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A central limit theorem for the spatial Λ-Fleming-Viot process with selection

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Abstract

We study the evolution of gene frequencies in a population living in $\mathbb{R}^d$, modelled by the spatial Λ-Fleming-Viot process with natural selection. We suppose that the population is divided into two genetic types, a and A, and consider the proportion of the population which is of type a at each spatial location. If we let both the selection intensity and the fraction of individuals replaced during reproduction events tend to zero, the process can be rescaled so as to converge to the solution to a reaction-diffusion equation (typically the Fisher-KPP equation). We show that the rescaled fluctuations converge in distribution to the solution to a linear stochastic partial differential equation. Depending on whether offspring dispersal is only local or if large scale extinction-recolonization events are allowed to take place, the limiting equation is either the stochastic heat equation with a linear drift term driven by space-time white noise or the corresponding fractional heat equation driven by a coloured noise which is white in time. If individuals are diploid (*i.e.* either AA, Aa or aa) and if natural selection favours heterozygous (Aa) individuals, a stable intermediate gene frequency is maintained in the population. We give estimates for the asymptotic effect of random fluctuations around the equilibrium frequency on the local average fitness in the population. In particular, we find that the size of this effect - known as the *drift load* - depends crucially on the dimension d of the space in which the population evolves, and is reduced relative to the case without spatial structure.

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Introduction

Consider a population distributed across a geographical space (typically of dimension one or two). Suppose that each individual carries one of several possible versions (or *alleles*) of a gene. How do the different allele frequencies evolve with time and how are they shaped by the main evolutionary forces, such as natural selection and migration? To answer this question, early models from population genetics were adapted by G. Malécot [Mal48], S. Wright [Wri43] and M. Kimura [Kim53] to include spatial structure. These spatial models either considered subdivided populations reproducing locally and exchanging migrants at each generation or made inconsistent assumptions about the distribution of individuals across space.

In this work, we focus on a mathematical model for populations evolving in a spatial continuum, the spatial Λ-Fleming-Viot process (SLFV for short), originally proposed in [Eth08]. The main feature of this model is that instead of each individual carrying exponential clocks determining its reproduction and death times, reproduction times are specified by a Poisson point process of extinction-recolonization events. At each of these events, some proportion - often denoted u - of the individuals present in the region affected by the event is replaced by the offspring of an individual (the *parent*) chosen within this region. (The proportion u which is replaced is called the *impact parameter*.) We shall only consider cases where the region affected is a (d-dimensional) ball, and the Poisson point process specifies the time, centre and radius of reproduction events. (Since we consider scaling limits, minor changes to this assumption would not change our results.)

We suppose that each individual in the population has a type taken from a compact space K. The state of the SLFV process at time t is then given by a map $\rho_t : \mathbb{R}^d \to M(K)$ defined Lebesgue almost everywhere, where $M(K)$ is the set of probability measures on the type space K. We think of $\rho_t(x)$ as the distribution of the type of an individual sampled from location x at time t. More precisely, the spatial Λ-Fleming-Viot process can be obtained as the high population density limit of an individual based model (see [BEV13a]) where the sequence of empirical measures of the individuals' location and type converges to the measure $\rho_t(x)\mathrm{d}x$. We thus sometimes use heuristics based on the behaviour of individuals in the prelimiting model even though one cannot speak of individuals in the SLFV.

Natural selection can be included in the SLFV by introducing an independent Poisson point process of selective events which give an advantage to a particular type. Multiple *potential parents* are sampled in the region affected by the event and one is chosen to be the parent and have offspring in a biased way depending on their types. The *selection parameter* determines the rate of this Poisson point process.

A comprehensive survey of recent developments related to the SLFV can be found in [BEV13a]. Several works have focussed on characterising the behaviour of this model over large space and time scales, in the special case where only two types (or two alleles) a and A are present in the population. In this case the state of the process is given by a map $q_t : \mathbb{R}^d \to [0, 1]$ defined Lebesgue almost everywhere, where $q_t(x)$ denotes the probability that a randomly chosen individual at location x and time t is of type a, or in other words the proportion of type a at location x and at time t. We shall first consider the simplest form of selection when individuals are *haploid*, i.e. each individual has one copy of the gene, and type A is favoured. At selective events, two potential parents are chosen and if their types are different, the parent is the one which has type A. In [EVY14], rescaling limits of this form of the spatial Λ-Fleming-Viot process with selection (SLFVS) have been obtained when both the impact parameter and the selection parameter tend to zero. Earlier results on the large scale behaviour of the SLFV had already been established in [BEV13b] in the neutral case (*i.e.* without selection), but keeping the impact parameter macroscopic. The behaviour of the SLFVS in the corresponding regime is studied in [EFS15] and [EFPS15].

The limiting process obtained by [EVY14] turns out to be deterministic as soon as $d \geq 2$, and, when the reproduction events have bounded radius, it is given by the celebrated Fisher-KPP equation,

$$\frac{\partial f_t}{\partial t} = \frac{1}{2}\Delta f_t - s f_t(1 - f_t). \tag{0.1}$$

This result fits the original interpretation of this equation proposed by R. A. Fisher as a model for the spread of advantageous genes in a spatially distributed population [Fis37]. The spatial Λ-Fleming-Viot process with selection (SLFVS) can thus be thought of as a refinement of the Fisher-KPP equation, combining spatial structure and a random sampling effect at each generation - what biologists call *genetic drift*.

In the present work we prove a slightly stronger form of convergence to this deterministic rescaling limit. We also study the fluctuations of the allele frequency about (an approximation of) $(f_t)_{t\geq 0}$. We find that if the impact parameter is sufficiently small compared to the selection parameter and the fluctuations are rescaled in the right way then in the limit they solve the following stochastic partial differential equation,

$$\mathrm{d}z_t = \left[\frac{1}{2}\Delta z_t - s(1 - 2f_t)z_t\right]\mathrm{d}t + \sqrt{f_t(1 - f_t)}\,\mathrm{d}W_t, \tag{0.2}$$

where W is space-time white noise, and f is the solution of (0.1). More detailed statements with the precise conditions on the parameters of the SLFVS are given in Section 2.

A very similar result was proved by F. Norman in the non-spatial setting [Nor75a] (see also [Nor74a], [Nor77] and [Nor75b]). Norman considered the Wright-Fisher model for a population of size N under natural selection (see [Eth11] for an introduction to such models). Let p_n^N denote the proportion of individuals not carrying the favoured allele at generation n, and suppose that the selection parameter is given by $s_N = \varepsilon_N s$, with $\varepsilon_N \to 0$ and $\varepsilon_N N \to \infty$ as $N \to \infty$. (At each generation, individuals choose a parent of the favoured type with probability $\frac{(1+s_N)(1-p_n^N)}{1+s_N(1-p_n^N)}$.) Norman showed that, as $N \to \infty$, $p^N_{\lfloor t/\varepsilon_N \rfloor}$ converges to g_t, which satisfies

$$\frac{\mathrm{d}g_t}{\mathrm{d}t} = -s g_t(1 - g_t).$$

(In the weak selection regime - *i.e.* when $N s_N = \mathcal{O}(1)$ - one recovers the classical Wright-Fisher diffusion, see [Eth11].) Furthermore, the fluctuations of p^N_{t/ε_N} around g_t are of order $(N\varepsilon_N)^{-1/2}$. More precisely, for $t = n\varepsilon_N$, $n \in \mathbb{N}$, set

$$Z^N(t) = (N\varepsilon_N)^{1/2}\left(p^N_{t/\varepsilon_N} - g_t\right),$$

and define $Z^N(t)$ for all $t \geq 0$ by linear interpolation. Theorem 2 in [Nor75a] states that, as $N \to \infty$, $\left(Z^N(t)\right)_{t\geq 0}$ converges to the solution of the following stochastic differential equation,

$$\mathrm{d}z_t = -s(1 - 2g_t)z_t\mathrm{d}t + \sqrt{g_t(1 - g_t)}\mathrm{d}B_t,$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion; note that $(z_t)_{t\geq 0}$ is a Gaussian diffusion. A similar regime in the case of a neutral model with mutations was already studied by W. Feller in [Fel51, Section 9], who identified the limiting diffusion for the fluctuations around the equilibrium frequency.

Norman's result can be extended to other classical models from population genetics, and in particular to continuous-time processes such as the Moran model and the (non-spatial) Λ-Fleming-Viot process (introduced in [BLG03]). The necessary tools can be found mainly in [EK86, Chapter 11] (see also Chapter 6 of the same book) and in [Kur71]. In this paper we adapt these methods to the setting of the spatial Λ-Fleming-Viot process, with the necessary tools for stochastic partial differential equations taken from [Wal86] (see also [MT95] and [DMFL86]).

We also consider a second regime for the SLFVS to allow large scale extinction-recolonization events; we let the radius of reproduction events follow an α-stable distribution truncated at zero. For this regime, as in [EVY14], we find the Fisher-KPP equation with non-local diffusion as a rescaling limit (*i.e.* with a fractional Laplacian instead of the usual Laplacian). The Laplacian is also replaced by a fractional Laplacian in (0.2), the equation satisfied by the limiting fluctuations, and the noise W becomes a coloured noise with spatial correlations of order $|x-y|^{-\alpha}$ (see Subsection 2.2).

These results are valid for a general class of selection mechanisms, with modified versions of (0.1) and (0.2) (and our proof will cover the general case). As an application of our results on the fluctuations, we turn to a particular kind of selection mechanism. Suppose a given gene is present in two different forms - denoted A_1 and A_2 - within a population. Suppose also that each individual carries two copies of this gene (each inherited from one of two parents). We say that individuals are *diploid*, and *homozygous* individuals are those who carry two copies of the same type (A_1A_1 or A_2A_2) while *heterozygous* individuals carry one copy of each type (A_1A_2). *Overdominance* occurs when the relative fitnesses of the three possible genotypes are as follows,

$$\begin{array}{ccc} A_1A_1 & A_1A_2 & A_2A_2 \\ 1-s_1 & 1 & 1-s_2, \end{array}$$

where $s_1,\ s_2 > 0$. In words, heterozygous individuals produce more offspring than both types of homozygous individuals. In this setting, in an infinite population a stable intermediate allele frequency is expected to be maintained, preventing either type from disappearing. If q is the frequency of type A_1 and $p = 1-q$ that of type A_2 and if mating is random, the respective proportions of the three genotypes will be q^2, $2qp$, p^2, hence the population cannot remain composed exclusively of heterozygous individuals. As a consequence, even when the stable equilibrium is reached, the mean fitness of the population will not be as high as the highest possible individual fitness (*i.e.* that of heterozygous individuals). This fitness reduction is referred to as the *segregation load*.

In finite populations, because of finite sample size, the allele frequency is never exactly at its optimum. This was the subject of a work by A. Robertson [Rob70] who considered this specific configuration of the relative fitnesses. He argued that the mean fitness in a *panmictic* population (*i.e.* one with no spatial structure) with finite but relatively large size N is reduced by a term of order $(4N)^{-1}$, irrespective of the strength of selection. This is due to a trade-off between genetic drift and natural selection. The stronger selection is, the quicker the allele frequency is pushed back to the equilibrium, but at the same time even a small step away from the optimal frequency is very costly in terms of mean fitness. On the other hand, if natural selection is relatively weak, the allele frequency can wander off more easily, but the mean fitness of the population decreases more slowly. This reduction in the mean fitness due to genetic drift - which is added to the reduction from the segregation load - is called the *drift load*.

Robertson's result can be made rigorous using tools found in [Nor74a] and [Nor74b]. We adapt these to our setting and study the same effect in spatially structured populations. We find that the spatial structure significantly reduces the drift load, in a way that depends crucially on dimension. It turns out that migration prevents the allele frequencies from straying too far away from the equilibrium frequency, because incoming migrants are on average close to this equilibrium.

The paper is laid out as follows. We define the spatial Λ-Fleming-Viot process for a haploid model with general frequency dependent selection and for a diploid model of overdominance in Section 1. In Section 2 we state the main convergence results for the SLFVS in the bounded radius and stable radius regimes and we present our estimate of the drift load in spatially structured populations. In Section 3, we present the main ingredient of the proof: a martingale problem satisfied by the SLFVS. At the end of Subsections 3.2 and 3.3, we state more general results on solutions to these martingale problems which imply our convergence results for the SLFVS. Most of the remainder of the paper is dedicated to the proofs of these results. The central limit theorem in the bounded radius case is proved in Section 4, while the stable regime is dealt with in Section 5 (the two proofs share the same structure, but differ in the details of the estimates). Finally, the asymptotics of the drift load are derived in Section 6.

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1 Definition of the model

1.1 The state space of the spatial Λ-Fleming-Viot process with selection

We now turn to a precise definition of the underlying model, the spatial Λ-Fleming-Viot process with selection on $\mathbb{R}^d$, starting with the state space of the process. At each time $t \geq 0$, $\{q_t(x) : x \in \mathbb{R}^d\}$ is a random function such that

$$q_t(x) := \text{proportion of type } a \text{ alleles at spatial position } x \text{ at time } t, \tag{1.1}$$

which is in fact defined up to a Lebesgue null set of $\mathbb{R}^d$. More precisely, let Ξ be the quotient of the space of Lebesgue-measurable maps $f : \mathbb{R}^d \to [0,1]$ by the equivalence relation

$$f \sim f' \iff Leb(\{x \in \mathbb{R}^d : f(x) \neq f'(x)\}) = 0.$$

We endow Ξ with the topology of vague convergence: letting $\langle f, \phi \rangle = \int_{\mathbb{R}^d} f(x)\phi(x)\mathrm{d}x$, a sequence $(f_n)_n$ converges vaguely to $f \in \Xi$ if and only if $\langle f_n, \phi \rangle \underset{n\to\infty}{\longrightarrow} \langle f, \phi \rangle$ for any continuous and compactly supported function $\phi : \mathbb{R}^d \to \mathbb{R}$. A convenient metric for this topology is given by choosing a separating family $(\phi_n)_{n\geq 1}$ of smooth, compactly supported functions which are uniformly bounded in $L^1(\mathbb{R}^d)$. Then for $f, g \in \Xi$,

$$d_\Xi(f,g) = \sum_{n\geq 1} \frac{1}{2^n} \left| \langle f, \phi_n \rangle - \langle g, \phi_n \rangle \right| \tag{1.2}$$

defines a metric for the topology of vague convergence on Ξ. The SLFVS up to time T is then going to be a $\mathrm{D}([0,T],\Xi)$-valued random variable: a Ξ-valued process with càdlàg paths.

Definition 1.1. *For $T > 0$, let $f, g \in \mathrm{D}([0,T],\Xi)$ be a pair of càdlàg maps $(f_t)_{0\leq t\leq T}$, $(g_t)_{0\leq t\leq T}$ from $[0,T]$ to Ξ. Then*

$$d(f,g) = \sup_{t\in[0,T]} d_\Xi(f_t, g_t)$$

is a metric for the topology of uniform convergence on $\mathrm{D}([0,T],\Xi)$.

For more details, see Section 2.2 of [VW15].

1.2 The spatial Λ-Fleming-Viot process with selection

Let us now define the dynamics of the process. Let $u \in (0,1]$ and $s \in [0,1]$, and let $\mu(dr)$ be a finite measure on $(0,\infty)$ satisfying

$$\int_0^\infty r^d \mu(\mathrm{d}r) < \infty. \tag{1.3}$$

For $m \in \mathbb{N}$ and $w \in [0,1]$, let $\vec{B}_w^m$ be a vector of m independent random variables taking the value a with probability w and A otherwise. Then let $F : [0,1] \to \mathbb{R}$ be a polynomial such that for some $m \in \mathbb{N}$ and $p : \{a, A\}^m \to [0,1]$, for each $w \in [0,1]$,

$$w - F(w) = \mathbb{E}\left[p(\vec{B}_w^m)\right]. \tag{1.4}$$

(The choice of p and m is not unique, but this will not matter.)

Definition 1.2 (SLFVS, haploid case with general frequency dependent selection)**.** *Let Π and Π^S be two independent Poisson point processes on $\mathbb{R}_+ \times \mathbb{R}^d \times (0,\infty)$ with intensity measures $(1-s)\, dt \otimes dx \otimes \mu(dr)$ and $s\, dt \otimes dx \otimes \mu(dr)$ respectively. The spatial Λ-Fleming-Viot process with selection for a haploid population in $\mathbb{R}^d$ with impact parameter u, radius of reproduction events given by $\mu(dr)$, selection parameter s and selection function F is defined as follows. If $(t,x,r) \in \Pi$, a neutral event occurs at time t within the ball $B(x,r)$:*

1. *Choose a location y uniformly at random in $B(x,r)$ and sample a parental type $k \in \{a, A\}$ according to $q_{t^-}(y)$ (i.e. $k = a$ with probability $q_{t^-}(y)$).*

2. *Update q as follows:*

$$\forall z \in \mathbb{R}^d, q_t(z) = q_{t^-}(z) + u \mathbb{1}_{|x-z|<r} (\mathbb{1}_{k=a} - q_{t^-}(z)). \tag{1.5}$$

Similarly, if $(t,x,r) \in \Pi^S$, a selective event occurs at time t inside $B(x,r)$:

1. *Choose m locations $y_1, \ldots, y_m$ independently uniformly at random in $B(x,r)$, sample a type k_i at each location y_i according to $q_{t^-}(y_i)$ and then let $k = a$ with probability $p(k_1, \ldots, k_m)$ and $k = A$ otherwise.*

2. *Update q as in* (1.5).

Note that if we let $w = |B(x,r)|^{-1} \int_{B(x,r)} q_{t^-}(z)\, dz$, then at a neutral reproduction event, $\mathbb{P}(k = a) = w$ and at a selective event, $\mathbb{P}(k = a) = w - F(w)$. This justifies the definition in terms of the selection function F (and the terminology for F), since the law of the process depends only on F, and not on the specific choice of p and m in (1.4).

Remark 1.3. The existence of a unique Ξ-valued process following these dynamics under condition (1.3) is proved in [EVY14, Theorem 1.2] in the special case $F(w) = w(1-w)$ (in the neutral case $s = 0$, this was done in [BEV10]). In our general case, the condition on $w - F(w)$ in (1.4) allows one to define a branching and coalescing dual process and hence prove existence and uniqueness in the same way as in [EVY14].

We shall consider two different distributions μ for the radii of events,

i) the fixed radius case : $\mu(\mathrm{d}r) = \delta_R(\mathrm{d}r)$ for some $R > 0$,

ii) the stable radius case : $\mu(\mathrm{d}r) = \frac{\mathbb{1}_{r \geq 1}}{r^{d+\alpha+1}} \mathrm{d}r$ for a fixed $\alpha \in (0, 2 \wedge d)$.

In each case, (1.3) is clearly satisfied.

We give two variants of this definition corresponding to the two selection mechanisms discussed in the introduction. We begin with a model for a selective advantage for A alleles in haploid reproduction.

Definition 1.4 (SLFVS, haploid model, genic selection)**.** *The spatial Λ-Fleming-Viot process with genic selection with impact parameter u, radius of reproduction events given by $\mu(dr)$ and selection parameter s is defined as in Definition 1.2 with $F(w) = w(1-w)$. In this case, $m = 2$ and the function p equals simply*

$$p(k_1, k_2) = \mathbb{1}_{k_1=k_2=a}.$$

In other words, during selective reproduction events, two types are sampled in $B(x,r)$ and $k = a$ if and only if both types are a.

1.3 The SLFVS with overdominance

We now define a variant of the SLFVS to model overdominance. Individuals are diploid and we study a gene which is present in two different forms within the population, denoted A_1 and A_2. For $t \geq 0$ and $x \in \mathbb{R}^d$, let

$$q_t(x) := \text{ the proportion of the allele type } A_1 \text{ at location } x \text{ at time } t.$$

(If p_1 is the proportion of A_1A_1 indiduals and p_H is the proportion of A_1A_2 heterozygous individuals, then $q = p_1 + \frac{1}{2}p_H$.) We assume that the relative fitnesses of the different genotypes are as follows:

$$\begin{array}{ccc} A_1A_1 & A_1A_2 & A_2A_2 \\ 1-s_1 & 1 & 1-s_2. \end{array}$$

In other words, for an event (t,x,r) in the SLFVS with $w = |B(x,r)|^{-1}\int_{B(x,r)} q_{t^-}(z)\,dz$, we want to choose parental types $(k_1,k_2) \in \{A_1,A_2\}^2$ at random with

$$\begin{aligned} \mathbb{P}\left(\{k_1,k_2\} = \{A_1,A_1\}\right) &= P_{11} := \tfrac{(1-s_1)w^2}{1-s_1w^2-s_2(1-w)^2}, \\ \mathbb{P}\left(\{k_1,k_2\} = \{A_1,A_2\}\right) &= P_{12} := \tfrac{2w(1-w)}{1-s_1w^2-s_2(1-w)^2}, \\ \mathbb{P}\left(\{k_1,k_2\} = \{A_2,A_2\}\right) &= P_{22} := \tfrac{(1-s_2)(1-w)^2}{1-s_1w^2-s_2(1-w)^2}. \end{aligned}$$

We also suppose that, with probability ν_1, the type A_1 alleles produced mutate to type A_2, and that, with probability ν_2, the type A_2 mutate to type A_1 (this is a technical assumption to ensure that $q_t(x) \notin \{0,1\}$; we shall assume that ν_1 and ν_2 are small compared to s_1 and s_2). This gives us the following modified probabilities for the parental types:

$$\begin{aligned} \mathbb{P}\left(\{k_1,k_2\} = \{A_1,A_1\}\right) &= (1-\nu_1)P_{11} + \nu_2(1-P_{11}), \\ \mathbb{P}\left(\{k_1,k_2\} = \{A_1,A_2\}\right) &= (1-\nu_1-\nu_2)P_{12}, \\ \mathbb{P}\left(\{k_1,k_2\} = \{A_2,A_2\}\right) &= (1-\nu_2)P_{22} + \nu_1(1-P_{22}). \end{aligned}$$

We are going to be interested in small values of s_i and ν_i, so we expand:

$$\begin{aligned} &\mathbb{P}\left(\{k_1,k_2\} = \{A_1,A_1\}\right) \\ &\quad = (1-\nu_1)(w^2(1-s_1+s_1w^2+s_2(1-w)^2) + \mathcal{O}\left(s^2\right)) + \nu_2(1-w^2+\mathcal{O}\left(s\right)) \\ &\quad = (1-s_1-s_2-\nu_1-\nu_2)w^2 + s_1w^4 + s_2w^2(1+(1-w)^2) + \nu_2 + \mathcal{O}\left(s^2+\nu s\right), \end{aligned} \tag{1.6}$$

where $s = s_1 + s_2$ and $\nu = \nu_1 + \nu_2$. Similarly, we have

$$\begin{aligned} \mathbb{P}\left(\{k_1,k_2\} = \{A_1,A_2\}\right) &= (1-s_1-s_2-\nu_1-\nu_2)2w(1-w) + s_12w(1-w)(1+w^2) \\ &\quad + s_22w(1-w)(1+(1-w)^2) + \mathcal{O}\left(s^2+\nu s\right), \\ \mathbb{P}\left(\{k_1,k_2\} = \{A_2,A_2\}\right) &= (1-s_1-s_2-\nu_1-\nu_2)(1-w)^2 + s_1(1-w)^2(1+w^2) \\ &\quad + s_2(1-w)^4 + \nu_1 + \mathcal{O}\left(s^2+\nu s\right). \end{aligned} \tag{1.7}$$

The following model results in the parental type probabilities given in (1.6) and (1.7), neglecting the $\mathcal{O}\left(s^2+\nu s\right)$ terms.

Definition 1.5 (SLFVS, overdominance). *Suppose that* $\nu_1+\nu_2+s_1+s_2<1$*. Let* Π, Π^{S_i} *and* Π^{ν_i}, $i=1,2$ *be five independent Poisson point processes on* $\mathbb{R}_+\times\mathbb{R}^d\times(0,\infty)$ *with respective intensity measures* $(1-s_1-s_2-\nu_1-\nu_2)\,dt\otimes dx\otimes\mu(dr)$, $s_i\,dt\otimes dx\otimes\mu(dr)$ *and* $\nu_i\,dt\otimes dx\otimes\mu(dr)$*. The spatial* Λ*-Fleming-Viot process with overdominance with impact parameter* u*, radius of reproduction events given by* μ*, selection parameters* s_1, s_2 *and mutation parameters* ν_1, ν_2 *is defined as follows. If* $(t,x,r)\in\Pi$*, a neutral event occurs at time* t *in* $B(x,r)$*:*

1. *Pick two locations* y_1 *and* y_2 *uniformly at random within* $B(x,r)$ *and sample one parental type* $k_i\in\{A_1,A_2\}$ *at each location according to* $q_{t^-}(y_i)$*, independently of each other.*

2. *Update* q *as follows:*

$$\forall z\in\mathbb{R}^d,\quad q_t(z)=q_{t^-}(z)+u\mathbb{1}_{|x-z|<r}\left(\tfrac{1}{2}(\mathbb{1}_{k_1=A_1}+\mathbb{1}_{k_2=A_1})-q_{t^-}(z)\right). \tag{1.8}$$

If $(t,x,r)\in\Pi^{S_i}$*, a selective event occurs at time* t *in* $B(x,r)$*:*

1. *Pick four locations uniformly at random within* $B(x,r)$ *and sample one type at each location, forming two pairs of types. If one pair is* $\{A_i,A_i\}$*, let* $\{k_1,k_2\}$ *be the other pair; otherwise pick one pair at random, each with probability* $1/2$*. (If the two sampled pairs are* $\{A_i,A_i\}$*, then* $\{k_1,k_2\}=\{A_i,A_i\}$*.)*

2. *Update* q *as in* (1.8).

If $(t,x,r)\in\Pi^{\nu_i}$*, a mutation event occurs at time* t *in* $B(x,r)$*:*

1. *Set* $\{k_1,k_2\}=\{A_{3-i},A_{3-i}\}$*, irrespective of the state of* q_{t^-}*. (In other words we suppose that the* A_i *genes of the offspring mutate to type* A_{3-i}*.)*

2. *Update* q *as in* (1.8).

Remark 1.6. Similarly to the haploid case, existence and uniqueness for this process can be proved as in [EVY14] using a dual process.

We shall see in Section 3 that this process satisfies essentially the same martingale problem as the general haploid process in Definition 1.2 with

$$F(w)=w(1-w)(w-\frac{s_2}{s_1+s_2})+\frac{\nu_1}{s_1+s_2}w-\frac{\nu_2}{s_1+s_2}(1-w).$$

2 Statement of the results

In this section, we present our main results. We consider the SLFVS as in Definitions 1.2 and 1.5, and we let the impact parameter and the selection and mutation parameters tend to zero. On a suitable space and time scale (depending on the regime of the radii of reproduction events) the process $\left(q_t^N\right)_{t\geq 0}$ converges to a deterministic process. We also characterise the limiting fluctuations of $\left(q_t^N\right)_{t\geq 0}$ about an approximation to this deterministic process as the solution to a stochastic partial differential equation.

2.1 Fixed radius of reproduction events

We begin by considering the regime in which the radii of the regions affected by reproduction events are bounded. We shall only give the proof in the case of fixed radius events; the proof for bounded radius events is the same but notationally awkward.

Fix $u, s \in (0,1]$ and $R > 0$, and choose $w_0 : \mathbb{R}^d \to [0,1]$ with uniformly bounded spatial derivatives of up to the fourth order. Take two sequences $(\varepsilon_N)_{N\geq 1}$, $(\delta_N)_{N\geq 1}$ of positive real numbers in $(0,1]$ decreasing to zero, and set

$$s_N = \delta_N^2 s, \qquad u_N = \varepsilon_N u, \qquad r_N = \delta_N R, \qquad q_0^N(x) = w_0(\delta_N x).$$

Let $\mu(dr) = \delta_R$, and let $F : \mathbb{R} \to \mathbb{R}$ be a smooth, bounded function with bounded derivatives of order up to four such that $F|_{[0,1]}$ satisfies (1.4) for some $m \in \mathbb{N}$ and $p : \{a, A\}^m \to [0,1]$. Then for $N \geq 1$, let $\left(q_t^N\right)_{t\geq 0}$ be the spatial Λ-Fleming-Viot process in $\mathbb{R}^d$ with selection following the dynamics of Definition 1.2 with impact parameter u_N, radius of reproduction events R, selection parameter s_N and selection function F started from the initial condition q_0^N.

Define the rescaled process $\left(\mathbf{q}_t^N\right)_{t\geq 0}$ by setting:

$$\forall x \in \mathbb{R}^d, t \geq 0,\ \mathbf{q}_t^N(x) = q^N_{t/(\varepsilon_N \delta_N^2)}(x/\delta_N). \tag{2.1}$$

We justify this scaling as follows. Recall from (1.1) that we think of $q_t^N(x)$ as denoting the proportion of the population at location x and time t which is of type a. Consider an individual randomly chosen from the population at location x at time t. It finds itself within a region affected by a reproduction event at rate $|B(0,R)|$. The probability that it dies and is replaced by a new individual is $u_N = \varepsilon_N u$, so, if we rescale time by $1/\varepsilon_N$, this will happen at rate $\mathcal{O}(1)$. Also, we are going to see later (see Section 3.2) that the reproduction events act like a discrete heat flow on the allele frequencies. We rescale time further by $1/\delta_N^2$ and space by $1/\delta_N$, which corresponds to the diffusive scaling of this discrete heat flow. Since selective events also take place at rate $\mathcal{O}\left(\delta_N^2\right)$, this is the right scaling to consider in order to observe the effects of both migration and selection in the limit. (Due to this diffusive scaling we shall refer to this regime as the Brownian case.)

We need to introduce some notation. Let $L^{1,\infty}(\mathbb{R}^d)$ denote the space of bounded and integrable real-valued functions on $\mathbb{R}^d$. For $r > 0$, we set $V_r = |B(0,r)|$ and, for $x, y \in \mathbb{R}^d$,

$$V_r(x,y) = |B(x,r) \cap B(y,r)|. \tag{2.2}$$

For $\phi : \mathbb{R}^d \to \mathbb{R}$ and $x \in \mathbb{R}^d$, set

$$\overline{\phi}(x,r) = \frac{1}{V_r}\int_{B(x,r)} \phi(y)\mathrm{d}y.$$

When there is no ambiguity, we shall not specify the radius r and simply write $\overline{\phi}(x)$. This notation will be used throughout this paper and formulae will routinely involve averages of averages, *etc*. For example we also write

$$\overline{\overline{\phi}}(x,r) = \frac{1}{V_r^2}\int_{B(x,r)}\int_{B(y,r)} \phi(z)\mathrm{d}z\mathrm{d}y. \tag{2.3}$$

Let us define a linear operator $\mathcal{L}^{(r)}$ by setting

$$\mathcal{L}^{(r)}\phi(x) = \frac{d+2}{2r^2}\left(\overline{\overline{\phi}}(x,r) - \phi(x)\right). \tag{2.4}$$

Finally let $\mathcal{S}(\mathbb{R}^d)$ denote the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^d$, whose derivatives of all orders are also rapidly decreasing. Accordingly, let $\mathcal{S}'(\mathbb{R}^d)$ denote the space of tempered distributions.

Lemma 2.1. *If $w_0 : \mathbb{R}^d \to [0,1]$ has uniformly bounded spatial derivatives of order up to four, then*

$$\begin{cases} \dfrac{\partial f_t^N}{\partial t} = uV_R \left[\dfrac{2R^2}{d+2} \mathcal{L}^{(r_N)} f_t^N - s\overline{F(\overline{f_t^N})}(r_N) \right], \\ f_0^N = w_0. \end{cases} \tag{2.5}$$

defines a unique (deterministic) function f^N in $L^\infty([0,T] \times \mathbb{R}^d)$. In addition, it admits spatial derivatives of order up to four which are all in $L^\infty([0,T] \times \mathbb{R}^d)$.

We prove this lemma in Appendix B with a Picard iteration.

As stated in the introduction, the spatial Λ-Fleming-Viot process with genic selection with fixed radius of reproduction events converges, under what can be considered a diffusive scaling, to the solution of the Fisher-KPP equation (as in [EVY14] for $d \geq 2$) while the limiting fluctuations are given by the solution to a stochastic partial differential equation which generalises the result obtained in [Nor75a]. We can now give a precise statement of this result for general frequency dependent selection. The same result holds for radius distributions given by a finite measure μ on a bounded interval.

Theorem 2.2 (Central Limit Theorem for the SLFVS in $\mathbb{R}^d$ with fixed radius of reproduction events)**.** *Let $\left(\mathbf{q}_t^N\right)_{t\geq 0}$ be defined as in (2.1). Suppose that $\varepsilon_N = o\left(\delta_N^{d+2}\right)$, then the process $\left(\mathbf{q}_t^N\right)_{t\geq 0}$ converges in L^1 and in probability (for the metric d of Definition 1.1) to the deterministic solution $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ of the following PDE,*

$$\begin{cases} \dfrac{\partial f_t}{\partial t} = uV_R \left[\dfrac{R^2}{d+2} \Delta f_t - sF(f_t) \right], \\ f_0 = w_0. \end{cases}$$

In addition,

$$Z_t^N = (\varepsilon_N \delta_N^{d-2})^{-1/2} (\mathbf{q}_t^N - f_t^N)$$

defines a sequence of distribution-valued processes converging in distribution in $\mathrm{D}([0,T],$ $\mathcal{S}'(\mathbb{R}^d))$ to the solution of the following stochastic partial differential equation,

$$\begin{cases} \mathrm{d}z_t = uV_R \left[\dfrac{R^2}{d+2} \Delta z_t - sF'(f_t) z_t \right] \mathrm{d}t + uV_R \sqrt{f_t(1-f_t)} \mathrm{d}W_t, \\ z_0 = 0, \end{cases}$$

where W is a space-time white noise.

Remark 2.3. The impact parameter u_N is inversely proportional to the neighbourhood size - *i.e.* the probability that two individuals have a common parent in the previous generation (see Section 3.6 of [BEV13a] for details). Hence, letting u_N tend to zero corresponds to letting the neighbourhood size grow to infinity.

We shall show in Section 3 that Theorem 2.2 is a consequence of Theorem 3.7. The latter is a result on sequences of solutions to a martingale problem and is proved in Section 4. In [EVY14], the authors already showed that in the special case of genic selection (as in Definition 1.4), for $d \geq 2$, the sequence of averages of $\left(\mathbf{q}_t^N\right)_{t\geq 0}$ over balls of radius r_N converges in distribution in $\mathrm{D}([0,\infty), \Xi)$ to the solution of the Fisher-KPP equation.

Remark 2.4. It would have been more natural to consider the fluctuations directly around the deterministic limit $(f_t)_{t\geq 0}$, but in fact the difference between f_N and f is too large (of order δ_N^2, see Proposition 4.7). We have that $|Z_t^N - (\varepsilon_N \delta_N^{d-2})^{-1/2} (\mathbf{q}_t^N - f_t)| = \mathcal{O}(\delta_N^2 (\varepsilon_N \delta_N^{d-2})^{-1/2})$ but if $\varepsilon_N = o\left(\delta_N^{d+2}\right)$ then $(\varepsilon_N \delta_N^{d-2})^{1/2} \delta_N^{-2} = o(\delta_N^{d-2})$ and so $(\varepsilon_N \delta_N^{d-2})^{-1/2} \delta_N^2 \to \infty$ as $N \to \infty$ as soon as $d \geq 2$.

2.2 Stable radii of reproduction events

In the previous subsection, we assumed that the radius of dispersion of the offspring produced at reproduction events was small. We now wish to allow large scale extinction-recolonization events to take place to illustrate the fact that "catastrophic" extinction events can occur, followed by a quick replacement of the dead individuals by the offspring of a small subset (here only one individual) of the survivors. To do so, we suppose that the intensity measure for the radius of reproduction events $\mu(\mathrm{d}r)$ has a power law behaviour, following the work in [EVY14]. The corresponding limiting behaviour is described by reaction-diffusion equations with non-local diffusion, studied for example in [Chm13, AK15]. Suppose that the measure $\mu(\mathrm{d}r)$ for the radius of reproduction events is given by

$$\mu(\mathrm{d}r) = \frac{\mathbb{1}_{r\geq 1}}{r^{d+\alpha+1}}\mathrm{d}r, \tag{2.6}$$

for some $\alpha \in (0, 2\wedge d)$. Fix $u, s \in (0,1]$ and choose $w_0 : \mathbb{R}^d \to [0,1]$ with uniformly bounded spatial derivatives of up to the second order. Again, take $(\varepsilon_N)_{N\geq 1}$ and $(\delta_N)_{N\geq 1}$ two sequences in $(0,1]$ decreasing to zero, and set

$$s_N = \delta_N^\alpha s, \qquad u_N = \varepsilon_N u, \qquad q_0^N(x) = w_0(\delta_N x).$$

Let $F : \mathbb{R} \to \mathbb{R}$ be a smooth, bounded function with bounded first and second derivatives such that $F|_{[0,1]}$ satisfies (1.4) for some $m \in \mathbb{N}$ and $p : \{a, A\}^m \to [0,1]$. Then for $N \geq 1$, let $\left(q_t^N\right)_{t\geq 0}$ be the spatial Λ-Fleming-Viot process with selection following the dynamics of Definition 1.2 with impact parameter u_N, radius of reproduction events given by $\mu(\mathrm{d}r)$ in (2.6), selection parameter s_N and selection function F started from the initial condition q_0^N.

