p_n < i < l_n$;

(b) the number of descendants of $u^{(n)}_{r_n}$ is not greater than $2\varepsilon n$, since, similarly, $\mathcal{W}^{(n)}_{j_n} < \mathcal{W}^{(n)}_{r_n}$ so that $S^{(n)}_{r_n} \le r_n - j_n \le r_n - p_n \le 2\varepsilon n$;

(c) the number of descendants of $u^{(n)}_{l_n}$ is not greater than $2\varepsilon n$ since

$$S^{(n)}_{l_n} \le \min\left\{ i \ge l_n : \mathcal{W}^{(n)}_i < \mathcal{W}^{(n)}_{l_n} \right\} - l_n \le nt - (t - 2\varepsilon)n \le 2\varepsilon n,$$

where we have used (4.4) for the second inequality.

(d) Fix $p_n \le i \le r_n - 1$. If $M^{(n)}_i$ denotes the size of the largest subtree grafted on a child of $u^{(n)}_i$, then $S^{(n)}_i - M^{(n)}_i \le 4\varepsilon n$. Indeed, note that this is trivial if $p_n < i < r_n - 1$ since we observed that $u^{(n)}_i$ then has only one child; in the two other cases, we have $S^{(n)}_i \le nt - p_n$ using (4.4), and in addition, $M^{(n)}_i \ge l_n - j_n$, so that

$$S^{(n)}_i - M^{(n)}_i \le (nt - l_n) + (j_n - p_n) \le 4\varepsilon n.$$

We finally show that the path $[\![u^{(n)}_{r_n}, u^{(n)}_{l_n}]\!]$ in θ_n is close to the chord $[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}]$ for the Hausdorff distance.

Step 1: Control of the positions of $u^{(n)}_{r_n}$ and $u^{(n)}_{l_n}$. We claim that

$$\left|\mathrm{e}^{-2\mathrm{i}\pi s} - u^{(n)}_{r_n}\right| \le 35\varepsilon \quad \text{and} \quad \left|\mathrm{e}^{-2\mathrm{i}\pi t} - u^{(n)}_{l_n}\right| \le 35\varepsilon. \tag{4.5}$$

Indeed, By Lemma 4.5 (i), we have $|\mathrm{e}^{-2\mathrm{i}\pi r_n/n} - u^{(n)}_{r_n}| \le 21\varepsilon$ by (b) and $|\mathrm{e}^{-2\mathrm{i}\pi l_n/n} - u^{(n)}_{l_n}| \le 21\varepsilon$ by (c). Our claim then follows by the triangular inequality since $|\mathrm{e}^{-2\mathrm{i}\pi r_n/n} - \mathrm{e}^{-2\mathrm{i}\pi s}| \le 7|r_n/n - s| \le 14\varepsilon$ and $|\mathrm{e}^{-2\mathrm{i}\pi l_n/n} - \mathrm{e}^{-2\mathrm{i}\pi t}| \le 7|l_n/n - t| \le 14\varepsilon$.

Step 2: Control of the path between $u^{(n)}_{r_n}$ and $u^{(n)}_{p_n}$. By (d), for every vertex $u^{(n)}_k \in [\![u^{(n)}_{p_n}, u^{(n)}_{r_n}[\![$ or, equivalently, for every $p_n \le k \le r_n - 1$, we have $S^{(n)}_k - M^{(n)}_k \le 4\varepsilon n$, so an application of Lemma 4.5 (ii) yields

$$\min\left(\left|\mathrm{e}^{-2\mathrm{i}\pi k/n} - u^{(n)}_k\right|, \left|\mathrm{e}^{-2\mathrm{i}\pi (k + S^{(n)}_k)/n} - u^{(n)}_k\right| \right) \le 35\varepsilon.$$

Note that $|\mathrm{e}^{-2\mathrm{i}\pi k/n} - \mathrm{e}^{-2\mathrm{i}\pi s}| \le 7|k/n - s| \le 7(r_n/n - s) \le 7(j_n/n - s) \le 14\varepsilon$. Also, $S^{(n)}_k + k \le l_n < nt$, so that $|\mathrm{e}^{-2\mathrm{i}\pi (k+S^{(n)}_k)/n} - \mathrm{e}^{-2\mathrm{i}\pi t}| \le 7|(k + S^{(n)}_k)/n - t| \le 7(t - l_n) \le 14\varepsilon$. Therefore

$$\min\left(\left|\mathrm{e}^{-2\mathrm{i}\pi s} - u^{(n)}_k\right|, \left|\mathrm{e}^{-2\mathrm{i}\pi t/n} - u^{(n)}_k\right| \right) \le 49\varepsilon. \tag{4.6}$$

Since $u^{(n)}_{l_n}$ is a child of $u^{(n)}_{p_n}$ by (a), we conclude from (4.5) and (4.6) that for every $u \in [\![u^{(n)}_{r_n}, u^{(n)}_{l_n}]\!]$,

$$\min\left(\left|\mathrm{e}^{-2\mathrm{i}\pi s} - u\right|, \left|\mathrm{e}^{-2\mathrm{i}\pi t/n} - u\right| \right) \le 49\varepsilon.$$

Therefore, letting $\mathcal{L}^{(n)}$ be the path $[\![u^{(n)}_{r_n}, u^{(n)}_{l_n}]\!]$ in the noncrossing tree, we get that $\mathcal{L}^{(n)} \subset [\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}]^{(49\varepsilon)}$. Since $\mathcal{L}^{(n)}$ is a union of finite segments joining $u^{(n)}_{r_n}$ to $u^{(n)}_{l_n}$, we get that $[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}] \subset (\mathcal{L}^{(n)})^{(49\varepsilon)} \subset \theta_n^{(49\varepsilon)}$, which establishes our original claim and completes the proof. □

4.2 The càdlàg case

Recall the definition of $p_s(\boldsymbol{\ell})$ from Sec. 3.3.

Proposition 4.6. *Let θ_n be a noncrossing tree with n vertices and shape τ_n. Denote by $\varnothing = u_0^{(n)} \prec u_1^{(n)} \prec \cdots \prec u_{n-1}^{(n)}$ the vertices of τ_n listed in lexicographical order, let $k_i^{(n)}$ be the number of children of $u_i^{(n)}$ and let $L_i^{(n)}$ be the number of children of $u_i^{(n)}$ lying to the "left" of $u_i^{(n)}$ in θ_n. Let $Z \in \mathbb{D}([0,1],\mathbb{R})$ be a càdlàg function satisfying (H0), ..., (H4). Assume that there exists a sequence $B_n \to \infty$ and a sequence $\boldsymbol{\ell} = (\ell_s : s \in J(Z))$ indexed by the jump times of Z such that the following properties hold:*

- *(i) We have $\mathsf{H}(\tau_n)/n \to 0$ as $n \to \infty$.*
- *(ii) The convergence $\mathcal{W}(\tau_n)/B_n \to Z$ holds for the Skorokhod J_1 topology.*
- *(iii) For every $s \in (0,1)$, if $i_n \in \{0,1,\dots,n-1\}$ is such that $\lim_{n\to\infty} k_{i_n}^{(n)}/B_n > 0$ and $i_n/n \to s$, then $L_{i_n}^{(n)}/k_{i_n}^{(n)} \to \ell_s$.*
- *(iv) For every $s \in J(Z)$, Z does not attain a local minimum at $p_s(\boldsymbol{\ell})$.*

Then $\theta_n \to L(Z,\boldsymbol{\ell})$ for the Hausdorff topology.

