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Transition density of a hyperbolic Bessel process*

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Abstract

We investigate the transition density of a hyperbolic Bessel process for integer dimensions and show a link between transition densities of a hyperbolic Brownian motion and a Bessel process in the same dimension. Using the so-called Millson's formula for the densities of hyperbolic Brownian motion we also show a link between transition density of n-dimensional hyperbolic Bessel process and 2-dimensional (if n is even) or 3-dimensional (if n is odd) hyperbolic Brownian motion. This helps us to get explicit formulas for the Bessel process transition density.

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1 Introduction

Hyperbolic Bessel process is one of many diffusion processes that are used to model different physical and economic phenomena (e.g. Asian options pricing, see [11]). This process has been recently investigated by Jakubowski and Wiśniewolski in a paper [9], earlier its properties were investigated by Gruet in [6], [7], [8] and by Borodin in [1]. These authors defined hyperbolic Bessel process for parameters $\nu > -\frac{1}{2}$ as a solution of a stochastic differential equation or a diffusion generated by a differential operator. They also gave some formulas for the transition density. In particular, Gruet in ([6], Remarque 3 and [7], Theorem 6.1), using spectral theory for Jacobi semigroups developed by Koornwinder [10], got the following formula for the transition density of a hyperbolic Bessel process with parameter $\nu > -\frac{1}{2}$: for $a,\, b > 0$

$$q_t^{(\nu)}(a,b) = k_\nu \int_0^\infty \exp\left(-\left(\left(\frac{2\nu+1}{2}\right)^2 + p^2\right)\frac{t}{2}\right)\phi_p^{(\nu)}(a)\phi_p^{(\nu)}(b)\left|\frac{\Gamma(ip+\nu+1/2)}{\Gamma(ip)}\right|^2\, dp,$$

where $k_\nu = \pi^{-\nu-1}2^{-2\nu-1}/\Gamma(\nu+1)$ and $\phi_p^{(\nu)}(r) = {}_2F_1\left(\frac{\nu+1/2-ip}{2}, \frac{\nu+1/2+ip}{2}; \nu+1, -\sinh^2 r\right)$.

This formula is very complicated, except only for $\nu = 1/2$, when hyperbolic Bessel process is a radial part of a hyperbolic Brownian motion in three-dimensional space. However, manageable formulas, at least for integer parameter ν, would be very useful

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for investigation of stable processes (and more general: subordinated Brownian motions) with values in hyperbolic spaces. Such investigations have just started, compare for instance [13].

In this paper we obtain such manageable formulas for the transition densities of hyperbolic Bessel processes of integer dimension, that is for the radial parts of hyperbolic Brownian motions. Using this connection between hyperbolic Brownian motion and hyperbolic Bessel process we describe the transition density of a Bessel process as an integral of a Brownian density. Moreover, application of the so-called Millson formula for hyperbolic Brownian density gives much simpler formulas for hyperbolic Bessel densities than all formulas known so far. For instance, in the case of a Bessel process of odd dimension, such transition density is a sum of elementary functions and error function Erf.

A transition density of a stochastic process with values in a subset of $\mathbb{R}^n$ can be taken with respect to the Lebesgue measure or, if this subset is a Riemannian manifold, with respect to the volume measure on this manifold. Moreover, a transition density of any linear diffusion process can be taken with respect to the Lebesgue measure or with respect to the speed measure. In order to avoid misunderstanding, all densities we consider in this paper, if taken with respect to the Lebesgue measure, will be denoted by p (with sub- or superscripts), otherwise they will be denoted with bar: $\bar{p}$.

2 Hyperbolic spaces and hyperbolic processes

There are several different models of n-dimensional hyperbolic space (see e.g. [14]). For our considerations the unit ball model of a hyperbolic space will be convenient but our results concerning hyperbolic Bessel process do not depend on the model.

2.1 Hyperbolic space

Let us define $\mathbb{D}^n$ as the unit ball in $\mathbb{R}^n$, that is $\mathbb{D}^n = \{x \in \mathbb{R}^n : |x| < 1\}$, where $|x|$ denotes the Euclidean norm. This set, equipped with the Riemannian metric $ds^2 = \frac{4|dx|^2}{(1-|x|^2)^2}$, is a model of n-dimensional hyperbolic space (compare e.g. [3] or [14]). The volume element in $\mathbb{D}^n$ is given by $dV_n = \frac{dx}{(1-|x|^2)^n}$ and the hyperbolic distance $d_{\mathbb{D}^n}(x,y)$ between points x and y is given by

$$\mathrm{ch}\,(d_{\mathbb{D}^n}(x,y)) = 1 + \frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)}. \tag{2.1}$$

If $x = \vec{0} = (0,...,0)$, formula (2.1) simplifies: for $y \in \mathbb{D}^n$ $d_{\mathbb{D}^n}(\vec{0},y) = \ln((1+|y|)/(1-|y|))$. This implies that the hyperbolic spheres with center $\vec{0}$ are ordinary Euclidean spheres, the same is true for all hyperbolic spheres (see e.g. [3], p. 156).

