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On hypoelliptic bridge*

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Abstract

A conditioned hypoelliptic process on a compact manifold, satisfying the strong Hörmander's condition, is a hypoelliptic bridge. If the Markov generator satisfies the two step strong Hörmander condition, the drift of the conditioned hypoelliptic bridge is integrable on $[0,1]$ and the hypoelliptic bridge is a continuous semi-martingale.

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1 Introduction

We are motivated by the path integration formula and also by the L^2 analysis on the space of pinned continuous curves where the Brownian bridge plays an important role. Let M be a smooth connected Riemannian manifold. Denote by $C([0,1];M)$ the space of continuous functions from $[0,1]$ to M and $C_{x_0,z_0}([0,1];M)$ its subspace of curves that begin at x_0 and end at z_0. If (x_t) is a Brownian motion with initial value x_0 and with infinite life time, a Brownian bridge $(b_t^{x_0,z_0}, 0 \le t \le 1)$ begins at x_0 and ends at z_0 is a stochastic process with probability distribution $P(\cdot|x_1 = z_0)$. By Girsanov transform, it is fairly easy to obtain information on the Brownian bridge over a compact interval of $[0,1)$, we are concerned to push these to the terminal time. If M is compact, the Brownian bridge is well known to induce a probability measure on $C_{x_0,z_0}([0,1];M)$.

For the L^2 analysis, it is standard to equip the space with the probability measure determined by the Brownian bridge, which fuelled the study of the logarithm of the heat kernel and their derivatives. However there is no particular strong argument for the use of Brownian bridges, and indeed one is tempted to explore. For example on a Lie group, a basic object is a diffusion operator built from a family of left invariant vector fields generated by elements of the Lie algebra. If $\{X_1,\dots,X_k\}$ is a Lie algebra generating subset of the Lie algebra, the sum of the squares of the corresponding vector fields $\sum_i (L_{X_i})^2$ is naturally hypoelliptic and we are lead to hypoelliptic bridges. Here L_v denotes Lie differentiation in the direction of a vector v.

If $\{X_i, i = 0,1,\dots,m\}$ is a family of smooth vector fields, let $\mathcal{L} = \frac{1}{2}\sum_{k=1}^m L_{X_k}L_{X_k} + L_{X_0}$. If the diffusion coefficients $\{X_1,\dots,X_m\}$ and their iterated Lie brackets span the tangent space T_xM at each x, $\mathcal{L}$ is said to satisfy *the strong Hörmander condition*. Denote by D_k the set of vector fields and their commutators up to level k. If $\mathcal{L}$ satisfies the strong Hörmander condition the minimal k needed to span T_xM is denoted by $l(x)$.

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If for all x, $l(x) \leq p$, $\mathcal{L}$ is said to satisfy the *p-step strong Hörmander condition*. If $\{X_j, [X_0, X_j], j = 1, \dots, m\}$ spans the tangent space at each point, $\mathcal{L}$ is said to satisfy the Hörmander condition. We assume that there exists a global parabolic integral kernel for $\mathcal{L}$, which holds if $\mathcal{L}$ is a sub-Laplacian and the sub-Riemannian distance is complete [35, L. Strichartz]; or is divergence free with respect to an auxiliary Riemannian volume measure and M is compact [22, D. Jerison, A. Sanchez-Calle] and [10, B. Davies]; or is uniformly hypoelliptic and $M = \mathbb{R}^n$ [26, S. Kusuoka, D. Stroock]. See also L. Rothschild and E. Stein (1976) and G. B. Folland (1975).

Given a hypoelliptic $\mathcal{L}$, the probability distribution of the $\mathcal{L}$ diffusion process conditioned to reach the terminal value y at time 1 is absolutely continuous on $[0, t]$, for any $t < 1$, with respect to that of the $\mathcal{L}$-diffusion. It is not so clear how it approaches the terminal value at the terminal time. As preliminary we describe this first using a time reversal and then using heat kernel estimates. If M is compact, $\mathcal{L}$ satisfies the two step strong Hörmander condition and $X_0 = \sum_{k=1}^m c_k X_k$, we make a simple observation on the hypoelliptic bridge (y_t): it has sample continuous paths until the terminal time and

$$\mathbf{E} \int_0^1 |d \log q_{1-s}(\cdot, z_0)(X_k(y_s))| \, ds < \infty.$$

In particular $(y_t, 0 \leq t \leq 1)$ is a continuous semi-martingale. The integral bound on the drift of the bridge is obtained from small time estimates on the fundamental solution and its gradient, the latter from [7, H. Cao, S.-T. Yau]. The Gaussian bounds for q_t depend on the volume of the intrinsic metric balls $B_x(\sqrt{t})$ for small time and on the Euclidean ball for large time. Around x the metric distance is comparable with $\rho^{\frac{1}{l(x)}}$ where ρ is the Riemannian distance. The larger is $l(x)$, the more singular is the heat kernel at 0. It is tempting to argue that the integral bound obtained here, for diffusions satisfying two-step Hörmander condition, fails when $l(x)$ is sufficiently large. On the other hand the following results are proved recently: the Brownian bridge concentrates on the sub-Riemannian geodesic at $t \to 0$. See [2, I. Bailleul, L. Mesnager, J. Norris] and [21, Y. Inahama]. Since the L^1 bound and the semi-martingale property depend on properties of the heat kernel for small time, and since the sub-Riemannian geodesic is horizontal in whose direction the singularity in t should be exactly $t^{-\frac{n}{2}}$, we tend to believe these conclusions hold much more generally.