Roughly speaking, Condition (iv) ensures that the special vertex from which each face is triangulated is not an endpoint of a chord of $L(Z,\boldsymbol{\ell})$ (but of course belongs to the closure of the endpoints of chords).

Proof. As in the proof of Proposition 4.1, we assume that θ_n converges towards a lamination L and we aim at showing that $L = L(Z,\boldsymbol{\ell})$. Again, since $L(Z,\boldsymbol{\ell})$ is maximal, it suffices to check that $L(Z,\boldsymbol{\ell}) \subset L$.

We first show that $L(Z) \subset L$. To this end, fix $\varepsilon > 0$ and choose $0 \le s < t \le 1$ such that $s \simeq^Z t$. If $\Delta Z(s) = 0$, then $Z(t) = Z(s) = \inf_{[s,t]} Z$ and $\Delta Z(t) = 0$. Arguments similar to those of the proof of Proposition 4.1 show that $[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}] \subset \theta_n^{(49\varepsilon)}$ for n sufficiently large. If $\Delta Z(s) > 0$, then $t = \inf\{u > s : Z(t) = Z(s-)\}$ and for every $\varepsilon > 0$ we have $\inf_{[s-\varepsilon,s]} Z < Z(s-)$ by (H3) and $\inf_{[t,t+\varepsilon]} Z < Z(t)$ by (H2). Using these inequalities, again similar arguments to those of the proof of Proposition 4.1 entail that $[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}] \subset \theta_n^{(49\varepsilon)}$ for n sufficiently large. We leave the (merely technical) details to the reader, and refer to [31, Proof of Theorem 7.1] for detailed arguments.

Next, let $s \in J(Z)$, set $s' = \inf\{t > s : Z(t) = Z(s-)\}$ and fix $t \in [s,s']$ such that $Z(t) = \inf_{[s,t]} Z$ (observe that (H3) implies $\Delta Z(t) = 0$). We shall show that $[\mathrm{e}^{-2\mathrm{i}\pi p_s(\boldsymbol{\ell})}, \mathrm{e}^{-2\mathrm{i}\pi t}] \subset \theta_n^{(\varepsilon)}$ for n sufficiently large. Let i_n as in (iii) and set

$$S_{i_n}^{(n)} = \min\left\{ j \ge i_n + 1 : \mathcal{W}_j^{(n)} = \mathcal{W}_{i_n+1}^{(n)} - L_{i_n}^{(n)} \right\} - i_n - 1,$$

the total number of (strict) descendants of the first $L_{i_n}^{(n)}$ children of $u_{i_n}^{(n)}$. Then, by definition of $L_{i_n}^{(n)}$,

$$\left| u_{i_n}^{(n)} - \mathrm{e}^{-2\mathrm{i}\pi(i_n + S_{i_n}^{(n)})/n} \right| \le 7 \frac{\mathsf{H}(\tau_n)}{n},$$

where the error term corresponds to the vertices belonging to $[\![\varnothing, u_{i_n}^{(n)}[\![$ which may be folded to the right of $u_{i_n}^{(n)}$ in θ_n. Since $k_{i_n}^{(n)}/B_n \to \Delta Z(s)$, we have $L_{i_n}^{(n)}/B_n \to \Delta Z(s)\ell_s$. In addition, $\mathcal{W}_{i_n}^{(n)}/B_n \to Z(s)$. By (iv), Z does not attain a local minimum at $p_s(\boldsymbol{\ell})$, so by continuity properties of first passage times for the Skorokhod topology,

$$n^{-1} \min\left\{ j \ge i_n : \mathcal{W}_j^{(n)} = \mathcal{W}_{i_n+1}^{(n)} - L_{i_n}^{(n)} \right\} \underset{n\to\infty}{\longrightarrow} \inf\left\{ t \ge s : Z_t = Z_s - \Delta Z(s)\ell_s \right\} = p_s(\boldsymbol{\ell}).$$

Therefore $n^{-1}(S^{(n)}_{i_n} + i_n) \to p_s(\boldsymbol{\ell})$, implying, by the previous bound and (i) that

$$\left| \mathrm{e}^{-2\mathrm{i}\pi p_s(\boldsymbol{\ell})} - u^{(n)}_{i_n} \right| \underset{n\to\infty}{\longrightarrow} 0. \tag{4.7}$$

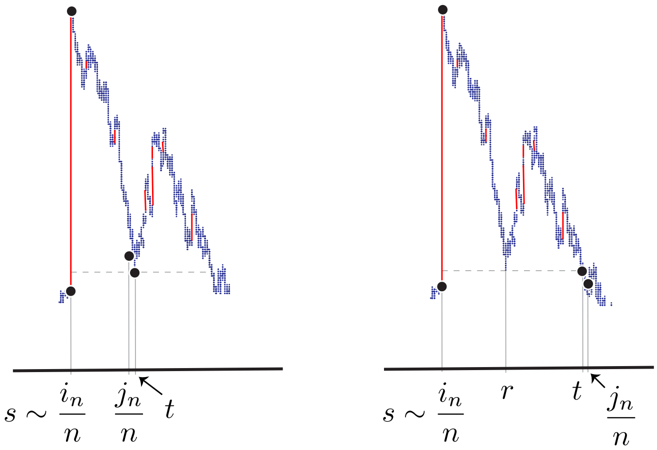

Figure 5: Illustration of the choice of j_n. On the left, the case where $Z(r) > Z(t)$ for every $r \in (s,t]$ and on the right, the case where there exists (a unique) $r \in (s,t)$ such that $Z(r) = Z(t)$.

We claim that there exists $j_n \in \{0, 1, \dots, n-1\}$ such that $j_n/n \to t$, $u^{(n)}_{j_n}$ is a child of $u^{(n)}_{i_n}$ and the number of descendants of $u^{(n)}_{j_n}$ is $o(n)$ as $n \to \infty$. Indeed, suppose first that $Z(r) > Z(t)$ for every $r \in (s,t)$. Fix $\varepsilon \in (0, t-s)$; from (H3), the infimum of Z over $[s, t-\varepsilon]$ is achieved at some point of this interval. Therefore, for n large enough, there exists an integer j_n such that $j_n/n \in [t-\varepsilon, t]$, $\mathcal{W}_m > \mathcal{W}_{j_n}$ for every integer $m \in [i_n+1, j_n-1]$, and $\inf\{l > j_n : \mathcal{W}_l = W_{j_n} - 1\} \le j_n + n\varepsilon$; the claim then follows. Suppose next that there exists $r \in (s,t)$ such that $Z(r) = Z(t) = \inf_{[s,t]} Z$; then note that r must be a time of local minimum by (H3), so this can only occur when $Z(t) > Z(s-)$ because otherwise it would contradict (H4), also t cannot be a time of a local minimum by (H1). We conclude that for every $\varepsilon > 0$, we can find $t' \in (t, t+\varepsilon)$ such that $Z(s) < Z(t') < Z(r)$ for every $r \in (s, t')$ and the previous approximation thus applies.

