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Self attracting diffusions on a sphere and application to a periodic case*

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Abstract

This paper proves almost-sure convergence for the self attracting diffusion on the unit sphere

$$dX_t = \nu \circ dW_t(X_t) - a\int_0^t \nabla_{\mathbb{S}^n} V_{X_s}(X_t)dsdt, \qquad X_0 = x \in \mathbb{S}^n,$$

where $\nu > 0$, $a < 0$, $V_y(x) = \langle x, y\rangle$ is the usual scalar product on $\mathbb{R}^{n+1}$, $\circ$ stands for the Stratonovich differential and $(W_t(.))_{t\geqslant 0}$ is a Brownian vector field on $\mathbb{S}^n$. From this we deduce the almost-sure convergence of the real-valued self attracting diffusion

$$d\vartheta_t = \nu dW_t + a\int_0^t \sin(c(\vartheta_t - \vartheta_s))dsdt,$$

where $(W_t)_{t\geqslant 0}$ is a real Brownian motion and $c > 0$.

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1 Introduction

In this paper, we are interested in the asymptotic behaviour of the solution of the stochastic differential equation (SDE)

$$dX_t = \nu \circ dW_t(X_t) - a\int_0^t \nabla_{\mathbb{S}^n} V_{X_s}(X_t)dsdt, \qquad X_0 = x \in \mathbb{S}^n, \tag{1.1}$$

where $\nu > 0$, $a \in \mathbb{R}$, $\circ$ stands for the Stratonovich differential, $(W_t(.))_{t\geqslant 0}$ is a Brownian vector field on $\mathbb{S}^n$, $\nabla_{\mathbb{S}^n}$ is the gradient on $\mathbb{S}^n$ and $V_y(x) = \langle x, y\rangle$ where $\langle .,.\rangle$ is the canonical scalar product on $\mathbb{R}^{n+1}$.

Let us start with a short heuristic description of the process. First of all, observe that for $x, y \in \mathbb{S}^n$, we have

$$\|x - y\|^2 = 2 - 2\langle x, y\rangle = 2 - 2\cos(D(x, y)), \tag{1.2}$$

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where $D(.,.)$ is the geodesic distance on $\mathbb{S}^n$ and $\|.\|$ is the standard Euclidean norm on $\mathbb{R}^{n+1}$. If $a > 0$, the drift term points in a direction that tends to increase the distance between X_t and its past positions. In other words X_t *is repelled by its past.* It follows from a more general result proved in [2] dealing with self repelling diffusions on a compact manifold that

Theorem 1.1 (Theorem 5, [2], Benaïm, Gauthier)**.** *If $a > 0$, the law of X_t converges to the uniform law on $\mathbb{S}^n$.*

If $a < 0$, X_t *is attracted by its past* and one may expect localization.The goal of this paper is to prove such a result.

Theorem 1.2. *If $a < 0$, there exists a random variable $X_\infty \in \mathbb{S}^n$ such that*

$$\|X_t - X_\infty\| = \begin{cases} O\big(t^{-1/2}\sqrt{\ln(t)}\big) & \text{if } n = 1 \\ O\big((\frac{\ln(t)}{t})^{-1/4}\big) & \text{otherwise} \end{cases}.$$

We point out that the self interacting diffusion (1.1) has already received some attention in 2002 by M.Benaïm, M.Ledoux and O.Raimond ([4]), *but* in the normalized case; that is, when $\int_0^t V_{X_s}(X_t)ds$ is replaced by $\frac{1}{t}\int_0^t V_{X_s}(X_t)ds$. The interpretation is therefore different. While the drift term of (1.5) can be "seen" as a summation over $[0,t]$ of the interaction between the current position X_t and its position at time s and thus an *accumulation of the interacting force*, their drift is then an *average of the interacting force*. The asymptotic behaviour is then given by the following Theorem.

Theorem 1.3 (Theorem 4.5, [4], Benaïm, Ledoux, Raimond)**.** *For $a \neq 0$, let $(X_t)_{t\geqslant 0}$ be the solution of the SDE*

$$dX_t = \circ dW_t(X_t) - \frac{a}{t}\int_0^t \nabla_{\mathbb{S}^n} V_{X_s}(X_t)dsdt, \qquad X_0 = x \in \mathbb{S}^n. \tag{1.3}$$

Set $\mu_t = \frac{1}{t}\int_0^t \delta_{X_s}ds$.

