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Pathwise uniqueness for an SPDE with Hölder continuous coefficient driven by α-stable noise*

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Abstract

In this paper we study the pathwise uniqueness of nonnegative solution to the following stochastic partial differential equation with Hölder continuous noise coefficient:

$$\frac{\partial X_t(x)}{\partial t} = \frac{1}{2}\Delta X_t(x) + G(X_t(x)) + H(X_{t-}(x))\dot{L}_t(x), \quad t > 0,\ x \in \mathbb{R},$$

where for $1 < \alpha < 2$ and $0 < \beta < 1$, $\dot{L}$ denotes an α-stable white noise on $\mathbb{R}_+ \times \mathbb{R}$ without negative jumps, G satisfies a condition weaker than Lipschitz and H is nondecreasing and β-Hölder continuous.

For $G \equiv 0$ and $H(x) = x^\beta$, a weak solution to the above stochastic heat equation was constructed in Mytnik (2002) and the pathwise uniqueness of the nonnegative solution was left as an open problem. In this paper we give an affirmative answer to this problem for certain values of α and β. In particular, for $\alpha\beta = 1$ the solution to the above equation is the density of a super-Brownian motion with α-stable branching (see Mytnik (2002)) and our result leads to its pathwise uniqueness for $1 < \alpha < \sqrt{5} - 1$.

The local Hölder continuity of the solution is also obtained in this paper for fixed time $t > 0$.

Keywords: stochastic partial differential equation; stochastic heat equation; stable white noise; pathwise uniqueness; Hölder continuity.
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1 Introduction

1.1 Background and motivation

It was first proved by Konno and Shiga (1988) [14] and by Reimers (1989) [28] that for an arbitrary initial measure the one-dimensional super-Brownian motion with binary

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branching has a jointly continuous density that is a random field $\{X_t(x) : t > 0, x \in \mathbb{R}\}$ satisfying the following continuous-type stochastic partial differential equation (SPDE):

$$\frac{\partial}{\partial t} X_t(x) = \frac{1}{2}\Delta X_t(x) + \sqrt{X_t(x)}\dot{W}_t(x), \quad t > 0,\ x \in \mathbb{R}, \tag{1.1}$$

where Δ denotes the one-dimensional Laplacian operator and $\{\dot{W}_t(x) : t > 0, x \in \mathbb{R}\}$ denotes the derivative of a space-time Gaussian white noise.

The weak uniqueness of solution to the above stochastic heat equation follows from the wellposedness of a martingale problem for the associated super-Brownian motion. The pathwise uniqueness of nonnegative solution to SPDE (1.1) remained open even though it had been studied by many authors. The main difficulty comes from the non-Lipschitz diffusion coefficient. Progresses have been made in considering modifications of the SPDE. When the random field $\{W_t(x) : t > 0, x \in \mathbb{R}\}$ is colored in space and white in time, the pathwise uniqueness of nonnegative solution to the SPDE was obtained by Mytnik *et al.* (2006) [23]. Further work can be found in Rippl and Sturm (2013) [29] and in Neuman (2014) [25]. Xiong (2013) [33] proved the pathwise uniqueness of a SPDE satisfied by "distribution function" of the super-Brownian on $\mathbb{R}$. When $\{W_t(x) : t > 0, x \in \mathbb{R}\}$ is a space-time Gaussian white noise, the solutions are allowed to take both positive and negative values and $\sqrt{X_t(x)}$ is replaced by $\sigma(t, x, X_t(x))$ in SPDE (1.1), the pathwise uniqueness of the solution was proved by Mytnik and Perkins (2011) [22] for $\sigma(\cdot, \cdot, u)$ with Hölder continuity in u of index $\beta_0 > 3/4$. Further work can be found in Mytnik and Neuman (2015) [20]. Recently, some negative results were obtained. When $\sqrt{X_t(x)}$ is replaced by $|X_t(x)|^{\beta_1}$ in the SPDE (1.1), Burdzy *et al.* (2010) [3] showed a non-uniqueness result for $0 < \beta_1 < 1/2$ and Mueller *et al.* (2014) [18] proved a non-uniqueness result for $1/2 \le \beta_1 < 3/4$.

Mytnik (2002) [19] considered the following jump-type SPDE and constructed a weak solution:

$$\frac{\partial X_t(x)}{\partial t} = \frac{1}{2}\Delta X_t(x) + X_{t-}(x)^\beta \dot{L}_t(x), \quad t > 0,\ x \in \mathbb{R}, \tag{1.2}$$

where $0 < \beta < 1$ and for $1 < \alpha < 2$, $\dot{L}$ is a one sided α-stable white noise on $\mathbb{R}_+ \times \mathbb{R}$ without negative jumps. Put $p := \alpha\beta < 2$. The solution to (1.2) with $p = 1$ is the density of a super-Brownian motion with α-stable branching and the weak uniqueness of the solution holds; see [19, Theorem 1.6]. But for the other values of p the uniqueness for (1.2) was left as an open problem; see [19, Remark 1.7]. During the past ten years there have been a number of very interesting results on the solution of SPDE (1.2) for $p = 1$. In particular, Mytnik and Perkins (2003) [21] showed that the solution has a continuous modification at any fixed time. Fleischmann *et al.* (2010) [8] showed that this continuous modification is locally Hölder continuous with index $\eta_c := 2/\alpha - 1$, and Fleischmann *et al.* (2011) [9] further showed that it is Hölder continuous with index $\bar{\eta}_c := (3/\alpha - 1) \wedge 1$ at any given spatial point. A more precise analysis on the regularity of the solution was given in Mytnik and Wachtel (2015) [24]. He *et al.* (2014) [11] showed that solution to another (1.2) related jump-type and distribution-function-valued SPDE is pathwise unique. For $p \neq 1$, the uniqueness of solution (including the weak uniqueness) to SPDE (1.2) and the regularities of the solution $X_t(\cdot)$ at a fixed time t are also left as open problems; see [19, Remark 5.9].

In this paper we want to establish the pathwise uniqueness of nonnegative solution to (1.2). For this purpose we consider a SPDE more general than (1.2):

$$\frac{\partial X_t(x)}{\partial t} = \frac{1}{2}\Delta X_t(x) + G(X_t(x)) + H(X_{t-}(x))\dot{L}_t(x), \quad t > 0,\ x \in \mathbb{R}, \tag{1.3}$$

where G and H are non-negative functions satisfying the following conditions:

(C1) (Linear growth condition) There is a constant C so that

$$0 \le G(x) \le C(x+1), \qquad x \ge 0.$$

(C2) Function G is continuous and there is a non-decreasing and concave function r_0 on $[0,\infty)$ so that $r_0(0)=0$, $\int_{0+} r_0(z)^{-1}dz = \infty$ and

$$\mathrm{sgn}(x-y)(G(x)-G(y)) \le r_0(|x-y|), \qquad x,y \ge 0,$$

where $\mathrm{sgn}(x) := 1_{(0,\infty)}(x) - 1_{(-\infty,0)}(x)$.

(C3) (β-Hölder continuity) There exist constants $0<\beta<1$ and $C>0$ so that

$$|H(x)-H(y)| \le C|x-y|^{\beta}, \qquad x,y \ge 0.$$

(C4) $H(x)$ is a nondecreasing function.

There have been many results on SPDEs driven by stable noises; see e.g. [1, 31, 17, 2, 4]. In [1], the existence and uniqueness were established for solutions of stochastic reaction equations driven by Poisson random measures. The existence of weak solutions and pathwise uniqueness for stochastic evolution equations driven by Lévy processes can be found in [4]. It was also shown in [4] that the pathwise uniqueness holds if the coefficient of the Lévy noise satisfies a condition weaker than Lipschitz continuity but stronger than Hölder continuity. The main results of [2, 31, 1] are the strong existence and uniqueness of solution to (1.3) with general Lévy noise $\dot{L}$ and Lipschitz continuous coefficient H. In this paper we use a Yamada-Watanabe argument that is different from [4], and we consider a stable noise without negative jumps. The stable noise had not been treated in the above mentioned papers although technically it is not hard to extend their results in that direction under the Lipschitz condition on H. One contribution of this paper is that we are able to relax the Lipschitz condition on H since we only need it to be Hölder continuous with its Hölder exponent within a certain range.

The SPDE (1.2) was studied in Mueller (1998) [17] for α-stable noise $\dot{L}$ with $0<\alpha<1$. We also refer to Peszat and Zabczyk (2007) [26] for early work on SPDEs driven by Lévy noises.

Throughout this paper, we always assume that $1<\alpha<2$, $0<\beta<1$ *and the solutions to* (1.2) *and* (1.3) *are nonnegative. Our goal* is to establish the pathwise uniqueness of solution to (1.3) under conditions (C1)–(C4) and further restrictions on α and β. In particular, for $p=1$ we show that the pathwise uniqueness holds for $1<\alpha<\sqrt{5}-1$. To prove the pathwise uniqueness we need to show a local Hölder continuity of the solution at fixed time $t>0$, which also extends the regularity results for super-Brownian motion with α-stable branching obtained in Fleischmann *et al.* (2010) [8].

To continue with the introduction we present some notation. Let $\mathscr{B}(\mathbb{R})$ be the set of Borel functions on $\mathbb{R}$. Let $B(\mathbb{R})$ denote the Banach space of bounded Borel functions on $\mathbb{R}$ furnished with the supremum norm $\|\cdot\|$. We use $C(\mathbb{R})$ to denote the subset of $B(\mathbb{R})$ of bounded continuous functions. For any integer $n\ge 1$ let $C^n(\mathbb{R})$ be the subset of $C(\mathbb{R})$ of functions with bounded continuous derivatives up to the nth order. Let $C_c^n(\mathbb{R})$ be the subset of $C^n(\mathbb{R})$ of functions with compact supports. We use the superscript "+" to denote the subsets of positive elements of the function spaces, e.g., $B(\mathbb{R})^+$. For $f,g\in\mathscr{B}(\mathbb{R})$ write $\langle f,g\rangle := \int_{\mathbb{R}} f(x)g(x)dx$ whenever it exists. Let $M(\mathbb{R})$ be the space of finite Borel measures on $\mathbb{R}$ equipped with the weak convergence topology. For $\mu\in M(\mathbb{R})$ and $f\in B(\mathbb{R})$ we also write $\mu(f) := \int f d\mu$.

Equation (1.3) is a formal SPDE that is understood in the following sense: For any $f\in\mathscr{S}(\mathbb{R})$, the (Schwartz) space of rapidly decreasing and infinitely differentiable functions on $\mathbb{R}$,

$$\langle X_t, f\rangle \;=\; X_0(f) + \frac{1}{2}\int_0^t \langle X_s, f''\rangle ds + \int_0^t ds \int_{\mathbb{R}} G(X_s(x))f(x)dx$$

$$+\int_0^t \int_{\mathbb{R}} H(X_{s-}(x))f(x)L(ds,dx), \qquad t \geq 0, \tag{1.4}$$

where $X_0 \in M(\mathbb{R})$ and $L(ds,dx)$ is the one-sided α-stable white noise on $\mathbb{R}_+ \times \mathbb{R}$ without negative jumps.

Definition 1.1. *SPDE* (1.4) *has a weak solution (X,L) with initial value $X_0 \in M(\mathbb{R})$ if there is a pair (X,L) defined on the same filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbf{P})$ satisfying the following conditions.*

(i) L is an α-stable white noise on $\mathbb{R}_+ \times \mathbb{R}$ without negative jumps.

(ii) The two-parameter nonnegative process $X = \{X_t(x) : t > 0, x \in \mathbb{R}\}$ is progressively measurable on $\mathbb{R}_+ \times \mathbb{R} \times \Omega$, and $\{1_{\{t=0\}}X_0(dx) + 1_{\{t>0\}}X_t(x)dx : t \geq 0\}$ is a $M(\mathbb{R})$-valued cádlág process.

(iii) (X,L) satisfies (1.4)*.*

The definition of this kind of α-stable white noise $L(ds,dx)$ and Definition 1.1 can be found in [19].

1.2 The main results and approaches

Given $t > 0$, we say $\tilde{X}_t$ is a *continuous modification* of X_t if $\tilde{X}_t(x)$ is continuous in x and $\mathbf{P}\{\tilde{X}_t(x) = X_t(x)\} = 1$ for all $x \in \mathbb{R}$. The following theorem shows the local Hölder continuity (in the spatial variable) for the continuous modification of the solution to (1.4).

Theorem 1.2. *(Local Hölder continuity) For any fixed $t > 0$, X_t has a continuous modification $\tilde{X}_t$. Moreover, for each $\eta < \eta_c := \frac{2}{\alpha} - 1$, with probability one the continuous modification $\tilde{X}_t$ is locally Hölder continuous of exponent η, i.e. for any compact set $\mathbb{K} \subset \mathbb{R}$,*

$$\sup_{x,z \in \mathbb{K}, x \neq z} \frac{|\tilde{X}_t(x) - \tilde{X}_t(z)|}{|x-z|^\eta} < \infty, \ \ \mathbf{P}\text{-a.s.} \tag{1.5}$$

In addition, if $\beta < 1/\alpha + (\alpha-1)/2$, then for each $T > 0$ and subsequence $\{n' : n' \geq 1\}$ of $\{n : n \geq 1\}$, we have

$$\liminf_{n' \to \infty} \frac{1}{2^{n'}} \sum_{k=1}^{2^{n'}} \sup_{x,z \in \mathbb{K}, x \neq z} \frac{|\tilde{X}_{n'_k T}(x) - \tilde{X}_{n'_k T}(z)|}{|x-z|^\eta} < \infty, \ \ \mathbf{P}\text{-a.s.}, \tag{1.6}$$

where ${n'}_k := \frac{k}{2^{n'}}$ for $1 \leq k \leq 2^{n'}$.

Remark 1.3. Theorem 1.2 gives an answer to [8, Conjecture 1.5] when the fractional Laplacian operator Δ_α is replaced by the Laplacian operator Δ and the function g there is replaced by H. It also gives an answer to the open problem of [19, Remark 5.9].

Assumption 1.4. *For $p := \alpha\beta > 1$, there is a constant $q > \frac{3p}{3-\alpha}$ so that for any weak solution (X,L) to* (1.4) *it holds that*

$$\mathbf{P}\Big\{ \int_0^t ds \int_{\mathbb{R}} X_s(x)^q dx < \infty \text{ for all } t > 0 \Big\} = 1.$$

Theorem 1.5. *(Pathwise uniqueness) Suppose that conditions (C1)–(C4) hold, and that*

$$2(\alpha-1)/(2-\alpha)^2 < \beta < 1/\alpha + (\alpha-1)/2. \tag{1.7}$$

For $p > 1$, we further assume that Assumption 1.4 holds. Then given any two weak solutions (X,L) and (Y,L), with $X_0 = Y_0 \in M(\mathbb{R})$, to equation (1.4) *defined on the same filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbf{P})$, we have with probability one, for each $t > 0$*

$$X_t(x) = Y_t(x), \qquad \lambda_0\text{-a.e. } x, \tag{1.8}$$

where λ_0 denotes the Lebesgue measure on $\mathbb{R}$.

Remark 1.6. (i) Since we assume that $\beta \in (0,1)$, it follows from the first inequality of (1.7) that the above theorem holds for $\alpha \in (1, 3-\sqrt{3})$.

(ii) Theorem 1.5 gives an affirmative answers to the open problem of [19, Remark 1.7] for α and β satisfying (1.7).

(iii) For $p=1$, inequality (1.7) is equivalent to $1<\alpha<\sqrt{5}-1$. So, for super-Brownian motion, i.e. $G \equiv 0$, $H(x)=x^\beta$ and $p=1$, Theorem 1.5 also leads to the pathwise uniqueness of (1.2) for $1<\alpha<\sqrt{5}-1$, which is a key result of this paper.

(iv) We stress here that in Theorem 1.5, Assumption 1.4 is not needed if $0<p=\alpha\beta\le 1$.

(v) If $G\equiv 0$ and $H(x)=x^\beta$, then SPDE (1.4) has a weak solution satisfying Assumption 1.4 by [19, Proposition 5.1] and the proof of [19, Theorem 1.5].

(vi) The non-negativity assumption on H and G is not necessary for the proof of uniqueness. But it makes the proof a bit simpler and may be needed to show the existence of solution to the SPDE.

To prove the uniqueness we need a local Hölder continuity of the solution at fixed time $t>0$ (Theorem 1.2). For super-Brownian motion, the proof for the local Hölder continuity of $X_t(x)$ is based on the following equation from Fleischmann *et al.* (2010) [8]:

$$\langle X_t, f\rangle = X_0(f) + \frac{1}{2}\int_0^t \langle X_s, f''\rangle ds + \int_0^t\int_0^\infty\int_{\mathbb{R}} f(x)zM(ds,dz,dx), \tag{1.9}$$

where $M(ds,dz,dx)$ is the compensated measure of an optional random measure on $(0,\infty)^2\times\mathbb{R}$ with compensator $\hat{M}(ds,dz,dx) := dsm_0(dz)X_s(x)dx$ for measure $m_0(dz) := c_0 z^{-1-\alpha}1_{\{z>0\}}dz$ with $c_0 := \alpha(\alpha-1)/\Gamma(2-\alpha)$ and Gamma function Γ. Equation (1.9) is established for super-Brownian motion. But for the other cases, the solution to (1.4) may not be a density of super-Brownian motion and we can not obtain the equivalent of equation (1.9). So, inspired by Dawson and Li (2006, 2012)[6, 7], we reformulate (1.4) as the following SPDE in Proposition 2.1:

$$\begin{aligned}\langle X_t, f\rangle &= X_0(f) + \frac{1}{2}\int_0^t \langle X_s, f''\rangle ds + \int_0^t ds\int_{\mathbb{R}} G(X_s(x))f(x)dx \\ &\quad + \int_0^t\int_0^\infty\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha} zf(u)\tilde{N}_0(ds,dz,du,dv),\end{aligned} \tag{1.10}$$

where $f\in\mathscr{S}(\mathbb{R})$ and $\tilde{N}_0(ds,dz,du,dv)$ is a compensated Poisson random measure on $(0,\infty)^2\times\mathbb{R}\times(0,\infty)$ with intensity $dsm_0(dz)dudv$. By modifying the proof of [8, Theroem 1.2(a)] and using (1.10), we can obtain Theorem 1.2. Notice that $\eta_c\uparrow 1$ as $\alpha\downarrow 1$, which is quite different from that of a continuous-type SPDE whose local Hölder index is typically smaller than $\frac{1}{2}$. This observation is key to proving the pathwise uniqueness.

We now outline our approach. By an infinite-dimensional version of the Yamada-Watanabe argument for ordinary stochastic differential equations (see Mytnik *et al.* (2006)), showing the pathwise uniqueness is reduced to showing that the analogue of the local time term is zero; see the proofs of Theorem 1.5 and Lemma 4.3. That is to show that

$$\mathbf{E}\{I_5^{m,n}(t\wedge\tau_k)\}\to 0 \tag{1.11}$$

as $m,n\to\infty$, where

$$I_5^{m,n}(t\wedge\tau_k) := \int_0^{t\wedge\tau_k} ds\int_0^\infty m_0(dz)\int_{\mathbb{R}}\langle D_n(\langle U_s,\Phi_\cdot^m\rangle, zV_s(y)\Phi_\cdot^m(y)),\Psi_s\rangle dy$$

for $\tau_k := \gamma_k\wedge\sigma_k$, and γ_k and σ_k are two stopping times to be defined later in (4.3) and (4.4), respectively. Here U_s is the difference of two weak solutions to (1.4), V_s denotes the difference of compositions of these two solutions into function H, respectively, Ψ is a test function, Φ_x^m is a mollifier, $D_n(y,z) := \phi_n(y+z)-\phi_n(y)-z\phi_n'(y)$, ϕ_n ($supp(\phi_n'') \subset (a_n, a_{n-1})$ and $a_n \downarrow 0$) is the function satisfying $\phi_n(x) \to |x|$ from Yamada-Watanabe's proof. To prove (1.11), we divide $\mathbf{E}\{I_5^{m,n}(t \wedge \tau_k)\}$ into two terms

$$I_{5,1}^{m,n,k,i}(t) := \mathbf{E}\Big\{ \int_0^{t\wedge\gamma_k} ds \int_0^{1/i} m_0(dz) \int_{\mathbb{R}} \langle D_n(\langle U_s, \Phi_\cdot^m\rangle, zV_s(y)\Phi_\cdot^m(y)), \Psi_s\rangle dy\Big\}$$

and

$$I_{5,2}^{m,n,k,i}(t) := \mathbf{E}\Big\{ \int_0^{t\wedge\sigma_k} ds \int_{1/i}^{\infty} m_0(dz) \int_{\mathbb{R}} \langle D_n(\langle U_s, \Phi_\cdot^m\rangle, zV_s(y)\Phi_\cdot^m(y)), \Psi_s\rangle dy\Big\},$$

so that $\mathbf{E}\{I_5^{m,n}(t\wedge\sigma_k)\} \le I_{5,1}^{m,n,k,i}(t) + I_{5,2}^{m,n,k,i}(t)$ for all $i \ge 1$.

Using the fact $\phi_n'' \le 2(na_n)^{-1}$ (see its definition at the beginning of Subsection 4.1), we can show that $I_{5,1}^{m,n,k,i}(t)$ goes to zero as $m,n,i \to \infty$ in a dependent way (see Lemma 4.4). So, the difficult part is to show that $I_{5,2}^{m,n,k,i}(t)$ goes to zero as $m,n,i \to \infty$. To this end we use the local Hölder continuity of the solutions and the monotonicity of H to estimate $I_{5,2}^{m,n,k,i}(t)$, which is elaborated in the following. The proof is inspired by an argument of Mytnik and Perkins (2011) [22], for fixed s, m and x, denote by $x_{s,m} \in [-1,1]$ a value satisfying

$$|\tilde{V}_s(x - \frac{x_{s,m}}{m})| = \inf_{y\in[-1,1]} |\tilde{V}_s(x-\frac{y}{m})|,$$

where $\tilde{V}_s$ and $\tilde{U}_s$ are the continuous modifications of V_s and U_s, respectively. The key to proving that $I_{5,2}^{m,n,k,i}(t)$ goes to zero is to split it into two terms again, where one term is bounded from above by

$$\begin{aligned} I_{5,2,1}^{m,n,k,i}(t) \;:=\; & \mathbf{E}\Big\{ \int_0^{t\wedge\sigma_k} ds \int_{-K}^{K} \Psi_s(x)dx \int_{1/i}^{\infty} zm_0(dz) \int_{-1}^{1} \Phi(y)dy \\ & \times \int_0^1 \Big|\tilde{D}_n\big(\langle \tilde{U}_s, \Phi_x^m\rangle, mzh\tilde{V}_s(x-\frac{y}{m})\big)[V_s(x-\frac{y}{m}) - \tilde{V}_s(x-\frac{x_{s,m}}{m})]\Big| dh\Big\}, \end{aligned}$$

and the other term is bounded from above by

$$\begin{aligned} I_{5,2,2}^{m,n,i}(t) \;:=\; & \mathbf{E}\Big\{ \int_0^{t} ds \int_{-K}^{K} \Psi_s(x)dx \int_{1/i}^{\infty} zm_0(dz) \int_{-1}^{1} \Phi(y)dy \\ & \times \int_0^1 \tilde{D}_n\big(\langle \tilde{U}_s, \Phi_x^m\rangle, mzh\tilde{V}_s(x-\frac{y}{m})\big)|\tilde{V}_s(x-\frac{x_{s,m}}{m})| 1_{\{\tilde{V}_s(x-\frac{x_{s,m}}{m})\neq 0\}} dh\Big\}, \end{aligned}$$

where $\tilde{D}_n(y,z) = \phi_n'(y+z) - \phi_n'(y)$.

The local Hölder continuity of the solutions is used to estimate $I_{5,2,1}^{m,n,k,i}(t)$ and the nondecreasingness of H is used to estimate $I_{5,2,2}^{m,n,i}(t)$. Observe that for fixed s, the continuous modification $\tilde{X}_s$ of the weak solution to (1.4) satisfies

$$\begin{aligned} & \sup_{|x|\le K, |y|\vee|v|\le 1} |\tilde{X}_s(x-\frac{y}{m}) - \tilde{X}_s(x-\frac{v}{m})|^\beta \\ & \qquad \le (2/m)^{\eta\beta} \sup_{|x|\le K, |y|\vee|v|\le 1, y\neq v} \frac{|\tilde{X}_s(x-\frac{y}{m}) - \tilde{X}_s(x-\frac{v}{m})|^\beta}{|y/m - v/m|^{\eta\beta}}, \end{aligned} \tag{1.12}$$

where $K > 0$ and $0 < \eta < \eta_c = 2/\alpha - 1$. So, it is natural to apply the Hölder continuity of $x \mapsto \tilde{X}_s(x)$ to find a collection of suitable stopping times $(\sigma_k)_{k\ge1}$ so that $\lim_{k\to\infty}\sigma_k = \infty$ almost surely, and using the β-Hölder continuity of H (condition (C3)), the term $I_{5,2,1}^{m,n,k,i}(t)$

can be bounded by $m^{-\eta\beta}i^{\alpha-1}$ which goes to 0 as m,n,i jointly go to infinity in a certain way. It is hard to show that the supremum or integral with respect to $s\in(0,T]$ on the right hand side of (1.12) is finite. To this end, the time σ_k is chosen so that a Riemann type "integral" of the right hand side of (1.12) over $s\in[0,\sigma_k]$ is finite. One can find the details in the Step 1 of the proof of Lemma 4.5.

Concerning the second term $I_{5,2,2}^{m,n,i}(t)$, if $\tilde{V}_s(x-\frac{x_{s,m}}{m})\neq 0$, then the function $[-1,1]\ni y\mapsto \tilde{V}_s(x-\frac{y}{m})$ is bounded away from zero. The nondecreasing condition (C4) on H ensures that $\tilde{V}_s(x-\frac{y}{m})$ and $\tilde{U}_s(x-\frac{y}{m})$ always have the same sign, which means $\tilde{D}_n\big(\langle\tilde{U}_s,\Phi_x^m\rangle, mzh\tilde{V}_s(x-\frac{y}{m})\big)=0$ for $|\langle\tilde{U}_s,\Phi_x^m\rangle|\geq a_{n-1}$ for all $z,h\geq 0$ (here we use the fact $supp(\phi_n'')\subset(a_n,a_{n-1})$ and $a_n\downarrow 0$), and this condition is only needed here. Thus by the β-Hölder continuity of H (condition (C3)),

$$|\tilde{V}_s(x-\frac{x_{s,m}}{m})|\leq C|\tilde{U}_s(x-\frac{x_{s,m}}{m})|^\beta\leq C|\langle\tilde{U}_s,\Phi_x^m\rangle|\leq Ca_{n-1}^\beta,$$

which implies that $I_{5,2,2}^{m,n,i}(t)$ also converges to zero as $m,n,i\to\infty$ under certain conditions of α and β; see the details in the Step 2 of the proof of Lemma 4.5.

1.3 Comments on the main results with general G and H

The main results, Theorems 1.2 and 1.5, also hold if functions $G(x)$ and $H(x)$ are replaced by $G(t,x,y)$ and $H(t,x,y)$, respectively, as in [22, 23]. More specifically, we can consider an SPDE more general than (1.3):

$$\frac{\partial X_t(x)}{\partial t}=\frac{1}{2}\Delta X_t(x)+G(t,x,X_t(x))+H(t,x,X_{t-}(x))\dot{L}_t(x),\quad t>0,\ x\in\mathbb{R},\tag{1.13}$$

where G and H satisfy the following growth and continuity conditions:

(1) The mapping $(t,x,y)\mapsto(G(t,x,y),H(t,x,y))$ is continuous and there is a constant C so that

$$|G(t,x,y)|+|H(t,x,y)|\leq C(1+y),\qquad t,y\geq 0,\ x\in\mathbb{R}.$$

(2) Let r_0 be the concave function defined in condition (C2). Then

$$\operatorname{sgn}(y_1-y_2)(G(t,x,y_1)-G(t,x,y_2))\leq r_0(|y_1-y_2|),\quad t,y_1,y_2\geq 0,\ x\in\mathbb{R}.$$

(3) (β-Hölder continuity) There exist constants $1<\beta<1$ and $C>0$ so that

$$|H(t,x,y_1)-H(t,x,y_2)|\leq C|y_1-y_2|^\beta,\qquad t,y_1,y_2\geq 0,\ x\in\mathbb{R}.$$

(4) For fixed $t\geq 0$ and $x\in\mathbb{R}$, $H(t,x,y)$ is nondecreasing in y.

Under the above conditions, by the same arguments in this paper, we can show that the results of Theorems 1.2 and 1.5 also hold for SPDE (1.13). For simplicity we only study the simplified version (1.3) in this paper.

The paper is organized as follows. In Section 2 we first present some properties of the weak solution to equation (1.4). The proofs of Theorems 1.2 and 1.5 are established in Sections 3 and 4, respectively. In Section 5, the proofs of Proposition 2.2 and Lemma 2.4 are presented.