This implies that $|\mathrm{e}^{-2\mathrm{i}\pi j_n/n} - u^{(n)}_{j_n}| \to 0$ by Lemma 4.5 (i), so that

$$\left| \mathrm{e}^{-2\mathrm{i}\pi t} - u^{(n)}_{j_n} \right| \underset{n\to\infty}{\longrightarrow} 0. \tag{4.8}$$

Combining (4.7) and (4.8), since $u^{(n)}_{j_n}$ is a child of $u^{(n)}_{i_n}$, we get that for every n sufficiently large

$$\left[\mathrm{e}^{-2\mathrm{i}\pi p_s(\boldsymbol{\ell})}, \mathrm{e}^{-2\mathrm{i}\pi t}\right] \subset \left[u^{(n)}_{i_n}, u^{(n)}_{j_n}\right]^{(\varepsilon)} \subset \theta^{(\varepsilon)}_n.$$

This completes the proof. □

4.3 The uniform stable triangulation

If τ is a plane tree with n vertices, we set $\Theta^U(\tau) = \Phi_n^{-1}(\tau, \mathcal{C})$, where $\mathcal{C}$ is a random element of $\mathbb{C}(\tau)$ chosen uniformly at random. In other words, $\Theta^U(\tau)$ is a noncrossing tree obtained by a "uniform" embedding of τ.

Our next result establishes an invariance principle for large critical Bienaymé–Galton–Watson trees in the domain of attraction of a stable law of index $\alpha \in (1,2)$ which are embedded uniformly in a noncrossing way. The distributional limit is the uniform stable triangulation, which was introduced in Sec. 3.3.

Theorem 4.7. *Fix $\alpha \in (1,2)$. For every critical offspring distribution μ belonging to the domain of attraction of a stable law of index α, if $\mathcal{T}_n$ is a Bienaymé–Galton–Watson tree with offspring distribution μ conditioned to have n vertices, the convergence*

$$\Theta^U(\mathcal{T}_n) \quad \xrightarrow[n\to\infty]{(d)} \quad \mathbf{L}_\alpha^U$$

holds in distribution for the Hausdorff distance on the space of all compact subsets of $\overline{\mathbb{D}}$.

Proof. We want to apply Skorokhod's representation theorem and Proposition 4.6 with $Z = X_\alpha^{\mathrm{ex}}$. Assumptions (i) and (ii) hold by (2.1) as well as the fact $\frac{B_n}{n}\mathsf{H}(\mathcal{T}_n)$ converges in distribution to a positive random variable as $n \to \infty$ [16]. To see that Assumption (iii) holds, denote by $\varnothing = u_0^{(n)} \prec u_1^{(n)} \prec \cdots \prec u_{n-1}^{(n)}$ the vertices of $\mathcal{T}_n$ listed in lexicographical order, let $k_i^{(n)}$ be the number of children of $u_i^{(n)}$ and let $L_i^{(n)}$ be the number of children of $u_i^{(n)}$ lying to the "left" of $u_i^{(n)}$ in $\Theta^U(\mathcal{T}_n)$. By definition, conditionally given $\mathcal{T}_n$, $L_i^{(n)}$ is uniformly distributed in $\{0,1,\ldots,k_i^{(n)}\}$, and the random variables $(L_i^{(n)} : 0 \leq i \leq n-1)$ are independent. In particular, conditionally on $k_{i_n}^{(n)} \to \infty$, $L_{i_n}^{(n)}/k_{i_n}^{(n)}$ converges in distribution to a uniform random variable on $[0,1]$. Finally, Assumption (iv) holds: almost surely, for every $s \in J(X_\alpha^{\mathrm{ex}})$, X_α^{ex} does not attain a local minimum at $p_s(\boldsymbol{\ell}^U)$, where, conditionally given X_α^{ex}, $\boldsymbol{\ell}^U = (\ell_s)_{s\in J(X_\alpha^{\mathrm{ex}})}$ is a sequence of i.i.d. uniform random variables on $[0,1]$. Indeed, almost surely, the times at which X_α^{ex} attains a local minimum are at most countable, so for every $s \in J(X_\alpha^{\mathrm{ex}})$, the probability that $p_s(\boldsymbol{\ell}^U)$ is such a time is zero and, almost surely, $J(X_\alpha^{\mathrm{ex}})$ is countable. □

5 Applications to simply generated noncrossing trees

In this section, we consider simply generated noncrossing trees, as defined by (1.1). We first prove that such trees are almost Bienaymé–Galton–Watson trees, and then establish Theorem 1.1 by using the invariance principles obtained in the previous section.

We denote by $\mathrm{BGW}^{\mu_\varnothing,\mu}$ the law of a modified Bienaymé–Galton–Watson tree, where the offspring distribution of the root is $\mu_\varnothing$, and that of the other vertices is μ. For every integer n, we denote by $\mathrm{BGW}_n^{\mu_\varnothing,\mu}$ the law of such a tree conditioned to have n vertices.

5.1 Simply generated noncrossing trees are almost Bienaymé–Galton–Watson trees

As we have seen, every noncrossing tree θ carries a planar structure, canonically rooted at the vertex corresponding to the complex number 1, which is called the shape of θ and is denoted by $S(\theta)$. If Θ_n a random noncrossing tree uniformly distributed on $\mathbb{NC}_n$, then Theorem 1 in [30] shows that $S(\Theta_n)$ is a modified Bienaymé–Galton–Watson tree, where the root has a different offspring distribution, conditioned to have size n. Our next result extends this to simply generated noncrossing trees.

Theorem 5.1. *Assume that*

$$\rho \quad := \quad \left(\limsup_{k\to\infty} w(k)^{1/k}\right)^{-1} > 0.$$

Fix $b \in (0,\rho)$, set

$$a = \left(\sum_{k=0}^{\infty}(k+1)w(k+1)b^k\right)^{-1} \qquad \textit{and} \qquad c = \left(\sum_{k=1}^{\infty} w(k)b^k\right)^{-1},$$

and define

$$\begin{cases} \mu(k) = a(k+1)w(k+1)b^k & (k \geq 0), \\ \mu_\varnothing(k) = cw(k)b^k & (k \geq 1). \end{cases} \tag{5.1}$$

Then the law of the shape of a noncrossing tree sampled according to $\mathbb{P}_n^w$ *is* $\mathrm{BGW}_n^{\mu_\varnothing,\mu}$.

Observe that

$$\sum_{j=1}^{\infty} j\mu_\varnothing(j) = \frac{bc}{a}, \qquad \text{whence} \qquad \frac{k\mu_\varnothing(k)}{\sum_{j=1}^{\infty} j\mu_\varnothing(j)} = \mu(k-1). \tag{5.2}$$

We shall see that the probability that the root of a modified Bienaymé–Galton–Watson tree conditioned to have n vertices has k children converges towards $\frac{k\mu_\varnothing(k)}{\sum_{j=1}^{\infty} j\mu_\varnothing(j)}$ as $n \to \infty$. The above identity then translates roughly the fact that in a large modified Bienaymé–Galton–Watson tree as above, the law of the degree of the root is close to that of the other vertices, as it is the case for a simply generated noncrossing tree.