The Laplace-Beltrami operator in $\mathbb{D}^n$ is given by

$$\Delta_{\mathbb{D}^n} = \frac{(1-|x|^2)^2}{4}\sum_{k=1}^{n}\frac{\partial^2}{\partial x_k^2} + (n-2)\frac{(1-|x|^2)}{2}\sum_{k=1}^{n} x_k\frac{\partial}{\partial x_k}.$$

We will use the following (hyperbolic) spherical coordinates in $\mathbb{D}^n$. Let $r = d_{\mathbb{D}^n}(\vec{0},x)$ be the radial coordinate of a point $x \in \mathbb{D}^n$ and let $\phi_1, \phi_2, \ldots \phi_{n-1}$ be the angular coordinates, where ϕ_{n-1} ranges over $[0,2\pi)$ and $\phi_1, \phi_2, \ldots \phi_{n-2}$ range over $[0,\pi]$. Observe that if $r = d_{\mathbb{D}^n}(\vec{0},(R,0,...,0)) = \log\frac{1+R}{1-R}$ then $R = \tanh\frac{r}{2}$, hence

$$\begin{aligned} x_1 &= \tanh\left(\frac{r}{2}\right)\cos(\phi_1), \qquad (2.2)\\ x_k &= \tanh\left(\frac{r}{2}\right)\sin(\phi_1)\ldots\sin(\phi_{k-1})\cos(\phi_k), \quad k = 2,3,...,n-1, \end{aligned}$$

$$x_n \quad = \quad \tanh\left(\frac{r}{2}\right)\sin(\phi_1)\ldots\sin(\phi_{n-2})\sin(\phi_{n-1}).$$

The volume element in the spherical coordinates is given by the Jacobi determinant

$$dV_n^{(spher)} = \mathrm{sh}^{\,n-1} r \sin^{n-2}(\phi_1)\sin^{n-3}(\phi_2)\ldots\sin(\phi_{n-2})\,dr\,d\phi_1\,d\phi_2\,\ldots\,d\phi_{n-2}\,d\phi_{n-1}.$$

Observe that

$$\int_0^\pi \sin^{n-2}(\phi_1)d\phi_1 \int_0^\pi \sin^{n-3}(\phi_2)\,d\phi_2\,\ldots \int_0^\pi \sin(\phi_{n-2})\,d\phi_{n-2}\int_0^{2\pi} d\phi_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)} = \Omega_{n-1}$$

is the area of the Euclidean unit sphere in $\mathbb{R}^n$. We also have $\Omega_{n-3} = \frac{n-2}{2\pi}\Omega_{n-1}$ and $\Omega_0 = 2$.

In polar coordinates the Laplace-Beltrami operator has the following form

$$\Delta_{\mathbb{D}^n} = \frac{\partial^2}{\partial r^2} + (n-1)\coth(r)\frac{\partial}{\partial r} + \frac{1}{\mathrm{sh}^{\,2} r}\Delta_S, \tag{2.3}$$

where Δ_S is the Laplace operator on the unit sphere (see [3], p. 158).

2.2 Hyperbolic Brownian motion and its density

The hyperbolic Brownian motion in $\mathbb{D}^n$ is defined as a diffusion generated by $\frac{1}{2}\Delta_{\mathbb{D}^n}$. Let us denote the hyperbolic Brownian motion in $\mathbb{D}^n$ by $X = (X_1, X_2, \ldots, X_n)$. It is well-known (see e.g. [3]) that the transition density $\bar{p}_n^{\mathrm{Br}}(t;x,y)$ of a hyperbolic Brownian motion in a real n-dimensional hyperbolic space (with respect to the volume element dV_n of this space) is a function of t and the distance r between x and y, hence a formula for $\bar{p}_n^{\mathrm{Br}}(t;r)$ does not depend on the model of a hyperbolic space.

The density $\bar{p}_n^{\mathrm{Br}}(t;r)$ can be expressed by the Gruet's formula ([6])

$$\bar{p}_n^{\mathrm{Br}}(t;r) = \frac{e^{-(n-1)^2 t/8}}{\pi(2\pi)^{n/2}t^{1/2}}\Gamma\left(\frac{n+1}{2}\right)\int_0^\infty \frac{e^{(\pi^2-b^2)/2t}\,\mathrm{sh}\,(b)\sin(\pi b/t)}{(\mathrm{ch}\,(b) + \mathrm{ch}\,(r))^{(n+1)/2}}db. \tag{2.4}$$

Being complicated, formula (2.4) is not very useful in applications. Fortunately, it is possible to get simpler formulas for the density in dimensions 2 and 3 (see [5] and references therein):

$$\bar{p}_2^{\mathrm{Br}}(t;r) = \frac{e^{\frac{-t}{8}}}{2(\pi t)^{\frac{3}{2}}}\int_r^\infty \frac{s e^{\frac{-s^2}{2t}}}{\sqrt{\mathrm{ch}\,s - \mathrm{ch}\,r}}ds,$$

$$\bar{p}_3^{\mathrm{Br}}(t;r) = \frac{1}{(2\pi t)^{3/2}}\frac{r}{\mathrm{sh}\,r}\exp\left(-\frac{t}{2} - \frac{r^2}{2t}\right).$$

Moreover, the following formula attributed to Millson (compare [5]): for $n = 1, 2, \ldots$

$$\bar{p}_{n+2}^{\mathrm{Br}}(t;r) = -\frac{e^{-nt/2}}{2\pi\,\mathrm{sh}\,(r)}\frac{\partial}{\partial r}\bar{p}_n^{\mathrm{Br}}(t;r), \tag{2.5}$$

gives a tool to compute $\bar{p}_n^{\mathrm{Br}}$ for all $n \geq 3$. Namely, iterating (2.5) one can get the following formulas, proved by Grigor'yan and Noguci in [5] by a different method: if $n = 2m+1$, then

$$\bar{p}_n^{\mathrm{Br}}(t;r) = \frac{(-1)^m}{(2\pi)^m}\frac{1}{\sqrt{2\pi t}}\left(\frac{1}{\mathrm{sh}\,r}\frac{\partial}{\partial r}\right)^m e^{\frac{-m^2 t}{2} - \frac{r^2}{2t}}, \tag{2.6}$$

and if $n = 2m+2$, then

$$\bar{p}_n^{\mathrm{Br}}(t;r) = \frac{(-1)^m}{(2\pi)^m}\frac{1}{2(\pi t)^{\frac{3}{2}}}e^{\frac{-(2m+1)^2 t}{8}}\left(\frac{1}{\mathrm{sh}\,r}\frac{\partial}{\partial r}\right)^m \int_r^\infty \frac{s e^{\frac{-s^2}{2t}}}{\sqrt{\mathrm{ch}\,s - \mathrm{ch}\,r}}ds. \tag{2.7}$$

Observe, that for odd dimensions the densities are elementary functions.