The main difference with the setting of Subsection 2.1 is that the flow resulting from the reproduction events is the α-stable version of the heat flow (see Section 3.3). Thus we apply a stable scaling of time by $1/\delta_N^\alpha$ and space by $1/\delta_N$ (after rescaling time by $1/\varepsilon_N$ as previously). Since we have chosen $s_N = \delta_N^\alpha s$, this is the right scaling to consider in order to observe both selection and migration in the limit. For all $x \in \mathbb{R}^d$ and $t \geq 0$, set

$$\mathbf{q}_t^N(x) = q^N_{t/(\varepsilon_N\delta_N^\alpha)}(x/\delta_N). \tag{2.7}$$

We need some more notation; recall the notation for double averages in (2.3). The following will take up the role played by $\overline{F(\overline{w})}$ in the fixed radius case. For $H : [0,1] \to \mathbb{R}$, $\delta > 0$, and $f \in \Xi$, set

$$H^{(\delta)}(f) : x \mapsto \alpha \int_1^\infty \overline{H(\overline{f})}(x, \delta r)\frac{\mathrm{d}r}{r^{\alpha+1}}. \tag{2.8}$$

Recalling the notation in (2.2), set, for $x, y \in \mathbb{R}^d$,

$$\Phi(|x-y|) = \int_{\frac{|x-y|}{2}}^\infty \frac{V_r(x,y)}{V_r}\frac{\mathrm{d}r}{r^{d+\alpha+1}}, \qquad \Phi^{(\delta)}(|x-y|) = \int_{\frac{|x-y|}{2}\vee\delta}^\infty \frac{V_r(x,y)}{V_r}\frac{\mathrm{d}r}{r^{d+\alpha+1}}.$$

For $\phi : \mathbb{R}^d \to \mathbb{R}$ which admits uniformly bounded spatial derivatives of order up to two and $\psi \in L^{1,\infty}(\mathbb{R}^d)$,

$$\mathcal{D}^\alpha\phi(x) = \int_{\mathbb{R}^d}\Phi(|x-y|)(\phi(y)-\phi(x))\mathrm{d}y, \quad \mathcal{D}^{\alpha,\delta}\psi(x) = \int_{\mathbb{R}^d}\Phi^{(\delta)}(|x-y|)(\psi(y)-\psi(x))\mathrm{d}y. \tag{2.9}$$

Remark 2.5. Up to a multiplicative constant, depending on d and α, $\mathcal{D}^\alpha$ is the fractional Laplacian (this can be seen via the Fourier transform, see [SKM93]).

We can now formulate our result for the stable radii regime. The main difference from Theorem 2.2 is that the Laplacian has to be replaced by the operator $\mathcal{D}^\alpha$ and that the noise driving the fluctuations is replaced by a coloured noise which is white in time and has spatial correlations which decay like $K_\alpha(z_1, z_2)$ as $|z_1 - z_2| \to \infty$, where

$$K_\alpha(z_1, z_2) = \int_{\frac{|z_1-z_2|}{2}}^{\infty} V_r(z_1, z_2) \frac{\mathrm{d}r}{r^{d+\alpha+1}} = \frac{C_{d,\alpha}}{|z_1 - z_2|^\alpha}. \tag{2.10}$$

We also set the following notation: for $f \in \Xi$,

$$[f]_\alpha(z_1, z_2) = \frac{\int_{\frac{|z_1-z_2|}{2}}^{\infty} \frac{\mathrm{d}r}{r^{d+\alpha+1}} \int_{B(z_1,r)\cap B(z_2,r)} \overline{f}(x,r)\mathrm{d}x}{\int_{\frac{|z_1-z_2|}{2}}^{\infty} V_r(z_1, z_2) \frac{\mathrm{d}r}{r^{d+\alpha+1}}}. \tag{2.11}$$

Note that if f denotes the frequency of type a in $\mathbf{q}_t^N$ immediately before a (neutral) reproduction event which hits both z_1 and z_2 with $|z_1 - z_2| \geq 2\delta_N$, then $[f]_\alpha(z_1, z_2)$ is the probability that the offspring produced in this event are of type a.

The following lemma provides a deterministic centering term f^N around which we consider the fluctuations of $\mathbf{q}^N$.

Lemma 2.6. *If $w_0 : \mathbb{R}^d \to [0, 1]$ has uniformly bounded spatial derivatives of order up to two, then*

$$\begin{cases} \dfrac{\partial f_t^N}{\partial t} = u \left[\mathcal{D}^{\alpha,\delta_N} f_t^N - \dfrac{V_1 s}{\alpha} F^{(\delta_N)}(f_t^N) \right], \\ f_0^N = w_0 \end{cases} \tag{2.12}$$

defines a unique (deterministic) function f^N in $L^\infty([0, T] \times \mathbb{R}^d)$. In addition, it admits spatial derivatives of order up to two which are all in $L^\infty([0, T] \times \mathbb{R}^d)$.

This lemma is proved in Appendix B.

Theorem 2.7 (Central Limit Theorem for the SLFVS in $\mathbb{R}^d$ with stable radii of reproduction events)**.** *Let $\left(\mathbf{q}_t^N\right)_{t\geq 0}$ be defined as in (2.7). Suppose that $\varepsilon_N = o\left(\delta_N^{2\alpha}\right)$; then $\left(\mathbf{q}_t^N\right)_{t\geq 0}$ converges in L^1 and in probability (for the metric d of Definition 1.1) to the deterministic solution $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ of the following PDE,*

$$\begin{cases} \dfrac{\partial f_t}{\partial t} = u \left[\mathcal{D}^\alpha f_t - \dfrac{sV_1}{\alpha} F(f_t) \right], \\ f_0 = w_0. \end{cases} \tag{2.13}$$

In addition,

$$Z_t^N = \varepsilon_N^{-1/2}(\mathbf{q}_t^N - f_t^N)$$

defines a sequence of distribution-valued processes, converging in distribution in $\mathrm{D}\left([0, T], \mathcal{S}'(\mathbb{R}^d)\right)$ to the solution of the following stochastic partial differential equation,

$$\begin{cases} \mathrm{d}z_t = u \left[\mathcal{D}^\alpha z_t - \dfrac{sV_1}{\alpha} F'(f_t) z_t \right] \mathrm{d}t + u \, \mathrm{d}W_t^\alpha \\ z_0 = 0, \end{cases}$$

where W^α is a coloured noise with covariation measure given by

$$\begin{aligned} Q^\alpha(\mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}s) = K_\alpha(z_1, z_2) \, ([f_s]_\alpha(z_1, z_2)(1 - f_s(z_1))(1 - f_s(z_2)) \\ +(1 - [f_s]_\alpha(z_1, z_2)) f_s(z_1) f_s(z_2)) \, \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}s. \end{aligned} \tag{2.14}$$

Remark 2.8. The fact that the correlations in the noise decay as $|z_1 - z_2|^{-\alpha}$ can be expected from the results in [BEK06] (see also [BEK10]). The authors prove that, if N is a Poisson point process on $\mathbb{R}^d \times \mathbb{R}_+$ whose intensity measure is of the form $\mathrm{d}x f(r)\mathrm{d}r$ with $f(r) \sim \frac{C}{r^{1+\alpha+d}}$, one can define a generalized random field X on the space of signed measures on $\mathbb{R}^d$ with finite total variation by

$$\langle X, \mu \rangle = \int_{\mathbb{R}^d \times \mathbb{R}_+} \mu(B(x,r)) N(\mathrm{d}x, \mathrm{d}r).$$

Under a suitable scaling of the radius and of the intensity measure, it is shown that the fluctuations of X converge (in the sense of finite dimensional distributions) to a centred Gaussian random linear functional W^α with

$$\mathbb{E}\left[W^\alpha(\mu) W^\alpha(\nu)\right] = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |z_1 - z_2|^{-\alpha} \mu(\mathrm{d}z_1) \nu(\mathrm{d}z_2).$$

(The notation has been changed so as to fit that of our setting; in [BEK06], $\beta = \alpha + d$.) The second factor in (2.14) (besides $K_\alpha(z_1, z_2)$) comes from the expression for the covariation between the jump in the allele frequency during a reproduction event at the two locations z_1 and z_2 (see (3.6) and the definition of $[f]_\alpha$ in (2.11)).

We show in Section 3 that Theorem 2.7 is a consequence of Theorem 3.10. The latter is a result on sequences of solutions to a martingale problem and is proved in Section 5. In [EVY14], the authors showed that in the special case of genic selection (as in Definition 1.4), for $d \geq 2$, a sequence of spatially averaged versions of $\left(\mathbf{q}_t^N\right)_{t\geq 0}$ converges in distribution in $\mathrm{D}([0,\infty), \Xi)$ to the solution of (2.13).

2.3 Drift load for a spatially structured population

We shall illustrate the application of our results by studying the drift load in the SLFVS with overdominance as in Definition 1.5, in the case of bounded radii.

As in Section 2.1, fix u, s_1, s_2, ν_1, ν_2 in $(0,1]$ and $R > 0$ such that $s_1 + s_2 + \nu_1 + \nu_2 < 1$, take two sequences $(\varepsilon_N)_{N\geq 1}$, $(\delta_N)_{N\geq 1}$ of positive real numbers in $(0,1]$ decreasing to zero, and set

$$u_N = \varepsilon_N u, \qquad r_N = \delta_N R \qquad s_{i,N} = \delta_N^2 s_i, \qquad \nu_{i,N} = \delta_N^2 \nu_i \tag{2.15}$$

for $i = 1, 2$. Then for $N \geq 1$, let $\left(q_t^N\right)_{t\geq 0}$ be the SLFVS following the dynamics of Definition 1.5 with impact parameter u_N, radius of reproduction events R, selection parameters $s_{i,N}$ and mutation parameters $\nu_{i,N}$, started from some initial condition q_0^N.

One thing to note is that for our results to hold, we need to make sure that the allele frequencies do not get "stuck" - even locally - at the boundaries (*i.e.* upon reaching 0 or 1), which could significantly slow down the convergence to the equilibrium frequency. For this reason we choose to assume that during some mutation reproduction events the type of the offspring can differ from that of its parent. This will not affect the results in any other way provided that the mutation parameters are negligible compared to the selection parameters.

Now let

$$F(w) = w(1-w)(w - \frac{s_2}{s_1+s_2}) + \frac{\nu_1}{s_1+s_2} w - \frac{\nu_2}{s_1+s_2}(1-w). \tag{2.16}$$

We shall see in Section 3 that this function plays the same role as in the haploid case. Note that F satisfies the following conditions:

$$\exists \lambda \in [0,1] : F(\lambda) = 0; \tag{2.17}$$

furthermore there is only one such λ and it satisfies

$$0 < \lambda < 1 \text{ and } F'(\lambda) > 0. \tag{2.18}$$

For the function F given in (2.16), λ is given by

$$\lambda = \frac{s_2}{s_1 + s_2} + \mathcal{O}\left(\frac{\nu_1 + \nu_2}{s_1 + s_2}\right) \tag{2.19}$$

(since the first term is a solution of $w(1-w)(w - \frac{s_2}{s_1+s_2}) = 0$).

Let us define $K^N(t,x)$, the local mean fitness at a point $x \in \mathbb{R}^d$, as the expected fitness of an individual formed by fusing two gametes chosen uniformly at random from $B(x,R)$ at time $t \geq 0$. In other words, its two copies of the gene are sampled independently by selecting two parental locations y_1 and y_2 uniformly at random in $B(x,R)$ and then types according to $q_t(y_1)$ and $q_t(y_2)$. Then (see [Rob70]),

$$\begin{aligned}
K^N(t,x) &= \mathbb{E}\left[(1-s_{1,N})\overline{q_t^N}(x,R)^2 + 2\overline{q_t^N}(x,R)(1-\overline{q_t^N}(x,R)) + (1-s_{2,N})(1-\overline{q_t^N}(x,R))^2\right] \\
&= 1 - \mathbb{E}\left[s_{1,N}\overline{q_t^N}(x,R)^2 + s_{2,N}(1-\overline{q_t^N}(x,R))^2\right] \\
&= 1 - \frac{s_{1,N}s_{2,N}}{s_{1,N}+s_{2,N}} - (s_{1,N}+s_{2,N})\mathbb{E}\left[\left(\overline{q_t^N}(x,R) - \frac{s_2}{s_1+s_2}\right)^2\right] \\
&= 1 - \frac{s_{1,N}s_{2,N}}{s_{1,N}+s_{2,N}} - (s_{1,N}+s_{2,N})\mathbb{E}\left[\left(\overline{q_t^N}(x,R) - \lambda\right)^2\right] \\
&\qquad - (s_{1,N}+s_{2,N})\left(\lambda - \frac{s_2}{s_1+s_2}\right)\left(2\mathbb{E}\left[\overline{q_t^N}(x,R)\right] - \lambda - \frac{s_2}{s_1+s_2}\right).
\end{aligned}$$

The first term $\frac{s_{1,N}s_{2,N}}{s_{1,N}+s_{2,N}}$ is the segregation load mentioned in the introduction, and it is of order δ_N^2. The second term is then the local drift load, which we aim to estimate at large times for large N. The last term is an error due to the mutation events; by (2.19), it is $\mathcal{O}\left(\delta_N^2(\nu_1+\nu_2)\right)$, and so negligible compared to the segregation load if $\frac{\nu_1+\nu_2}{s_1+s_2}$ is small. Let us set

$$\Delta^N(t,x) = (s_{1,N}+s_{2,N})\mathbb{E}\left[\left(\overline{q_t^N}(x,R) - \lambda\right)^2\right]. \tag{2.20}$$

The following theorem is proved in Section 6 using some of the intermediate results used to prove Theorem 2.2.

Theorem 2.9. *Suppose that $q_0^N(x) = \lambda$ for all x and assume that $\varepsilon_N = o\left(\delta_N^4\right)$. There exists a constant $C > 0$, depending only on the dimension d, such that, for all $x \in \mathbb{R}^d$, as $N, t \to \infty$, if t grows fast enough that $\varepsilon_N t \to \infty$ if $d \geq 3$ and $\varepsilon_N \delta_N^2 t \to \infty$ if $d \leq 2$,*

$$\Delta^N(t,x) \underset{N,t\to\infty}{\sim} C\varepsilon_N\delta_N^2 c_N,$$

where

$$c_N = \begin{cases} 1 & \text{if } d \geq 3, \\ \left|\log \delta_N^2\right| & \text{if } d = 2, \\ \delta_N^{-1} & \text{if } d = 1. \end{cases} \tag{2.21}$$

Assumption (2.17)-(2.18) is crucial in [Nor74a], which serves as a basis for this result. In fact this condition ensures that λ is the only equilibrium point for the allele frequency, and that it is stable.

Remark 2.10. We chose to start the process from the equilibrium frequency λ - *i.e.* very near stationarity - but we need not do so. The same result can be obtained starting from an arbitrary initial condition, provided we let t grow sufficiently fast that the process reaches stationarity quickly enough. The corresponding centering term f^N is then defined as in (2.5), and (2.17)-(2.18) ensures that it converges to λ exponentially quickly. Starting from λ simplifies the proof as in this case, for all $t \geq 0$, $f_t^N = \lambda$.

In the non-spatial setting of the Λ-Fleming Viot process, a simplified version of the proof of Theorem 2.9 shows that the drift load is asymptotically proportional to u_N. We can see u_N as being inversely proportional to the neighbourhood size, in other words the probability that two individuals had a common parent in the previous generation (see [BEV13a] for details). This agrees with Robertson's estimate [Rob70] of $(4N)^{-1}$, where N is the total population size in a panmictic population. Note that this estimate is independent of the strength of selection. This can be seen as the result of a trade off between selection and genetic drift: if selection is weak, the allele frequency can be far from the equilibrium whereas if selection is stronger, the allele frequency stays nearer to the equilibrium and in both cases the mean fitness of the population is the same.

For spatially structured populations, however, Theorem 2.9 shows that the local drift load is significantly smaller than in the non-spatial setting and does depend on the strength of natural selection. For example, if a population lives in a geographical space of dimension 2, the corresponding drift load will be of order $\varepsilon_N \delta_N^2 \left|\log \delta_N^2\right|$, and since $u_N = u\varepsilon_N$ and $s_N := s_{1,N} + s_{2,N} = \delta_N^2(s_1 + s_2)$, it is of order $u_N s_N \left|\log s_N\right|$. Moreover, we see a strong effect of dimension on this estimate. Populations living in a space with a higher dimension have a reduced drift load compared to populations evolving in smaller dimensions. This result illustrates the fact that, in a higher dimension, migration is more efficient at preventing the allele frequencies from being locally far from the equilibrium frequency. It turns out from the proof that this is linked to the recurrence properties of Brownian motion.

Remark 2.11 (Drift load in the stable case)**.** If one considers instead the SLFVS with stable radii of reproduction events, under similar conditions to those in Theorem 2.7, one finds that for all $d \geq 1$ and $\alpha \in (0, 2 \wedge d)$, $\Delta^N(t,x)$ is asymptotically equivalent to a constant times $u_N s_N \left|\log s_N\right|$.

3 Martingale problems for the SLFVS

This section provides the basic ingredients for the proofs of Theorems 2.2 and 2.7. In Subsection 3.1, we prove that the SLFVS satisfies a martingale problem. In Subsections 3.2 and 3.3, we study the martingale problem for the rescaled version of this process, in the fixed radius case and in the stable radii case, and state general convergence results for processes satisfying these martingale problems. Theorems 2.2 and 2.7 are direct consequences of these results.

3.1 The martingale problem for the SLFVS

Let $\left(q_t^N\right)_{t\geq 0}$ be defined (as in Sections 2.1 and 2.2) as the SLFVS as in Definition 1.2 with impact parameter u_N, distribution of reproduction event radii given by $\mu(dr)$, selection parameter s_N and selection function F. Let $(\mathcal{F}_t)_{t\geq 0}$ denote the natural filtration of this process.

For $p > 0$ and $\phi : \mathbb{R}^d \to \mathbb{R}$, let $\|\phi\|_p = (\int_{\mathbb{R}^d} |\phi(x)|^p \mathrm{d}x)^{1/p}$.

Proposition 3.1. *Suppose that $\int_0^\infty V_r^2 \mu(\mathrm{d}r) < \infty$. For any $\phi : \mathbb{R}^d \to \mathbb{R}$ in $L^{1,\infty}(\mathbb{R}^d)$,*

$$\left\langle q_t^N, \phi \right\rangle - \left\langle q_0^N, \phi \right\rangle - \int_0^t \int_0^\infty u_N V_r \left\{ \left\langle q_s^N, \overline{\overline{\phi}}(r) - \phi \right\rangle - s_N \left\langle \overline{F(\overline{q_s^N})}(r), \phi \right\rangle \right\} \mu(\mathrm{d}r)\mathrm{d}s \tag{3.1}$$

defines a (mean zero) square integrable $\mathcal{F}_t$-martingale with (predictable) variation *process*

$$\int_0^t \int_0^\infty u_N^2 V_r^2 \int_{(\mathbb{R}^d)^2} \phi(z_1)\phi(z_2)\sigma^{(r)}_{z_1,z_2}(q_s^N)\mathrm{d}z_1\mathrm{d}z_2\mu(\mathrm{d}r)\mathrm{d}s + \mathcal{O}\left(tu_N^2 s_N \left\|\phi\right\|_2^2\right), \tag{3.2}$$

where

$$\sigma^{(r)}_{z_1,z_2}(q) = \frac{1}{V_r^2}\int_{B(z_1,r)\cap B(z_2,r)} [\overline{q}(x,r)(1-q(z_1))(1-q(z_2)) + (1-\overline{q}(x,r))q(z_1)q(z_2)]\mathrm{d}x. \tag{3.3}$$

Proposition 3.1 can be seen as a way to write q_t as the sum of the effects of the different evolutionary forces at play in this model. The term $\overline{\overline{\phi}} - \phi$ represents migration, while the term involving the function F in (3.1) accounts for the bias introduced during selective events. As for the martingale term, it corresponds to the stochasticity at each reproduction event, which is called *genetic drift*.

Proof of Proposition 3.1. We drop the superscript N from q^N in this proof. Let $\mathbb{P}_{t,x,r}$ (resp. $\mathbb{P}^S_{t,x,r}$) denote the distribution of the parental type k at a reproduction event $(t,x,r) \in \Pi$ (resp. in Π^S). Then, from the definition of $(q_t)_{t\geq 0}$,

$$\begin{aligned}\lim_{\delta t\downarrow 0}\frac{1}{\delta t}\mathbb{E}\left[\langle q_{t+\delta t},\phi\rangle - \langle q_t,\phi\rangle \mid q_t = q\right] = \\ \int_{\mathbb{R}^d}\mathrm{d}x\int_0^\infty \mu(\mathrm{d}r)\int_{\mathbb{R}^d}\phi(z)u_N\mathbb{1}_{|x-z|<r}\big\{(1-s_N)\mathbb{E}_{t,x,r}\left[\mathbb{1}_{k=a} - q_t(z) \mid q_t = q\right] \\ + s_N\,\mathbb{E}^S_{t,x,r}\left[\mathbb{1}_{k=a} - q_t(z) \mid q_t = q\right]\big\}\mathrm{d}z.\end{aligned} \tag{3.4}$$

Recall from Definition 1.2 that $\mathbb{P}_{t,x,r}\left(k=a \mid q_t = q\right) = \overline{q}(x,r)$ and

$$\mathbb{P}^S_{t,x,r}\left(k=a \mid q_t = q\right) = \overline{q}(x,r) - F(\overline{q}(x,r)).$$

Integrating with respect to the variable x over $B(z,r)$ then yields

$$\begin{aligned}&\lim_{\delta t\downarrow 0}\frac{1}{\delta t}\mathbb{E}\left[\langle q_{t+\delta t},\phi\rangle - \langle q_t,\phi\rangle \mid q_t = q\right] \\ &\qquad = \int_0^\infty \mu(\mathrm{d}r)u_N V_r\int_{\mathbb{R}^d}\phi(z)\left\{(\overline{\overline{q}}(z,r) - q(z)) - s_N\overline{F(\overline{q})}(z,r)\right\}\mathrm{d}z.\end{aligned}$$

Thus (3.1) indeed defines a martingale - see for example [EK86, Proposition 4.1.7] (we can change the order of integration to do the averaging on ϕ instead of q in the first term). To compute its variation process, write

$$\begin{aligned}&\lim_{\delta t\downarrow 0}\frac{1}{\delta t}\mathbb{E}\left[\left(\langle q_{t+\delta t},\phi\rangle - \langle q_t,\phi\rangle\right)^2 \,\middle|\, q_t = q\right] \\ &\quad = \int_{\mathbb{R}^d}\int_0^\infty\int_{(\mathbb{R}^d)^2}\phi(z_1)\phi(z_2)u_N^2\mathbb{1}_{\substack{|z_1-x|<r\\|z_2-x|<r}} \\ &\qquad \big\{(1-s_N)\mathbb{E}_{t,x,r}\left[(\mathbb{1}_{k=a} - q_t(z_1))(\mathbb{1}_{k=a} - q_t(z_2)) \mid q_t = q\right] \\ &\qquad + s_N\,\mathbb{E}^S_{t,x,r}\left[(\mathbb{1}_{k=a} - q_t(z_1))(\mathbb{1}_{k=a} - q_t(z_2)) \mid q_t = q\right]\big\}\mathrm{d}z_1\mathrm{d}z_2\mu(\mathrm{d}r)\mathrm{d}x.\end{aligned} \tag{3.5}$$

But

$$\begin{aligned}&\mathbb{E}_{t,x,r}\left[(\mathbb{1}_{k=a} - q_t(z_1))(\mathbb{1}_{k=a} - q_t(z_2)) \mid q_t = q\right] \\ &\qquad = \overline{q}(x,r)(1-q(z_1))(1-q(z_2)) + (1-\overline{q}(x,r))q(z_1)q(z_2),\end{aligned} \tag{3.6}$$

and the other term within the curly brackets is $\mathcal{O}(s_N)$. Thus, integrating with respect to x and using (3.3), we recover

$$\begin{aligned}\lim_{\delta t\downarrow 0}\frac{1}{\delta t}\mathbb{E}\left[\left(\langle q_{t+\delta t},\phi\rangle-\langle q_t,\phi\rangle\right)^2\,\middle|\, q_t=q\right]&\\ =\int_0^\infty \mu(\mathrm{d}r)u_N^2V_r^2\int_{(\mathbb{R}^d)^2}\phi(z_1)\phi(z_2)\sigma^{(r)}_{z_1,z_2}(q)\mathrm{d}z_1\mathrm{d}z_2&\\ +\mathcal{O}(s_N)\int_0^\infty\mu(\mathrm{d}r)u_N^2V_r^2\int_{\mathbb{R}^d}\left(\frac{1}{V_r}\int_{B(x,r)}\phi(z)dz\right)^2dx.&\quad (3.7)\end{aligned}$$

By Jensen's inequality, $\int_{\mathbb{R}^d}\left(\frac{1}{V_r}\int_{B(x,r)}\phi(z)dz\right)^2dx\le\|\phi\|_2^2$ and the result follows from the assumption that $\int_0^\infty V_r^2\mu(\mathrm{d}r)<\infty$. □

Now let $\left(q_t^N\right)_{t\ge 0}$ denote the SLFVS with overdominance as defined in Definition 1.5 with impact parameter u_N, radius of reproduction events R, selection parameters $s_{i,N}$ and mutation parameters $\nu_{i,N}$ defined in (2.15). Recall the definition of F in (2.16) and let $(\mathcal{F}_t)_{t\ge 0}$ denote the natural filtration of this process.

Proposition 3.2. *Let $s=s_1+s_2$ (and $s_N=s_{1,N}+s_{2,N}$). For any $\phi:\mathbb{R}^d\to\mathbb{R}$ in $L^{1,\infty}(\mathbb{R}^d)$,*

$$\left\langle q_t^N,\phi\right\rangle-\left\langle q_0^N,\phi\right\rangle-\int_0^t u_NV_r\left\{\left\langle q_s^N,\overline{\overline{\phi}}(R)-\phi\right\rangle-s_N\left\langle\overline{F(\overline{q_s^N})}(R),\phi\right\rangle\right\}\mathrm{d}s \quad (3.8)$$

defines a (mean zero) square integrable $\mathcal{F}_t$-martingale with (predictable) variation process

$$\int_0^t u_N^2V_R^2\int_{(\mathbb{R}^d)^2}\phi(z_1)\phi(z_2)\rho^{(R)}_{z_1,z_2}(q_s^N)\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}s+\mathcal{O}\left(tu_N^2\delta_N^2\|\phi\|_2^2\right), \quad (3.9)$$

where

$$\begin{aligned}\rho^{(r)}_{z_1,z_2}(q)=\frac{1}{V_r^2}\int_{B(z_1,r)\cap B(z_2,r)}&[\overline{q}(x,r)^2(1-q(z_1))(1-q(z_2))\\ &+2\overline{q}(x,r)(1-\overline{q}(x,r))(\tfrac12-q(z_1))(\tfrac12-q(z_2))+(1-\overline{q}(x,r))^2q(z_1)q(z_2)]\mathrm{d}x. \quad (3.10)\end{aligned}$$

Proof. Suppose a reproduction event hits the ball $B(x,r)$ at time t, and let $w=\overline{q_{t^-}^N}(x,r)$. Then,

$$\begin{aligned}\mathbb{P}\left(\{k_1,k_2\}=\{A_1,A_1\}\right)&=(1-s_{1,N}-s_{2,N}-\nu_{1,N}-\nu_{2,N})w^2+s_{1,N}w^4\\ &\quad+s_{2,N}w^2(1+(1-w)^2)+\nu_{2,N},\\ \mathbb{P}\left(\{k_1,k_2\}=\{A_1,A_2\}\right)&=(1-s_{1,N}-s_{2,N}-\nu_{1,N}-\nu_{2,N})2w(1-w)\\ &\quad+s_{1,N}2w(1-w)(1+w^2)+s_{2,N}2w(1-w)(1+(1-w)^2),\\ \mathbb{P}\left(\{k_1,k_2\}=\{A_2,A_2\}\right)&=(1-s_{1,N}-s_{2,N}-\nu_{1,N}-\nu_{2,N})(1-w)^2+s_{1,N}(1-w)^2(1+w^2)\\ &\quad+s_{2,N}(1-w)^4+\nu_{1,N}.\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{E}&\left[\tfrac12\left(\mathbb{1}_{k_1=A_1}+\mathbb{1}_{k_2=A_1}\right)\,\middle|\,\overline{q_{t^-}}(x,r)=w\right]\\ &=w+w(1-w)(-s_{1,N}w+s_{2,N}(1-w))-\nu_{1,N}w+\nu_{2,N}(1-w)\\ &=w-s_NF(w),\end{aligned}$$

recalling the definition of F in (2.16) and $s_{i,N}$, $\nu_{i,N}$ in (2.15) and that $s_N=s_{1,N}+s_{2,N}$. Therefore

$$\mathbb{E}_{t,x,r}\left[\tfrac12\left(\mathbb{1}_{k_1=A_1}+\mathbb{1}_{k_2=A_1}\right)-q_{t^-}^N(z)\,\middle|\,q_{t^-}^N=q\right]=\overline{q}(x,r)-q(z)-s_NF(\overline{q}(x,r)).$$

It follows as in the proof of Proposition 3.1 that (3.8) is a martingale. The result for the variation process also follows as in the proof of Proposition 3.1 (all terms containing either $s_{i,N}$ or $\nu_{i,N}$ are collected in the error term in (3.9)). Note that $\sigma^{(r)}$ is replaced by $\rho^{(r)}$ in order to account for the fact that (1.5) is replaced by (1.8). □

Remark 3.3. If q were continuous then as $r \to 0$, $\sigma^{(r)}_{z_1,z_2}(q) \to \delta_{z_1=z_2} q(z_1)(1-q(z_1))$ and $\rho^{(r)}_{z_1,z_2}(q) \to \frac{1}{2}\delta_{z_1=z_2} q(z_1)(1-q(z_1))$. The factor of $1/2$ represents the doubling of effective population size for a diploid population compared to a haploid one.

3.2 The rescaled martingale problem - Fixed radius case

As at the start of Subsection 2.1, let $(\varepsilon_N)_{N\geq 1}$, $(\delta_N)_{N\geq 1}$ be sequences in $(0,1]$ decreasing towards zero, and let $F : \mathbb{R} \to \mathbb{R}$.

Definition 3.4 (Martingale Problem (M1)). *Given $(\varepsilon_N)_{N\geq 1}$, $(\delta_N)_{N\geq 1}$ and F, let $\eta_N = \varepsilon_N \delta_N^2$, $\tau_N = \varepsilon_N^2 \delta_N^d$ and $r_N = \delta_N R$. Then for $N \geq 1$, we say that a Ξ-valued process $(w_t^N)_{t\geq 0}$ satisfies the martingale problem (M1) if for all ϕ in $L^{1,\infty}(\mathbb{R}^d)$,*

$$\left\langle w_t^N, \phi \right\rangle - \langle w_0, \phi \rangle - \eta_N u V_R \int_0^t \left\{ \frac{2R^2}{d+2} \left\langle w_s^N, \mathcal{L}^{(r_N)}\phi \right\rangle - s \left\langle \overline{F(\overline{w_s^N})}(r_N), \phi \right\rangle \right\} \mathrm{d}s \tag{3.11}$$

defines a (mean zero) square-integrable martingale with (predictable) variation process

$$\tau_N u^2 V_R^2 \int_0^t \int_{(\mathbb{R}^d)^2} \phi(z_1)\phi(z_2)\sigma^{(r_N)}_{z_1,z_2}(w_s^N)\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}s + \mathcal{O}\left(t\tau_N\delta_N^2 \|\phi\|_2^2\right). \tag{3.12}$$

Remark 3.5. Of course, one cannot expect uniqueness to hold for this martingale problem, due to the unspecified error term in (3.12). In the limit when $N \to \infty$, however, the error terms will vanish.

Let $\left(q_t^N\right)_{t\geq 0}$ be defined as at the start of Section 2.1. Set $w_t^N(x) = q_t^N(x/\delta_N)$.

Proposition 3.6. *For each N, the process $\left(w_t^N\right)_{t\geq 0}$ satisfies the martingale problem (M1).*

Proof. From Proposition 3.1, we know that, for $\phi \in L^{1,\infty}(\mathbb{R}^d)$,

$$\left\langle q_t^N, \phi \right\rangle = \left\langle q_0^N, \phi \right\rangle + u_N V_R \int_0^t \left\{ \left\langle q_s^N, \overline{\overline{\phi}}(R) - \phi \right\rangle - s_N \left\langle \overline{F(\overline{q_s^N})}(R), \phi \right\rangle \right\} \mathrm{d}s + \mathcal{M}_t^N(\phi),$$

where $\mathcal{M}_t^N(\phi)$ is a martingale. By a change of variables,

$$\left\langle w_t^N, \phi \right\rangle = \delta_N^d \left\langle q_t^N, \phi^{(\delta_N)} \right\rangle, \tag{3.13}$$

with $\phi^{(\delta)}(x) = \phi(\delta x)$. Also,

$$\delta_N^d \left\langle q_s^N, \overline{\overline{\phi^{(\delta_N)}}}(R) \right\rangle = \left\langle w_s^N, \overline{\overline{\phi}}(\delta_N R) \right\rangle \quad \text{and} \quad \delta_N^d \left\langle \overline{F(\overline{q_s^N})}(R), \phi^{(\delta_N)} \right\rangle = \left\langle \overline{F(\overline{w_s^N})}(\delta_N R), \phi \right\rangle. \tag{3.14}$$

Thus, recalling the definition of the operator $\mathcal{L}^{(r)}$ in (2.4) and the initial condition $q_0^N(x) = w_0(\delta_N x)$, we have

$$\left\langle w_t^N, \phi \right\rangle = \langle w_0, \phi \rangle + \varepsilon_N \delta_N^2 u V_R \int_0^t \left\{ \frac{2R^2}{d+2} \left\langle w_s^N, \mathcal{L}^{(\delta_N R)}\phi \right\rangle - s \left\langle \overline{F(\overline{w_s^N})}(\delta_N R), \phi \right\rangle \right\} \mathrm{d}s + \delta_N^d \mathcal{M}_t^N(\phi^{(\delta_N)}).$$

Moreover, by a change of variables in the variation process given in (3.2),

$$\delta_N^{2d}\left\langle \mathcal{M}^N(\phi^{(\delta_N)})\right\rangle_t = \varepsilon_N^2 u^2 V_R^2 \int_0^t \int_{(\mathbb{R}^d)^2} \phi(z_1)\phi(z_2)\sigma^{(R)}_{z_1/\delta_N, z_2/\delta_N}(q_s^N)\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}s + \mathcal{O}\left(t\varepsilon_N^2\delta_N^2\delta_N^d\left\|\phi\right\|_2^2\right), \quad (3.15)$$

and

$$\sigma^{(R)}_{z_1/\delta_N, z_2/\delta_N}(q_s^N) = \delta_N^d \sigma^{(\delta_N R)}_{z_1,z_2}(w_s^N).$$

Hence w^N satisfies the martingale problem (M1). □

Proposition 3.6 is the main ingredient in the proof of Theorem 2.2. In fact we shall now see that under suitable conditions on the parameters $(\varepsilon_N)_{N\geq 1}$ and $(\delta_N)_{N\geq 1}$, the function F and the initial condition w_0, any sequence of processes $\left(w_t^N\right)_{t\geq 0}$ satisfying the martingale problem (M1) in Definition 3.4 will also satisfy a result analogous to Theorem 2.2. If τ_N is of a smaller order than η_N, w^N can be expected to be asymptotically deterministic (on a suitable time-scale), and we can study its fluctuations around a deterministic centering term. Define $f^N : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ as in (2.5). Quite naturally, this corresponds to equating (3.11) to zero and making its time-scale fit that of the limiting process.

Since the operator $\mathcal{L}^{(r)}$ approximates the Laplacian as $r \to 0$ (see Proposition A.2 in the appendix), f_t^N converges to $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ as $N \to \infty$, where f_t is the solution of the following equation,

$$\begin{cases} \dfrac{\partial f_t}{\partial t} = uV_R\left(\dfrac{R^2}{d+2}\Delta f_t - sF(f_t)\right), \\ f_0 = w_0. \end{cases} \quad (3.16)$$

(See Proposition 4.7 for a precise statement.) The following result is proved in Section 4.