Remark 5.2. *The condition* (1.2) *appearing in Theorem 1.1 is equivalent to the fact that the probability measure* μ *defined by* (5.1) *can be chosen to be critical; in this case, it is unique. Indeed, consider the function*

$$\Psi : x \in [0, \rho) \quad \longmapsto \quad \frac{\sum_{k=0}^{\infty} k(k+1)w(k+1)x^k}{\sum_{k=0}^{\infty}(k+1)w(k+1)x^k}. \tag{5.3}$$

Janson [23, Lemma 3.1] observed that Ψ *is null at* 0, *continuous and increasing. Therefore, for every value* $m \in (0, \Psi(\rho))$, *where* $\Psi(\rho) \coloneqq \lim_{x\uparrow\rho}\Psi(x)$, *there exists a unique probability measure* μ *of the form* (5.1) *with expectation* m. *In particular, one can choose* μ *to be critical if and only if*

$$\lim_{x\uparrow\rho}\Psi(x) \geq 1,$$

in which case, $b > 0$ *is the unique number such that*

$$\sum_{k=0}^{\infty}(k+1)(k-1)w(k+1)b^k = 0. \tag{5.4}$$

Remark 5.3. *Consider the uniform distribution on noncrossing trees:* $w(k) = 1$ *for every* $k \geq 1$. *Then* (5.4) *holds with* $b = 1/3$. *A simple calculation yields* $a = 4/9$ *and* $c = 2$, *so that* (5.1) *reads*

$$\begin{cases} \mu(k) = 4(k+1)3^{-(k+2)} & (k \geq 0), \\ \mu_\varnothing(k) = 2 \times 3^{-k} & (k \geq 1). \end{cases} \tag{5.5}$$

In particular, Theorem 5.1 recovers the special case of Marckert & Panholzer [30, Theorem 1].

Proof of Theorem 5.1. Fix $n \geq 1$ and denote by $\mathbb{Q}_n^w$ the law of the shape of a random noncrossing tree sampled according to $\mathbb{P}_n^w$. We aim at showing that $\mathbb{Q}_n^w = \mathrm{BGW}_n^{\mu_\varnothing,\mu}$. To this end, fix $\tau \in \mathbb{T}_n$, and let $k_0, k_1, \dots, k_{n-1}$ be the number of children of its vertices listed in lexicographical order (in particular, k_0 is the number of children of its root). By definition,

$$\mathrm{BGW}^{\mu_\varnothing,\mu}(\tau) = \mu_\varnothing(k_0)\prod_{i=1}^{n-1}\mu(k_i) = cw(k_0)b^{k_0}\prod_{i=1}^{n-1}a(k_i+1)w(k_i+1)b^{k_i}.$$

Note that $\sum_{i=0}^{n-1} k_i = n-1$, whence

$$\mathrm{BGW}_n^{\mu_\varnothing,\mu}(\tau) = \frac{c(ab)^{n-1}}{\mathrm{BGW}^{\mu_\varnothing,\mu}(\mathbb{T}_n)}w(k_0)\prod_{i=1}^{n-1}(k_i+1)w(k_i+1).$$

Next, observe that $\mathbb{P}_n^w(\theta)$ only depends on $S(\theta)$ and that $\#\{\theta \in \mathbb{NC}_n : S(\theta) = \tau\} = \#\mathbb{C}(\tau) = \prod_{i=1}^{n-1}(k_i+1)$ by Proposition 2.1. It follows that

$$\mathbb{Q}_n^w(\tau) = \sum_{\theta \in \mathbb{NC}_n : S(\theta)=\tau} \mathbb{P}_n^w(\theta) = \frac{1}{Z_n^w} \#\mathbb{C}(\tau) \prod_{u \in \tau} w(\deg u) = \frac{1}{Z_n^w} w(k_0) \prod_{i=1}^{n-1} (k_i+1) w(k_i+1).$$

Since $\mathbb{Q}_n^w$ and $\mathrm{BGW}_n^{\mu_\varnothing,\mu}$ are both probability measures on $\mathbb{T}_n$, we conclude that we have the identity $c(ab)^{n-1}/\mathrm{BGW}^{\mu_\varnothing,\mu}(\mathbb{T}_n) = 1/Z_n^w$ and the claim follows. □

5.2 Largest subtree of the root of large modified Bienaymé–Galton–Watson trees

Finally, Theorem 1.1 will readily follow from Propositions 4.1 and 4.6, as in the proof of Theorem 4.7, using the next convergence, which extends Duquesne's theorem (2.1) to modified Bienaymé–Galton–Watson trees.

Theorem 5.4. *Fix $\alpha \in (1,2]$. Let $\mu_\varnothing$ be a probability measure on $\mathbb{N}$ with finite mean and μ a probability measure on $\mathbb{Z}_+$ which is critical and belongs to the domain of attraction of a stable law with index α. For every integer $n \geq 1$, sample $\mathcal{T}_n$ according to $\mathrm{BGW}_n^{\mu_\varnothing,\mu}$ (provided that $\mathrm{BGW}_n^{\mu_\varnothing,\mu}$ is well defined). Then the convergence in distribution*

$$\left(\frac{1}{B_n} \mathcal{W}_{\lfloor ns \rfloor}(\mathcal{T}_n) : s \in [0,1] \right) \quad \xrightarrow[n\to\infty]{(d)} \quad (X_\alpha^{\mathrm{ex}}(s) : s \in [0,1]),$$

holds in the space $\mathbb{D}([0,1],\mathbb{R})$, where $(B_n)_{n\geq 1}$ is the same sequence as in (2.1).

Marckert & Panholzer [30] obtained this limit theorem in the case where $\mu_\varnothing$ and μ are given by (5.5). We follow the same approach in the general case, which roughly speaking consists in comparing $\mathrm{BGW}_n^{\mu_\varnothing,\mu}$ and BGW_n^{μ}. However, Marckert & Panholzer crucially use the fact that the support of μ and that of $\mu_\varnothing$ differ only at 0. This is not the case when $\mu_\varnothing$ and μ are given by (5.1) as soon as $w(k) = 0$ for some $k \geq 1$, so some care is needed (see Remark 5.8). Our approach also gives a limit theorem for the size of the maximal subtree grafted on the root of a size-conditioned (possibly modified) Bienaymé–Galton–Watson tree.

We start by proving Theorem 1.1, assuming that Theorem 5.4 holds.

Proof of Theorem 1.1. Recall that the Hausdorff dimension of $\mathbf{L}_\alpha^U$ has been computed in Theorem 3.4. Define μ and $\mu_\varnothing$ by (5.1), so that the shape of Θ_n has law $\mathrm{BGW}_n^{\mu_\varnothing,\mu}$ by Theorem 5.1. In addition, the proof of Theorem 5.1 also shows that conditionally given the shape $S(\Theta_n)$, the random variable $C(\Theta_n)$ is uniformly distributed on the set of all its possible values. Under the assumption of Theorem 1.1, μ is critical and in the domain of attraction of a stable law of index α. Since $\mu_\varnothing$ has finite mean by (5.2), we can apply Theorem 5.4; moreover we will see in Remark 5.9 that the height of the shape of Θ_n is small compared to n so we conclude as in the proofs of Corollary 4.2 and Theorem 4.7. □

Remark 5.5. *If $k \mapsto w(k+1)$ is a critical probability distribution on $\mathbb{Z}_+$ belonging to the domain of attraction of a stable law of index $\alpha \in (1,2)$, a simply generated noncrossing tree with weights w will converge to the Brownian triangulation (and its shape to the Brownian CRT), but a simply generated plane tree with weights w will converge, appropriately rescaled, to the α-stable random tree, and embedded in a uniform manner it will converge to the uniform α-stable triangulation.*

We fix for the rest of this section $\mu_\varnothing$ a probability measure on $\mathbb{N}$ with finite mean and μ a probability measure on $\mathbb{Z}_+$ which is critical and belongs to the domain of attraction of a stable law with index $\alpha \in (1,2]$. We further assume that μ is aperiodic to avoid unnecessary complications, meaning that $\gcd\{i > 0 : \mu(i) > 0\} = 1$ so that $\mathrm{BGW}^{\mu}(|\mathcal{T}| = n) > 0$ for every n sufficiently large. The key estimate in order to prove Theorem 5.4 is the following, which may be of independent interest. Denote by $N_0(\tau)$ the number of children of the root of a plane tree τ and by $M(\tau)$ the size of the largest subtree that stems from one of these children.