Notation: Throughout this paper, we adopt the conventions

$$\int_x^y=\int_{(x,y]}\ \text{ and }\ \int_x^\infty=\int_{(x,\infty)}$$

for any $y\geq x\geq 0$. Let C denote a positive constant whose value might change from line to line. We write C_ε or C'_ε if the constant depends on another value $\varepsilon\geq 0$. Write $\mathbb{Q}$

for the set of rational numbers. We sometimes write $\mathbb{R}_+$ for $[0,\infty)$. Let $(P_t)_{t\geq 0}$ denote the transition semigroup of a one-dimensional Brownian motion. For $t>0$ and $x\in\mathbb{R}$ write $p_t(x) := (2\pi t)^{-\frac{1}{2}}\exp\{-x^2/(2t)\}$. We always use $N_0(ds,dz,du,dv)$ to denote the Poisson random measure corresponding to the compensated Poisson random measure $\tilde{N}_0(ds,dz,du,dv)$.

2 Properties of the weak solution

In this section we establish some properties of the weak solution to (1.4), which will be used in the next two sections. Recall the measure $m_0(dz) = c_0 z^{-1-\alpha}1_{\{z>0\}}dz$ for $c_0 = \alpha(\alpha-1)/\Gamma(2-\alpha)$ and Gamma function Γ. By the proof of Theorem 1.1(a) of Mytnik and Perkins (2003), there is a Poisson random measure $N(ds,dz,dx)$ on $(0,\infty)^2\times\mathbb{R}$ with intensity $dsm_0(dz)dx$ so that

$$L(ds,dx) = \int_0^\infty z\tilde{N}(ds,dz,dx), \tag{2.1}$$

where $\tilde{N}(ds,dz,du)$ is the compensated Poisson random measure for $N(ds,dz,dx)$. Thus, if $\{X_t : t\geq 0\}$ is a weak solution of (1.4), then for each $f\in\mathscr{S}(\mathbb{R})$ we have

$$\begin{aligned}\langle X_t,f\rangle &= X_0(f) + \frac{1}{2}\int_0^t\langle X_s,f''\rangle ds + \int_0^t ds\int_{\mathbb{R}} G(X_s(x))f(x)dx\\ &\quad + \int_0^t\int_0^\infty\int_{\mathbb{R}} H(X_{s-}(x))f(x)z\tilde{N}(ds,dz,dx), \quad t>0,\end{aligned} \tag{2.2}$$

which will be used to obtain (1.10). For this we need Assumption 1.4 on the weak solution of (1.4) for the case $p>1$. For $0<p\leq 1$, by Definition 1.1 and the Hölder inequality it is easy to check that the Itô integrals in (1.4) and (2.2) are well defined. For $1<p<2$, under Assumption 1.4 and by a similar argument it is easy to check that the Itô integrals in (1.4) and (2.2) are also well defined; see the details in Lemma 2.3. By [19, Proposition 5.1] and the proof of [19, Theorem 1.5], for $G\equiv 0$ and $H(x)=x^\beta$, the solution to (1.4) exists and satisfies Assumption 1.4. In the following proposition we always assume that conditions (C1) and (C3) are satisfied and Assumption 1.4 holds for the weak solution (X,L) to (1.4).

Proposition 2.1. *(i) If (X,L) is a weak solution to (1.4), then there is, on an enlarged probability space, a compensated Poisson random measure $\tilde{N}_0(ds,dz,du,dv)$ on $(0,\infty)^2\times\mathbb{R}\times(0,\infty)$ with intensity $dsm_0(dz)dudv$ so that (1.10) holds. (ii) Conversely, if X satisfies (1.10), then there is an α-stable white noise $L(ds,dx)$ on $\mathbb{R}_+\times\mathbb{R}$ without negative jumps so that (1.4) holds.*

Proof. (i) Suppose that (X,L) is a weak solution of (1.4). Then by the argument at the beginning of this section, (2.2) holds. Define a predictable $(0,\infty)\times(\mathbb{R}\cup\{\infty\})$-valued process $\theta(s,z,u,v)$ by $\theta(s,z,u,v) = (\theta_1(s,z,u),\theta_2(s,u,v))$ with

$$\theta_1(s,z,u) := \frac{z}{H(X_{s-}(u))}1_{\{H(X_{s-}(u))\neq 0\}} + z1_{\{H(X_{s-}(u))=0\}}$$

and

$$\theta_2(s,u,v) := \tilde{\theta}(s,u,v)1_{\{H(X_{s-}(u))\neq 0\}} + \bar{\theta}(u,v)1_{\{H(X_{s-}(u))=0\}},$$

where

$$\tilde{\theta}(s,u,v) := \begin{cases} u, & v\leq H(X_{s-}(u))^\alpha\\ \infty, & v>H(X_{s-}(u))^\alpha\end{cases}, \qquad \bar{\theta}(u,v) := \begin{cases} u, & v\in(0,1)\\ \infty, & v\in(0,1)^c\end{cases}$$

and we use the convention that $0 \cdot \infty = 0$. Then for all $B \in \mathscr{B}(0,\infty)$ and $a \le b \in \mathbb{R}$,

$$\begin{aligned}
& 1_{B\times(a,b]}(\theta(s,z,u,v)) \\
= \ & 1_{B\times(a,b]}(\theta_1(s,z,u),\theta_2(s,u,v)) \\
= \ & 1_{\{H(X_{s-}(u))\neq 0, u\in(a,b], v\le H(X_{s-}(u))^\alpha\}} 1_B\Big(\frac{z}{H(X_{s-}(u))}\Big) \\
& + 1_{\{H(X_{s-}(u))=0, z\in B, u\in(a,b], v\in(0,1)\}}.
\end{aligned}$$

Moreover, recalling $m_0(dz) = c_0 z^{-1-\alpha} 1_{\{z>0\}} dz$, by a change of variable it is easy to see that

$$\begin{aligned}
& \int_0^\infty \int_{\mathbb{R}} \int_0^\infty 1_{B\times(a,b]}(\theta(s,z,u,v)) m_0(dz) du dv \\
& \quad = \int_0^\infty m_0(dz) \int_a^b du \int_0^{H(X_{s-}(u))^\alpha} 1_{\{H(X_{s-}(u))\neq 0\}} 1_B\Big(\frac{z}{H(X_{s-}(u))}\Big) dv \\
& \qquad + \int_0^\infty \int_a^b 1_{\{H(X_{s-}(u))=0\}} 1_B(z) m_0(dz) du \\
& \quad = \int_0^\infty \int_{\mathbb{R}} 1_{B\times(a,b]}(z,u) m_0(dz) du.
\end{aligned}$$

Then by [12, p.93], on an extension of the probability space, there exists a Poisson random measure $N_0(ds,dz,du,dv)$ on $(0,\infty)^2 \times \mathbb{R} \times (0,\infty)$ with intensity $ds m_0(dz) du dv$ so that

$$N((0,t]\times B\times(a,b]) = \int_0^t \int_0^\infty \int_{\mathbb{R}} \int_0^\infty 1_{B\times(a,b]}(\theta(s,z,u,v)) N_0(ds,dz,du,dv).$$

Let $\tilde{N}_0(ds,dz,du,dv) = N_0(ds,dz,du,dv) - ds m_0(dz) du dv$. Then by (2.1) it is easy to see that for each $f \in \mathscr{S}(\mathbb{R})$,

$$\int_0^t \int_{\mathbb{R}} H(X_{s-}(u)) f(u) L(ds,du) = \int_0^t \int_0^\infty \int_{\mathbb{R}} \int_0^{H(X_{s-}(u))^\alpha} z f(u) \tilde{N}_0(ds,dz,du,dv).$$

(ii) The proof is essentially the same as that of [15, Theorem 9.32]. Suppose that $\{X_t : t > 0, x \in \mathbb{R}\}$ satisfies (1.10). Define the random measure $N(ds,dz,du)$ on $(0,\infty)^3$ by

$$\begin{aligned}
& N((0,t]\times B\times(a,b]) \\
& \quad := \int_0^t \int_0^\infty \int_a^b \int_0^{H(X_{s-}(u))^\alpha} 1_{\{H(X_{s-}(u))\neq 0\}} 1_B\Big(\frac{z}{H(X_{s-}(u))}\Big) N_0(ds,dz,du,dv) \\
& \qquad + \int_0^t \int_0^\infty \int_a^b \int_0^1 1_{\{H(X_{s-}(u))=0\}} 1_B(z) N_0(ds,dz,du,dv).
\end{aligned}$$

It is easy to see that $N(ds,dz,du)$ has a predictable compensator

$$\begin{aligned}
& \hat{N}((0,t]\times B\times(a,b]) \\
= \ & \int_0^t \int_0^\infty \int_a^b \int_0^{H(X_{s-}(u))^\alpha} 1_{\{H(X_{s-}(u))\neq 0\}} 1_B\Big(\frac{z}{H(X_{s-}(u))}\Big) ds m_0(dz) du dv \\
& + \int_0^t \int_0^\infty \int_a^b \int_0^1 1_{\{H(X_{s-}(u))=0\}} 1_B(z) ds m_0(dz) du dv \\
= \ & \int_0^t \int_0^\infty \int_a^b 1_B(z) ds m_0(dz) du.
\end{aligned}$$

Then $N(ds,dz,du)$ is a Poisson random measure with intensity $ds m_0(dz) du$; see [13, Theorems II.1.8 and II.4.8]. Define the α-stable white noise L by

$$L_t(a,b] = \int_0^t \int_0^\infty \int_a^b z \tilde{N}(ds,dz,du).$$

We then have

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}} H(X_{s-}(u))f(u)L(ds,du)\\ =\;&\int_0^t\int_0^\infty\int_{\mathbb{R}} H(X_{s-}(u))f(u)z\tilde{N}(ds,dz,du)\\ =\;&\int_0^t\int_0^\infty\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha} zf(u)\tilde{N}_0(ds,dz,du,dv)\end{aligned}$$

for each $f\in\mathscr{S}(\mathbb{R})$. (X,L) is thus a weak solution to (1.4). □

In the rest of this section, we always assume that conditions (C1) and (C3) are satisfied and (X,L) is a weak solution to (1.4) with deterministic initial value $X_0\in M(\mathbb{R})$ and with Assumption 1.4 satisfied. Then it follows from Proposition 2.1, $\{X_t(x):t>0,x\in\mathbb{R}\}$ satisfies (1.10). Recall that $(P_t)_{t\geq 0}$ is the transition semigroup of a one-dimensional Brownian motion and $p_t(x)=(2\pi t)^{-\frac{1}{2}}\exp\{-x^2/(2t)\}$ for $t>0$ and $x\in\mathbb{R}$.

Proposition 2.2. *For any $t>0$ and $f\in B(\mathbb{R})$ satisfying $\lambda_0(|f|)<\infty$ we have*

$$\begin{aligned}\langle X_t,f\rangle &= X_0(P_tf)+\int_0^t ds\int_{\mathbb{R}} G(X_s(x))P_{t-s}f(x)dx\\ &\quad+\int_0^t\int_0^\infty\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha} zP_{t-s}f(u)\tilde{N}_0(ds,dz,du,dv),\quad \mathbf{P}\text{-a.s.}\end{aligned}\tag{2.3}$$

Moreover, for each $t>0$, we have $\mathbf{P}$-a.s., λ_0-a.e. x,

$$\begin{aligned}X_t(x) &= \int_{\mathbb{R}} p_t(x-z)X_0(dz)+\int_0^t ds\int_{\mathbb{R}} p_{t-s}(x-z)G(X_s(z))dz\\ &\quad+\int_0^t\int_0^\infty\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha} zp_{t-s}(x-u)\tilde{N}_0(ds,dz,du,dv),\end{aligned}\tag{2.4}$$

The proof is given in the Appendix.

We refer to [26, Theorem 8.23(i)(p. 129)] and [12, p. 62] for the stochastic integration with respect to a Poisson random measure. For $k>0$ let $\tilde{\tau}_k$ be a stopping time defined by

$$\tilde{\tau}_k:=\inf\{t:F(t)>k\}\tag{2.5}$$

with the convention $\inf\emptyset=\infty$, where $F(t):=(\int_0^t ds\int_{\mathbb{R}} X_s(x)^q dx)\vee\langle X_t,1\rangle$ for the case $p>1$ and $F(t):=\langle X_t,1\rangle$ for the case $p\leq 1$. Then it follows from Definition 1.1 and Assumption 2.1 that

$$\lim_{k\to\infty}\tilde{\tau}_k=\infty,\qquad \mathbf{P}\text{-a.s.}\tag{2.6}$$

The following lemma says that Assumption 1.4 also assures that the Itô integrals in (2.3) and (2.4) are well defined.

Lemma 2.3. *If Assumption 1.4 holds, the Itô integrals in* (2.3) *and* (2.4) *are well defined.*

Proof. Since the reasoning for (2.3) is similar, we only show that (2.4) is well defined. We first consider the case $0<p\leq 1$. Since $u^p\leq u+1$ for $u\geq 0$, then for each $1\leq\bar{\alpha}<2$ and any $x\in\mathbb{R}$,

$$\begin{aligned}&\mathbf{E}\Big\{\int_0^{t\wedge\tilde{\tau}_k} ds\int_{\mathbb{R}} X_s(u)^p p_{t-s}(x-u)^{\bar{\alpha}}du\Big\}\\ &\quad\leq\mathbf{E}\Big\{\int_0^t[2\pi(t-s)]^{-\frac{\bar{\alpha}-1}{2}}ds\int_{\mathbb{R}}[X_s(u)+1]p_{t-s}(x-u)1_{\{s\leq\tilde{\tau}_k\}}du\Big\}\\ &\quad\leq\int_0^t[2\pi(t-s)]^{-\frac{\bar{\alpha}-1}{2}}\mathbf{E}\Big\{1+[2\pi(t-s)]^{-1/2}\langle X_s,1\rangle 1_{\{s\leq\tilde{\tau}_k\}}\Big\}ds\end{aligned}$$

$$\leq \int_0^t [2\pi(t-s)]^{-\frac{\bar{\alpha}-1}{2}} \Big\{1 + [2\pi(t-s)]^{-1/2} k\Big\} ds < \infty. \tag{2.7}$$

Therefore, by condition (C3), for any $k \geq 1$,

$$\begin{aligned}\mathbf{E}\Big\{ &\int_0^{t\wedge\tilde{\tau}_k} ds \int_1^\infty z m_0(dz) \int_{\mathbb{R}} du \int_0^{H(X_{s-}(u))^\alpha} p_{t-s}(x-u) dv\Big\} \\ &\leq C\,\mathbf{E}\Big\{ \int_0^{t\wedge\tilde{\tau}_k} ds \int_1^\infty z m_0(dz) \int_{\mathbb{R}} [1 + X_s(u)^p] p_{t-s}(x-u) du\Big\} < \infty\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}\Big\{ &\int_0^{t\wedge\tilde{\tau}_k} ds \int_0^1 z^{\bar{\bar{\alpha}}} m_0(dz) \int_{\mathbb{R}} du \int_0^{H(X_{s-}(u))^\alpha} p_{t-s}(x-u)^{\bar{\bar{\alpha}}} dv\Big\} \\ &\leq C\,\mathbf{E}\Big\{ \int_0^{t\wedge\tilde{\tau}_k} ds \int_0^1 z^{\bar{\bar{\alpha}}} m_0(dz) \int_{\mathbb{R}} [1 + X_s(u)^p] p_{t-s}(x-u)^{\bar{\bar{\alpha}}} du\Big\} < \infty\end{aligned}$$

for $\alpha < \bar{\bar{\alpha}} < 2$, which ensures that the stochastic integral

$$\int_0^{t\wedge\tilde{\tau}_k} \int_0^1 \int_{\mathbb{R}} \int_0^{H(X_{s-}(u))^\alpha} z p_{t-s}(x-u) \tilde{N}_0(ds, dz, du, dv)$$

and

$$\begin{aligned}&\int_0^{t\wedge\tilde{\tau}_k} \int_1^\infty \int_{\mathbb{R}} \int_0^{H(X_{s-}(u))^\alpha} z p_{t-s}(x-u) \tilde{N}_0(ds, dz, du, dv) \\ &\quad = \int_0^{t\wedge\tilde{\tau}_k} \int_1^\infty \int_{\mathbb{R}} \int_0^{H(X_{s-}(u))^\alpha} z p_{t-s}(x-u) N_0(ds, dz, du, dv) \\ &\qquad - \int_0^{t\wedge\tilde{\tau}_k} ds \int_1^\infty m_0(dz) \int_{\mathbb{R}} du \int_0^{H(X_{s-}(u))^\alpha} z p_{t-s}(x-u) dv\end{aligned}$$

are well defined by [26, Theorem 8.23(i)(p. 129)] and [12, p. 62], respectively. Then one can see that the stochastic integral (2.4) is well defined.

In the following we consider the case $p > 1$. Observe that $\frac{q}{p} > \frac{3}{3-\alpha} > \frac{3}{2}$, which implies $\frac{1}{2}(\frac{q}{q-p} - 1) < 1$. Thus by the Hölder inequality,

$$\begin{aligned}\mathbf{E}\Big\{ &\int_0^{t\wedge\tilde{\tau}_k} ds \int_{\mathbb{R}} X_s(u)^p p_{t-s}(x-u) du\Big\} \\ &\leq \Big\{\mathbf{E}\Big[\int_0^{t\wedge\tilde{\tau}_k} ds \int_{\mathbb{R}} X_s(u)^q du\Big]\Big\}^{\frac{p}{q}} \Big\{ \int_0^t ds \int_{\mathbb{R}} p_{t-s}(x-u)^{\frac{q}{q-p}} du\Big\}^{1-\frac{p}{q}} \\ &\leq k^{\frac{p}{q}} \Big\{ \int_0^t [2\pi(t-s)]^{-\frac{1}{2}(\frac{q}{q-p}-1)} ds \int_{\mathbb{R}} p_{t-s}(x-u) du\Big\}^{1-\frac{p}{q}} \\ &= k^{\frac{p}{q}} \Big\{ \int_0^t [2\pi(t-s)]^{-\frac{1}{2}(\frac{q}{q-p}-1)} ds\Big\}^{1-\frac{p}{q}} < \infty.\end{aligned} \tag{2.8}$$

Observe that $q > 3p/(3-\alpha)$ implies $q\alpha/(q-p) < 3$. Then similar to (2.8), there is a constant $\alpha < \bar{\alpha}' < 2$ so that $q\bar{\alpha}'/(q-p) < 3$ and

$$\mathbf{E}\Big\{ \int_0^{t\wedge\tilde{\tau}_k} ds \int_{\mathbb{R}} X_s(u)^p p_{t-s}(x-u)^{\bar{\alpha}'} du\Big\} \leq k^{\frac{p}{q}} \Big\{ \int_0^t [2\pi(t-s)]^{-\frac{1}{2}(\frac{q\bar{\alpha}'}{q-p}-1)} ds\Big\}^{1-\frac{p}{q}} < \infty. \tag{2.9}$$

Therefore, by condition (C3) again, for any $k \geq 1$,

$$\mathbf{E}\Big\{ \int_0^{t\wedge\tilde{\tau}_k} ds \int_1^\infty z m_0(dz) \int_{\mathbb{R}} du \int_0^{H(X_{s-}(u))^\alpha} p_{t-s}(x-u) dv\Big\}$$

$$\leq C\,\mathbf{E}\Big\{\int_0^{t\wedge\tilde{\tau}_k} ds \int_1^\infty zm_0(dz)\int_{\mathbb{R}}[1+X_s(u)^p]p_{t-s}(x-u)du\Big\} < \infty$$

and

$$\mathbf{E}\Big\{\int_0^{t\wedge\tilde{\tau}_k} ds \int_0^1 z^{\bar{\alpha}'}m_0(dz)\int_{\mathbb{R}} du\int_0^{H(X_{s-}(u))^\alpha} p_{t-s}(x-u)^{\bar{\alpha}'}dv\Big\}$$
$$\leq C\,\mathbf{E}\Big\{\int_0^{t\wedge\tilde{\tau}_k} ds \int_0^1 z^{\bar{\alpha}'}m_0(dz)\int_{\mathbb{R}}[1+X_s(u)^p]p_{t-s}(x-u)^{\bar{\alpha}'}du\Big\} < \infty.$$

So, the stochastic integral in (2.4) is also well defined by [26, Theorem 8.23(i)(p. 129)] and [12, p. 62] again. □

Lemma 2.4. *Let $0<\bar{p}<\alpha$ be fixed. Then for any $T>0$ and any $0<t\leq T$, there is a set $K_t\subset\mathbb{R}$ of Lebesgue measure zero so that*

$$\mathbf{E}\{X_t(x)^{\bar{p}}\}\leq C_T t^{-\frac{\bar{p}}{2}},\qquad x\in\mathbb{R}\backslash K_t. \tag{2.10}$$

The proof is also given in the Appendix.

Lemma 2.5. *Suppose that $T>0$, $\delta\in(1,\alpha)$, $\delta_1\in(\alpha,2)$ and $0<r<\min\{1,\frac{3-\delta_1}{\delta_1}\}$. Then for each $0<t\leq T$ and the set $K_t\subset\mathbb{R}$ from Lemma 2.4 we have*

$$\mathbf{E}\{|X_t(x_1)-X_t(x_2)|^\delta\}\leq C_T t^{-\frac{(r+1)\delta}{2}}|x_1-x_2|^{r\delta},\qquad x_1,x_2\in\mathbb{R}\backslash K_t. \tag{2.11}$$

Proof. For $t>0$ and $x\in\mathbb{R}$ let

$$Z_1(t,x):=\int_0^t\int_0^1\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha} zp_{t-s}(x-u)\tilde{N}_0(ds,dz,du,dv)$$

and

$$Z_2(t,x):=\int_0^t\int_1^\infty\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha} zp_{t-s}(x-u)\tilde{N}_0(ds,dz,du,dv).$$

By (2.4e) in [30], for all $t>0$, $\theta\in[0,1]$ and $u\in\mathbb{R}$ we have

$$|p_t(x_1-u)-p_t(x_2-u)|\leq C|x_1-x_2|^\theta t^{-\theta/2}[p_t(x_1-u)+p_t(x_2-u)], \tag{2.12}$$

which implies

$$|p_t(x_1-u)-p_t(x_2-u)|^{\delta_1}du\leq C|x_1-x_2|^{r\delta_1}t^{-\frac{r\delta_1+\delta_1-1}{2}}[p_t(x_1-u)+p_t(x_2-u)]. \tag{2.13}$$

Then by (1.6) in [31], condition (C3) and Lemma 2.4,

$$\begin{aligned}
&\mathbf{E}\{|Z_1(t,x_1)-Z_1(t,x_2)|^{\delta_1}\}\\
\leq\;& C\int_0^1 z^{\delta_1}m_0(dz)\int_0^t ds\int_{\mathbb{R}}\mathbf{E}\{H(X_{s-}(u))^\alpha\}|p_{t-s}(x_1-u)-p_{t-s}(x_2-u)|^{\delta_1}du\\
\leq\;& C\int_0^1 z^{\delta_1}m_0(dz)\int_0^t ds\int_{\mathbb{R}}\mathbf{E}\{1+X_s(u)^p\}|p_{t-s}(x_1-u)-p_{t-s}(x_2-u)|^{\delta_1}du\\
\leq\;& C_T|x_1-x_2|^{r\delta_1}\int_0^t[1+s^{-p/2}](t-s)^{-\frac{r\delta_1+\delta_1-1}{2}}ds\int_{\mathbb{R}}[p_{t-s}(x_1-u)+p_{t-s}(x_2-u)]du\\
\leq\;& C_T|x_1-x_2|^{r\delta_1}\int_0^t[1+s^{-p/2}](t-s)^{-\frac{r\delta_1+\delta_1-1}{2}}ds.
\end{aligned} \tag{2.14}$$

It follows from the Hölder inequality that

$$\mathbf{E}\{|Z_1(t,x_1)-Z_1(t,x_2)|^\delta\}\leq\Big\{\mathbf{E}\big[|Z_1(t,x_1)-Z_1(t,x_2)|^{\delta_1}\big]\Big\}^{\frac{\delta}{\delta_1}}. \tag{2.15}$$

Similar to (2.14) we have

$$\mathbf{E}\{|Z_2(t,x_1)-Z_2(t,x_2)|^\delta\} \le C_T|x_1-x_2|^{r\delta}\int_0^t [1+s^{-p/2}](t-s)^{-\frac{r\delta+\delta-1}{2}}ds. \tag{2.16}$$

Combining (2.14)–(2.16) one has

$$\mathbf{E}\Big\{|Z_1(t,x_1)-Z_1(t,x_2)|^\delta + |Z_2(t,x_1)-Z_2(t,x_2)|^\delta\Big\} \le C_T t^{-\frac{(r+1)\delta}{2}}|x_1-x_2|^{r\delta}. \tag{2.17}$$

By the Hölder inequality and condition (C1),

$$\begin{aligned}
&\mathbf{E}\Big\{\Big|\int_0^t ds\int_{\mathbb{R}} p_{t-s}(x_1-z)G(X_s(z))dz - \int_0^t ds\int_{\mathbb{R}} p_{t-s}(x_2-z)G(X_s(z))dz\Big|^\delta\Big\}\\
&\quad\le 2\,\mathbf{E}\Big\{\int_0^t ds\int_{\mathbb{R}} |p_{t-s}(x_1-z)-p_{t-s}(x_2-z)|G(X_s(z))^\delta dz\Big\}\\
&\quad\le C\,\mathbf{E}\Big\{\int_0^t ds\int_{\mathbb{R}} |p_{t-s}(x_1-z)-p_{t-s}(x_2-z)|[1+X_s(z)^\delta]dz\Big\}\\
&\quad\le C|x_1-x_2|^{r\delta}\int_0^t (t-s)^{-r\delta/2}[1+s^{-\delta/2}]ds\int_{\mathbb{R}}[p_{t-s}(x_1-u)+p_{t-s}(x_2-u)]du\\
&\quad\le C_T|x_1-x_2|t^{-\frac{(r+1)\delta}{2}},
\end{aligned} \tag{2.18}$$

where Lemma 2.4 and (2.12) was used in the third inequality. By (2.12) again we have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}} p_t(x_1-y)X_0(dy) - \int_{\mathbb{R}} p_t(x_2-y)X_0(dy)\Big|\\
&\qquad\le \int_{\mathbb{R}} |p_t(x_1-y)-p_t(x_2-y)|X_0(dy) \le C|x_1-x_2|^r t^{-\frac{r+1}{2}}X_0(1),
\end{aligned}$$

which together with (2.4) and (2.17)–(2.18) implies (2.11). □

Lemma 2.6. *For each $t>0$ and $t_n>0$ satisfying $t_n\to t$ as $n\to\infty$, there is a set $K_t\subset\mathbb{R}$ of Lebesgue measure zero so that*

$$\lim_{n\to\infty}\mathbf{E}\{|X_{t_n}(x)-X_t(x)|\}=0, \qquad x\in\mathbb{R}\backslash K_t.$$

Proof. For $t_0,t>0$, by (2.4),

$$\begin{aligned}
&|X_{t_0+t}(x)-X_{t_0}(x)|\\
&\quad\le \int_{\mathbb{R}}|p_{t_0+t}(x-y)-p_{t_0}(x-y)|X_0(dy) + \int_{t_0}^{t_0+t}ds\int_{\mathbb{R}}p_{t_0+t-s}(x-z)G(X_s(z))dz\\
&\qquad+\int_0^{t_0}ds\int_{\mathbb{R}}\big|p_{t_0+t-s}(x-z)-p_{t_0-s}(x-z)\big|G(X_s(z))dz\\
&\qquad+\Big|\int_{t_0}^{t_0+t}\int_0^\infty\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha} zp_{t_0+t-s}(x-u)\tilde N_0(ds,dz,du,dv)\Big|\\
&\qquad+\Big|\int_0^{t_0}\int_0^\infty\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha} z[p_{t_0+t-s}(x-u)-p_{t_0-s}(x-u)]\tilde N_0(ds,dz,du,dv)\Big|\\
&\quad=: I_1(t_0,t)+I_2(t_0,t)+I_3(t_0,t)+|I_4(t_0,t)|+|I_5(t_0,t)|.
\end{aligned}$$

By the dominated convergence, $I_1(t_0,t)$ tends to zero as $t\to0$.

By condition (C1), Lemma 2.4 and the dominated convergence, for $0<t_0\le T$ we have that both

$$\mathbf{E}\{I_2(t_0,t)\}\le C\,\mathbf{E}\Big\{\int_{t_0}^{t_0+t}ds\int_{\mathbb{R}}p_{t_0+t-s}(x-z)[1+X_s(z)]dz\Big\}\le C\int_{t_0}^{t_0+t}(1+s^{-\frac12})ds$$

and

$$\begin{aligned}\mathbf{E}\{I_3(t_0,t)\} &\le C\,\mathbf{E}\Big\{\int_0^{t_0} ds \int_{\mathbb{R}} |p_{t_0+t-s}(x-z)-p_{t_0-s}(x-z)|[1+X_s(z)]dz\Big\}\\ &\le C_T \int_0^{t_0}(1+s^{-\frac{1}{2}})ds\int_{\mathbb{R}}|p_{t_0+t-s}(x-z)-p_{t_0-s}(x-z)|dz\end{aligned}$$

go to 0 as $t\to 0$.