Theorem 3.7. *Suppose that $\left(w_t^N\right)_{t\geq 0}$ is a Ξ-valued process which satisfies the martingale problem (M1) in Definition 3.4 for some smooth, bounded $F : \mathbb{R} \to \mathbb{R}$ with bounded derivatives of order up to four and $(\delta_N)_N$, $(\varepsilon_N)_N$ converging to zero as $N \to \infty$. Moreover, suppose*

$$\tau_N/\eta_N = o\left(\delta_N^{2d}\right). \quad (3.17)$$

Suppose also that w_0 has uniformly bounded derivatives of up to the fourth order and that there exists α_N such that the jumps of $\left(w_t^N\right)_{t\geq 0}$ are (almost surely) dominated by

$$\sup_{t\geq 0}\left|\left\langle w_t^N, \phi\right\rangle - \left\langle w_{t^-}^N, \phi\right\rangle\right| \leq \alpha_N \left\|\phi\right\|_1 \quad (3.18)$$

for every $\phi \in L^{1,\infty}(\mathbb{R}^d)$, with $\alpha_N^2 = o(\tau_N/\eta_N)$. Then

$$\left(w^N_{t/\eta_N}\right)_{t\geq 0} \underset{N\to\infty}{\overset{L^1, P}{\longrightarrow}} (f_t)_{t\geq 0} \quad (3.19)$$

in $(\mathrm{D}([0,T],\Xi), d)$ for every $T > 0$ with d given by Definition 1.1. In addition,

$$Z_t^N = (\eta_N/\tau_N)^{1/2}(w^N_{t/\eta_N} - f_t^N)$$

defines a sequence of distribution-valued processes which converges in distribution in $\mathrm{D}\left([0,T], \mathcal{S}'(\mathbb{R}^d)\right)$ to the solution of the following stochastic partial differential equation,

$$\begin{cases} \mathrm{d}z_t = uV_R[\dfrac{R^2}{d+2}\Delta z_t - sF'(f_t)z_t]\mathrm{d}t + uV_R\sqrt{f_t(1-f_t)}\cdot \mathrm{d}W_t, \\ z_0 = 0, \end{cases} \quad (3.20)$$

W being a space-time white noise.

Theorem 2.2 is now a direct consequence.

Proof of Theorem 2.2. Recall that $(\mathbf{q}_t^N)_{t\geq 0}$ is defined in (2.1) as a rescaling of $\left(q_t^N\right)_{t\geq 0}$, and that by Proposition 3.6, letting $w_t^N(x) = q_t^N(x/\delta_N)$, $\left(w_t^N\right)_{t\geq 0}$ satisfies the martingale problem (M1). Also $\tau_N/\eta_N = o\left(\delta_N^{2d}\right)$ follows from $\varepsilon_N = o(\delta_N^{d+2})$, and the bound on the jumps (3.18) holds with $\alpha_N = \varepsilon_N u$ by (1.5). Hence Theorem 3.7 applies and the result follows by noting that $w_{t/\eta_N}^N = \mathbf{q}_t^N$. □

The proof of Theorem 3.7 can be found in full detail in Section 4, but, in order to shed some light on the limiting equations that we obtain and to identify the difficulties in proving this result, let us outline the first calculations involved in the proof. As in [Kur71], we use bounds on the martingale (3.11) to show the convergence of $\left(w_{t/\eta_N}^N\right)_{t\geq 0}$. When properly rescaled, this martingale converges to a continuous Gaussian martingale, implying the convergence of the fluctuation process $\left(Z_t^N\right)_{t\geq 0}$.

For ease of notation, we shall set the constants uV_R, $2R^2/(d+2)$ and s to 1 in the definition of (M1). Let $\mathbb{M}_t^N(\phi)$ denote $\tau_N^{-1/2}$ times the martingale defined in (3.11). Formally, we can then write (M1) as

$$\mathrm{d}w_t^N = \eta_N \left[\mathcal{L}^{(r_N)} w_t^N - \overline{F(\overline{w_t^N})}(r_N)\right] \mathrm{d}t + \tau_N^{1/2} \mathrm{d}\mathbb{M}_t^N.$$

Now set

$$M_t^N(\phi) = \eta_N^{1/2} \mathbb{M}_{t/\eta_N}^N(\phi).$$

(This Brownian scaling is not surprising since in the SLFVS case $\mathbb{M}^N$ is essentially an integral against a compensated Poisson point process, and we expect M^N to converge to an integral against white noise.) Replacing t by t/η_N above, we have

$$\mathrm{d}w_{t/\eta_N}^N = \left[\mathcal{L}^{(r_N)} w_{t/\eta_N}^N - \overline{F(\overline{w_{t/\eta_N}^N})}(r_N)\right] \mathrm{d}t + (\tau_N/\eta_N)^{1/2} \mathrm{d}M_t^N.$$

Subtracting the equation

$$\mathrm{d}f_t^N = \left[\mathcal{L}^{(r_N)} f_t^N - \overline{F(\overline{f_t^N})}(r_N)\right] \mathrm{d}t,$$

and multiplying by $(\eta_N/\tau_N)^{1/2}$ on both sides, we obtain

$$\mathrm{d}Z_t^N = \left[\mathcal{L}^{(r_N)} Z_t^N - (\eta_N/\tau_N)^{1/2} \overline{\left(F(\overline{w_{t/\eta_N}^N}) - F(\overline{f_t^N})\right)}(r_N)\right] \mathrm{d}t + \mathrm{d}M_t^N. \tag{3.21}$$

Since the function $F : \mathbb{R} \to \mathbb{R}$ is smooth, for $k \in \{1, 2\}$ and $x, y \in [0, 1]$, we can define the following:

$$R_k(x, y) = \int_0^1 \frac{t^{k-1}}{(k-1)!} F^{(k)}(x + t(y - x)) \mathrm{d}t. \tag{3.22}$$

Then R_k is continuous and bounded by $\frac{1}{k!}\left\|F^{(k)}\right\|_\infty$. In addition, by Taylor's formula,

$$F(x) = F(y) + (x - y) R_1(x, y), \tag{3.23}$$
$$F(x) = F(y) + (x - y) F'(y) + (x - y)^2 R_2(x, y). \tag{3.24}$$

Substituting the second relation into (3.21) yields

$$\mathrm{d}Z_t^N = \left[\mathcal{L}^{(r_N)} Z_t^N - \overline{\overline{Z_t^N} F'(\overline{f_t^N})}(r_N) - (\tau_N/\eta_N)^{1/2} \overline{(\overline{Z_t^N})^2 R_2(\overline{w_{t/\eta_N}^N}, \overline{f_t^N})}(r_N)\right] \mathrm{d}t + \mathrm{d}M_t^N.$$

In fact, this equality holds in mild form,

$$\left\langle Z_t^N, \phi \right\rangle = \int_0^t \left\langle Z_s^N, \mathcal{L}^{(r_N)}\phi - \overline{F'(\overline{f_s^N})\overline{\phi}}(r_N) \right\rangle \mathrm{d}s \\ - (\tau_N/\eta_N)^{1/2} \int_0^t \left\langle (\overline{Z_s^N})^2, R_2(\overline{w_{s/\eta_N}^N}, \overline{f_s^N})\overline{\phi}(r_N) \right\rangle \mathrm{d}s + M_t^N(\phi). \quad (3.25)$$

(In other words, every step above can be done using the integral form, yielding (3.25).) We can see M^N as a martingale measure and, from a change of variables in (3.12), its covariation measure is given by

$$Q^N(\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}s) = \sigma_{z_1,z_2}^{(r_N)}(w_{s/\eta_N}^N)\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}s + \mathcal{O}\left(\delta_N^2\right)\delta_{z_1=z_2}(\mathrm{d}z_1\mathrm{d}z_2)\mathrm{d}s. \quad (3.26)$$

Accordingly, we will sometimes write $M_t^N(\phi)$ as a stochastic integral (as defined in [Wal86, Chapter 2]),

$$M_t^N(\phi) = \int_0^t \int_{\mathbb{R}^d} \phi(x) M^N(\mathrm{d}x\mathrm{d}s).$$

Note that we have linearised the drift term in (3.11) around the deterministic centering term, and that the remaining term (where R_2 appears) is the error due to this linearisation. The main difficulty in proving the convergence of Z^N is to control this error. At first sight, it would seem that the factor $(\tau_N/\eta_N)^{1/2}$ in front of it is enough to make it vanish in the limit. However, some care is needed in dealing with the quadratic term in the spatial integral. Since Z^N is going to converge as a distribution-valued process, its square does not make sense in the limit. The control of this term is achieved through Lemma 4.6, where we bound the square of the average of Z_t^N over a ball of radius r_N. It is for this purpose that we require that $\tau_N/\eta_N = o\left(r_N^{2d}\right)$.

Once this is done, we will be in a good position to prove the convergence of Z^N. Indeed, as r_N tends to zero, $\mathcal{L}^{(r_N)}\phi - \overline{F'(\overline{f_s^N})\overline{\phi}}(r_N)$ is well approximated by $\frac{1}{2}\Delta\phi - F'(f_s)\phi$ (see Proposition A.2). We also prove that M^N converges to $\sqrt{f_t(1-f_t)} \cdot W_t$ (as defined in [Wal86, Chapter 2]) using the expression (3.26) for its covariance.

The proof of convergence of Z^N follows the classical strategy of proving that the sequence is tight before uniquely characterising its possible limit points. We are outside the safe borders of real-valued processes, but the theory presented in [Wal86] provides the main tools needed for the proof of our result. In particular, the argument relies heavily on Mitoma's Theorem (Theorem 6.13 in [Wal86]), which states that a sequence of processes $(X_t^n)_{t\geq 0}$, $n \geq 1$ with sample paths in $\mathrm{D}\left([0,T], \mathcal{S}'(\mathbb{R}^d)\right)$ a.s. is tight if and only if, for each $\phi \in \mathcal{S}(\mathbb{R}^d)$, the sequence of real-valued processes $\left(\langle X^n, \phi\rangle\right)_{n\geq 1}$ is tight in $\mathrm{D}\left([0,T], \mathbb{R}\right)$ (see also Theorem 4.1).

3.3 The rescaled martingale problem - Stable radii case

For $\phi \in L^{1,\infty}(\mathbb{R}^d)$, and $\alpha \in (0, d\wedge 2)$, define the following norm

$$\|\phi\|_{(\alpha)}^2 = \int_{(\mathbb{R}^d)^2} \phi(z_1)\phi(z_2)\left|z_1 - z_2\right|^{-\alpha} \mathrm{d}z_1\mathrm{d}z_2. \quad (3.27)$$

Let $(\varepsilon_N)_{N\geq 1}$, $(\delta_N)_{N\geq 1}$ be sequences in $(0,1]$ decreasing towards zero, and let $F : \mathbb{R} \to \mathbb{R}$.

Definition 3.8 (Martingale Problem (M2))**.** *Given $(\varepsilon_N)_{N\geq 1}$, $(\delta_N)_{N\geq 1}$ and F, let $\eta_N = \varepsilon_N \delta_N^\alpha$ and $\tau_N = \varepsilon_N^2\delta_N^\alpha$. Then for $N \geq 1$, we say that a Ξ-valued process $(w_t^N)_{t\geq 0}$ satisfies the martingale problem (M2) if for all ϕ in $L^{1,\infty}(\mathbb{R}^d)$,*

$$\left\langle w_t^N, \phi \right\rangle - \langle w_0, \phi\rangle - \eta_N u \int_0^t \left\{ \left\langle w_s^N, \mathcal{D}^{\alpha,\delta_N}\phi \right\rangle - \frac{sV_1}{\alpha}\left\langle F^{(\delta_N)}(w_s^N), \phi \right\rangle \right\} \mathrm{d}s \quad (3.28)$$

defines a (mean zero) square-integrable martingale with (predictable) variation process

$$\tau_N u^2 \int_0^t \int_{(\mathbb{R}^d)^2} \phi(z_1)\phi(z_2)\sigma^{(\alpha,\delta_N)}_{z_1,z_2}(w^N_s)\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}s + \mathcal{O}\left(t\tau_N\delta_N^\alpha \|\phi\|^2_{(\alpha)}\right), \tag{3.29}$$

where, for $\sigma^{(r)}$ *defined as in* (3.3),

$$\sigma^{(\alpha,\delta)}_{z_1,z_2}(w) = \int_{\frac{|z_1-z_2|}{2}\vee\delta}^\infty V_r^2\sigma^{(r)}_{z_1,z_2}(w)\frac{\mathrm{d}r}{r^{d+\alpha+1}}. \tag{3.30}$$

(Note that the remark about uniqueness made after Definition 3.4 also applies to the martingale problem (M2).)

Let $\left(q^N_t\right)_{t\geq 0}$ be defined as at the start of Section 2.2. Set $w^N_t(x) = q^N_t(x/\delta_N)$.

Proposition 3.9. *For each* $N \geq 1$ *the process* $\left(w^N_t\right)_{t\geq 0}$ *satisfies the martingale problem (M2).*

Proof. This is proved in a similar way to Proposition 3.6. Note that we cannot apply Proposition 3.1 directly, since in the stable case, $\int_0^\infty V_r^2\mu(\mathrm{d}r) = \infty$ (recall that $V_r = |B(0,r)|$). However, its proof carries over with small adjustments.

For any $\phi \in L^{1,\infty}(\mathbb{R}^d)$,

$$\left\langle q^N_t, \phi\right\rangle = \left\langle q^N_0, \phi\right\rangle + \int_0^t\int_1^\infty u_N V_r\left\{\left\langle q^N_s, \overline{\overline{\phi}}(r) - \phi\right\rangle - s_N\left\langle\overline{F(\overline{q^N_s})}(r), \phi\right\rangle\right\}\frac{\mathrm{d}r}{r^{d+\alpha+1}}\mathrm{d}s + \mathcal{M}^N_t(\phi),$$

where $\mathcal{M}^N_t(\phi)$ is a martingale. Using (3.13) and (3.14), it follows that

$$\begin{aligned}\left\langle w^N_t, \phi\right\rangle &= \left\langle w_0, \phi\right\rangle \\ &\quad + \varepsilon_N u\int_0^t\int_1^\infty V_r\Big\{\left\langle w^N_s, \overline{\overline{\phi}}(\delta_N r) - \phi\right\rangle - s_N\left\langle\overline{F(\overline{w^N_s})}(\delta_N r), \phi\right\rangle\Big\}\frac{\mathrm{d}r}{r^{d+\alpha+1}}\mathrm{d}s \\ &\quad + \delta_N^d\mathcal{M}^N_t(\phi^{(\delta_N)}).\end{aligned}$$

By the definition of $\mathcal{D}^{\alpha,\delta}$ in (2.9) and $V_r(x,y)$ in (2.2),

$$\begin{aligned}\int_1^\infty V_r(\overline{\overline{\phi}}(x,\delta_N r) - \phi(x))\frac{\mathrm{d}r}{r^{d+\alpha+1}} &= \delta_N^\alpha\int_{\delta_N}^\infty\int_{\mathbb{R}^d}\frac{V_r(x,y)}{V_r}(\phi(y)-\phi(x))\mathrm{d}y\frac{\mathrm{d}r}{r^{d+\alpha+1}} \\ &= \delta_N^\alpha\mathcal{D}^{\alpha,\delta_N}\phi(x).\end{aligned}$$

Further, by (2.8),

$$\int_1^\infty V_r\overline{F(\overline{w^N_s})}(\delta_N r)\frac{\mathrm{d}r}{r^{d+\alpha+1}} = \frac{V_1}{\alpha}F^{(\delta_N)}(w^N_s).$$

As a result,

$$\left\langle w^N_t, \phi\right\rangle = \left\langle w_0, \phi\right\rangle + \varepsilon_N\delta_N^\alpha u\int_0^t\left\{\left\langle w^N_s, \mathcal{D}^{\alpha,\delta_N}\phi\right\rangle - \frac{sV_1}{\alpha}\left\langle F^{(\delta_N)}(w^N_s), \phi\right\rangle\right\}\mathrm{d}s + \delta_N^d\mathcal{M}^N_t(\phi^{(\delta_N)}).$$

For the predictable variation process, the term from the second line of (3.5) in the proof of Proposition 3.1 can be bounded by

$$\mathcal{O}\left(s_N\right)u_N^2\int_{(\mathbb{R}^d)^2}\int_0^\infty\phi(z_1)\phi(z_2)V_r(z_1,z_2)\frac{\mathrm{d}r}{r^{1+d+\alpha}}\mathrm{d}z_1\mathrm{d}z_2.$$

We recover the error term in (3.29) since $V_r(z_1,z_2) \leq r^d\mathbb{1}_{r\geq\frac{1}{2}|z_1-z_2|}$. The first term in (3.29) follows from the definition of $\sigma^{(\alpha,r)}$ in (3.30). □

As in Subsection 3.2, we can now state a general result for a sequence of processes satisfying (M2) which implies Theorem 2.7. Let f^N be defined as in (2.12) and define f as the solution to

$$\begin{cases} \dfrac{\partial f_t}{\partial t} = u(\mathcal{D}^\alpha f_t - \frac{sV_1}{\alpha} F(f_t)), \\ f_0 = w_0. \end{cases} \tag{3.31}$$

The following result is proved in Section 5.

Theorem 3.10. *Suppose that $\left(w_t^N\right)_{t\geq 0}$ satisfies the martingale problem (M2) in Definition 3.8 for some smooth, bounded function $F : \mathbb{R} \to \mathbb{R}$ with bounded first and second derivatives and $(\delta_N)_N$, $(\varepsilon_N)_N$ converging to zero as $N \to \infty$. Moreover, suppose*

$$\tau_N/\eta_N = o\left(\delta_N^{2\alpha}\right). \tag{3.32}$$

Suppose also that w_0 has uniformly bounded derivatives of up to the second order and that there exists α_N such that the jumps of $\left(w_t^N\right)_{t\geq 0}$ are dominated by

$$\sup_{t\geq 0} \left| \left\langle w_t^N, \phi \right\rangle - \left\langle w_{t^-}^N, \phi \right\rangle \right| \leq \alpha_N \left\|\phi\right\|_1$$

for every $\phi \in L^{1,\infty}(\mathbb{R}^d)$, with $\alpha_N^2 = o\left(\tau_N/\eta_N\right)$. Then

$$\left(w_{t/\eta_N}^N\right)_{t\geq 0} \underset{N\to\infty}{\overset{L^1,\, P}{\longrightarrow}} (f_t)_{t\geq 0}$$

in $\left(\mathrm{D}\left([0,T], \Xi\right), d\right)$. In addition,

$$Z_t^N = (\eta_N/\tau_N)^{1/2}(w_{t/\eta_N}^N - f_t^N)$$

defines a sequence of distribution-valued processes which converges in distribution in $\mathrm{D}\left([0,T], \mathcal{S}'(\mathbb{R}^d)\right)$ to the solution of the following stochastic partial differential equation,

$$\begin{cases} \mathrm{d}z_t = u[\mathcal{D}^\alpha z_t - \frac{sV_1}{\alpha} F'(f_t) z_t]\mathrm{d}t + u \cdot \mathrm{d}W_t^\alpha \\ z_0 = 0, \end{cases} \tag{3.33}$$

where W^α is a coloured noise with covariation measure given by (2.14).

Theorem 2.7 is now a direct consequence.

Proof of Theorem 2.7. Recall that $(\mathbf{q}_t^N)_{t\geq 0}$ is defined in (2.7) as a rescaling of $\left(q_t^N\right)_{t\geq 0}$, and that by Proposition 3.9, letting $w_t^N(x) = q_t^N(x/\delta_N)$, $\left(w_t^N\right)_{t\geq 0}$ satisfies the martingale problem (M2). Also $\tau_N/\eta_N = o\left(\delta_N^{2\alpha}\right)$ follows from $\varepsilon_N = o(\delta_N^{2\alpha})$, and the bound on the jumps (3.18) holds with $\alpha_N = \varepsilon_N u$ by (1.5). We conclude by applying Theorem 3.10 to $w_{t/\eta_N}^N = \mathbf{q}_t^N$. □

The proof of Theorem 3.10 will make use of the same ideas as in the proof of Theorem 3.7 and, to improve readability, the steps of the proof which are most similar to those in the Brownian case will be dealt with more quickly, going into details only when the two arguments differ.

4 The Brownian case - proof of Theorem 3.7

As in the sketch of the proof in Subsection 3.2, for ease of notation, we shall set the constants uV_R, $2R^2/(d+2)$ and s to 1 in the definition of (M1). Recall the expression for $\left\langle Z_t^N, \phi\right\rangle$ in (3.25); the next subsection shows how time-dependent test functions can be used to write $\left\langle Z_t^N, \phi\right\rangle$ as the sum of a stochastic integral against a martingale measure and a non-linear term. Subsection 4.2 will provide a bound on this quadratic term using a Gronwall estimate. We can then prove the convergence of the process $\left(w^N_{t/\eta_N}\right)_{t\geq 0}$ to $(f_t)_{t\geq 0}$ in Subsection 4.3.

The following result is used to reduce the convergence of distribution-valued processes to the convergence of a family of real-valued processes; it is a direct corollary of Mitoma's theorem [Wal86, Theorem 6.13].

Theorem 4.1 ([Wal86, Theorem 6.15])**.** *Let $(X^n)_{n\geq 1}$ be a sequence of processes with sample paths in $\mathrm{D}\left([0,T],\mathcal{S}'(\mathbb{R}^d)\right)$. Suppose*

i) for each $\phi\in\mathcal{S}(\mathbb{R}^d)$, $(\langle X^n,\phi\rangle)_{n\geq 1}$ is tight,

ii) for each $\phi_1,\dots,\phi_k$ in $\mathcal{S}(\mathbb{R}^d)$ and $t_1,\dots,t_k$ in $[0,T]$, the distribution of $(\left\langle X^n_{t_1},\phi_1\right\rangle,\dots,\left\langle X^n_{t_k},\phi_k\right\rangle)$ converges weakly on $\mathbb{R}^k$.

Then there exists a process $(X_t)_{t\geq 0}$ with sample paths in $\mathrm{D}\left([0,T],\mathcal{S}'(\mathbb{R}^d)\right)$ such that X^n converges in distribution to X.

In order to apply this result to the sequence of distribution-valued processes $\left(Z^N\right)_{N\geq 1}$, we need to check that the two conditions (i) and (ii) are satisfied. The first one is proved in Subsection 4.4, thus implying the tightness of the sequence by Mitoma's theorem. Subsection 4.5 deals with the convergence of the martingale measure M^N (again as a distribution valued process, so this subsection will use Theorem 4.1). Finally condition (ii) is checked in Subsection 4.6.

In this section, in order to simplify the notation we often drop the sub- and superscripts N when there is no ambiguity; for instance, $\mathcal{L}$ should always be read $\mathcal{L}^{(r)}$, with $r=r_N$, and $w^N_{t/\eta}$ should be read w^N_{t/η_N}.

4.1 Time dependent test functions

Fix $\phi\in\mathcal{S}(\mathbb{R}^d)$. We consider time dependent test functions

$$\varphi:\mathbb{R}^d\times\{(s,t):0\leq s\leq t\leq T\}\to\mathbb{R}$$

such that (with a slight abuse of notation) $\varphi(s,t)\in L^\infty(\mathbb{R}^d)$ for all $0\leq s\leq t$ and φ is continuously differentiable with respect to the time variables. Let us recall equation (3.25):

$$\left\langle Z^N_t,\phi\right\rangle=\int_0^t\left\langle Z^N_s,\mathcal{L}^{(r_N)}\phi-\overline{F'(\overline{f^N_s})\overline{\phi}}(r_N)\right\rangle\mathrm{d}s \\ -(\tau_N/\eta_N)^{1/2}\int_0^t\left\langle(\overline{Z^N_s})^2,R_2(\overline{w^N_{s/\eta_N}},\overline{f^N_s})\overline{\phi}(r_N)\right\rangle\mathrm{d}s+M^N_t(\phi).$$

Adapting Exercise 5.1 of [Wal86], we obtain that for any time dependent test function φ,

$$\left\langle Z^N_t,\varphi(t,t)\right\rangle=\int_0^t\left\langle Z^N_s,\partial_s\varphi(s,t)+\mathcal{L}^{(r_N)}\varphi(s,t)-\overline{F'(\overline{f^N_s})\overline{\varphi(s,t)}}(r_N)\right\rangle\mathrm{d}s \\ -(\tau_N/\eta_N)^{1/2}\int_0^t\left\langle(\overline{Z^N_s})^2,R_2(\overline{w^N_{s/\eta_N}},\overline{f^N_s})\overline{\varphi(s,t)}(r_N)\right\rangle\mathrm{d}s+\int_0^t\int_{\mathbb{R}^d}\varphi(x,s,t)M^N(\mathrm{d}x\mathrm{d}s). \tag{4.1}$$

(To see this, use (3.25) to get an expression for $\left\langle Z^N_s,\partial_s\varphi(s,t)\right\rangle$ and integrate over s, using Fubini's theorem, then apply (3.25) again with $\phi=\varphi(t,t)$; see also Theorem 2.6 in [Wal86].) Suppose then that φ^N solves

$$\begin{cases}\partial_s\varphi^N(x,s,t)+\mathcal{L}^{(r_N)}\varphi^N(x,s,t)-\overline{F'(\overline{f^N_s})\overline{\varphi^N(s,t)}}(x,r_N)=0,\\ \varphi^N(x,t,t)=\phi(x).\end{cases} \tag{4.2}$$

Equations (4.1) and (3.25) then yield

$$\left\langle Z_t^N, \phi \right\rangle = -(\tau_N/\eta_N)^{1/2} \int_0^t \left\langle (\overline{Z_s^N})^2, R_2(\overline{w_{s/\eta}^N}, \overline{f_s^N}) \overline{\varphi^N(s,t)} \right\rangle \mathrm{d}s + \int_0^t \int_{\mathbb{R}^d} \varphi^N(x,s,t) M^N(\mathrm{d}x\mathrm{d}s). \quad (4.3)$$

Here we see that in the special case where F is linear, $R_2 = 0$ and it remains to prove the convergence of the stochastic integral of φ^N against the martingale measure M^N. Define φ as the solution to

$$\begin{cases} \partial_s \varphi(x,s,t) + \frac{1}{2}\Delta\varphi(x,s,t) - F'(f_s(x))\varphi(x,s,t) = 0, \\ \varphi(x,t,t) = \phi(x). \end{cases} \quad (4.4)$$

For a multi-index $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}_0^d$, let $|\beta| = \beta_1 + \ldots + \beta_d$ and for $g : \mathbb{R}^d \to \mathbb{R}$, let the derivative with respect to β be given by $\partial_\beta g(x) = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \ldots \partial x_d^{\beta_d}} g(x_1, \ldots, x_d)$.

Lemma 4.2. *Fix $T > 0$ and take $\phi \in \mathcal{S}(\mathbb{R}^d)$. There exists a unique solution φ^N to (4.2) in $L^\infty(\mathbb{R}^d \times \{(s,t) : 0 \le s \le t \le T\})$ which admits spatial derivatives of order up to four. Moreover, for any multi-index β with $0 \le |\beta| \le 4$,*

$$\sup_{0 \le s \le t \le T} \left\| \partial_\beta \varphi^N(s,t) \right\|_q < \infty \qquad \text{and} \qquad \sup_{0 \le s \le t \le T} \left\| \varphi(s,t) \right\|_q < \infty \quad (4.5)$$

for all $q \in [1, \infty]$.

A proof of this lemma is given in Appendix C. The following lemma, whose proof is also given in Appendix C, shows that φ^N converges to φ as $N \to \infty$ and provides uniform bounds on $\partial_\beta \varphi^N$.

Lemma 4.3. *For $T > 0$, $\phi \in \mathcal{S}(\mathbb{R}^d)$, there exists a constant K_1 such that, for all $N \ge 1$ and for $q \in \{1, 2\}$,*

$$\sup_{0 \le s \le t \le T} \left\| \varphi^N(s,t) - \varphi(s,t) \right\|_q \le K_1 r_N^2.$$

In addition, there exist constants K_2 and K_3 such that, for any multi-index β with $0 < |\beta| \le 4$, for all $N \ge 1$,

$$\sup_{0 \le s \le t \le T} \left\| \varphi^N(s,t) \right\|_q \le K_2 \left\| \phi \right\|_q, \qquad \text{and} \qquad \sup_{0 \le s \le t \le T} \left\| \partial_\beta \varphi^N(s,t) \right\|_q \le K_3.$$

and K_2 does not depend on ϕ.

We shall see in Subsection 4.5 that M^N converges weakly in $\mathrm{D}\left([0,T], \mathcal{S}'(\mathbb{R}^d)\right)$, and hence in Subsection 4.6 (using results of [Wal86]) that the second term in (4.3) converges. However, in the general case where F is not linear, the first term in (4.3) has to be controlled.

Remark 4.4. Recall the definition of R_1 in (3.23); coming back to equation (3.25), and using (3.23) instead of (3.24), we write

$$\left\langle Z_t^N, \phi \right\rangle = \int_0^t \left\{ \left\langle \mathcal{L}^{(r_N)} Z_s^N, \phi \right\rangle - \left\langle \overline{\overline{Z_s^N} R_1(\overline{w_{s/\eta}^N}, \overline{f_s^N})}, \phi \right\rangle \right\} \mathrm{d}s + \int_0^t \int_{\mathbb{R}^d} \phi(y) M^N(\mathrm{d}y\mathrm{d}s).$$

Then by the same argument as for (4.1), for a time dependent test function φ,

$$\left\langle Z_t^N, \varphi(t,t) \right\rangle = \int_0^t \left\langle Z_s^N, \partial_s \varphi(s,t) + \mathcal{L}^{(r_N)} \varphi(s,t) - \overline{R_1(\overline{w_{s/\eta}^N}, \overline{f_s^N}) \overline{\varphi(s,t)}}(r_N) \right\rangle \mathrm{d}s + \int_0^t \int_{\mathbb{R}^d} \varphi(x,s,t) M^N(\mathrm{d}x\mathrm{d}s). \quad (4.6)$$

It is tempting to try to define φ^N as the solution to

$$\begin{cases} \partial_s \varphi^N(x,s,t) + \mathcal{L}^{(r_N)}\varphi^N(x,s,t) - \overline{R_1(\overline{w^N_{s/\eta}},\overline{f^N_s})\overline{\varphi^N(s,t)}}(x,r_N) = 0, \\ \varphi^N(x,t,t) = \phi(x). \end{cases}$$

In this way, we would get rid of the first integral in (4.6). However, in this case, $s \mapsto \varphi^N(\cdot,s,\cdot)$ is not adapted to the canonical filtration of our process and the stochastic integral with respect to the martingale measure M^N is not well defined.

4.2 Regularity estimate

The following result is an easy consequence of the definition of M^N.

Lemma 4.5. *There exists a constant K_4 such that if for all $0 \le t \le T$, $\phi_t : \mathbb{R}^d \to \mathbb{R}$ is in $L^2(\mathbb{R}^d)$, then*

$$\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} \phi_s(x) M^N(\mathrm{d}x\mathrm{d}s)\right)^2\right] \le K_4 \int_0^t \|\phi_s\|_2^2 \,\mathrm{d}s.$$

Proof. From the definition of Q^N in (3.26) and the definition of $\sigma^{(r)}_{z_1,z_2}$ in (3.3), letting $r = r_N$,

$$\begin{aligned} &\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} \phi_s(x) M^N(\mathrm{d}x\mathrm{d}s)\right)^2\right] \\ &\quad = \mathbb{E}\left[\int_0^t \int_{(\mathbb{R}^d)^2} \phi_s(z_1)\phi_s(z_2)\sigma_{z_1,z_2}(w^N_{s/\eta})\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}s\right] + \mathcal{O}\left(\delta_N^2 \int_0^t \|\phi_s\|_2^2 \, ds\right) \\ &\quad \le \int_0^t \int_{(\mathbb{R}^d)^3} \frac{1}{V_r^2} \mathbb{1}_{\substack{|x-z_1|<r \\ |x-z_2|<r}} |\phi_s(z_1)|\,|\phi_s(z_2)|\,\mathrm{d}x\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}s + \mathcal{O}\left(\delta_N^2 \int_0^t \|\phi_s\|_2^2 \, ds\right) \\ &\quad = \int_0^t \int_{\mathbb{R}^d} \left(\frac{1}{V_r}\int_{B(x,r)} |\phi_s(z)|\,\mathrm{d}z\right)^2 \mathrm{d}x\mathrm{d}s + \mathcal{O}\left(\delta_N^2 \int_0^t \|\phi_s\|_2^2 \, ds\right) \\ &\quad \le K_4 \int_0^t \|\phi_s\|_2^2 \,\mathrm{d}s. \end{aligned}$$

(We have used Jensen's inequality in the last line.) □

For $t > 0$ and $x \in \mathbb{R}^d$, let

$$G_t(x) = (2\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{2t}\right)$$

be the fundamental solution to the heat equation on $\mathbb{R}^d$; $\phi \mapsto G_t * \phi$ is then the semigroup of standard Brownian motion. Then f_t as defined in (3.16) satisfies

$$f_t(x) = G_t * w_0(x) - \int_0^t G_{t-s} * F(f_s)(x)\mathrm{d}s.$$

Likewise, for $r > 0$, recall the definition of $\mathcal{L}^{(r)}$ in (2.4) and let $\left(\xi^{(r)}_t\right)_{t\ge 0}$ be a symmetric Lévy process on $\mathbb{R}^d$ with generator $\phi \mapsto \mathcal{L}^{(r)}\phi$. Let $\phi \mapsto G^{(r)}_t * \phi$ be the corresponding semigroup. Note that since $\xi^{(r)}_t = 0$ with positive probability, $G^{(r)}_t$ is not a well-defined function, but we do have $x \mapsto \overline{G^{(r)}_t}(x,r) \in L^{1,\infty}$. Then f^N as defined in (2.5) satisfies

$$f^N_t(x) = G^{(r_N)}_t * w_0(x) - \int_0^t G^{(r_N)}_{t-s} * \overline{F(\overline{f^N_s})}(x)\mathrm{d}s. \tag{4.7}$$

The following provides a bound on the second moment of $\overline{Z_t^N}$, which allows us to control the quadratic term in (4.3). Note that $x \mapsto \overline{Z_t^N}(x, r_N)$ is a well defined function (despite the fact that $w_{t/\eta}^N$ is only defined up to a Lebesgue-null set).

Lemma 4.6. *For $T > 0$, there exists a constant $K_5 > 0$, independent of N, such that for $0 \le t \le T$,*

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}\left[\overline{Z_t^N}(x, r_N)^2\right] \le \frac{K_5}{V_{r_N}}.$$

The proof of this result mirrors that of Theorem 1 in [Nor75a], although it is more technical because of the Laplacian and the various spatial averages.