Proposition 5.6. *Let $\mathcal{N}$ be a random variable with law given by*

$$\mathbb{P}\left(\mathcal{N} = k\right) = \frac{k\mu_{\varnothing}(k)}{\sum_{j \geq 1} j\mu_{\varnothing}(j)} \qquad (k \geq 1)$$

and let $(Y_i)_{i \geq 1}$ be an independent sequence of i.i.d. random variables having the law of the total size of a BGW^{μ} tree. Then, for every $k \geq 0$ and $L \geq 1$,

$$\mathrm{BGW}_n^{\mu_{\varnothing},\mu}(n-1-M = k, N_0 = L) \quad \underset{n\to\infty}{\longrightarrow} \quad \mathbb{P}\left(Y_1 + Y_2 + \cdots + Y_{L-1} = k, \mathcal{N} = L\right).$$

Note that this implies that for every $k \geq 0$,

$$\mathrm{BGW}_n^{\mu_{\varnothing},\mu}(n-1-M = k) \quad \underset{n\to\infty}{\longrightarrow} \quad \mathbb{P}\left(Y_1 + Y_2 + \cdots + Y_{\mathcal{N}-1} = k\right).$$

In particular, under $\mathrm{BGW}_n^{\mu_{\varnothing},\mu}$, M/n converges to 1 in probability as $n \to \infty$, which was proved by Marckert & Panholzer when $\mu_{\varnothing}$ and μ are given by (5.5). Note also that this result covers the case of Bienaymé–Galton–Watson trees by taking $\mu_{\varnothing} = \mu$.

We establish Proposition 5.6 in several steps and first introduce some notation. Let $S = (S_n)_{n \geq 0}$ be the random walk starting from 0 with step distribution $(\mu(k+1) : k \geq -1)$. Observe that S is an aperiodic centred random walk with step distribution in the domain of attraction of a stable law with index α. Recall the spectrally positive Lévy process X_{α} introduced in Sec. 2.1 and denote by p_1 the density of $X_{\alpha}(1)$; the latter is known to be positive, continuous and bounded (see e.g. Zolotarev [40, I. 4]). We will use the local limit theorem (see Ibragimov & Linnik [22, Theorem 4.2.1]), which tells us that

$$\sup_{k \in \mathbb{Z}} \left| B_n \mathbb{P}(S_n = k) - p_1(B_n^{-1}k) \right| \quad \underset{n\to\infty}{\longrightarrow} \quad 0. \tag{5.6}$$

For every $k \geq 1$, denote by T_{-k} the first hitting time of $-k$ by the random walk S. We will need Kemperman's formula, which states that

$$\mathbb{P}\left(T_{-k} = n\right) = \frac{k}{n} \cdot \mathbb{P}\left(S_n = -k\right) \tag{5.7}$$

for every $k \geq 1$ and $n \geq 1$ (see e.g. [38, Chap. 6]). In particular, the total size Y_1 of a BGW^{μ} tree belongs to the domain of attraction of a stable law of index $1/\alpha$, since $\mathbb{P}\left(Y_1 = n\right) = \mathbb{P}\left(T_{-1} = n\right) = n^{-1}\mathbb{P}\left(S_n = -1\right) \sim (nB_n)^{-1} p_1(0)$ as $n \to \infty$.

The main tool to prove prove Proposition 5.6 is the following lemma.

Lemma 5.7. *(i) We have*

$$\mathrm{BGW}^{\mu_{\varnothing},\mu}(|\mathcal{T}| = n) \quad \underset{n\to\infty}{\sim} \quad \frac{1}{|\Gamma(-1/\alpha)|} \cdot \left(\sum_{k \geq 1} k\mu_{\varnothing}(k) \right) \cdot \frac{1}{n \cdot B_n}.$$

(ii) We have

$$\mathrm{BGW}_n^{\mu_{\varnothing},\mu}(N_0 = k) \quad \underset{n\to\infty}{\longrightarrow} \quad \frac{k\mu_{\varnothing}(k)}{\sum_{j \geq 1} j\mu_{\varnothing}(j)} \qquad \textit{uniformly in } k. \tag{5.8}$$

(iii) *Fix $k \geq 1$; for every $n \geq k$, consider a forest of k independent Bienaymé–Galton–Watson trees with offspring distribution μ, conditioned to have total size n and denote by $M_n^{\mu,k}$ the size of the largest tree. Then, as $n \to \infty$, $n - M_n^{\mu,k}$ converges in distribution to the total size of $k-1$ independent Bienaymé–Galton–Watson trees with offspring distribution μ.*

In particular, with the notation of (iii), $n^{-1} M_n^{\mu,k} \to 1$ in probability as $n \to \infty$.

Proof. Observe that under $\mathrm{BGW}^{\mu_\varnothing,\mu}$, the Łukasiewicz path associated with the tree is distributed as a random walk issued from 0, with first step distributed as $(\mu_\varnothing(k+1) : k \geq 0)$ and the next ones as $(\mu(k+1) : k \geq -1)$, stopped at its first hitting time of -1. As a consequence, by decomposing the Łukasiewicz path after the first step, for every $k \geq 1$ we have:

$$\mathrm{BGW}^{\mu_\varnothing,\mu}(|\mathcal{T}| = n) = \sum_{k=1}^{n-1} \mu_\varnothing(k) \mathbb{P}\left(T_{-k} = n-1\right) = \sum_{k=1}^{n-1} \frac{k\mu_\varnothing(k)}{n-1} \mathbb{P}\left(S_{n-1} = -k\right), \tag{5.9}$$

where we have used Kemperman's formula (5.7) for the last equality. Next note that for every fixed $k \geq 1$, we have

$$k\mu_\varnothing(k) B_{n-1} \mathbb{P}(S_{n-1} = -k) = k\mu_\varnothing(k) \left(p_1(-B_{n-1}^{-1} k) + o(1)\right) \quad \underset{n\to\infty}{\longrightarrow} \quad k\mu_\varnothing(k) p_1(0).$$

Since $\sum_{k \geq 1} k\mu_\varnothing(k) < \infty$ and p_1 is bounded, the above convergence yields also

$$\sum_{k\geq 1} k\mu_\varnothing(k) B_{n-1} \mathbb{P}(S_{n-1} = -k) \quad \underset{n\to\infty}{\longrightarrow} \quad \sum_{k\geq 1} k\mu_\varnothing(k) p_1(0).$$

Recall from [18, Lemma XVII.6.1] that $p_1(0) = |\Gamma(-1/\alpha)|^{-1}$. Moreover, since $(B_n)_{n\geq 1}$ is regularly varying with index $1/\alpha$, we have $B_{n-1}/B_n \to 1$ as $n \to \infty$. Assertion (i) then follows.