2.3 Hyperbolic Bessel process of integer dimension

We are interested in hyperbolic Bessel processes of dimensions greater than one, because one-dimensional hyperbolic Bessel process is simply a reflected one-dimensional hyperbolic Brownian motion (compare [7]).

We use the following definition of the hyperbolic Bessel process $Z_t^{(n)}$ of dimension n:

$$Z_t^{(n)} = d_{\mathbb{D}^n}(\vec{0}, X_t),$$

that is, $Z_t^{(n)}$ is the hyperbolic distance of X_t from its starting point $\vec{0} = (0, 0, \dots, 0)$.

$Z_t^{(n)}$ is a one-dimensional diffusion (with values in $[0, \infty)$) generated by $\frac{1}{2}\Delta_{\mathbb{D}^n}^{(r)} = \frac{1}{2}\frac{\partial^2}{\partial r^2} + \frac{n-1}{2}\coth r \frac{\partial}{\partial r}$, the radial part of $\frac{1}{2}\Delta_{\mathbb{D}^n}$. By the general theory of linear diffusions (see e.g. [2]) we infer that its speed measure is equal to $m_n(dx) = 2\,\mathrm{sh}^{\,n-1} x\, dx$, its scale function is equal to $s_n(x) = \int_c^x \mathrm{sh}^{\,1-n} u\, du$ and that $Z_t^{(n)}$ has a density of the transition probability (with respect to the speed measure) with the following property:

$$\bar{p}_n^{\text{Bess}}(t; y, x) = \bar{p}_n^{\text{Bess}}(t; x, y). \tag{2.8}$$

Because for every $n = 2, 3, ...$ the hyperbolic Brownian motion with probability one does not hit its starting point for $t > 0$, the hyperbolic Bessel process does not hit zero for $n \geq 2$. In the next section, using Millson formula for the transition density of the hyperbolic Brownian motion, we will compute $p_n^{\text{Bess}}(t; a, b)$ for all $a,\, b > 0$. Using (2.8), the above-mentioned symmetry of $\bar{p}_n^{\text{Bess}}$, we may and do assume that $a \leq b$.

3 Transition density of the hyperbolic Bessel process

All formulas for densities $p_n^{\text{Bess}}(t; a, b)$ of hyperbolic Bessel processes we obtain in this section, are computed with respect to the Lebesgue measure db on the real line. In order to obtain the symmetric form of the density of the hyperbolic Bessel process in dimension n (that is, the density with respect to the speed measure) we have to divide by $2\,\mathrm{sh}^{\,n-1} b$ the density computed with respect to the Lebesgue measure. Such normalized density $\bar{p}_n^{\text{Bess}}(t; a, b) = p_n^{\text{Bess}}(t, a, b)/(2\,\mathrm{sh}^{\,n-1}(b))$ has property (2.8) and fulfills equation

$$\frac{\partial \bar{p}_n^{\text{Bess}}}{\partial t} = \frac{1}{2}\left(\frac{\partial^2 \bar{p}_n^{\text{Bess}}}{\partial b^2} + (n-1)\coth(b)\frac{\partial \bar{p}_n^{\text{Bess}}}{\partial b}\right).$$

Let $S(\vec{0}, r)$ denote a hyperbolic sphere in $\mathbb{D}^n$ with center $\vec{0}$ and hyperbolic radius $r > 0$. We will use the following simple fact: hyperbolic Bessel process of dimension n moves from point $a > 0$ to $b > 0$ if and only if the hyperbolic Brownian motion in $\mathbb{D}^n$ moves from $S(\vec{0}, a)$ to $S(\vec{0}, b)$.

Let $S(\vec{a}, r)$ denote a sphere in $\mathbb{D}^n$ with Euclidean center $\vec{a} = (\tanh(a/2), 0, \dots, 0)$ and hyperbolic radius r for any $a \geq 0$ and $r > 0$. Fix $b \geq a$ and observe that spheres $S(\vec{0}, b)$ and $S(\vec{a}, r)$ intersect only if $b - a \leq r \leq a + b$. Then their intersection is a sphere in $\mathbb{R}^{n-1}$ or a single point:

$$S(\vec{a}, r) \cap S(\vec{0}, b) = \left\{ x \in \mathbb{D}^n : d_{\mathbb{D}^n}(\vec{0}, x) = b,\ x_1 = x^{a,b}(r) := \frac{\tanh^2 \frac{b}{2} + \tanh \frac{a+r}{2} \tanh \frac{a-r}{2}}{\tanh \frac{a+r}{2} + \tanh \frac{a-r}{2}} \right\}.$$

Now, for $0 < h < a + b - r$, let us compute $A_{a,b}(r, h)$, the area of this part of the sphere $S(\vec{0}, b)$, that is contained between spheres $S(\vec{a}, r)$ and $S(\vec{a}, r + h)$. For this purpose we will integrate the Jacobi determinant $dV_n^{(spher)}$ over that part of $S(\vec{0}, b)$, where x_1 varies from $x^{a,b}(r + h)$ to $x^{a,b}(r)$, thus ϕ_1 varies from $\arccos\left(\frac{x^{a,b}(r+h)}{\tanh\left(\frac{b}{2}\right)}\right)$ to $\arccos\left(\frac{x^{a,b}(r)}{\tanh\left(\frac{b}{2}\right)}\right)$. We have:

$$A_{a,b}(r,h) = \Omega_{n-2}\,\mathrm{sh}^{\,n-1} b \int_{\arccos\left(\frac{x^{a,b}(r)}{\tanh\left(\frac{b}{2}\right)}\right)}^{\arccos\left(\frac{x^{a,b}(r+h)}{\tanh\left(\frac{b}{2}\right)}\right)} \sin^{n-2}\phi_1 d\phi_1,$$

where Ω_{n-2} is the area of the unit sphere in $\mathbb{R}^{n-1}$.