Let

$$I_{4,1}(t_0,t) := \int_{t_0}^{t_0+t}\int_0^1\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha} zp_{t_0+t-s}(x-u)\tilde{N}_0(ds,dz,du,dv)$$

and

$$I_{5,1}(t_0,t) := \int_0^{t_0}\int_0^1\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha} z[p_{t_0+t-s}(x-u)-p_{t_0-s}(x-u)]\tilde{N}_0(ds,dz,du,dv).$$

Let $I_{4,2}(t_0,t) := I_4(t_0,t) - I_{4,1}(t_0,t)$ and $I_{5,2}(t_0,t) := I_5(t_0,t)-I_{5,1}(t_0,t)$. Then by the Hölder continuity of H, Lemma 2.4 and the dominated convergence again, for $0<t_0\le T$, both

$$\begin{aligned}\mathbf{E}\{I_{4,1}(t_0,t)^2\} &= \int_0^1 z^2 m_0(dz)\int_{t_0}^{t_0+t} ds\int_{\mathbb{R}}\mathbf{E}\{H(X_s(u))^\alpha\}p_{t_0+t-s}(x-u)^2du\\ &\le C\int_0^1 z^2m_0(dz)\int_{t_0}^{t_0+t} ds\int_{\mathbb{R}}\mathbf{E}\{1+X_s(u)^p\}p_{t_0+t-s}(x-u)^2du\\ &\le C_T\int_{t_0}^{t_0+t}[1+s^{-\frac{p}{2}}](t_0+t-s)^{-\frac{1}{2}}ds\end{aligned}$$

and

$$\begin{aligned}&\mathbf{E}\{I_{5,1}(t_0,t)^2\}\\ &= \int_0^1 z^2m_0(dz)\int_0^{t_0}ds\int_{\mathbb{R}}\mathbf{E}\{H(X_s(u))^\alpha\}[p_{t_0+t-s}(x-u)-p_{t_0-s}(x-u)]^2du\\ &\le C_T\int_0^{t_0}[1+s^{-\frac{p}{2}}]ds\int_{\mathbb{R}}[p_{t_0+t-s}(x-u)-p_{t_0-s}(x-u)]^2du\end{aligned}$$

go to 0 as $t\to 0$.

Similarly, both

$$\begin{aligned}&\mathbf{E}\{|I_{4,2}(t_0,t)|\}\\ &\quad\le \mathbf{E}\Big\{\Big|\int_{t_0}^{t_0+t}\int_1^\infty\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha} zp_{t_0+t-s}(x-u)N_0(ds,dz,du,dv)\Big|\Big\}\\ &\qquad+\mathbf{E}\Big\{\int_1^\infty zm_0(dz)\int_{t_0}^{t_0+t}ds\int_{\mathbb{R}}H(X_{s-}(u))^\alpha p_{t_0+t-s}(x-u)du\Big\}\\ &\quad\le 2\int_1^\infty zm_0(dz)\int_{t_0}^{t_0+t}ds\int_{\mathbb{R}}\mathbf{E}\{H(X_s(u))^\alpha\}p_{t_0+t-s}(x-u)du\end{aligned}$$

and

$$\mathbf{E}\{|I_{5,2}(t_0,t)|\}\le 2\int_1^\infty zm_0(dz)\int_0^{t_0}ds\int_{\mathbb{R}}\mathbf{E}\{H(X_s(u))^\alpha\}|p_{t_0+t-s}(x-u)-p_{t_0-s}(x-u)|du$$

go to 0 as $t\to 0$. The proof is thus completed. □

3 Proof of Theorem 1.2

In this section we establish the proof of Theorem 1.2. Throughout this section we always assume that conditions (C1) and (C3) hold and that (X,L) is a weak solution to

(1.4) satisfying Assumption 1.4 with deterministic initial value $X_0 \in M(\mathbb{R})$. For $n \geq 1$ and $0 \leq k \leq 2^n$, put $n_k := k/2^n$. Define n'_k similarly for $n' \geq 1$.

For any $x \in \mathbb{R}$ and $s > 0$ let

$$Z_s(x) = \int_0^s \int_0^\infty \int_{\mathbb{R}} \int_0^{H(X_{s_1-}(u))^\alpha} z p_{s-s_1}(x-u) \tilde{N}_0(ds_1, dz, du, dv). \tag{3.1}$$

Before presenting the proof of Theorem 1.2, we first establish a weaker version of the result which will be used to conclude the proof of Theorem 1.2.

Lemma 3.1. *The results of Theorem 1.2 hold with η_c replaced by $\eta'_c = \eta_c 1_{\{\alpha \geq \frac{3}{2}\}} + \frac{\alpha-1}{\alpha} 1_{\{\alpha < \frac{3}{2}\}}$.*

Proof. Since the proof of (1.5) is essentially the same as that of [8, Remark 2.10], we only present the proof of (1.6). Let r, δ and δ_1 satisfy the conditions in Lemma 2.5 and $r\delta > 1$. By (2.17) and the proof of [32, Corollary 1.2(ii)], for each $0 < \varepsilon < r - 1/\delta$ and $T > 0$, there is a constant C_T independent of $t \in (0, T]$ so that

$$\mathbf{E}\Big\{ \sup_{x,y \in \mathbb{K}, x \neq y} \frac{|\tilde{Z}_t(x) - \tilde{Z}_t(y)|}{|x-y|^\varepsilon} \Big\} \leq C_T t^{-\frac{r+1}{2}},$$

where $\tilde{Z}_t(x)$ denotes a continuous modification of $Z_t(x)$ for each $t > 0$. Then by Fatou's lemma, for each subsequence $\{n' : n' \geq 1\}$ of $\{n : n \geq 1\}$,

$$\begin{aligned}
&\mathbf{E}\Big\{ \liminf_{n \to \infty} \frac{1}{2^n} \sum_{k=1}^{2^n} \sup_{x,y \in \mathbb{K}, x \neq y} \frac{|\tilde{Z}_{n_k T}(x) - \tilde{Z}_{n_k T}(y)|}{|x-y|^\varepsilon} \Big\} \\
&\leq \liminf_{n \to \infty} \mathbf{E}\Big\{ \frac{1}{2^n} \sum_{k=1}^{2^n} \sup_{x,y \in \mathbb{K}, x \neq y} \frac{|\tilde{Z}_{n_k T}(x) - \tilde{Z}_{n_k T}(y)|}{|x-y|^\varepsilon} \Big\} \\
&= \liminf_{n \to \infty} \frac{1}{2^n} \sum_{k=1}^{2^n} \mathbf{E}\Big\{ \sup_{x,y \in \mathbb{K}, x \neq y} \frac{|\tilde{Z}_{n_k T}(x) - \tilde{Z}_{n_k T}(y)|}{|x-y|^\varepsilon} \Big\} \\
&\leq C_T \liminf_{n \to \infty} \frac{1}{2^n} \sum_{k=1}^{2^n} (n_k T)^{-\frac{r+1}{2}} = C_T \int_0^T s^{-\frac{r+1}{2}} ds < \infty,
\end{aligned}$$

which implies

$$\liminf_{n' \to \infty} \frac{1}{2^{n'}} \sum_{k=1}^{2^{n'}} \sup_{x,z \in \mathbb{K}, x \neq z} \frac{|\tilde{Z}_{n'_k T}(x) - \tilde{Z}_{n'_k T}(z)|}{|x-z|^\varepsilon} < \infty, \quad \mathbf{P}\text{-a.s.}$$

Let

$$\tilde{X}_t(x) = \int_{\mathbb{R}} p_t(x-u) X_0(du) + \int_0^t ds \int_{\mathbb{R}} p_{t-s}(x-z) G(X_s(z)) dz + \tilde{Z}_t(x).$$

Then it follows from (2.4) that $\tilde{X}_t(x)$ is a continuous modification of $X_t(x)$ for each fixed $t > 0$. We first show that (1.6) holds for $\eta = \varepsilon \in (0, r - 1/\delta)$ in the following. By (2.12) and condition (C1) for each $\varepsilon' \in (0,1)$,

$$\int_{\mathbb{R}} |p_t(x-u) - p_t(y-u)| X_0(du) \leq C|x-y|^{\varepsilon'} t^{-\frac{\varepsilon'+1}{2}} X_0(1)$$

and

$$\int_0^t ds \int_{\mathbb{R}} |p_{t-s}(x-z) - p_{t-s}(y-z)| G(X_s(z)) dz$$

$$
\begin{aligned}
&\le C\int_0^t ds\int_{\mathbb{R}} |p_{t-s}(x-z)-p_{t-s}(y-z)|[X_s(z)+1]dz\\
&\le C_T|x-y|^{\varepsilon'}\int_0^t (t-s)^{-\frac{\varepsilon'+1}{2}}[\langle X_s,1\rangle+1]ds
\end{aligned}
$$

for each $t>0$. Then

$$
\begin{aligned}
&\limsup_{n'\to\infty}\frac{1}{2^{n'}}\sum_{k=1}^{2^{n'}}\sup_{x,y\in\mathbb{K},x\neq y}\frac{\int_{\mathbb{R}}|p_{n'_kT}(x-u)-p_{n'_kT}(y-u)|X_0(du)}{|x-y|^{\varepsilon'}}\\
&\quad\le \limsup_{n'\to\infty}\frac{C}{2^{n'}}\sum_{k=1}^{2^{n'}}(n'_kT)^{-\frac{\varepsilon'+1}{2}}X_0(1)=CX_0(1)\int_0^T t^{-\frac{\varepsilon'+1}{2}}ds<\infty
\end{aligned}
\tag{3.2}
$$

and

$$
\begin{aligned}
&\limsup_{n'\to\infty}\frac{1}{2^{n'}}\sum_{k=1}^{2^{n'}}\sup_{x,y\in\mathbb{K},x\neq y}\frac{1}{|x-y|^{\varepsilon'}}\int_0^{n'_kT}ds\int_{\mathbb{R}}|p_{n'_kT-s}(x-z)\\
&\qquad\qquad -p_{n'_kT-s}(y-z)|G(X_s(z))dz\\
&\quad\le \limsup_{n'\to\infty}\frac{C}{2^{n'}}\sum_{k=1}^{2^{n'}}\int_0^{n'_kT}(n'_kT-s)^{-\frac{\varepsilon'+1}{2}}[\langle X_s,1\rangle+1]ds\\
&\quad= C_T\int_0^T dt\int_0^t (t-s)^{-\frac{\varepsilon'+1}{2}}[\langle X_s,1\rangle+1]ds\\
&\quad= C_T\big[\sup_{s\in(0,T]}\langle X_s,1\rangle+1\big]\int_0^T dt\int_0^t (t-s)^{-\frac{\varepsilon'+1}{2}}ds<\infty,\quad \mathbf{P}\text{-a.s.},
\end{aligned}
\tag{3.3}
$$

where the fact $\sup_{s\in(0,T]}\langle X_s,1\rangle<\infty$ $\mathbf{P}$-a.s. was used in the last inequality. Therefore, (1.6) holds for $\eta<r-1/\delta$. Let $\delta=\alpha-\sigma$ and $\delta_1=\alpha+\sigma$ for small enough $\sigma>0$. This means that (1.6) holds for

$$
\eta<\min\Big\{1,\frac{3}{\alpha+\sigma}-1\Big\}-\frac{1}{\alpha-\sigma}=\min\Big\{1-\frac{1}{\alpha-\sigma},\frac{3}{\alpha+\sigma}-\frac{1}{\alpha-\sigma}-1\Big\}.
$$

Letting $\sigma\to 0$ one can finish the proof. □

Lemma 3.2. *For any fixed $t>0$, let $\tilde X_t$ be a continuous modification of X_t. Then for any compact subset $\mathbb{K}$ of $\mathbb{R}$ and $\delta\in(1,\alpha)$,*

$$
\sup_{t\in(0,T]} t^{\frac{\delta}{2}}\,\mathbf{E}\Big\{\sup_{x\in\mathbb{K}}\tilde X_t(x)^\delta\Big\}<\infty.
\tag{3.4}
$$

Proof. By Lemma 2.4, for each $t\in(0,T]$, there is a sequence $\{y_t(n):n\ge 1\}\subset\mathbb{K}\cup[-1,1]$ so that $y_t(n)\to 0$ as $n\to\infty$ and

$$
\begin{aligned}
\mathbf{E}\{\tilde X_t(0)^\delta\} &= \mathbf{E}\Big\{\lim_{n\to\infty}\tilde X_t(y_t(n))^\delta\Big\}\le\liminf_{n\to\infty}\mathbf{E}\Big\{\tilde X_t(y_t(n))^\delta\Big\}\\
&= \liminf_{n\to\infty}\mathbf{E}\Big\{X_t(y_t(n))^\delta\Big\}\le C_Tt^{-\frac{\delta}{2}},
\end{aligned}
$$

which implies

$$
\sup_{t\in(0,T]} t^{\frac{\delta}{2}}\,\mathbf{E}\big\{\tilde X_t(0)^\delta\big\}<\infty.
\tag{3.5}
$$

Then the desired result follows from (3.5), Lemma 2.5 and [32, Corollary 1.2(iii)]. □

By Proposition 2.1, $\{X_t(x):t>0,x\in\mathbb{R}\}$ satisfies (1.10) with $X_0\in M(\mathbb{R})$. Similar to [8, Lemma 2.12] we can prove the following lemma.

Lemma 3.3. *Fix $\delta, \delta' \in [1,3)$, $r, r' \in [0,1]$ with $r < \frac{3-\delta}{\delta}$ and $\frac{\alpha(r'\delta'+\delta')}{2(\alpha+1-p\vee 1)} < 1$, and a nonempty compact set $\mathbb{K} \subset \mathbb{R}$. Define*

$$\bar{\mathcal{V}}_n := \int_0^1 \frac{1}{2^n} \sum_{k=1}^{2^n} 1_{(n_{k-1},n_k]}(s) \sum_{i=k}^{2^n} (n_i - s)^{-\frac{r\delta+\delta-1}{2}} \bar{\mathcal{V}}_{n,i}(s) ds$$

and

$$\bar{\mathcal{U}}_n := \sup_{1\le i\le 2^n} \int_0^{n_i} (n_i - s)^{-\frac{r'\delta'+\delta'-1}{2}} \bar{\mathcal{V}}_{n,i}(s) ds$$

with

$$\bar{\mathcal{V}}_{n,i}(s) := \sup_{x_1,x_2\in\mathbb{K}} \int_{\mathbb{R}} H(X_s(u))^\alpha |p_{n_i-s}(x_1-u) + p_{n_i-s}(x_2-u)| du.$$

Then for any $\varepsilon > 0$, there exits $C_\varepsilon > 0$ so that

$$\sup_{n\ge 1} \mathbf{P}[\bar{\mathcal{V}}_n \ge C_\varepsilon] \le \varepsilon, \quad \sup_{n\ge 1} \mathbf{P}[\bar{\mathcal{U}}_n \ge C_\varepsilon] \le \varepsilon, \qquad \varepsilon > 0. \tag{3.6}$$

Moreover, for each $x_1, x_2 \in \mathbb{K}$ and $n \ge 1$ we have

$$\begin{aligned}\int_0^1 ds \int_{\mathbb{R}} H(X_s(u))^\alpha \frac{1}{2^n} \sum_{k=1}^{2^n} 1_{(n_{k-1},n_k]}(s) \sum_{i=k}^{2^n} |p_{n_i-s}(x_1-u) \\ -p_{n_i-s}(x_2-u)|^\delta du \le C\bar{\mathcal{V}}_n |x_1-x_2|^{r\delta}\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\int_0^{n_i} ds \int_{\mathbb{R}} H(X_s(u))^\alpha |p_{n_i-s}(x_1-u) \\ -p_{n_i-s}(x_2-u)|^{\delta'} du \le C\bar{\mathcal{U}}_n |x_1-x_2|^{r'\delta'}, \quad 1 \le i \le 2^n.\end{aligned} \tag{3.8}$$

Proof. We assume that $\mathbb{K} \subset [0,1]$ for simplicity. Observe that $\mathbf{P}$-a.s.,

$$\begin{aligned} & \bar{\mathcal{V}}_{n,i}(s) \\ \le\ & C \sup_{x_1,x_2\in[0,1]} \int_{\mathbb{R}} [1 + X_s(y)^p] \Big[p_{n_i-s}(x_1-y) + p_{n_i-s}(x_2-y)\Big] dy \\ \le\ & C + C \sup_{x_1,x_2\in[0,1]} \int_{|y|\ge 2} X_s(y)^p \Big[p_{n_i-s}(x_1-y) + p_{n_i-s}(x_2-y)\Big] dy + C \sup_{|y|\le 2} \tilde{X}_s(y)^p \\ \le\ & C + C\int_{\mathbb{R}} X_s(y)^p \Big[p_{n_i-s}(y+1) + p_{n_i-s}(y-1)\Big] dy + C \sup_{|y|\le 2} \tilde{X}_s(y)^p, \ s \in (0,1]. \end{aligned} \tag{3.9}$$

Then by Lemmas 2.4 and 3.2,

$$\mathbf{E}\{\bar{\mathcal{V}}_{n,i}(s)\} \le C[1 + s^{-\frac{p}{2}}], \qquad s \in (0,1].$$

It is elementary to check that

$$\begin{aligned} \sup_{n\ge 1} \mathbf{E}\{\bar{\mathcal{V}}_n\} &\le \sup_{n\ge 1} C \int_0^1 \frac{1}{2^n} \sum_{k=1}^{2^n} 1_{(n_{k-1},n_k]}(s) \sum_{i=k}^{2^n} (n_i-s)^{-\frac{r\delta+\delta-1}{2}} [1 + s^{-\frac{p}{2}}] ds \\ &\le \sup_{n\ge 1} \frac{C}{2^n} \sum_{k=1}^{2^n} \sum_{i=k}^{2^n} \int_{n_{k-1}}^{n_k} (n_i-s)^{-\frac{r\delta+\delta-1}{2}} [1 + s^{-\frac{p}{2}}] ds \\ &= \sup_{n\ge 1} \frac{C}{2^n} \sum_{i=1}^{2^n} \int_0^{n_i} (n_i-s)^{-\frac{r\delta+\delta-1}{2}} [1 + s^{-\frac{p}{2}}] ds < \infty. \end{aligned}$$

Then the first assertion of (3.6) follows from the Markov inequality. Using (2.13) one gets (3.7) and (3.8).

We now prove the second assertion of (3.6). Observe that for each constant $\theta > 0$ and $n_{i-1} \le s < n_i$,

$$\begin{aligned}
&(n_i - s)^{-\theta} \int_{y\ge 2} X_s(y)^p p_{n_i-s}(y-1)dy \\
&\quad \le C(n_i - s)^{-\theta-\frac{1}{2}} e^{-\frac{(y-1)^2}{4(n_i-s)}} \int_{\mathbb{R}} X_s(y)^p p_2(y-1)dy \\
&\quad \le C \int_{\mathbb{R}} X_s(y)^p p_2(y-1)dy
\end{aligned}$$

and

$$(n_i - s)^{-\theta} \int_{y\le -2} X_s(y)^p p_{n_i-s}(y+1)dy \le C \int_{\mathbb{R}} X_s(y)^p p_2(y+1)dy.$$

Then it is easy to check that

$$\begin{aligned}
&(n_i - s)^{-\frac{r'\delta'+\delta'-1}{2}} \bar{\mathcal{V}}_{n,i}(s) \\
\le\ & C(n_i - s)^{-\frac{r'\delta'+\delta'-1}{2}} \Big\{ 1 + \int_{y\ge 2} X_s(y)^p p_{n_i-s}(y-1)dy + \int_{y\le -2} X_s(y)^p p_{n_i-s}(y+1)dy \\
&+ \sup_{x_1,x_2\in[0,1]} \int_{|y|\le 2} X_s(y)^p \big[p_{n_i-s}(x_1+y) + p_{n_i-s}(x_2+y)\big] dy \Big\} \\
\le\ & C(n_i - s)^{-\frac{r'\delta'+\delta'-1}{2}} + C \int_{\mathbb{R}} X_s(y)^p [p_2(y+1) + p_2(y-1)]dy \\
&+(n_i - s)^{-\frac{r'\delta'+\delta'-1}{2}} \sup_{x_1,x_2\in[0,1]} \int_{|y|\le 2} X_s(y)^p \big[p_{n_i-s}(x_1+y) + p_{n_i-s}(x_2+y)\big] dy \Big\} \qquad (3.10)
\end{aligned}$$

for each $1 \le i \le 2^n$ and $n_{i-1} \le s < n_i$. Observe that for each $x \in [0,1]$ and $n_{i-1} \le s < n_i$,

$$\int_{|y|\le 2} X_s(y)^p p_{n_i-s}(x+y)dy \le \Big| \int_{|y|\le 2} X_s(y) p_{n_i-s}(x+y)dy \Big|^p \le [2\pi(n_i - s)^{-\frac{1}{2}} \langle X_s, 1\rangle]^p$$

for $0 < p \le 1$ and

$$\int_{|y|\le 2} X_s(y)^p p_{n_i-s}(x+y)dy \le [2\pi(n_i - s)]^{-\frac{1}{2}} \langle X_s, 1\rangle \sup_{|y|\le 2} \tilde{X}_s(y)^{p-1}, \ \ \mathbf{P}\text{-a.s.}$$

for $1 < p < 2$. Combining with (3.10) we have

$$\begin{aligned}
\bar{\mathcal{U}}_n \ \le\ & C \sup_{1\le i\le 2^n} \int_0^{n_i} (n_i - s)^{-\frac{r'\delta'+\delta'-1}{2}} ds + C \int_0^1 ds \int_{\mathbb{R}} X_s(y)^p [p_2(y+1) + p_2(y-1)]dy \\
&+C \sup_{s\in(0,1]} \langle X_s, 1\rangle^p \sup_{1\le i\le 2^n} \int_0^{n_i} (n_i - s)^{-\frac{r'\delta'+\delta'+p-1}{2}} ds \\
\le\ & C + C \int_0^1 ds \int_{\mathbb{R}} X_s(y)^p [p_2(y+1) + p_2(y-1)]dy + C \sup_{s\in(0,1]} \langle X_s, 1\rangle^p \qquad (3.11)
\end{aligned}$$

for $0 < p \le 1$ and

$$\begin{aligned}
\bar{\mathcal{U}}_n \ \le\ & C \sup_{1\le i\le 2^n} \int_0^{n_i} (n_i - s)^{-\frac{r'\delta'+\delta'-1}{2}} ds + C \int_0^1 ds \int_{\mathbb{R}} X_s(y)^p [p_2(y+1) + p_2(y-1)]dy \\
&+C \sup_{s\in(0,1]} \langle X_s, 1\rangle \sup_{1\le i\le 2^n} \int_0^{n_i} (n_i - s)^{-\frac{r'\delta'+\delta'}{2}} \sup_{|y|\le 2} \tilde{X}_s(y)^{p-1} ds \\
\le\ & C + C \int_0^1 ds \int_{\mathbb{R}} X_s(y)^p [p_2(y+1) + p_2(y-1)]dy \\
&+C \sup_{s\in(0,1]} \langle X_s, 1\rangle \sup_{1\le i\le 2^n} \int_0^{n_i} (n_i - s)^{-\frac{r'\delta'+\delta'}{2}} \sup_{|y|\le 2} \tilde{X}_s(y)^{p-1} ds, \ \ \mathbf{P}\text{-a.s.}
\end{aligned}$$

for $1 < p < 2$. Taking $\alpha_- \in (1, \alpha)$ with $\frac{\alpha_-(r'\delta'+\delta')}{2(\alpha_-+1-p)} < 1$, we have

$$\begin{aligned}
&\int_0^{n_i}(n_i-s)^{-\frac{r'\delta'+\delta'}{2}}\sup_{|y|\le 2}\tilde{X}_s(y)^{p-1}ds\\
&\le \Big|\int_0^{n_i}(n_i-s)^{-\frac{\alpha_-(r'\delta'+\delta')}{2(\alpha_-+1-p)}}ds\Big|^{\frac{\alpha_-+1-p}{\alpha_-}}\Big|\int_0^{n_i}\sup_{|y|\le 2}\tilde{X}_s(y)^{\alpha_-}ds\Big|^{\frac{p-1}{\alpha_-}}\\
&\le C\Big|\int_0^{1}\sup_{|y|\le 2}\tilde{X}_s(y)^{\alpha_-}ds\Big|^{\frac{p-1}{\alpha_-}}
\end{aligned}$$

for the case $1<p<2$. It then follows that

$$\begin{aligned}
\bar{\mathcal{U}}_n &\le C+C\int_0^1 ds\int_{\mathbb{R}}X_s(y)^p[p_2(y+1)+p_2(y-1)]dy\\
&\quad +C\sup_{s\in(0,1]}\langle X_s,1\rangle\Big|\int_0^1\sup_{|y|\le 2}X_s(y)^{\alpha_-}ds\Big|^{\frac{p-1}{\alpha_-}}
\end{aligned}\tag{3.12}$$

for the case $1<p<2$. By Lemmas 2.4 and 3.2,

$$\mathbf{E}\Big\{\int_0^1 ds\int_{\mathbb{R}}X_s(y)^p[p_2(y+1)+p_2(y-1)]dy\Big\}\le C\int_0^1 s^{-\frac{p}{2}}ds<\infty$$

and

$$\mathbf{E}\Big\{\int_0^1\sup_{|y|\le 2}X_s(y)^{\alpha_-}ds\Big\}\le\int_0^1\mathbf{E}\Big\{\sup_{|y|\le 2}X_s(y)^{\alpha_-}\Big\}ds\le C\int_0^1 s^{-\alpha_-/2}ds<\infty,$$

which imply

$$\int_0^1 ds\int_{\mathbb{R}}X_s(y)^p[p_2(y+1)+p_2(y-1)]dy+\int_0^1\sup_{|y|\le 2}X_s(y)^{\alpha_-}ds<\infty,\quad \mathbf{P}\text{-a.s.}$$

Combining with (3.11)–(3.12) and the fact

$$\sup_{s\in(0,1]}\langle X_s,1\rangle<\infty,\quad \mathbf{P}\text{-a.s.},$$

we have

$$\sup_{n\ge 1}\bar{\mathcal{U}}_n<\infty,\quad \mathbf{P}\text{-a.s.},$$

which implies the second assertion of (3.6). □

For $t\ge 0$ and $\psi\in B(\mathbb{R})$ define discontinuous martingales

$$t\mapsto M_t^1(\psi):=\int_0^t\int_0^\infty\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha}z\psi(u)1_{\{|u|\le\mathbb{K}_1\}}\tilde{N}_0(ds,dz,du,dv)$$

and

$$t\mapsto M_t^2(\psi):=\int_0^t\int_0^\infty\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha}\frac{z\psi(u)}{|u|}1_{\{|u|>\mathbb{K}_1\}}\tilde{N}_0(ds,dz,du,dv),$$

where $\mathbb{K}_1:=\mathbb{K}_0+1$ with $\mathbb{K}_0:=\sup_{x\in\mathbb{K}}|x|$. For $i=1,2$ let $\Delta M_s^i(y)$ denote the jumps of $M^i(ds,dy)$. Similar to [8, Lemma 2.14] one can show the following result.