We now establish (ii). As in the proof of (i), also using (5.7), we have

$$\mathrm{BGW}_n^{\mu_\varnothing,\mu}(N_0 = k) = \frac{\mu_\varnothing(k)\mathbb{P}(T_{-k} = n-1)}{\mathrm{BGW}^{\mu_\varnothing,\mu}(|\mathcal{T}| = n)} = k\mu_\varnothing(k) \cdot \frac{B_n \mathbb{P}\left(S_{n-1} = -k\right)}{(n-1) B_n \mathrm{BGW}^{\mu_\varnothing,\mu}(|\mathcal{T}| = n)}$$

By (i) and the local limit theorem (5.6), the convergence in (5.8) therefore holds for every k fixed. To obtain a uniform convergence, fix any $\varepsilon > 0$ and let $K \geq 1$ be such that $\sum_{j \geq K} j\mu_\varnothing(j) < \varepsilon$. Then

$$\sup_{1\leq k\leq K} \left| k\mu_\varnothing(k) B_n \mathbb{P}(S_{n-1} = -k) - p_1(0) k\mu_\varnothing(k) \right| \quad \underset{n\to\infty}{\longrightarrow} \quad 0,$$

and, from (5.6),

$$\sup_{k\geq K} \left| k\mu_\varnothing(k) B_n \mathbb{P}(S_{n-1} = -k) - p_1(0) k\mu_\varnothing(k) \right| \leq \varepsilon(2\|p_1\| + o(1)),$$

which establishes (ii).

We finally prove (iii). Let $\mathcal{T}_1, \dots, \mathcal{T}_k$ be k independent Bienaymé–Galton–Watson trees with offspring distribution μ. To simplify notation, set $Z_j = \sum_{i=1}^{j} |\mathcal{T}_i|$ for $1 \leq j \leq k$. Fix $m \geq 0$. Note that, for $n > 4m$,

$$\left\{ \sup_{1\leq i\leq k} |\mathcal{T}_i| = n-m, Z_k = n \right\} = \bigcup_{i=1}^{k} \left\{ |\mathcal{T}_i| = n-m, Z_k = n \right\},$$

where the union is taken on disjoint events. As a consequence, by exchangeability of the vector $(|\mathcal{T}_1|, \ldots, |\mathcal{T}_k|)$ under the conditional distribution $\mathbb{P}(\,\cdot \mid Z_k = n)$, we have

$$\mathbb{P}\left(\sup_{1 \le i \le k} |\mathcal{T}_i| = n - m \;\middle|\; Z_k = n \right) = \sum_{i=1}^{k} \mathbb{P}(|\mathcal{T}_i| = n - m \mid Z_k = n)$$
$$= k \cdot \mathbb{P}(Z_1 = n - m \mid Z_k = n).$$

Next, we have for $n > 4m$,

$$k \cdot \mathbb{P}(Z_1 = n - m \mid Z_k = n) = k \cdot \frac{\mathbb{P}(Z_1 = n - m)\mathbb{P}(Z_{k-1} = m)}{\mathbb{P}(Z_k = n)}.$$

Since Z_k has the same law as the first hitting time of $-k$ by the random walk S, Kemperman's formula (5.7) yields

$$k \cdot \frac{\mathbb{P}(Z_1 = n - m)}{\mathbb{P}(Z_k = n)} = k \cdot \frac{\frac{1}{n-m}\mathbb{P}(S_{n-m} = -1)}{\frac{k}{n}\mathbb{P}(S_n = -k)} \quad \underset{n \to \infty}{\longrightarrow} \quad 1,$$

where the convergence follows from the local limit theorem (5.6) and $B_{n-m}/B_n \to 1$. It follows that

$$\mathbb{P}\left(\sup_{1 \le i \le k} |\mathcal{T}_i| = n - m \;\middle|\; Z_k = n \right) \quad \underset{n \to \infty}{\longrightarrow} \quad \mathbb{P}(Z_{k-1} = m)$$

for every $m \ge 0$, which completes the proof. □

We finally prove Proposition 5.6 and Theorem 5.4.

Proof of Proposition 5.6. Observe that for every $L \ge 1$ fixed, under the conditional distribution $\mathrm{BGW}_n^{\mu_\varnothing,\mu}(\,\cdot \mid N_0 = L)$, the L subtrees of the root are distributed as a forest of L independent Bienaymé–Galton–Watson trees with the same offspring distribution μ, conditioned to have total size $n - 1$. Therefore, with the notation of Lemma 5.7 and its proof, for every $L \ge 1$ and $k \ge 0$, the quantity

$$\mathrm{BGW}_n^{\mu_\varnothing,\mu}(M = n - 1 - k, N_0 = L)$$

is equal to

$$\mathbb{P}(M_{n-1}^{\mu,L} = n - 1 - k) \cdot \mathrm{BGW}_n^{\mu_\varnothing,\mu}(N_0 = L),$$

which, thanks to Lemma 5.7 (ii) and (iii), converges as $n \to \infty$ to

$$\mathbb{P}(Z_{L-1} = k) \cdot \mathbb{P}\left(\mathcal{N} = L\right).$$

This completes the proof. □

Remark 5.8. *In order to prove that under* $\mathrm{BGW}_n^{\mu_\varnothing,\mu}$*,* $M/n \to 1$ *in probability as* $n \to \infty$ *when* $\mu_\varnothing$ *and* μ *are given by* (5.5)*, Marckert & Panholzer crucially use the fact that for every* $k \ge 1$*, conditionally given* $N_0 = k$*, the laws* $\mathrm{BGW}_n^{\mu_\varnothing,\mu}$ *and* BGW_n^{μ} *are the same. However, in the general case,* $\mu_\varnothing$ *and* μ *may have different supports. For this reason, we use an additional idea which consists in estimating the size of the largest tree in a forest of Bienaymé–Galton–Watson trees (Lemma 5.7 (iii)) and which also allows us to obtain a joint convergence in distribution in Proposition 5.6.*

Proof of Theorem 5.4. We see from Proposition 5.6 that under $\mathrm{BGW}_n^{\mu_\varnothing,\mu}$, with probability tending to 1 as $n \to \infty$, the root has one subtree, say τ_n, of size $M_n = n - o(n)$. Furthermore, conditional on M_n, this subtree is distributed as $\mathrm{BGW}_{M_n}^{\mu}$. We conclude from (2.1) that its associated rescaled Łukasiewicz path $(B_{M_n}^{-1}\mathcal{W}_{\lfloor M_n s \rfloor}(\tau_n), s \in [0,1])$ converges in distribution towards to $(X_\alpha^{\mathrm{ex}}(s) : s \in [0,1])$ as $n \to \infty$. Since all the other subtrees have total size $o(n)$ with high probability, their contribution does not affect the limit by standard properties of the Skorokhod topology, and the claim follows. □

Remark 5.9. *As in [30, Section 3.4], under the assumptions of Theorem 5.4, we have in fact the joint convergence in distribution of the rescaled Łukasiewicz path, the height process and the contour process of the trees to $(X^{\mathrm{ex}}_\alpha, H^{\mathrm{ex}}_\alpha, H^{\mathrm{ex}}_\alpha)$. Indeed, more than* (2.1)*, Duquesne [16] obtained this convergence for (non-modified) conditioned Bienaymé–Galton–Watson trees and the above argument extends verbatim. A consequence is for example that the height of the shape of Θ_n is of order n/B_n, which was crucial for the proof of Theorem 1.1.*

5.3 Application to degree-constrained noncrossing trees

Our goal is now to prove Theorem 1.4. Recall that $\mathbb{NC}^{\mathcal{A}}_n$ is the set of all noncrossing trees having n vertices and with degrees only belonging to $\mathcal{A} \subset \mathbb{N}$. Recall also from Section 2 the notation $\mathbb{C}(\tau)$ for a plane tree τ and the bijection Φ_n between $\mathbb{NC}_n$ and $\mathbb{T}^{\mathsf{dec}}_n$. We first introduce some notation. Denote by $\mathbb{T}^{\mathcal{A}}_n$ the set of all plane trees having n vertices and with degrees only belonging to $\mathcal{A}$ and set $\mathbb{T}^{\mathcal{A},\mathsf{dec}}_n = \{(\tau, \mathbf{c}) \in \mathbb{T}^{\mathsf{dec}}_n : \tau \in \mathbb{T}^{\mathcal{A}}_n\}$. It is clear that Φ_n also yields a bijection between $\mathbb{NC}^{\mathcal{A}}_n$ and $\mathbb{T}^{\mathcal{A},\mathsf{dec}}_n$.