2 Preliminaries

To condition a diffusion process from x_0 to reach y_0 at 1, it is natural to assume there is a control path reaching y_0 from x_0 and the transition probability measures have positive densities, q_t, with respect to a Riemannian volume measure dx. Hence it is reasonable to assume the strong Hörmander condition on its Markov generator. The purpose for this section is to familiarize ourselves with the basic properties of hypoelliptic bridges. The following consistent family of finite dimensional probability densities,

$$q^{x_0, y_0}_{t_1, \dots, t_n} = \frac{q_{t_1}(x_0, x_1) \dots q_{t_n - t_{n-1}}(x_{n-1}, x_n) q_{1-t_n}(x_n, y_0)}{q(x_0, y_0)}, \quad t_i < 1, \tag{2.1}$$

determine a probability Borel measure on $M^{[0,1]}$. If the finite dimensional distributions of (y_t) are given by (2.1) and $\lim_{t \to 1} y_t = y_0$, it is said to be the *hypoelliptic bridge*. If for a positive number $a < 1$, $\sup_{a \leq t \leq 1} |q_t(x, y_0)|_\infty < \infty$ then $\lim_{t \to 1} y_t = \delta_{y_0}$, weakly. If M is compact, $\mathbf{E}\rho^2(y_t, y_0) \to 0$.

It is well known that, at least when M is a compact manifold, the conditioned Brownian motion induces a measure on the space of continuous paths. This is noted in

[13, J. Eells, K. D. Elworthy], [6, J.-M. Bismut], [28, P. Malliavin, M.-P. Malliavin], and [12, B. Driver]. They were interested in relating the Wiener and pinned Wiener measures to the topology and geometry of the path space over a manifold which later involves the quest for an L^2 Hodge theory, see e.g. [15, 16, K. D. Elworthy, Xue-Mei Li], and the quasi-invariance of the pinned Brownian motion measure. An alternative proof for the quasi-invariance theorem of Malliavin and Malliavin is given in [17, M. Gordina].

For a hypoelliptic diffusion we discuss two cases: in the first $\mathcal{L}$ has an invariant measure μ, i.e. $\int \mathcal{L} f d\mu = 0$ for any $f \in C_K^\infty$, the space of smooth functions with compact supports, and in the second we assume estimates on the heat kernel. We begin with the first case. In general we do not know there is a globally solution to $\mathcal{L}^*\mu = 0$. If $\mathcal{L}$ satisfies the strong Hörmander condition, and M is compact or $\mathcal{L}$ is in divergence form with respect to any measure, the $\mathcal{L}$-diffusion (x_t) has a finite invariant measure. If $\{X_1, \dots, X_m\}$ are linearly independent they determine a sub-Riemannian metric. The sub-elliptic Laplacian Δ_H is defined to be $-\operatorname{div} \nabla^H$ where ∇^H is the sub-Riemannian gradient and the divergence is with respect to a volume form dx. Then $\Delta_H = \sum_{i=1}^m L_{X_i} L_{X_i} + X_0$ where $X_0 = -\sum_{i=1}^m \operatorname{div}_\mu(X_i) X_i$. Suppose that in local coordinates $\mu = G dx$ is a measure with G a smooth density and suppose that M is complete in the sub-Riemannian metric and $\mathcal{L}$ satisfies the strong Hörmander condition and is formally symmetric with respect to μ, then Δ_H with initial domain C_K^∞ is essentially self adjoint on $L^2(M;\mu)$. See [35, R. Strichartz]. In this paper we do not use sub-Riemannian structures.

Throughout this paper x_t is assumed to be conservative, otherwise the set of paths considered would exclude the paths with life time less than 1, which we are not willing to compromise. For simplicity we drop the subscript 1 in q_1. If $f : M \to \mathbb{R}$ is a differentiable function we define its horizontal gradient to be $\nabla^H f = \sum_{i=1}^m (X_i f) X_i$. Let $\hat{\mathcal{L}}$ denote the adjoint operator with respect to a, not necessarily finite, invariant measure μ, i.e. $\int \mathcal{L} f g d\mu = \int f \hat{\mathcal{L}} g d\mu$. Denote by $\hat{x}_t$ the adjoint process.

Proposition 2.1. *Let M be a C^∞ manifold and let $\mathcal{L}$ be a diffusion operator satisfying the strong Hörmander condition and s.t. the $\mathcal{L}$-diffusion is conservative. If $\mathcal{L}^*\mu = 0$ has a solution and the adjoint process is conservative, the hypoelliptic bridge determines a probability measure on $C_{x_0,y_0}([0,1];M)$ and the hypoelliptic bridge (y_t) has a continuous modification.*