Lemma 3.4. *Let $\gamma\in(0,\alpha^{-1})$ and $\lambda:=\alpha^{-1}-\gamma$. Then for each $\varepsilon>0$ there exists a constant $C_\varepsilon>0$ independent of n so that*

$$\mathbf{P}\Big(\bigcup_{k=1}^{2^n}\Big\{\Delta M_s^1(y)>2^{\lambda n}C_\varepsilon(n_k-s)^\lambda\text{ for some }s\in[n_{k-1},n_k)\text{ and }|y|\le\mathbb{K}_1\Big\}\Big)\le\varepsilon$$

and

$$\mathbf{P}\Big(\bigcup_{k=1}^{2^n}\Big\{\Delta M_s^2(y) > 2^{\lambda n}C_\varepsilon(n_k - s)^\lambda \text{ for some } s\in[n_{k-1},n_k) \text{ and } |y|>\mathbb{K}_1\Big)\le\varepsilon.$$

Proof. Since the proofs are similar, we only present the first one. Let $c>0$. For $n\geq 1$ and $1\leq k\leq 2^n$ put

$$Y_{n,k}^1 := N_0\Big((s,z,u,v): s\in[n_{k-1},n_k), z\geq 2^{\lambda n}c(n_k-s)^\lambda, |u|\leq\mathbb{K}_1, v\leq H(X_{s-}(u))^\alpha\Big).$$

Then by the Markov inequality,

$$\mathbf{P}\Big\{\frac{\Delta M_s^1(y)}{2^{\lambda n}(n_k-s)^\lambda} > c \text{ for some } s\in[n_{k-1},n_k) \text{ and } |y|\leq\mathbb{K}_1\Big\} = \mathbf{P}\{Y_{n,k}^1\geq 1\}\leq\mathbf{E}\{Y_{n,k}^1\}.$$

By Lemma 2.4,

$$\begin{aligned}
\mathbf{E}\{Y_{n,k}^1\} &= \mathbf{E}\Big\{\int_{n_{k-1}}^{n_k}ds\int_0^\infty m_0(dz)\int_{-\mathbb{K}_1}^{\mathbb{K}_1}du\int_0^{H(X_s(u))^\alpha}1_{\{z\geq c2^{\lambda n}(n_k-s)^\lambda\}}dv\Big\}\\
&= Cc^{-\alpha}\int_{n_{k-1}}^{n_k}ds\int_{-\mathbb{K}_1}^{\mathbb{K}_1}\mathbf{E}\{H(X_s(u))^\alpha\}2^{-\alpha\lambda n}(n_k-s)^{-\alpha\lambda}du\\
&\leq Cc^{-\alpha}\int_{n_{k-1}}^{n_k}ds\int_{-\mathbb{K}_1}^{\mathbb{K}_1}\mathbf{E}\{1+X_s(u)^p\}2^{-\alpha\lambda n}(n_k-s)^{-\alpha\lambda}du\\
&\leq Cc^{-\alpha}2^{-\alpha\lambda n}\int_{n_{k-1}}^{n_k}(n_k-s)^{-\alpha\lambda}[1+s^{-\frac{p}{2}}]ds\leq Cc^{-\alpha}n_{k-1}^{-p/2}2^{-n}
\end{aligned}$$

for $2\leq k\leq 2^n$. Similarly

$$\mathbf{E}\{Y_{n,1}^1\}\leq Cc^{-\alpha}2^{-\alpha\lambda n}\int_0^{n_1}(n_1-s)^{-\alpha\lambda}[1+s^{-\frac{p}{2}}]ds\leq Cc^{-\alpha}n_1^{-p/2}2^{-n},$$

Thus

$$\sum_{k=1}^{2^n}\mathbf{E}\{Y_{n,k}^1\}\leq Cc^{-\alpha}.$$

The desired result then follows. □

Let $\mathscr{L}$ be the space of measurable functions ψ on $\mathbb{R}_+\times\mathbb{R}$ so that

$$\int_0^t s^{-\frac{p}{2}}ds\int_{\mathbb{R}}\Big[|\psi(s,x)|+|\psi(s,x)|^2\Big]dx<\infty,\qquad t>0.$$

Similar to [8, Lemma 2.15], we have the next result.

Lemma 3.5. *Given $\psi\in\mathscr{L}$ with $\psi\geq 0$, there exist a spectrally positive α-stable process $\{L_t: t\geq 0\}$ so that for $t\geq 0$,*

$$Z_t(\psi) := \int_0^t\int_0^\infty\int_{\mathbb{R}}\int_0^\infty 1_{\{v\leq H(X_s(u))^\alpha\}}z\psi(s,u)\tilde{N}_0(ds,dz,du,dv) = L_{T'(t)},$$

where

$$T'(t) := \int_0^t ds\int_{\mathbb{R}}[H(X_s(u))\psi(s,u)]^\alpha du.$$

The proof is similar to that of [8, Lemma 2.15].

Proof of Theorem 1.2. By Lemma 3.1 we only consider the case $\alpha<3/2$. Since the proof of (1.5) is essentially the same as that of [8, Theroem 1.2(a)] but based on equation (1.10), we only present the proof of (1.6), which is a modification of that of [8, Theroem 1.2(a)] and proceeds as follows. Let $X_0\in M(\mathbb{R})$ be fixed. We assume that $T=1$ in this proof. Recall (3.1) and $\lambda=1/\alpha-\gamma$ with $\gamma\in(0,\alpha^{-1})$. Also recall that $n_k=k/2^n$ for $n\geq 1$

and $0 \le k \le 2^n$. Let $f(x)^+$ and $f(x)^-$ be, respectively, the positive part and the negative part of $f(x)$. For $x_1, x_2 \in \mathbb{R}$ and $s \in [0,1]$ define

$$\mathcal{U}_s(x_1,x_2) := \int_0^s \int_0^\infty \int_{\mathbb{R}} \int_0^{H(X_{s-}(u))^\alpha} z[p_{s-s_1}(x_1-u) - p_{s-s_1}(x_2-u)]^+ \tilde{N}_0(ds_1,dz,du,dv).$$

Let $\mathcal{V}_s(x_1,x_2)$ be defined as $\mathcal{U}_s(x_1,x_2)$ with $[p_{s-s_1}(x_1-u) - p_{s-s_1}(x_2-u)]^+$ replaced by $[p_{s-s_1}(x_1-u) - p_{s-s_1}(x_2-u)]^-$. Then

$$Z_s(x_1) - Z_s(x_2) = \mathcal{U}_s(x_1,x_2) - \mathcal{V}_s(x_1,x_2). \tag{3.13}$$

Observe that

$$\begin{aligned}
&\frac{1}{2^n}\sum_{k=1}^{2^n}\mathcal{U}_{n_k}(x_1,x_2)\\
=\ & \frac{1}{2^n}\sum_{k=1}^{2^n}\int_0^{n_k}\int_0^\infty\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha} z[p_{n_k-s}(x_1-u)-p_{n_k-s}(x_2-u)]^+\tilde{N}_0(ds,dz,du,dv)\\
=\ & \int_0^1\int_0^\infty\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha}\frac{1}{2^n}\sum_{k=1}^{2^n}1_{(0,n_k]}(s)z[p_{n_k-s}(x_1-u)\\
&\qquad -p_{n_k-s}(x_2-u)]^+\tilde{N}_0(ds,dz,du,dv)\\
=\ & \int_0^1\int_0^\infty\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha}\frac{1}{2^n}\sum_{k=1}^{2^n}1_{(n_{k-1},n_k]}(s)\sum_{i=k}^{2^n}z[p_{n_i-s}(x_1-u)\\
&\qquad -p_{n_i-s}(x_2-u)]^+\tilde{N}_0(ds,dz,du,dv)\\
=\ & \int_0^1\int_0^\infty\int_{\mathbb{R}}\int_0^\infty 1_{\{v\le H(X_{s-}(u))^\alpha\}}z\psi_n(s,u)\tilde{N}_0(ds,dz,du,dv),
\end{aligned}$$

where

$$\psi_n(s,u) := \frac{1}{2^n}\sum_{k=1}^{2^n}1_{(n_{k-1},n_k]}(s)\sum_{i=k}^{2^n}[p_{n_i-s}(x_1-u)-p_{n_i-s}(x_2-u)]^+.$$

One can see that $\psi_n(s,u)$ satisfies the assumptions of Lemma 3.5, and there is a stable process $\{L_t : t \ge 0\}$ so that

$$\frac{1}{2^n}\sum_{k=1}^{2^n}\mathcal{U}_{n_k}(x_1,x_2) = L_{T_n}, \tag{3.14}$$

where

$$T_n := \int_0^1 ds\int_{\mathbb{R}}[H(X_s(u))\psi_n(s,u)]^\alpha du.$$

Let $\varepsilon \in (0,1)$ be fixed. Let $\bar{\mathcal{V}}_n$ and $\bar{\mathcal{U}}_n$ be defined in Lemma 3.3. Then

$$\sup_{n\ge1}\Big\{\mathbf{P}(\bar{\mathcal{V}}_n > C_\varepsilon) + \mathbf{P}(\bar{\mathcal{U}}_n > C_\varepsilon)\Big\} \le 2\varepsilon. \tag{3.15}$$

Set

$$\begin{aligned}
A_n^\varepsilon\ :=\ \bigcap_{k=1}^{2^n}\Bigg(&\Big\{\frac{\Delta M_s^1(y)}{2^{\lambda n}(n_k-s)^\lambda}\le C_\varepsilon \text{ for all } s\in[n_{k-1},n_k) \text{ and } |y|\le \mathbb{K}_1\Big\}\\
&\bigcap\Big\{\frac{\Delta M_s^2(y)}{2^{\lambda n}(n_k-s)^\lambda}\le C_\varepsilon \text{ for all } s\in[n_{k-1},n_k) \text{ and } |y|> \mathbb{K}_1\Big\}\Bigg)
\end{aligned}$$

$$\bigcap\Big\{\bar{\mathcal{V}}_n \le C_\varepsilon, \bar{\mathcal{U}}_n \le C_\varepsilon\Big\}.$$

By Lemmas 3.3 and 3.4,

$$\sup_{n\ge 1} \mathbf{P}(A_n^{\varepsilon,c}) \le 4\varepsilon,$$

where $A_n^{\varepsilon,c}$ denotes the complement of A_n^ε. Define $\mathcal{U}_s^{n,\varepsilon}(x_1,x_2) := \mathcal{U}_s(x_1,x_2)1_{A_n^\varepsilon}$.

In order to complete our proof we need to establish the following two lemmas.

Lemma 3.6. *For each $n \ge 1$ and $r > 0$,*

$$\mathbf{P}\Big\{\frac{1}{2^n}\sum_{k=1}^{2^n}\mathcal{U}_{n_k}^{n,\varepsilon}(x_1,x_2) \ge r|x_1-x_2|^\eta\Big\} \le (C_\varepsilon r^{-1}|x_1-x_2|)^{C'_\varepsilon r|x_1-x_2|^{(\eta-\eta_c)/2}}.$$

Proof. By (3.14), we have

$$\mathbf{P}\Big\{\frac{1}{2^n}\sum_{k=1}^{2^n}\mathcal{U}_{n_k}^{n,\varepsilon}(x_1,x_2) \ge r|x_1-x_2|^\eta\Big\} = \mathbf{P}\Big\{L_{T_n} > r|x_1-x_2|^\eta, A_n^\varepsilon\Big\}. \tag{3.16}$$

Note that on event A_n^ε the jumps of $M_s^1(x)$ do not exceed

$$C_\varepsilon 2^{\lambda n}(n_k - s)^\lambda, \qquad s \in [n_{k-1}, n_k).$$

Then, on A_n^ε, the jumps of

$$(0,1) \ni l \mapsto \int_0^l\int_0^\infty\int_{\mathbb{R}}\int_0^\infty 1_{\{v\le H(X_{s-}(u))^\alpha, |u|\le \mathbb{K}_1\}} z\psi_n(s,u)\tilde{N}_0(ds,dz,du,dv)$$

are bounded by

$$\begin{aligned} I_n &:= C_\varepsilon 2^{\lambda n}\sup_{1\le k\le 2^n}\sup_{(s,y)\in[n_{k-1},n_k)\times\mathbb{R}}(n_k-s)^\lambda\psi_n(s,y) \\ &\le C_\varepsilon 2^{\lambda n}\sup_{1\le k\le 2^n}\sup_{(s,y)\in[n_{k-1},n_k)\times\mathbb{R}}\frac{1}{2^n}\sum_{i=k}^{2^n}(n_k-s)^\lambda|p_{n_i-s}(x_1-y) \\ &\qquad -p_{n_i-s}(x_2-y)|. \end{aligned}\tag{3.17}$$

Applying (2.12) with $\theta = \eta_c - 2\gamma$ gives

$$\begin{aligned} &\sup_{y\in\mathbb{R}}\frac{1}{2^n}\sum_{i=k}^{2^n}(n_k-s)^\lambda|p_{n_i-s}(x_1-y)-p_{n_i-s}(x_2-y)| \\ &\le C|x_1-x_2|^{\eta_c-2\gamma}\frac{1}{2^n}\sum_{i=k}^{2^n}(n_i-s)^{-\eta_c/2+\gamma}(n_k-s)^\lambda\sup_{y\in\mathbb{R}}p_{n_i-s}(y) \\ &\le C|x_1-x_2|^{\eta_c-2\gamma}\frac{1}{2^n}\sum_{i=k}^{2^n}\Big(\frac{n_k-s}{n_i-s}\Big)^\lambda \\ &\le C|x_1-x_2|^{\eta_c-2\gamma}\frac{1}{2^n}\sum_{i=k}^{2^n}\Big(\frac{1}{i-k+1}\Big)^\lambda \le C|x_1-x_2|^{\eta_c-2\gamma}2^{-\lambda n} \end{aligned}$$

for $s \in [n_{k-1}, n_k)$. This implies

$$I_n \le C_\varepsilon|x_1-x_2|^{\eta_c-2\gamma}. \tag{3.18}$$

Similarly, on A_n^ε, one can see that the jumps of

$$(0,1) \ni l \mapsto \int_0^l\int_0^\infty\int_{\mathbb{R}}\int_0^\infty 1_{\{v\le H(X_{s-}(u))^\alpha, |u|> \mathbb{K}_1\}} z\psi_n(s,u)\tilde{N}_0(ds,dz,du,dv)$$

are bounded by

$$C_\varepsilon |x_1 - x_2|^{\eta_c - 2\gamma}.$$

Combining with (3.18) we conclude that the jumps of

$$(0,1) \ni l \mapsto \int_0^l \int_0^\infty \int_{\mathbb{R}} \int_0^\infty 1_{\{v \le H(X_{s-}(u))^\alpha\}} z\psi_n(s,u)\tilde{N}_0(ds,dz,du,dv)$$

on A_n^ε are bounded by

$$C_\varepsilon |x_1 - x_2|^{\eta_c - 2\gamma}.$$

Observe that

$$\begin{aligned}
&\mathbf{P}\Big\{L_{T_n} \ge r|x_1 - x_2|^\eta, A_n^\varepsilon\Big\}\\
&\quad = \mathbf{P}\Big\{L_{T_n} \ge r|x_1 - x_2|^\eta, \sup_{u<T_n} \Delta L_u \le C_\varepsilon |x_1 - x_2|^{\eta_c - 2\gamma}, A_n^\varepsilon\Big\}\\
&\quad \le \mathbf{P}\Big\{\sup_{v \le T_n} L_v 1_{\{\sup_{u<v} \Delta L_u \le C_\varepsilon |x_1 - x_2|^{\eta_c - 2\gamma}\}} \ge r|x_1 - x_2|^\eta, A_n^\varepsilon\Big\}. \qquad (3.19)
\end{aligned}$$

Moreover,

$$\begin{aligned}
T_n &\le \int_0^1 ds \int_{\mathbb{R}} H(X_s(u))^\alpha \sum_{k=1}^{2^n} 1_{(n_{k-1},n_k]}(s) \Big| \frac{1}{2^n} \sum_{i=k}^{2^n} |p_{n_i - s}(x_1 - u) - p_{n_i - s}(x_2 - u)| \Big|^\alpha du\\
&\le \int_0^1 ds \int_{\mathbb{R}} H(X_s(u))^\alpha \sum_{k=1}^{2^n} 1_{(n_{k-1},n_k]}(s) \frac{1}{2^n} \sum_{i=k}^{2^n} |p_{n_i - s}(x_1 - u) - p_{n_i - s}(x_2 - u)|^\alpha du.
\end{aligned}$$

Applying Lemma 3.3 with $\delta = \alpha$ and $r = 1$ one gets

$$T_n \le C_\varepsilon |x_1 - x_2|^\alpha \qquad \text{on } \{\bar{\mathcal{V}}_n \le C_\varepsilon\},$$

which combined with (3.19) implies that

$$\begin{aligned}
&\mathbf{P}\Big\{L_{T_n} \ge r|x_1 - x_2|^\eta, A_n^\varepsilon\Big\}\\
&\quad \le \mathbf{P}\Big\{\sup_{v \le C_\varepsilon |x_1 - x_2|^\alpha} L_v 1_{\{\sup_{u<v} \Delta L_u \le C_\varepsilon |x_1 - x_2|^{\eta_c - 2\gamma}\}} \ge r|x_1 - x_2|^\eta\Big\}.
\end{aligned}$$

By (3.14) in [8], and [8, Lemma 2.3] with $\kappa = \alpha$, $t = C_\varepsilon |x_1 - x_2|^\alpha$, $x = r|x_1 - x_2|^\eta$, and $y = C_\varepsilon |x_1 - x_2|^{\eta_c - 2\gamma}$, one obtains

$$\mathbf{P}\Big\{L_{T_n} \ge r|x_1 - x_2|^\eta, A_n^\varepsilon\Big\} \le \Big(C_\varepsilon r^{-1} |x_1 - x_2|^{2\alpha - 2}\Big)^{C'_\varepsilon r |x_1 - x_2|^{\eta - \eta_c + 2\gamma}}. \qquad (3.20)$$

Taking $\gamma := \frac{\eta_c - \eta}{4}$, we have

$$\mathbf{P}\Big\{L_{T_n} \ge r|x_1 - x_2|^\eta, A_n^\varepsilon\Big\} \le (C_\varepsilon r^{-1} |x_1 - x_2|)^{C'_\varepsilon r |x_1 - x_2|^{(\eta - \eta_c)/2}},$$

which together with (3.16) proves the lemma. □

Lemma 3.7. *For each $n \ge 1$ and $1 \le i \le 2^n$, let $\{L_{n_i}(t) : t \ge 0\}$ be a spectrally positive α-stable process. Let $L_{n_i}(t)^-$ be the negative part of $L_{n_i}(t)$, and $T(t)$ be defined as in Lemma 3.5 with $\psi(s,u)$ replaced by $[p_{t-s}(x_1 - u) - p_{t-s}(x_2 - u)]^+$. Then for each $x, \varepsilon > 0$ and $n \ge 1$,*

$$\mathbf{P}\Big\{\frac{1}{2^n} \sum_{i=1}^{2^n} L_{n_i}(T(n_i))^- > x, \bar{\mathcal{U}}_n \le C_\varepsilon\Big\} \le C_\varepsilon \exp\Big\{-\frac{C'_\varepsilon x^{\alpha/(\alpha-1)}}{|x_1 - x_2|^{r'\delta'/(\alpha-1)}}\Big\},$$

where r' and δ' are defined in Lemma 3.3.

Proof. It is easy to see for all $h > 0$,

$$\begin{aligned}
&\mathbf{P}\Big\{\frac{1}{2^n}\sum_{i=1}^{2^n} L_{n_i}(T(n_i))^- > x, \bar{\mathcal{U}}_n \le C_\varepsilon\Big\} \\
&\qquad = \mathbf{P}\Big\{\exp\Big[\frac{h}{2^n}\sum_{i=1}^{2^n} L_{n_i}(T(n_i))^-\Big] > e^{hx}, \bar{\mathcal{U}}_n \le C_\varepsilon\Big\} \\
&\qquad \le e^{-hx}\mathbf{E}\Big\{\exp\Big[\frac{h}{2^n}\sum_{i=1}^{2^n} L_{n_i}(T(n_i))^-\Big] 1_{\{\bar{\mathcal{U}}_n \le C_\varepsilon\}}\Big\} \\
&\qquad \le e^{-hx}\prod_{i=1}^{2^n}\Big|\mathbf{E}\Big[\exp\Big(hL_{n_i}(T(n_i))^-\Big)1_{\{\bar{\mathcal{U}}_n \le C_\varepsilon\}}\Big]\Big|^{\frac{1}{2^n}}. \qquad (3.21)
\end{aligned}$$

Observe that

$$\begin{aligned}
& \mathbf{E}\Big[\exp\Big(hL_{n_i}(T(n_i))^-\Big)1_{\{\bar{\mathcal{U}}_n \le C_\varepsilon\}}\Big] \\
= \; & \mathbf{P}\Big[L_{n_i}(T(n_i))^- = 0, \bar{\mathcal{U}}_n \le C_\varepsilon\Big] + \mathbf{E}\Big[\exp\Big(hL_{n_i}(T(n_i))^-\Big)1_{\{\bar{\mathcal{U}}_n \le C_\varepsilon, L_{n_i}(T(n_i))^- > 0\}}\Big] \\
\le \; & 1 + \int_0^\infty e^{hy}\mathbf{P}\big[L_{n_i}(T(n_i))^- > y, \bar{\mathcal{U}}_n \le C_\varepsilon\big]dy. \qquad (3.22)
\end{aligned}$$

By Lemma 3.3,

$$T(n_i) \le C\bar{\mathcal{U}}_n|x_1 - x_2|^{r'\delta'} \le C_\varepsilon|x_1 - x_2|^{r'\delta'}$$

on $\{\bar{\mathcal{U}}_n \le C_\varepsilon\}$. Then using [8, Lemma 2.4], for each $y > 0$,

$$\begin{aligned}
&\mathbf{P}\big\{L_{n_i}(T(n_i))^- > y, \bar{\mathcal{U}}_n \le C_\varepsilon\big\} \\
&\quad = \mathbf{P}\Big\{L_{n_i}(T(n_i)) < -y, \bar{\mathcal{U}}_n \le C_\varepsilon\Big\} \\
&\quad \le \mathbf{P}\Big\{\inf_{u \le C_\varepsilon|x_1 - x_2|^{r'\delta'}} L_{n_i}(u) < -y\Big\} \\
&\quad \le \exp\Big\{-C_\varepsilon y^{\alpha/(\alpha-1)}|x_1 - x_2|^{-r'\delta'/(\alpha-1)}\Big\}. \qquad (3.23)
\end{aligned}$$

Since for all $a, b \ge 0$,

$$ab \le (1 - \alpha^{-1})a^{\alpha/(\alpha-1)} + \alpha^{-1}b^\alpha,$$

then

$$\begin{aligned}
hy \; &= \; \Big[\Big|\frac{C_\varepsilon}{2(1-\alpha^{-1})}\Big|^{\frac{\alpha-1}{\alpha}} y|x_1 - x_2|^{-r'\delta'/\alpha}\Big] \times \Big[\Big|\frac{C_\varepsilon}{2(1-\alpha^{-1})}\Big|^{\frac{1-\alpha}{\alpha}} h|x_1 - x_2|^{r'\delta'/\alpha}\Big] \\
&\le \; \frac{C_\varepsilon}{2}y^{\alpha/(\alpha-1)}|x_1 - x_2|^{-r'\delta'/(\alpha-1)} + \alpha^{-1}\Big|\frac{C_\varepsilon}{2(1-\alpha^{-1})}\Big|^{1-\alpha}h^\alpha|x_1 - x_2|^{r'\delta'}.
\end{aligned}$$

Thus combining (3.22) and (3.23) we have

$$\begin{aligned}
&\mathbf{E}\Big[\exp\Big(hL_{n_i}(T(n_i))^-\Big)1_{\{\bar{\mathcal{U}}_n \le C_\varepsilon\}}\Big] \\
&\quad \le 1 + \int_0^\infty e^{hy}\exp\Big\{-C_\varepsilon y^{\alpha/(\alpha-1)}|x_1 - x_2|^{-r'\delta'/(\alpha-1)}\Big\}dy \\
&\quad \le 1 + \exp\big\{C'_\varepsilon|x_1 - x_2|^{r'\delta'}h^\alpha\big\}\int_0^\infty \exp\Big\{-2^{-1}C_\varepsilon y^{\alpha/(\alpha-1)}|x_1 - x_2|^{-r'\delta'/(\alpha-1)}\Big\}dy \\
&\quad \le 1 + C_\varepsilon\exp\big\{C'_\varepsilon|x_1 - x_2|^{r'\delta'}h^\alpha\big\} \le C_\varepsilon\exp\big\{C'_\varepsilon|x_1 - x_2|^{r'\delta'}h^\alpha\big\}.
\end{aligned}$$

Then it follows from (3.21) that

$$\mathbf{P}\Big\{\frac{1}{2^n}\sum_{i=1}^{2^n} L_{n_i}(T(n_i))^- > x, \bar{\mathcal{U}}_n \le C_\varepsilon\Big\} \le C_\varepsilon\exp\big\{C'_\varepsilon|x_1 - x_2|^{r'\delta'}h^\alpha - hx\big\}.$$

Minimizing the function $h \mapsto C'_\varepsilon|x_1 - x_2|^{r'\delta'}h^\alpha - hx$, the desired result follows. □

We now return to the proof of Theorem 1.2. By Lemma 3.5, for each $0 < s \leq 1$, there exists a spectrally positive α-stable processes $\{L_s(t) : t \geq 0\}$ so that

$$\mathcal{U}_s(x_1, x_2) = L_s(T(s)),$$

where

$$T(s) = \int_0^s ds_1 \int_{\mathbb{R}} \Big[H(X_{s_1}(u))[p_{s-s_1}(x_1 - u) - p_{s-s_1}(x_2 - u)]^+\Big]^\alpha du.$$

Since $p < 1 + \alpha(\alpha - 1)/2$, applying Lemma 3.7 with $r' = 2 - \alpha$ and $\delta' = 1$, we get

$$\mathbf{P}\Big\{\frac{1}{2^n}\sum_{i=1}^{2^n}\mathcal{U}_{n_i}(x_1, x_2)^- > r|x_1 - x_2|^\eta, A_n^\varepsilon\Big\} \leq C_\varepsilon \exp\Big\{-\frac{C'_\varepsilon r^{\alpha/(\alpha-1)}}{|x_1 - x_2|^{(2-\alpha-\eta\alpha)/(\alpha-1)}}\Big\}, \quad (3.24)$$

where $\mathcal{U}_{n_i}(x_1, x_2)^-$ denotes the negative part of $\mathcal{U}_{n_i}(x_1, x_2)$. Observe that

$$\frac{1}{2^n}\sum_{i=1}^{2^n}|\mathcal{U}_{n_i}(x_1, x_2)| = \frac{1}{2^n}\sum_{i=1}^{2^n}\mathcal{U}_{n_i}(x_1, x_2) + \frac{2}{2^n}\sum_{i=1}^{2^n}\mathcal{U}_{n_i}(x_1, x_2)^-,$$

which together with Lemma 3.6 and (3.24) implies

$$\begin{aligned}
&\mathbf{P}\Big\{\frac{1}{2^n}\sum_{i=1}^{2^n}|\mathcal{U}_{n_i}(x_1, x_2)| > 2r|x_1 - x_2|^\eta, A_n^\varepsilon\Big\}\\
\leq\ &\mathbf{P}\Big\{\frac{1}{2^n}\sum_{i=1}^{2^n}\mathcal{U}_{n_i}(x_1, x_2) > r|x_1 - x_2|^\eta, A_n^\varepsilon\Big\} + \mathbf{P}\Big\{\frac{2}{2^n}\sum_{i=1}^{2^n}\mathcal{U}_{n_i}(x_1, x_2)^- > r|x_1 - x_2|^\eta, A_n^\varepsilon\Big\}\\
\leq\ &(C_\varepsilon r^{-1}|x_1 - x_2|)^{C'_\varepsilon r|x_1-x_2|^{(\eta-\eta_c)/2}} + C_\varepsilon \exp\Big\{-\frac{C'_\varepsilon r^{\alpha/(\alpha-1)}}{|x_1 - x_2|^{(2-\alpha-\eta\alpha)/(\alpha-1)}}\Big\}.
\end{aligned}$$

By the same argument we can also obtain the same estimation for $\mathcal{V}_s(x_1, x_2)$. It then follows from (3.13) that

$$\begin{aligned}
&\mathbf{P}\Big\{\frac{1}{2^n}\sum_{i=1}^{2^n}|Z_s(x_1) - Z_s(x_2)| > 8r|x_1 - x_2|^\eta, A_n^\varepsilon\Big\}\\
&\leq (C_\varepsilon r^{-1}|x_1 - x_2|)^{C'_\varepsilon r|x_1-x_2|^{(\eta-\eta_c)/2}} + C_\varepsilon \exp\Big\{-\frac{C'_\varepsilon r^{\alpha/(\alpha-1)}}{|x_1 - x_2|^{(2-\alpha-\eta\alpha)/(\alpha-1)}}\Big\}.
\end{aligned}$$

Define $Z_t^{n,\varepsilon} := Z_t 1_{A_n^\varepsilon}$. By the proof of Lemma 3.1, Z_t has a continuous modification $\tilde{Z}_t$ for fixed $t > 0$. Then $\tilde{Z}_t^{n,\varepsilon} := \tilde{Z}_t 1_{A_n^\varepsilon}$ is a continuous modification of $Z_t^{n,\varepsilon}$ for fixed $t > 0$. By [10, Lemma III.5.1], it is easy to see that

$$\mathbf{P}\Big\{\frac{1}{2^n}\sum_{k=1}^{2^n}\sup_{x,z\in\mathbb{K},|x-z|\leq\delta}|\tilde{Z}_{n_k}^{n,\varepsilon}(x) - \tilde{Z}_{n_k}^{n,\varepsilon}(z)| \geq r\mathcal{E}\Big(\Big[\log_2\frac{\mathbb{K}_0}{2\delta}\Big]\Big)\Big\} \leq Q\Big(\Big[\log_2\frac{\mathbb{K}_0}{2\delta}\Big], r\Big) \quad (3.25)$$

for all $n \geq 1$ and $\delta > 0$, where

$$\mathcal{E}(m) := \sum_{l=m}^\infty 8(2^{-l}\mathbb{K}_0)^\eta = \frac{8\mathbb{K}_0^\eta}{1 - 2^{-\eta}}2^{-\eta m}$$

and

$$Q(m, r) := \sum_{l=m}^\infty 2^{l+1}\Big[(C_\varepsilon r^{-1}2^{-l}\mathbb{K}_0)^{C'_\varepsilon r(2^{-l}\mathbb{K}_0)^{(\eta-\eta_c)/2}} + C_\varepsilon \exp\Big(-\frac{C'_\varepsilon r^{\alpha/(\alpha-1)}}{(2^{-l}\mathbb{K}_0)^{(2-\alpha-\eta\alpha)/(\alpha-1)}}\Big)\Big].$$