Proof of Theorem 1.4. It is clear that $\mathcal{A} \neq \{1\}$, otherwise $\mathbb{NC}^{\mathcal{A}}_n = \varnothing$ for every $n \geq 2$. We first construct a uniform element of $\mathbb{NC}^{\mathcal{A}}_n$ as follows. Set $w(k) = \mathbb{1}_{k\in\mathcal{A}}$. Recalling the definition of Ψ in (5.3), we have

$$\Psi(1) = \frac{\sum_{k\in\mathcal{A},k>1}(k-1)k}{1+\sum_{k\in\mathcal{A},k>1} k}.$$

Then note that

$$\sum_{k\in\mathcal{A},k>1}(k-1)k - \sum_{k\in\mathcal{A},k>1} k = \sum_{k\in\mathcal{A},k>1} k(k-2) > 1,$$

since $\mathcal{A} \neq \{1,2\}$. As a consequence, there exists $b \in (0,1)$ such that (1.2) holds, and we can consider the probability measures $\mu^{\mathcal{A}}$ and $\mu^{\mathcal{A}}_\varnothing$ given by Theorem 5.1. More precisely,

$$\mu^{\mathcal{A}}(k) = a(k+1)b^k \mathbb{1}_{k+1\in\mathcal{A}}, \qquad \mu^{\mathcal{A}}_\varnothing(k) = cb^k \mathbb{1}_{k\in\mathcal{A}},$$

with $a = (\sum_{i+1\in\mathcal{A}}(i+1)b^i)^{-1}$ and $c = (\sum_{i\in\mathcal{A}} b^i)^{-1}$. Let $\mathcal{T}_n$ having the law $\mathrm{BGW}^{\mu^{\mathcal{A}}_\varnothing,\mu^{\mathcal{A}}}_n$ and conditionally given $\mathcal{T}_n$, let $\mathcal{C}(\mathcal{T}_n)$ be a uniform element of $\mathbb{C}(\mathcal{T}_n)$. Finally, set $\Theta^{\mathcal{A}}_n = \Phi_n^{-1}((\mathcal{T}_n, \mathcal{C}(\mathcal{T}_n)))$. Then $\Theta^{\mathcal{A}}_n$ is uniformly distributed in $\mathbb{NC}^{\mathcal{A}}_n$. Indeed, this simply follows from the fact that Φ_n is a bijection between $\mathbb{NC}^{\mathcal{A}}_n$ and $\mathbb{T}^{\mathcal{A},\mathsf{dec}}_n$ and that $\mathcal{T}_n$ is uniformly distributed on $\mathbb{T}^{\mathcal{A}}_n$ by Theorem 5.1.

Now fix $\tau \in \mathbb{T}^{\mathcal{A}}_n$ and $\mathbf{c} \in \mathbb{C}(\tau)$. By the previous discussion, we have

$$\begin{aligned}
\frac{1}{\#\mathbb{NC}^{\mathcal{A}}_n} &= \mathbb{P}\left((\mathcal{T}_n, \mathcal{C}(\mathcal{T}_n)) = (\tau, \mathbf{c})\right) \\
&= \mathbb{P}\left(\mathcal{T}_n = \tau\right) \cdot \frac{1}{\#\mathbb{C}(\tau)} \\
&= \frac{\mathrm{BGW}^{\mu^{\mathcal{A}}_\varnothing,\mu^{\mathcal{A}}}(\mathcal{T} = \tau)}{\mathrm{BGW}^{\mu^{\mathcal{A}}_\varnothing,\mu^{\mathcal{A}}}(|\mathcal{T}| = n)} \cdot \frac{1}{\prod_{u\in\tau\setminus\{\varnothing\}}(k_u+1)}.
\end{aligned}$$

However, by definition,

$$\mathrm{BGW}^{\mu^{\mathcal{A}}_\varnothing,\mu^{\mathcal{A}}}(\mathcal{T} = \tau) = cb^{k_\varnothing} \cdot \prod_{u\in\tau\setminus\{\varnothing\}} a(k_u+1)b^{k_u} = c \cdot (ab)^{n-1} \cdot \prod_{u\in\tau\setminus\{\varnothing\}} (k_u+1).$$

As a consequence $\#\mathbb{NC}^{\mathcal{A}}_n = c^{-1} \cdot (ab)^{-(n-1)} \cdot \mathrm{BGW}^{\mu^{\mathcal{A}}_\varnothing,\mu^{\mathcal{A}}}(|\mathcal{T}| = n)$. Since $\mu^{\mathcal{A}}$ has finite variance, say, $\sigma^2_{\mathcal{A}}$, an adaptation of Lemma 5.7 (i) to the possibly periodic case yields

$$\mathrm{BGW}^{\mu_\varnothing,\mu}(|\mathcal{T}| = n) \quad \underset{n\to\infty}{\sim} \quad \gcd(\mathcal{A}-1) \cdot \frac{1}{\sqrt{4\pi}} \cdot \left(\sum_{k\geq 1} k\mu_\varnothing(k)\right) \cdot \frac{1}{n \cdot \sigma_{\mathcal{A}}\sqrt{n/2}},$$

where n is chosen so that $n \equiv 2 \pmod{\gcd(\mathcal{A}-1)}$. Hence

$$\#\mathbb{NC}_n^{\mathcal{A}} \quad \underset{n\to\infty}{\sim} \quad \gcd(\mathcal{A}-1)\frac{1}{\sqrt{2\pi\sigma_{\mathcal{A}}^2}} \cdot \left(\sum_{k\geq 1} k\mu_{\varnothing}^{\mathcal{A}}(k)\right) \cdot \frac{1}{c} \cdot (ab)^{-(n-1)} \cdot n^{-3/2}.$$

The conclusion follows by definition of a and c. □

6 Iterating laminations, *ad libitum*

Recall that in Section 3.3, we have constructed a triangulation $L(X_\alpha^{\mathrm{ex}}, \boldsymbol{\ell})$ from the stable lamination $L(X_\alpha^{\mathrm{ex}})$ by triangulating each face. In the last part of this paper, we propose other ways to fill-in the faces of stable laminations.

The study of multiple iterated real-valued processes has been triggered by the work of Curien & Konstantopoulos [8], which were motivated by the iteration of two Brownian motions considered by Burdzy [6]. Casse & Marckert [7] recently studied the iteration of reflected Brownian motion as well as the iteration of stable processes. Here we propose to iterate laminations, in a sense that will be made precise in the following lines.