1. *If $a \geqslant -(n+1)/2$, then μ_t converges almost surely (for the topology of weak* convergence) toward the Riemannian probability measure on $\mathbb{S}^n$.*
2. *If $a < -(n+1)/2$, then there exists a random variable $\varsigma \in \mathbb{S}^n$ such that μ_t converges almost surely toward the measure*

$$\mu_{c,\varsigma}(dx) = \frac{\exp(\beta(a)\langle x, \varsigma\rangle)}{Z_a},$$

where Z_a is the normalization constant, $\beta(a)$ is the unique positive solution to the implicit equation

$$2a\Lambda_n'(\beta) + \beta = 0,$$

where $\Lambda_n(\beta) = \log(\int_0^\pi \exp(-\beta\cos(x))\lambda_n(dx))$ and $\lambda_n(dx) = \frac{(\sin(x))^{n-1}}{\int_0^\pi (\sin(x))^{n-1}dx}dx$.

An intermediate framework between those considered in Theorem 1.2 and Theorem 1.3 is to add a time-dependent weight $g(t)$ to the normalized case that increases to infinity, but "not too fast", when time increases. In that case, O.Raimond proved the following Theorem.

Theorem 1.4 (Theorem 3.1, [12], Raimond)**.** *Let $(X_t)_{t\geqslant 0}$ be the solution of the SDE*

$$dX_t = \circ dW_t(X_t) - \frac{g(t)}{t}\int_{\mathbb{S}^n} V_{X_s}(X_t)dsdt, \qquad X_0 = x \in \mathbb{S}^n, \tag{1.4}$$

where g is an increasing function such that $\lim_{t\to\infty} g(t) = \infty$. Assume that there exists positive constants c, t_0 such that for $t \geqslant t_0$, $g(t) \leqslant c\log(t)$ and $|g'(t)| = O(t^{-\gamma})$, with $\gamma \in]0,1]$.

Then, there exists a random variable X_∞ in $\mathbb{S}^n$ such that almost surely, $\mu_t = \frac{1}{t}\int_0^t \delta_{X_s} ds$ converges weakly towards δ_{X_∞}.

As an easy application of Theorem 1.2, we obtain the almost-sure convergence with a rate of convergence of the solution of the real-valued SDE

$$d\vartheta_t = \nu dW_t + a\int_0^t \sin(c(\vartheta_t - \vartheta_s))dsdt, \qquad \vartheta_0 = 0, \tag{1.5}$$

where $(W_t)_t$ is a real Brownian motion, $a < 0$ and $\nu, c > 0$.

In 1995, M.Cranston and Y. Le Jan proved an almost-sure convergence result in [5] in the cases where $a\sin(cx)$ is replaced by $f(x) = ax$ (linear case) or $f(x) = a \times sgn(x)$ (constant case) with $a < 0$. This last case was extended in all dimension by O.Raimond in [11] in 1997. A few years later, S.Herrmann and B.Roynette weakened the condition of the profile function f around 0 and were still able to get almost-sure convergence (see [6]) for the solution of the stochastic differential equation

$$d\vartheta_t = \nu dW_t + \int_0^t f(\vartheta_t - \vartheta_s)dsdt. \tag{1.6}$$

Rate of convergence were given in [7] by S.Herrmann and M.Scheutzow. For the linear case, they proved that the optimal rate of convergence is $O\big(t^{-1/2}\sqrt{\log(t)}\big)$ (Proposition 4 in [7]).

However, a common fundamental property of these three papers lies in the fact that the associated profile function f is monotone.