Proof. Let (x_t) be an $\mathcal{L}$-diffusion and (y_t) the conditioned bridge process. Restricted to an interval $[0, 3/4]$, y_t is a 'Doob transform' of (x_t). Let $\{w_t^i\}$ be a family of real valued independent one dimensional Brownian motions. Then x_t and y_t can be represented as solutions to the equations with initial values $x_0 = y_0$,

$$\begin{aligned} dx_t &= \sum_{i=1}^m X_i(x_t) \circ dw_t^i + X_0(x_t) dt, \\ dy_t &= \sum_{i=1}^m X_i(y_t) \circ dw_t^i + X_0(y_t) dt + \nabla^H \log q_{1-t}(y_t, y_0) dt, \end{aligned} \tag{2.2}$$

where the gradient is with respect to the first variable. We set

$$\begin{aligned} \tilde{w}_t^i &= w_t^i - \int_0^t d \log q_{1-s}(\cdot, y_0)(X_i(x_s)) ds, \\ N_t &= \sum_{i=1}^m \int_0^t d \log q_{1-s}(\cdot, y_0)(X_i(x_s)) dw_s^i - \frac{1}{2} \sum_{i=1}^m \int_0^t \left| d \log q_{1-s}(\cdot, y_0)(X_i(x_s)) \right|^2 ds. \end{aligned}$$

Let dx be the volume measure of a Riemannian metric, ∇ its Levi-Civita connection and $Z = \frac{1}{2} \sum_{i=1}^n \nabla X_i(X_i) + X_0$. Then

$$\frac{\partial}{\partial s}\log q_{1-s}+\frac{1}{2}\sum_{i=1}^{m}\nabla^2\log q_{1-s}(X_i,X_i)+L_Z\log q_{1-s}=-\frac{1}{2}\sum_i|d\log q_{1-s}(X_i,X_i)|^2,$$

from which we obtain:

$$\begin{aligned}\log q_{1-t}(x_t,y_0)=&\log q(x_0,y_0)+\sum_{i=1}^{m}\int_0^t d\log q_{1-s}(\cdot,y_0)(X_i(x_s))dw_s^i\\&-\frac{1}{2}\sum_{i=1}^{m}\int_0^t|d\log q_{1-s}(\cdot,y_0)(X_i(x_s))|^2\,ds.\end{aligned}$$

Plugging this back into the formula for N_t, we see $\exp(N_t)=\frac{q_{1-t}(x_t,y_0)}{q(x_0,y_0)}$. Since $\mathbf{E}\frac{q_{1-t}(x_t,y_0)}{q(x_0,y_0)}=1$, $(\exp(N_s),0\le s\le t)$ is a martingale for any $t<1$. If F is supported on continuous paths defined up to a time $t<1$, then $\mathbf{E}F(y_\cdot)=\mathbf{E}F(x_\cdot)e^{N_t}$. From this we see that the finite dimensional distributions of (y_t) agree with that of the conditioned process, when restricted to $[0,t]$. Since (x_t) admits a continuous modification and hence determines a probability measure on $C([0,3/4];M)$, so does (y_t).

Let us define a function $V=\mathrm{div}(X_0)-\frac{1}{2}\sum_{i=1}^m L_{X_i}(\mathrm{div}(X_i))+\frac{1}{2}\sum_{i=1}^m(\mathrm{div}(X_i))^2$ and a vector field $Y=-X_0-\sum_{i=1}^m\mathrm{div}(X_i)X_i$. The invariant measure μ is a distributional solution to $\mathcal{L}^*\mu=0$ where $\mathcal{L}^*=\frac{1}{2}\sum_{i=1}^m L_{X_i}L_{X_i}+L_Y+V$ is the L^2 adjoint of $\mathcal{L}$ with respect to dx, with respect to which the divergence is also taken. Since $\mathcal{L}$ satisfies the strong Hörmander condition so does $\mathcal{L}^*$. By a theorem of L. Hörmander [20] any distributional solution to $\mathcal{L}^*\mu=0$ has a strictly positive smooth density m w.r.t. dx.

If $\hat{x}_t$ is adjoint to (x_t), with respect to m, its Markov generator has the same leading term as $\mathcal{L}$ and, by the same argument as above, satisfies also the strong Hörmander condition. We denote by $\hat{q}_t$ its smooth density and there is the following identity: $m(x)q_t(x,y)=m(y)\hat{q}_t(y,x)$. Since the $\hat{\mathcal{L}}$ diffusion is conservative, we condition $\hat{x}_t$ to reach x from y in time 1. The corresponding process is denoted by $\hat{y}_t$. Then $\hat{y}_{1-t}$ has the same distribution as y_t. This follows from

$$q^{x_0,y_0}_{t_1,\dots,t_n}=\frac{q_{t_1}(x_0,x_1)\dots q_{t_n-t_{n-1}}(x_{n-1},x_n)q_{1-t_n}(x_n,y_0)}{q(x_0,y_0)},\quad t_i<1,$$

in which we replace q by $\hat{q}$. By the same argument as above, we see that $\hat{y}_t$ has a continuous modification on $[0,3/4]$. Thus x_t determines a probability measure on $C_{x_0,y_0}([0,1];M)$. The probability measure on the Borel σ-algebra of $M^{[0,1]}$, agrees with those determined by the continuous modification of x_t and $\hat{x}_t$ respectively, when restricted to paths on $[0,3/4]$ and $[1/4,1]$. The required conclusion follows. □

Remark 2.2. 1. The conclusion of the proposition holds in particular for a diffusion operator on a compact manifold satisfying the strong Hörmander condition.