It is easy to check that for

$$Q(r):=\sum_{m=0}^{\infty}Q(m,r)<\infty,\quad Q(r)\to 0\quad\text{as}\quad r\to\infty. \tag{3.26}$$

Observe that for each $m,n\geq 1$ and $1\leq k\leq 2^n$

$$\sup_{x,z\in\mathbb{K},\frac{\delta}{2^{m+1}}<|x-z|\leq\frac{\delta}{2^m}}\frac{|\tilde{Z}^{n,\varepsilon}_{n_k}(x)-\tilde{Z}^{n,\varepsilon}_{n_k}(z)|}{|x-z|^\eta}\leq\sup_{x,z\in\mathbb{K},|x-z|\leq\frac{\delta}{2^m}}|\tilde{Z}^{n,\varepsilon}_{n_k}(x)-\tilde{Z}^{n,\varepsilon}_{n_k}(z)|\Big(\frac{\delta}{2^{m+1}}\Big)^{-\eta}.$$

This implies

$$\begin{aligned}&\Big\{\frac{1}{2^n}\sum_{k=1}^{2^n}\sup_{x,z\in\mathbb{K},\frac{\delta}{2^{m+1}}<|x-z|\leq\frac{\delta}{2^m}}\frac{|\tilde{Z}^{n,\varepsilon}_{n_k}(x)-\tilde{Z}^{n,\varepsilon}_{n_k}(z)|}{|x-z|^\eta}\geq r\Big\}\\&\quad\subset\Big\{\frac{1}{2^n}\sum_{k=1}^{2^n}\sup_{x,z\in\mathbb{K},|x-z|\leq\frac{\delta}{2^m}}|\tilde{Z}^{n,\varepsilon}_{n_k}(x)-\tilde{Z}^{n,\varepsilon}_{n_k}(z)|\geq r\Big(\frac{\delta}{2^{m+1}}\Big)^{\eta}\Big\}\\&\quad\subset\Big\{\frac{1}{2^n}\sum_{k=1}^{2^n}\sup_{x,z\in\mathbb{K},|x-z|\leq\frac{\delta}{2^m}}|\tilde{Z}^{n,\varepsilon}_{n_k}(x)-\tilde{Z}^{n,\varepsilon}_{n_k}(z)|\geq rc_1\mathcal{E}\Big(\Big[\log_2\frac{\mathbb{K}_0}{2^{-m}\delta}\Big]\Big)\Big\},\end{aligned}$$

where $c_1:=\inf_{m\geq 0}\mathcal{E}\Big(\Big[\log_2\frac{\mathbb{K}_0}{2^{-m}\delta}\Big]\Big)^{-1}\Big(\frac{\delta}{2^{m+1}}\Big)^{\eta}>0$. It follows from (3.25) that for each $m\geq 0$,

$$\begin{aligned}&\mathbf{P}\Big\{\frac{1}{2^n}\sum_{k=1}^{2^n}\sup_{x,z\in\mathbb{K},\frac{\delta}{2^{m+1}}<|x-z|\leq\frac{\delta}{2^m}}\frac{|\tilde{Z}^{n,\varepsilon}_{n_k}(x)-\tilde{Z}^{n,\varepsilon}_{n_k}(z)|}{|x-z|^\eta}\geq r\Big\}\\&\quad\leq\mathbf{P}\Big\{\frac{1}{2^n}\sum_{k=1}^{2^n}\sup_{x,z\in\mathbb{K},|x-z|\leq\frac{\delta}{2^m}}|\tilde{Z}^{n,\varepsilon}_{n_k}(x)-\tilde{Z}^{n,\varepsilon}_{n_k}(z)|\geq rc_1\mathcal{E}\Big(\Big[\log_2\frac{\mathbb{K}_0}{2^{-m}\delta}\Big]\Big)\Big\}\\&\quad\leq Q\Big(\Big[\log_2\frac{\mathbb{K}_0}{2^{-m}\delta}\Big],rc_1\Big),\end{aligned}$$

which implies

$$\mathbf{P}\Big\{\frac{1}{2^n}\sum_{k=1}^{2^n}\sup_{x,z\in\mathbb{K},0<|x-z|\leq\delta}\frac{|\tilde{Z}^{n,\varepsilon}_{n_k}(x)-\tilde{Z}^{n,\varepsilon}_{n_k}(z)|}{|x-z|^\eta}\geq r\Big\}\leq Q(rc_1).$$

Then by Fatou's lemma and (3.15) for each subsequence $\{n':n'\geq 1\}$ of $\{n:n\geq 1\}$,

$$\begin{aligned}&\mathbf{P}\Big\{\liminf_{n'\to\infty}\frac{1}{2^{n'}}\sum_{k=1}^{2^{n'}}\sup_{x,z\in\mathbb{K}\cap\mathbb{Q},0<|x-z|\leq\delta}\frac{|Z_{n'_k}(x)-Z_{n'_k}(z)|}{|x-z|^\eta}\geq r\Big\}\\&\quad\leq\liminf_{n'\to\infty}\mathbf{P}\Big\{\frac{1}{2^{n'}}\sum_{k=1}^{2^{n'}}\sup_{x,z\in\mathbb{K}\cap\mathbb{Q},0<|x-z|\leq\delta}\frac{|Z_{n'_k}(x)-Z_{n'_k}(z)|}{|x-z|^\eta}\geq r\Big\}\\&\quad\leq\liminf_{n'\to\infty}\Big\{\mathbf{P}\Big(\frac{1}{2^{n'}}\sum_{k=1}^{2^{n'}}\sup_{x,z\in\mathbb{K},0<|x-z|\leq\delta}\frac{|\tilde{Z}^{n',\varepsilon}_{n'_k}(x)-\tilde{Z}^{n',\varepsilon}_{n'_k}(z)|}{|x-z|^\eta}\geq r\Big)+\mathbf{P}(A^{\varepsilon,c}_{n'})\Big\}\\&\quad\leq Q(rc_1)+4\varepsilon.\end{aligned}$$

By first letting $r\to\infty$ and then letting $\varepsilon\to 0$, we immediately have

$$\liminf_{n'\to\infty}\frac{1}{2^{n'}}\sum_{k=1}^{2^{n'}}\sup_{x,z\in\mathbb{K}\cap\mathbb{Q},0<|x-z|\leq\delta}\frac{|Z_{n'_k}(x)-Z_{n'_k}(z)|}{|x-z|^\eta}<\infty,\quad\mathbf{P}\text{-a.s.}$$

by (3.26). Letting $\delta \to \infty$ we have

$$\liminf_{n'\to\infty} \frac{1}{2^{n'}} \sum_{k=1}^{2^{n'}} \sup_{x,z\in\mathbb{K}\cap\mathbb{Q}} \frac{|Z_{n'_k}(x) - Z_{n'_k}(z)|}{|x-z|^\eta} < \infty, \qquad \mathbf{P}\text{-a.s.} \tag{3.27}$$

Thus (1.6) follows from (2.4) and (3.2)–(3.3). This completes the proof. □

4 Proof of Theorem 1.5

4.1 The proof

In this subsection we prove the pathwise uniqueness of solution for (1.4). Throughout this subsection we always assume that the assumptions of Theorem 1.5 hold. For the proof of Theorem 1.5 we adopt the arguments from [23, 22]. By conditioning we may assume that the initial states X_0 and Y_0 are both deterministic. For $n \geq 1$ define

$$a_n := \exp\{-n(n+1)/2\}.$$

Then $a_{n+1} = a_n a_n^{2/n}$. Let $\psi_n \in C_c^\infty(\mathbb{R})$ satisfy $\mathrm{supp}(\psi_n) \subset (a_n, a_{n-1})$, $\int_{a_n}^{a_{n-1}} \psi_n(x)dx = 1$, and $0 \leq \psi_n(x) \leq 2/(nx) \leq 2/(na_n)$ for all $x > 0$ and $n \geq 1$. For $x \in \mathbb{R}$ and $n \geq 1$ let

$$\phi_n(x) := \int_0^{|x|} dy \int_0^y \psi_n(z)dz.$$

Then $\|\phi_n'\| \leq 1$, $\phi_n(x) \to |x|$, and $\phi_n'(x) \to \mathrm{sgn}(x)$ for $x \in \mathbb{R}$ as $n \to \infty$. For $n \geq 1$ and $y, z \in \mathbb{R}$ put

$$D_n(y,z) := \phi_n(y+z) - \phi_n(y) - z\phi_n'(y) \ \text{ and } \ H_n(y,z) := \phi_n(y+z) - \phi_n(y).$$

Let $\Phi \in C_c^\infty(\mathbb{R})$ satisfy $0 \leq \Phi \leq 1$, $\mathrm{supp}(\Phi) \subset (-1,1)$ and $\int_{\mathbb{R}} \Phi(x)dx = 1$. Let $\Phi_x^m(y) = \Phi^m(x,y) := m\Phi(m(x-y))$ for $x, y \in \mathbb{R}$ and $m \geq 1$. For $t \geq 0$ and $y \in \mathbb{R}$ let $U_t(y) := X_t(y) - Y_t(y)$, $V_t(y) := H(X_t(y)) - H(Y_t(y))$ and $R_t(y) := G(X_t(y)) - G(Y_t(y))$. By the argument in Section 2, both $\{X_t : t \geq 0\}$ and $\{Y_t : t \geq 0\}$ satisfy equation (2.2). Using (2.2) and Itô's formula we have

$$\begin{aligned}
\phi_n(\langle U_t, \Phi_x^m\rangle) \ = \ & \frac{1}{2}\int_0^t \phi_n'(\langle U_s, \Phi_x^m\rangle)\langle U_s, \Delta\Phi_x^m\rangle ds + \int_0^t \phi_n'(\langle U_s, \Phi_x^m\rangle)\langle R_s, \Phi_x^m\rangle ds \\
& + \int_0^t \int_0^\infty \int_{\mathbb{R}} H_n(\langle U_{s-}, \Phi_x^m\rangle, zV_{s-}(y)\Phi_x^m(y))\tilde{N}(ds,dz,dx) \\
& + \int_0^t ds \int_0^\infty m_0(dz) \int_{\mathbb{R}} D_n(\langle U_s, \Phi_x^m\rangle, zV_s(y)\Phi_x^m(y))dy.
\end{aligned} \tag{4.1}$$

For $t > 0$ let $\tilde{X}_t$ and $\tilde{Y}_t$ denote the continuous modifications of X_t and Y_t, respectively. Let $\tilde{U}_t(y) := \tilde{X}_t(y) - \tilde{Y}_t(y)$, $\tilde{V}_t(y) := H(\tilde{X}_t(y)) - H(\tilde{Y}_t(y))$ and $\tilde{R}_t(y) := G(\tilde{X}_t(y)) - G(\tilde{Y}_t(y))$.

For $T, K > 0$, let Ψ be a nonnegative and compactly supported infinitely differentiable function on $[0,T] \times \mathbb{R}$ satisfying

$$\Psi_s(x) = 0 \text{ for all } (s,x) \in [0,T] \times [-K,K]^c.$$

By (4.1) and a stochastic Fubini's theorem, it is easy to see that

$$\begin{aligned}
& \langle \phi_n(\langle U_t, \Phi_\cdot^m\rangle), \Psi_t\rangle \\
= \ & \sum_{i=1}^k [\langle \phi_n(\langle U_{t_i}, \Phi_\cdot^m\rangle), \Psi_{t_i}\rangle - \langle \phi_n(\langle U_{t_{i-1}}, \Phi_\cdot^m\rangle), \Psi_{t_{i-1}}\rangle]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^{k}[\langle \phi_n(\langle U_{t_i}, \Phi^m_\cdot\rangle), \Psi_{t_{i-1}}\rangle - \langle \phi_n(\langle U_{t_{i-1}}, \Phi^m_\cdot\rangle), \Psi_{t_{i-1}}\rangle] \\
&\quad + \sum_{i=1}^{k}[\langle \phi_n(\langle U_{t_i}, \Phi^m_\cdot\rangle), \Psi_{t_i} - \Psi_{t_{i-1}}\rangle] \\
&= \frac{1}{2}\int_0^t \sum_{i=1}^{k} I_i(s)\langle \phi_n'(\langle U_s, \Phi^m_\cdot\rangle)\langle U_s, \Delta\Phi^m_\cdot\rangle, \Psi_{t_{i-1}}\rangle ds \\
&\quad + \int_0^t \sum_{i=1}^{k} I_i(s)\langle \phi_n'(\langle U_s, \Phi^m_\cdot\rangle)\langle R_s, \Phi^m_\cdot\rangle, \Psi_{t_{i-1}}\rangle ds + \int_0^t \sum_{i=1}^{k} I_i(s)\langle \phi_n(\langle U_{t_i}, \Phi^m_\cdot\rangle), \dot{\Psi}_s\rangle ds \\
&\quad + \int_0^t \int_0^\infty \int_{\mathbb{R}} \sum_{i=1}^{k} I_i(s)\langle H_n(\langle U_{s-}, \Phi^m_\cdot\rangle, zV_{s-}(y)\Phi^m_\cdot(y)), \Psi_{t_{i-1}}\rangle \tilde{N}(ds, dz, dx) \\
&\quad + \int_0^t ds \int_0^\infty m_0(dz) \int_{\mathbb{R}} \sum_{i=1}^{k} I_i(s)\langle D_n(\langle U_s, \Phi^m_\cdot\rangle, zV_s(y)\Phi^m_\cdot(y)), \Psi_{t_{i-1}}\rangle dy,
\end{aligned}$$

where $0 = t_0 < t_1 < \cdots < t_k = t$ and $I_i(s) := 1_{(t_{i-1}, t_i]}(s)$. Letting $\max_{1\le i\le k}(t_i - t_{i-1})$ converge to zero we have $\mathbf{P}$-a.s.

$$\begin{aligned}
\langle \phi_n(\langle U_t, \Phi^m_\cdot\rangle), \Psi_t\rangle &= \frac{1}{2}\int_0^t \langle \phi_n'(\langle U_s, \Phi^m_\cdot\rangle)\langle U_s, \Delta\Phi^m_\cdot\rangle, \Psi_s\rangle ds \\
&\quad + \int_0^t \langle \phi_n'(\langle U_s, \Phi^m_\cdot\rangle)\langle R_s, \Phi^m_\cdot\rangle, \Psi_s\rangle ds + \int_0^t \langle \phi_n(\langle U_s, \Phi^m_\cdot\rangle), \dot{\Psi}_s\rangle ds \\
&\quad + \int_0^t \int_0^\infty \int_{\mathbb{R}} \langle H_n(\langle U_{s-}, \Phi^m_\cdot\rangle, zV_{s-}(y)\Phi^m_\cdot(y)), \Psi_s\rangle \tilde{N}(ds, dz, dy) \\
&\quad + \int_0^t ds \int_0^\infty m_0(dz) \int_{\mathbb{R}} \langle D_n(\langle U_s, \Phi^m_\cdot\rangle, zV_s(y)\Phi^m_\cdot(y)), \Psi_s\rangle dy \\
&=: I_1^{m,n}(t) + I_2^{m,n}(t) + I_3^{m,n}(t) + I_4^{m,n}(t) + I_5^{m,n}(t), \qquad t \ge 0. \qquad (4.2)
\end{aligned}$$

For $k \ge 1$ define a stopping time γ_k by

$$\gamma_k := \inf\Big\{t \in (0, T] : \langle X_t, 1\rangle + \langle Y_t, 1\rangle > k\Big\} \qquad (4.3)$$

with the convention $\inf\emptyset = \infty$. By Definition 1.1,

$$\{1_{\{t=0\}}X_0(1) + 1_{\{t>0\}}\langle X_t, 1\rangle : t \ge 0\} \quad \text{and} \quad \{1_{\{t=0\}}Y_0(1) + 1_{\{t>0\}}\langle Y_t, 1\rangle : t \ge 0\}$$

are cádlág processes. Thus

$$\sup_{t\in(0,T]}[\langle X_t, 1\rangle + \langle Y_t, 1\rangle] < \infty, \quad \mathbf{P}\text{-a.s.},$$

which implies $\lim_{k\to\infty}\gamma_k = \infty$ almost surely. Let $\{l' : l' \ge 1\}$ be the subsequence of $\{l : l \ge 1\}$ that will be determined later in Lemma 4.6. For $0 \le i \le 2^{l'}$, let $l'_i = i/2^{l'}$. For any nonnegative function f define

$$\bar{\int}_{(0,t]} f(s)ds := \liminf_{l'\to\infty}\int_0^t \sum_{i=1}^{2^{l'}} 1_{(l'_{i-1}T, l'_iT]}(s) f(l'_iT)ds, \qquad t > 0$$

and

$$\bar{\int}_{(0,t)} f(s)ds := \lim_{t'\uparrow t}\bar{\int}_{(0,t']} f(s)ds.$$

Then $\bar{\int}_{(0,t]} f(s)ds \leq \bar{\int}_{(0,T]} f(s)ds$. For fixed $K > 0$ and $0 < \eta < \eta_c = \frac{2}{\alpha} - 1$ define σ_k by

$$\sigma_k = \inf\Big\{t \in (0,T] : \bar{\int}_{(0,t]} \sup_{\substack{x \neq z,\\ x,z \in [-(K+1),K+1] \cap \mathbb{Q}}} \frac{|X_s(x) - X_s(z)| \vee |Y_s(x) - Y_s(z)|}{|x-z|^\eta} ds > k\Big\}. \tag{4.4}$$

By Theorem 1.2,

$$\bar{\int}_{(0,T]} \sup_{\substack{x \neq z,\\ x,z \in [-(K+1),K+1] \cap \mathbb{Q}}} \frac{|X_s(x) - X_s(z)| \vee |Y_s(x) - Y_s(z)|}{|x-z|^\eta} ds < \infty, \quad \mathbf{P}\text{-a.s.},$$

which implies $\lim_{k\to\infty} \sigma_k = \infty$, $\mathbf{P}$-a.s.

In the rest of this subsection we always write

$$\tau_k := \min\{\gamma_k, \sigma_k\}.$$

Before proving Theorem 1.5, we state three important lemmas. Similar to [23, Lemma 2.2(b)] we have the following result.

Lemma 4.1. *For any stopping time τ and $t > 0$, we have*

$$\limsup_{m,n\to\infty} \mathbf{E}\{I_1^{m,n}(t\wedge\tau)\} \leq \frac{1}{2}\mathbf{E}\Big\{\int_0^{t\wedge\tau} ds \int_{\mathbb{R}} |U_s(x)|\Delta\Psi_s(x)dx\Big\}, \tag{4.5}$$

$$\lim_{m,n\to\infty} \mathbf{E}\{I_2^{m,n}(t\wedge\tau)\} = \mathbf{E}\Big\{\int_0^{t\wedge\tau} ds \int_{\mathbb{R}} \mathrm{sgn}(U_s(x))R_s(x)\Psi_s(x)dx\Big\}$$

and

$$\lim_{m,n\to\infty} \mathbf{E}\{I_3^{m,n}(t\wedge\tau)\} = \mathbf{E}\Big\{\int_0^{t\wedge\tau} ds \int_{\mathbb{R}} |U_s(x)|\dot{\Psi}_s(x)dx\Big\}.$$

Lemma 4.2. *For any stopping time τ, any $t > 0$ and $m, n \geq 1$, we have*

$$\mathbf{E}\{I_4^{m,n}(t\wedge\tau)\} = 0. \tag{4.6}$$

The first inequality of (1.7) is equivalent to $\beta > \frac{(\alpha-1)(\eta_c+1)}{(2-\alpha)\eta_c}$, which is also equivalent to $\eta_c^{-1} < \frac{(2-\alpha)\beta}{\alpha-1} - 1$. Thus there exist constants $\varepsilon, \delta > 0$ satisfying

$$\eta_c^{-1} < \delta < \frac{(2-\alpha)\beta}{\alpha-1} - 1 \;\text{ and }\; \frac{\delta+1}{2-\alpha} < \varepsilon < \frac{\beta}{\alpha-1},$$

which implies

$$\frac{\delta+1}{2-\alpha} < \varepsilon < \frac{\delta\eta_c\beta}{\alpha-1} \;\text{ and }\; \varepsilon < \frac{\beta}{\alpha-1}. \tag{4.7}$$

Lemma 4.3. *If $m = a_{n-1}^{-\delta}$ for the δ in (4.7), then for each $t > 0$ and $k \geq 1$,*

$$\lim_{n\to\infty} \mathbf{E}\{I_5^{m,n}(t\wedge\tau_k)\} = 0.$$

Deferring the proofs of Lemmas 4.3–4.5, we first present the main proof.

Proof of Theorem 1.5. By the continuity of $x \mapsto \tilde{U}_t(x)$, for each $x \in \mathbb{R}$ and $t > 0$,

$$\lim_{m\to\infty} \langle \tilde{U}_t, \Phi_x^m\rangle = \lim_{m\to\infty} \int_{\mathbb{R}} \tilde{U}_t(x - \frac{y}{m})\Phi(y)dy = \tilde{U}_t(x). \tag{4.8}$$

Note that $\|\phi_n'\| \leq 1$. Then for all $x_m \to x$ as $m \to \infty$, we have that

$$|\phi_n(x_m) - |x|| \leq |\phi_n(x_m) - \phi_n(x)| + |\phi_n(x) - |x|| \leq |x_m - x| + |\phi_n(x) - |x||,$$

which converges to zero as $m,n\to\infty$. Now by (4.8) and Fatou's lemma

$$\begin{aligned}&\mathbf{E}\{\langle|U_t|,\Psi_t\rangle 1_{\{t\le\tau_k\}}\}=\mathbf{E}\Big\{\langle|\tilde U_t|,\Psi_t\rangle 1_{\{t\le\tau_k\}}\Big\}\\ =&\ \mathbf{E}\Big\{\big\langle\lim_{m,n\to\infty}\phi_n(\langle\tilde U_t,\Phi^m_\cdot\rangle),\Psi_t\big\rangle 1_{\{t\le\tau_k\}}\Big\}\le\liminf_{m,n\to\infty}\mathbf{E}\Big\{\langle\phi_n(\langle\tilde U_t,\Phi^m_\cdot\rangle),\Psi_t\rangle 1_{\{t\le\tau_k\}}\Big\}\\ =&\ \liminf_{m,n\to\infty}\mathbf{E}\Big\{\langle\phi_n(\langle U_t,\Phi^m_\cdot\rangle),\Psi_t\rangle 1_{\{t\le\tau_k\}}\Big\}\le\liminf_{m,n\to\infty}\mathbf{E}\Big\{\langle\phi_n(\langle U_{t\wedge\tau_k},\Phi^m_\cdot\rangle),\Psi_{t\wedge\tau_k}\rangle\Big\}.\end{aligned}$$

Together with (4.2) and Lemmas 4.1–4.3 we have

$$\begin{aligned}\mathbf{E}\{\langle|U_t|,\Psi_t\rangle 1_{\{t\le\tau_k\}}\}\ \le\ &\mathbf{E}\Big\{\int_0^{t\wedge\tau_k}ds\int_{\mathbb{R}}|U_s(x)|\big[\frac{1}{2}\Delta\Psi_s(x)+\dot\Psi_s(x)\big]dx\Big\}\\ &+\mathbf{E}\Big\{\int_0^{t\wedge\tau_k}ds\int_{\mathbb{R}}\mathrm{sgn}(U_s(x))R_s(x)\Psi_s(x)dx\Big\}.\end{aligned}$$

Letting $k\to\infty$ in the above inequality we have

$$\begin{aligned}\mathbf{E}\{\langle|U_t|,\Psi_t\rangle\}\ \le\ &\int_0^t ds\int_{\mathbb{R}}\mathbf{E}[|U_s(x)|]\big[\frac{1}{2}\Delta\Psi_s(x)+\dot\Psi_s(x)\big]dx\\ &+\int_0^t ds\int_{\mathbb{R}}\mathbf{E}\big[\mathrm{sgn}(U_s(x))R_s(x)\big]\Psi_s(x)dx.\end{aligned}$$

This is similar to (34) in [23]. Then by the same argument as [23, Theorem 1.6], for any fixed $t>0$ and nonnegative $f\in C_c^\infty(\mathbb{R})$, with $\Psi_s(x)$ replaced by $\bar\Psi_N(s,x):=(P_{t-s}f(x))g_N(x)$ for a proper sequence of functions $(g_N)_{N\ge 1}$ so that $g_N(x)\to 1$ for all $x\in\mathbb{R}$ and the first term on the right hand side of the above inequality goes to zero as $N\to\infty$. Thus, we have

$$\langle\mathbf{E}[|U_t|],f\rangle\le\int_0^t ds\int_{\mathbb{R}}\mathbf{E}\big[\mathrm{sgn}(U_s(x))R_s(x)\big]P_{t-s}f(x)dx\le\int_0^t\langle\mathbf{E}[r_0(|U_s|)],P_{t-s}f\rangle ds,$$

where condition (C2) is needed for the last inequality. It is elementary to check that the above inequality holds for each $f\in\mathscr{B}(\mathbb{R})^+$ satisfying $\lambda_0(f)<\infty$. This means that for each $f\in\mathscr{B}(\mathbb{R})^+$ satisfying $\lambda_0(f)=1$,

$$\langle\mathbf{E}[|U_t|],P_{T-t}f\rangle\le\int_0^t\langle\mathbf{E}[r_0(|U_s|)],P_{T-s}f\rangle ds.$$

Then by the concaveness of $x\mapsto r_0(x)$ and Jensen's inequality,

$$\begin{aligned}\langle\mathbf{E}[|U_t|],P_{T-t}f\rangle&\le\int_{\mathbb{R}}\mathbf{E}[|U_t(x)|]P_{T-t}f(x)dx\\ &=\int_0^t ds\int_{\mathbb{R}}r_0\big(\mathbf{E}[|U_s(x)|]\big)P_{T-s}f(x)dx\le\int_0^t r_0\big(\langle\mathbf{E}[|U_s|],P_{T-s}f\rangle\big)ds.\end{aligned}\tag{4.9}$$

Since $\int_{0+}r_0(z)^{-1}dz=\infty$, the above inequality implies that $\langle\mathbf{E}[|U_t|],P_{T-t}f\rangle=0$ for all $t>0$. Thus

$$\mathbf{P}\{X_t(x)=Y_t(x)\text{ for }\lambda_0\text{-a.e. }x\}=1$$

for all $t>0$. It follows that $\langle X_t,f\rangle=\langle Y_t,f\rangle$ $\mathbf{P}$-a.s. for all $t>0$ and $f\in\mathscr{S}(\mathbb{R})$. By the right-continuities of $t\mapsto\langle X_t,f\rangle$ and $t\mapsto\langle Y_t,f\rangle$ we have $\mathbf{P}\{\langle X_t,f\rangle=\langle Y_t,f\rangle$ for all $t>0\}=1$ for all $f\in\mathscr{S}(\mathbb{R})$. Considering a suitable sequence $\{f_1,f_2,\cdots\}\subset\mathscr{S}(\mathbb{R})$ we can conclude (1.8). □

We now present the proofs of Lemmas 4.1–4.3.