1.1 Reformulation of the problem

From now on, we assume that $a < 0$ and that n is fixed. Since the values of ν and a do not play any particular role, we assume without loss of generality that $\nu = 1$ and $a = -1$. Thus (1.1) becomes

$$dX_t = \circ dW_t(X_t) + \int_0^t \nabla_{\mathbb{S}^n} V_{X_s}(X_t)dsdt, \qquad X_0 = x \in \mathbb{S}^n \tag{1.7}$$

with $V_y(x) = \langle x, y\rangle =: V(x,y)$. Because the law of the process $(X_t)_{t\geqslant 0}$ is the same for any Brownian vector field on $\mathbb{S}^n$, we assume from now on and without loss of generality that $W_t(x) = B_t - \langle x, B_t\rangle x$, where $(B_t)_{t\geqslant 0}$ is a standard Brownian motion on $\mathbb{R}^{n+1}$.

Since V satisfies Hypothesis 1.3 and 1.4 in [4], then (1.7) admits a unique strong solution by Proposition 2.5 in [4]. We recall that for a function $F : \mathbb{R}^{n+1} \to \mathbb{R}$, we have

$$\nabla_{\mathbb{S}^n}(F_{|\mathbb{S}^n})(x) = \nabla_{\mathbb{R}^{n+1}} F(x) - \langle x, \nabla_{\mathbb{R}^{n+1}} F(x)\rangle x; \; x \in \mathbb{S}^n. \tag{1.8}$$

For $x \in \mathbb{S}^n$, we let $u \mapsto P(x,u)$ be the orthogonal projection on $T_x\mathbb{S}^n$ given by

$$P(x,u) = u - \langle x, u\rangle x.$$

Following the same idea as in [2], we set $U_t := \int_0^t X_s ds \in \mathbb{R}^{n+1}$ in order to get the SDE on $\mathbb{S}^n \times \mathbb{R}^{n+1}$:

$$\begin{cases} dX_t = P(X_t, \circ dB_t + U_t dt) \\ dU_t = X_t dt \end{cases} \tag{1.9}$$

with initial condition $(X_0, U_0) = (x, 0)$.

Remark 1.5. We have $P(X_t, \circ dB_t + U_t dt) = P(X_t, dB_t + U_t dt) - \frac{n}{2}X_t dt$,

The paper is organised as follows. In Section 2, we present the detailed strategy used for proving Theorem 1.2 and prove the application to a periodic case whereas the more technical proofs are presented in Section 3.

2 Guideline of the proof of Theorem 1.2

Set $R_t = \|U_t\|$ and define $V_t \in \mathbb{S}^n$ and $C_t \in [-1,1]$ as follows:

$$V_t = \begin{cases} U_t/R_t \text{ if } R_t > 0 \\ X_t \text{ otherwise} \end{cases} \tag{2.1}$$

and

$$C_t = \langle V_t, X_t \rangle. \tag{2.2}$$

With these notations, we have

$$R_t C_t = \langle U_t, X_t \rangle. \tag{2.3}$$

Since the coordinates functions

$$e_j : \mathbb{S}^n \subset \mathbb{R}^{n+1} \to \mathbb{R} : x \mapsto x_j, \text{ for } j = 1, \cdots, n+1,$$

are eigenfunctions for the Laplacian operator on $\mathbb{S}^n$ associated to the eigenvalue $-n$ (see Chapter 3, Section C in [1]), then by Lemma 3 and Lemma 5 in section 4 of [2], the system (1.9) satisfies the Hörmander condition (also called condition (E) in [2] and [8]).

Thus, for all $t > 0$, the law of (X_t, U_t) has a smooth density with respect to the Lebesgue measure on $\mathbb{S}^n \times \mathbb{R}^{n+1}$ (see Theorem 3.(i) in [8]). Hence for all $t > 0$,

$$\mathbb{P}\Big(C_t^2 = 1 \text{ or } R_t = 0\Big) = \mathbb{P}\Big(U_t \text{ is parallel to } X_t\Big) = 0. \tag{2.4}$$

Since $\int_0^t \langle P(X_s, dB_s), V_s \rangle$ is a martingale whose quadratic variation is $\int_0^t (1 - C_s^2) ds$, then the process $(W_t)_{t \geqslant 0}$ defined by $W_0 = 0$ and, for $t > 0$, by

$$W_t = \int_0^t \mathbf{1}_{\{C_s^2 < 1 \text{ and } R_s > 0\}} \frac{\langle P(X_s, dB_s), V_s \rangle}{\sqrt{1 - C_s^2}}, \tag{2.5}$$

is a standard Brownian motion on $\mathbb{R}$.