Thus $dS_{a,b}(r)$, the surface element (on $S(\vec{0},b)$) of intersection $S(\vec{0},b)$ with $S(\vec{a},r)$, is given by

$$\begin{aligned} & dS_{a,b}(r) = \qquad (3.1)\\ = \; & \lim_{h\to 0}\frac{A_{a,b}(r,h)}{h}dr = \Omega_{n-2}\,\mathrm{sh}^{\,n-1} b\left(\lim_{h\to 0}\frac{1}{h}\int_{\arccos\left(\frac{x^{a,b}(r)}{\tanh\left(\frac{b}{2}\right)}\right)}^{\arccos\left(\frac{x^{a,b}(r+h)}{\tanh\left(\frac{b}{2}\right)}\right)} \sin^{n-2}\phi_1 d\phi_1\right) dr = \\ = \; & \Omega_{n-2}\,\mathrm{sh}^{\,n-1} b \sin^{n-2}\left(\arccos\left(\frac{x^{a,b}(r)}{\tanh\left(\frac{b}{2}\right)}\right)\right)\frac{\partial}{\partial r}\arccos\left(\frac{x^{a,b}(r)}{\tanh\left(\frac{b}{2}\right)}\right) dr = \\ = \; & \Omega_{n-2}\,\mathrm{sh}^{\,n-1} b \sin^{n-2}\left(\arccos\left(\frac{x^{a,b}(r)}{\tanh\left(\frac{b}{2}\right)}\right)\right)\frac{-1}{\sin\left(\arccos\left(\frac{x^{a,b}(r)}{\tanh\left(\frac{b}{2}\right)}\right)\right)}\frac{-\,\mathrm{sh}\, r}{\mathrm{sh}\, a\,\mathrm{sh}\, b}dr = \\ = \; & \frac{\Omega_{n-2}\,\mathrm{sh}^{\,n-2} b\,\mathrm{sh}\, r}{\mathrm{sh}\, a}\left(1-\frac{(\,\mathrm{ch}\, a\,\mathrm{ch}\, b-\,\mathrm{ch}\, r)^2}{\mathrm{sh}^{\,2} a\,\mathrm{sh}^{\,2} b}\right)^{\frac{n-3}{2}} dr. \end{aligned}$$

But

$$\left(1-\frac{(\,\mathrm{ch}\, a\,\mathrm{ch}\, b-\,\mathrm{ch}\, r)^2}{\mathrm{sh}^{\,2} a\,\mathrm{sh}^{\,2} b}\right) = \frac{(\,\mathrm{ch}\,(b+a)-\,\mathrm{ch}\, r)(\,\mathrm{ch}\, r-\,\mathrm{ch}\,(b-a))}{\mathrm{sh}^{\,2} a\,\mathrm{sh}^{\,2} b},$$

hence

$$dS_{a,b}(r) = \frac{\Omega_{n-2}\,\mathrm{sh}\, b}{\mathrm{sh}^{\,n-2} a}\left[(\,\mathrm{ch}\,(b+a)-\,\mathrm{ch}\, r)(\,\mathrm{ch}\, r-\,\mathrm{ch}\,(b-a))\right]^{\frac{n-3}{2}}\,\mathrm{sh}\, r\, dr.$$

Now $p_n^{\mathrm{Bess}}(t;a,b)$, the transition density from a to b of the hyperbolic Bessel process, is an integral of the Brownian transition density $\bar{p}_n^{\mathrm{Br}}(t;r)$ with respect to the measure $dS_{a,b}(r)$ over the whole sphere $S(0,b)$:

$$p_n^{\mathrm{Bess}}(t,a,b) = \int_{b-a}^{b+a}\bar{p}_n^{\mathrm{Br}}(t,r)dS_{a,b}(r).$$

In this way we get our first result.

Theorem 3.1. *For any $n\geq 2$, $t>0$, $a>0$ and $b\geq a$ we have*

$$p_n^{Bess}(t;a,b) = \frac{\Omega_{n-2}\,sh\, b}{sh^{\,n-2} a}\int_{b-a}^{b+a}\left[(\,ch\,(b+a)-\,ch\, r)(\,ch\, r-\,ch\,(b-a))\right]^{\frac{n-3}{2}}\bar{p}_n^{\mathrm{Br}}(t,r)\,sh\, r dr. \qquad (3.2)$$

Theorem 1 describes a relation between transition densities $p_n^{\mathrm{Bess}}(t;a,b)$ and $\bar{p}_n^{\mathrm{Br}}(t,r)$. Using Millson's formula (2.5) and integrating by parts, it is possible to get a relation between $p_n^{\mathrm{Bess}}(t;a,b)$ and $\bar{p}_{n-2}^{\mathrm{Br}}(t,r)$: for $n\geq 4$