2. The strong Hörmander condition in the proposition can be replaced by the Hörmander condition plus the condition that the solution to $\mathcal{L}^*m=0$ is strictly positive. The same proof is valid following the following observation. Let $Y=-\sum_i(\mathrm{div}\,X_i)X_i-X_0$. Since Y is the sum of X_0 and a linear combination of the diffusion vector fields, $\mathcal{L}^*=\frac{1}{2}\sum_i L_{X_i}L_{X_i}+L_Y+V$ satisfies the Hörmander condition if $\mathcal{L}$ does. A simple computation shows that $\hat{\mathcal{L}}=\frac{1}{2}\sum_i L_{X_i}L_{X_i}+L_Y$ satisfies also the Hörmander condition.
3. If $\mathcal{L}=-\frac{1}{2}\sum_i(L_{X_i})^*L_{X_i}$ is in the divergence form, with respect to a measure $\mu=mdx$ where dx is a Riemannian volume measure and m a smooth function, then μ is an invariant measure. More generally, $\mathcal{L}^*g=\frac{1}{2}\sum_i L_{X_i}L_{X_i}g+L_Yg+Vg=0$, where $V=\frac{1}{2}(\mathrm{div}\,X_i)^2-\frac{1}{2}L_{X_i}\,\mathrm{div}(X_i)+\mathrm{div}(X_0)$ is a smooth function, has a solution

on any compact set. If V vanishes identically then the constant functions are solutions. The existence problem for a globally defined non-trivial solution to the Schrödinger equation in the context of PDE is beyond the scope of the current article. However we should mention the possibility to explore the transition probabilities $Q_t(x, A)$ of a small set A (Doeblin's conditions) or a Lyapunov function for a specific dynamic.

We move on to results based on heat kernel estimates and begin with reviewing Gaussian upper bounds for the fundamental solutions. The Markov generator for an elliptic diffusion is necessarily of the form $\frac{1}{2}\Delta + Z$ where Δ is the Laplace-Beltrami operator for some Riemannian metric on M and Z is a vector field, in which case the diffusion is a Brownian motion with drift Z. Once we understand the case of $\mathcal{L} = \frac{1}{2}\Delta$, an additional (well behaved) drift vector field Z can be taken care of. For a detailed review on heat kernel upper bounds see [33, L. Saloff-Coste]. Take first $\mathcal{L} = \frac{1}{2}\Delta$. If the Ricci curvature of the manifold is bounded from below by $-K$ where K is a positive number, then $p_t(x, x) \sim t^{-\frac{n}{2}}$ where $n = \dim(M)$ and $t \in (0, 1)$. This is a theorem of P. Li and S.-T. Yau [27], extending the result of J. Cheeger and S.-T. Yau [9]. In general if there exists an increasing function $\beta : (0, \infty) \to \mathbb{R}_+$ such that for all $t > 0$ there is the on diagonal estimate $p_t(x, x) \le \frac{1}{\beta(t)}$ and if β satisfies the doubling property, $\beta(2t) \le A\beta(t)$ for all $t > 0$ and some number A, then for some constant D, δ, and C,

$$p(t, x, y) \le \frac{C}{\beta(\delta t)} e^{-\frac{\rho^2(x,y)}{2Dt}}. \tag{2.3}$$

See [18, A. Grigoryan] and [5, A. Bendikov, L. Saloff-Coste] for detailed accounts. If $M = \mathbb{R}^n$, a Sobolev inequality implies Nash's inequality which in turn implies an on diagonal estimate with $\beta(t) = t^{\frac{n}{2}}$, see [31, J. Nash]. Conversely by a theorem in [36, N. Varopoulos], generalised in [8, E. Carlen, S. Kusuoka, D. Stroock], the on diagonal estimate implies Sobolev's inequality.

If $\mathcal{L} = \sum_{k=1}^m L_{X_k} L_{X_k} + L_{X_0}$ is not elliptic, but satisfies Hörmander condition, the bounds on the fundamental solution have different orders depending on whether the time is small or large. To use Kolmogorov's Theorem, it is for the small time we need the more refined upper bound. Under Hörmander condition the fundamental solution q_t of the parabolic equation $\frac{\partial}{\partial t} = \mathcal{L}$ is expected to admit a Gaussian upper bound. For small time, it is better to use the *intrinsic metric distance* d defined by the formula:

$$d(x, y) = \inf \left\{ l \mid \gamma : [0, l] \to M, \dot{\gamma} = \sum_{i=1}^m a_i X_i, \sum_{i=1}^m (a_i(s))^2 \le 1 \right\},$$

where γ is taken over all Lipschitz continuous curves on a compact interval connecting x to y. This intrinsic distance is a natural distance for $\mathcal{L}$, i.e. d induces the original topology of the manifold.

For diffusions on a compact manifold satisfying the strong Hörmander's condition and with the drift X_0 vanishing identically, there is the following estimates in terms of the volume of the metric ball $B_x(r\sqrt{t})$ centred at x:

$$\frac{C_1}{\mathrm{vol}(B_x(\sqrt{t}))} e^{-\frac{C_3 d^2(x,y)}{t}} \le q_t(x, y) \le \frac{C_2}{\mathrm{vol}(B_x(\sqrt{t}))} e^{-\frac{C_4 d^2(x,y)}{t}}, \tag{2.4}$$

for all $x, y \in M$ and all $t > 0$. This is a theorem of D. Jerison and A. Sanchez-Calle [22]. In [34, A. Sanchez-Calle], this upper bound is obtained for (x, y) satisfying the relation $d(x, y) \le \sqrt{t}$ and $t \le 1$. Estimates in (2.4) for the heat kernel is effective only for small times. Indeed, as $q_t(x, y)$ is smooth and strictly positive, we obtain trivial upper and lower constant bounds for q_t. It is another matter to obtain the best constants.