Lemma 2.1. $((C_t, R_t))_{t \geqslant 0}$ *is solution to*

$$\begin{cases} dC_t = \sqrt{1 - C_t^2} dW_t + [(R_t + \frac{1}{R_t})(1 - C_t^2) - \frac{n}{2} C_t] dt \\ dR_t = C_t dt \end{cases} \tag{2.6}$$

whenever $R_t > 0$.

Proof. Since $dR_t^2 = 2\langle U_t, dU_t \rangle = 2R_t C_t dt$, then, as long as $R_t > 0$, we have

$$dR_t = C_t dt. \tag{2.7}$$

Hence,

$$dV_t = \frac{1}{R_t}(X_t - C_t V_t) dt. \tag{2.8}$$

Therefore, by Itô's formula

$$\begin{aligned} dC_t &= \langle X_t, dV_t \rangle + \langle V_t, P(X_t, dB_t + U_t dt) \rangle - \frac{n}{2} \langle V_t, X_t \rangle dt \\ &= \sqrt{1 - C_t^2} dW_t + (R_t + \frac{1}{R_t})(1 - C_t^2) dt - \frac{n}{2} C_t dt. \end{aligned} \tag{2.9}$$

□

A first important result, whose proof is postponed to Section 3, is

Lemma 2.2. *One has* $\liminf_{t \to \infty} \frac{R_t}{\sqrt{t}} \geqslant \frac{2}{\sqrt{n}}$ *almost surely.*

From this lemma, we prove in Section 3

Lemma 2.3. *The processes $(C_t)_{t\geqslant 0}$ and $(\frac{R_t}{t})_{t>0}$ converge almost surely to 1. Furthermore*

$$|C_t - 1| = \begin{cases} O\big(\frac{\ln(t)}{t}\big) & \text{if } n = 1 \\ O\big(\sqrt{\frac{\ln(t)}{t}}\big) & \text{otherwise} \end{cases} .$$

Thanks to this lemma, we obtain

Lemma 2.4. *V_t converges almost surely.*

Proof. Since

$$\|X_t - C_t V_t\| = \sqrt{1 - C_t^2}, \tag{2.10}$$

it follows from Lemma 2.3

$$\frac{1}{R_t}\|X_t - C_t V_t\| = O\big(t^{-5/4}\ln^{1/4}(t)\big), \tag{2.11}$$

which is an integrable quantity. Hence the result follows from (2.8) and (2.11). □

We can now prove the main result.

Proof of Theorem 1.2

By Lemmas 2.3 and 2.4, there exists a random variable $X_\infty \in \mathbb{S}^n$ such that $\lim_{t\to\infty} C_t V_t = X_\infty$.

The rate of convergence follows from the triangle inequality, (2.8), (2.10), (2.11) and Lemma 2.3.

As an application of Theorem 1.2, we have the following result.

Theorem 2.5. *Let $(\vartheta_t)_{t\geqslant 0}$ be the solution of the SDE*

$$d\vartheta_t = \nu dW_t + a\int_0^t \sin(c(\vartheta_t - \vartheta_s))ds dt, \qquad \vartheta_0 = 0, \tag{2.12}$$

where $(W_t)_t$ is a real Brownian motion, $a < 0$ and $\nu, c > 0$. Then there exists a random variable ϑ_∞ such that $|\vartheta_t - \vartheta_\infty| = O(\sqrt{\frac{\ln(t)}{t}})$.

Proof. First of all (2.12) admits a unique strong solution because the function $\sin(.)$ is Lipschitz continuous (see for example Proposition 1 in [6]).