$$\begin{aligned} & p_n^{\mathrm{Bess}}(t;a,b) = \qquad (3.3)\\ & = \tfrac{\Omega_{n-2}\,\mathrm{sh}\, b}{\mathrm{sh}^{\,n-2} a}\int_{b-a}^{b+a}\left[(\,\mathrm{ch}\,(b+a)-\,\mathrm{ch}\, r)(\,\mathrm{ch}\, r-\,\mathrm{ch}\,(b-a))\right]^{\frac{n-3}{2}}\bar{p}_n^{\mathrm{Br}}(t,r)\,\mathrm{sh}\, r dr = \\ & = -\tfrac{e^{-\frac{(n-2)t}{2}}}{2\pi}\tfrac{\Omega_{n-2}\,\mathrm{sh}\, b}{\mathrm{sh}^{\,n-2} a}\int_{b-a}^{b+a}\left[(\,\mathrm{ch}\,(b+a)-\,\mathrm{ch}\, r)(\,\mathrm{ch}\, r-\,\mathrm{ch}\,(b-a))\right]^{\frac{n-3}{2}}\tfrac{\partial}{\partial r}\bar{p}_{n-2}^{\mathrm{Br}}(t,r)dr = \\ & = -\tfrac{e^{-\frac{(n-2)t}{2}}}{2\pi}\tfrac{\Omega_{n-2}\,\mathrm{sh}\, b}{\mathrm{sh}^{\,n-2} a}\left[\left[(\,\mathrm{ch}\,(b+a)-\,\mathrm{ch}\, r)(\,\mathrm{ch}\, r-\,\mathrm{ch}\,(b-a))\right]^{\frac{n-3}{2}}\bar{p}_{n-2}^{\mathrm{Br}}(t,r)\right]_{r=b-a}^{r=b+a} + \\ & \tfrac{e^{-\frac{(n-2)t}{2}}}{2\pi}\tfrac{\Omega_{n-2}\,\mathrm{sh}\, b}{\mathrm{sh}^{\,n-2} a}\int_{b-a}^{b+a}\tfrac{\partial}{\partial r}\left[(\,\mathrm{ch}\,(b+a)-\,\mathrm{ch}\, r)(\,\mathrm{ch}\, r-\,\mathrm{ch}\,(b-a))\right]^{\frac{n-3}{2}}\bar{p}_{n-2}^{\mathrm{Br}}(t,r)dr. \end{aligned}$$

Clearly $(\operatorname{ch}(b+a)-\operatorname{ch} r)(\operatorname{ch} r-\operatorname{ch}(b-a))$ is equal to 0 for both $r=b-a$ and $r=b+a$, thus the formula above reduces to the second term only:

$$p_n^{\text{Bess}}(t;a,b)= \\ =\frac{e^{-\frac{(n-2)t}{2}}}{2\pi}\frac{\Omega_{n-2}\operatorname{sh} b}{\operatorname{sh}^{n-2}a}\int_{b-a}^{b+a}\frac{\partial}{\partial r}\left[(\operatorname{ch}(b+a)-\operatorname{ch} r)(\operatorname{ch} r-\operatorname{ch}(b-a))\right]^{\frac{n-3}{2}}\bar{p}_{n-2}^{\text{Br}}(t,r)dr.$$

After differentiation under the integral sign, we get

$$p_n^{\text{Bess}}(t;a,b)=\frac{(n-3)\Omega_{n-2}e^{-\frac{(n-2)t}{2}}\operatorname{sh} b}{2\pi\operatorname{sh}^{n-2}a}\times \\ \times\int_{b-a}^{b+a}\left[(\operatorname{ch}(b+a)-\operatorname{ch} r)(\operatorname{ch} r-\operatorname{ch}(b-a))\right]^{\frac{n-5}{2}}(\operatorname{ch} a\operatorname{ch} b-\operatorname{ch} r)\bar{p}_{n-2}^{\text{Br}}(t,r)\operatorname{sh} r\,dr. \tag{3.4}$$

When we use equality $\frac{(n-3)\Omega_{n-2}}{2\pi}=\Omega_{n-4}$, we get the following theorem.

Theorem 3.2. *For any $n\geq 4$, $t>0$, $a>0$ and $b>a$ there holds*

$$p_n^{\text{Bess}}(t;a,b)=\frac{\Omega_{n-4}e^{-\frac{(n-2)t}{2}}\operatorname{sh} b}{\operatorname{sh}^{n-2}a}\times \\ \times\int_{b-a}^{b+a}\left[(\operatorname{ch}(b+a)-\operatorname{ch} r)(\operatorname{ch} r-\operatorname{ch}(b-a))\right]^{\frac{n-5}{2}}(\operatorname{ch} a\operatorname{ch} b-\operatorname{ch} r)\bar{p}_{n-2}^{\text{Br}}(t,r)\operatorname{sh} r\,dr. \tag{3.5}$$

We could repeat this procedure until we get a formula with $\bar{p}_2^{\text{Br}}$ or $\bar{p}_3^{\text{Br}}$ under the integral sign. Instead, we put (2.6) into (3.2) and get for odd dimensions $n=2m+1$

$$p_n^{\text{Bess}}(t;a,b)=\frac{(-1)^m\Omega_{n-2}\operatorname{sh} b}{(2\pi)^m\sqrt{2\pi t}\operatorname{sh}^{n-2}a}\times \\ \times\int_{b-a}^{b+a}\operatorname{sh} r\left[(\operatorname{ch}(b+a)-\operatorname{ch} r)(\operatorname{ch} r-\operatorname{ch}(b-a))\right]^{\frac{n-3}{2}}\left(\frac{1}{\operatorname{sh} r}\frac{\partial}{\partial r}\right)^m\left(e^{\frac{-m^2t}{2}-\frac{r^2}{2t}}\right)dr. \tag{3.6}$$

As we did it before, we may compute the integral in the above formula by parts:

$$\begin{aligned} p_n^{\text{Bess}}(t;a,b)&=\frac{(-1)^m\Omega_{n-2}\operatorname{sh} b}{(2\pi)^m\sqrt{2\pi t}\operatorname{sh}^{n-2}a}\times \\ &\times\Bigg(\left[\left[(\operatorname{ch}(b+a)-\operatorname{ch} r)(\operatorname{ch} r-\operatorname{ch}(b-a))\right]^{\frac{n-3}{2}}\left(\frac{1}{\operatorname{sh} r}\frac{\partial}{\partial r}\right)^{m-1}e^{\frac{-m^2t}{2}-\frac{r^2}{2t}}\right]_{r=b-a}^{r=b+a}+ \\ &-\int_{b-a}^{b+a}\left(\frac{\partial}{\partial r}\frac{1}{\operatorname{sh} r}\right)\left(\operatorname{sh} r\left[(\operatorname{ch}(b+a)-\operatorname{ch} r)(\operatorname{ch} r-\operatorname{ch}(b-a))\right]^{\frac{n-3}{2}}\right)\times \\ &\times\left(\frac{1}{\operatorname{sh} r}\frac{\partial}{\partial r}\right)^{m-1}\left(e^{\frac{-m^2t}{2}-\frac{r^2}{2t}}\right)dr\Bigg). \end{aligned} \tag{3.7}$$