Proof. We take $r = r_N$, $\eta = \eta_N$ and $\mathcal{L} = \mathcal{L}^{(r_N)}$ throughout the proof. Use equation (4.6) with the time-dependent test function $\varphi(s,t) = G_{t-s}^{(r)} * \phi$, yielding

$$\left\langle Z_t^N, \phi \right\rangle = -\int_0^t \left\langle G_{t-s}^{(r)} * \left(\overline{\overline{Z_s^N} R_1(\overline{w_{s/\eta}^N}, \overline{f_s^N})}\right), \phi \right\rangle \mathrm{d}s + \int_0^t \int_{\mathbb{R}^d} G_{t-s}^{(r)} * \phi(y) M^N(\mathrm{d}y \mathrm{d}s).$$

Now take $\phi(y) = \frac{1}{V_r} \mathbb{1}_{|x-y|<r}$, and use Proposition A.1 in Appendix A to obtain

$$\begin{aligned}
\overline{Z_t^N}(x) &= -\int_0^t G_{t-s}^{(r)} * \left(\overline{\overline{\overline{Z_s^N} R_1(\overline{w_{s/\eta}^N}, \overline{f_s^N})}}\right)(x) \mathrm{d}s + \int_0^t \int_{\mathbb{R}^d} \overline{G_{t-s}^{(r)}}(x-y) M^N(\mathrm{d}y \mathrm{d}s) \\
&= -\int_0^t \int_{\mathbb{R}^d} \overline{\overline{G_{t-s}^{(r)}}}(x-y) \overline{Z_s^N}(y) R_1(\overline{w_{s/\eta}^N}(y), \overline{f_s^N}(y)) \mathrm{d}y \mathrm{d}s \\
&\qquad + \int_0^t \int_{\mathbb{R}^d} \overline{G_{t-s}^{(r)}}(x-y) M^N(\mathrm{d}y \mathrm{d}s).
\end{aligned}$$

We now want to apply Gronwall's lemma, but the last term must be controlled carefully. Taking the square of both sides and using $(a+b)^2 \le 2(a^2+b^2)$, we have

$$\begin{aligned}
\overline{Z_t^N}(x)^2 \le 2 &\left(\int_0^t \int_{\mathbb{R}^d} \overline{\overline{G_{t-s}^{(r)}}}(x-y) \overline{Z_s^N}(y) R_1(\overline{w_{s/\eta}^N}(y), \overline{f_s^N}(y)) \mathrm{d}y \mathrm{d}s \right)^2 \\
&+ 2 \left(\int_0^t \int_{\mathbb{R}^d} \overline{G_{t-s}^{(r)}}(x-y) M^N(\mathrm{d}y \mathrm{d}s) \right)^2.
\end{aligned}$$

By Jensen's inequality (and noting that $\int_{\mathbb{R}^d} \overline{G_t^{(r)}}(x) \mathrm{d}x = 1$) and the bound $\|R_k\|_\infty \le \frac{1}{k!} \left\| F^{(k)} \right\|_\infty$ from (3.22), we have

$$\begin{aligned}
\overline{Z_t^N}(x)^2 \le 2t &\int_0^t \int_{\mathbb{R}^d} \overline{\overline{G_{t-s}^{(r)}}}(x-y) \left\| F' \right\|_\infty^2 \overline{Z_s^N}(y)^2 \mathrm{d}y \mathrm{d}s \\
&+ 2 \left(\int_0^t \int_{\mathbb{R}^d} \overline{G_{t-s}^{(r)}}(x-y) M^N(\mathrm{d}y \mathrm{d}s) \right)^2.
\end{aligned}$$

Taking expectations on both sides and using Fubini's theorem, we obtain

$$\begin{aligned}
\mathbb{E}\left[\overline{Z_t^N}(x)^2\right] \le 2t \left\| F' \right\|_\infty^2 &\int_0^t \int_{\mathbb{R}^d} \overline{\overline{G_{t-s}^{(r)}}}(x-y) \mathbb{E}\left[\overline{Z_s^N}(y)^2\right] \mathrm{d}y \mathrm{d}s \\
&+ 2\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} \overline{G_{t-s}^{(r)}}(x-y) M^N(\mathrm{d}y \mathrm{d}s) \right)^2 \right].
\end{aligned}$$

From Lemma 4.5, we have

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} \overline{G_{t-s}^{(r)}}(x-y) M^N(\mathrm{d}y\mathrm{d}s)\right)^2\right] &\leq K_4 \int_0^t \left\|\overline{G_{t-s}^{(r)}}(x-\cdot)\right\|_2^2 \mathrm{d}s \\
&= K_4 \int_0^t \int_{\mathbb{R}^d} \mathbb{E}_0\left[\frac{1}{V_r}\mathbb{1}_{\left|\xi_{t-s}^{(r)}-(x-y)\right|<r}\right]^2 \mathrm{d}y\mathrm{d}s \\
&\leq K_4 \int_0^t \mathbb{E}_0\left[\int_{\mathbb{R}^d}\left(\frac{1}{V_r}\mathbb{1}_{\left|\xi_{t-s}^{(r)}-(x-y)\right|<r}\right)^2 \mathrm{d}y\right] \mathrm{d}s \\
&= \frac{K_4}{V_r} t. \qquad (4.8)
\end{aligned}$$

($\mathbb{E}_0[\cdot]$ denotes the expectation with respect to the law of $\left(\xi_t^{(r)}\right)_{t\geq 0}$ started from the origin.) In addition, we note that $\mathbb{E}\left[\overline{Z_s^N}(y)^2\right] \leq \sup_{x\in\mathbb{R}^d} \mathbb{E}\left[\overline{Z_s^N}(x)^2\right]$ and, combined with the fact that $\int_{\mathbb{R}^d} \overline{G_t^{(r)}}(x)\mathrm{d}x = 1$, this yields

$$\mathbb{E}\left[\overline{Z_t^N}(x)^2\right] \leq 2t\,\|F'\|_\infty^2 \int_0^t \sup_{y\in\mathbb{R}^d} \mathbb{E}\left[\overline{Z_s^N}(y)^2\right] \mathrm{d}s + 2\frac{K_4}{V_r} t.$$

The right hand side does not depend on x, so we can take the supremum over $x \in \mathbb{R}^d$ on the left and write for $0 \leq t \leq T$

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}\left[\overline{Z_t^N}(x)^2\right] \leq 2T\,\|F'\|_\infty^2 \int_0^t \sup_{x\in\mathbb{R}^d} \mathbb{E}\left[\overline{Z_s^N}(x)^2\right] \mathrm{d}s + 2\frac{K_4}{V_r} T.$$

For each $N \geq 1$, for any $t \in [0,T]$ and $x \in \mathbb{R}^d$, by Lemma 2.1,

$$\overline{Z_t^N}(x) \leq (\eta_N/\tau_N)^{1/2}\left(1 + \sup_{t\in[0,T]} \left\|f_t^N\right\|_\infty\right) < \infty.$$

As a result, $t \mapsto \sup_{x\in\mathbb{R}^d} \mathbb{E}\left[\overline{Z_t^N}(x)^2\right]$ is bounded on $[0,T]$. Hence we can apply Gronwall's lemma (see e.g. Theorem 5.1 in [EK86]) to deduce that

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}\left[\overline{Z_t^N}(x)^2\right] \leq 2\frac{K_4}{V_r} T e^{2Tt\|F'\|^2} \leq \frac{K_5}{V_r}.$$

□

4.3 Convergence to the deterministic limit

The following result, proved in Appendix B, shows that f^N converges to f.

Proposition 4.7. *For $T > 0$, there exist constants K_6 and K_7 such that, for all $N \geq 1$,*

$$\sup_{0\leq t\leq T} \left\|f_t^N - f_t\right\|_\infty \leq K_6 r_N^2,$$

and for all multi-indices $\beta \in \mathbb{N}_0^d$ with $0 \leq |\beta| \leq 4$,

$$\sup_{0\leq t\leq T} \left\|\partial_\beta f_t^N\right\|_\infty \leq K_7.$$

We are now in a position to prove the first statement of Theorem 3.7, namely the convergence of the process $(w_{t/\eta_N}^N)_{t\geq 0}$. We are going to prove the following lemma.

Lemma 4.8. *For $T > 0$, there exists a constant K_8 such that for all $N \geq 1$ and for any function ϕ satisfying $\|\phi\|_q \leq 1$ and $\max_{|\beta|=2} \|\partial_\beta \phi\|_q \leq 1$ for $q \in \{1, 2\}$,*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \left|\left\langle Z_t^N, \phi\right\rangle\right|\right] \leq K_8. \tag{4.9}$$

Before we prove Lemma 4.8, we show that it implies the convergence of $\left(w_{t/\eta}^N\right)_{t\geq 0}$. We can choose a separating family $(\phi_n)_{n\geq 1}$ of compactly supported smooth functions satisfying $\|\phi_n\|_q \leq 1$ and $\max_{|\beta|=2} \|\partial_\beta \phi_n\|_q \leq 1$ for $q \in \{1, 2\}$, and define d as in (1.2) using this family. Then

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T} d(w_{t/\eta}^N, f_t)\right] \\
&\qquad \leq \sum_{n\geq 1} \frac{1}{2^n} \left\{\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\left\langle w_{t/\eta}^N, \phi_n\right\rangle - \left\langle f_t^N, \phi_n\right\rangle\right|\right] + \sup_{0\leq t\leq T}\left|\left\langle f_t^N, \phi_n\right\rangle - \langle f_t, \phi_n\rangle\right|\right\} \\
&\qquad \leq \sum_{n\geq 1} \frac{1}{2^n} \left\{(\tau_N/\eta_N)^{1/2}\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\left\langle Z_t^N, \phi_n\right\rangle\right|\right] + \sup_{0\leq t\leq T}\left\|f_t^N - f_t\right\|_\infty \|\phi_n\|_1\right\} \\
&\qquad \leq \sum_{n\geq 1} \frac{1}{2^n}\left\{K_8(\tau_N/\eta_N)^{1/2} + K_6 r_N^2\right\} = K_8(\tau_N/\eta_N)^{1/2} + K_6 r_N^2,
\end{aligned}$$

where the last line follows by Proposition 4.7 and Lemma 4.8. The right-hand-side converges to zero as $N \to \infty$, yielding the uniform convergence (on compact time intervals) of $(w_{t/\eta_N}^N)_{t\geq 0}$ to $(f_t)_{t\geq 0}$, the solution of equation (3.16). Note that, as soon as $d \geq 2$, r_N^2 is the leading order on the right-hand-side (see (3.17)).

Proof of Lemma 4.8. We are going to make use of (3.25) and apply Doob's maximal inequality to the martingale part. Let us first show that there exist two constants K and K' such that, for $t \in [0, T]$,

$$\mathbb{E}\left[\left|\left\langle Z_t^N, \phi\right\rangle\right|\right] \leq K\|\phi\|_2 + K' \frac{(\tau_N/\eta_N)^{1/2}}{V_{r_N}} \|\phi\|_1. \tag{4.10}$$

(From now on we shall write $\tau = \tau_N$, $\eta = \eta_N$ and $r = r_N$). Indeed, taking the expectation of the absolute value of both sides of (4.3) and using Lemma 4.5, we have

$$\begin{aligned}
\mathbb{E}\left[\left|\left\langle Z_t^N, \phi\right\rangle\right|\right] &\leq (\tau/\eta)^{1/2}\frac{1}{2}\|F''\|_\infty \int_0^t \left\langle \mathbb{E}\left[(\overline{Z_s^N})^2\right], \left|\overline{\varphi^N(s,t)}\right|\right\rangle \mathrm{d}s \\
&\qquad\qquad + \left(K_4 \int_0^t \left\|\varphi^N(s,t)\right\|_2^2 \mathrm{d}s\right)^{1/2} \\
&\leq \frac{1}{2}\|F''\|_\infty K_5 T \frac{(\tau/\eta)^{1/2}}{V_r} K_2 \|\phi\|_1 + K_4^{1/2} T^{1/2} K_2 \|\phi\|_2,
\end{aligned}$$

where we used Lemmas 4.6 and 4.3 in the last line. We have thus proved (4.10). Recalling (3.25) and the notation $M_t^N(\phi) = \int_0^t \int_{\mathbb{R}^d} \phi(x) M^N(\mathrm{d}x\mathrm{d}s)$, we write

$$\begin{aligned}
\sup_{t\in[0,T]}\left|\left\langle Z_t^N, \phi\right\rangle\right| &\leq \int_0^T \left|\left\langle Z_s^N, \mathcal{L}\phi - \overline{F'(\overline{f_s^N})\overline{\phi}}\right\rangle\right| \mathrm{d}s \\
&\qquad + \frac{1}{2}\|F''\|_\infty (\tau/\eta)^{1/2} \int_0^T \left\langle (\overline{Z_s^N})^2, \left|\overline{\phi}\right|\right\rangle \mathrm{d}s + \sup_{t\in[0,T]}\left|M_t^N(\phi)\right|.
\end{aligned}$$

Taking expectations on both sides, we use Lemma 4.6 and apply (4.10) with ϕ replaced by $\mathcal{L}\phi - \overline{F'(\overline{f_s^N})\overline{\phi}}$ to write

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|\left\langle Z_t^N, \phi\right\rangle\right|\right] \\ \leq \int_0^T \left\{ K(\|\mathcal{L}\phi\|_2 + \|F'\|_\infty \|\phi\|_2) + K' \frac{(\tau/\eta)^{1/2}}{V_r} (\|\mathcal{L}\phi\|_1 + \|F'\|_\infty \|\phi\|_1) \right\} \mathrm{d}s \\ + \frac{1}{2} \|F''\|_\infty K_5 T \frac{(\tau/\eta)^{1/2}}{V_r} \|\phi\|_1 + \mathbb{E}\left[\sup_{t\in[0,T]} \left|M_t^N(\phi)\right|^2\right]^{1/2}. \quad (4.11)$$

By Doob's inequality and Lemma 4.5,

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|M_t^N(\phi)\right|^2\right] \leq 4K_4 T \|\phi\|_2^2.$$

Furthermore, $\|\mathcal{L}\phi\|_q \leq \frac{d(d+2)}{2} \max_{|\beta|=2} \|\partial_\beta \phi\|_q$ by Proposition A.2.i in Appendix A, and $\frac{(\tau/\eta)^{1/2}}{V_r}$ tends to zero as $N \to \infty$ due to assumption (3.17). Hence, if $\|\phi\|_q \leq 1$ and $\max_{|\beta|=2} \|\partial_\beta \phi\|_q \leq 1$ for $q \in \{1, 2\}$, the right-hand-side of (4.11) is bounded by some constant independent of N and ϕ. □

4.4 Tightness

To prove that the sequence $\left(Z^N\right)_{N\geq 1}$ is tight in $\mathrm{D}\left([0,T], \mathcal{S}'(\mathbb{R}^d)\right)$, we adapt the argument from the proof of Theorem 7.13 in [Wal86].

Proposition 4.9. *For any $\phi \in \mathcal{S}(\mathbb{R}^d)$, for any arbitrary sequence $(T_N, \rho_N)_{N\geq 1}$ such that T_N is a stopping time (with respect to the natural filtration of the process $\left(Z_t^N\right)_{t\geq 0}$) with values in $[0,T]$ for all N and ρ_N is a deterministic sequence of positive numbers decreasing to zero as $N \to \infty$,*

$$\left\langle Z_{T_N+\rho_N}^N, \phi\right\rangle - \left\langle Z_{T_N}^N, \phi\right\rangle \to 0 \quad (4.12)$$

in probability as $N \to \infty$.

By Aldous' criterion ([Ald78] and [Wal86, Theorem 6.8]), Proposition 4.9 together with Lemma 4.8 imply that the sequence of real-valued processes $\left(\left\langle Z^N, \phi\right\rangle\right)_{N\geq 1}$ is tight in $\mathrm{D}\left([0,T], \mathbb{R}\right)$ for any $\phi \in \mathcal{S}(\mathbb{R}^d)$. In turn, Mitoma's theorem [Wal86, Theorem 6.13] implies the tightness of $\left(Z^N\right)_{N\geq 1}$ in $\mathrm{D}\left([0,T], \mathcal{S}'(\mathbb{R}^d)\right)$.

The proof of Proposition 4.9 requires the three following lemmas (the first two are proved in Appendix C). Extend φ^N to $\mathbb{R}^d \times [0,T]^2$ by setting, for $s, t \in [0,T]$,

$$\varphi^N(x, s, t) := \varphi^N(x, s \wedge t, t). \quad (4.13)$$

In other words, for $s > t$, $\varphi^N(s,t)$ equals ϕ.

Lemma 4.10. *For $T > 0$, there exists a constant K_9 such that, for all $N \geq 1$ and for $q \in \{1, 2\}$,*

$$\forall s, t, t' \in [0,T], \quad \left\|\varphi^N(s, t') - \varphi^N(s,t)\right\|_q \leq K_9 \left|t' - t\right|.$$

Lemma 4.11. *For $T > 0$, there exists a constant K_{10} such that, for all $s \in [0,T]$,*

$$\left\|\sup_{t\in[s,T]} \left|\varphi^N(s,t)\right|\right\|_1 \leq K_{10}.$$

Define

$$V_t^N = \int_0^T \int_{\mathbb{R}^d} \varphi^N(x,s,t) M^N(\mathrm{d}x\mathrm{d}s).$$

Lemma 4.12. *For $T > 0$ and for any $0 < \beta < 1/2$, there exists a random variable Y_N such that*

$$\forall t, t' \in [0,T], \quad \left| V_{t'}^N - V_t^N \right| \leq Y_N \left| t' - t \right|^\beta, \tag{4.14}$$

almost surely, and $\mathbb{E}\left[Y_N^2\right] \leq C'$ for all $N \geq 1$.

Proof. By Lemma 4.5 and then Lemma 4.10,

$$\begin{aligned}
\mathbb{E}\left[\left|V_{t'}^N - V_t^N\right|^2\right] &= \mathbb{E}\left[\left(\int_0^T \int_{\mathbb{R}^d} (\varphi^N(x,s,t') - \varphi^N(x,s,t)) M^N(\mathrm{d}x\mathrm{d}s)\right)^2\right]. \\
&\leq K_4 \int_0^T \left\| \varphi^N(s,t') - \varphi^N(s,t) \right\|_2^2 \mathrm{d}s \\
&\leq (K_9)^2 T K_4 \left| t' - t \right|^2.
\end{aligned}$$

The result follows by Kolmogorov's continuity theorem [Wal86, Corollary 1.2]. □

Proof of Proposition 4.9. We are going to treat each term in (4.3) separately. The first one converges to zero in L^1, uniformly on $[0,T]$, as a consequence of Lemma 4.6. The second one is dealt with as in [Wal86, Theorem 7.13]. From (4.3), write

$$\begin{aligned}
\left\langle Z_{T_N+\rho_N}^N, \phi \right\rangle - \left\langle Z_{T_N}^N, \phi \right\rangle &= (\tau_N/\eta_N)^{1/2} \int_0^{T_N} \left\langle (\overline{Z_s^N})^2, R_2(\overline{w_{s/\eta}^N}, \overline{f_s^N}) \overline{\varphi^N(s,T_N)} \right\rangle \mathrm{d}s \\
&\quad - (\tau_N/\eta_N)^{1/2} \int_0^{T_N+\rho_N} \left\langle (\overline{Z_s^N})^2, R_2(\overline{w_{s/\eta}^N}, \overline{f_s^N}) \overline{\varphi^N(s,T_N+\rho_N)} \right\rangle \mathrm{d}s \\
&\quad + \left(V_{T_N+\rho_N}^N - V_{T_N}^N \right) + \int_{T_N}^{T_N+\rho_N} \int_{\mathbb{R}^d} \phi(x) M^N(\mathrm{d}x\mathrm{d}s).
\end{aligned} \tag{4.15}$$

Let us deal with each term separately. The first two are similar so we need only consider the first one. Since inside the integral $s \leq T_N \leq T$, $\left|\varphi^N(s,T_N)\right| \leq \sup_{t\in[s,T]} \left|\varphi^N(s,t)\right|$ and we have

$$\left| \int_0^{T_N} \left\langle (\overline{Z_s^N})^2, R_2(\overline{w_{s/\eta}^N}, \overline{f_s^N}) \overline{\varphi^N(s,T_N)} \right\rangle \mathrm{d}s \right| \leq \frac{1}{2} \left\| F'' \right\|_\infty \int_0^T \left\langle (\overline{Z_s^N})^2, \overline{\sup_{t\in[s,T]} \left|\varphi^N(s,t)\right|} \right\rangle \mathrm{d}s.$$

Taking the expectation on both sides, we get

$$\begin{aligned}
&\mathbb{E}\left[\left| \int_0^{T_N} \left\langle (\overline{Z_s^N})^2, R_2(\overline{w_{s/\eta}^N}, \overline{f_s^N}) \overline{\varphi^N(s,T_N)} \right\rangle \mathrm{d}s \right|\right] \\
&\qquad \leq \frac{1}{2} \left\| F'' \right\|_\infty \int_0^T \left\langle \mathbb{E}\left[(\overline{Z_s^N})^2\right], \overline{\sup_{t\in[s,T]} \left|\varphi^N(s,t)\right|} \right\rangle \mathrm{d}s \\
&\qquad \leq \frac{1}{2} \left\| F'' \right\|_\infty \frac{K_5}{V_{r_N}} \int_0^T \left\| \sup_{t\in[s,T]} \left|\varphi^N(s,t)\right| \right\|_1 \mathrm{d}s \\
&\qquad \leq \frac{1}{2} \left\| F'' \right\|_\infty \frac{K_5}{V_{r_N}} T K_{10}.
\end{aligned} \tag{4.16}$$

where the second line follows by Lemma 4.6 and the third line follows by Lemma 4.11. Recall that we assumed in (3.17) that $\tau_N/\eta_N = o\left(r_N^{2d}\right)$; hence the first term on the right-hand-side of (4.15) converges to zero in L^1. By Lemma 4.12, we have, almost surely,

$$\left|V^N_{T_N+\rho_N} - V^N_{T_N}\right| \le Y_N \rho_N^{1/4}.$$

Taking the expectation of the square of both sides, we write

$$\mathbb{E}\left[\left|V^N_{T_N+\rho_N} - V^N_{T_N}\right|^2\right] \le C' \rho_N^{1/2}.$$

Hence the third term converges to zero in L^2 and in probability as $N \to \infty$. Finally, since T_N is a stopping time, we can apply Lemma 4.5 to the fourth term,

$$\begin{aligned}\mathbb{E}\left[\left(\int_{T_N}^{T_N+\rho_N}\int_{\mathbb{R}^d}\phi(x)M^N(\mathrm{d}x\mathrm{d}s)\right)^2\right] &\le K_4 \mathbb{E}\left[\int_{T_N}^{T_N+\rho_N}\|\phi\|_2^2\,\mathrm{d}s\right] \\ &\le K_4\|\phi\|_2^2\,\rho_N.\end{aligned}$$

This concludes the proof of Proposition 4.9. □

4.5 Convergence of the martingale measure M^N

The next step is to show that the martingale measure M^N converges weakly in $\mathrm{D}\left([0,T],\mathcal{S}'(\mathbb{R}^d)\right)$ as $N \to \infty$ to M, where $M_t = \sqrt{f_t(1-f_t)} \cdot W_t$ is a stochastic integral (as defined in [Wal86, Chapter 2]) against the space-time white noise W and f is the solution of (3.16). We will naturally use Theorem 4.1, along with the following result on convergence to Gaussian martingales (which is a consequence of Lévy's characterisation of Brownian motion).

For any $\mathbb{R}^d$-valued càdlàg process $(Y_t)_{t\ge 0}$, define $\Delta Y_t = Y_t - Y_{t^-}$.

Theorem 4.13 ([JS03, Theorem VIII 3.11])**.** *Fix $T > 0$ and suppose $(X_t)_{t\ge0} = (X^1_t, \ldots, X^d_t)_{t\ge0}$ is a continuous d-dimensional Gaussian martingale and for each $n \ge 1$, $(X^n_t)_{t\ge0} = (X^{n,1}_t, \ldots, X^{n,d}_t)_{t\ge0}$ is a càdlàg, locally square-integrable martingale such that*

(i) $|\Delta X^n_t|$ is bounded uniformly in n for all t, and $\sup_{t\le T}|\Delta X^n_t| \xrightarrow[n\to\infty]{P} 0$.

(ii) For each $t \in \mathbb{Q}\cap[0,T]$, $\left\langle X^{n,i}, X^{n,j}\right\rangle_t \xrightarrow[n\to\infty]{P} \left\langle X^i, X^j\right\rangle_t$.

Then X^n converges in distribution to X in $D([0,T],\mathbb{R}^d)$.

In our setting, the limiting process $(M_t(\phi))_{t\ge0}$ is a continuous martingale with quadratic variation

$$\langle M(\phi)\rangle_t = \int_0^t\int_{\mathbb{R}^d}\phi(x)^2 f_s(x)(1-f_s(x))\mathrm{d}x\mathrm{d}s.$$

(See [Wal86, Theorem 2.5].) Since this quantity is deterministic, $(M_t(\phi))_{t\ge0}$ is Gaussian, and we can apply the result above. The following lemma is then enough to conclude that M^N converges to M.

Lemma 4.14. *For any $\phi \in \mathcal{S}(\mathbb{R}^d)$,*

i) For all $t \ge 0$, $\left|\Delta M^N_t(\phi)\right| \le K$ for some constant K, and $\sup_{0\le t\le T}\left|\Delta M^N_t(\phi)\right| \xrightarrow[N\to\infty]{P} 0$.

ii) For each $t \in [0,T]$, $\left\langle M^N(\phi)\right\rangle_t \xrightarrow[N\to\infty]{P} \langle M(\phi)\rangle_t$.

Indeed, by polarisation, we can recover $\left\langle M^N(\phi_i), M^N(\phi_j)\right\rangle_t$ from $\left\langle M^N(\phi_i+\phi_j)\right\rangle_t$ and $\langle M^N(\phi_i - \phi_j)\rangle_t$, and (ii) of Theorem 4.13 is satisfied by vectors of the form $(M_t^N(\phi_1), \ldots, M_t^N(\phi_p))_{t\geq 0}$. As a result, for any $\phi_1, \ldots, \phi_p$ in $\mathcal{S}(\mathbb{R}^d)$, $\left(M_t^N(\phi_1), \ldots, M_t^N(\phi_p)\right)_{t\geq 0}$ converges in distribution to $(M_t(\phi_1), \ldots, M_t(\phi_p))_{t\geq 0}$ in $\mathrm{D}\left([0,T], \mathbb{R}^d\right)$. In particular, for any $\phi\in\mathcal{S}(\mathbb{R}^d)$, $\left(M^N(\phi)\right)_{N\geq 1}$ is tight, and M^N satisfies the assumptions of Theorem 4.1, hence M^N converges in distribution to M as $N\to\infty$ in $\mathrm{D}\left([0,T], \mathcal{S}'(\mathbb{R}^d)\right)$.

Proof of Lemma 4.14. By the definition of $M^N(\phi)$,

$$\begin{aligned} M_t^N(\phi) - M_{t^-}^N(\phi) &= \left\langle Z_t^N, \phi\right\rangle - \left\langle Z_{t^-}^N, \phi\right\rangle \\ &= (\eta_N/\tau_N)^{1/2}\left(\left\langle w_{t/\eta}^N, \phi\right\rangle - \left\langle w_{t^-/\eta}^N, \phi\right\rangle\right). \end{aligned}$$

The bound on the jumps of $\langle w_t, \phi\rangle$ in (3.18) implies

$$\begin{aligned} \sup_{t\geq 0}\left|\Delta M_t^N(\phi)\right| &\leq (\eta_N/\tau_N)^{1/2}\sup_{t\geq 0}\left|\left\langle w_t^N - w_{t^-}^N, \phi\right\rangle\right| \\ &\leq \alpha_N(\eta_N/\tau_N)^{1/2}\left\|\phi\right\|_1. \end{aligned}$$

But we have assumed that $\alpha_N^2 = o\left(\tau_N/\eta_N\right)$, so (i) is satisfied. To prove (ii), recall from (3.26) that

$$\left\langle M^N(\phi)\right\rangle_t = \int_0^t \int_{(\mathbb{R}^d)^2} \phi(z_1)\phi(z_2)\sigma_{z_1,z_2}^{(r_N)}(w_{s/\eta}^N)\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}s + \mathcal{O}\left(\delta_N^2 t\left\|\phi\right\|_2^2\right).$$

The rationale here is to show that the main contribution to this term comes from the diagonal $\{(z_1,z_2) : z_1 = z_2\}$ when $r_N\to 0$. From the definition of $\sigma^{(r_N)}$ in (3.3), letting $r = r_N$,

$$\begin{aligned} &\int_{(\mathbb{R}^d)^2}\phi(z_1)\phi(z_2)\sigma_{z_1,z_2}^{(r)}(w_{s/\eta}^N)\mathrm{d}z_1\mathrm{d}z_2 \\ &\qquad = \frac{1}{V_r^2}\int_{(\mathbb{R}^d)^3}\phi(z_1)\phi(z_2)[\overline{w_{s/\eta}^N}(x)(1-w_{s/\eta}^N(z_1))(1-w_{s/\eta}^N(z_2)) \\ &\qquad\qquad + (1-\overline{w_{s/\eta}^N}(x))w_{s/\eta}^N(z_1)w_{s/\eta}^N(z_2)]\mathbb{1}_{\substack{|z_1-x|<r\\|z_2-x|<r}}\mathrm{d}x\mathrm{d}z_1\mathrm{d}z_2. \end{aligned}$$

Changing the order of integration gives

$$\int_{(\mathbb{R}^d)^2}\phi(z_1)\phi(z_2)\sigma_{z_1,z_2}^{(r)}(w_{s/\eta}^N)\mathrm{d}z_1\mathrm{d}z_2 = \left\langle\overline{w_{s/\eta}^N}, \left(\overline{\phi(1-w_{s/\eta}^N)}\right)^2\right\rangle + \left\langle 1-\overline{w_{s/\eta}^N}, \left(\overline{\phi\, w_{s/\eta}^N}\right)^2\right\rangle. \tag{4.17}$$

We are left with showing that the right-hand-side of (4.17) converges in probability to

$$\left\langle f_s, (\phi(1-f_s))^2\right\rangle + \left\langle 1-f_s, (\phi f_s)^2\right\rangle = \left\langle f_s(1-f_s), \phi^2\right\rangle.$$

To do this, we first justify that ϕ can be let out of the average, we use Lemma 4.6 to argue that we can replace $w_{s/\eta}^N$ by f_s^N, then the regularity of f^N allows us to remove the averages and finally we know from Proposition 4.7 that f^N converges to f. First note that

$$\overline{\phi w}(x) - \phi(x)\overline{w}(x) = \frac{1}{V_r}\int_{B(x,r)} w(y)(\phi(y)-\phi(x))\mathrm{d}y.$$

Since $0\leq w(y)\leq 1$ a.e., we have

$$\left|\overline{\phi w}(x) - \phi(x)\overline{w}(x)\right| \leq \frac{1}{V_r}\int_{B(x,r)}\left|\phi(y)-\phi(x)\right|\mathrm{d}y \leq \frac{1}{V_r}\int_{B(x,r)}\sum_{i=1}^d\left\|\partial_i\phi\right\|_\infty|y-x|_i\,\mathrm{d}y.$$

Hence
$$\left\|\overline{\phi w}-\phi\overline{w}\right\|_\infty \le d\, r_N \max_i \|\partial_i\phi\|_\infty .$$
As a consequence,
$$\begin{aligned}\left\langle 1-\overline{w},(\overline{\phi w})^2\right\rangle - \left\langle 1-\overline{w},\phi^2\overline{w}^2\right\rangle &= \left\langle 1-\overline{w},(\overline{\phi w}-\phi\overline{w})(\overline{\phi w}+\phi\overline{w})\right\rangle\\ \left|\left\langle 1-\overline{w},(\overline{\phi w})^2\right\rangle - \left\langle 1-\overline{w},\phi^2\overline{w}^2\right\rangle\right| &\le 2\,\|\phi\|_1 \left\|\overline{\phi w}-\phi\overline{w}\right\|_\infty\\ &\le 2d\,\|\phi\|_1\, r_N \max_i \|\partial_i\phi\|_\infty .\end{aligned}$$
By the same argument (replacing w by $1-w$), we can also let ϕ out of the average in the first term on the right-hand-side of (4.17), and the problem reduces to showing the convergence of
$$\left\langle \overline{w^N_{s/\eta}},\phi^2(1-\overline{w^N_{s/\eta}})^2\right\rangle + \left\langle 1-\overline{w^N_{s/\eta}},\phi^2\overline{w^N_{s/\eta}}^2\right\rangle = \left\langle \overline{w^N_{s/\eta}}(1-\overline{w^N_{s/\eta}}),\phi^2\right\rangle .$$
We now see that it is enough to show that uniformly for $s\in[0,T]$,
$$\sup_{x\in\mathbb{R}^d}\mathbb{E}\left[\left|\overline{w^N_{s/\eta}}(x)-f_s(x)\right|\right] \underset{N\to\infty}{\longrightarrow} 0.$$
But, by the triangle inequality,
$$\begin{aligned}\mathbb{E}\left[\left|\overline{w^N_{s/\eta}}(x)-f_s(x)\right|\right] &\le (\tau_N/\eta_N)^{1/2}\mathbb{E}\left[\overline{Z^N_s}(x)^2\right]^{1/2} + \left\|\overline{f^N_s}-f^N_s\right\|_\infty + \left\|f^N_s-f_s\right\|_\infty\\ &\le \frac{(\tau_N/\eta_N)^{1/2}}{V_{r_N}^{1/2}}K_5^{1/2} + \frac{d}{2}r_N^2K_7 + K_6r_N^2.\end{aligned}$$
(We have used Lemma 4.6, Proposition A.2 in Appendix A and Proposition 4.7.) Note that this bound is uniform for $s\in[0,T]$. The right-hand-side converges to zero as $N\to\infty$ (due to assumption (3.17)), providing the desired result. From all this we conclude
$$\int_{(\mathbb{R}^d)^2}\phi(z_1)\phi(z_2)\sigma^{(r_N)}_{z_1,z_2}(w^N_{s/\eta})\mathrm{d}z_1\mathrm{d}z_2 \overset{L^1}{\underset{N\to\infty}{\longrightarrow}} \int_{\mathbb{R}^d}\phi(x)^2f_s(x)(1-f_s(x))\mathrm{d}x,$$
uniformly for $s\in[0,T]$, which gives us (ii). □

4.6 Conclusion of the proof

We are almost done. We have proved that the sequence of processes $\left(Z^N\right)_{N\ge1}$ is tight, and we need only characterise its potential limit points. Recall the following expression for $\left\langle Z^N_t,\phi\right\rangle$ from (4.3):
$$\begin{aligned}\left\langle Z^N_t,\phi\right\rangle = -(\tau_N/\eta_N)^{1/2}\int_0^t \left\langle (\overline{Z^N_s})^2, R_2(\overline{w^N_{s/\eta}},\overline{f^N_s})\overline{\varphi^N(s,t)}\right\rangle \mathrm{d}s\\ + \int_0^t\int_{\mathbb{R}^d}\varphi^N(x,s,t)M^N(\mathrm{d}x\mathrm{d}s).\end{aligned}$$
The first term converges to zero in L^1 from (4.16). Also,
$$\int_0^t\int_{\mathbb{R}^d}\varphi^N(x,s,t)M^N(\mathrm{d}x\mathrm{d}s) - \int_0^t\int_{\mathbb{R}^d}\varphi(x,s,t)M^N(\mathrm{d}x\mathrm{d}s) \overset{L^2}{\underset{N\to\infty}{\longrightarrow}} 0,$$
since, from Lemma 4.5 and Lemma 4.3,
$$\begin{aligned}\mathbb{E}\left[\left(\int_0^t\int_{\mathbb{R}^d}(\varphi^N(x,s,t)-\varphi(x,s,t))M^N(\mathrm{d}x\mathrm{d}s)\right)^2\right] &\le K_4\int_0^t \left\|\varphi^N(s,t)-\varphi(s,t)\right\|_2^2\mathrm{d}s\\ &\le K_1^2TK_4r_N^4.\end{aligned}$$

For $\phi_1,\dots,\phi_p$ in $\mathcal{S}(\mathbb{R}^d)$, let $\varphi_1,\dots,\varphi_p$ be the corresponding solutions of (4.4) with $\phi=\phi_i$. Since we showed in Section 4.5 that M^N converges weakly to M, by [Wal86, Proposition 7.12], for $t_1,\dots,t_p\in[0,T]$,

$$\left(\int_0^{t_1}\int_{\mathbb{R}^d}\varphi_1(x,s,t_1)M^N(\mathrm{d}x\mathrm{d}s),\dots,\int_0^{t_k}\int_{\mathbb{R}^d}\varphi_k(x,s,t_k)M^N(\mathrm{d}x\mathrm{d}s)\right)$$
$$\xrightarrow[N\to\infty]{d}\left(\int_0^{t_1}\int_{\mathbb{R}^d}\varphi_1(x,s,t_1)M(\mathrm{d}x\mathrm{d}s),\dots,\int_0^{t_k}\int_{\mathbb{R}^d}\varphi_k(x,s,t_k)M(\mathrm{d}x\mathrm{d}s)\right).$$

This uniquely characterises the potential limit points of $\left(Z^N\right)_{N\geq1}$. By Theorem 4.1, $\left(Z_t^N\right)_{t\geq0}$ converges in distribution to a distribution-valued process $(z_t)_{t\geq0}$ given by

$$\langle z_t,\phi\rangle=\int_0^t\int_{\mathbb{R}^d}\varphi(x,s,t)M(\mathrm{d}x\mathrm{d}s),\tag{4.18}$$

where φ satisfies the backwards heat equation (4.4) with terminal condition ϕ,

$$\begin{cases}\partial_s\varphi(x,s,t)+\dfrac{1}{2}\Delta\varphi(x,s,t)-F'(f_s(x))\varphi(x,s,t)=0,\\ \varphi(x,t,t)=\phi(x).\end{cases}$$

It is an easy exercise to prove that z_t satisfies

$$\langle z_t,\phi\rangle=\int_0^t\left\langle z_s,\frac{1}{2}\Delta\phi-F'(f_s)\phi\right\rangle\mathrm{d}s+\int_0^t\int_{\mathbb{R}^d}\phi(x)M(\mathrm{d}x\mathrm{d}s).\tag{4.19}$$

(See the proof of [Wal86, Theorem 5.2].) In other words, $(z_t)_{t\geq0}$ is the (mild) solution of (3.20) (recall that $M_t=\sqrt{f_t(1-f_t)}\cdot W_t$) and Theorem 3.7 is proved.