Proof of Lemma 4.1. By the same argument as [23, Lemma 2.2(b)],

$$\limsup_{m,n\to\infty} \mathbf{E}\{I_1^{m,n}(t\wedge\tau)\} \\ \leq \limsup_{m,n\to\infty} \frac{1}{2}\,\mathbf{E}\Big\{\int_0^{t\wedge\tau} ds \int_{\mathbb{R}} \phi_n'(\langle U_s, \Phi_x^m\rangle)\langle U_s, \Phi_x^m\rangle \Delta\Psi_s(x)dx\Big\}. \tag{4.10}$$

Now using (4.8) and the fact that $\phi_n'(x)\to \mathrm{sgn}(x)$ as $n\to\infty$, we have

$$\lim_{m,n\to\infty} \phi_n'(\langle \tilde{U}_s, \Phi_x^m\rangle)\langle \tilde{U}_s, \Phi_x^m\rangle = \lim_{m,n\to\infty} \phi_n'(\langle \tilde{U}_s, \Phi_x^m\rangle) \int_{-1}^1 \tilde{U}_s(x-\frac{y}{m})\Phi(y)dy = |\tilde{U}_s(x)|.$$

Observe that $\|\phi_n'\|\leq 1$ for all $n\geq 1$ and

$$0\leq \phi_n'(\langle \tilde{U}_s, \Phi_x^m\rangle)\langle \tilde{U}_s, \Phi_x^m\rangle = \phi_n'(\langle \tilde{U}_s, \Phi_x^m\rangle)\int_{\mathbb{R}} \tilde{U}_s(x-\frac{y}{m})\Phi(y)dy \leq \sup_{|y|\leq K+1} |\tilde{X}_s(y)+\tilde{Y}_s(y)|.$$

Then by (4.10), (3.4) and the dominated convergence

$$\begin{aligned}\limsup_{m,n\to\infty} \mathbf{E}\{I_1^{m,n}(t\wedge\tau)\} &\leq \limsup_{m,n\to\infty} \frac{1}{2}\int_0^t ds\int_{\mathbb{R}} \mathbf{E}\Big\{\phi_n'(\langle \tilde{U}_s, \Phi_x^m\rangle)\langle \tilde{U}_s, \Phi_x^m\rangle\Delta\Psi_s(x)1_{\{s\leq\tau\}}\Big\}dx \\ &\leq \frac{1}{2}\int_0^t ds\int_{\mathbb{R}} \mathbf{E}\Big\{\limsup_{m,n\to\infty}\phi_n'(\langle \tilde{U}_s, \Phi_x^m\rangle)\langle \tilde{U}_s, \Phi_x^m\rangle\Delta\Psi_s(x)1_{\{s\leq\tau\}}\Big\}dx \\ &= \frac{1}{2}\int_0^t ds\int_{\mathbb{R}} \mathbf{E}\Big\{|\tilde{U}_s(x)|\Delta\Psi_s(x)1_{\{s\leq\tau\}}\Big\}dx \\ &= \frac{1}{2}\,\mathbf{E}\Big\{\int_0^{t\wedge\tau} ds\int_{\mathbb{R}} |U_s(x)|\Delta\Psi_s(x)dx\Big\}.\end{aligned}$$

By the continuity of $x\mapsto \tilde{U}_s(x)$ and $x\mapsto \tilde{R}_s(x)$, one also sees that

$$\lim_{m,n\to\infty} \phi_n'(\langle \tilde{U}_s, \Phi_x^m\rangle)\langle \tilde{R}_s, \Phi_x^m\rangle \\ = \lim_{m,n\to\infty} \phi_n'(\langle \tilde{U}_s, \Phi_x^m\rangle)\int_{-1}^1 \tilde{R}_s(x-\frac{y}{m})\Phi(y)dy = \mathrm{sgn}(\tilde{U}_s(x))\tilde{R}_s(x).$$

By condition (C1) and the fact $\|\phi_n'\|\leq 1$ we have

$$|\phi_n'(\langle \tilde{U}_s, \Phi_x^m\rangle)\langle \tilde{R}_s, \Phi_x^m\rangle| \leq |\langle \tilde{R}_s, \Phi_x^m\rangle| = \Big|\int_{\mathbb{R}} \tilde{R}_s(x-\frac{y}{m})\Phi(y)dy\Big| \\ \leq C\int_{\mathbb{R}} [\tilde{X}_s(x-\frac{y}{m})+\tilde{Y}_s(x-\frac{y}{m})]\Phi(y)dy + C \leq C\sup_{|y|\leq K+1}|\tilde{X}_s(y)+\tilde{Y}_s(y)| + C.$$

By the dominated convergence again,

$$\begin{aligned}\lim_{m,n\to\infty} \mathbf{E}\{I_2^{m,n}(t\wedge\tau)\} &= \int_0^{t\wedge\tau} ds\int_{\mathbb{R}} \mathbf{E}\Big\{\lim_{m,n\to\infty}\phi_n'(\langle \tilde{U}_s, \Phi_x^m\rangle)\langle \tilde{R}_s, \Phi_x^m\rangle\Psi_s(x)\Big\}dx \\ &= \int_0^{t\wedge\tau} ds\int_{\mathbb{R}} \mathbf{E}\big\{\mathrm{sgn}(U_s(x))R_s(x)\Psi_s(x)\big\}dx.\end{aligned}$$

By the fact $\|\phi_n'\|\leq 1$ again,

$$\Big|\mathbf{E}\{I_3^{m,n}(t\wedge\tau)\} - \mathbf{E}\Big\{\int_0^{t\wedge\tau} ds\int_{\mathbb{R}} |U_s(x)|\dot{\Psi}_s(x)dx\Big\}\Big| \\ \leq \int_0^t ds\int_{\mathbb{R}} \mathbf{E}\{|\phi_n(\langle U_s, \Phi_x^m\rangle) - |U_s(x)||\}|\dot{\Psi}_s(x)|dx \\ \leq \int_0^t ds\int_{\mathbb{R}} \mathbf{E}\Big\{|\phi_n(\langle U_s, \Phi_x^m\rangle) - \phi_n(U_s(x))| + |\phi_n(U_s(x)) - |U_s(x)||\Big\}|\dot{\Psi}_s(x)|dx$$

$$
\begin{aligned}
&\leq \int_0^t ds \int_{\mathbb{R}} |\dot{\Psi}_s(x)| dx \int_{\mathbb{R}} \mathbf{E}\Big\{|U_s(x-\frac{y}{m}) - U_s(x)|\Big\}\Phi(y)dy \\
&\quad + \int_0^t ds \int_{\mathbb{R}} \mathbf{E}\{|\phi_n(U_s(x)) - |U_s(x)||\}|\dot{\Psi}_s(x)|dx.
\end{aligned}
$$

Now by (2.11) and the dominated convergence we finish the proof. □

Proof of Lemma 4.2. For $t \geq 0$ and $m, n \geq 1$ let

$$
I_{4,1}^{m,n}(t) := \int_0^t \int_0^1 \int_{\mathbb{R}} \langle H_n(\langle U_{s-}, \Phi_\cdot^m\rangle, zV_{s-}(y)\Phi_\cdot^m(y)), \Psi_s\rangle \tilde{N}(ds, dz, dy)
$$

and $I_{4,2}^{m,n}(t) := I_4^{m,n}(t) - I_{4,1}^{m,n}(t)$. By the Burkholder-Davis-Gundy inequality (see [27, p.195]), for $\bar{\alpha} \in (\alpha, \frac{\alpha}{\beta} \wedge 2)$ and $T > 0$,

$$
\begin{aligned}
&\mathbf{E}\Big\{\sup_{t\in[0,T]} |I_{4,1}^{m,n}(t\wedge\tau)|^{\bar{\alpha}}\Big\} \\
&\leq C\,\mathbf{E}\Big\{\Big[\int_0^T \int_0^1 \int_{\mathbb{R}} |\langle H_n(\langle U_{s-}, \Phi_\cdot^m\rangle, zV_{s-}(y)\Phi_\cdot^m(y)), \Psi_s\rangle|^2 N(ds, dz, dy)\Big]^{\frac{\bar{\alpha}}{2}}\Big\} \\
&\leq C\,\mathbf{E}\Big\{\int_0^T \int_0^1 \int_{\mathbb{R}} |\langle H_n(\langle U_{s-}, \Phi_\cdot^m\rangle, zV_{s-}(y)\Phi_\cdot^m(y)), \Psi_s\rangle|^{\bar{\alpha}} N(ds, dz, dy)\Big\},
\end{aligned}
$$

where for the last inequality we used the fact that

$$
\Big|\sum_{i=1}^n x_i^2\Big|^{\frac{\bar{\alpha}}{2}} \leq \sum_{i=1}^n |x_i|^{\bar{\alpha}}
$$

for all $x_i \in \mathbb{R}$ and $n \geq 1$. Since $|H_n(y,z)| \leq |z|$ for all $y, z \in \mathbb{R}$, and Ψ is continuous and compactly supported, combining the Hölder inequality, condition (C3) and Lemma 2.4 we have

$$
\begin{aligned}
&\mathbf{E}\Big\{\sup_{t\in[0,T]} |I_{4,1}^{m,n}(t\wedge\tau)|^{\bar{\alpha}}\Big\} \\
&\leq C\,\mathbf{E}\Big\{\int_0^T \int_0^1 \int_{\mathbb{R}} \Big|z\int_{\mathbb{R}} V_{s-}(y)\Phi_x^m(y)\Psi_s(x)dx\Big|^{\bar{\alpha}} N(ds, dz, dy)\Big\} \\
&\leq C\,\mathbf{E}\Big\{\int_0^T \int_0^1 \int_{\mathbb{R}} z^{\bar{\alpha}}\Big[\int_{\mathbb{R}} |V_{s-}(y)\Phi_x^m(y)\Psi_s(x)|^{\bar{\alpha}}dx\Big] N(ds, dz, dy)\Big\} \\
&= C\int_0^T ds \int_0^1 z^{\bar{\alpha}} m_0(dz) \int_{\mathbb{R}} \Psi_s(x)^{\bar{\alpha}} dx \int_{\mathbb{R}} \mathbf{E}\{|V_s(y)|^{\bar{\alpha}}\}\Phi_x^m(y)^{\bar{\alpha}} dy \\
&\leq C\int_0^T ds \int_{\mathbb{R}} \Psi_s(x)^{\bar{\alpha}} dx \int_{\mathbb{R}} \mathbf{E}\{|X_s(y) - Y_s(y)|^{\bar{\alpha}\beta}\}\Phi_x^m(x)^{\bar{\alpha}} dy \\
&\leq C\int_0^T ds \int_{\mathbb{R}} \Psi_s(x)^{\bar{\alpha}} dx \int_{\mathbb{R}} \mathbf{E}\{X_s(y)^{\bar{\alpha}\beta} + Y_s(y)^{\bar{\alpha}\beta}\}\Phi_x^m(x)^{\bar{\alpha}} dy < \infty.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\mathbf{E}\Big\{\sup_{t\in[0,T]} |I_{4,2}^{m,n}(t\wedge\tau)|\Big\} \\
&\leq \mathbf{E}\Big\{\int_0^T \int_1^\infty \int_{\mathbb{R}} |\langle H_n(\langle U_{s-}, \Phi_\cdot^m\rangle, zV_{s-}(y)\Phi_\cdot^m(y)), \Psi_s\rangle| N(ds, dz, dy)\Big\} \\
&\quad + \mathbf{E}\Big\{\int_0^T ds \int_1^\infty m_0(dz) \int_{\mathbb{R}} |\langle H_n(\langle U_{s-}, \Phi_\cdot^m\rangle, zV_{s-}(y)\Phi_\cdot^m(y)), \Psi_s\rangle| dy\Big\} \\
&= 2\int_0^T ds \int_1^\infty m_0(dz) \int_{\mathbb{R}} \mathbf{E}\Big\{|\langle H_n(\langle U_{s-}, \Phi_\cdot^m\rangle, zV_{s-}(y)\Phi_\cdot^m(y)), \Psi_s\rangle|\Big\} dy
\end{aligned}
$$

$$\leq 2\int_0^T ds \int_1^\infty z m_0(dz) \int_{\mathbb{R}} \Psi_s(x)dx \int_{\mathbb{R}} \mathbf{E}\{|V_s(y)|\}|\Phi_x^m(y)|dy < \infty.$$

It follows that for $T > 0$,

$$\mathbf{E}\Big\{ \sup_{t\in[0,T]} |I_4^{m,n}(t\wedge\tau)|\Big\} < \infty.$$

Then by [27, p.38], $t \mapsto I_4^{m,n}(t\wedge\tau)$ is a martingale, which implies (4.6). □

To prove Lemma 4.3, we only need to show the following two lemmas.

Lemma 4.4. *For* $m, n, k, i \geq 1$ *and* $t \in [0, T]$ *let*

$$I_{5,1}^{m,n,k,i}(t) := \mathbf{E}\Big\{ \int_0^{t\wedge\gamma_k} ds \int_0^{1/i} m_0(dz) \int_{\mathbb{R}} \langle D_n(\langle U_s, \Phi_\cdot^m\rangle, zV_s(y)\Phi_\cdot^m(y)), \Psi_s\rangle dy\Big\}.$$

Then

$$I_{5,1}^{m,n,k,i}(t) \leq C_T km(na_n)^{-1} i^{\alpha-2}, \qquad t \in [0,T].$$

Lemma 4.5. *For* $m, n, k, i \geq 1$ *and* $t \in [0, T]$*, let*

$$I_{5,2}^{m,n,k,i}(t) := \mathbf{E}\Big\{ \int_0^{t\wedge\sigma_k} ds \int_{1/i}^{\infty} m_0(dz) \int_{\mathbb{R}} \langle D_n(\langle U_s, \Phi_\cdot^m\rangle, zV_s(y)\Phi_\cdot^m(y)), \Psi_s\rangle dy\Big\}.$$

Then

$$I_{5,2}^{m,n,k,i}(t) \leq C_T[k^\beta m^{-\eta\beta} + a_{n-1}^\beta] i^{\alpha-1}, \qquad t \in [0,T].$$

Proof of Lemma 4.3. Recall that δ, ε satisfy (4.7). Choose $i = a_{n-1}^{-\varepsilon}$ and $\eta < \eta_c$ satisfying (4.7) with η replaced by η_c. Then

$$\mathbf{E}\{I_5^{m,n}(t\wedge\tau_k)\} \leq I_{5,1}^{m,n,k,i}(t) + I_{5,2}^{m,n,k,i}(t)$$

which converges to zero as $n \to \infty$ by Lemmas 4.4 and 4.5. □

We first present the proof for Lemma 4.4.

Proof of Lemma 4.4. Recall that $\psi_n(x) \leq 2(na_n)^{-1}$. Then by (3.3) in [16] and condition (C3),

$$\begin{aligned}
& m^{-1} D_n(\langle U_s, \Phi_x^m\rangle, zmV_s(x-y/m)\Phi(y)) \\
= \;& mz^2 V_s(x-y/m)^2 \Phi(y)^2 \int_0^1 \psi_n\big(|\langle U_s, \Phi_x^m\rangle + zhmV_s(x-y/m)\Phi(y)|\big)(1-h)dh \\
\leq \;& Cm(na_n)^{-1} z^2 |U_s(x-y/m)|^{2\beta}\Phi(y) \\
\leq \;& Cm(na_n)^{-1} z^2 |X_s(x-y/m) + Y_s(x-y/m)|^{((2\beta)\vee 1)-1} \\
& \quad \times |1 + X_s(x-y/m) + Y_s(x-y/m)|\Phi(y).
\end{aligned}$$

It follows that $\mathbf{P}$-a.s.

$$\begin{aligned}
&\int_{-K}^K \Psi_s(x)dx \int_{-1}^1 m^{-1} D_n(\langle U_s, \Phi_x^m\rangle, zmV_s(x-y/m)\Phi(y))dy \\
&\quad \leq \frac{Cc_2 mz^2 \tilde{K}_s}{na_n} \int_{-K}^K dx \int_{-1}^1 |1 + X_s(x-y/m) + Y_s(x-y/m)|\Phi(y)dy \\
&\quad \leq \frac{Cc_2 mz^2 \tilde{K}_s}{na_n}[1 + \langle X_s, 1\rangle + \langle Y_s, 1\rangle] \leq \frac{Cc_2 kmz^2 \tilde{K}_s}{na_n}
\end{aligned} \tag{4.11}$$

on $\{s \le \gamma_k\}$, where

$$c_2 := \sup_{(s,x)\in[0,T]\times[-K,K]} \Psi_s(x), \quad \tilde{K}_s := \sup_{|u|\le K+1} |\tilde{X}_s(u) + \tilde{Y}_s(u)|^{((2\beta)\vee 1)-1}.$$

Since $0 < \beta < 1$, $2\beta - 1 < \alpha_-$ for each $\alpha_- \in (1,\alpha)$. Then by (3.4), for each $0 < s \le T$,

$$\mathbf{E}\{\tilde{K}_s\} \le 2\mathbf{E}\Big\{1 + \sup_{|u|\le K+1} |\tilde{X}_s(u) \vee \tilde{Y}_s(u)|^{\alpha_-}\Big\} \le C_T s^{-\frac{\alpha_-}{2}}.$$

The above inequality together with (4.11) leads to

$$I_{5,1}^{m,n,k,i}(t) \le Ckm(na_n)^{-1} \int_0^{1/i} z^2 m_0(dz) \le Ckm(na_n)^{-1} i^{\alpha-2},$$

which finishes the proof. □

For $m,n,k,i \ge 1$ and $t > 0$ define

$$\begin{aligned} J_{m,n,k,i}(t) \;:=\; & \int_{\mathbb{R}} \Psi_t(x)dx \int_{1/i}^{\infty} z m_0(dz) \int_{\mathbb{R}} \Phi(y)dy \\ & \times \int_0^1 \tilde{D}_n\big(\langle U_t, \Phi_x^m\rangle, mzhV_t(x-\frac{y}{m})\big)V_t(x-\frac{y}{m})dh, \end{aligned}$$

where $\tilde{D}_n(y,z) = \phi_n'(y+z) - \phi_n'(y)$ for all $y,z \in \mathbb{R}$ and $n \ge 1$.

To show Lemma 4.5 we need to show two more lemmas.

Lemma 4.6. *There is a subsequence $\{l' : l' \ge 1\}$ of $\{l : l \ge 1\}$ so that for each $m,n,k,i \ge 1$, $\mathbf{P}$-a.s.*

$$\lim_{l'\to\infty} \int_0^t \sum_{j=1}^{2^{l'}} 1_{(l'_{j-1}T, l'_jT]}(s)|J_{m,n,k,i}(l'_jT) - J_{m,n,k,i}(s)|ds = 0, \qquad t \in (0,T]. \tag{4.12}$$

Proof. Observe that $\|\phi_n'\| \le 1$. Then by condition (C3)

$$|M_{m,n,k}(x,y,z,h,t)| := |\tilde{D}_n\big(\langle U_t, \Phi_x^m\rangle, mzhV_t(x-\frac{y}{m})\big)V_t(x-\frac{y}{m})| \le C|U_t(x-\frac{y}{m})|^{\beta}$$

and

$$J_{m,n,k,i}(t) \le c_2 C i^{\alpha-1} \int_{-K}^{K} dx \int_{-1}^{1} |U_t(x-\frac{y}{m})|^{\beta} dy,$$

where $c_2 = \sup_{(s,x)\in[0,T]\times[-K,K]} \Psi_s(x)$.

Now by Lemma 2.4, there is a constant $\delta \in (1,\alpha)$ so that for each $t \in (0,T]$, there is a set $K_t \subset \mathbb{R}$ of Lebesgue measure zero satisfying

$$\mathbf{E}\big\{|M_{m,n,k}(x,y,z,h,t)|^{\delta}\big\} \le C_T t^{-\beta\delta/2}, \qquad x,y \in \mathbb{R}\backslash K_t, \tag{4.13}$$

and

$$\mathbf{E}\{J_{m,n,k,i}(t)^{\delta}\} \le C_T i^{\delta(\alpha-1)} t^{-\beta\delta/2}. \tag{4.14}$$

By Lemma 2.6 for each $t > 0$ and $t_j \to t$ as $j \to \infty$, there is a set $\bar{K}_t \subset \mathbb{R}$ of Lebesgue measure zero so that for each $x \in \mathbb{R}\backslash\bar{K}_t$, both

$$X_{t_j}(x) \to X_t(x) \quad \text{and} \quad Y_{t_j}(x) \to Y_t(x)$$

in probability as $j \to \infty$. Then for each $x, y \in \mathbb{R}\backslash\bar{K}_t$,

$$M_{m,n,k}(x,y,z,h,t_j) \to M_{m,n,k}(x,y,z,h,t)$$

in probability as $j \to \infty$. Together with (4.13) we have

$$\lim_{j\to\infty} \mathbf{E}\Big\{\big|M_{m,n,k}(x,y,z,h,t_j) - M_{m,n,k}(x,y,z,h,t)\big|\Big\} = 0, \qquad x, y \in \mathbb{R}\backslash(K_t \cup \bar{K}_t).$$

Using (4.14) and [5, Theorem 4.5.2] we then have

$$\lim_{t\downarrow u} \mathbf{E}\Big\{|J_{m,n,k,i}(t) - J_{m,n,k,i}(u)|\Big\} = 0. \tag{4.15}$$

Using (4.14) again,

$$\begin{aligned}&\sup_{l\geq 1} \int_0^T \Big|\sum_{j=1}^{2^l} 1_{(l_{j-1}T, l_jT]}(s)\mathbf{E}\{|J_{m,n,k,i}(l_jT) - J_{m,n,k,i}(s)|\}\Big|^\delta ds\\ &\quad\leq \sup_{l\geq 1} \int_0^T \sum_{j=1}^{2^l} 1_{(l_{j-1}T, l_jT]}(s)\mathbf{E}\{J_{m,n,k,i}(l_jT)^\delta + J_{m,n,k,i}(s)^\delta\}ds < \infty.\end{aligned}$$

It then follows from (4.15) and [5, Theorem 4.5.2] again that

$$\begin{aligned}&\lim_{l\to\infty} \mathbf{E}\Big\{\int_0^T \sum_{j=1}^{2^l} 1_{(l_{j-1}T, l_jT]}(s)|J_{m,n,k,i}(l_jT) - J_{m,n,k,i}(s)|ds\Big\}\\ &\quad= \lim_{l\to\infty} \int_0^T \sum_{j=1}^{2^l} 1_{(l_{j-1}T, l_jT]}(s)\mathbf{E}\{|J_{m,n,k,i}(l_jT) - J_{m,n,k,i}(s)|\}ds = 0,\end{aligned}$$

which implies that

$$\int_0^T \sum_{i=1}^{2^l} 1_{(l_{j-1}T, l_jT]}(s)|J_{m,n,k,i}(l_jT) - J_{m,n,k,i}(s)|ds$$

goes to 0 in probability as $l \to \infty$. Then for each $m, n, k, i \geq 1$, there is a subsequence $\{l' := l'(m,n,k,i) : l' \geq 1\}$ of $\{l : l \geq 1\}$ so that (4.12) holds. Therefore, one can choose a proper subsequence of $\{l : l \geq 1\}$ which is independent of m, n, k, i so that (4.12) holds. □

Lemma 4.7. *Let $\{l' : l' \geq 1\}$ be the subsequence of $\{l : l \geq 1\}$ in Lemma 4.6. Then $\mathbf{P}$-a.s.*

$$\lim_{l'\to\infty} \int_0^t \sum_{j=1}^{2^{l'}} 1_{(l'_{j-1}T, l'_jT]}(s)J_{m,n,k,i}(l'_jT)1_{\{l'_jT<\sigma_k\}}ds = \int_0^t J_{m,n,k,i}(s)1_{\{s<\sigma_k\}}ds, \quad t \in (0,T].$$

Proof. It follows from (4.14) that

$$\mathbf{E}\Big\{\int_0^T J_{m,n,k,i}(s)ds\Big\} < \infty,$$

which implies that $\mathbf{P}$-a.s.

$$\int_0^T J_{m,n,k,i}(s)ds < \infty.$$

Then by Lemma 4.6 and the dominated convergence one obtains $\mathbf{P}$-a.s.

$$\lim_{l'\to\infty} \int_0^t \sum_{j=1}^{2^{l'}} 1_{(l'_{j-1}T, l'_jT]}(s)|J_{m,n,k,i}(l'_jT)1_{\{l'_jT<\sigma_k\}} - J_{m,n,k,i}(s)1_{\{s<\sigma_k\}}|ds$$

$$\leq \lim_{l'\to\infty}\int_0^t \sum_{j=1}^{2^{l'}} 1_{(l'_{j-1}T, l'_jT]}(s)|J_{m,n,k,i}(l'_jT) - J_{m,n,k,i}(s)|1_{\{l'_jT<\sigma_k\}}ds$$
$$+\lim_{l'\to\infty}\int_0^t \sum_{j=1}^{2^{l'}} 1_{(l'_{j-1}T, l'_jT]}(s)J_{m,n,k,i}(s)|1_{\{l'_jT<\sigma_k\}} - 1_{\{s<\sigma_k\}}|ds = 0,$$

which completes the proof. □

We are now ready to show Lemma 4.5.

Proof of Lemma 4.5. In the following let $t > 0$ and $m, n, k \geq 1$ be fixed. By Taylor's formula, dominated convergence and Lemma 4.7 we have

$$\begin{aligned} I_{5,2}^{m,n,k,i}(t) &= \mathbf{E}\Big\{\int_0^{t\wedge\sigma_k} ds \int_{-K}^{K} \Psi_s(x)dx \int_{1/i}^{\infty} zm_0(dz)\int_{-1}^{1}\Phi(y)dy \\ &\qquad \times \int_0^1 \tilde{D}_n\big(\langle U_s, \Phi_x^m\rangle, mzhV_s(x-\frac{y}{m})\big)V_s(x-\frac{y}{m})dh\Big\} \\ &= \mathbf{E}\Big\{\int_0^t J_{m,n,k,i}(s)1_{\{s<\sigma_k\}}ds\Big\} = \mathbf{E}\Big\{\bar{\int}_{(0,t)} J_{m,n,k,i}(s)1_{\{s<\sigma_k\}}ds\Big\}, \end{aligned} \tag{4.16}$$

where recall that $\tilde{D}_n(y,z) = \phi_n'(y+z) - \phi_n'(y)$. Let

$$\begin{aligned} \tilde{J}_{m,n,k,i}(t) &:= \int_{\mathbb{R}} \Psi_t(x)dx \int_{1/i}^{\infty} zm_0(dz)\int_{\mathbb{R}}\Phi(y)dy \\ &\qquad \times \int_0^1 \tilde{D}_n\big(\langle \tilde{U}_t, \Phi_x^m\rangle, mzh\tilde{V}_t(x-\frac{y}{m})\big)\tilde{V}_t(x-\frac{y}{m})dh. \end{aligned}$$

Observe that for each fixed $t > 0$, $\tilde{U}_t$ and $\tilde{V}_t$ are the continuous modifications of U_t and V_t, respectively. Then it is elementary to check that for each $t > 0$, $\langle|\tilde{U}_t - U_t|, 1\rangle = \langle|\tilde{V}_t - V_t|, 1\rangle = 0$, $\mathbf{P}$-a.s. This implies $\tilde{J}_{m,n,k,i}(t) = J_{m,n,k,i}(t)$ for all $t \in (0,\infty)\cap\mathbb{Q}$, $\mathbf{P}$-a.s. Together with (4.16) we have $\mathbf{P}$-a.s.

$$I_{5,2}^{m,n,k,i}(t) \leq \mathbf{E}\Big\{\bar{\int}_{(0,t)} \tilde{J}_{m,n,k,i}(s)1_{\{s<\sigma_k\}}ds\Big\}. \tag{4.17}$$

For fixed s and x let $x_{s,m} \in [-1,1]$ be a value satisfying

$$|\tilde{V}_s(x-\frac{x_{s,m}}{m})| = \inf_{y\in[-1,1]}\{|\tilde{V}_s(x-\frac{y}{m})|\}.$$

It follows from (4.17) that

$$\begin{aligned} & I_{5,2}^{m,n,k,i}(t) \\ \leq\ & \mathbf{E}\Big\{\int_0^{t\wedge\sigma_k} ds\int_{-K}^{K}\Psi_s(x)dx\int_{1/i}^{\infty} zm_0(dz)\int_{-1}^{1}\Phi(y)dy \\ & \quad \times \int_0^1 \Big|\tilde{D}_n\big(\langle\tilde{U}_s,\Phi_x^m\rangle, mzh\tilde{V}_s(x-\frac{y}{m})\big)[\tilde{V}_s(x-\frac{y}{m}) - \tilde{V}_s(x-\frac{x_{s,m}}{m})]\Big|dh\Big\} \\ & +\mathbf{E}\Big\{\int_0^t ds\int_{-K}^{K}\Psi_s(x)dx\int_{1/i}^{\infty} zm_0(dz)\int_{-1}^{1}\Phi(y)dy \\ & \quad \times \int_0^1 \Big|\tilde{D}_n\big(\langle\tilde{U}_s,\Phi_x^m\rangle, mzh\tilde{V}_s(x-\frac{y}{m})\big)\tilde{V}_s(x-\frac{x_{s,m}}{m})\Big|1_{\{\tilde{V}_s(x-\frac{x_{s,m}}{m})\neq 0\}}dh\Big\} \\ =:\ & I_{5,2,1}^{m,n,k,i}(t) + I_{5,2,2}^{m,n,i}(t). \end{aligned} \tag{4.18}$$

We can finish the proof in two steps.