Again, if $n>3$ then the first expression is equal to 0 for both $r=b-a$ and $r=b+a$, hence

$$\begin{aligned} p_n^{\text{Bess}}(t;a,b) &= \frac{(-1)^{m+1}\Omega_{n-1}\operatorname{sh} b}{(2\pi)^m\sqrt{2\pi t}\operatorname{sh}^{n-2}a}\times \\ &\times\int_{b-a}^{b+a}\left(\frac{\partial}{\partial r}\right)\left(\left[(\operatorname{ch}(b+a)-\operatorname{ch} r)(\operatorname{ch} r-\operatorname{ch}(b-a))\right]^{\frac{n-3}{2}}\right)\times \\ &\times\left(\frac{1}{\operatorname{sh} r}\frac{\partial}{\partial r}\right)^{m-1}\left(e^{\frac{-m^2t}{2}-\frac{r^2}{2t}}\right)\operatorname{sh} r dr. \end{aligned}$$

After $(m-1)$ such integrations by parts, we express $p_n^{\text{Bess}}(t;a,b)$ as an integral of $\left(\frac{1}{\operatorname{sh} r}\frac{\partial}{\partial r}\right)\left(e^{-r^2/(2t)}\right)=-\frac{r}{t\operatorname{sh} r}e^{-r^2/(2t)}$, and this function, multiplied by $\frac{-e^{-t/2}}{(2\pi)^{3/2}t^{1/2}}$, is the

Brownian transition density $\bar{p}_3^{\mathrm{Br}}(t;r)$. Every integration by parts changes the sign of the expression, so that after $(m-1)$ such operations and after additional multiplication by (-1), factor $(-1)^m$ in (3.6) turns into $(-1)^{m+(m-1)+1}=1$.

Evidently the same may be done in even dimensions $n=2m+2$, using formula (2.7). This time, however, we transfer all m differentiations from the second factor to the first one. In this way we get the following result, which may be seen as a Bessel counterpart of Grigor'yan-Noguchi result ([5]):

Theorem 3.3. *For any $n\geq 2$, $t>0$, $a>0$ and $b>a$: if $n=2m+1$, then $m-1=\frac{n-3}{2}$ and*

$$p_n^{Bess}(t;a,b)=\frac{\Omega_{n-2}e^{-\frac{m^2t}{2}}\,sh\,b}{t(2\pi)^m\sqrt{2\pi t}\,sh^{\,n-2}a}\times \tag{3.8}$$
$$\times\int_{b-a}^{b+a}\left(\frac{\partial}{\partial r}\frac{1}{sh\,r}\right)^{m-1}\left\{sh\,r\left[(\,ch\,(b+a)-\,ch\,r)(\,ch\,r-\,ch\,(b-a))\right]^{\frac{n-3}{2}}\right\}\frac{r}{sh\,r}e^{-\frac{r^2}{2t}}\,dr,$$

and if $n=2m+2$, then

$$p_n^{Bess}(t;a,b)=\frac{\Omega_{n-2}e^{\frac{-(2m+1)^2t}{8}}\,sh\,b}{(2\pi)^m2(\pi t)^{\frac{3}{2}}\,sh^{\,n-2}a}\times \tag{3.9}$$
$$\times\int_{b-a}^{b+a}\left[\left(\frac{\partial}{\partial r}\frac{1}{sh\,r}\right)^{m}\left\{sh\,r\left[(\,ch\,(b+a)-\,ch\,r)(\,ch\,r-\,ch\,(b-a))\right]^{\frac{n-3}{2}}\right\}\int_r^{\infty}\frac{se^{\frac{-s^2}{2t}}}{\sqrt{ch\,s-ch\,r}}ds\right]dr.$$

3.1 General form of the density for odd dimensions

We will now investigate the expression

$$\left(\frac{\partial}{\partial r}\frac{1}{\operatorname{sh} r}\right)^{m-1}\left\{\operatorname{sh} r\left[(\operatorname{ch}(b+a)-\operatorname{ch} r)(\operatorname{ch} r-\operatorname{ch}(b-a))\right]^{m-1}\right\},$$

which appears in formula (3.8). Define the following function

$$f_{a,b}(r)=(\operatorname{ch}(b+a)-\operatorname{ch} r)(\operatorname{ch} r-\operatorname{ch}(b-a)).$$

It is easy to see that the part of $(f_{a,b}(r))^{m-1}$ which depends on r, consists only of expressions of the form $\operatorname{ch}^k r$ with multiplicative constants:

$$(f_{a,b}(r))^{m-1}\operatorname{sh} r=\operatorname{sh} r\sum_{k=0}^{2m-2}C_k\operatorname{ch}^k r,$$

where C_k depends only on k, a and b.