5 The stable case - proof of Theorem 3.10

Turning to the proof of the central limit theorem in the stable case, we warn that its overall structure is the same as that in the Brownian case. Whenever the details of the argument are exactly the same as previously, we simply mention intermediate results without detailing their proof. Some steps need a different treatment however, and we explain those in more detail. To simplify our formulas, we use the following notation:

$$a_n\lesssim b_n\Leftrightarrow\exists K>0:\forall n\geq1,\ 0\leq a_n\leq K\,b_n.\tag{5.1}$$

The specific constants can always be retrieved from Section 4 or from a trivial calculation. Also as in Section 4 we set the constants u and $(sV_1)/\alpha$ to 1 in the martingale problem (M2) defined in Definition 3.8. Let us write (M2) as

$$\mathrm{d}w_t^N=\eta_N\left[\mathcal{D}^{\alpha,\delta_N}w_t^N-F^{(\delta_N)}(w_t^N)\right]\mathrm{d}t+\tau_N^{1/2}\mathrm{d}\mathbb{M}_t^N.$$

Recall that $Z_t^N=(\eta_N/\tau_N)^{1/2}(w_{t/\eta}^N-f_t^N)$. Setting $M_t^N(\phi)=\eta_N^{1/2}\mathbb{M}_{t/\eta_N}^N(\phi)$ and using the definition of $F^{(\delta_N)}$ in (2.8), we have, by the same argument as for (3.21),

$$\mathrm{d}Z_t^N=\left[\mathcal{D}^{\alpha,\delta_N}Z_t^N-\alpha(\eta_N/\tau_N)^{1/2}\int_1^\infty\overline{\left(F(\overline{w_{t/\eta}^N})-F(\overline{f_t^N})\right)}(\delta_Nr)\frac{\mathrm{d}r}{r^{1+\alpha}}\right]\mathrm{d}t+\mathrm{d}M_t^N.\tag{5.2}$$

Using the definition of R_2 in (3.24) as for (3.25), one obtains (in mild form)

$$\begin{aligned}\left\langle Z_t^N,\phi\right\rangle=&\int_0^t\left\langle Z_s^N,\mathcal{D}^{\alpha,\delta_N}\phi-\alpha\int_1^\infty\overline{F'(\overline{f_s^N})\overline{\phi}}(\delta_Nr)\frac{\mathrm{d}r}{r^{\alpha+1}}\right\rangle\mathrm{d}s\\&-\left(\frac{\tau_N}{\eta_N}\right)^{\frac{1}{2}}\alpha\int_0^t\int_1^\infty\left\langle(\overline{Z_s^N}(\delta_Nr))^2,R_2(\overline{w_{s/\eta}^N},\overline{f_s^N})\overline{\phi}(\delta_Nr)\right\rangle\frac{\mathrm{d}r}{r^{\alpha+1}}\mathrm{d}s+M_t^N(\phi),\end{aligned}\tag{5.3}$$

and the covariation measure of M^N is given by

$$Q^N(\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}s) = (\sigma^{(\alpha,\delta_N)}_{z_1,z_2}(w^N_{s/\eta}) + |z_1 - z_2|^{-\alpha}\,\mathcal{O}\,(\delta_N^\alpha))\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}s. \tag{5.4}$$

5.1 Time dependent test functions

Note that the analogue of (4.1) holds in the stable setting, that is, for any time dependent test function φ, by (5.3) and the same argument as for (4.1),

$$\begin{aligned}\left\langle Z^N_t, \varphi(t,t)\right\rangle = &\int_0^t \left\langle Z^N_s, \partial_s\varphi(s,t) + \mathcal{D}^{\alpha,\delta_N}\varphi(s,t) - \alpha\int_1^\infty \overline{F'(\overline{f^N_s})\overline{\varphi(s,t)}}(\delta_N r)\frac{\mathrm{d}r}{r^{\alpha+1}}\right\rangle \mathrm{d}s \\ &- \left(\frac{\tau_N}{\eta_N}\right)^{\frac{1}{2}} \alpha\int_0^t\int_1^\infty \left\langle (\overline{Z^N_s}(\delta_N r))^2, R_2(\overline{w^N_{s/\eta}}, \overline{f^N_s})\overline{\varphi(s,t)}(\delta_N r)\right\rangle \frac{\mathrm{d}r}{r^{\alpha+1}}\mathrm{d}s \\ &+ \int_0^t\int_{\mathbb{R}^d} \varphi(x,s,t)M^N(\mathrm{d}x\mathrm{d}s).\end{aligned} \tag{5.5}$$

For $\phi \in \mathcal{S}(\mathbb{R}^d)$, suppose φ^N solves

$$\begin{cases}\partial_s\varphi^N(x,s,t) + \mathcal{D}^{\alpha,\delta_N}\varphi^N(x,s,t) - \alpha\displaystyle\int_1^\infty \overline{F'(\overline{f^N_s})\overline{\varphi^N(s,t)}}(x,\delta_N r)\frac{\mathrm{d}r}{r^{\alpha+1}} = 0 \\ \varphi^N(x,t,t) = \phi(x).\end{cases} \tag{5.6}$$

By (5.5), we have

$$\begin{aligned}\left\langle Z^N_t, \phi\right\rangle = &-\left(\frac{\tau_N}{\eta_N}\right)^{1/2} \alpha\int_0^t\int_1^\infty \left\langle (\overline{Z^N_s}(\delta_N r))^2, R_2(\overline{w^N_{s/\eta}}, \overline{f^N_s})\overline{\varphi^N(s,t)}(\delta_N r)\right\rangle \frac{\mathrm{d}r}{r^{\alpha+1}}\mathrm{d}s \\ &+ \int_0^t\int_{\mathbb{R}^d} \varphi^N(x,s,t)M^N(\mathrm{d}x\mathrm{d}s).\end{aligned} \tag{5.7}$$

We are thus left with finding a suitable way to bound the first term above and showing the convergence of the stochastic integral against M^N. The convergence of the martingale measure M^N is going to involve slightly different calculations compared to the previous case as the limiting noise is not a space-time white noise. The convergence of φ^N, however, is proved in a similar way to before. For $\phi \in \mathcal{S}(\mathbb{R}^d)$, define φ as the solution to the following

$$\begin{cases}\partial_s\varphi(x,s,t) + \mathcal{D}^\alpha\varphi(x,s,t) - F'(f_s(x))\varphi(x,s,t) = 0 \\ \varphi(x,t,t) = \phi(x).\end{cases} \tag{5.8}$$

Lemma 5.1. *Fix $T > 0$ and take $\phi \in \mathcal{S}(\mathbb{R}^d)$. There exists a unique solution φ^N to* (5.6) *in $L^\infty(\mathbb{R}^d \times \{(s,t) : 0 \le s \le t \le T\})$ which admits spatial derivatives of order up to two. Moreover, for any multi-index β with $0 \le |\beta| \le 2$,*

$$\sup_{0\le s\le t\le T} \left\|\partial_\beta\varphi^N(s,t)\right\|_q < \infty \qquad \text{and} \qquad \sup_{0\le s\le t\le T} \left\|\varphi(s,t)\right\|_q < \infty \tag{5.9}$$

for $q \in \{1, \infty\}$.

The proof of Lemma 5.1 is a straightforward adaptation of that of Lemma 4.2. The following lemma, whose proof is given in Appendix C, provides the convergence of φ^N to φ.

Lemma 5.2. *For $T > 0$, $\phi \in \mathcal{S}(\mathbb{R}^d)$ and for $q \in \{1, \infty\}$,*

$$\sup_{0\le s\le t\le T} \left\|\varphi^N(s,t) - \varphi(s,t)\right\|_q \lesssim \delta_N^{\alpha\wedge(2-\alpha)}.$$

In addition, for $0 < |\beta| \le 2$,

$$\sup_{0\le s\le t\le T} \left\|\varphi^N(s,t)\right\|_q \lesssim \|\phi\|_q \qquad \text{and} \qquad \sup_{0\le s\le t\le T} \left\|\partial_\beta\varphi^N(s,t)\right\|_q \lesssim 1.$$

5.2 Regularity estimate

Let us first state the following L^2 bound for the stochastic integral.

Lemma 5.3. *For $0 \le t \le T$ and $\alpha < d$, suppose $\phi_t : \mathbb{R}^d \to \mathbb{R}$ is in $L^{1,\infty}(\mathbb{R}^d)$; then*

$$\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} \phi_s(x) M^N(\mathrm{d}x\mathrm{d}s)\right)^2\right] \lesssim \int_0^t \int_{(\mathbb{R}^d)^2} |\phi_s(z_1)|\,|\phi_s(z_2)|\,(\delta_N \vee \tfrac{|z_1-z_2|}{2})^{-\alpha}\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}s \tag{5.10}$$

$$\lesssim \int_0^t \|\phi_s\|_1(\|\phi_s\|_\infty + \|\phi_s\|_1)\mathrm{d}s. \tag{5.11}$$

The proof uses the following lemma, which is proved in Appendix C.

Lemma 5.4. *For $\alpha < d$, then for $f, g \in L^{1,\infty}(\mathbb{R}^d)$*

$$\left|\int_{(\mathbb{R}^d)^2} f(z_1)g(z_2)\,|z_1 - z_2|^{-\alpha}\,\mathrm{d}z_1\mathrm{d}z_2\right| \le \|f\|_1\,(\frac{dV_1}{d-\alpha}\,\|g\|_\infty + \|g\|_1).$$

Proof of Lemma 5.3. From the expression for the covariation measure in (5.4),

$$\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} \phi_s(x) M^N(\mathrm{d}x\mathrm{d}s)\right)^2\right] = \mathbb{E}\left[\int_0^t \int_{(\mathbb{R}^d)^2} \phi_s(z_1)\phi_s(z_2)\sigma^{\alpha,\delta_N}_{z_1,z_2}(w^N_{s/\eta})\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}s\right] + \mathcal{O}(\delta_N^\alpha)\int_0^t \int_{(\mathbb{R}^d)^2} \phi_s(z_1)\phi_s(z_2)|z_1-z_2|^{-\alpha}\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}s.$$

But, by the definition of $\sigma^{(\alpha,\delta)}$ in (3.30),

$$\begin{aligned}\left|\sigma^{\alpha,\delta}_{z_1,z_2}(w)\right| &\le \int_{\delta\vee\frac{|z_1-z_2|}{2}}^\infty V_r(z_1,z_2)\frac{\mathrm{d}r}{r^{d+\alpha+1}} \\ &\lesssim V_1 \int_{\delta\vee\frac{|z_1-z_2|}{2}}^\infty \frac{\mathrm{d}r}{r^{\alpha+1}} \\ &\lesssim \left(\delta \vee \frac{|z_1-z_2|}{2}\right)^{-\alpha},\end{aligned}$$

yielding (5.10). Since $(\delta_N \vee \frac{|z_1-z_2|}{2})^{-\alpha} \le (\frac{|z_1-z_2|}{2})^{-\alpha}$, inequality (5.11) is obtained from (5.10) and Lemma 5.4. ☐

Let $\mathcal{G}^{(\alpha)}$ (resp. $\mathcal{G}^{(\alpha,\delta)}$) denote the fundamental solution to the fractional heat equation with the operator $\mathcal{D}^\alpha$ (resp. the fractional heat equation with the truncated operator $\mathcal{D}^{\alpha,\delta}$). Then the centering term f^N as defined in (2.12) can be written as

$$f^N_t(x) = \mathcal{G}^{(\alpha,\delta_N)}_t * w_0(x) - \int_0^t \mathcal{G}^{(\alpha,\delta_N)}_{t-s} * F^{(\delta_N)}(f^N_s)(x)\mathrm{d}s. \tag{5.12}$$

Likewise, using the definition of f_t in (3.31),

$$f_t(x) = \mathcal{G}^{(\alpha)}_t * w_0(x) - \int_0^t \mathcal{G}^{(\alpha)}_{t-s} * F(f_s)(x)\mathrm{d}s.$$

We now prove the following counterpart of the regularity estimate (Lemma 4.6), which allows us to bound the quadratic error term in (5.7).

Lemma 5.5. *Fix $T > 0$; for $0 \le t \le T$,*

$$\int_1^\infty \sup_{x\in\mathbb{R}^d} \mathbb{E}\left[\overline{Z^N_t}(x,\delta_N r)^2\right]\frac{\mathrm{d}r}{r^{\alpha+1}} \lesssim \frac{1}{\delta_N^\alpha}.$$

Proof. From (5.2) and the definition of R_1 in (3.23), we have

$$\langle Z_t^N, \phi \rangle = \int_0^t \left\langle Z_s^N, \mathcal{D}^{\alpha,\delta_N}\phi - \alpha \int_1^\infty \overline{R_1(\overline{w_{s/\eta}^N}, \overline{f_s^N})\overline{\phi}}(\delta_N r)\frac{\mathrm{d}r}{r^{\alpha+1}} \right\rangle \mathrm{d}s + \int_0^t \int_{\mathbb{R}^d} \phi(y) M^N(\mathrm{d}y\mathrm{d}s).$$

By the same argument as for (4.1), for a time dependent test function φ,

$$\begin{aligned}\langle Z_t^N, \varphi(t,t) \rangle &= \int_0^t \left\langle Z_s^N, \partial_s \varphi(s,t) + \mathcal{D}^{\alpha,\delta_N}\varphi(s,t) - \alpha \int_1^\infty \overline{R_1(\overline{w_{s/\eta}^N}, \overline{f_s^N})\overline{\varphi(s,t)}}(\delta_N r)\frac{\mathrm{d}r}{r^{\alpha+1}} \right\rangle \mathrm{d}s \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \varphi(x,s,t) M^N(\mathrm{d}x\mathrm{d}s).\end{aligned}$$

Applying this with $\varphi(x,s,t) = \mathcal{G}_{t-s}^{(\alpha,\delta_N)} * \phi(x)$ yields

$$\langle Z_t^N, \phi \rangle = -\alpha \int_0^t \int_1^\infty \left\langle \mathcal{G}_{t-s}^{(\alpha,\delta_N)} * \overline{\overline{Z_s^N} R_1(\overline{w_{s/\eta}^N}, \overline{f_s^N})}(\delta_N r), \phi \right\rangle \frac{\mathrm{d}r}{r^{\alpha+1}}\mathrm{d}s + \int_0^t \int_{\mathbb{R}^d} \mathcal{G}_{t-s}^{(\alpha,\delta_N)} * \phi(y) M^N(\mathrm{d}y\mathrm{d}s).$$

Now we take $\phi(y) = \frac{1}{V_R}\mathbb{1}_{|x-y|<R}$ to obtain

$$\overline{Z_t^N}(x,R) = -\alpha \int_0^t \int_1^\infty \left(\overline{\mathcal{G}_{t-s}^{(\alpha,\delta_N)}}(R) * \overline{\overline{Z_s^N} R_1(\overline{w_{s/\eta}^N}, \overline{f_s^N})}(\delta_N r)\right)(x)\frac{\mathrm{d}r}{r^{\alpha+1}}\mathrm{d}s + \int_0^t \int_{\mathbb{R}^d} \overline{\mathcal{G}_{t-s}^{(\alpha,\delta_N)}}(x-y,R) M^N(\mathrm{d}y\mathrm{d}s).$$

Repeating the same steps as in the proof of Lemma 4.6 and using Jensen's inequality, we get

$$\left(\overline{Z_t^N}(x,R)\right)^2 \lesssim \alpha \int_0^t \int_1^\infty \int_{\mathbb{R}^d} \overline{\overline{\mathcal{G}_{t-s}^{(\alpha,\delta_N)}}}(x-y,R,\delta_N r)\left(\overline{Z_s^N}(y,\delta_N r)\right)^2 \mathrm{d}y \frac{\mathrm{d}r}{r^{\alpha+1}}\mathrm{d}s + \left(\int_0^t \int_{\mathbb{R}^d} \overline{\mathcal{G}_{t-s}^{(\alpha,\delta_N)}}(x-y,R) M^N(\mathrm{d}y\mathrm{d}s)\right)^2.$$

Using the first inequality of Lemma 5.3 and bounding $\left(\delta_N \vee \frac{|z_1-z_2|}{2}\right)^{-\alpha}$ by $\delta_N^{-\alpha}$, we have, for $0 \leq t \leq T$,

$$\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} \overline{\mathcal{G}_{t-s}^{(\alpha,\delta_N)}}(x-y,R) M^N(\mathrm{d}y\mathrm{d}s)\right)^2\right] \lesssim \delta_N^{-\alpha} \int_0^t \left(\int_{\mathbb{R}^d} \overline{\mathcal{G}_{t-s}^{(\alpha,\delta_N)}}(x-y,R)\mathrm{d}y\right)^2 \mathrm{d}s \lesssim \delta_N^{-\alpha}.$$

As a result

$$\mathbb{E}\left[\left(\overline{Z_t^N}(x,R)\right)^2\right] \lesssim \int_0^t \int_1^\infty \int_{\mathbb{R}^d} \overline{\overline{\mathcal{G}_{t-s}^{(\alpha,\delta_N)}}}(x-y,R,\delta_N r)\mathbb{E}\left[\left(\overline{Z_s^N}(y,\delta_N r)\right)^2\right] \mathrm{d}y \frac{\mathrm{d}r}{r^{\alpha+1}}\mathrm{d}s + \delta_N^{-\alpha}.$$

Taking the supremum of $\mathbb{E}\left[\overline{Z_s^N}(y,\delta_N r)^2\right]$ over y inside the integral on the right-hand-side, the function $\overline{\mathcal{G}^{(\alpha,\delta)}}$ integrates to 1, yielding

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}\left[\left(\overline{Z_t^N}(x,R)\right)^2\right] \lesssim \int_0^t \int_1^\infty \sup_{x\in\mathbb{R}^d} \mathbb{E}\left[\left(\overline{Z_s^N}(x,\delta_N r)\right)^2\right] \frac{\mathrm{d}r}{r^{\alpha+1}}\mathrm{d}s + \delta_N^{-\alpha}.$$

Integrating over R, we get

$$\int_1^\infty \sup_{x\in\mathbb{R}^d} \mathbb{E}\left[\left(\overline{Z_t^N}(x,\delta_N r)\right)^2\right] \frac{\mathrm{d}r}{r^{\alpha+1}} \lesssim \int_0^t \int_1^\infty \sup_{x\in\mathbb{R}^d} \mathbb{E}\left[\left(\overline{Z_s^N}(x,\delta_N r)\right)^2\right] \frac{\mathrm{d}r}{r^{\alpha+1}} \mathrm{d}s + \delta_N^{-\alpha}.$$

Moreover, for each $N \geq 1$ and for all $t \in [0,T]$,

$$\int_1^\infty \sup_{x\in\mathbb{R}^d} \mathbb{E}\left[\left(\overline{Z_t^N}(x,\delta_N r)\right)^2\right] \frac{\mathrm{d}r}{r^{\alpha+1}} \leq \frac{\eta_N}{\alpha\tau_N}(1 + \sup_{t\in[0,T]} \left\|f_t^N\right\|_\infty)^2$$

which is bounded on $[0,T]$ by Lemma 2.6. Hence, by Gronwall's inequality, for $0 \leq t \leq T$,

$$\int_1^\infty \sup_{x\in\mathbb{R}^d} \mathbb{E}\left[\left(\overline{Z_t^N}(x,\delta_N r)\right)^2\right] \frac{\mathrm{d}r}{r^{\alpha+1}} \lesssim \delta_N^{-\alpha}.$$

□

5.3 Convergence to the deterministic limit

The following result is proved in Appendix B.

Proposition 5.6. *For $T > 0$,*

$$\sup_{0\leq t\leq T} \left\|f_t^N - f_t\right\|_\infty \lesssim \delta_N^{\alpha\wedge(2-\alpha)},$$

and for $0 \leq |\beta| \leq 2$,

$$\sup_{0\leq t\leq T} \left\|\partial_\beta f_t^N\right\|_\infty \lesssim 1.$$

By the same argument as in Section 4.3, choosing a separating family $(\phi_n)_{n\geq 1}$ of compactly supported smooth functions satisfying $\|\phi_n\|_q \leq 1$ and $\max_{|\beta|=2} \|\partial_\beta \phi_n\|_q \leq 1$ for $q \in \{1,\infty\}$ and using the corresponding metric d on Ξ,

$$\begin{aligned}\mathbb{E}&\left[\sup_{0\leq t\leq T} d(w_{t/\eta}^N, f_t^N)\right] \\ &\leq \sum_{n\geq 1} \frac{1}{2^n}\left\{(\tau_N/\eta_N)^{1/2}\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\left\langle Z_t^N,\phi_n\right\rangle\right|\right] + \sup_{0\leq t\leq T}\left\|f_t^N - f_t\right\|_\infty \|\phi_n\|_1\right\}.\end{aligned}$$

The convergence of $w_{t/\eta}^N$ to f_t in L^1 follows from Proposition 5.6 and Lemma 5.7 below.

Lemma 5.7. *For any function ϕ satisfying $\|\phi\|_q \leq 1$ and $\max_{|\beta|=2} \|\partial_\beta\phi\|_q \leq 1$ for $q \in \{1,\infty\}$,*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\left\langle Z_t^N,\phi\right\rangle\right|\right] \lesssim 1.$$

As a result,

$$\mathbb{E}\left[\sup_{0\leq t\leq T} d(w_{t/\eta}^N, f_t^N)\right] \lesssim (\tau_N/\eta_N)^{1/2} + \delta_N^{\alpha\wedge(2-\alpha)}.$$

From (3.32), it can be seen that the leading term on the right-hand-side is $\delta_N^{\alpha\wedge(2-\alpha)}$, which goes to zero as $N \to \infty$, yielding the convergence of $\left(w_{t/\eta}^N\right)_{t\geq 0}$. The following lemma is needed for the proof of Lemma 5.7 and is proved in the same manner as (4.10) in Section 4.3.

Lemma 5.8. *For $\phi \in L^{1,\infty}(\mathbb{R}^d)$ and $t \in [0,T]$,*

$$\mathbb{E}\left[\left|\left\langle Z_t^N,\phi\right\rangle\right|\right] \lesssim \|\phi\|_1 + \|\phi\|_\infty.$$

Proof. Taking expectations on both sides of (5.7),

$$\mathbb{E}\left[\left|\left\langle Z_t^N,\phi\right\rangle\right|\right] \lesssim (\tau_N/\eta_N)^{1/2}\int_0^t\int_1^\infty \left\langle \mathbb{E}\left[\left(\overline{Z_s^N}(\delta_N r)\right)^2\right], \overline{|\varphi^N(s,t)|}(\delta_N r)\right\rangle \frac{\mathrm{d}r}{r^{\alpha+1}}\mathrm{d}s + \mathbb{E}\left[\left(\int_0^t\int_{\mathbb{R}^d}\varphi^N(x,s,t)M^N(\mathrm{d}x\mathrm{d}s)\right)^2\right]^{1/2}. \quad (5.13)$$

In the first integral, we have

$$\left\langle \mathbb{E}\left[\left(\overline{Z_s^N}(\delta_N r)\right)^2\right], \overline{|\varphi^N(s,t)|}(\delta_N r)\right\rangle \leq \left\|\varphi^N(s,t)\right\|_1 \sup_{x\in\mathbb{R}^d}\mathbb{E}\left[\left(\overline{Z_s^N}(x,\delta_N r)\right)^2\right].$$

Hence, applying Lemma 5.5 to the first term and Lemma 5.3 to the second term on the right-hand-side of (5.13) yields

$$\begin{aligned}
\mathbb{E}\left[\left|\left\langle Z_t^N,\phi\right\rangle\right|\right] &\lesssim \frac{(\tau_N/\eta_N)^{1/2}}{\delta_N^\alpha}\int_0^t \left\|\varphi^N(s,t)\right\|_1 \mathrm{d}s \\
&\quad + \left(\int_0^t \left\|\varphi^N(s,t)\right\|_1 (\left\|\varphi^N(s,t)\right\|_\infty + \left\|\varphi^N(s,t)\right\|_1)\mathrm{d}s\right)^{1/2} \\
&\lesssim \frac{(\tau_N/\eta_N)^{1/2}}{\delta_N^\alpha}\left\|\phi\right\|_1 + \left(\left\|\phi\right\|_1(\left\|\phi\right\|_\infty + \left\|\phi\right\|_1)\right)^{1/2} \\
&\lesssim \left\|\phi\right\|_1 + \left\|\phi\right\|_\infty.
\end{aligned}$$

We have used the fact that (by Lemma 5.2) $\left\|\varphi^N(s,t)\right\|_q \lesssim \left\|\phi\right\|_q$ to pass from the first line to the second. The third line follows since $\tau_N/\eta_N = o\left(\delta_N^{2\alpha}\right)$ by (3.32). ☐

Proof of Lemma 5.7. The proof of Lemma 5.7 is similar to the proof of Lemma 4.8. Setting $\psi_s = \mathcal{D}^{\alpha,\delta_N}\phi - \alpha\int_1^\infty \overline{F'(\overline{f_s^N})\overline{\phi}}(\delta_N r)\frac{\mathrm{d}r}{r^{\alpha+1}}$ and using (5.3),

$$\sup_{0\leq t\leq T}\left|\left\langle Z_t^N,\phi\right\rangle\right| \lesssim \int_0^T \left|\left\langle Z_s^N,\psi_s\right\rangle\right|\mathrm{d}s + (\tau_N/\eta_N)^{1/2}\left\|F''\right\|_\infty \int_0^T\int_1^\infty \left\langle \left(\overline{Z_s^N}(\delta_N r)\right)^2, \overline{|\phi|}(\delta_N r)\right\rangle \frac{\mathrm{d}r}{r^{\alpha+1}}\mathrm{d}s + \sup_{0\leq t\leq T}\left|M_t^N(\phi)\right|.$$

Taking the expectation on both sides, Lemma 5.8 can be used in the first term, and Lemma 5.5 in the second one, to yield

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\left\langle Z_t^N,\phi\right\rangle\right|\right] \lesssim \int_0^T (\left\|\psi_s\right\|_1 + \left\|\psi_s\right\|_\infty)\mathrm{d}s + \frac{(\tau_N/\eta_N)^{1/2}}{\delta_N^\alpha}\left\|F''\right\|_\infty \left\|\phi\right\|_1 + \mathbb{E}\left[\sup_{0\leq t\leq T}\left|M_t^N(\phi)\right|^2\right]^{1/2}.$$

But $\left\|\psi_s\right\|_q \lesssim \left\|\mathcal{D}^{\alpha,\delta}\phi\right\|_q + \left\|F'\right\|_\infty \left\|\phi\right\|_q$ and, by Proposition A.3.i in Appendix A, $\left\|\mathcal{D}^{\alpha,\delta}\phi\right\|_q \lesssim \left\|\phi\right\|_q + \max_{|\beta|=2}\left\|\partial_\beta\phi\right\|_q$. In addition, by Doob's inequality, and using Lemma 5.3,

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq t\leq T}\left|M_t^N(\phi)\right|^2\right] &\lesssim \mathbb{E}\left[M_T^N(\phi)^2\right] \\
&\lesssim \left\|\phi\right\|_1 (\left\|\phi\right\|_1 + \left\|\phi\right\|_\infty).
\end{aligned}$$

As a result, if $\|\phi\|_q \leq 1$ and $\max_{|\beta|=2} \|\partial_\beta \phi\|_q \leq 1$ for $q \in \{1, \infty\}$,

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \left|\left\langle Z_t^N, \phi\right\rangle\right|\right] \lesssim 1.$$

□

5.4 Tightness

The overall argument for the tightness of the sequence $\left(Z_t^N\right)_{t\geq 0}$ is the same as in Section 4.4.

Proposition 5.9. *For any $\phi \in \mathcal{S}(\mathbb{R}^d)$ and for any sequence $(T_N, \rho_N)_{N\geq 1}$ such that T_N is a stopping time with values in $[0, T]$ for every $N \geq 1$ and $\rho_N \downarrow 0$ as $N \to \infty$,*

$$\left\langle Z^N_{T_N+\rho_N}, \phi\right\rangle - \left\langle Z^N_{T_N}, \phi\right\rangle \xrightarrow[N\to\infty]{P} 0. \tag{5.14}$$

Tightness of $\left(Z^N\right)_{N\geq 1}$ in $\mathrm{D}\left([0, T], \mathcal{S}'(\mathbb{R}^d)\right)$ then follows from Aldous' criterion [Ald78] and Mitoma's theorem [Wal86, Theorem 6.13].

Extend φ^N to $\mathbb{R}^d \times [0, T]^2$ as in (4.13); we need estimates on φ^N as in Lemmas 4.10 and 4.11. The proof of the following lemma is in Appendix C.

Lemma 5.10. *For $T > 0$, $q \in \{1, \infty\}$ and for all $s, t, t' \in [0, T]$,*

$$\left\|\varphi^N(s, t') - \varphi^N(s, t)\right\|_q \lesssim |t - t'|\,.$$

In addition, for all $s \in [0, T]$,

$$\left\|\sup_{t\in[s,T]} \left|\varphi^N(s, t)\right|\right\|_1 \lesssim 1.$$

Proof of Proposition 5.9. We only detail how the quadratic part of (5.7) can be bounded using Lemma 5.5, and refer to Section 4.4 for the rest of the proof of Proposition 5.9. For T_N a stopping time with values in $[0, T]$, write

$$\left|\int_0^{T_N}\int_1^\infty \left\langle (\overline{Z_s^N}(\delta_N r))^2, R_2(\overline{w^N_{s/\eta}}, \overline{f_s^N})\overline{\varphi^N(s, T_N)}(\delta_N r)\right\rangle \frac{\mathrm{d}r}{r^{\alpha+1}}\mathrm{d}s\right|$$
$$\lesssim \|F''\|_\infty \int_0^T\int_1^\infty \left\langle (\overline{Z_s^N}(\delta_N r))^2, \overline{\sup_{t\in[s,T]} |\varphi^N(s, t)|}(\delta_N r)\right\rangle \frac{\mathrm{d}r}{r^{\alpha+1}}\mathrm{d}s.$$

Taking the expectation on both sides and the supremum inside the spatial integral against φ^N, we get

$$\mathbb{E}\left[\left|\int_0^{T_N}\int_1^\infty \left\langle (\overline{Z_s^N}(\delta_N r))^2, R_2(\overline{w^N_{s/\eta}}, \overline{f_s^N})\overline{\varphi^N(s, T_N)}(\delta_N r)\right\rangle \frac{\mathrm{d}r}{r^{\alpha+1}}\mathrm{d}s\right|\right]$$
$$\lesssim \|F''\|_\infty \int_0^T\int_1^\infty \sup_{x\in\mathbb{R}^d} \mathbb{E}\left[\overline{Z_s^N}(x, \delta_N r)^2\right] \left\|\sup_{t\in[s,T]} \left|\varphi^N(s, t)\right|\right\|_1 \frac{\mathrm{d}r}{r^{\alpha+1}}\mathrm{d}s$$
$$\lesssim \delta_N^{-\alpha},$$

by Lemma 5.5 and Lemma 5.10. The other terms in (5.14) are bounded as in the proof of Proposition 4.9 in Section 4.4, using Lemmas 5.10 and 5.3. □

5.5 Convergence of the martingale measure M^N

The convergence of M^N relies on applying Theorem 4.13 to vectors of the form $\left(M_t^N(\phi_1),\dots,\ M_t^N(\phi_p)\right)_{t\geq 0}$, although the details differ from the proof in the Brownian case (in Section 4.5). Indeed, M^N no longer converges to a stochastic integral against a space-time white noise, but to W^α, a coloured Gaussian noise such that

$$\langle W^\alpha(\phi)\rangle_t = \int_0^t \int_{(\mathbb{R}^d)^2} \phi(z_1)\phi(z_2)\sigma^\alpha_{z_1,z_2}(f_s)\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}s$$

with

$$\sigma^\alpha_{z_1,z_2}(f) = \int_{\frac{|z_1-z_2|}{2}}^\infty \frac{\mathrm{d}r}{r^{d+\alpha+1}} \int_{B(z_1,r)\cap B(z_2,r)} \left[\overline{f}(x,r)(1-f(z_1)-f(z_2)) + f(z_1)f(z_2)\right]\mathrm{d}x.$$

Hence the weak convergence of M^N to W^α in $\mathrm{D}\left([0,T],\mathcal{S}'(\mathbb{R}^d)\right)$ will follow (as in Section 4.5) from the following lemma.

Lemma 5.11. *For any $\phi\in\mathcal{S}(\mathbb{R}^d)$,*

i) For all $t\geq 0$, $\left|\Delta M_t^N(\phi)\right| \lesssim 1$, and $\sup_{0\leq t\leq T}\left|\Delta M_t^N(\phi)\right| \xrightarrow[N\to\infty]{P} 0$.

ii) For each $t\in[0,T]$, $\left\langle M^N(\phi)\right\rangle_t \xrightarrow[N\to\infty]{P} \langle W^\alpha(\phi)\rangle_t$.

Proof. The proof of the first part is the same as for Lemma 4.14:

$$\sup_{t\geq 0}\left|\Delta M_t^N(\phi)\right| \leq \alpha_N(\eta_N/\tau_N)^{1/2}\left\|\phi\right\|_1,$$

which tends to zero since $\alpha_N^2 = o\left(\tau_N/\eta_N\right)$. For the second part of the statement, we first show that

$$\left|\int_{(\mathbb{R}^d)^2}\phi(z_1)\phi(z_2)\sigma^{(\alpha,\delta_N)}_{z_1,z_2}(w^N_{s/\eta})\mathrm{d}z_1\mathrm{d}z_2 - \int_{(\mathbb{R}^d)^2}\phi(z_1)\phi(z_2)\sigma^{(\alpha,\delta_N)}_{z_1,z_2}(f^N_s)\mathrm{d}z_1\mathrm{d}z_2\right| \xrightarrow[N\to\infty]{L^1} 0. \tag{5.15}$$

From the definition of $\sigma^{(\alpha,\delta_N)}_{z_1,z_2}$ in (3.30),

$$\sigma^{(\alpha,\delta_N)}_{z_1,z_2}(w) = \int_{\delta_N\vee\frac{|z_1-z_2|}{2}}^\infty \Big\{(1-w(z_1)-w(z_2))\int_{B(z_1,r)\cap B(z_2,r)}\overline{w}(x,r)\mathrm{d}x \\ + V_r(z_1,z_2)w(z_1)w(z_2)\Big\}\frac{\mathrm{d}r}{r^{d+\alpha+1}}.$$

Subtracting the corresponding expressions with $w^N_{s/\eta}$ and f^N_s and reordering terms, we write

$$\begin{aligned}
&\sigma^{(\alpha,\delta_N)}_{z_1,z_2}(w^N_{s/\eta}) - \sigma^{(\alpha,\delta_N)}_{z_1,z_2}(f^N_s) \\
&\quad = \int_{\delta_N\vee\frac{|z_1-z_2|}{2}}^\infty \Big\{(1-w^N_{s/\eta}(z_1)-w^N_{s/\eta}(z_2))\int_{B(z_1,r)\cap B(z_2,r)}\left(\overline{w^N_{s/\eta}}(x,r)-\overline{f^N_s}(x,r)\right)\mathrm{d}x \\
&\qquad + (f^N_s(z_1)-w^N_{s/\eta}(z_1)+f^N_s(z_2)-w^N_{s/\eta}(z_2))\int_{B(z_1,r)\cap B(z_2,r)}\overline{f^N_s}(x,r)\mathrm{d}x \\
&\quad + V_r(z_1,z_2)\left(w^N_{s/\eta}(z_1)(w^N_{s/\eta}(z_2)-f^N_s(z_2)) + f^N_s(z_2)(w^N_{s/\eta}(z_1)-f^N_s(z_1))\right)\Big\}\frac{\mathrm{d}r}{r^{d+\alpha+1}}.
\end{aligned} \tag{5.16}$$

We shall deal with the terms from each of the three lines separately, so let us call them $A(z_1,z_2)$, $B(z_1,z_2)$ and $C(z_1,z_2)$ (they are in fact defined for a.e. z_1 and z_2, and so is all that follows, but this is not a problem since what we really show is (5.15)). For the first term write

$$\begin{aligned}
&\mathbb{E}\left[|A(z_1,z_2)|\right] \\
&\quad\leq (\tau_N/\eta_N)^{1/2}\int_{\delta_N\vee\frac{|z_1-z_2|}{2}}^{\infty}\int_{B(z_1,r)\cap B(z_2,r)}\mathbb{E}\left[\left|\overline{Z_s^N}(x,r)\right|\right]\mathrm{d}x\frac{\mathrm{d}r}{r^{d+\alpha+1}} \\
&\quad\leq (\tau_N/\eta_N)^{1/2}\int_{\delta_N\vee\frac{|z_1-z_2|}{2}}^{\infty}V_r(z_1,z_2)\sup_{x\in\mathbb{R}^d}\mathbb{E}\left[\left|\overline{Z_s^N}(x,r)\right|\right]\frac{\mathrm{d}r}{r^{d+\alpha+1}} \\
&\quad\leq (\tau_N/\eta_N)^{1/2}V_1\int_{\delta_N}^{\infty}\mathbb{1}_{2r>|z_1-z_2|}\sup_{x\in\mathbb{R}^d}\mathbb{E}\left[\left|\overline{Z_s^N}(x,r)\right|^2\right]^{1/2}\frac{\mathrm{d}r}{r^{\alpha+1}} \\
&\quad\leq (\tau_N/\eta_N)^{1/2}\frac{V_1}{\alpha^{1/2}}\left(\delta_N\vee\frac{|z_1-z_2|}{2}\right)^{-\alpha/2}\left(\int_{\delta_N}^{\infty}\sup_{x\in\mathbb{R}^d}\mathbb{E}\left[\left|\overline{Z_s^N}(x,r)\right|^2\right]\frac{\mathrm{d}r}{r^{\alpha+1}}\right)^{1/2}.
\end{aligned}$$

(We have used the Cauchy-Schwartz inequality in the last line.) In addition, by Lemma 5.5,

$$\begin{aligned}
\int_{\delta_N}^{\infty}\sup_{x\in\mathbb{R}^d}\mathbb{E}\left[\left|\overline{Z_s^N}(x,r)\right|^2\right]\frac{\mathrm{d}r}{r^{\alpha+1}} &= \delta_N^{-\alpha}\int_1^{\infty}\sup_{x\in\mathbb{R}^d}\mathbb{E}\left[\left|\overline{Z_s^N}(x,\delta_N r)\right|^2\right]\frac{\mathrm{d}r}{r^{\alpha+1}} \\
&\lesssim \delta_N^{-2\alpha}.
\end{aligned}$$

Hence

$$\mathbb{E}\left[|A(z_1,z_2)|\right]\lesssim(\tau_N/\eta_N)^{1/2}\delta_N^{-\alpha}\left|z_1-z_2\right|^{-\alpha/2},$$

and, using Lemma 5.4 and (3.32),

$$\int_{(\mathbb{R}^d)^2}\phi(z_1)\phi(z_2)A(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2\xrightarrow[N\to\infty]{L^1}0. \tag{5.17}$$

For the second term, by symmetry,

$$\left|\int_{(\mathbb{R}^d)^2}\phi(z_1)\phi(z_2)B(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2\right|\leq 2(\tau_N/\eta_N)^{1/2}\int_{\mathbb{R}^d}|\phi(z_2)|\left|\left\langle Z_s^N,\psi_{z_2}^N\right\rangle\right|\mathrm{d}z_2,$$

where

$$\psi_{z_2}^N(z_1)=\phi(z_1)\int_{\delta_N\vee\frac{|z_1-z_2|}{2}}^{\infty}\int_{B(z_1,r)\cap B(z_2,r)}\overline{f_s^N}(x,r)\mathrm{d}x\frac{\mathrm{d}r}{r^{d+\alpha+1}}.$$

In particular, by Proposition 5.6 $\left\|\psi_{z_2}^N\right\|_q\lesssim\delta_N^{-\alpha}\left\|\phi\right\|_q$ for $q\in\{1,\infty\}$ and, since $\psi_{z_2}^N$ is deterministic, by Lemma 5.8

$$\mathbb{E}\left[\left|\int_{(\mathbb{R}^d)^2}\phi(z_1)\phi(z_2)B(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2\right|\right]\lesssim(\tau_N/\eta_N)^{1/2}\delta_N^{-\alpha}\left\|\phi\right\|_1(\left\|\phi\right\|_1+\left\|\phi\right\|_\infty).$$

Hence, by (3.32),

$$\int_{(\mathbb{R}^d)^2}\phi(z_1)\phi(z_2)B(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2\xrightarrow[N\to\infty]{L^1}0. \tag{5.18}$$

The third term is controlled in a similar way, this time setting

$$\psi_{z_2}^N(z_1)=\phi(z_1)\int_{\delta_N\vee\frac{|z_1-z_2|}{2}}^{\infty}V_r(z_1,z_2)\frac{\mathrm{d}r}{r^{d+\alpha+1}},$$

which satisfies the same inequalities as the previous $\psi_{z_2}^N$. The bound on $\left\|f_s^N\right\|_\infty$ from Proposition 5.6 yields

$$\int_{(\mathbb{R}^d)^2}\phi(z_1)\phi(z_2)C(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2\xrightarrow[N\to\infty]{L^1}0. \tag{5.19}$$

The convergence (5.15) follows from (5.17), (5.18), (5.19) and (5.16).