Step 1. We first estimate $I^{m,n,k,i}_{5,2,1}(t)$. Since for fixed $s > 0$, $\tilde{X}_s$ and $\tilde{Y}_s$ are the continuous modifications of X_s and Y_s, respectively, then we have $\mathbf{P}$-a.s.

$$X_s(x) = \tilde{X}_s(x),\ Y_s(x) = \tilde{Y}_s(x), \qquad x \in \mathbb{R} \cap \mathbb{Q},\ s \in (0,\infty) \cap \mathbb{Q}.$$

Combining this with the definition of σ_k and $\overline{\int}_{(0,t]}$, we have $\mathbf{P}$-a.s.

$$\begin{aligned} \sigma_k &= \tilde{\sigma}_k := \inf\Big\{t \in (0,T] : \overline{\int}_{(0,t]} \sup_{x,z \in [-(K+1),K+1]\cap\mathbb{Q}, x \neq z} \\ &\qquad\qquad \frac{|\tilde{X}_s(x) - \tilde{X}_s(z)| \vee |\tilde{Y}_s(x) - \tilde{Y}_s(z)|}{|x-z|^\eta} ds > k\Big\} \\ &= \inf\Big\{t \in (0,T] : \overline{\int}_{(0,t]} \sup_{x,z \in [-(K+1),K+1], x \neq z} \\ &\qquad\qquad \frac{|\tilde{X}_s(x) - \tilde{X}_s(z)| \vee |\tilde{Y}_s(x) - \tilde{Y}_s(z)|}{|x-z|^\eta} ds > k\Big\}. \end{aligned} \tag{4.19}$$

By the Hölder inequality,

$$\begin{aligned} &\overline{\int}_{(0,t\wedge\tilde{\sigma}_k)} \sup_{|x|\le K, |y|\vee|v|\le 1} |\tilde{X}_s(x - \frac{y}{m}) - \tilde{X}_s(x - \frac{v}{m})|^\beta ds \\ &\le t^{1-\beta} \Big[\overline{\int}_{(0,t\wedge\tilde{\sigma}_k)} \sup_{|x|\le K, |y|\vee|v|\le 1} |\tilde{X}_s(x - \frac{y}{m}) - \tilde{X}_s(x - \frac{v}{m})| ds\Big]^\beta \\ &\le t^{1-\beta}(2/m)^{\eta\beta} \Big[\overline{\int}_{(0,t\wedge\tilde{\sigma}_k)} \sup_{|x|\le K, |y|\vee|v|\le 1, y\neq v} \frac{|\tilde{X}_s(x - \frac{y}{m}) - \tilde{X}_s(x - \frac{v}{m})|}{|y/m - v/m|^\eta} ds\Big]^\beta \\ &\le 2^{\eta\beta} t^{1-\beta} m^{-\eta\beta} k^\beta, \end{aligned}$$

and the same estimation holds for $\tilde{Y}$. Then by (4.19) we have $\mathbf{P}$-a.s.

$$\begin{aligned} &\overline{\int}_{(0,t\wedge\sigma_k)} \sup_{|x|\le K, |y|\vee|v|\le 1} |\tilde{V}_s(x - \frac{y}{m}) - \tilde{V}_s(x - \frac{v}{m})| ds \\ \le\ & C \overline{\int}_{(0,t\wedge\tilde{\sigma}_k)} \sup_{|x|\le K, |y|\vee|v|\le 1} \Big[|\tilde{X}_s(x - \frac{y}{m}) - \tilde{X}_s(x - \frac{v}{m})|^\beta \\ &\qquad\qquad + |\tilde{Y}_s(x - \frac{y}{m}) - \tilde{Y}_s(x - \frac{v}{m})|^\beta\Big] ds \\ \le\ & 2^{\eta\beta+1} C t^{1-\beta} m^{-\eta\beta} k^\beta. \end{aligned}$$

Observe that $|\tilde{D}_n(y,z)| \le 2$ for all $n \ge 1$ and $y, z \in \mathbb{R}$. It then follows that

$$\begin{aligned} I^{m,n,k,i}_{5,2,1}(t) &\le \frac{2c_2 i^{\alpha-1}}{\alpha - 1} \mathbf{E}\Big\{\overline{\int}_{(0,t\wedge\sigma_k)} ds \int_{-K}^K dx \int_{-1}^1 |\tilde{V}_s(x - \frac{y}{m}) - \tilde{V}_s(x - \frac{x_{s,m}}{m})|\Big\} \Phi(y) dy \\ &\le \frac{2c_2 i^{\alpha-1}}{\alpha - 1} \int_{-K}^K dx \int_{-1}^1 \mathbf{E}\Big\{\overline{\int}_{(0,t\wedge\sigma_k)} \\ &\qquad \sup_{|x|\le K, |y|\vee|v|\le 1} |\tilde{V}_s(x - \frac{y}{m}) - \tilde{V}_s(x - \frac{v}{m})| ds\Big\} \Phi(y) dy \\ &\le 2^{\eta\beta+2} k^\beta c_2 C K t^{1-\beta} m^{-\eta\beta} i^{\alpha-1} (\alpha-1)^{-1}, \end{aligned} \tag{4.20}$$

where $c_2 = \sup_{(s,x)\in[0,T]\times[-K,K]} \Psi_s(x)$.

Step 2. We then estimate $I^{m,n,i}_{5,2,2}(t)$. Since $\mathrm{supp}(\phi_n'') \subset (a_n, a_{n-1})$, then $\tilde{D}_n\big(y,z\big) = 0$ for $y \ge a_{n-1}$ and $z \ge 0$. It then follows that for each $y, z \ge 0$,

$$\tilde{D}_n(y,z) = \tilde{D}_n(y,z) 1_{\{|y| < a_{n-1}\}}. \tag{4.21}$$

One can also get (4.21) for the case $y, z \le 0$.

By the Hölder continuity of H, there is a constant $c_3 > 0$ so that $|\tilde{U}_s(x - \frac{u}{m})| \ge c_3 |\tilde{V}_s(x - \frac{u}{m})|^{1/\beta}$ for all $u \in [-1,1]$. Then

$$\{|\langle\tilde{U}_s,\Phi_x^m\rangle|<a_{n-1}\}\subset\Big\{|\tilde{V}_s(x-\frac{x_{s,m}}{m})|<(c_3^{-1}a_{n-1})^\beta\Big\}. \tag{4.22}$$

To verify (4.22), if

$$|\tilde{V}_s(x-\frac{x_{s,m}}{m})|\geq(c_3^{-1}a_{n-1})^\beta,$$

then $\tilde{V}_s(x-\frac{x_{s,m}(x)}{m})\neq 0$. This implies that $\tilde{V}_s(x-\frac{u}{m})\neq 0$ for all $u\in[-1,1]$. Then by the continuity of $u\mapsto\tilde{V}_s(u)$ and the mean value theorem, $\tilde{V}_s(x-\frac{u}{m})>0$ for all $u\in[-1,1]$, or $\tilde{V}_s(x-\frac{u}{m})<0$ for all $u\in[-1,1]$. On the other hand, H is a nondecreasing function (condition (C4) in Section 1). Then $\tilde{U}_s(x-\frac{u}{m})>0$ as $\tilde{V}_s(x-\frac{u}{m})>0$ and $\tilde{U}_s(x-\frac{u}{m})<0$ as $\tilde{V}_s(x-\frac{u}{m})<0$. Therefore,

$$\begin{aligned}\Big|\int_{\mathbb{R}}\tilde{U}_s(x-\frac{u}{m})\Phi(u)du\Big|&=\int_{-1}^1|\tilde{U}_s(x-\frac{u}{m})|\Phi(u)du\\&\geq c_3\int_{-1}^1|\tilde{V}_s(x-\frac{u}{m})|^{\frac{1}{\beta}}\Phi(u)du\geq c_3|\tilde{V}_s(x-\frac{x_{s,m}(x)}{m})|^{\frac{1}{\beta}}\geq a_{n-1},\end{aligned}$$

which implies (4.22).

By (4.21) one can also see that

$$\begin{aligned}&\tilde{D}_n\big(\langle\tilde{U}_s,\Phi_x^m\rangle,mzh\tilde{V}_s(x-\frac{x_{s,m}}{m})\Phi(y)\big)1_{\{\tilde{V}_s(x-\frac{x_{s,m}}{m})\neq 0\}}\\&=\tilde{D}_n\big(\langle\tilde{U}_s,\Phi_x^m\rangle,mzh\tilde{V}_s(x-\frac{x_{s,m}}{m})\Phi(y)\big)1_{\{|\tilde{V}_s(x-\frac{x_{s,m}}{m})\neq 0,\,\langle\tilde{U}_s,\Phi_x^m\rangle|<a_{n-1}\}}.\end{aligned}$$

Putting together (4.22) with the fact $|\tilde{D}_n(y,z)|\leq 2$ for all $y,z\in\mathbb{R}$ and $m\geq 1$, we have

$$\begin{aligned}&\Big|\tilde{D}_n\big(\langle\tilde{U}_s,\Phi_x^m\rangle,mzh\tilde{V}_s(x-\frac{x_{s,m}}{m})\Phi(y)\big)\tilde{V}_s(x-\frac{x_{s,m}}{m})\Big|\\&\leq 2|\tilde{V}_s(x-\frac{x_{s,m}}{m})|1_{\{\langle\tilde{U}_s,\Phi_x^m\rangle|<a_{n-1}\}}\leq 2(c_3^{-1}a_{n-1})^\beta.\end{aligned}$$

Then

$$I_{5,2,2}^{m,n,i}(t)\leq 2(c_3^{-1}a_{n-1})^\beta\int_0^t ds\int_{-K}^K\Psi_s(x)dx\int_{1/i}^\infty zm_0(dz)\leq C_Ta_{n-1}^\beta i^{\alpha-1},\qquad t\in[0,T].$$

Combining with (4.18) and (4.20), we finish the proof. □

4.2 A remark on the proof

We remark that we have not considered the increased Hölder regularity near its zero (used in Mytnik and Perkins (2011) in proving the pathwise uniqueness of SPDE driven by Gaussian white noise), i.e. the difference of two solutions is jointly Hölder continuous with the Hölder exponent in space in (0,1) and with Hölder exponent in time in (0,1/2) when the difference is close to zero. But for the SPDE (1.4), it is hard to establish the similar result of Hölder regularity for the difference of two solutions because the regularities of the solutions in time at fixed spacial point could be bad. For example, for super-Brownian motion (i.e. $G\equiv 0$, $H(x)=x^\beta$ and $p=1$ in (1.4)) it was proved in [21, Theorem 1.2] that for any $t,\delta>0$ and almost every spatial point $x\in\mathbb{R}$ fixed, the essential supremum of the solution over time interval $(t,t+\delta)$ is infinity.

In this paper the proof of pathwise uniqueness for the solution to (1.3) relies on the Hölder continuity of the solution at a fixed time. If one uses the Hölder continuity at any given spatial point where the Hölder exponent $\bar{\eta}_c=(3/\alpha-1)\wedge 1$ is bigger than η_c, it appears that the criterion for pathwise uniqueness could be improved with

$\beta > \frac{(\alpha-1)(\bar\eta_c+1)}{(2-\alpha)\bar\eta_c}$ in Theorem 1.5 (with η_c replaced by $\bar\eta_c$ in (4.7) and $1<\alpha<(\sqrt{17}+1)/4$ for the case $p=1$). But there is a problem with this approach, which we explain below. For the stoping time σ_k one gets an equation similar to (4.1) with t replaced by $t\wedge\sigma_k(x)$. Then by the same argument as in (4.2) we have

$$\begin{aligned}
&\langle\phi_n(\langle U_{t\wedge\tau_k(\cdot)},\Phi^m_\cdot\rangle),\Psi_t\rangle\\
=\;& \frac{1}{2}\int_0^t\langle\phi_n'(\langle U_s,\Phi^m_\cdot\rangle)\langle U_s,\Delta\Phi^m_\cdot\rangle 1_{\{s\le\tau_k(\cdot)\}},\Psi_s\rangle ds\\
&+\int_0^t\langle\phi_n'(\langle U_s,\Phi^m_\cdot\rangle)\langle R_s,\Phi^m_\cdot\rangle 1_{\{s\le\tau_k(\cdot)\}},\Psi_s\rangle ds+\int_0^t\langle\phi_n(\langle U_{s\wedge\tau_k(\cdot)},\Phi^m_\cdot\rangle),\dot\Psi_s\rangle ds\\
&+\int_0^t\int_0^\infty\int_{\mathbb{R}}\langle H_n(\langle U_{s-},\Phi^m_\cdot\rangle,zV_{s-}(y)\Phi^m_\cdot(y))1_{\{s\le\tau_k(\cdot)\}},\Psi_s\rangle\tilde N(ds,dz,dy)\\
&+\int_0^t ds\int_0^\infty m_0(dz)\int_{\mathbb{R}}\langle D_n(\langle U_s,\Phi^m_\cdot\rangle,zV_s(y)\Phi^m_\cdot(y))1_{\{s\le\tau_k(\cdot)\}},\Psi_s\rangle dy\\
=:\;&\hat I_1^{m,n}(t,k)+\hat I_2^{m,n}(t,k)+\hat I_3^{m,n}(t,k)+\hat I_4^{m,n}(t,k)+\hat I_5^{m,n}(t,k),
\end{aligned}$$

where $U_t,V_t,\Phi^m_x,D_n,\phi_n,\Psi_t,\gamma_k$ defined below (1.11), $\tau_k(x):=\gamma_k\wedge\sigma_k(x)$, $H_n(y,z):=\phi_n(y+z)-\phi_n(y)$ and R_s denotes the difference of compositions of the two solutions into function G. As in Lemmas 4.1–4.2, for each $k\ge1$ one can get

$$\lim_{m,n\to\infty}\mathbf{E}\{\hat I_2^{m,n}(t,k)\}=\mathbf{E}\Big\{\int_0^t ds\int_{\mathbb{R}}\mathrm{sgn}(U_s(x))Q_s(x)\Psi_s(x)1_{\{s\le\tau_k(x)\}}dx\Big\}$$

and

$$\lim_{m,n\to\infty}\mathbf{E}\{\hat I_3^{m,n}(t,k)\}=\mathbf{E}\Big\{\int_0^t ds\int_{\mathbb{R}}|U_{s\wedge\tau_k(x)}(x)|\dot\Psi_s(x)dx\Big\},\quad \mathbf{E}\{\hat I_4^{m,n}(t,k)\}=0.$$

Similar to Lemma 4.3, we also have that if $m=a_{n-1}^{-\delta}$ for $\delta>0$, then for each $t>0$ and $k\ge1$,

$$\lim_{m,n\to\infty}\mathbf{E}\{\hat I_5^{m,n}(t,k)\}=0.$$

But it is hard to deal with

$$\mathbf{E}\{\hat I_1^{m,n}(t,k)\}.$$

The difficulty comes from the fact that $x\mapsto 1_{\{s\le\tau_k(x)\}}$ is not continuous. So we cannot use the same argument as [23, Lemma 2.2(b)] to obtain an inequality like (4.10).

5 Appendix: proofs of Proposition 2.2 and Lemma 2.4

Before proving Proposition 2.2, we state a lemma.

Lemma 5.1. *Let $t\in[0,T]$ be fixed. For any $k\ge1$, $\lambda>0$ and $f\in C(\mathbb{R})$ satisfying $\lambda_0(|f|)<\infty$ we have $\mathbf{P}$-a.s.*

$$\begin{aligned}
\langle X_{t\wedge\tilde\tau_k},P_{t-(t\wedge\tilde\tau_k)+\lambda}f\rangle&=X_0(P_{t+\lambda}f)+\int_0^t\langle G(X_s),P_{t-s+\lambda}f\rangle 1_{\{s\le\tilde\tau_k\}}ds\\
&+\int_0^t\int_0^\infty\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha}zP_{t-s+\lambda}f(u)1_{\{s\le\tilde\tau_k\}}\tilde N_0(ds,dz,du,dv),
\end{aligned}\tag{5.1}$$

where $\tilde\tau_k$ is the stopping time defined in (2.5).

Proof. We consider a partition $\Delta_n := \{0 = t_0 < t_1 < \cdots < t_n = t\}$ of $[0,t]$. Let $|\Delta_n| := \max_{1\leq i\leq n}|t_i - t_{i-1}|$. Let $f_\lambda := P_\lambda f$. It is clear that $\frac{dP_s f_\lambda(x)}{ds} = \frac{1}{2}P_s f''_\lambda(x)$ for $s\geq 0$. For $k\geq 1$ and $s\in[0,T]$, let $Z_k(s) = X_{s\wedge\tilde\tau_k}$. By Proposition 2.1

$$\begin{aligned}\langle Z_k(t), f_\lambda\rangle &= X_0(f_\lambda) + \frac{1}{2}\int_0^t \langle X_s, f''_\lambda\rangle 1_{\{s\leq\tilde\tau_k\}}ds + \int_0^t \langle G(X_s), f_\lambda\rangle 1_{\{s\leq\tilde\tau_k\}}ds\\ &\quad + \int_0^t\int_0^\infty\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha} z f_\lambda(u) 1_{\{s\leq\tilde\tau_k\}}\tilde N_0(ds,dz,du,dv).\end{aligned}$$

It follows that

$$\begin{aligned}&\langle Z_k(t), P_{t-(t\wedge\tilde\tau_k)}f_\lambda\rangle\\ =\;& X_0(P_{t+\lambda}f) + \sum_{i=1}^n\langle Z_k(t_i), P_{t-(t_i\wedge\tilde\tau_k)}f_\lambda - P_{t-(t_{i-1}\wedge\tilde\tau_k)}f_\lambda\rangle\\ &+\sum_{i=1}^n\left[\langle Z_k(t_i), P_{t-(t_{i-1}\wedge\tilde\tau_k)}f_\lambda\rangle - \langle Z_k(t_{i-1}), P_{t-(t_{i-1}\wedge\tilde\tau_k)}f_\lambda\rangle\right]\\ =\;& X_0(P_{t+\lambda}f) + \frac{1}{2}\sum_{i=1}^n\int_{t-(t_{i-1}\wedge\tilde\tau_k)}^{t-(t_i\wedge\tilde\tau_k)}\langle Z_k(t_i), P_s f''_\lambda\rangle ds\\ &+\frac{1}{2}\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\langle X_s, P_{t-(t_{i-1}\wedge\tilde\tau_k)}f''_\lambda\rangle 1_{\{s\leq\tilde\tau_k\}}ds\\ &+\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\langle G(X_s), P_{t-(t_{i-1}\wedge\tilde\tau_k)+\lambda}f\rangle 1_{\{s\leq\tilde\tau_k\}}ds\\ &+\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\int_0^\infty\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha} zP_{t-(t_{i-1}\wedge\tilde\tau_k)+\lambda}f(u)1_{\{s\leq\tilde\tau_k\}}\tilde N_0(ds,dz,du,dv)\\ =\;& X_0(P_{t+\lambda}f) + \frac{1}{2}\int_0^t\sum_{i=1}^n I_i(s)\left[\langle X_s, P_{t-(t_{i-1}\wedge\tilde\tau_k)}f''_\lambda\rangle - \langle Z_k(t_i), P_{t-s}f''_\lambda\rangle\right]1_{\{s\leq\tilde\tau_k\}}ds\\ &+\int_0^t\sum_{i=1}^n I_i(s)\langle G(X_s), P_{t-(t_{i-1}\wedge\tilde\tau_k)+\lambda}f\rangle 1_{\{s\leq\tilde\tau_k\}}ds\\ &+\int_0^t\int_0^\infty\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha} z\sum_{i=1}^n I_i(s)P_{t-(t_{i-1}\wedge\tilde\tau_k)+\lambda}f(u)\\ &\qquad\qquad\times 1_{\{s\leq\tilde\tau_k\}}\tilde N_0(ds,dz,du,dv),\end{aligned}\tag{5.2}$$

where $I_i(s) := 1_{(t_{i-1},t_i]}(s)$.

Since $\|f''_\lambda\|<\infty$ and $\langle X_s, 1\rangle\leq k$ on $\{s\leq\tilde\tau_k\}$, then by the dominated convergence, $\mathbf{P}$-a.s.

$$\begin{aligned}&\lim_{|\Delta_n|\to 0}\int_0^t\sum_{i=1}^n I_i(s)|\langle X_s, P_{t-(t_{i-1}\wedge\tilde\tau_k)}f''_\lambda\rangle - \langle Z_k(t_i), P_{t-s}f''_\lambda\rangle|1_{\{s\leq\tilde\tau_k\}}ds\\ &\leq \lim_{|\Delta_n|\to 0}\int_0^t\sum_{i=1}^n I_i(s)|\langle X_s, P_{t-(t_{i-1}\wedge\tilde\tau_k)}f''_\lambda - P_{t-s}f''_\lambda\rangle|1_{\{s\leq\tilde\tau_k\}}ds\\ &\quad+\lim_{|\Delta_n|\to 0}\int_0^t\sum_{i=1}^n I_i(s)|\langle X_s, P_{t-s}f''_\lambda\rangle - \langle X_{t_i\wedge\tilde\tau_k}, P_{t-s}f''_\lambda\rangle|1_{\{s\leq\tilde\tau_k\}}ds\\ &\leq\int_0^t\lim_{|\Delta_n|\to 0}\sum_{i=1}^n I_i(s)|\langle X_s, P_{t-(t_{i-1}\wedge\tilde\tau_k)}f''_\lambda - P_{t-s}f''_\lambda\rangle|1_{\{s\leq\tilde\tau_k\}}ds\\ &\quad+\int_0^t\lim_{|\Delta_n|\to 0}\sum_{i=1}^n I_i(s)|\langle X_s, P_{t-s}f''_\lambda\rangle - \langle X_{t_i\wedge\tilde\tau_k}, P_{t-s}f''_\lambda\rangle|1_{\{s\leq\tilde\tau_k\}}ds = 0,\end{aligned}\tag{5.3}$$

where the right continuities of $t'\mapsto P_{t'}f''_\lambda$ and $t'\mapsto\langle X_{t'}, P_{t-s}f''_\lambda\rangle$ were used in the last equation.

By the Lipschitz condition on G and the dominated convergence we can also have $\mathbf{P}$-a.s.

$$\begin{aligned}&\lim_{|\Delta_n|\to 0}\int_0^t\sum_{i=1}^n I_i(s)|\langle G(X_s),P_{t-(t_{i-1}\wedge\tilde\tau_k)+\lambda}f\rangle-\langle G(X_s),P_{t-s+\lambda}f\rangle|1_{\{s\le\tilde\tau_k\}}ds\\ &\le\lim_{|\Delta_n|\to 0}\int_0^t\sum_{i=1}^n I_i(s)\langle G(X_s),|P_{t-(t_{i-1}\wedge\tilde\tau_k)+\lambda}f-P_{t-s+\lambda}f|\rangle|1_{\{s\le\tilde\tau_k\}}ds=0.\end{aligned}\tag{5.4}$$

Observe that for $s\in[0,T]$

$$f(s,u,\lambda,n,k):=\sum_{i=1}^n I_i(s)|P_{t-(t_{i-1}\wedge\tilde\tau_k)+\lambda}f(u)-P_{t-s+\lambda}f(u)|\le 2\|f\|$$

and $f(s,u,\lambda,n,k)$ converges to zero by the right continuity of $t'\mapsto P_{t'}f$ for $s\le\tilde\tau_k$ as $\Delta_n\to 0$. By the same argument as in (2.7) and (2.8),

$$\mathbf{E}\Big\{\int_0^{t\wedge\tilde\tau_k}ds\int_{\mathbb{R}}[1+X_s(u)^p]P_{\lambda+T}f(u)du\Big\}<\infty$$

for each $k\ge 1$. Then by the dominated convergence and Burkholder-Davis-Gundy inequality it is easy to see that as $\Delta_n\to 0$,

$$\begin{aligned}&\mathbf{E}\Big\{\Big|\int_0^t\int_1^\infty\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha}z\Big[\sum_{i=1}^n I_i(s)P_{t-(t_{i-1}\wedge\tilde\tau_k)+\lambda}f(u)\\ &\qquad\qquad -P_{t-s+\lambda}f(u)\Big]1_{\{s\le\tilde\tau_k\}}\tilde N_0(ds,dz,du,dv)\Big|\Big\}\\ &\le\ 2\int_1^\infty zm_0(dz)\mathbf{E}\Big\{\int_0^t ds\int_{\mathbb{R}}H(X_s(u))^\alpha f(s,u,\lambda,n,k)1_{\{s\le\tilde\tau_k\}}du\Big\}\\ &\le\ 2C\int_1^\infty zm_0(dz)\,\mathbf{E}\Big\{\int_0^t ds\int_{\mathbb{R}}[1+X_s(u)^p]f(s,u,\lambda,n,k)1_{\{s\le\tilde\tau_k\}}du\Big\}\to 0\end{aligned}\tag{5.5}$$

and

$$\begin{aligned}&\mathbf{E}\Big\{\Big|\int_0^t\int_0^1\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha}z\Big[\sum_{i=1}^n I_i(s)P_{t-(t_{i-1}\wedge\tilde\tau_k)+\lambda}f(u)\\ &\qquad\qquad -P_{t-s+\lambda}f(u)\Big]1_{\{s\le\tilde\tau_k\}}\tilde N_0(ds,dz,du,dv)\Big|^2\Big\}\\ &\le\ C\|f\|\int_0^1 z^2m_0(dz)\mathbf{E}\Big\{\int_0^t ds\int_{\mathbb{R}}[1+X_s(u)^p]f(s,u,\lambda,n,k)1_{\{s\le\tilde\tau_k\}}du\Big\}\to 0.\end{aligned}\tag{5.6}$$

Now it is obvious that (5.1) follows from (5.2)–(5.4) and (5.5)–(5.6). □

Now we are ready to present proof of Proposition 2.2.