Let A denote the following operator: $A=\left(\frac{\partial}{\partial r}\frac{1}{\operatorname{sh} r}\right)$. Applying $(m-1)$ times A to $(f_{a,b}(r))^{m-1}\operatorname{sh} r$, where $A^0g(r)=g(r)$, we get

$$A^{m-1}\left((f_{a,b}(r))^{m-1}\operatorname{sh} r\right)=\operatorname{sh} r\sum_{k=0}^{m-1}D_k\operatorname{ch}^k r, \tag{3.10}$$

where D_k also depends only on k, a and b. Thus, in order to compute the transition density of the hyperbolic Bessel process in odd dimensions, by (3.8) and (3.10) one has only to compute the integrals of the form

$$\int_{b-a}^{b+a}r\operatorname{ch}^k re^{-\frac{r^2}{2t}}\,dr.$$

But

$$\operatorname{ch}^k r=\left(\frac{e^r+e^{-r}}{2}\right)^k=\frac{1}{2^k}\sum_{l=0}^{k}\binom{k}{l}e^{(2l-k)r}.$$

Now, let j be an integer and compute the integral $\int_{b-a}^{b+a} re^{jr}e^{\frac{-r^2}{2t}}dr$. By substitution $s=\frac{r-jt}{\sqrt{2t}}$, we get

$$\begin{aligned}\int_{b-a}^{b+a} re^{jr}e^{\frac{-r^2}{2t}}dr &= \int_{\frac{b-a-jt}{\sqrt{2t}}}^{\frac{b+a-jt}{\sqrt{2t}}} \sqrt{2t}\left(\sqrt{2t}s+jt\right)e^{-s^2+\frac{j^2t}{2}}ds = \\ &= e^{\frac{j^2t}{2}}\left[2t\int_{\frac{b-a-jt}{\sqrt{2t}}}^{\frac{b+a-jt}{\sqrt{2t}}} se^{-s^2}ds+\sqrt{2}jt^{\frac{3}{2}}\int_{\frac{b-a-jt}{\sqrt{2t}}}^{\frac{b+a-jt}{\sqrt{2t}}} e^{-s^2}ds\right] = \\ = e^{\frac{j^2t}{2}}\left[t\left(e^{\frac{-(b-a-jt)^2}{2t}}-e^{\frac{-(b+a-jt)^2}{2t}}\right)\right. &\left.+\sqrt{\tfrac{\pi}{2}}jt^{\frac{3}{2}}\left(\mathrm{Erf}\left(\tfrac{b+a-jt}{\sqrt{2t}}\right)-\mathrm{Erf}\left(\tfrac{b-a-jt}{\sqrt{2t}}\right)\right)\right],\end{aligned} \tag{3.11}$$

where $\mathrm{Erf}(x)$ is the error function, $\mathrm{Erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-s^2}ds$. Thus for odd n the density $p_n^{\mathrm{Bess}}(t;a,b)$ can be expressed as a sum of elementary functions and the error function.

Corollary 3.4. *Let $m\in\mathbb{N}$ and $n=2m+1$. Then for $t>0$ and $b>a>0$ the integral in (3.8) is equal to*

$$\begin{aligned}\sum_{k=0}^{m-1}\Bigg\{&A_k e^{\frac{k^2t}{2}}\left[t\left(e^{\frac{-(b-a-kt)^2}{2t}}-e^{\frac{-(b+a-kt)^2}{2t}}\right)+\right.\\ &\left.+\sqrt{\frac{\pi}{2}}kt^{\frac{3}{2}}\left(\mathrm{Erf}\left(\frac{b+a-kt}{\sqrt{2t}}\right)-\mathrm{Erf}\left(\frac{b-a-kt}{\sqrt{2t}}\right)\right)\right]+\\ &+B_k e^{\frac{k^2t}{2}}\left[t\left(e^{\frac{-(b-a+kt)^2}{2t}}-e^{\frac{-(b+a+kt)^2}{2t}}\right)+\right.\\ &\left.+\sqrt{\frac{\pi}{2}}kt^{\frac{3}{2}}\left(\mathrm{Erf}\left(\frac{b+a+kt}{\sqrt{2t}}\right)-\mathrm{Erf}\left(\frac{b-a+kt}{\sqrt{2t}}\right)\right)\right]\Bigg\},\end{aligned} \tag{3.12}$$

where A_k and B_k are constants independent of a and b.

3.2 Examples

In principle, formula (3.8) allows us to express the transition density of the hyperbolic Bessel process explicitly for odd dimensions. However, formulas get more and more complicated in higher dimensions. For $n=3$ formula (3.8) simplifies (compare e.g. [12], [7] or [4]):

$$p_3^{\mathrm{Bess}}(t;a,b)=\frac{e^{\frac{-t}{2}}\,\mathrm{sh}\,b}{\sqrt{2\pi t}\,\mathrm{sh}\,a}\int_{b-a}^{b+a}\frac{r}{t}e^{\frac{-r^2}{2t}}dr=\frac{e^{\frac{-t}{2}}\,\mathrm{sh}\,b}{\sqrt{2\pi t}\,\mathrm{sh}\,a}\left(e^{\frac{-(b-a)^2}{2t}}-e^{\frac{-(b+a)^2}{2t}}\right).$$

For $n=5$ formula (3.8) gives

$$p_5^{\mathrm{Bess}}(t;a,b)=\frac{\Omega_3 e^{-2t}\,\mathrm{sh}\,b}{t4\pi^2\sqrt{2\pi t}\,\mathrm{sh}^3 a}\int_{b-a}^{b+a}\frac{\frac{\partial}{\partial r}\left[(\mathrm{ch}\,(b+a)-\mathrm{ch}\,r)(\mathrm{ch}\,r-\mathrm{ch}\,(b-a))\right]re^{-\frac{r^2}{2t}}}{\mathrm{sh}\,r}dr.$$