Recall the definition of $\sigma^{(\alpha,\delta)}$ in (3.30) and write

$$\left|\sigma^{(\alpha,\delta_N)}_{z_1,z_2}(f^N_s) - \sigma^{\alpha}_{z_1,z_2}(f^N_s)\right| \lesssim \mathbb{1}_{|z_1-z_2|\le 2\delta_N} \int_{\frac{|z_1-z_2|}{2}}^{\delta_N} V_r(z_1,z_2)\frac{\mathrm{d}r}{r^{d+\alpha+1}}$$
$$\lesssim \mathbb{1}_{|z_1-z_2|\le 2\delta_N} |z_1-z_2|^{-\alpha}.$$

Hence

$$\begin{aligned}\left|\int_{(\mathbb{R}^d)^2}\phi(z_1)\phi(z_2)(\sigma^{(\alpha,\delta_N)}_{z_1,z_2}(f^N_s) - \sigma^{\alpha}_{z_1,z_2}(f^N_s))\mathrm{d}z_1\mathrm{d}z_2\right| \\ &\lesssim \int_{(\mathbb{R}^d)^2}|\phi(z_1)|\,|\phi(z_2)|\,\mathbb{1}_{|z_1-z_2|\le 2\delta_N}|z_1-z_2|^{-\alpha}\mathrm{d}z_1\mathrm{d}z_2 \\ &\lesssim \|\phi\|_\infty\int_{\mathbb{R}^d}|\phi(z_1)|\int_0^{2\delta_N} r^{-\alpha+d-1}\mathrm{d}r\mathrm{d}z_1 \\ &\lesssim \|\phi\|_\infty \|\phi\|_1\,\delta_N^{d-\alpha} \underset{N\to\infty}{\longrightarrow} 0.\end{aligned} \tag{5.20}$$

Finally, proceeding as in (5.16) with f_s instead of $w^N_{s/\eta}$ and σ^α instead of $\sigma^{(\alpha,\delta_N)}$, we write

$$\begin{aligned}&\sigma^{\alpha}_{z_1,z_2}(f_s) - \sigma^{\alpha}_{z_1,z_2}(f^N_s) \\ &\quad= \int_{\frac{|z_1-z_2|}{2}}^{\infty}\Big\{(1-f_s(z_1)-f_s(z_2))\int_{B(z_1,r)\cap B(z_2,r)}\left(\overline{f_s}(x,r)-\overline{f^N_s}(x,r)\right)\mathrm{d}x \\ &\qquad + (f^N_s(z_1)-f_s(z_1)+f^N_s(z_2)-f_s(z_2))\int_{B(z_1,r)\cap B(z_2,r)}\overline{f^N_s}(x,r)\mathrm{d}x \\ &\qquad + V_r(z_1,z_2)\left(f_s(z_1)(f_s(z_2)-f^N_s(z_2)) + f^N_s(z_2)(f_s(z_1)-f^N_s(z_1))\right)\Big\}\frac{\mathrm{d}r}{r^{d+\alpha+1}}.\end{aligned}$$

Therefore

$$\begin{aligned}\left|\sigma^{\alpha}_{z_1,z_2}(f^N_s) - \sigma^{\alpha}_{z_1,z_2}(f_s)\right| &\le (4+3\sup_{s\in[0,T]}\left\|f^N_s\right\|_\infty)\left\|f^N_s - f_s\right\|_\infty\int_{\frac{|z_1-z_2|}{2}}^{\infty} V_r(z_1,z_2)\frac{\mathrm{d}r}{r^{1+d+\alpha}} \\ &\lesssim |z_1-z_2|^{-\alpha}\,\delta_N^{\alpha\wedge(2-\alpha)}\end{aligned}$$

by Proposition 5.6. It follows from Lemma 5.4 that

$$\left|\int_{(\mathbb{R}^d)^2}\phi(z_1)\phi(z_2)(\sigma^{\alpha}_{z_1,z_2}(f^N_s) - \sigma^{\alpha}_{z_1,z_2}(f_s))\mathrm{d}z_1\mathrm{d}z_2\right| \underset{N\to\infty}{\longrightarrow} 0. \tag{5.21}$$

By (5.15), (5.20) and (5.21), we have shown that for all $t\in[0,T]$,

$$\left\langle M^N(\phi)\right\rangle_t \overset{L^1,P}{\underset{N\to\infty}{\longrightarrow}} \langle W^\alpha(\phi)\rangle_t.$$

□

5.6 Conclusion of the proof

We can now conclude the proof of Theorem 3.10. We have proved that the sequence $\left(Z^N\right)_{N\ge1}$ is tight and we can characterise its potential limit points using the convergence of M^N. Recall the following expression for $\left\langle Z^N_t,\phi\right\rangle$ from (5.7) :

$$\begin{aligned}\left\langle Z^N_t,\phi\right\rangle = -\left(\frac{\tau_N}{\eta_N}\right)^{1/2}\alpha\int_0^t\int_1^\infty\left\langle(\overline{Z^N_s}(\delta_N r))^2, R_2(\overline{w^N_{s/\eta}},\overline{f^N_s})\overline{\varphi^N(s,t)}(\delta_N r)\right\rangle\frac{\mathrm{d}r}{r^{\alpha+1}}\mathrm{d}s \\ + \int_0^t\int_{\mathbb{R}^d}\varphi^N(x,s,t)M^N(\mathrm{d}x\mathrm{d}s).\end{aligned}$$

In Section 5.4, we showed that the first term converges to zero in L^1. In addition, by Lemmas 5.2 and 5.3,

$$\int_0^t \int_{\mathbb{R}^d} \varphi^N(x,s,t) M^N(\mathrm{d}x\mathrm{d}s) - \int_0^t \int_{\mathbb{R}^d} \varphi(x,s,t) M^N(\mathrm{d}x\mathrm{d}s) \xrightarrow[N\to\infty]{L^2} 0.$$

For $\phi_1, \dots, \phi_p$ in $\mathcal{S}(\mathbb{R}^d)$, let $\varphi_1, \dots, \varphi_p$ be the corresponding solutions of (5.8) with $\phi = \phi_i$. Since M^N converges weakly to W^α, by [Wal86, Proposition 7.12], for $t_1, \dots, t_p \in [0,T]$

$$\left(\int_0^{t_1} \int_{\mathbb{R}^d} \varphi_1(x,s,t_1) M^N(\mathrm{d}x\mathrm{d}s), \dots, \int_0^{t_k} \int_{\mathbb{R}^d} \varphi_k(x,s,t_k) M^N(\mathrm{d}x\mathrm{d}s)\right) \\ \xrightarrow[N\to\infty]{d} \left(\int_0^{t_1} \int_{\mathbb{R}^d} \varphi_1(x,s,t_1) W^\alpha(\mathrm{d}x\mathrm{d}s), \dots, \int_0^{t_k} \int_{\mathbb{R}^d} \varphi_k(x,s,t_k) W^\alpha(\mathrm{d}x\mathrm{d}s)\right).$$

Hence the same convergence holds (in distribution) for $\left(\left\langle Z^N_{t_1}, \phi_1\right\rangle, \dots, \left\langle Z^N_{t_p}, \phi_p\right\rangle\right)$ and this characterises the potential limit points of $\left(Z^N\right)_{N\geq 1}$. By Theorem 4.1, $\left(Z^N_t\right)_{t\geq 0}$ converges in distribution to a distribution-valued process $(z_t)_{t\geq 0}$ which satisfies

$$\langle z_t, \phi\rangle = \int_0^t \int_{\mathbb{R}^d} \varphi(x,s,t) W^\alpha(\mathrm{d}x\mathrm{d}s).$$

By the same argument as in Section 4.6, $(z_t)_{t\geq 0}$ solves the stochastic PDE (3.33), which concludes the proof.

6 Drift load - proof of Theorem 2.9

Recall the definition of F and $\rho^{(r_N)}_{z_1,z_2}$ in (2.16) and (3.10) respectively.

Definition 6.1 (Martingale Problem (M3))**.** *Given $(\varepsilon_N)_{N\geq 1}$, $(\delta_N)_{N\geq 1}$ and $F : \mathbb{R} \to \mathbb{R}$, let $\eta_N = \varepsilon_N \delta_N^2$, $\tau_N = \varepsilon_N^2 \delta_N^d$ and $r_N = \delta_N R$. Then for $N \geq 1$, we say that a Ξ-valued process $(w^N_t)_{t\geq 0}$ satisfies the martingale problem (M3) if for all ϕ in $L^{1,\infty}(\mathbb{R}^d)$,*

$$\left\langle w^N_t, \phi\right\rangle - \langle w_0, \phi\rangle - \eta_N u V_R \int_0^t \left\{\frac{2R^2}{d+2}\left\langle w^N_s, \mathcal{L}^{(r_N)}\phi\right\rangle - s\left\langle \overline{F(\overline{w^N_s})}(r_N), \phi\right\rangle\right\} \mathrm{d}s \qquad (6.1)$$

defines a (mean zero) square-integrable martingale with (predictable) variation process

$$\tau_N u^2 V_R^2 \int_0^t \int_{(\mathbb{R}^d)^2} \phi(z_1)\phi(z_2)\rho^{(r_N)}_{z_1,z_2}(w^N_s)\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}s + \mathcal{O}\left(t\tau_N\delta_N^2 \left\|\phi\right\|_2^2\right). \qquad (6.2)$$

(Again, uniqueness does not hold for this martingale problem, but we will not require it.)

Let q^N_t denote the SLFVS with overdominance defined in Definition 1.5 with parameters as defined in (2.15) in Section 2.3. As in Subsection 3.2, we consider the rescaled process $w^N_t(x) = q^N_t(x/\delta_N)$. By Proposition 3.2, using the same rescaling argument as in Proposition 3.6, we have the following result.

Proposition 6.2. *The process $\left(w^N_t\right)_{t\geq 0}$ satisfies the martingale problem (M3).*

As in Theorem 2.2, we define the process of rescaled fluctuations by

$$Z^N_t = (\eta_N/\tau_N)^{1/2}\left(w^N_{t/\eta} - \lambda\right). \qquad (6.3)$$

(Recall that since $w_0 = \lambda$, the centering term is constant and equals λ.) Then by the definition of Δ^N in (2.20),

$$\Delta^N(t,x) = \delta_N^2(s_1+s_2)\varepsilon_N\delta_N^{d-2}\mathbb{E}\left[\overline{Z^N_{\eta_N t}}(\delta_N x, r_N)^2\right].$$

Let us define the following notation for any $\phi \in L^{1,\infty}(\mathbb{R}^d)$,

$$\phi_r(x) = \frac{1}{r^d}\phi(x/r). \tag{6.4}$$

Theorem 2.9 is then a direct consequence of the following theorem.

Theorem 6.3. *Suppose that $\tau_N/\eta_N = o\left(r_N^{d+2}\right)$. Then for all $\phi \in L^{1,\infty}(\mathbb{R}^d)$, there exists a constant $C > 0$ - depending on the dimension d - such that, as $N \to \infty$ with $t \to \infty$ for $d \leq 2$ and $t\delta_N^{-2} \to \infty$ for $d \geq 3$,*

$$\mathbb{E}\left[\left\langle Z_t^N, \phi_{r_N}\right\rangle^2\right] \underset{N,t\to\infty}{\sim} C\delta_N^{2-d}c_N.$$

Proof of Theorem 2.9. Setting $\phi = \mathbb{1}_{|x|\leq 1}$ gives the result for $\Delta^N(t,0)$; the general result follows by translation invariance. □

Note that the only difference between the martingale problems (M1) and (M3) in Definitions 3.4 and 6.1 is that $\sigma_{z_1,z_2}^{(r_N)}$ is replaced by $\rho_{z_1,z_2}^{(r_N)}$. Hence it is easy to see that Lemma 4.5 and Lemma 4.6 also hold in this case (with different constants). It is also possible to adapt the proofs in Section 4.5 to show that on compact time intervals, $\left(Z_t^N\right)_{t\geq 0}$ converges to the solution of the following SPDE,

$$\mathrm{d}z_t = \left[\frac{1}{2}\Delta z_t - F'(\lambda)z_t\right]\mathrm{d}t + \sqrt{\frac{1}{2}\lambda(1-\lambda)}\mathrm{d}W_t.$$

This process admits a stationary distribution, under which $\langle z_t, \phi\rangle$ is a Gaussian random variable with variance

$$\frac{1}{2}\lambda(1-\lambda)\int_0^\infty \int_{\mathbb{R}^d} e^{-2F'(\lambda)t}G_t * \phi(x)^2\mathrm{d}x\mathrm{d}t.$$

We can thus hope to extend the convergence of $\left(Z_t^N\right)_{t\geq 0}$ to the whole real line (as in [Nor77]), and use the above expression to estimate the second moment of $\left\langle Z_t^N, \phi_{r_N}\right\rangle$ for large times. Some care is needed though, as we are letting the support of the test function vanish as $N \to \infty$.

Proof of Theorem 6.3. Since $q_0^N = \lambda$, by the same argument as for (3.25),

$$\mathrm{d}Z_t^N = \left[\mathcal{L}^{(r_N)}Z_t^N - (\eta_N/\tau_N)^{1/2}\overline{\left(F(\overline{w_{t/\eta}^N}) - F(\lambda)\right)}(r_N)\right]\mathrm{d}t + \mathrm{d}M_t^N \tag{6.5}$$

$$= \left[\mathcal{L}^{(r_N)}Z_t^N - F'(\lambda)\overline{\overline{\overline{Z_t^N}}}(r_N) - (\tau_N/\eta_N)^{1/2}\,\overline{(\overline{Z_t^N})^2R_2(\overline{w_{t/\eta}^N},\lambda)}(r_N)\right]\mathrm{d}t + \mathrm{d}M_t^N, \tag{6.6}$$

where M^N is a martingale measure with covariation measure Q^N given by

$$Q^N(dz_1dz_2ds) = \rho_{z_1,z_2}^{(r_N)}(w_{s/\eta}^N)dz_1dz_2ds + \mathcal{O}\left(\delta_N^2\right)\delta_{z_1=z_2}(\mathrm{d}z_1\mathrm{d}z_2)\mathrm{d}s. \tag{6.7}$$

Consider a time dependent test function φ^N which solves

$$\begin{cases} \partial_s\varphi^N(x,s,t) + \mathcal{L}^{(r_N)}\varphi^N(x,s,t) - F'(\lambda)\overline{\overline{\overline{\varphi^N(s,t)}}}(x,r_N) = 0, \\ \varphi^N(x,t,t) = \phi(x). \end{cases} \tag{6.8}$$

Then, by (6.6), by the same argument as for (4.1),

$$\left\langle Z_t^N, \phi\right\rangle = -(\tau_N/\eta_N)^{1/2}\int_0^t \left\langle (\overline{Z_s^N})^2, R_2(\overline{w_{s/\eta}^N},\lambda)\overline{\varphi^N(s,t)}(r_N)\right\rangle\mathrm{d}s$$
$$+ \int_0^t\int_{\mathbb{R}^d}\varphi^N(x,s,t)M^N(\mathrm{d}x\mathrm{d}s). \tag{6.9}$$

The remainder of the proof now consists of proving that the main contribution to the variance of $\left\langle Z_t^N, \phi_{r_N}\right\rangle$ is made by the last term on the right-hand-side and then estimating this contribution. Note that φ^N is given explicitly by

$$\varphi^N(x,s,t) = e^{-F'(\lambda)(t-s)} G^{(r_N)}_{D_N(t-s)} * \phi(x), \tag{6.10}$$

with $D_N = 1 - F'(\lambda)\frac{2r_N^2}{d+2}$. To see this, differentiate with respect to s and write

$$\partial_s \varphi^N(x,s,t) = F'(\lambda)\varphi^N(x,s,t) - \left(1 - F'(\lambda)\frac{2r_N^2}{d+2}\right)\mathcal{L}^{(r_N)}\varphi^N(x,s,t).$$

Since by (2.4), $\frac{2r_N^2}{d+2}\mathcal{L}^{(r_N)}\varphi^N = \overline{\overline{\varphi^N}}(r_N) - \varphi^N$, we have a solution to (6.8). In particular,

$$\left\|\varphi^N(s,t)\right\|_q \leq \|\phi\|_q \, e^{-F'(\lambda)(t-s)}. \tag{6.11}$$

The following lemma extends the result of Lemma 4.6 to arbitrarily large times, and will be proved in Subsection 6.1.

Lemma 6.4. *There exist constants K_1' and K_2' such that, for all $x \in \mathbb{R}^d$ and all $t \geq 0$,*

$$\mathbb{E}\left[\overline{Z_t^N}(x,r_N)^2\right] \leq \frac{K_1'}{r_N^d}, \qquad \text{and} \qquad \mathbb{E}\left[\overline{Z_t^N}(x,r_N)^4\right] \leq \frac{K_2'}{r_N^{2d}}.$$

Using the expression for φ^N in (6.10) and then the Cauchy-Schwartz inequality,

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t \left\langle (\overline{Z_s^N})^2, R_2(\overline{w_{s/\eta}^N},\lambda)\overline{\varphi^N(s,t)}\right\rangle \mathrm{d}s\right)^2\right] \\
&\quad \leq \frac{1}{4}\|F''\|_\infty^2 \, \mathbb{E}\left[\left(\int_0^t e^{-F'(\lambda)(t-s)/2}\left(e^{-F'(\lambda)(t-s)/2}\left\langle (\overline{Z_s^N})^2, \left|\overline{G^{(r_N)}_{D_N(t-s)} * \phi}\right|\right\rangle\right)\mathrm{d}s\right)^2\right] \\
&\quad \leq \frac{1}{4}\|F''\|_\infty^2 \, \frac{1-e^{-F'(\lambda)t}}{F'(\lambda)}\mathbb{E}\left[\int_0^t e^{-F'(\lambda)(t-s)}\left\langle (\overline{Z_s^N})^2, \left|\overline{G^{(r_N)}_{D_N(t-s)} * \phi}\right|\right\rangle^2 \mathrm{d}s\right].
\end{aligned}$$

Another use of the Cauchy-Schwartz inequality yields

$$\left\langle (\overline{Z_s^N})^2, \left|\overline{G^{(r_N)}_{D_N(t-s)} * \phi}\right|\right\rangle^2 \leq \left\|G^{(r_N)}_{D_N(t-s)} * \phi\right\|_1 \left\langle (\overline{Z_s^N})^4, \left|\overline{G^{(r_N)}_{D_N(t-s)} * \phi}\right|\right\rangle.$$

Hence, using Lemma 6.4 and the fact that $\left\|G_t^{(r)} * \phi\right\|_1 \leq \|\phi\|_1$,

$$\mathbb{E}\left[\left\langle (\overline{Z_s^N})^2, \left|\overline{G^{(r_N)}_{D_N(t-s)} * \phi}\right|\right\rangle^2\right] \leq \|\phi\|_1^2 \, \frac{K_2'}{r_N^{2d}}.$$

As a result,

$$\mathbb{E}\left[\left(\int_0^t \left\langle (\overline{Z_s^N})^2, R_2(\overline{w_{s/\eta}^N},\lambda)\overline{\varphi^N(s,t)}\right\rangle \mathrm{d}s\right)^2\right] \lesssim \|\phi\|_1^2 \, r_N^{-2d}, \tag{6.12}$$

uniformly in $t \in \mathbb{R}_+$. We now move on to estimating the contribution of the second term in (6.9). The following lemma will be proved in Subsection 6.1.

Lemma 6.5. *As $N \to \infty$,*

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} \varphi^N(x,s,t) M^N(\mathrm{d}x\mathrm{d}s)\right)^2\right] &= \frac{1}{2}\lambda(1-\lambda)\int_0^t \left\|\overline{\varphi^N(s,t)}(r_N)\right\|_2^2 \mathrm{d}s \\
&\qquad + o\left(r_N\right)\int_0^t \left\|\varphi^N(s,t)\right\|_2^2 \mathrm{d}s.
\end{aligned}$$

As we shall see in Subsection 6.1, this is a consequence of the fact that in the expression for Q^N in (6.7), w^N can be replaced by λ. As a result, using (6.10) and (6.12) in (6.9) and since $\tau_N/\eta_N = o\left(r_N^{d+2}\right)$, we have

$$\mathbb{E}\left[\left\langle Z_t^N, \phi_{r_N}\right\rangle^2\right] = \frac{1}{2}\lambda(1-\lambda)\int_0^t e^{-2F'(\lambda)s}\left\|\overline{G_{D_N s}^{(r_N)} * \phi_{r_N}}(r_N)\right\|_2^2 \mathrm{d}s \\ + o\left(r_N\right)\int_0^t e^{-2F'(\lambda)s}\left\|G_{D_N s}^{(r_N)} * \phi_{r_N}\right\|_2^2 \mathrm{d}s + o\left(r_N^{2-d}\right). \quad (6.13)$$

To study the asymptotic behaviour of the first integral, we use the scaling properties of the function $G^{(r)}$. Recall that $\left(\xi_t^{(r)}\right)_{t\geq 0}$ is a Lévy process with infinitesimal generator $\mathcal{L}^{(r)}$; it is not difficult to show that it satisfies the following scaling property:

$$\forall c > 0, \quad \mathbb{E}_x\left[\phi(\xi_t^{(r)})\right] = \mathbb{E}_{x/c}\left[\phi(c\,\xi_{t/c^2}^{(r/c)})\right]. \quad (6.14)$$

(Simply look at the infinitesimal generator of both processes.) Hence

$$G_t^{(r_N)} * \phi_{r_N}(x) = r_N^{-d} G_{t/r_N^2}^{(1)} * \phi_1(x/r_N).$$

Set $f(t) = \left\|\overline{G_t^{(1)} * \phi}(1)\right\|_2^2$; it follows that

$$\left\|\overline{G_{D_N s}^{(r_N)} * \phi_{r_N}}(r_N)\right\|_2^2 = r_N^{-d} f(D_N s/r_N^2). \quad (6.15)$$

We shall show that, as $N, t \to \infty$, there is a constant $\tilde{C} > 0$ such that

$$\int_0^t e^{-2F'(\lambda)s} f(D_N s/r_N^2)\mathrm{d}s \sim \tilde{C} r_N^2 c_N. \quad (6.16)$$

Theorem 6.3 then follows from (6.16) and (6.13). To prove (6.16) we need the following estimate of $f(t)$ when $t \to \infty$.

Lemma 6.6. *For $\phi \geq 0$, as $t \to \infty$,*

$$f(t) \sim (4\pi t)^{-d/2}\left\|\phi\right\|_1^2. \quad (6.17)$$

For the proof of this estimate we will use the following properties of the semigroup $G^{(r)}$, which will be proved in Appendix D.

Lemma 6.7. *For any $r > 0$ and $t > 0$, the law of $\xi_t^{(r)}$ takes the form*

$$G_t^{(r)}(\mathrm{d}x) = e^{-\frac{(d+2)}{2r^2}t}\delta_0(\mathrm{d}x) + g_t^{(r)}(x)\mathrm{d}x.$$

Furthermore, $g_t^{(r)}$ is continuous on $\mathbb{R}^d$, is invariant under rotations which fix the origin and $g_t^{(r)}(y)$ is a decreasing function of $|y|$.

Proof of Lemma 6.6. By the semigroup property of $\phi \mapsto G_t^{(r)} * \phi$, $f(t)$ can also be written $\left\langle G_{2t}^{(1)} * \overline{\phi}(1), \overline{\phi}(1)\right\rangle$. In addition, by the scaling property of $\left(\xi_t^{(r)}\right)_{t\geq 0}$ in (6.14) and using Lemma 6.7,

$$\begin{aligned} G_{2t}^{(1)} * \phi(x) &= \mathbb{E}_0\left[\phi(x + \sqrt{t}\xi_2^{(1/\sqrt{t})})\right] \\ &= \phi(x)e^{-(d+2)t} + \int_{\mathbb{R}^d}\phi(x+\sqrt{t}y)g_2^{(1/\sqrt{t})}(y)\mathrm{d}y \\ &= \phi(x)e^{-(d+2)t} + t^{-d/2}\int_{\mathbb{R}^d}\phi(x+y)g_2^{(1/\sqrt{t})}(y/\sqrt{t})\mathrm{d}y. \end{aligned}$$

By Proposition A.2.ii and Theorem 4.8.2 in [EK86], the finite dimensional distributions of $\left(\xi_t^{(r)}\right)_{t\geq 0}$ converge to those of standard Brownian motion as $r \to 0$. In particular, $\xi_2^{(r)} \xrightarrow[r\to 0]{d} \mathcal{N}(0,2)$, and $g_2^{(r)}(x) \to G_2(x)$ as $r \to 0$ for almost every $x \in \mathbb{R}^d$ (the probability that $\xi_t^{(r)} = 0$ vanishes as $r \to 0$ for any $t > 0$). Since G_2 is continuous on $\mathbb{R}^d$ and $g_2^{(r)}$ is decreasing as a function of the modulus, this convergence takes place uniformly on compact sets by Dini's second theorem. So, fixing $\epsilon > 0$, for any $R > 0$, for r small enough,

$$\sup_{|x|<R} \left| g_2^{(r)}(x) - G_2(x) \right| \leq \epsilon.$$

As a result, using the continuity of G_2, for any y, for t large enough,

$$\left| g_2^{(1/\sqrt{t})}(y/\sqrt{t}) - G_2(0) \right| \leq 2\epsilon.$$

Hence, since $g_t^{(r)}(y) \leq g_t^{(r)}(0)$, by dominated convergence,

$$\int_{\mathbb{R}^d} \phi(x+y) g_2^{(1/\sqrt{t})}(y/\sqrt{t}) \mathrm{d}y \underset{t\to\infty}{\longrightarrow} (4\pi)^{-d/2} \int_{\mathbb{R}^d} \phi(y) \mathrm{d}y. \tag{6.18}$$

From the above expression for f,

$$f(t) = e^{-(d+2)t} \int_{\mathbb{R}^d} \overline{\phi}(x,1)^2 \mathrm{d}x + t^{-d/2} \int_{(\mathbb{R}^d)^2} g_2^{(1/\sqrt{t})}(y/\sqrt{t}) \overline{\phi}(x+y,1) \overline{\phi}(x,1) \mathrm{d}y \mathrm{d}x.$$

Replacing ϕ with $\overline{\phi}(1)$ in (6.18) and letting $t \to \infty$ yields the result. □

Furthermore, $0 \leq f(t) \leq \|\phi\|_2^2$ for all $t \geq 0$, and thus f is integrable on $(0,\infty)$ if and only if $d \geq 3$.

Remark 6.8. This is in fact a consequence of the fact that $\left(\xi_t^{(1)}\right)_{t\geq 0}$ is transient if and only if $d \geq 3$ (as with Brownian motion). The function f can be expressed in terms of the probability of $\xi_{2t}^{(1)}$ being in a ball of radius 1, which is integrable on $(0,\infty)$ if and only if $\left(\xi_t^{(1)}\right)_{t\geq 0}$ is transient.

We now prove (6.16) separately for each regime.

High dimension If $d \geq 3$, change the variable of integration to write

$$\int_0^t e^{-2F'(\lambda)s} f(D_N s/r_N^2) \mathrm{d}s = r_N^2 \int_0^{D_N t/r_N^2} e^{-2F'(\lambda) D_N^{-1} r_N^2 s} f(s) \mathrm{d}s.$$

Since f is integrable, by dominated convergence, and since $t\delta_N^{-2} \to \infty$,

$$\int_0^{D_N t/r_N^2} e^{-2F'(\lambda) D_N^{-1} r_N^2 s} f(s) \mathrm{d}s \underset{N,t\to\infty}{\longrightarrow} \int_0^\infty f(s) \mathrm{d}s.$$

(Also recall that $D_N = 1 + \mathcal{O}\left(r_N^2\right)$.)

Dimension 1 If $d = 1$, however, from (6.17), we see that, as $N \to \infty$, $\frac{1}{r_N} f(s/r_N^2) \to (4\pi s)^{-1/2} \|\phi\|_1^2$, so, by dominated convergence,

$$\int_0^t e^{-2F'(\lambda)s} f(D_N s/r_N^2) \mathrm{d}s \underset{N,t\to\infty}{\sim} r_N \|\phi\|_1^2 \hat{C},$$

for some constant $\hat{C} > 0$.

Dimension 2 If $d = 2$, let T_1 and T_2 be two positive constants. For $t \geq T_2$ and N large enough such that $r_N^2 T_1 \leq T_2$, we split the integral as follows :

$$\int_0^t e^{-2F'(\lambda)s} f(D_N s/r_N^2)\mathrm{d}s = \int_0^{r_N^2 T_1} e^{-2F'(\lambda)s} f(D_N s/r_N^2)\mathrm{d}s + \int_{r_N^2 T_1}^{T_2} e^{-2F'(\lambda)s} f(D_N s/r_N^2)\mathrm{d}s + \int_{T_2}^{t} e^{-2F'(\lambda)s} f(D_N s/r_N^2)\mathrm{d}s.$$

We first show that the first and last terms are of order r_N^2. Since $0 \leq f(t) \leq \|\phi\|_2^2$ for all $t \geq 0$,

$$\left| \int_0^{r_N^2 T_1} e^{-2F'(\lambda)s} f(D_N s/r_N^2)\mathrm{d}s \right| \leq r_N^2 T_1 \left\|\phi\right\|_2^2,$$

and by (6.17)

$$\left| \int_{T_2}^{t} e^{-2F'(\lambda)s} f(D_N s/r_N^2)\mathrm{d}s \right| \lesssim r_N^2 \int_{T_2}^{\infty} e^{-2F'(\lambda)s} \frac{\mathrm{d}s}{s}.$$

For the middle term, by (6.17), $\frac{1}{r_N^2} f(s/r_N^2) \underset{N\to\infty}{\longrightarrow} (4\pi s)^{-1} \left\|\phi\right\|_1^2$, and $D_N = 1 + \mathcal{O}\left(r_N^2\right)$ so as $N \to \infty$, by dominated convergence,

$$\int_{r_N^2 T_1}^{T_2} e^{-2F'(\lambda)s} f(D_N s/r_N^2)\mathrm{d}s \sim r_N^2 (4\pi)^{-1} \left\|\phi\right\|_1^2 \int_{r_N^2 T_1}^{T_2} e^{-2F'(\lambda)s} \frac{\mathrm{d}s}{s}.$$

Further

$$\left| \int_{T_1 r_N^2}^{T_2} e^{-2F'(\lambda)s} \frac{\mathrm{d}s}{s} - \int_{T_1 r_N^2}^{T_2} \frac{\mathrm{d}s}{s} \right| \leq 2F'(\lambda) \int_{T_1 r_N^2}^{T_2} s \frac{\mathrm{d}s}{s} \leq 2F'(\lambda) T_2,$$

and

$$\int_{T_1 r_N^2}^{T_2} \frac{\mathrm{d}s}{s} = \log\left(\frac{T_2}{T_1 r_N^2}\right) \sim \left|\log r_N^2\right|.$$

As a result

$$\int_0^t e^{-2F'(\lambda)s} f(D_N s/r_N^2)\mathrm{d}s \sim \frac{\left\|\phi\right\|_1^2}{4\pi} r_N^2 \left|\log r_N^2\right|,$$

as $N, t \to \infty$. We have thus proved (6.16), and the result. □

6.1 Proofs of Lemmas 6.4 and 6.5

The proof of Lemma 6.4 requires the following two technical lemmas, which are proved in Appendix D.