For two points x, y close to each other,

$$\frac{1}{c}\rho(x,y) \le d(x,y) \le c\rho(x,y)^{\frac{1}{l(x)}}, \tag{2.5}$$

where $l(x)$ is the length in the strong Hörmander condition, assuming that the intrinsic sub-Riemannian metric associated with $\{X_1, \ldots, X_m\}$ agrees with the restriction of the Riemannian metric defining ρ. If M is compact and the vector fields are C^∞, then d and ρ are equivalent. The upper bound for d comes from the fact that any point in a small neighbourhood of a point x, of a uniform size, can be reached from x by a controlled path. This is essentially the Box-ball theorem of A. Nagel, E. Stein S. Wainger [30]. See [29, R. Montgomery]. For symmetric diffusions on $\mathbb{R}^n$ satisfying a 'uniform Hörmander's condition' and t small, estimates of the above form were proved in [24, S. Kusuoka, D. Stroock]. For large t the Euclidean metric is more relevant, see [25, S. Kusuoka, D. Stroock]. We do not need sharp estimates on the heat kernel.

Although an estimate of the type (2.4) is sufficient for us, the intrinsic distance is not easy to use. The fundamental solution q_t is the density of the probability distribution of the $\mathcal{L}$-diffusion evaluated at t with respect to the volume measure. In geodesics coordinates we easily integrate a function of ρ, not so easily a function of d. For this reason it is convenient to use the argument that established (2.5) to convert the quantities involving d^2 to ρ^2. Let us consider the volume of the metric ball centred at x with radius $\sqrt{t}$. When t is sufficiently small, one could apply (2.5) for crude estimates. A much refined estimate is given by G. Ben Arous,R. Léandre in [1]. For example we know that for x, y not in each other's cut locus, as $t \to 0$ $q_t(x,y) \sim \frac{C(x,y)}{t^{\frac{n}{2}}} e^{-\frac{d^2(x,y)}{2t}}$. On the diagonal $q_t(x,x) \sim c(x) t^{-\frac{Q(x)}{2}}$ for a number $Q(x)$ relating to $l(x)$, which holds also if X_0 is in the span of the diffusion vector fields and their first order Lie brackets. They also give an example where $X_0 \neq 0$ and q_t decreases exponentially on the diagonal.

Proposition 2.3. *Let M be a smooth manifold with an auxiliary Riemannian metric. Suppose that $\mathcal{L}$-diffusion is conservative, has a smooth density q_t and*

1. *For any $a_0 > 0$, $\sup_{a_0 \le t \le T} \sup_{x,y} q_t(x,y) < \infty$.*
2. *There exists positive numbers δ_0, a and $p > 1$, s.t. for all $0 \le s < t < T$,*

$$\begin{aligned} \sup_{s > \frac{1}{4}, |t-s| < t_0} \frac{\int_{M \times M} \rho^p(x,y) q_s(x_0, x) q_{t-s}(x,y)\, dy dx}{|t-s|^{1+\delta_0}} \le C; \\ \sup_{0 < t < \frac{3}{4}, |t-s| < t_0} \frac{\int_{M \times M} \rho^p(x,y) q_{t-s}(x,y) q_{1-t}(y, y_0)\, dx\, dy}{|t-s|^{1+\delta_0}} \le C. \end{aligned} \tag{2.6}$$

Then there exist positive constants t_0 and C such that for $|t-s| \le t_0$, $\mathbf{E}\rho^p(y_s, y_t) \le C|s-t|^{1+\delta}$.

Note we do not assume the diffusion is symmetric. By (2.3) the lemma applies to $\mathcal{L} = \frac{1}{2}\Delta$ on a complete Riemannian manifold whose Ricci curvature is bounded from below. The proof for the Lemma is included for reader's convenience.

Proof. We may assume $t_0 < 1/4$ and consider the following cases: $0 < s < t < \frac{3}{4}$; $0 < \frac{1}{4} < s < t$; $s = 0$; $t = 1$. We begin with the last case.

$$\begin{aligned} \mathbf{E}\rho^p(y_s, y_0) &= \frac{1}{q(x_0, y_0)} \int_M \rho^p(x, y_0) q_s(x_0, x) q_{1-s}(x, y_0)\, dx \\ &\le \frac{\sup_{s \ge \frac{1}{4}} \sup_y q_s(x, y_0)}{q(x_0, y_0)} \int_M \rho^p(x, y_0) q_{1-s}(x, y_0)\, dx. \end{aligned}$$

If $0 < s < t < \frac{3}{4}$,

$$\begin{aligned}\mathbf{E}\rho^p(y_s, y_t) &= \int_M q_{1-t}(y, y_0) \int_M \frac{\rho^p(x,y) q_s(x_0, x) q_{t-s}(x,y)}{q(x_0, y_0)} dxdy \\ &\le \frac{\sup_{t<\frac{3}{4}} \sup_y q_{1-t}(y, y_0)}{q(x_0, y_0)} \int_M \int_M q_s(x_0, x) \rho^p(x,y) q_{t-s}(x,y) dydx,\end{aligned}$$

concluding the estimates. The estimation for the other cases are similar. To show that the finite dimensional distributions $q_t^{x_0,y_0}$ determines a probability measure on $C([0,1]; M)$ it is sufficient to prove that there exist $p > 1$, $\delta_0 > 0$, and $t_0 > 0$ such that if $|t-s| < t_0$ and $0 \le s \le t \le 1$, $\mathbf{E}\rho(y_t, y_s)^p \le C|t-s|^{1+\delta_0}$. This completes the proof. □