Proof of (2.3). Recall that the stopping time $\tilde\tau_k$ is defined in (2.5). Let $f\in B(\mathbb{R})$ with $\lambda_0(|f|)<\infty$ in this step. For each $n\ge 1$ and $x\in\mathbb{R}$ define $f_n(x)=n\int_{x-1/n}^x f(y)dy$. Then $f_n\in C(\mathbb{R})$ and $\lambda_0(|f_n|)\le\lambda_0(|f|)<\infty$ by integration by parts. Then (5.1) holds with f replaced by f_n by Lemma 5.1. By the right continuity of $t'\mapsto P_{t'}f_n$ and the same argument in (5.5) and (5.6),

$$\mathbf{E}\Big\{\Big|\int_0^t\int_0^\infty\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha}z[P_{t-s+\lambda}f_n(u)-P_{t-s}f_n(u)]1_{\{s\le\tilde\tau_k\}}\tilde N_0(ds,dz,du,dv)\Big|\Big\}\to 0$$

as $\lambda\to 0$. Since (5.1) holds with f replaced by f_n, taking $\lambda\to 0$ we get

$$\langle X_{t\wedge\tilde\tau_k},P_{t-(t\wedge\tilde\tau_k)}f_n\rangle=X_0(P_tf_n)+\int_0^t\langle G(X_s),P_{t-s}f_n\rangle 1_{\{s\le\tilde\tau_k\}}ds$$

$$+\int_0^t\int_0^\infty\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha} zP_{t-s}f_n(u)1_{\{s\le\tilde\tau_k\}}\tilde N_0(ds,dz,du,dv). \tag{5.7}$$

Observe that $\|f_n\|\le\|f\|<\infty$ and $\lim_{n\to\infty}f_n(x)=f(x)$, λ_0-a.e. x. Then letting $n\to\infty$ in (5.7), by the dominated convergence and the same argument in (5.5)–(5.6) again, we obtain

$$\begin{aligned}\langle X_{t\wedge\tilde\tau_k},P_{t-(t\wedge\tilde\tau_k)}f\rangle &= X_0(P_tf)+\int_0^t\langle G(X_s),P_{t-s}f\rangle 1_{\{s\le\tilde\tau_k\}}ds\\ &\quad+\int_0^t\int_0^\infty\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha} zP_{t-s}f(u)1_{\{s\le\tilde\tau_k\}}\tilde N_0(ds,dz,du,dv),\end{aligned} \tag{5.8}$$

which implies (2.3) by taking $k\to\infty$. □

Proof of (2.4). Let $t>0$ and $f\in B(\mathbb{R})$ with $\lambda_0(|f|)<\infty$ be fixed. By Fubini's theorem,

$$\begin{aligned}&\int_0^{t\wedge\tilde\tau_k}\int_1^\infty\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha} zP_{t-s}f(u)\tilde N_0(ds,dz,du,dv)\\ =&\int_0^{t\wedge\tilde\tau_k}\int_1^\infty\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha} zP_{t-s}f(u)N_0(ds,dz,du,dv)\\ &-\int_0^{t\wedge\tilde\tau_k}ds\int_1^\infty m_0(dz)\int_{\mathbb{R}}du\int_0^{H(X_{s-}(u))^\alpha} zP_{t-s}f(u)dv\\ =&\int_{\mathbb{R}}f(x)\Big[\int_0^{t\wedge\tilde\tau_k}\int_1^\infty\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha} zp_{t-s}(x-u)N_0(ds,dz,du,dv)\Big]dx\\ &-\int_{\mathbb{R}}f(x)\Big[\int_0^{t\wedge\tilde\tau_k}ds\int_1^\infty m_0(dz)\int_{\mathbb{R}}du\int_0^{H(X_{s-}(u))^\alpha} zp_{t-s}(x-u)dv\Big]dx\\ =&\int_{\mathbb{R}}f(x)\Big[\int_0^{t\wedge\tilde\tau_k}\int_1^\infty\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha} zp_{t-s}(x-u)\tilde N_0(ds,dz,du,dv)\Big]dx.\end{aligned} \tag{5.9}$$

By stochastic Fubini's theorem (see e.g. [15, Theorem 7.24]), to prove $\mathbf{P}$-a.s.,

$$\begin{aligned}&\int_0^{t\wedge\tilde\tau_k}\int_0^1\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha} zP_{t-s}f(u)\tilde N_0(ds,dz,du,dv)\\ =&\int_{\mathbb{R}}f(x)\Big[\int_0^{t\wedge\tilde\tau_k}\int_0^1\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha} zp_{t-s}(x-u)\tilde N_0(ds,dz,du,dv)\Big]dx,\end{aligned} \tag{5.10}$$

we only need to verify

$$\begin{aligned}&\mathbf{E}\Big\{\int_{\mathbb{R}}f(x)dx\int_0^{t\wedge\tilde\tau_k}ds\int_0^1 m_0(dz)\int_{\mathbb{R}}du\int_0^{H(X_{s-}(u))^\alpha} z^2p_{t-s}(x-u)^2dv\Big\}\\ &\quad\le C\int_0^1 z^2m_0(dz)\mathbf{E}\Big\{\int_0^{t\wedge\tilde\tau_k}ds\int_{\mathbb{R}}|f(x)|dx\\ &\qquad\times\int_{\mathbb{R}}[1+X_s(u)^p]p_{t-s}(x-u)^2du\Big\}<\infty.\end{aligned} \tag{5.11}$$

Indeed, for the case $0<p<1$, by an argument similar to (2.7),

$$\begin{aligned}&\int_0^{t\wedge\tilde\tau_k}ds\int_{\mathbb{R}}|f(x)|dx\int_{\mathbb{R}}X_s(u)^pp_{t-s}(x-u)^2du\\ &\qquad\le\lambda_0(|f|)\int_0^t[2\pi s]^{-\frac{\bar\alpha-1}{2}}[1+(2\pi s)^{-1/2}k]ds<\infty.\end{aligned}$$

For the case $p=1$,

$$\int_0^{t\wedge\tilde\tau_k}ds\int_{\mathbb{R}}|f(x)|dx\int_{\mathbb{R}}X_s(u)p_{t-s}(x-u)^2du\le k\|f\|\int_0^t[2\pi(t-s)^{-1/2}]ds<\infty.$$

Observe that

$$X_s(u)^p p_{t-s}(x-u)^2 = [X_s(u)^{p-\delta} p_{t-s}(x-u)^{1-\delta}] \times [X_s(u)^{\delta} p_{t-s}(x-u)^{1+\delta}].$$

For the case $1<p<2$ choosing $\delta = (q-p)/(q-1) \in (0,1)$ for q given in Assumption 1.4, it is easy to see that $q = \frac{p-\delta}{1-\delta}$. Then by the Hölder inequality,

$$\begin{aligned}
&\int_0^{t\wedge\tilde\tau_k} ds \int_{\mathbb{R}} |f(x)|dx \int_{\mathbb{R}} X_s(u)^p p_{t-s}(x-u)^2 du \\
&\le \|f\| \Big[\int_0^{t\wedge\tilde\tau_k} ds \int_{\mathbb{R}} dx \int_{\mathbb{R}} X_s(u)^q p_{t-s}(x-u) du \Big]^{1-\delta} \\
&\quad \times \Big[\int_0^{t\wedge\tilde\tau_k} ds \int_{\mathbb{R}} dx \int_{\mathbb{R}} X_s(u) p_{t-s}(x-u)^{1+1/\delta} du \Big]^{\delta} \\
&\le \|f\| \Big[\int_0^{t\wedge\tilde\tau_k} ds \int_{\mathbb{R}} X_s(u)^q du \Big]^{1-\delta} \times \Big| \int_0^{t\wedge\tilde\tau_k} [2\pi(t-s)^{-1/2}]^{1/\delta} \langle X_s, 1\rangle ds \Big|^{\delta} \\
&\le k\|f\| [2\pi]^{-1} \Big[\int_0^t s^{-1/(2\delta)} ds \Big]^{\delta} < \infty,
\end{aligned}$$

which implies (5.11).

Combining (5.8)–(5.10), we have $\mathbf{P}$-a.s.

$$\begin{aligned}
&\int_{\mathbb{R}} X_{t\wedge\tilde\tau_k}(x) P_{t-(t\wedge\tilde\tau_k)} f(x) dx \\
= &\int_{\mathbb{R}} \Big[\int_{\mathbb{R}} p_t(x-z) X_0(dz) + \int_0^{t\wedge\tilde\tau_k} ds \int_{\mathbb{R}} p_{t-s}(x-z) G(X_s(z)) dz \\
&+ \int_0^{t\wedge\tilde\tau_k} \int_0^{\infty} \int_{\mathbb{R}} \int_0^{H(X_{s-}(u))^{\alpha}} z p_{t-s}(x-u) \tilde N_0(ds,dz,du,dv) \Big] f(x) dx.
\end{aligned}$$

Letting $k\to\infty$ one completes the proof. □

Proof of Lemma 2.4. If (2.10) holds for $1\vee p < \bar p < \alpha$, the rest can be given by the Jensen inequality. So in the following we always assume that $1\vee p < \bar p < \alpha$.

Step 1. Note that

$$\begin{aligned}
&\Big| \int_0^t ds \int_{\mathbb{R}} p_{t-s}(x-z) G(X_s(z)) dz \Big|^{\bar p} \\
&\le C t^{\bar p} \int_0^t ds \int_{\mathbb{R}} p_{t-s}(x-z) G(X_s(z))^{\bar p} dz \\
&\le C t^{\bar p} \int_0^t ds \int_{\mathbb{R}} p_{t-s}(x-z) [1 + X_s(z)^{\bar p}] dz. \qquad (5.12)
\end{aligned}$$

Recalling the stopping time $\tilde\tau_k$ defined in (2.5), one can see that

$$\begin{aligned}
\bar Z_k(t,x) &:= \int_0^{t\wedge\tilde\tau_k} \int_0^{\infty} \int_{\mathbb{R}} \int_0^{H(X_{s-}(u))^{\alpha}} z p_{t-s}(x-u) \tilde N_0(ds,dz,du,dv) \\
&= \int_0^t \int_0^1 \int_{\mathbb{R}} \int_0^{H(X_{s-}(u))^{\alpha}} z p_{t-s}(x-u) 1_{\{s\le\tilde\tau_k\}} \tilde N_0(ds,dz,du,dv) \\
&\quad + \int_0^t \int_1^{\infty} \int_{\mathbb{R}} \int_0^{H(X_{s-}(u))^{\alpha}} z p_{t-s}(x-u) 1_{\{s\le\tilde\tau_k\}} \tilde N_0(ds,dz,du,dv) \\
&=: \bar Z_{k,1}(t,x) + \bar Z_{k,2}(t,x).
\end{aligned}$$

By (1.6) in [31] and the fact $u^p \le u^{\bar p} + 1$ for $u\ge 0$, for $\alpha < \hat p < 2$ we have

$$\begin{aligned}
\mathbf{E}\Big\{ |\bar Z_{k,1}(t,x)|^{\hat p} \Big\} &\le C \int_0^1 z^{\hat p} m_0(dz) \mathbf{E}\Big\{ \int_0^t ds \int_{\mathbb{R}} H(X_{s-}(u))^{\alpha} p_{t-s}(x-u)^{\hat p} 1_{\{s\le\tilde\tau_k\}} du \Big\} \\
&\le C\, \mathbf{E}\Big\{ \int_0^t (t-s)^{-\frac{\hat p-1}{2}} ds \int_{\mathbb{R}} [1 + X_s(u)^p] p_{t-s}(x-u) 1_{\{s\le\tilde\tau_k\}} du \Big\}
\end{aligned}$$

$$\leq \; C\,\mathbf{E}\Big\{\int_0^t (t-s)^{-\frac{\hat{p}-1}{2}} ds \int_{\mathbb{R}} [1+X_s(u)^{\bar{p}}] p_{t-s}(x-u) 1_{\{s\leq\tilde{\tau}_k\}} du\Big\}$$

and

$$\begin{aligned}
\mathbf{E}\Big\{|\bar{Z}_{k,2}(t,x)|^{\bar{p}}\Big\} \;&\leq\; C\int_1^\infty z^{\bar{p}} m_0(dz)\mathbf{E}\Big\{\int_0^t ds\int_{\mathbb{R}} H(X_{s-}(u))^\alpha p_{t-s}(x-u)^{\bar{p}} 1_{\{s\leq\tilde{\tau}_k\}} du\Big\}\\
&\leq\; C\,\mathbf{E}\Big\{\int_0^t (t-s)^{-\frac{\bar{p}-1}{2}} ds\int_{\mathbb{R}} [1+X_s(u)^p] p_{t-s}(x-u) 1_{\{s\leq\tilde{\tau}_k\}} du\Big\}\\
&\leq\; C\,\mathbf{E}\Big\{\int_0^t (t-s)^{-\frac{\bar{p}-1}{2}} ds\int_{\mathbb{R}} [1+X_s(u)^{\bar{p}}] p_{t-s}(x-u) 1_{\{s\leq\tilde{\tau}_k\}} du\Big\}.
\end{aligned}$$

Then one obtains that

$$\begin{aligned}
&\mathbf{E}\{|\bar{Z}_k(t,x)|^{\bar{p}}\} \leq 2\mathbf{E}\{|\bar{Z}_{k,1}(t,x)|^{\bar{p}}\} + 2\mathbf{E}\{|\bar{Z}_{k,2}(t,x)|^{\bar{p}}\}\\
&\quad\leq C\Big\{\mathbf{E}[|\bar{Z}_{k,1}(t,x)|^{\hat{p}}] + \mathbf{E}[|\bar{Z}_{k,2}(t,x)|^{\bar{p}}] + 1\Big\}\\
&\quad\leq C\,\mathbf{E}\Big\{\int_0^t [(t-s)^{-\frac{\hat{p}-1}{2}} + (t-s)^{-\frac{\bar{p}-1}{2}}] ds\int_{\mathbb{R}} [1+X_s(u)^{\bar{p}}] p_{t-s}(x-u) 1_{\{s\leq\tilde{\tau}_k\}} du\Big\} + C.
\end{aligned}$$

Combining this with (2.4) and (5.12), we have for any $T>0$,

$$\begin{aligned}
&\mathbf{E}\Big\{\int_0^T dt\int_{\mathbb{R}} X_t(y)^{\bar{p}} p_{T-t}(x-y) 1_{\{t\leq\tilde{\tau}_k\}} dy\Big\}\\
\leq\;& 3\int_0^T dt\int_{\mathbb{R}} X_0(p_t(y-\cdot))^{\bar{p}} p_{T-t}(x-y) dy + 3\int_0^T dt\int_{\mathbb{R}} \mathbf{E}\{|\bar{Z}_k(t,y)|^{\bar{p}}\} p_{T-t}(x-y) dy\\
&+3\,\mathbf{E}\Big\{\int_0^T dt\int_{\mathbb{R}} \Big|\int_0^t ds\int_{\mathbb{R}} p_{t-s}(y-u) G(X_s(u)) du\Big|^{\bar{p}} p_{T-t}(x-y) 1_{\{t\leq\tilde{\tau}_k\}} dy\Big\}\\
\leq\;& C\int_0^T dt\int_0^t [t^{\bar{p}} + (t-s)^{-\frac{\hat{p}-1}{2}} + (t-s)^{-\frac{\bar{p}-1}{2}}]\\
&\qquad\times\mathbf{E}\Big\{\int_{\mathbb{R}} [1+X_s(u)^{\bar{p}}] p_{T-s}(x-u) 1_{\{s\leq\tilde{\tau}_k\}} du\Big\} ds + CX_0(1)^{\bar{p}} T^{\frac{2-\bar{p}}{2}} + CT\\
=\;& C\,\mathbf{E}\Big\{\int_0^T ds\int_{\mathbb{R}} [1+X_s(u)^{\bar{p}}] p_{T-s}(x-u) 1_{\{s\leq\tilde{\tau}_k\}} du\\
&\qquad\times\int_s^T [t^{\bar{p}} + (t-s)^{-\frac{\hat{p}-1}{2}} + (t-s)^{-\frac{\bar{p}-1}{2}}] dt\Big\} + CX_0(1)^{\bar{p}} T^{\frac{2-\bar{p}}{2}} + CT\\
\leq\;& C(T^{\bar{p}+1} + T^{\frac{3-\hat{p}}{2}} + T^{\frac{3-\bar{p}}{2}})\mathbf{E}\Big\{\int_0^T ds\int_{\mathbb{R}} [1+X_s(u)^{\bar{p}}] p_{T-s}(x-u) 1_{\{s\leq\tilde{\tau}_k\}} du\Big\}\\
&+CX_0(1)^{\bar{p}} T^{\frac{2-\bar{p}}{2}} + CT.
\end{aligned}\tag{5.13}$$

In view of (2.7) and (2.8),

$$\mathbf{E}\Big\{\int_0^T dt\int_{\mathbb{R}} X_t(y)^{\bar{p}} p_{T-t}(x-y) 1_{\{t\leq\tilde{\tau}_k\}} dy\Big\} < \infty, \qquad T>0.$$

Taking $\tilde{T}_0 > 0$ satisfying $K' := C(\tilde{T}_0^{\bar{p}+1} + \tilde{T}_0^{\frac{3-\hat{p}}{2}} + \tilde{T}_0^{\frac{3-\bar{p}}{2}}) < 1$, for all $T\in[0,\tilde{T}_0]$ and $k\geq 1$ we have

$$\mathbf{E}\Big\{\int_0^T dt\int_{\mathbb{R}} X_t(y)^{\bar{p}} p_{T-t}(x-y) 1_{\{t\leq\tilde{\tau}_k\}} dy\Big\} \leq (1-K')^{-1}\Big[CX_0(1)^{\bar{p}} T^{\frac{2-\bar{p}}{2}} + CT\Big].$$

Then by the monotone convergence theorem

$$\sup_{T\in[0,\tilde{T}_0]} \mathbf{E}\Big\{\int_0^T dt\int_{\mathbb{R}} X_t(x)^{\bar{p}} p_{T-t}(x-y) dx\Big\} < \infty. \tag{5.14}$$

Step 2. In this step we prove that (2.10) holds with T replaced by the $\tilde{T}_0$ specified in Step 1. We assume that $0<T\leq\tilde{T}_0$ in the following of this step. Observe that for $0<r<1$,

$$\begin{aligned}&\int_0^T (T-t)^{-\frac{r}{2}}dt\int_{\mathbb{R}} X_0(p_t(y-\cdot))^{\bar p}p_{T-t}(x-y)dy\\&\quad\le[(2\pi)^{-1}X_0(1)]^{\bar p}\int_0^T (T-t)^{-\frac{r}{2}}t^{-\frac{\bar p}{2}}dt\le[(2\pi)^{-1}X_0(1)]^{\bar p}T^{1-\frac{r}{2}-\frac{\bar p}{2}}dt. \end{aligned}\tag{5.15}$$

For $r\in(0,1)$ and $\delta\in[1,2)$,

$$\begin{aligned}&\int_0^T (T-t)^{-\frac{r}{2}}dt\int_{\mathbb{R}} p_{T-t}(x-y)dy\int_0^t ds\int_{\mathbb{R}} X_s(u)^{\bar p}p_{t-s}(y-u)^{\delta}du\\&\quad\le C\int_0^T (T-t)^{-\frac{r}{2}}dt\int_0^t (t-s)^{-\frac{\delta-1}{2}}ds\int_{\mathbb{R}} X_s(u)^{\bar p}p_{T-s}(x-u)du\\&\quad= C\int_0^T ds\int_{\mathbb{R}} X_s(u)^{\bar p}p_{T-s}(x-u)du\int_s^T (T-t)^{-\frac{r}{2}}(t-s)^{-\frac{\delta-1}{2}}dt\\&\quad\le CT^{\frac{3-r-\delta}{2}}\int_0^T ds\int_{\mathbb{R}} X_s(u)^{\bar p}p_{T-s}(x-u)du.\end{aligned}\tag{5.16}$$

Similar to the argument in (5.13), combining (2.4) and (5.14)–(5.16), it is easy to see that for $0<r<1$,

$$\sup_{T\in[0,\tilde T_0]} T^{\frac{r}{2}}\mathbf{E}\Big\{\int_0^T (T-t)^{-\frac{r}{2}}dt\int_{\mathbb{R}} X_t(y)^{\bar p}p_{T-t}(x-y)dy\Big\}<\infty.\tag{5.17}$$

By (1.6) of [31] again we have

$$\begin{aligned}&\mathbf{E}\Big\{\Big|\int_0^t\int_0^1\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha} zp_{t-s}(x-u)\tilde N_0(ds,dz,du,dv)\Big|^{\hat p}\Big\}\\&\quad\le C\int_0^1 z^{\hat p}m_0(dz)\,\mathbf{E}\Big\{\int_0^t ds\int_{\mathbb{R}} H(X_{s-}(u))^\alpha p_{t-s}(x-u)^{\hat p}du\Big\}\\&\quad\le C\int_0^1 z^{\hat p}m_0(dz)\,\mathbf{E}\Big\{\int_0^t ds\int_{\mathbb{R}}[1+X_s(u)^p]p_{t-s}(x-u)^{\hat p}du\Big\}\\&\quad\le C\,\mathbf{E}\Big\{\int_0^t (t-s)^{-\frac{\hat p-1}{2}}ds\int_{\mathbb{R}}[1+X_s(u)^{\bar p}]p_{t-s}(x-u)du\Big\}\end{aligned}$$

for $\alpha<\hat p<2$ and

$$\begin{aligned}&\mathbf{E}\Big\{\Big|\int_0^t\int_1^\infty\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha} zp_{t-s}(x-u)\tilde N_0(ds,dz,du,dv)\Big|^{\bar p}\Big\}\\&\quad\le C\int_1^\infty z^{\bar p}m_0(dz)\,\mathbf{E}\Big\{\int_0^t ds\int_{\mathbb{R}} H(X_{s-}(u))^\alpha p_{t-s}(x-u)^{\bar p}du\Big\}\\&\quad\le C\int_1^\infty z^{\bar p}m_0(dz)\,\mathbf{E}\Big\{\int_0^t ds\int_{\mathbb{R}}[1+X_s(u)^{\bar p}]p_{t-s}(x-u)^{\bar p}du\Big\}\\&\quad\le C\,\mathbf{E}\Big\{\int_0^t (t-s)^{-\frac{\bar p-1}{2}}ds\int_{\mathbb{R}}[1+X_s(u)^{\bar p}]p_{t-s}(x-u)du\Big\},\end{aligned}$$

which implies

$$\begin{aligned}&\mathbf{E}\Big\{\Big|\int_0^t\int_0^\infty\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha} zp_{t-s}(x-u)\tilde N_0(ds,dz,du,dv)\Big|^{\bar p}\Big\}\\&\le C\,\mathbf{E}\Big\{\int_0^t\big[(t-s)^{-\frac{\bar p-1}{2}}+(t-s)^{-\frac{\hat p-1}{2}}\big]ds\int_{\mathbb{R}} p_{t-s}(x-y)[1+X_s(y)^{\bar p}]dy\Big\}+C.\end{aligned}$$

By (2.4) and (5.12) again, we have

$$\mathbf{E}\{X_t(x)^{\bar p}\}$$

$$
\begin{aligned}
&\le\; Ct^{-\frac{\bar p}{2}} + C\,\mathbf{E}\Big\{\Big|\int_0^t ds\int_{\mathbb{R}} p_{t-s}(x-z)G(X_s(z))dz\Big|^{\bar p}\Big\}\\
&\quad +C\,\mathbf{E}\Big\{\Big|\int_0^t\int_0^\infty\int_{\mathbb{R}}\int_0^{H(X_{s-}(u))^\alpha} zp_{t-s}(x-u)\tilde N_0(ds,dz,du,dv)\Big|^{\bar p}\Big\}\\
&\le\; C\,\mathbf{E}\Big\{\int_0^t\big[t^{\bar p}+(t-s)^{-\frac{\hat p-1}{2}}+(t-s)^{-\frac{\hat p-1}{2}}\big]ds\int_{\mathbb{R}} p_{t-s}(x-y)[1+X_s(y)^{\bar p}]dy\Big\}\\
&\quad +Ct^{-\frac{\bar p}{2}}+C. \qquad (5.18)
\end{aligned}
$$

Then by (5.17) one sees that (2.10) holds with T replaced by $\tilde T_0$.

Step 3. Similar to Step 1, for $\tilde\gamma\in(0,1)$ and $0\le\tilde T_1\le\tilde T_0\wedge\tilde T_2$ with $\tilde\gamma^{\bar p+1}\tilde T_2^{\bar p}(\tilde T_2-\tilde T_1)\le\tilde T_0^{\bar p+1}$ and $\tilde T_2-\tilde T_1\le\tilde T_0$,

$$
\begin{aligned}
&\mathbf{E}\Big\{\int_{\tilde\gamma\tilde T_1}^{\tilde\gamma\tilde T_2}dt\int_{\mathbb{R}} X_t(y)^{\bar p}p_{\tilde\gamma\tilde T_2-t}(x-y)1_{\{t\le\tilde\tau_k\}}dy\Big\}\\
&\le\; C\int_{\tilde\gamma\tilde T_1}^{\tilde\gamma\tilde T_2}dt\int_0^t[t^{\bar p}+(t-s)^{-\frac{\hat p-1}{2}}+(t-s)^{-\frac{\bar p-1}{2}}]\mathbf{E}\Big\{\int_{\mathbb{R}}[1+X_s(u)^{\bar p}]\\
&\quad\times p_{\tilde\gamma\tilde T_2-s}(x-u)1_{\{s\le\tilde\tau_k\}}du\Big\}ds+CX_0(1)^{\bar p}\tilde T_2^{\frac{2-\bar p}{2}}+C\tilde T_2\\
&\le\; C\int_{\tilde\gamma\tilde T_1}^{\tilde\gamma\tilde T_2}\mathbf{E}\Big\{\int_{\mathbb{R}}[1+X_s(u)^{\bar p}]p_{\tilde\gamma\tilde T_2-s}(x-u)1_{\{s\le\tilde\tau_k\}}du\Big\}ds\\
&\qquad\qquad\times\int_s^{\tilde\gamma\tilde T_2}[t^{\bar p}+(t-s)^{-\frac{\hat p-1}{2}}+(t-s)^{-\frac{\bar p-1}{2}}]dt\\
&\quad +C\int_0^{\tilde\gamma\tilde T_1}ds\int_{\mathbb{R}}[1+\mathbf{E}\{X_s(u)^{\bar p}\}]p_{\tilde\gamma\tilde T_2-s}(x-u)du\\
&\qquad\times\int_{\tilde\gamma\tilde T_1}^{\tilde\gamma\tilde T_2}[t^{\bar p}+(t-s)^{-\frac{\hat p-1}{2}}+(t-s)^{-\frac{\bar p-1}{2}}]dt+CX_0(1)^{\bar p}\tilde T_0^{\frac{2-\bar p}{2}}+C\tilde T_0\\
&\le\; C[(\tilde\gamma\tilde T_2)^{\bar p}\tilde\gamma(\tilde T_2-\tilde T_1)+\tilde T_0^{\frac{3-\hat p}{2}}+\tilde T_0^{\frac{3-\bar p}{2}}]\\
&\qquad\times\mathbf{E}\Big\{\int_{\tilde\gamma\tilde T_1}^{\tilde\gamma\tilde T_2}ds\int_{\mathbb{R}}[1+X_s(u)^{\bar p}]p_{\tilde\gamma\tilde T_2-s}(x-u)1_{\{s\le\tilde\tau_k\}}du\Big\}\\
&\quad +C[\tilde\gamma(1+\tilde\gamma)^{\bar p}\tilde T_0^{\bar p+1}+\tilde T_0^{\frac{3-\hat p}{2}}+\tilde T_0^{\frac{3-\bar p}{2}}]\int_0^{\tilde\gamma\tilde T_1}ds\int_{\mathbb{R}}[1+s^{-\frac{\bar p}{2}}]p_{\tilde\gamma\tilde T_2-s}(x-u)du+C_{\tilde T_0}\\
&\le\; C[\tilde\gamma^{\bar p+1}\tilde T_2^{\bar p}(\tilde T_2-\tilde T_1)+\tilde T_0^{\frac{3-\hat p}{2}}+\tilde T_0^{\frac{3-\bar p}{2}}]\mathbf{E}\Big\{\int_{\tilde\gamma\tilde T_1}^{\tilde\gamma\tilde T_2}ds\int_{\mathbb{R}}[1+X_s(u)^{\bar p}]\\
&\qquad\times p_{\tilde\gamma\tilde T_2-s}(x-u)1_{\{s\le\tilde\tau_k\}}du\Big\}+C_{\tilde T_0}\\
&\le\; K'\,\mathbf{E}\Big\{\int_{\tilde\gamma\tilde T_1}^{\tilde\gamma\tilde T_2}ds\int_{\mathbb{R}}[1+X_s(u)^{\bar p}]p_{\tilde\gamma\tilde T_2-s}(x-u)1_{\{s\le\tilde\tau_k\}}du\Big\}+C_{\tilde T_0}, \qquad (5.19)
\end{aligned}
$$

where the assertion in Step 2 was used in the third inequality. This implies

$$
\sup_{(\tilde T_1,\tilde T_2)\in\mathbb{B}}\mathbf{E}\Big\{\int_{\tilde\gamma\tilde T_1}^{\tilde\gamma\tilde T_2}dt\int_{\mathbb{R}} X_t(y)^{\bar p}p_{\tilde\gamma\tilde T_2-t}(x-y)dy\Big\}<\infty, \qquad (5.20)
$$

where

$$
\mathbb{B}:=\{(\tilde T_1,\tilde T_2):\tilde T_1\in[0,\tilde T_0],\tilde T_2-\tilde T_1\le\tilde T_0,\tilde\gamma^{\bar p+1}\tilde T_2^{\bar p}(\tilde T_2-\tilde T_1)\le\tilde T_0^{\bar p+1}\}.
$$

Then by the assertion in Step 2 again,

$$
\sup_{(\tilde T_1,\tilde T_2)\in\mathbb{B}}\mathbf{E}\Big\{\int_0^{\tilde\gamma\tilde T_2}dt\int_{\mathbb{R}} X_t(y)^{\bar p}p_{\tilde\gamma\tilde T_2-t}(x-y)dy\Big\}<\infty.
$$

Similar to (5.17) we have

$$\sup_{(\tilde{T}_1,\tilde{T}_2)\in\mathbb{B}} \mathbf{E}\Big\{\int_0^{\tilde{\gamma}\tilde{T}_2}(\tilde{\gamma}\tilde{T}_2-t)^{-\frac{r}{2}}dt\int_{\mathbb{R}}p_{\tilde{\gamma}\tilde{T}_2-t}(x-y)X_t(y)^{\bar{p}}dy\Big\}<\infty \tag{5.21}$$

for $r\in(0,1)$. This together this with (5.18) shows that (2.10) holds with T replaced by $\tilde{\gamma}\tilde{T}_2$, where $\tilde{\gamma}^{\bar{p}+1}\tilde{T}_2^{\bar{p}}(\tilde{T}_2-\tilde{T}_0)\le\tilde{T}_0^{\bar{p}+1}$ and $\tilde{T}_2\le 2\tilde{T}_0$.

Step 4. Since $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\le 1+\ln n$, one can chose $\tilde{\gamma}\in(0,1)$ so that $\sup_{n\ge1}\tilde{\gamma}^{\bar{p}+1}(1+\ln n)^{\bar{p}}/n\le 1$, which implies

$$\sup_{n\ge1}\frac{\tilde{\gamma}^{\bar{p}+1}}{n}\big[1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\big]^{\bar{p}}\le 1.$$

Observe that Step 2 proves that (2.10) holds with T replaced by $\tilde{\gamma}\tilde{T}_0$. With $\tilde{T}_1$ and $\tilde{T}_2$ replaced by $\tilde{T}_1'(0\le\tilde{T}_1'\le\tilde{T}_0)$ and $(1+\frac{1}{2})\tilde{T}_1'$, respectively, in (5.19)–(5.20), we get (2.10) with T replaced by $\tilde{\gamma}(1+\frac{1}{2})\tilde{T}_0$. Repeating the above argument, for each $n\ge1$, with $\tilde{T}_1$ and $\tilde{T}_2$ replaced by $(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1})\tilde{T}_1'$ and $(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n})\tilde{T}_1'$, respectively, in (5.19)–(5.20), we can get (2.10) with T replaced by $\tilde{\gamma}(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n})\tilde{T}_0$, which completes the proof. □

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