Lemma 6.9. *Let $\phi : \mathbb{R}^d \to \mathbb{R}$, $r > 0$ and suppose that $g : \mathbb{R}^d \to \mathbb{R}$ satisfies $0 < \gamma \leq g(x) \leq 1$ for all $x \in \mathbb{R}^d$. Then*

$$2\phi(x)\mathcal{L}^{(r)}\phi(x) - 2\phi(x)\overline{\overline{\phi\, g}}(x,r) \leq \mathcal{L}^{(r)}\phi^2(x) - 2\left(\gamma - \frac{r^2}{d+2}\right)\phi(x)^2.$$

Further, for some constant $c > 0$, for r small enough,

$$4\phi(x)^3\mathcal{L}^{(r)}\phi(x) - 4\phi(x)^3\overline{\overline{\phi\, g}}(x,r) \leq \mathcal{L}^{(r)}\phi^4(x) - 4(\gamma - c\, r^2)\phi(x)^4.$$

Lemma 6.10. *Suppose $h : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ is a function that is continuously differentiable with respect to the time variable t and which satisfies the following differential inequality for some positive α :*

$$\partial_t h_t(x) \leq \mathcal{L} h_t(x) - \alpha h_t(x) + g_t(x).$$

Then for all $0 \leq s \leq t$ *and for any* $1 \leq q \leq \infty$*,*

$$\|h_t\|_q \leq e^{-\alpha(t-s)} \|h_s\|_q + \frac{1}{\alpha} \sup_{u\in[s,t]} \|g_u\|_q .$$

Proof of Lemma 6.4. Set

$$h(t,x) = \mathbb{E}\left[\overline{Z_t^N}(x,r_N)^2\right].$$

We are going to make use of Lemma 6.10, so we want to obtain a differential inequality for h. To this end, average (6.5) on $B(x,r_N)$ and use the expression for R_1 in (3.23) to get

$$\mathrm{d}\overline{Z_s^N}(x,r_N) = \left[\mathcal{L}^{(r_N)}\overline{Z_s^N}(x,r_N) - \overline{\overline{\overline{Z_s^N}\, R_1(\overline{w_{s/\eta}^N},\lambda)}}(x,r_N)\right]\mathrm{d}s + \frac{1}{V_{r_N}}\mathrm{d}M_s^N(B(x,r_N)).$$

(From now on all averages will be over radius r_N.) By the generalised Itô formula, noting $\Delta Y_s = Y_s - Y_{s^-}$,

$$\begin{aligned}\mathrm{d}\left(\overline{Z_s^N}(x)\right)^2 &= 2\overline{Z_s^N}(x)\mathrm{d}\overline{Z_s^N}(x) + \mathrm{d}\left[\overline{Z_\cdot^N}(x)\right]_s \\ &\quad + \left(\overline{Z_{s^-}^N}(x) + \Delta\overline{Z_s^N}(x)\right)^2 - \left(\overline{Z_{s^-}^N}(x)\right)^2 - 2\overline{Z_{s^-}^N}(x)\Delta\overline{Z_s^N}(x) - \left(\Delta\overline{Z_s^N}(x)\right)^2.\end{aligned}$$

Expanding the brackets, the terms on the second line cancel and, integrating for $s \in [0,t]$, we have

$$\begin{aligned}\overline{Z_t^N}(x)^2 &= 2\int_0^t \overline{Z_s^N}(x)\left[\mathcal{L}^{(r_N)}\overline{Z_s^N}(x) - \overline{\overline{\overline{Z_s^N}\, R_1(\overline{w_{s/\eta}^N},\lambda)}}(x)\right]\mathrm{d}s \\ &\quad + \frac{2}{V_{r_N}}\int_0^t \overline{Z_s^N}(x)\mathrm{d}M_s^N(B(x,r_N)) + \frac{1}{V_{r_N}^2}\left[M^N(B(x,r_N))\right]_t.\end{aligned}$$

Taking expectations on both sides, since the second term is a martingale,

$$\begin{aligned}h(t,x) &= 2\int_0^t \mathbb{E}\left[\overline{Z_s^N}(x)\mathcal{L}^{(r_N)}\overline{Z_s^N}(x) - \overline{Z_s^N}(x)\overline{\overline{\overline{Z_s^N}\, R_1(\overline{w_{s/\eta}^N},\lambda)}}(x)\right]\mathrm{d}s \\ &\quad + \frac{1}{V_{r_N}^2}\mathbb{E}\left[\left\langle M^N(B(x,r_N))\right\rangle_t\right].\end{aligned}$$

Differentiating yields

$$\begin{aligned}\frac{\partial h}{\partial t}(t,x) &= 2\mathbb{E}\left[\overline{Z_t^N}(x)\mathcal{L}^{(r_N)}\overline{Z_t^N}(x) - \overline{Z_t^N}(x)\overline{\overline{\overline{Z_t^N}\, R_1(\overline{w_{t/\eta}^N},\lambda)}}(x)\right] \\ &\quad + \frac{1}{V_{r_N}^2}\mathbb{E}\left[\int_{B(x,r_N)^2}\rho_{z_1,z_2}^{(r_N)}(w_{t/\eta}^N)\mathrm{d}z_1\mathrm{d}z_2\right] + \mathcal{O}\left(\frac{\delta_N^2}{V_{r_N}}\right).\end{aligned}$$

The second term is bounded by $\frac{1}{V_{r_N}}$, and the first one has the same form as the left-hand-side of the first statement of Lemma 6.9. In [Nor74b] (at the beginning of the proof of Theorem 3.2), it is proved that the conditions on F in (2.17)-(2.18) imply

$$\inf_{x\in[0,1]} R_1(x,\lambda) =: \gamma > 0. \tag{6.19}$$

Then, taking $\phi = \overline{Z_t^N}$ and $g = R_1(\overline{w_{s/\eta}^N},\lambda)$, Lemma 6.9 implies that, for all $t \geq 0$,

$$\frac{\partial h}{\partial t}(t,x) \leq \mathcal{L}h(t,x) - \alpha_N h(t,x) + \frac{1+\mathcal{O}\left(\delta_N^2\right)}{V_{r_N}},$$

with $\alpha_N = \gamma + \mathcal{O}\left(r_N^2\right)$. Using Lemma 6.10 (with $s = 0$ and $q = \infty$) we can now write, since $Z_0^N = 0$,

$$\mathbb{E}\left[\overline{Z_t^N}(x)^2\right] \leq \frac{1 + \mathcal{O}\left(\delta_N^2\right)}{\alpha_N V_{r_N}} \lesssim \frac{1}{r_N^d}. \tag{6.20}$$

The second inequality is proved in essentially the same way, although the computations become more involved. We compute the fourth moment of $\overline{Z_t^N}$ with Itô's formula, as before:

$$\begin{aligned} \mathrm{d}\left(\overline{Z_t^N}(x)\right)^4 &= 4(\overline{Z_t^N}(x))^3 \mathrm{d}\overline{Z_t^N}(x) + \frac{1}{2} 4 \times 3 (\overline{Z_t^N}(x))^2 \mathrm{d}\left[\overline{Z_\cdot^N}\right]_t \\ &+ \left(\overline{Z_{t^-}^N}(x) + \Delta\overline{Z_t^N}(x)\right)^4 - \left(\overline{Z_{t^-}^N}(x)\right)^4 - 4(\overline{Z_{t^-}^N}(x))^3 \Delta\overline{Z_t^N}(x) - \frac{1}{2} 3 \times 4 (\overline{Z_{t^-}^N}(x))^2 (\Delta\overline{Z_t^N}(x))^2. \end{aligned}$$

Hence, taking expectations, the martingale terms can be dropped and we write:

$$\begin{aligned} \mathbb{E}\left[\left(\overline{Z_t^N}(x)\right)^4\right] &= 4\int_0^t \mathbb{E}\left[\overline{Z_s^N}(x)^3 \mathcal{L}^{(r_N)} \overline{Z_s^N}(x) - \overline{Z_s^N}(x)^3 \overline{\overline{\overline{Z_s^N}\, R_1(\overline{w_{s/\eta}^N}, \lambda)}}(x)\right] \mathrm{d}s \\ &+ 6\frac{1}{V_{r_N}^2} \int_0^t \int_{B(x,r_N)^2} \mathbb{E}\left[\overline{Z_s^N}(x)^2 \rho_{z_1,z_2}^{(r_N)}(w_{s/\eta}^N)\right] \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}s + \mathcal{O}\left(\delta_N^2\right) \frac{1}{V_{r_N}} \int_0^t \mathbb{E}\left[\overline{Z_s^N}(x)^2\right] \mathrm{d}s \\ &+ \mathbb{E}\left[\sum_{s\leq t} \left\{4\overline{Z_{s^-}^N}(x)(\Delta\overline{Z_s^N}(x))^3 + (\Delta\overline{Z_s^N}(x))^4\right\}\right], \end{aligned}$$

where the sum is over jump times for the process $(\overline{Z_t^N}(x))_{t\geq 0}$. We can bound the size of the jumps $\Delta\overline{Z_s^N}(x)$ by a deterministic constant. By the definition of the SLFVS with overdominance in Definition 1.5,

$$\sup_{t\geq 0} \left|\langle q_t^N, \phi\rangle - \langle q_{t^-}^N, \phi\rangle\right| \leq u\varepsilon_N \|\phi\|_1 .$$

Hence $\left|\Delta\overline{Z_s^N}(x)\right| \leq u\varepsilon_N(\eta_N/\tau_N)^{1/2} = u\varepsilon_N^{1/2}\delta_N^{1-d/2}$. As a result

$$\begin{aligned} &\mathbb{E}\left[\sum_{s\leq t} \left\{4\overline{Z_{s^-}^N}(x)(\Delta\overline{Z_s^N}(x))^3 + (\Delta\overline{Z_s^N}(x))^4\right\}\right] \\ &\qquad \leq \mathbb{E}\left[\sum_{s\leq t} \left\{4(u\varepsilon_N^{1/2}\delta_N^{1-d/2})^3 \left|\overline{Z_{s^-}^N}(x)\right| + (u\varepsilon_N^{1/2}\delta_N^{1-d/2})^4\right\}\right], \end{aligned}$$

where the sum is still over the jump times of $\overline{Z_s^N}(x)$. These jumps occur according to a Poisson process with rate $V_{2R}\,\eta_N^{-1}$, so, using (6.20) to bound $\mathbb{E}\left[\left|\overline{Z_{s^-}^N}(x)\right|\right]$, we obtain

$$\begin{aligned} &\mathbb{E}\left[\sum_{s\leq t} \left\{4\overline{Z_{s^-}^N}(x)(\Delta\overline{Z_s^N}(x))^3 + (\Delta\overline{Z_s^N}(x))^4\right\}\right] \\ &\quad \leq V_{2R}\,\eta_N^{-1}\left\{4(u\varepsilon_N^{1/2}\delta_N^{1-d/2})^3 \mathbb{E}\left[\int_0^t \left|\overline{Z_{s^-}^N}(x)\right| \mathrm{d}s\right] + t(u\varepsilon_N^{1/2}\delta_N^{1-d/2})^4\right\} = o\left(r_N^{-2d}\right). \end{aligned}$$

Now note that

$$\begin{aligned} \int_{B(x,r_N)^2} \mathbb{E}\left[\overline{Z_s^N}(x)^2 \rho_{z_1,z_2}^{(r)}(w_{s/\eta}^N)\right] \mathrm{d}z_1 \mathrm{d}z_2 &\lesssim \frac{1}{r_N^d} \int_{B(x,r_N)^2} \frac{V_{r_N}(z_1,z_2)}{V_{r_N}^2} \mathrm{d}z_1 \mathrm{d}z_2 \\ &\lesssim 1. \end{aligned}$$

Hence, setting $h(t,x) = \mathbb{E}\left[(\overline{Z_t^N}(x))^4\right]$,

$$\frac{\partial h}{\partial t}(t,x) = 4\mathbb{E}\left[\overline{Z_t^N}(x)^3 \mathcal{L}^{(r_N)}\overline{Z_t^N}(x) - \overline{Z_t^N}(x)^3 \overline{\overline{Z_t^N}\, R_1(\overline{w_{t/\eta}^N}, \lambda)}\right] + \frac{g_t(x)}{r_N^{2d}},$$

where $|g_t(x)| \lesssim 1$. Now the second statement of Lemma 6.9 yields :

$$\frac{\partial h}{\partial t}(t,x) \le \mathcal{L}h(t,x) - 4(\gamma - cr_N^2)h(t,x) + \frac{g_t(x)}{r_N^{2d}},$$

and by Lemma 6.10, we have

$$h(t,x) \lesssim \frac{1}{r_N^{2d}},$$

uniformly in $t \ge 0$. □

The following lemma is needed in the proof of Lemma 6.5.

Lemma 6.11. *The following holds uniformly for all $t \ge 0$:*

$$\mathbb{E}\left[\left|\left\langle Z_t^N, \phi\right\rangle\right|\right] \lesssim r_N^{1-d/2} c_N^{1/2}(\|\phi\|_1 + r_N^{d/2}\|\phi\|_2).$$

Proof. Recall the expression for $\left\langle Z_t^N, \phi\right\rangle$ in (6.9) and the expression for φ^N in (6.10); using Lemma 6.4 and Lemma 4.5, we can write

$$\mathbb{E}\left[\left|\left\langle Z_t^N, \phi\right\rangle\right|\right] \lesssim \frac{(\tau_N/\eta_N)^{1/2}}{r_N^d}\int_0^t \|\phi\|_1 e^{-F'(\lambda)(t-s)}\mathrm{d}s + \left(\int_0^t e^{-2F'(\lambda)(t-s)}\left\|G_{D_N(t-s)}^{(r_N)} * \phi\right\|_2^2 \mathrm{d}s\right)^{1/2}.$$

Replacing ϕ by $(\phi_{1/r_N})_{r_N}$ - as defined in (6.4) - to use (6.15) and then looking at the proof of (6.16) in the proof of Theorem 6.3, we see that

$$\int_0^t e^{-2F'(\lambda)(t-s)}\left\|G_{D_N(t-s)}^{(r_N)} * \phi\right\|_2^2 \mathrm{d}s \lesssim r_N^{2-d}c_N(\left\|\phi_{1/r_N}\right\|_1^2 + \left\|\phi_{1/r_N}\right\|_2^2).$$

But $\left\|\phi_{1/r_N}\right\|_1 = \|\phi\|_1$ and $\left\|\phi_{1/r_N}\right\|_2 = r_N^{d/2}\|\phi\|_2$, hence

$$\mathbb{E}\left[\left|\left\langle Z_t^N, \phi\right\rangle\right|\right] \lesssim \|\phi\|_1 \frac{(\tau_N/\eta_N)^{1/2}}{r_N^d} + r_N^{1-d/2}c_N^{1/2}(\|\phi\|_1 + r_N^{d/2}\|\phi\|_2),$$

and we have the required result since $\tau_N/\eta_N = o\left(r_N^{d+2}\right)$. □

Proof of Lemma 6.5. We drop the superscript N from φ^N throughout the proof and take averages over the radius $r := r_N$. Recall from the expressions for Q^N in (6.7) and $\rho^{(r)}$ in (3.10) that the variance of the stochastic integral $\int_0^t\int_{\mathbb{R}^d}\varphi(x,s,t)\, M^N(\mathrm{d}x\mathrm{d}s)$ is given by

$$\int_0^t\int_{(\mathbb{R}^d)^3}\frac{1}{V_r^2}\mathbb{1}_{\substack{|x-z_1|<r\\|x-z_2|<r}}\varphi(z_1,s,t)\varphi(z_2,s,t)\mathbb{E}\Big[\overline{w_{s/\eta}^N}(x,r_N)^2(1-w_{s/\eta}^N(z_1))(1-w_{s/\eta}^N(z_2)) + 2\overline{w_{s/\eta}^N}(x,r_N)(1-\overline{w_{s/\eta}^N}(x,r_N))(\tfrac{1}{2}-w_{s/\eta}^N(z_1))(\tfrac{1}{2}-w_{s/\eta}^N(z_2)) + (1-\overline{w_{s/\eta}^N}(x,r_N))^2 w_{s/\eta}^N(z_1)w_{s/\eta}^N(z_2)\Big]\mathrm{d}x\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}s + \mathcal{O}\left(\delta_N^2\right)\int_0^t\|\varphi(s,t)\|_2^2\,\mathrm{d}s,$$

which can also be written

$$\int_0^t \mathbb{E}\left[\left\langle (\overline{w^N_{s/\eta}})^2, \left(\overline{(1-w^N_{s/\eta})\varphi(s,t)}\right)^2\right\rangle + \left\langle 2\overline{w^N_{s/\eta}}(1-\overline{w^N_{s/\eta}}), \left(\overline{(\tfrac{1}{2}-w^N_{s/\eta})\varphi(s,t)}\right)^2\right\rangle \right.$$
$$\left. + \left\langle (1-\overline{w^N_{s/\eta}})^2, \left(\overline{w^N_{s/\eta}\varphi(s,t)}\right)^2\right\rangle + \mathcal{O}\left(\delta_N^2 \left\|\varphi(s,t)\right\|_2^2\right)\right] \mathrm{d}s. \tag{6.21}$$

We want to show that in this expression, $w^N_{s/\eta}$ can (asymptotically) be replaced by λ, hence we write

$$\left\langle (\overline{w^N_{t/\eta}})^2, \left(\overline{(1-w^N_{t/\eta})\varphi}\right)^2\right\rangle - \left\langle \lambda^2, (1-\lambda)^2\overline{\varphi}^2\right\rangle$$
$$= \left\langle (\overline{w^N_{t/\eta}})^2 - \lambda^2, \left(\overline{(1-w^N_{t/\eta})\varphi}\right)^2\right\rangle + \left\langle \lambda^2, \left(\overline{(1-w^N_{t/\eta})\varphi}\right)^2 - (1-\lambda)^2\overline{\varphi}^2\right\rangle.$$

Since $(\overline{w^N_{t/\eta}})^2 - \lambda^2 = (\tau_N/\eta_N)^{1/2}\overline{Z^N_t}(\overline{w^N_{t/\eta}} + \lambda)$, using Lemma 6.4,

$$\mathbb{E}\left[\left|\left\langle (\overline{w^N_{t/\eta}})^2 - \lambda^2, \left(\overline{(1-w^N_{t/\eta})\varphi}\right)^2\right\rangle\right|\right] \leq 2(\tau_N/\eta_N)^{1/2}\left\langle \mathbb{E}\left[(\overline{Z^N_t})^2\right]^{1/2}, \overline{|\varphi|}^2\right\rangle$$
$$\lesssim \frac{(\tau_N/\eta_N)^{1/2}}{r_N^{d/2}}\left\|\varphi\right\|_2^2 = o\left(r_N\left\|\varphi\right\|_2^2\right).$$

In addition,

$$\left\langle \lambda^2, \left(\overline{(1-w^N_{t/\eta})\varphi}\right)^2 - (1-\lambda)^2\overline{\varphi}^2\right\rangle$$
$$= \lambda^2\int_{(\mathbb{R}^d)^3}\frac{1}{V_r^2}\mathbb{1}_{\substack{|x-z_1|<r\\|x-z_2|<r}}\varphi(z_1)\varphi(z_2)\Big\{(1-w^N_{t/\eta}(z_1))(1-w^N_{t/\eta}(z_2)) - (1-\lambda)^2$$
$$+ (1-\lambda)(1-w^N_{t/\eta}(z_1)) - (1-\lambda)(1-w^N_{t/\eta}(z_2))\Big\}\mathrm{d}x\mathrm{d}z_1\mathrm{d}z_2$$
$$= \lambda^2\int_{(\mathbb{R}^d)^3}\frac{1}{V_r^2}\mathbb{1}_{\substack{|x-z_1|<r\\|x-z_2|<r}}\varphi(z_1)\varphi(z_2)((1-w^N_{t/\eta}(z_1))-(1-\lambda))(1-w^N_{t/\eta}(z_2)+1-\lambda)\mathrm{d}x\mathrm{d}z_1\mathrm{d}z_2.$$

(The last two terms inside the curly braces cancel out by permuting z_1 and z_2.) Thus,

$$\left|\left\langle \lambda^2, \left(\overline{(1-w^N_{t/\eta})\varphi}\right)^2 - (1-\lambda)^2\overline{\varphi}^2\right\rangle\right| \leq 2\lambda^2(\tau_N/\eta_N)^{1/2}\int_{\mathbb{R}^d}|\varphi(z_2)|\left|\left\langle Z^N_t, \psi^N_{z_2}\right\rangle\right|\mathrm{d}z_2,$$

where $\psi^N_{z_2}(z_1) = \frac{V_r(z_1,z_2)}{V_r^2}\varphi(z_1)$. In particular,

$$\left\|\psi^N_{z_2}\right\|_1 = \overline{\overline{|\varphi|}}(z_2, r_N), \qquad \text{and} \qquad \left\|\psi^N_{z_2}\right\|_2^2 \leq \frac{1}{V_{r_N}}\overline{\overline{|\varphi|^2}}(z_2, r_N). \tag{6.22}$$

By Lemma 6.11, we get

$$\mathbb{E}\left[\int_{\mathbb{R}^d}|\varphi(z_2)|\left|\left\langle Z^N_t, \psi_{z_2}\right\rangle\right|\mathrm{d}z_2\right] \lesssim r_N^{1-d/2}c_N^{1/2}\int_{\mathbb{R}^d}|\varphi(z_2)|\left(\left\|\psi^N_{z_2}\right\|_1 + r_N^{d/2}\left\|\psi^N_{z_2}\right\|_2\right)\mathrm{d}z_2$$
$$\lesssim r_N^{1-d/2}c_N^{1/2}\left\|\varphi\right\|_2\left(\int_{\mathbb{R}^d}(\left\|\psi^N_{z_2}\right\|_1^2 + r_N^d\left\|\psi^N_{z_2}\right\|_2^2)\mathrm{d}z_2\right)^{1/2},$$

using the Cauchy-Schwartz inequality in the second line. By (6.22),

$$\left(\int_{\mathbb{R}^d}(\left\|\psi^N_{z_2}\right\|_1^2 + r_N^d\left\|\psi^N_{z_2}\right\|_2^2)\mathrm{d}z_2\right)^{1/2} \leq \left(\int_{\mathbb{R}^d}\left(\overline{\overline{|\phi|^2}}(z_2, r_N) + \frac{1}{V_1}\overline{\overline{|\phi|^2}}(z_2, r_N)\right)\mathrm{d}z_2\right)^{1/2}$$
$$\lesssim \left\|\varphi\right\|_2.$$

Since $\tau_N/\eta_N = o\left(r_N^{d+2}\right)$,

$$\mathbb{E}\left[\left|\left\langle \lambda^2, \left(\overline{(1-w_{t/\eta}^N)\varphi}\right)^2 - (1-\lambda)^2\overline{\varphi}^2\right\rangle\right|\right] = o\left(r_N^2 c_N^{1/2} \left\|\varphi\right\|_2^2\right).$$

We use a similar argument for the other terms in (6.21) to show that replacing $w_{s/\eta}^N$ by λ makes a difference of $o\left(r_N^2 c_N^{1/2} \left\|\varphi\right\|_2^2\right)$. We have thus shown that, since $r_N c_N^{1/2} \underset{N\to\infty}{\longrightarrow} 0$,

$$\mathbb{E}\Bigg[\left\langle (\overline{w_{s/\eta}^N})^2, \left(\overline{(1-w_{s/\eta}^N)\varphi}\right)^2\right\rangle + \left\langle 2\overline{w_{s/\eta}^N}(1-\overline{w_{s/\eta}^N}), \left(\overline{(\tfrac{1}{2}-w_{s/\eta}^N)\varphi}\right)^2\right\rangle$$
$$+ \left\langle (1-\overline{w_{s/\eta}^N})^2, \left(\overline{w_{s/\eta}^N\varphi}\right)^2\right\rangle\Bigg] = \tfrac{1}{2}\lambda(1-\lambda)\left\|\overline{\varphi}\right\|_2^2 + o\left(r_N\left\|\varphi\right\|_2^2\right),$$

uniformly in $s \geq 0$. The result follows from the bound on $\left\|\varphi^N\right\|_q$ in (6.11). ☐

A Approximating the (fractional) Laplacian

We start by stating some basic properties of averaged functions which are used throughout the paper.

Proposition A.1. *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ and $\psi : \mathbb{R}^d \to \mathbb{R}$ be in $L^{1,\infty}(\mathbb{R}^d)$. Then*

i) $\overline{\phi * \psi} = \overline{\phi} * \psi = \phi * \overline{\psi}$,

ii) $\langle\overline{\phi}, \psi\rangle = \langle\phi, \overline{\psi}\rangle$.

iii) If in addition $\phi \in L^q(\mathbb{R}^d)$, by Jensen's inequality, $\left\|\overline{\phi}\right\|_q \leq \left\|\phi\right\|_q$.

iv) If β is a multi-index in $\mathbb{N}_0^d$, and ϕ is differentiable enough that $\partial_\beta\phi$ is well defined on $\mathbb{R}^d$, then $\partial_\beta\overline{\phi} = \overline{\partial_\beta\phi}$.

v) Also, $\partial_\beta(\psi * \phi) = \psi * \partial_\beta\phi$.

We use here the notation $\lesssim$ defined in (5.1).

Proposition A.2. *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be twice continuously differentiable and suppose that $\left\|\partial_\beta\phi\right\|_q < \infty$ for $0 \leq |\beta| \leq 2$ and $1 \leq q \leq \infty$. Then*

i) $\left\|\overline{\phi}(r) - \phi\right\|_q \leq \frac{d}{2}r^2 \max_{|\beta|=2}\left\|\partial_\beta\phi\right\|_q$.

If in addition, ϕ admits $\left\|\cdot\right\|_q$-bounded derivatives of up to the fourth order,

ii) $\left\|\overline{\overline{\phi}}(r) - \phi - \frac{r^2}{d+2}\Delta\phi\right\|_q \leq \frac{d^3}{3}r^4 \max_{|\beta|=4}\left\|\partial_\beta\phi\right\|_q$.

Proof of Proposition A.2. By Taylor's theorem,

$$\phi(y) = \phi(x) + \sum_{i=1}^d \partial_i\phi(x)(y-x)_i + \sum_{i,j} R_{ij}(y)(y-x)_{ij},$$

where $R_{ij}(y) = \int_0^1 (1-t)\partial_{ij}\phi(x+t(y-x))\mathrm{d}t$ (we use the notation $x_{i_1\ldots i_k} = x_{i_1}\ldots x_{i_k}$). By symmetry, the average of the first sum over a ball of centre x and radius r vanishes, and

$$\left|\overline{\phi}(x,r) - \phi(x)\right| \leq \sum_{i,j}\frac{1}{V_r}\int_{B(x,r)} |R_{ij}(y)|\,|y-x|_{ij}\,\mathrm{d}y. \tag{A.1}$$

If $q=\infty$, then $|R_{ij}(y)| \leq \frac{1}{2}\left\|\partial_{ij}\phi\right\|_\infty$ and we write

$$\left\|\overline{\phi}(r)-\phi\right\|_\infty \leq \frac{1}{2} d \max_{|\beta|=2} \left\|\partial_\beta\phi\right\|_\infty \frac{1}{V_r}\int_{B(0,r)} |y|^2\,\mathrm{d}y = \frac{d^2}{2(d+2)} r^2 \max_{|\beta|=2}\left\|\partial_\beta \phi\right\|_\infty .$$

If instead $1 \leq q < \infty$, write

$$\begin{aligned}\left\|\overline{\phi}(r)-\phi\right\|_q &\leq \sum_{i,j}\left(\int_{\mathbb{R}^d}\left(\frac{1}{V_r}\int_{B(0,r)} |R_{ij}(x+y)|\,|y|_{ij}\,\mathrm{d}y\right)^q \mathrm{d}x\right)^{1/q}\\ &\leq \sum_{i,j}\left(\int_{\mathbb{R}^d}\left(\frac{1}{V_r}\int_{B(0,r)}|y|_{ij}\,\mathrm{d}y\right)^{q-1}\frac{1}{V_r}\int_{B(0,r)}|R_{ij}(x+y)|^q\,|y|_{ij}\,\mathrm{d}y\mathrm{d}x\right)^{1/q},\end{aligned}$$

by Jensen's inequality. But, by the definition of R_{ij}

$$\begin{aligned}\int_{\mathbb{R}^d}|R_{ij}(x+y)|^q\,\mathrm{d}x &\leq \frac{1}{2^{q-1}}\int_0^1 (1-t)\int_{\mathbb{R}^d}|\partial_{ij}\phi(x+ty)|^q\,\mathrm{d}x\mathrm{d}t\\ &= \frac{1}{2^q}\left\|\partial_{ij}\phi\right\|_q^q .\end{aligned}$$

Plugging this into the previous inequality, we get

$$\begin{aligned}\left\|\overline{\phi}(r)-\phi\right\|_q &\leq \sum_{i,j}\frac{1}{2}\left\|\partial_{ij}\phi\right\|_q\left(\left(\frac{1}{V_r}\int_{B(0,r)}|y|_{ij}\,\mathrm{d}y\right)^{q-1}\frac{1}{V_r}\int_{B(0,r)}|y|_{ij}\,\mathrm{d}y\right)^{1/q}\\ &\leq \frac{1}{2} d \max_{|\beta|=2}\left\|\partial_\beta\phi\right\|_q \frac{1}{V_r}\int_{B(0,r)}|y|^2\,\mathrm{d}y\\ &\leq \frac{d}{2} r^2 \max_{|\beta|=2}\left\|\partial_\beta\phi\right\|_q .\end{aligned}$$

The second inequality is proved in essentially the same way. We expand ϕ according to Taylor's theorem to the fourth order:

$$\begin{aligned}\phi(y) = \phi(x) &+ \sum_i \partial_i\phi(x)(y-x)_i + \frac{1}{2}\sum_{i,j}\partial_{ij}\phi(x)(y-x)_{ij}\\ &+ \frac{1}{3!}\sum_{i,j,k}\partial_{ijk}\phi(x)(y-x)_{ijk} + \sum_{ijkl} R_{ijkl}(y)(y-x)_{ijkl},\end{aligned}$$

where $R_{ijkl}(y) = \frac{1}{3!}\int_0^1 (1-t)^3 \partial_{ijkl}\phi(x+t(y-x))\mathrm{d}t$. Integrating, all the antisymmetric terms vanish and we obtain

$$\begin{aligned}\overline{\overline{\phi}}(x,r)-\phi(x) = \frac{1}{2}\sum_i \partial_{ii}\phi(x)\frac{1}{V_r^2}&\int_{(\mathbb{R}^d)^2}(y-x)_{ii}\mathbb{1}_{\substack{|x-z|<r\\|y-z|<r}}\mathrm{d}z\mathrm{d}y\\ &+\sum_{ijkl}\frac{1}{V_r^2}\int_{(\mathbb{R}^d)^2} R_{ijkl}(y)(y-x)_{ijkl}\mathbb{1}_{\substack{|x-z|<r\\|y-z|<r}}\mathrm{d}z\mathrm{d}y.\end{aligned}$$

We begin by calculating the first term before bounding the second one. Note that, by symmetry, the integral of $(y-x)_{ii}$ does not depend on i, so the first sum above can be written as

$$\frac{1}{2}\Delta\phi(x)\frac{1}{dV_r^2}\int_{(\mathbb{R}^d)^2}|y-x|^2\,\mathbb{1}_{\substack{|x-z|<r\\|y-z|<r}}\mathrm{d}z\mathrm{d}y.$$

By the parallelogram identity, $|y-x|^2 = 2(|x-z|^2 + |y-z|^2) - |2z-(x+y)|^2$. Integrating, we see that

$$\frac{1}{V_r^2}\int_{(\mathbb{R}^d)^2} |y-x|^2 \mathbb{1}_{\substack{|x-z|<r\\|y-z|<r}} \mathrm{d}z\mathrm{d}y = 4\frac{1}{V_r}\int_{B(0,r)} |y|^2\,\mathrm{d}y - \frac{1}{V_r^2}\int_{(\mathbb{R}^d)^2} |(2z-y)-x|^2 \mathbb{1}_{\substack{|x-z|<r\\|(2z-y)-z|<r}} \mathrm{d}z\mathrm{d}y.$$

Setting $y' = 2z-y$ in the rightmost integral and moving this term to the left-hand side, we obtain

$$\frac{1}{V_r^2}\int_{(\mathbb{R}^d)^2} |y-x|^2 \mathbb{1}_{\substack{|x-z|<r\\|y-z|<r}} \mathrm{d}z\mathrm{d}y = \frac{2}{V_r}\int_{B(0,r)} |y|^2\,\mathrm{d}y = \frac{2d}{d+2}r^2.$$

Replacing this term in the equation above, we can write

$$\left|\overline{\overline{\phi}}(x,r) - \phi(x) - \frac{r^2}{d+2}\Delta\phi(x)\right| \le \sum_{ijkl}\frac{1}{V_r^2}\int_{(\mathbb{R}^d)^2} |R_{ijkl}(y)|\,|y-x|_{ijkl}\,\mathbb{1}_{\substack{|x-z|<r\\|y-z|<r}} \mathrm{d}z\mathrm{d}y.$$

Proceeding exactly as before and writing $|y|_{ijkl} \le \frac{1}{4}(|y_i|^4 + |y_j|^4 + |y_k|^4 + |y_l|^4)$, one shows that

$$\left\|\overline{\overline{\phi}}(r) - \phi - \frac{r^2}{d+2}\Delta\phi\right\|_q \le \frac{d^3}{4!}\max_{|\beta|=4}\|\partial_\beta\phi\|_q \sum_i \frac{1}{V_r^2}\int_{(\mathbb{R}^d)^2} |y_i|^4\,\mathbb{1}_{\substack{|-z|<r\\|y-z|<r}}\,\mathrm{d}z\mathrm{d}y.$$

Note that $\sum_i |y_i|^4 \le |y|^4$, and by the parallelogram identity, $|y|^4 + |2z-y|^4 \le 8(|z|^4 + |z-y|^4)$. As before, we can integrate on both sides:

$$\frac{1}{V_r^2}\int_{(\mathbb{R}^d)^2} |y|^4\,\mathbb{1}_{\substack{|-z|<r\\|y-z|<r}}\,\mathrm{d}z\mathrm{d}y + \frac{1}{V_r^2}\int_{(\mathbb{R}^d)^2} |2z-y|^4\,\mathbb{1}_{\substack{|-z|<r\\|(2z-y)-z|<r}}\,\mathrm{d}z\mathrm{d}y \le 16\frac{1}{V_r}\int_{B(0,r)} |y|^4\,\mathrm{d}y.$$

Hence,

$$\frac{1}{V_r^2}\int_{(\mathbb{R}^d)^2} |y|^4\,\mathbb{1}_{\substack{|-z|<r\\|y-z|<r}}\,\mathrm{d}z\mathrm{d}y \le 8\frac{1}{V_r}\int_{B(0,r)} |y|^4\,\mathrm{d}y = \frac{8d}{d+4}r^4.$$

□

Proposition A.3. *Take $\phi : \mathbb{R}^d \to \mathbb{R}$ to be twice continuously differentiable and suppose that $\|\partial_\beta\phi\|_q < \infty$ for $0 \le |\beta| \le 2$ and $q \in \{1,\infty\}$. Then*

i) $\left\|\mathcal{D}^{\alpha,\delta}\phi\right\|_q \lesssim \|\phi\|_q + \max_{|\beta|=2}\|\partial_\beta\phi\|_q$,

ii) $\left\|\mathcal{D}^{\alpha,\delta}\phi - \mathcal{D}^{\alpha}\phi\right\|_q \lesssim \delta^{2-\alpha}\max_{|\beta|=2}\|\partial_\beta\phi\|_q$.

Further if $0 \le \phi \le 1$,

iii) $\left\|F^{(\delta)}(\phi) - F(\phi)\right\|_\infty \lesssim \delta^\alpha\left(1 + \max_{|\beta|=1}\|\partial_\beta\phi\|_\infty^2 + \max_{|\beta|=2}\|\partial_\beta\phi\|_\infty\right)$.

Proof of Proposition A.3. From the definition of $\mathcal{D}^{\alpha,\delta}$ and $\Phi^{(\delta)}$ in (2.9), and then changing the order of integration,

$$\begin{aligned}\mathcal{D}^{\alpha,\delta}\phi(x) &= \int_{\mathbb{R}^d}\int_{\frac{|x-y|}{2}\vee\delta}^{\infty} \frac{V_r(x,y)}{V_r}(\phi(y)-\phi(x))\frac{\mathrm{d}r}{r^{d+\alpha+1}}\mathrm{d}y \\ &= \int_\delta^\infty \int_{\mathbb{R}^d} \mathbb{1}_{|x-y|<2r}\frac{1}{V_r r^d}\int_{\mathbb{R}^d}\mathbb{1}_{\substack{|z-x|<r\\|z-y|<r}}(\phi(y)-\phi(x))\mathrm{d}z\mathrm{d}y\frac{\mathrm{d}r}{r^{\alpha+1}} \\ &= V_1\int_\delta^\infty\left(\overline{\overline{\phi}}(x,r) - \phi(x)\right)\frac{\mathrm{d}r}{r^{\alpha+1}}.\end{aligned}$$

The last line follows by noting that $\mathbb{1}_{|x-y|<2r}\mathbb{1}_{|x-z|<r,\,|y-z|<r} = \mathbb{1}_{|x-z|<r,\,|y-z|<r}$, and then changing the order of integration again to integrate first with respect to y and then with respect to z. Assume that $\delta < 1$ (otherwise simply ignore the first term below). Then for $q \in \{1, \infty\}$, by Proposition A.1.iii,

$$\left\|\mathcal{D}^{\alpha,\delta}\phi\right\|_q \leq V_1 \int_\delta^1 \left\|\overline{\overline{\phi}}(r) - \phi\right\|_q \frac{\mathrm{d}r}{r^{\alpha+1}} + 2V_1 \left\|\phi\right\|_q \int_1^\infty \frac{\mathrm{d}r}{r^{\alpha+1}}.$$

By the triangular inequality and Proposition A.1.iii,

$$\begin{aligned}\left\|\overline{\overline{\phi}} - \phi\right\|_q &\leq \left\|\overline{\overline{\phi} - \phi}\right\|_q + \left\|\overline{\phi} - \phi\right\|_q \\ &\leq 2\left\|\overline{\phi} - \phi\right\|_q.\end{aligned}$$

Hence by Proposition A.2,

$$\begin{aligned}\left\|\mathcal{D}^{\alpha,\delta}\phi\right\|_q &\lesssim \max_{|\beta|=2} \left\|\partial_\beta \phi\right\|_q \int_\delta^1 r^2 \frac{\mathrm{d}r}{r^{\alpha+1}} + \left\|\phi\right\|_q \int_1^\infty \frac{\mathrm{d}r}{r^{\alpha+1}} \\ &\lesssim \max_{|\beta|=2} \left\|\partial_\beta \phi\right\|_q + \left\|\phi\right\|_q.\end{aligned}$$

Likewise, we have

$$\mathcal{D}^\alpha \phi(x) - \mathcal{D}^{\alpha,\delta}\phi(x) = V_1 \int_0^\delta \left(\overline{\overline{\phi}}(x,r) - \phi(x)\right) \frac{\mathrm{d}r}{r^{\alpha+1}}.$$

By Proposition A.2, we then write

$$\begin{aligned}\left\|\mathcal{D}^\alpha\phi - \mathcal{D}^{\alpha,\delta}\phi\right\|_q &\leq V_1 \int_0^\delta \left\|\overline{\overline{\phi}}(r) - \phi\right\|_q \frac{\mathrm{d}r}{r^{\alpha+1}} \\ &\lesssim \max_{|\beta|=2} \left\|\partial_\beta\phi\right\|_q \int_0^\delta r^2 \frac{\mathrm{d}r}{r^{\alpha+1}} \\ &\lesssim \delta^{2-\alpha} \max_{|\beta|=2} \left\|\partial_\beta\phi\right\|_q.\end{aligned}$$

The third statement is a rewording of the first one in a slightly different setting. Indeed by (2.8),

$$F^{(\delta)}(\phi)(x) - F(\phi(x)) = \alpha\delta^\alpha \int_\delta^\infty \left(\overline{F(\overline{\phi})}(x,r) - F(\phi(x))\right) \frac{\mathrm{d}r}{r^{\alpha+1}}.$$

Hence as in the proof of (i)

$$\left\|F^{(\delta)}(\phi) - F(\phi)\right\|_\infty \lesssim \delta^\alpha \left(\left\|F(\phi)\right\|_\infty + \max_{|\beta|=2}\left\|\partial_\beta F(\overline{\phi})\right\|_\infty + \left\|F'\right\|_\infty \max_{|\beta|=2}\left\|\partial_\beta\phi\right\|_\infty\right).$$

The last term appears because there is an average inside the function F. The result then follows from the fact that $\partial_{ij}F(\phi) = \partial_{ij}\phi\, F'(\phi) + \partial_i\phi\,\partial_j\phi\, F''(\phi)$. □

B The centering term

B.1 The Brownian case

Proof of Lemma 2.1. Fix $N \geq 1$ and let $r = r_N$. Define an operator $S : L^\infty([0,T]\times\mathbb{R}^d) \to L^\infty([0,T]\times\mathbb{R}^d)$ by

$$S(g)(t,x) = G_t^{(r)} * w_0(x) - \int_0^t G_{t-s}^{(r)} * \overline{F(\overline{g(s)})}(x,r)\mathrm{d}s.$$