If q is a continuous and M is compact, assumption (1) is automatic. We look into condition (2) in more detail. Denote μ the Euclidean surface measure on S^n, $c_x(\xi)$ the distance to the cut point of x along the geodesic $\gamma_x(\xi)$ in the direction of $\xi \in T_xM$. Denote ST_xM the unit sphere in T_xM and set

$$\begin{aligned} D_x &= \{t\xi : \xi \in ST_xM, t \in [0, c(\xi))\} = T_xM \setminus C_x \\ D_x(r) &= \{\xi \in ST_xM : r < c(\xi)\}. \end{aligned}$$

where C_x is the Riemannian cut locus at x. Note that $D_x(r)$ decrease with r. On D_x, $\exp_x$ is a diffeomorphism onto its image. Denote $J_x(v)$ the determinant of $(d\exp_x)_v$ identifying the tangent spaces of T_xM with itself. Furthermore we denote $A_x(r)$ the lower area function:

$$A(x,r) = \int_{D_x(r)} J_x(r\xi) d\mu(\xi) = \frac{1}{r^{n-1}} \int_{D_x} J_x(\eta) d\mu(\eta).$$

If $A(y_0, r)$ is bounded then the last inequality in the Lemma below holds trivially.

Lemma 2.4. *Suppose that there exist positive constants $C_1, C_2, C_3, \alpha, a, t_0 < 1$, positive increasing real valued functions β_i decaying at most polynomially near 0, such that the following estimates hold for $t < t_0$,*

$$\begin{aligned} &q_t(x,y) \le \frac{C_1}{\beta_2(t)}, \; q_t(x,y) \le \frac{C_1}{\beta_1(t)} e^{-\frac{C_2 \rho^{2\alpha}(x,y)}{t}} \;\; \textit{when} \quad \rho(x,y) \ge a\sqrt{t}; \\ &\sup_{u \ge 0} \int_{au}^\infty r^{\frac{p+n}{\alpha}} e^{-C_2 r^2} A(x, r^{\frac{1}{\alpha}} u^{\frac{1}{\alpha}}) dr < \infty. \end{aligned}$$

Then assumption (2) of Proposition 2.3 holds.

Proof. Let us consider $p > 1$, $0 \le s \le t \le \frac{3}{4}$ and $|t-s| \le t_0$. The other cases are similar. Working in polar coordinates we see that

$$\begin{aligned} &\int_M q_s(x_0, x) \int_M \rho^p(x,y) q_{t-s}(x,y) dydx \\ &= \int_M q_s(x_0, x) \int_0^\infty r^p \int_{D_x(r)} q_{t-s}(y, \exp_x(r\xi)) J_x(r\xi) \mu(d\xi) r^{n-1} drdx. \end{aligned}$$

We plug in the assumed upper bounds for the heat kernel in the respective regions to see the right hand side is bounded by:

$$\begin{aligned} &\int_M q_s(x_0, x) \int_0^{a\sqrt{t-s}} r^{n+p-1} \frac{C_1}{\beta_2(t-s)} \int_{D_x(r)} J_x(r\xi) \mu(d\xi) \, drdx \\ &+ \int_M q_s(x_0, x) \frac{C_1}{\beta_1(t-s)} \int_{a\sqrt{t-s}}^\infty r^{n+p-1} e^{-\frac{C_2 r^{2\alpha}}{t-s}} \int_{D_x(r)} J_x(r\xi) \mu(d\xi) \, drdx, \end{aligned}$$

which is further bounded by

$$\frac{C_1}{\beta_2(t-s)}a^{n+p-1}(t-s)^{\frac{n+p-1}{2}}\int_M q_s(x_0,x)dx\int_0^{a\sqrt{t-s}}A(x,r)dr$$
$$+\frac{C_1}{\beta_1(t-s)}\int_M dxq_s(x_0,x)\int_{a\sqrt{t-s}}^{\infty}r^{p+n-1}e^{-\frac{C_2r^{2\alpha}}{t-s}}A(x,r)dr.$$

This means,

$$\int_M q_s(x_0,x)\int_M \rho^p(x,y)q_{t-s}(x,y)dydx$$
$$\le \frac{C_1a^{n+p-1}(t-s)^{\frac{n+p-1}{2}}}{\beta_2(t-s)}\int_M q_s(x_0,x)\int_0^{a\sqrt{t_0}}A(x,r)drdx$$
$$+\frac{C_1(t-s)^{\frac{p+n}{2\alpha}}}{\beta_1(t-s)}\int_M dxq_s(x_0,x)\int_{a\sqrt{t-s}}^{\infty}r^{\frac{p+n}{\alpha}}e^{-C_2r^2}A(x,r^{\frac{1}{\alpha}}(t-s)^{\frac{1}{2\alpha}})dr.$$

Since $\beta_1(t),\beta_2(t)$ decays at most polynomially near 0, we may choose p and $\delta>0$ such that the assumption (2) of Proposition 2.3 holds. □

A diffusion operator $\mathcal{L}$ on $\mathbb{R}^n$ is said to satisfy the uniform Hörmander's condition, of Kusuoka and Stroock, if the following holds: There exists an integer l_0 such that $l(x)\le l_0$. The vector fields $\{X_1,\dots,X_m\}$ and their iterated bracket up to order l_0 give rise to a $n\times n$ symmetric matrix that is uniformly elliptic on $\mathbb{R}^n$. Also X_0 is in the linear span of $\{X_1,\dots,X_m\}$.