But

$$\frac{\partial}{\partial r}\left[(\mathrm{ch}\,(b+a)-\mathrm{ch}\,r)(\mathrm{ch}\,r-\mathrm{ch}\,(b-a))\right]=2\,\mathrm{sh}\,r(\mathrm{ch}\,a\,\mathrm{ch}\,b-\mathrm{ch}\,r)$$

and $\Omega_3=2\pi^2$, hence by Corollary 3.4

$$\begin{aligned}p_5^{\mathrm{Bess}}(t;a,b)&=\frac{e^{-2t}\,\mathrm{sh}\,b}{t\sqrt{2\pi t}\,\mathrm{sh}^3 a}\int_{b-a}^{b+a}r(\mathrm{ch}\,a\,\mathrm{ch}\,b-\mathrm{ch}\,r)e^{-\frac{r^2}{2t}}dr=\\ &\frac{e^{-2t-a-b}e^{-(a+b)^2/(2t)}\,\mathrm{sh}\,b}{4\sqrt{2\pi t}\,\mathrm{sh}^3 a}(e^{2a}-1)(e^{2b}-1)(e^{2ab/t}+1)+\\ -\frac{e^{-3t/2}\,\mathrm{sh}\,b}{4\,\mathrm{sh}^3 a}&\left(\mathrm{Erf}\left(\frac{b+a-t}{\sqrt{2t}}\right)-\mathrm{Erf}\left(\frac{b-a-t}{\sqrt{2t}}\right)-\mathrm{Erf}\left(\frac{b+a+t}{\sqrt{2t}}\right)+\mathrm{Erf}\left(\frac{b-a+t}{\sqrt{2t}}\right)\right).\end{aligned}$$

If we divide this density by $m(db) = 2\,\mathrm{sh}^4 b$, we get the density $\bar{p}_5^{\mathrm{Bess}}(t;a,b)$ with property (2.8).

Observe, that both parts (elementary and "Erf-part") of $p_5^{\mathrm{Bess}}(t;a,b)$ are positive functions for $b > a > 0$ and $t > 0$. This is obvious for the elementary part. In order to prove positivity of the "Erf-part" let us write it down in the integral form:

$$\mathrm{Erf}(\frac{b+a-t}{\sqrt{2t}}) - \mathrm{Erf}(\frac{b-a-t}{\sqrt{2t}}) + \mathrm{Erf}(\frac{b-a+t}{\sqrt{2t}}) - \mathrm{Erf}(\frac{b+a+t}{\sqrt{2t}}) =$$

$$\frac{2}{\sqrt{\pi}}\left(\int_{(b-t-a)/\sqrt{2t}}^{(b-t+a)/\sqrt{2t}} e^{-x^2}\,dx - \int_{(b+t-a)/\sqrt{2t}}^{(b+t+a)/\sqrt{2t}} e^{-x^2}\,dx\right).$$

Both integrals are over intervals of length $2a/\sqrt{2t}$. Because function e^{-x^2} is even and decreasing for $x > 0$, and $|b-t-a| < b+t+a$ for $0 < a < b$ and $t > 0$, hence the difference of the integrals is positive.

In a similar way, using (3.8), we compute

$$p_7^{\mathrm{Bess}}(t;a,b) = \tfrac{e^{-(a+b)^2/(2t)}e^{-9t/2}(e^{2ab/t}-1)\,\mathrm{sh}^3 b}{\sqrt{2\pi t}\,\mathrm{sh}^3 a} + \tag{3.13}$$
$$\tfrac{3e^{-4t}\,\mathrm{sh}\,b}{8\,\mathrm{sh}^5 a}\left[e^{3t/2}\left(\mathrm{Erf}(\tfrac{b-a+2t}{\sqrt{2t}}) - \mathrm{Erf}(\tfrac{b-a-2t}{\sqrt{2t}}) + \mathrm{Erf}(\tfrac{a+b-2t}{\sqrt{2t}}) - \mathrm{Erf}(\tfrac{a+b+2t}{\sqrt{2t}})\right) + \right.$$
$$\left. -2\,\mathrm{ch}\,a\,\mathrm{ch}\,b\left(\mathrm{Erf}(\tfrac{b-a+t}{\sqrt{2t}}) - \mathrm{Erf}(\tfrac{b-a-t}{\sqrt{2t}}) + \mathrm{Erf}(\tfrac{a+b-t}{\sqrt{2t}}) - \mathrm{Erf}(\tfrac{a+b+t}{\sqrt{2t}})\right)\right].$$

Despite the fact, that they are more complicated than in the case of odd dimension, our formulas for even n, given by (3.9), can also be useful. For instance formula (3.9) gives for $n = 2$ (that is, for $m = 0$) the following:

$$p_2^{\mathrm{Bess}}(t;a,b) = \frac{e^{-t/8}\,\mathrm{sh}\,b}{(\pi t)^{3/2}} \times$$
$$\times \int_{b-a}^{b+a}\left(\frac{\mathrm{sh}\,r}{\sqrt{(\mathrm{ch}\,(b+a) - \mathrm{ch}\,r)(\mathrm{ch}\,r - \mathrm{ch}\,(b-a))}}\int_r^\infty \frac{se^{-s^2/(2t)}}{\sqrt{\mathrm{ch}\,s - \mathrm{ch}\,r}}\,ds\right)dr.$$

For $n = 4$ we get

$$p_4^{\mathrm{Bess}}(t;a,b) = \frac{e^{-9t/8}\,\mathrm{sh}\,b}{(\pi t)^{3/2}\,\mathrm{sh}^2 a} \times$$
$$\times \int_{b-a}^{b+a}\left(\frac{\mathrm{sh}\,r(\mathrm{ch}\,a\,\mathrm{ch}\,b - \mathrm{ch}\,r)}{\sqrt{(\mathrm{ch}\,(b+a) - \mathrm{ch}\,r)(\mathrm{ch}\,r - \mathrm{ch}\,(b-a))}}\int_r^\infty \frac{se^{-s^2/(2t)}}{\sqrt{\mathrm{ch}\,s - \mathrm{ch}\,r}}\,ds\right)dr.$$

Function $h(r) = \int_r^\infty \frac{se^{-s^2/(2t)}}{\sqrt{\mathrm{ch}\,s - \mathrm{ch}\,r}}\,ds$ is not elementary, nevertheless it is very regular: a formula for the density $\bar{p}_2^{\mathrm{Br}}$ and (2.5) imply that it is strictly positive and decreasing. Using its approximation by elementary functions, one can obtain (at least numerically) good approximations of graphs of p_2^{Bess} or p_4^{Bess}.

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