Definition 6.1. *Let V be a face of a lamination of $\overline{\mathbb{D}}$. If V is a triangle, we say that V is decorated by convention. Otherwise, a decoration of V is an order preserving surjection $\phi_V : \mathbb{S}^1 \to \partial V \cap \mathbb{S}^1$. Intuitively, we can view ϕ_V as an inverse of the evolution of the "number" of vertices belonging to $\partial V \cap \mathbb{S}^1$ as one goes around $\mathbb{S}^1$. A decorated lamination is by definition a lamination with a decoration associated with every face.*

Let (V, ϕ_V) be a decorated face and L be a lamination of $\overline{\mathbb{D}}$. If F is a face of L, set

$$V_F = \overline{\bigcup_{[u,v]\in\partial F} [\phi_V(u), \phi_V(v)]}$$

and then

$$V(L) = \overline{V} \cup \bigcup_{F \text{ face of } L} V_F,$$

which is a lamination such that every face of $V(L)$ is the "interior" of V_F for some face F of L. In addition, if L is a decorated *lamination of $\overline{\mathbb{D}}$, $V(L)$ can be seen as a decorated lamination by setting $\phi_{V_F} = \phi_V \circ \phi_F$ for every decorated face (F, ϕ_F) of L.*

Now let L^0 be a decorated lamination, and let $\mathcal{L} = (L_V)_{V \text{ face of } L^0}$ be a collection of laminations indexed by the faces of L^0. Then set

$$\mathcal{L} \circ L^0 = \bigcup_{V \text{ face of } L^0} V(L_V).$$

It is possible to check that $\mathcal{L} \circ L^0$ is a lamination. Intuitively, it is obtained from L^0 by inserting the lamination L_V inside each face V of L^0. In addition, if $\mathcal{L} = (L_V)_{V \text{ face of } L^0}$ is a collection of decorated *laminations, then $\mathcal{L} \circ L^0$ is a decorated lamination.*

An important example is the α-stable lamination $L(X_\alpha^{\mathrm{ex}})$, which can be seen as a decorated lamination: if $\alpha \in (1,2)$ and if u is a jump time of X_α^{ex}, the bijection p_u defined by (3.3) is a decoration of the face coded by u (with the usual identification of $\mathbb{S}^1$ with $[0,1]$). It is actually possible to check that given a stable lamination $\mathbf{L}_\alpha$, we can recover the decorations p_u in a measurable way up to scaling factors by using approximations of local times, but we do not enter into the details since we do not require this fact.

Definition 6.2. *Fix $n \geq 1$ and let $\alpha_1, \ldots, \alpha_{n-1} \in (1,2)$ and $\alpha_n \in (1,2]$. Set $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$. Then $\mathbf{L}_{\boldsymbol{\alpha}}$ is the random decorated lamination defined recursively as follows. First, $\mathbf{L}_{(\alpha_1)}$ is just the α_1-stable lamination (which is a decorated lamination as seen above). Next, conditionally given $\mathbf{L}_{(\alpha_1,\ldots,\alpha_{n-1})}$, let $\mathcal{L}_{\alpha_n} = (\mathbf{L}_{\alpha_n}^F)_{F \text{ face of } \mathbf{L}_{(\alpha_1,\ldots,\alpha_{n-1})}}$ be a* *collection of independent α_n stable laminations indexed by the faces of $\mathbf{L}_{(\alpha_1,\dots,\alpha_{n-1})}$, which we view as decorated as explained above. Then set*

$$\mathbf{L}_{(\alpha_1,\dots,\alpha_{n-1},\alpha_n)} = \mathcal{L}_{\alpha_n} \circ \mathbf{L}_{(\alpha_1,\dots,\alpha_{n-1})}.$$

Intuitively, $\mathbf{L}_{(\alpha_1,\dots,\alpha_{n-1},\alpha_n)}$ is obtained from $\mathbf{L}_{(\alpha_1,\dots,\alpha_{n-1})}$ by inserting independent α_n-stable laminations inside every face of $\mathbf{L}_{(\alpha_1,\dots,\alpha_{n-1})}$.

Note that the lamination $\mathbf{L}_{(\alpha_1,\dots,\alpha_n)}$ is maximal if and only if $\alpha_n = 2$. We believe that the Hausdorff dimension $\dim(\mathbf{L}_{(\alpha_1,\dots,\alpha_n)})$ is almost surely equal to

$$\max\left(2-\frac{1}{\alpha_1}, 1+\frac{1}{\alpha_1}\left(1-\frac{1}{\alpha_2}\right), \dots, 1+\frac{1}{\alpha_1\alpha_2\cdots\alpha_{n-1}}\left(1-\frac{1}{\alpha_n}\right)\right). \tag{6.1}$$

Indeed, the decorations of the faces of $\mathbf{L}_{(\alpha_1,\dots,\alpha_k)}$ are closely related to the iteration of stable subordinators of indices $1/\alpha_1, 1/\alpha_2, \dots, 1/\alpha_k$, and one should be able to adapt [26, Section 5] to show that the the boundaries of the faces of $\mathbf{L}_{(\alpha_1,\dots,\alpha_k)}$ restricted to $\mathbb{S}^1$ have Hausdorff dimension $(\alpha_1\cdots\alpha_k)^{-1}$, so that $\mathbf{L}_{(\alpha_1,\dots,\alpha_k)} \setminus \mathbf{L}_{(\alpha_1,\dots,\alpha_{k-1})}$ has Hausdorff dimension $1+\frac{1}{\alpha_1\alpha_2\cdots\alpha_{k-1}}\left(1-\frac{1}{\alpha_k}\right)$. However, we have not worked out the details.

Question 6.3. If $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}'$, is it true that the laws of $\mathbf{L}_{\boldsymbol{\alpha}}$ and $\mathbf{L}_{\boldsymbol{\alpha}'}$ are singular with respect to each other?

If $(\alpha_1,\alpha_2) \neq (\alpha_1',\alpha_2')$, assuming that (6.1) holds, one can check that $\dim(\mathbf{L}_{(\alpha_1,\alpha_2)}) \neq \dim(\mathbf{L}_{(\alpha_1,\alpha_2)})$. However, still assuming that (6.1) is true, we have $\dim(\mathbf{L}_{(1.1,1.2,2)}) = \dim(\mathbf{L}_{(1.2,1.1,2)})$. Another direction would be to find out what happens to $\mathbf{L}_{(\alpha_1,\dots,\alpha_n)}$ as $n \to \infty$.

We believe that $\mathbf{L}_{(\alpha_1,\dots,\alpha_n)}$ is the scaling limit of a modified version of random dissections considered in [26]: instead of just choosing a random dissection of a large polygon according to critical Boltzmann weights in the domain of attraction of a stable law, first sample a random dissection with such Boltzmann weights in the domain of attraction of an α_1-stable law, then inside each face of the dissection independently sample again a random dissection with Boltzmann weights in the domain of attraction of an α_2-stable law, and so on. Similarly, as in [27], one can consider a random noncrossing partition with Boltzmann weights in the domain of attraction of an α_1-stable law, then partition each block independently at random using a noncrossing partition with Boltzmann weights in the domain of attraction of an α_2-stable law, and so on.

Question 6.4. In a certain sense, the α-stable random lamination can be seen as the dual of the α-stable tree. As was suggested to us by Nicolas Curien, iterating stable laminations can be alternatively seen as iterating stable trees. Roughly speaking, start with a stable tree of index α_1, and then “explode” each branch point by gluing inside a stable tree of index α_2, and so on. What is the Hausdorff dimension of the random tree constructed in this way? What happens as the number of iterations tends to infinity? We hope to investigate this in a future work.

Note that if one starts with a stable tree and explodes each branch point by simply gluing inside a “loop”, one gets the so-called stable looptrees which were introduced and studied in [10]. More generally, one can imagine exploding branchpoints in stable trees and glue inside any compact metric space equipped with a homeomorphism with $[0,1]$.

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