Corollary 2.5. *Under one of the following conditions, there exist positive constants t_0, δ_0, and C such that $\mathbf{E}\rho^p(y_s,y_t)\le C|s-t|^{1+\delta_0}$ for $|t-s|\le t_0$.*

1. *$\mathcal{L}=\sum_{i=1}^m (X_i)^2$ satisfies strong Hörmander condition, M is compact.*
2. *$M=\mathbb{R}^n$, $\mathcal{L}$ satisfies Kusuoka-Stroock's uniform Hörmander's condition.*
3. *$\mathcal{L}=\frac{1}{2}\Delta$, M is complete Riemannian with Ricci curvature bounded from below.*

Proof. (1) In the compact case we use (2.4) and (2.5), the latter holds globally. (2) By [24, S. Kusuoka, D. Stroock], there exists constants $M>1$ and r_0 such that for any $t\in(0,1]$ and $x,y\in\mathbb{R}^n$, $q_t(x,y)\le\frac{M}{\mathrm{vol}(B_x(\sqrt{t}))}e^{\frac{-d^2(x,y)}{Mt}}$. On $\mathbb{R}^n$ the lower surface function $A(x,r)$ is bounded by a constant, the last inequality in Lemma 2.4 is satisfied. Thus assumption (2) in Proposition 2.3 holds. For $t\ge 1$, we use the following from [25, S. Kusuoka and D. Stroock]: $q(t,x,y)\le Mt^{-\frac{n}{2}}e^{-\frac{|y-x|^2}{Mt}}$, which ensures assumption (1) in Proposition 2.3. (3) follows from the classical estimate (2.3), where the heat kernel upper bounds are of the same order for small time and for large time. □

3 L^1 integrability and the semi-martingale Property

Let $x_0,z_0\in M$ and $(y_t,0\le t<1)$ be the solution of the following equation

$$dy_t=\sum_{i=1}^m X_i(y_t)\circ dw_t^i+X_0(y_t)dt+\nabla^H\log q_{1-t}(\cdot,z_0)(y_t)dt,\quad y_0(\omega)=x_0.$$

Theorem 3.1. *Suppose M is compact, X_0 is divergence free, and $\mathcal{L}=\frac{1}{2}\sum_{i=1}^m L_{X_i}L_{X_i}+L_{X_0}$ satisfies the two step strong Hörmander condition. Then y_t has a sample continuous modification, $\lim_{t\to1}y_t=z_0$ a.s. and for each $i=1,\dots,m$,*

$$\mathbf{E}\int_0^1|d\log q_{1-s}(\cdot,z_0)(X_i(y_s))|\,ds<\infty.$$

If $\mathcal{L} = \frac{1}{2}\Delta$, this is well known. The standard proof relies on the following estimate on the heat kernel: $|\nabla_x \log p_t(x,y)| \le C(\frac{1}{\sqrt{t}} + \frac{\rho(x,y)}{t})$, which can be proved probabilistically or follows from the Gaussian upper and lower bounds and Hamilton's estimate for the heat kernel [19, R. Hamilton]: $s|\nabla_x \log p_s(\cdot,y)|^2 \le C_1 \log(\frac{C_2}{s^{\frac{n}{2}} p_s(\cdot,y_0)})$. See [12, B. Driver]. A Hamilton's type inequality is given in [23, Prop. 5.2, B. Kim] for certain sub-elliptic operators, however it is on the wrong side of critical integrability at $t=0$ for our application.

We give some examples where the assumptions are satisfied. (1) $M = SU(2)$, and X_1^*, X_2^* are left invariant vector fields generated by two Pauli matrices. (2) M is the torus, $X_1(x,y) = \frac{\partial}{\partial x}$ and $X_2(x,y) = \sin(2\pi x)\frac{\partial}{\partial y}$. (3) $M = G/Z^3$ where G is the Heisenberg group and $X_1(x,y,z) = \frac{\partial}{\partial x}$ and $X_2(x,y,z) = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$.

Proof. By the proof of Proposition 2.1, the distributions of (y_t) on $[0,1)$ are given by (2.1), which determine a probability measure on $C_{x_0,z_0}([0,1];M)$, and $\lim_{t\to 1} y_t = z_0$.

For the L^1 bound it is sufficient to prove that $\int_0^1 \sqrt{\mathbf{E}|\nabla \log q_{1-s}(y_s,z_0)|^2} ds < \infty$. We use the following theorem from [7, H. Cao, S. T. Yau]. Let $X_0, X_1, \dots, X_m$ be smooth vector fields on a compact manifold such that $X_0 = \sum_{k=1}^m c_k X_k$ for a set of smooth real valued functions c_k on M. Likewise suppose that for every set of $i,j,k = 1,\dots,m$, $[[X_i,X_j],X_k](x)$ can be expressed as a linear combination of vector fields from $\{X_{i'}, [X_{j'}, X_{k'}], i',j',k' = 1,\dots,m\}$. If u_t is a positive solution to the equation $\frac{\partial}{\partial t} u_t = \sum_i L_{X_i} L_{X_i} + L_{X_0}$, there exists a constant $\delta_0 > 1$, such that for all $\delta > \delta_0$ and $t > 0$,

$$\frac{1}{u^2}\sum_i |L_{X_i} u|^2 \le \delta \frac{L_{X_0} u}{u} + \delta \frac{1}{u}\frac{\partial u}{\partial t} + \frac{C_1}{t} + C_2,$$

where C_1, C_2 are constants depending on $\mathcal{L}$ and δ_0. Applying this to the fundamental solution q_t, we see that

$$\mathbf{E}|\nabla \log q_{1-s}(y_s,z_0)|^2 \le \delta \mathbf{E}\frac{L_{X_0} q_{1-s}(\cdot,z_0)}{q_{1-s}(\cdot,z_0)}(y_s) + \delta \mathbf{E}\frac{\frac{\partial q_{1-s}(\cdot,z_0)}{\partial s}(y_s)}{q_{1-s}(y_s,z_0)} + \frac{C_1}{1-s} + C_2.$$