Define the norm $\|g\|_{[0,T]} = \sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d}|g(t,x)|$; then S is Lipschitz on $L^\infty([0,T]\times\mathbb{R}^d)$ with respect to this norm, since (as $\left\|G_{t-s}^{(r)}*\phi\right\|_\infty \le \|\phi\|_\infty$ and by Proposition A.1.iii)

$$\|S(f)-S(g)\|_{[0,T]} \le T\|F'\|_\infty \|f-g\|_{[0,T]}. \tag{B.1}$$

Let us choose for now $T>0$ small enough that $k=T\|F'\|_\infty<1$. The results can be extended to arbitrarily large time intervals by iterating the argument on $[T,2T]$, $[2T,3T]$ and so on. The operator S is then a contraction in $L^\infty([0,T]\times\mathbb{R}^d)$ and admits a unique fixed point in this space. This fixed point is precisely f^N (see (4.7)). Define a sequence $(g_n)_{n\ge 0}$ of functions in $L^\infty([0,T]\times\mathbb{R}^d)$ by

$$\begin{cases} g_{n+1} = S(g_n), \\ g_0(t,x) = w_0(x). \end{cases}$$

Note that since w_0 admits spatial derivatives of order up to four, so does g_n for each n. A Picard iteration argument then yields the convergence of g_n to f^N in $L^\infty([0,T]\times\mathbb{R}^d)$. More precisely,

$$\left\|g_n-f^N\right\|_{[0,T]} \le k\left\|g_{n-1}-f^N\right\|_{[0,T]} \le \ldots \le k^n\left\|w_0-f^N\right\|_{[0,T]}. \tag{B.2}$$

Fix $1\le i\le d$ and set, for g,h in $L^\infty([0,T],\mathbb{R}^d)$,

$$S^1(g,h)(t,x) = G_t^{(r)}*\partial_i w_0(x) - \int_0^t G_{t-s}^{(r)}*\overline{F'(\overline{g(s)})\overline{h(s)}}(x)\mathrm{d}s.$$

Then by Proposition A.1.v, $\partial_i g_{n+1} = S^1(g_n,\partial_i g_n)$. In addition, for h_1,h_2,h_3 in $L^\infty([0,T]\times\mathbb{R}^d)$,

$$\left\|S^1(h_1,h_2)-S^1(h_1,h_3)\right\|_{[0,T]} \le T\|F'\|_\infty\|h_2-h_3\|_{[0,T]}, \tag{B.3}$$
$$\left\|S^1(h_1,h_2)-S^1(h_3,h_2)\right\|_{[0,T]} \le T\|F''\|_\infty\|h_2\|_{[0,T]}\|h_1-h_3\|_{[0,T]}. \tag{B.4}$$

Hence $S^1(g,\cdot)$ is a contraction in $L^\infty([0,T]\times\mathbb{R}^d)$ for any g in $L^\infty([0,T]\times\mathbb{R}^d)$. Let us call $\tilde g$ the unique fixed point of $S^1(f^N,\cdot)$ in this space. We shall now show that $\partial_i f^N$ exists and is equal to $\tilde g\in L^\infty([0,T]\times\mathbb{R}^d)$. Adapting the argument of the Picard iteration and using the inequalities (B.3) and (B.4) above, we write for $n\ge 1$,

$$\begin{aligned}\|\partial_i g_n-\tilde g\|_{[0,T]} &= \left\|S^1(g_{n-1},\partial_i g_{n-1})-S^1(f^N,\tilde g)\right\|_{[0,T]}\\ &\le T\|F'\|_\infty\|\partial_i g_{n-1}-\tilde g\|_{[0,T]} + T\|F''\|_\infty\|\tilde g\|_{[0,T]}\left\|g_{n-1}-f^N\right\|_{[0,T]}\\ &\le k\|\partial_i g_{n-1}-\tilde g\|_{[0,T]} + T\|F''\|_\infty\|\tilde g\|_{[0,T]}k^{n-1}\left\|w_0-f^N\right\|_{[0,T]},\end{aligned}$$

where $k=T\|F'\|_\infty<1$ and the last term is bounded using (B.2). Iterating yields

$$\|\partial_i g_n-\tilde g\|_{[0,T]} \le k^n\|\partial_i w_0-\tilde g\|_{[0,T]} + nk^{n-1}T\|F''\|_\infty\|\tilde g\|_{[0,T]}\left\|w_0-f^N\right\|_{[0,T]}.$$

Hence $\partial_i g_n$ converges to $\tilde g$ uniformly on $[0,T]\times\mathbb{R}^d$ (recall that we assumed $\partial_i w_0\in L^\infty(\mathbb{R}^d)$). Since we already showed in (B.2) that g_n converges uniformly to f^N, this implies that $\partial_i f^N=\tilde g\in L^\infty([0,T]\times\mathbb{R}^d)$. The proof for higher order derivatives of f^N is similar and we omit the details. □

Proof of Proposition 4.7. Recall the following expression for f^N from (4.7),

$$f_t^N(x) = G_t^{(r)}*w_0(x) - \int_0^t G_{t-s}^{(r)}*\overline{F(\overline{f_s^N})}(x)\mathrm{d}s. \tag{B.5}$$

Since $\left\|G_t^{(r)} * \phi\right\|_\infty \leq \|\phi\|_\infty$, it follows that $\left\|f_t^N\right\|_\infty \leq \|w_0\|_\infty + T\|F\|_\infty$ for $t \leq T$. We can now prove the second part of the statement by induction on $|\beta|$. (Recall that β is a multi-index $(\beta_1, \dots, \beta_d)$ in $\mathbb{N}_0^d$ and that $|\beta| = \beta_1 + \dots + \beta_d$.) Suppose that for every β' with $0 \leq |\beta'| < k \leq 4$, there exists a constant $K_{\beta'} < \infty$ independent of N such that $\sup_{0\leq t\leq T} \left\|\partial_{\beta'} f_t^N\right\|_\infty \leq K_{\beta'}$; take β such that $|\beta| = k$. (From now on we omit the superscript N in the induction proof.) Note that for some constants $C_{\alpha_1,\dots,\alpha_i} \in \mathbb{N}$,

$$\partial_\beta F(f) = \sum_{i\geq 1} F^{(i)}(f) \left(\sum_{\alpha_1+\dots+\alpha_i=\beta} C_{\alpha_1,\dots,\alpha_i} \partial_{\alpha_1} f \dots \partial_{\alpha_i} f \right)$$

where the second sum is over all possible multisets with i elements of non-zero multi-indices $(\alpha_1, \dots, \alpha_i)$ in $\mathbb{N}^d \setminus \{(0, \dots, 0)\}$ such that $\alpha_1 + \dots + \alpha_i = \beta$ (computing the sum coordinate by coordinate). Also, w_0 is assumed to have uniformly bounded derivatives of up to the fourth order. Using Proposition A.1 (iv and v), we can differentiate on both sides of (B.5) and we obtain

$$\begin{aligned}\partial_\beta f_t(x) &= G_t^{(r)} * \partial_\beta w_0(x) \\ &\quad - \int_0^t G_{t-s}^{(r)} * \left(C_\beta \overline{F'(\overline{f_s})\overline{\partial_\beta f_s}} + \sum_{\substack{\alpha_1+\dots+\alpha_i=\beta \\ i\geq 2}} C_{\alpha_1,\dots,\alpha_i} \overline{F^{(i)}(\overline{f_s})\overline{\partial_{\alpha_1} f_s} \dots \overline{\partial_{\alpha_i} f_s}} \right)(x)\mathrm{d}s.\end{aligned}$$

The sum is uniformly bounded by a constant K by the induction hypothesis, and so, using the fact that $\left\|G_t^{(r)} * \phi\right\|_\infty \leq \|\phi\|_\infty$,

$$\|\partial_\beta f_t\|_\infty \leq \|\partial_\beta w_0\|_\infty + TK + C_\beta \|F'\|_\infty \int_0^t \|\partial_\beta f_s\|_\infty \,\mathrm{d}s.$$

The function $t \mapsto \|\partial_\beta f_t\|_\infty$ is bounded on $[0, T]$ by Lemma 2.1. We can therefore apply Gronwall's inequality to conclude

$$\|\partial_\beta f_t\|_\infty \leq \left(\|\partial_\beta w_0\|_\infty + TK\right) e^{C_\beta \|F'\| T},$$

where the right hand side is independent of both $t \in [0, T]$ and $N \geq 1$. We can now prove the first statement using Gronwall's inequality again, together with Proposition A.2 and the first part of the proof.

Recall that G_t denotes the fundamental solution to the heat equation. Recalling that we set the constants uV_R, $2R^2/(d + 2)$ and s to 1, equations (2.5) and (3.16) can be written as

$$f_t^N(x) = G_t * w_0(x) + \int_0^t G_{t-s} * \left(\mathcal{L} f_s^N - \frac{1}{2}\Delta f_s^N - \overline{F(\overline{f_s^N})} \right)(x)\mathrm{d}s,$$

and

$$f_t(x) = G_t * w_0(x) + \int_0^t G_{t-s} * F(f_s)\mathrm{d}s.$$

Recall the definition of $\mathcal{L}^{(r)}$ in (2.4); by Proposition A.2,

$$\left\| \mathcal{L} f_s^N - \frac{1}{2}\Delta f_s^N \right\|_\infty \leq \frac{d^3(d+2)}{6} r_N^2 \max_{|\beta|=4} \left\|\partial_\beta f_s^N\right\|_\infty \lesssim r_N^2,$$

since $\max_{|\beta|=4} \left\|\partial_\beta f_s^N\right\|_\infty$ is uniformly bounded from the previous argument. Also by Proposition A.2,

$$\left\| \overline{F(\overline{f_s^N})} - F(f_s^N) \right\|_\infty \leq \frac{d}{2} r_N^2 \left(\max_{|\beta|=2} \left\|\partial_\beta F(\overline{f_s^N})\right\|_\infty + \|F'\|_\infty \max_{|\beta|=2} \left\|\partial_\beta f_s^N\right\|_\infty \right) \lesssim r_N^2.$$

(The term within brackets is uniformly bounded from the first part of the proof.) Finally, we also have

$$\left\|F(f_s^N) - F(f_s)\right\|_\infty \leq \|F'\|_\infty \left\|f_s^N - f_s\right\|_\infty .$$

Hence, using the fact that $\|G_t * \phi\|_\infty \leq \|\phi\|_\infty$, there exists a constant $C > 0$ such that, for $t \in [0, T]$,

$$\left\|f_t^N - f_t\right\|_\infty \leq C r_N^2 + \|F'\|_\infty \int_0^t \left\|f_s^N - f_s\right\|_\infty \mathrm{d}s.$$

Applying Gronwall's inequality (the function $t \mapsto \left\|f_t^N - f_t\right\|_\infty$ is bounded on $[0, T]$ by Lemma 2.1),

$$\left\|f_t^N - f_t\right\|_\infty \leq C e^{\|F'\| T} r_N^2.$$

□

B.2 The stable case

Proof of Lemma 2.6. Lemma 2.6 is proved exactly as Lemma 2.1 in the Brownian case. The corresponding operator $S : L^\infty([0, T] \times \mathbb{R}^d) \to L^\infty([0, T] \times \mathbb{R}^d)$ is given by

$$S(g)(t, x) = \mathcal{G}_t^{(\alpha,\delta_N)} * w_0(x) - \int_0^t \mathcal{G}_{t-s}^{(\alpha,\delta_N)} * F^{(\delta_N)}(f_s^N)(x) \mathrm{d}s.$$

It satisfies the same contraction property as (B.1), yielding the existence of a solution to (2.12) in $L^\infty([0, T] \times \mathbb{R}^d)$. The argument for the existence of spatial derivatives in $L^\infty([0, T] \times \mathbb{R}^d)$ is the same as in the Brownian case, with S^1 given by

$$S^1(g, h)(t, x) = \mathcal{G}_t^{(\alpha,\delta_N)} * \partial_i w_0(x) - \alpha \int_0^t \int_1^\infty \mathcal{G}_{t-s}^{(\alpha,\delta_N)} * \overline{F'(\overline{g(s)})\overline{h(s)}}(x, \delta_N r) \frac{\mathrm{d}r}{r^{1+\alpha}} \mathrm{d}s.$$

□

Proof of Proposition 5.6. The proof of the convergence of the centering term in the stable case goes along the same lines as in the Brownian case of Proposition 4.7. Differentiating (5.12) yields (dropping superscripts N)

$$\partial_\beta f_t(x) = \mathcal{G}_t^{(\alpha,\delta)} * \partial_\beta w_0(x) - \alpha \int_0^t \int_1^\infty \mathcal{G}_{t-s}^{(\alpha,\delta)} * \Bigg(C_\beta \overline{F'(\overline{f_s})\overline{\partial_\beta f_s}}(\delta r) + \sum_{\substack{\alpha_1 + \ldots + \alpha_k = \beta \\ k \geq 2}} C_{\alpha_1,\ldots,\alpha_k} \overline{F^{(k)}(\overline{f_s})\overline{\partial_{\alpha_1} f_s} \ldots \overline{\partial_{\alpha_k} f_s}}(\delta r) \Bigg)(x) \frac{\mathrm{d}r}{r^{\alpha+1}} \mathrm{d}s.$$

One can then proceed by induction as previously to show

$$\|\partial_\beta f_t\|_\infty \lesssim \|\partial_\beta w_0\|_\infty + T + \|F'\|_\infty \int_0^t \|\partial_\beta f_s\|_\infty \mathrm{d}s,$$

and Gronwall's inequality (using Lemma 2.6) yields the second part of the statement. For the first part, the proof is identical to that in the Brownian case, one simply has to replace the operators $\frac{1}{2}\Delta$ and $\mathcal{L}^{(r)}$ by $\mathcal{D}^\alpha$ and $\mathcal{D}^{\alpha,\delta}$, respectively, and likewise replace $\overline{F(\overline{f_t})}$ by $F^{(\delta)}(f_t)$. Proposition A.3 then yields the correct estimates on the corresponding error terms. □

C Time dependent test functions

C.1 The Brownian case

Proof of Lemma 4.2. In the spirit of the proof of Lemma 2.1, we characterize φ^N as the fixed point of a contraction in $L^\infty(\mathbb{R}^d \times \{(s,t) : 0 \leq s \leq t \leq T\})$. By the definition of φ^N in (4.2),

$$\varphi^N(x,s,t) = G^{(r_N)}_{t-s} * \phi(x) - \int_s^t G^{(r_N)}_{u-s} * \overline{F'(\overline{f^N_u})\overline{\varphi^N(u,t)}}(x)\mathrm{d}u. \tag{C.1}$$

In other words, φ^N is a fixed point of the following operator,

$$S(g)(x,s,t) = G^{(r_N)}_{t-s} * \phi(x) - \int_s^t G^{(r_N)}_{u-s} * \overline{F'(\overline{f^N_u})\overline{g(u,t)}}(x)\mathrm{d}u.$$

For $q \in [1,\infty]$, define the norm $\|g\|_{q,[0,T]} = \sup_{0\leq s\leq t\leq T} \|g(s,t)\|_q$; then since $G^{(r)}_{u-s}$ is a contraction in L^q and by Proposition A.1.iii,

$$\|S(h) - S(g)\|_{q,[0,T]} \leq T \|F'\|_\infty \|h - g\|_{q,[0,T]}$$

so for T small enough, S is a contraction. Note that the space of (equivalence classes of) measurable functions $g : \{(s,t) : 0 \leq s \leq t \leq T\} \to L^q(\mathbb{R}^d)$ such that $\|g\|_{q,[0,T]} < \infty$ is a Bochner space (and therefore a Banach space). Hence for each $q \in [1,\infty]$, there exists a unique fixed point of S which is uniformly bounded in $L^q(\mathbb{R}^d)$, obtained as the limit of the sequence

$$\begin{cases} g_{n+1} = S(g_n), \\ g_0(x,s,t) = \phi(x). \end{cases}$$

Since this sequence does not depend on q, the fixed point is the same for all q. This fixed point is φ^N. Proceeding as in the proof of Lemma 2.1, one shows that the spatial derivatives of g_n (of order up to four) converge uniformly to some function which is uniformly bounded in $L^q(\mathbb{R}^d)$ for all $q \in [1,\infty]$. As a result φ^N admits spatial derivatives of order up to four which are uniformly bounded in $L^q(\mathbb{R}^d)$ for $q \in [1,\infty]$. □

Proof of Lemma 4.3. The proof of Lemma 4.3 is similar in spirit to that of Proposition 4.7. We start by proving the bound on the derivatives of φ^N. Using the fact that $G^{(r)}_t$ is a contraction in L^q, we have, using (C.1), for $q = 1, 2$,

$$\begin{aligned} \left\|\varphi^N(s,t)\right\|_q^q &\leq 2^{q-1} \|\phi\|_q^q + (2(t-s))^{q-1} \int_s^t \left\|\overline{F'(\overline{f^N_u})\overline{\varphi^N(u,t)}}\right\|_q^q \mathrm{d}u \\ &\leq 2^{q-1} \|\phi\|_q^q + (2(t-s))^{q-1} \|F'\|_\infty^q \int_s^t \left\|\varphi^N(u,t)\right\|_q^q \mathrm{d}u, \end{aligned}$$

by Proposition A.1.iii. In addition, by Lemma 4.2, the function $s \mapsto \left\|\varphi^N(s,t)\right\|_q$ is bounded on $[0,t]$. By Gronwall's inequality, we conclude that

$$\left\|\varphi^N(s,t)\right\|_q \leq 2^{(q-1)/q} \|\phi\|_q \, e^{\frac{2^{q-1}}{q} T^q \|F'\|^q}.$$

Thus the statement holds for $\beta = 0$. We can then proceed by induction on $|\beta|$ as in the proof of Proposition 4.7 to show that the same holds for every $0 \leq |\beta| \leq 4$ (making use of the fact that by Proposition 4.7, f^N has uniformly bounded derivatives). We omit the details.

We are left with proving the convergence estimate for φ^N which is again a Gronwall estimate. As in the proof of Proposition 4.7, write (4.2) and (4.4) as

$$\begin{aligned} \varphi^N(x,s,t) &= G_{t-s} * \phi(x) \\ &\quad + \int_s^t G_{u-s} * \left(\mathcal{L}^{(r_N)} \varphi^N(u,t) - \frac{1}{2}\Delta\varphi^N(u,t) - \overline{F'(\overline{f^N_u})\overline{\varphi^N(u,t)}} \right)(x)\mathrm{d}u, \end{aligned} \tag{C.2}$$

and

$$\varphi(x,s,t) = G_{t-s} * \phi(x) - \int_s^t G_{u-s} * \left(F'(f_u)\varphi(u,t)\right)(x)\mathrm{d}u. \tag{C.3}$$

By Proposition A.2 and the bound on the spatial derivatives of φ^N,

$$\left\|\mathcal{L}^{(r_N)}\varphi^N(u,t) - \frac{1}{2}\Delta\varphi^N(u,t)\right\|_q \lesssim r_N^2.$$

Still by Proposition A.2, (omitting superscripts N and time variables)

$$\begin{aligned}&\left\|\overline{F'(\overline{f})\overline{\varphi}} - F'(f)\varphi\right\|_q \\ &\quad \leq \frac{d}{2}r_N^2\left(\max_{|\beta|=2}\left\|\partial_\beta(F'(\overline{f})\overline{\varphi})\right\|_q + \|F'\|_\infty \max_{|\beta|=2}\|\partial_\beta\varphi\|_q + \|\varphi\|_q\,\|F''\|_\infty \max_{|\beta|=2}\|\partial_\beta f\|_\infty\right).\end{aligned}$$

The last term inside the brackets is uniformly bounded by Proposition 4.7 and the second to last is bounded as a consequence of the first part of the proof. Also, $\partial_{ij}(F'(\overline{f})\overline{\varphi})$ is dominated by a linear combination of (averages of) derivatives of both f and φ. The latter are bounded in L^q while the former are bounded in L^∞, hence the first term within the brackets is also uniformly bounded. To sum up,

$$\left\|\overline{F'(\overline{f_u^N})\overline{\varphi^N(u,t)}} - F'(f_u^N)\varphi^N(u,t)\right\|_q \lesssim r_N^2. \tag{C.4}$$

Finally, by Proposition 4.7,

$$\left\|F'(f_u^N) - F'(f_u)\right\|_q \lesssim r_N^2.$$

Hence, subtracting (C.3) from (C.2) and using Jensen's inequality as above with the L^q-contraction property of G_t, we have, for $t \in [0,T]$,

$$\left\|\varphi^N(s,t) - \varphi(s,t)\right\|_q^q \lesssim r_N^{2q} + \int_s^t \left\|\varphi^N(u,t) - \varphi(u,t)\right\|_q^q \mathrm{d}u.$$

Also, by Lemma 4.2, the function $s \mapsto \left\|\varphi^N(s,t) - \varphi(s,t)\right\|_q$ is bounded on $[0,t]$. We conclude with Gronwall's inequality, yielding the first statement of Lemma 4.3. □

Proof of Lemma 4.10. We can assume that $t' > t \geq s$ (if $t' \geq s \geq t$, then $\varphi^N(s,t) = \phi = \varphi^N(s,s)$ and the problem reduces to bounding $\varphi^N(s,t') - \varphi^N(s,s)$). Using (C.1), we write

$$\begin{aligned}\varphi^N(x,s,t') - \varphi^N(x,s,t) &= G_{t'-s}^{(r_N)} * \phi(x) - G_{t-s}^{(r_N)} * \phi(x) - \int_t^{t'} G_{u-s}^{(r_N)} * \overline{F'(\overline{f_u^N})\overline{\varphi^N(u,t')}}(x)\mathrm{d}u \\ &\quad - \int_s^t G_{u-s}^{(r_N)} * \left(\overline{F'(\overline{f_u^N})(\overline{\varphi^N(u,t')} - \overline{\varphi^N(u,t)})}\right)(x)\mathrm{d}u.\end{aligned}$$

From the way we extended φ^N in (4.13), we see that for $u \geq t'$, $\varphi^N(u,t') - \varphi^N(u,t) = 0$ and for $t \leq u \leq t'$, $\varphi^N(u,t') - \varphi^N(u,t) = \varphi^N(u,t') - \phi$, so (omitting superscripts N)

$$\begin{aligned}\varphi(x,s,t') - \varphi(x,s,t) &= G_{t'-s}^{(r)} * \phi(x) - G_{t-s}^{(r)} * \phi(x) - \int_t^{t'} G_{u-s}^{(r)} * \overline{F'(\overline{f_u})\overline{\phi}}(x)\mathrm{d}u \\ &\quad - \int_s^T G_{u-s}^{(r)} * \left(\overline{F'(\overline{f_u})(\overline{\varphi(u,t')} - \overline{\varphi(u,t)})}\right)(x)\mathrm{d}u.\end{aligned}$$

Again, we use the L^q-contraction property of $G_t^{(r)}$ to write

$$\begin{aligned}\|\varphi(s,t') - \varphi(s,t)\|_q^q &\leq 3^{q-1}\left\|G_{t'-s}^{(r)} * \phi - G_{t-s}^{(r)} * \phi\right\|_q^q + 3^{q-1}\,|t'-t|^q\,\|F'\|_\infty^q\,\|\phi\|_q^q \\ &\quad + (3(T-s))^{q-1}\,\|F'\|_\infty^q \int_s^T \|\varphi(u,t') - \varphi(u,t)\|_q^q\,\mathrm{d}u. \end{aligned} \tag{C.5}$$

We need a bound on the first term; recalling the definition of $G^{(r)}$ in Subsection 4.2, we have

$$G^{(r)}_{t'-s} * \phi(x) - G^{(r)}_{t-s} * \phi(x) = \int_t^{t'} G^{(r)}_{u-s} * \mathcal{L}^{(r)}\phi(x)\mathrm{d}u.$$

By Jensen's inequality,

$$\begin{aligned}\left\|G^{(r)}_{t'-s} * \phi - G^{(r)}_{t-s} * \phi\right\|_q^q &\le |t'-t|^{q-1}\int_t^{t'} \left\|\mathcal{L}^{(r)}\phi\right\|_q^q \mathrm{d}u \\ &\lesssim |t'-t|^q,\end{aligned}$$

by Proposition A.2. Hence, returning to (C.5),

$$\|\varphi(s,t') - \varphi(s,t)\|_q^q \lesssim |t'-t|^q + \int_s^T \|\varphi(u,t')-\varphi(u,t)\|_q^q \,\mathrm{d}u.$$

Noting that from Lemma 4.2, we know that $s \mapsto \|\varphi(s,t') - \varphi(s,t)\|_q$ is bounded on $[0,T]$, Gronwall's inequality yields the result. □

Proof of Lemma 4.11. By the definition of $G^{(r)}$,

$$G^{(r_N)}_{t-s} * \phi(x) = \phi(x) + \int_s^t G^{(r_N)}_{u-s} * \mathcal{L}^{(r_N)}\phi(x)\mathrm{d}u.$$

Hence

$$\left\|\sup_{t\in[s,T]} G^{(r_N)}_{t-s} * \phi\right\|_1 \le \|\phi\|_1 + (T-s)\left\|\mathcal{L}^{(r_N)}\phi\right\|_1 \le \|\phi\|_1 + T\frac{d(d+2)}{2}\max_{|\beta|=2}\|\partial_\beta\phi\|_1 \tag{C.6}$$

(we have used Proposition A.2.i to bound $\left\|\mathcal{L}^{(r_N)}\phi\right\|_1$ independently of N). Recall from (C.1) that

$$\varphi^N(x,s,t) = G^{(r_N)}_{t-s} * \phi(x) - \int_s^t G^{(r_N)}_{u-s} * \overline{F'(\overline{f_u^N})\overline{\varphi^N(u,t)}}(x)\mathrm{d}u.$$

Within the second integral, $u \le t$, so we can write $\left|\varphi^N(u,t)\right| \le \sup_{t'\in[u,T]}\left|\varphi^N(u,t')\right|$. Thus (omitting superscripts and subscripts) by Proposition A.1.i,

$$\sup_{t\in[s,T]}|\varphi(x,s,t)| \le \sup_{t\in[s,T]}|G^{(r)}_{t-s} * \phi(x)| + \|F'\|_\infty \int_s^T \overline{\overline{G^{(r)}_{u-s}}} * \sup_{t\in[u,T]}|\varphi(u,t)|(x)\mathrm{d}u.$$

Integrating with respect to the variable $x \in \mathbb{R}^d$ yields

$$\left\|\sup_{t\in[s,T]}|\varphi(s,t)|\right\|_1 \le \|\phi\|_1 + T\frac{d(d+2)}{2}\max_{|\beta|=2}\|\partial_\beta\phi\|_1 + \|F'\|_\infty\int_s^T \left\|\sup_{t\in[u,T]}|\varphi(u,t)|\right\|_1 \mathrm{d}u.$$

Consider the space X of functions $g : \mathbb{R}^d \times [0,T]^2 \to \mathbb{R}$ such that $g(x,s,t) = g(x,t,t)$ for $s \ge t$ and the norm

$$\sup_{s\in[0,T]}\left\|\sup_{t\in[0,T]}|g(s,t)|\right\|_1$$

is finite. (As a closed subspace of a Bochner space, X is complete with respect to this norm.) Looking at the proof of Lemma 4.2, extend S to an operator on X by setting $S(g)(x,s,t) = \phi(x)$ for $s \ge t$. Then $S : X \to X$ (using (C.6)) and by the same argument as in the proof of Lemma 4.2, for T sufficiently small, S is a contraction on X. As a result we obtain that $s \mapsto \left\|\sup_{t\in[s,T]}|\varphi(s,t)|\right\|_1$ is bounded on $[0,T]$. Hence, by Gronwall's inequality,

$$\left\|\sup_{t\in[s,T]}|\varphi(s,t)|\right\|_1 \le \left(\|\phi\|_1 + T\frac{d(d+2)}{2}\max_{|\beta|=2}\|\partial_\beta\phi\|_1\right)e^{\|F'\|(T-s)}.$$

□

C.2 The stable case

Proof of Lemma 5.2. By the definition of φ^N in (5.6),

$$\varphi^N(x,s,t) = \mathcal{G}_{t-s}^{(\alpha,\delta_N)} * \phi(x) - \alpha \int_s^t \int_1^\infty \mathcal{G}_{u-s}^{(\alpha,\delta_N)} * \overline{F'(\overline{f_u^N})\overline{\varphi^N(u,t)}}(\delta_N r)(x) \frac{\mathrm{d}r}{r^{\alpha+1}} \mathrm{d}u. \quad (C.7)$$

Note that since we are only considering $q \in \{1, \infty\}$, we have

$$\left\| \int_1^\infty f(\cdot, r) \mathrm{d}r \right\|_q \leq \int_1^\infty \|f(\cdot, r)\|_q \, \mathrm{d}r.$$

Hence the bound on the derivatives of φ^N can be proved following the same argument as in the proof of Lemma 4.3 in the Brownian case, using Lemma 5.1 in place of Lemma 4.2. By the definition of φ,

$$\varphi(x,s,t) = \mathcal{G}_{t-s}^{(\alpha)} * \phi(x) - \int_s^t \mathcal{G}_{u-s}^{(\alpha)} * \left(F'(f_u)\varphi(u,t)\right)(x) \mathrm{d}u.$$

By Proposition A.3 and by the bound on the spatial derivatives of φ^N,

$$\left\| \mathcal{D}^{\alpha,\delta_N}\varphi^N(u,t) - \mathcal{D}^\alpha \varphi^N(u,t) \right\|_q \lesssim \delta_N^{2-\alpha}.$$

Using (C.4) (which is still true in this case by the bound on the derivatives of φ^N), we have

$$\begin{aligned} \int_1^\infty \left\| \overline{F'(\overline{f_u^N})\overline{\varphi^N(u,t)}}(\delta_N r) - F'(f_u^N)\varphi^N(u,t) \right\|_q \frac{\mathrm{d}r}{r^{\alpha+1}} &\lesssim \delta_N^2 \int_1^{\delta^{-1}} r^2 \frac{\mathrm{d}r}{r^{\alpha+1}} + \int_{\delta_N^{-1}}^\infty \frac{\mathrm{d}r}{r^{\alpha+1}} \\ &\lesssim \delta_N^\alpha. \end{aligned}$$

Finally, by Proposition 5.6,

$$\left\| F'(f_u^N) - F'(f_u) \right\|_q \lesssim \delta_N^{\alpha \wedge (2-\alpha)}.$$

As a result, by the same argument as in the proof of Lemma 4.3, by Gronwall's inequality,

$$\left\| \varphi^N(s,t) - \varphi(s,t) \right\|_q \lesssim \delta_N^{\alpha \wedge (2-\alpha)}.$$

□

Proof of Lemma 5.10. The argument for the continuity estimate is the same as in the proof of Lemma 4.10, using Proposition A.3. For the second bound, we use the same argument as in Lemma 4.11, again using Proposition A.3. □

Proof of Lemma 5.4. Splitting the integral with respect to z_2, we have

$$\begin{aligned} \int_{(\mathbb{R}^d)^2} |f(z_1)|\,|g(z_2)|\,|z_1 - z_2|^{-\alpha} \mathrm{d}z_1 \mathrm{d}z_2 \leq \|g\|_\infty \int_{\mathbb{R}^d} |f(z_1)| \int_{B(z_1,1)} |z_1 - z_2|^{-\alpha} \mathrm{d}z_2 \mathrm{d}z_1 \\ + \int_{\mathbb{R}^d} |f(z_1)| \int_{\mathbb{R}^d \setminus B(z_1,1)} |g(z_2)| \mathrm{d}z_2 \mathrm{d}z_1 \end{aligned}$$

But $\int_{B(0,1)} |y|^{-\alpha} \, \mathrm{d}y = \frac{dV_1}{d-\alpha}$ and we have :

$$\int_{(\mathbb{R}^d)^2} |f(z_1)|\,|g(z_2)|\,|z_1 - z_2|^{-\alpha} \mathrm{d}z_1 \mathrm{d}z_2 \leq \|g\|_\infty \frac{dV_1}{d-\alpha} \int_{\mathbb{R}^d} |f(z_1)| \, \mathrm{d}z_1 + \|g\|_1 \int_{\mathbb{R}^d} |f(z_1)| \, \mathrm{d}z_1.$$

□

D Estimates for drift load proofs

Proof of Lemma 6.7. For all $t > 0$, $\xi_t^{(r)}$ can be written as $\xi_t^{(r)} = \sum_{k=1}^{N_t} Y_k$, where $(N_t)_{t\geq 0}$ is a Poisson process with intensity $\frac{(d+2)}{2r^2}$ and $(Y_k)_{k\geq 1}$ is a sequence of independent and identically distributed random variables with density $\psi(y) = \frac{V_r(0,y)}{V_r^2}$. As a result, the law of $\xi_t^{(r)}$ can be written

$$G_t^{(r)}(\mathrm{d}x) = e^{-\frac{d+2}{2r^2}t}\delta_0(\mathrm{d}x) + e^{-\frac{d+2}{2r^2}t}\sum_{n\geq 1}\frac{\left(\frac{d+2}{2r^2}t\right)^n}{n!}\psi^{*n}(x)\mathrm{d}x.$$

Since ψ is continuous on $\mathbb{R}^d$, so is ψ^{*n} for any $n \geq 1$. In addition, $\psi(y)$ is decreasing as a function of $|y|$, and $\phi * \psi(x) = \overline{\overline{\phi}}(x,r)$ so, by induction it follows that $\psi^{*n}(y)$ is also decreasing as a function of $|y|$. Since the above sum converges uniformly, we can conclude that $g_t^{(r)}$ is continuous on $\mathbb{R}^d$ and that $g_t^{(r)}(y)$ is rotation invariant and is a decreasing function of $|y|$. □

Proof of Lemma 6.9. By some elementary algebra,

$$\begin{aligned}
\phi(y)^2 - \phi(x)^2 - 2\phi(x)(\phi(y)-\phi(x)) + 2\frac{2r^2}{d+2}\phi(x)(\phi(y)-\phi(x))g(y) \\
&= \left(\phi(y)-\phi(x)+\frac{2r^2}{d+2}\phi(x)g(y)\right)^2 - \left(\frac{2r^2}{d+2}\right)^2\phi(x)^2g(y)^2 \\
&\geq -\left(\frac{2r^2}{d+2}\right)^2\phi(x)^2,
\end{aligned}$$

since $g(y)^2 \leq 1$. Averaging the above inequality in y twice around x and multiplying by $\frac{d+2}{2r^2}$ yields

$$\mathcal{L}^{(r)}\phi^2(x) - 2\phi(x)\mathcal{L}^{(r)}\phi(x) + 2\phi(x)\overline{\overline{\phi\, g}}(x,r) - 2\phi(x)^2\overline{\overline{g}}(x,r) \geq -\frac{2r^2}{d+2}\phi(x)^2.$$

The first result then follows from the fact that $\gamma \leq g$. For the second inequality, set $a = \phi(y)$, $\epsilon = \frac{2r^2}{d+2}g(y)$ and $b = (1-\epsilon)^{1/3}\phi(x)$; then

$$\begin{aligned}
\phi(y)^4 - \phi(x)^4 - 4\left(1-\frac{2r^2}{d+2}g(y)\right)\phi(x)^3(\phi(y)-\phi(x)) \\
= a^4 - b^4 - 4b^3(a-b) + b^4 - \phi(x)^4 - 4b^3(b-\phi(x)).
\end{aligned}$$

By convexity of the function $x \mapsto x^4$, $a^4 - b^4 - 4b^3(a-b) \geq 0$, so the above expression is greater than

$$\phi(x)^4\left[(1-\epsilon)^{4/3} - 1 - 4(1-\epsilon)((1-\epsilon)^{1/3}-1)\right] \underset{\epsilon\to 0}{\sim} -\frac{2}{3}\phi(x)^4\epsilon^2.$$

Hence there exists c such that, for r small enough,

$$\phi(y)^4 - \phi(x)^4 - 4\left(1-\frac{2r^2}{d+2}g(y)\right)\phi(x)^3(\phi(y)-\phi(x)) \geq -4c\,r^4\phi(x)^4.$$

Averaging in y twice around x as above yields the second statement. □

Proof of Lemma 6.10. We define the following :

$$\mathcal{H}(x,u,t) = e^{-\alpha(t-u)}G_{t-u}^{(r)} * h_u(x).$$

Differentiating with respect to u yields

$$\frac{\partial \mathcal{H}}{\partial u}(x,u,t) = e^{-\alpha(t-u)} G^{(r)}_{t-u} * (\partial_u h_u - \mathcal{L} h_u + \alpha h_u)(x) \\ \leq e^{-\alpha(t-u)} G^{(r)}_{t-u} * g_u(x). \tag{D.1}$$

Integrating (D.1) over $u \in [s,t]$, we have

$$h_t(x) \leq e^{-\alpha(t-s)} G^{(r)}_{t-s} * h_s(x) + \int_s^t e^{-\alpha(t-u)} G^{(r)}_{t-u} * g_u(x) \mathrm{d}u. \tag{D.2}$$

By Jensen's inequality,

$$\left(\int_s^t e^{-\alpha(t-u)} G^{(r)}_{t-u} * g_u(x) \mathrm{d}u\right)^q \leq \left(\int_s^t e^{-\alpha(t-u)} \mathrm{d}u\right)^{q-1} \int_s^t e^{-\alpha(t-u)} \left(G^{(r)}_{t-u} * g_u(x)\right)^q \mathrm{d}u \\ \leq \frac{1}{\alpha^{q-1}} \int_s^t e^{-\alpha(t-u)} G^{(r)}_{t-u} * g_u^q(x) \mathrm{d}u.$$

The result follows by taking $\|\cdot\|_q$ norms on each side of (D.2). □

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