Using the explicit formula for the probability density of y_t, we see that for any $s<1$,

$$\begin{aligned}&\mathbf{E}\left(\frac{\frac{\partial}{\partial s} q_{1-s}(\cdot,z_0)(y_s)}{q_{1-s}(y_s,z_0)}\right) = \int_M \frac{\frac{\partial}{\partial s} q_{1-s}(x,z_0) q_s(x_0,x)}{q(x_0,z_0)} dx \\ &= \int_M \frac{\frac{\partial}{\partial s}(q_{1-s}(x,z_0)q_s(x_0,x)) - q_{1-s}(x,z_0)\frac{\partial}{\partial s} q_s(x_0,x)}{q(x_0,z_0)} dx = -\int_M \frac{q_{1-s}(x,z_0)\frac{\partial}{\partial s} q_s(x_0,x)}{q(x_0,z_0)} dx.\end{aligned}$$

Since the divergence of X_0 vanishes, the same reasoning leads to the following identities:

$$\begin{aligned}&\mathbf{E}\left(\frac{L_{X_0} q_{1-s}(\cdot,z_0)}{q_{1-s}(\cdot,z_0)}(y_s)\right) = \int_M \frac{L_{X_0} q_{1-s}(x,z_0) q_s(x_0,x)}{q(x_0,z_0)} dx \\ &= \int_M \frac{L_{X_0}(q_{1-s}(x,z_0)q_s(x_0,x)) - q_{1-s}(x,z_0) L_{X_0} q_s(x_0,x)}{q(x_0,z_0)} dx \\ &= \int_M \frac{-q_{1-s}(x,z_0) L_{X_0} q_s(x_0,x)}{q(x_0,z_0)} dx\end{aligned}$$

Let us consider the integral from $\frac{1}{2}$ to 1.

$$\begin{aligned}&\int_{\frac{1}{2}}^1 \sqrt{\mathbf{E}|\nabla \log q_{1-s}(y_s,z_0)|^2} ds \\ &\le \int_{\frac{1}{2}}^1 \left(\int_M \left|\frac{q_{1-s}(x,z_0)(L_{X_0} q_s(x_0,x) + \frac{\partial}{\partial s} q_s(x_0,x))}{q(x_0,z_0)}\right| dx + \frac{C_1}{1-s} + C_2\right)^{\frac{1}{2}} ds\end{aligned}$$

Since q_t is smooth and the manifold is compact, there is a constant C_3 such that

$$\sup_{s\in[\frac{1}{2},1]}\left|L_{X_0}q_s(x_0,x)+\frac{\partial}{\partial s}q_s(x_0,x))\right|\le C_3,$$

$$\int_{\frac{1}{2}}^1 \sqrt{\mathbf{E}|\nabla \log q_{1-s}(y_s,z_0)|^2}ds \le \int_{\frac{1}{2}}^1 \sqrt{\frac{C_3}{q(x_0,z_0)}+\frac{C_1}{1-s}+C_2}\, ds<\infty.$$

The same reasoning shows that $\int_0^{\frac{1}{2}} \sqrt{\mathbf{E}|\nabla \log q_{1-s}(y_s,z_0)|^2}ds$ is finite. □

Corollary 3.2. *If M is a complete Riemannian manifold with Ricci curvature bounded from below and $\mathcal{L}=\frac{1}{2}\Delta$, then the conclusion of the theorem holds.*

This follows from the Harnack inequality, [27, P. Li and S.-T. Yau] and [11, B. Davies]: for a constant $\alpha>1$, $\frac{|\nabla u|^2}{u^2}-\alpha\frac{u_t}{u}\le \alpha^2\frac{n}{2t}$.

Remark 3.3. (1) Two step Hörmander condition is used in [32, J. Picard], for a different problem. (2) It is also interesting to explore the Cameron-Martin quasi-invariance theorem in this context and prove the flow of the SDE is quasi invariant under a Girsanov-Martin shift. This should be fairly straight forward if the shift is induced from special vector fields of the form $\int_0^\cdot X^i(x)h_s^i ds$. The quasi-invariance of the conditioned hypoelliptic measure is now known in some sub-Riemannian case, see [4, F. Baudoin, M. Gordina, T. Melcher] for Heisenberg type Lie groups. (3) Finally we remark that a Li-Yau type inequality was extended to certain sub-Riemmanian situation [3, F. Baudoin, N. Garofalo], we have not yet managed to use it to our advantage, and this will be for a future study. For semigroups of Hörmander type second order differential operators, not necessarily satisfying Hörmander condition, see [14, K. D. Elworthy, Y. LeJan, Xue-Mei Li].

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