Set $\vartheta_t^{(c)} = c\vartheta_t$. Hence $(\vartheta_t^{(c)})_{t\geqslant 0}$ solves the SDE

$$d\vartheta_t^{(c)} = c\nu dW_t + ac\int_0^t \sin(\vartheta_t^{(c)} - \vartheta_s^{(c)})ds dt, \qquad \vartheta_0^{(c)} = 0. \tag{2.13}$$

Letting $X_t = \big(\cos(\vartheta_t^{(c)}), \sin(\vartheta_t^{(c)})\big)$, it follows that $(X_t)_{t\geqslant 0}$ is a solution of (1.1) when $n = 1$. Because $ac < 0$, there exists, by Theorem 1.2, $X_\infty \in \mathbb{S}^1$ such that

$$\|X_t - X_\infty\| = O\big(t^{-1/2}\sqrt{\log(t)}\big).$$

The result follows from the continuity of $t \mapsto \vartheta_t^{(c)}$. □

3 Proofs of Lemma 2.2 and Lemma 2.3

3.1 Proof of Lemma 2.2

Set $M_t = -2\int_0^t R_s\sqrt{1 - C_s^2}dW_s$, where W_t is defined by (2.5), and let

$$\mathcal{E}_M(t) = \exp\big(M_t - \frac{1}{2}\langle M\rangle_t\big). \tag{3.1}$$

Because

$$\langle M\rangle_t = 4\int_0^t R_s^2(1-C_s^2)ds \leqslant 4\int_0^t R_s^2 ds \leqslant 4\int_0^t s^2 ds = \frac{4t^3}{3}, \tag{3.2}$$

M_t satisfies the Novikov Condition (see [10], Chapter V, section D, page 198). Therefore $\mathcal{E}_M(t)$ is a positive martingale having 1 as expectation. Thus, it converges almost surely to a nonnegative integrable random variable $\mathcal{E}_M(\infty)$.

Hence, there exists a random variable $K < \infty$, such that almost surely, for all $t \geqslant 0$,

$$\ln(\mathcal{E}_M(t)) \leqslant 2K.$$

By Itô's formula and Lemma 2.1, we have

$$\begin{aligned} d(R_tC_t) &= C_t^2dt + R_t\sqrt{1-C_t}dW_t + (R_t^2+1)(1-C_t^2)dt - \frac{n}{2}R_tC_tdt \\ &= -\frac{1}{2}dM_t + \frac{1}{4}d\langle M\rangle_t - \frac{n}{2}R_tC_tdt + dt. \end{aligned} \tag{3.3}$$

Since $dR_t^2 = 2R_tC_tdt$, we obtain

$$\begin{aligned} R_tC_t + \frac{n}{4}R_t^2 &= -\frac{1}{2}\ln(\mathcal{E}_M(t)) + t \\ &\geqslant t - K. \end{aligned} \tag{3.4}$$

Because $C_t \leqslant 1$, we have for $t > K$

$$\frac{n}{2}R_t \geqslant -1 + \sqrt{n(t-K)}. \tag{3.5}$$

This completes the proof.

3.2 Proof of Lemma 2.3

Before starting the proof of Lemma 2.3, let us recall the Definition of an *asymptotic pseudotrajectory* introduced by Benaïm and Hirsch in [3].

Definition 3.1. *Let (M,d) be a metric space and Φ a semiflow; that is*

$$\Phi : \mathbb{R}_+ \times M \to M : (t,x) \mapsto \Phi(t,x) = \Phi_t(x)$$

is a continuous map such that

$$\Phi_0 = Id \text{ and } \Phi_{t+s} = \Phi_t \circ \Phi_s$$

for all $s,t \in \mathbb{R}_+$.

A continuous function $X : \mathbb{R}_+ \to M$ is an asymptotic pseudotrajectory for Φ if

$$\lim_{t\to\infty} \sup_{0\leqslant h\leqslant T} d(X_{t+h}, \Phi_h(X_t)) = 0 \tag{3.6}$$

for any $T > 0$. In other words, it means that for each fixed $T > 0$, the curve $X : [0,T] \to M : h \mapsto X_{t+h}$ shadows the Φ-trajectory over the interval $[0,T]$ with arbitrary accuracy for sufficiently large t.

If X is a continuous random process, then X is an almost sure asymptotic pseudotrajectory for Φ if (3.6) holds almost surely.

Theorem 3.2 (Theorem 1.2 in [3])**.** *Suppose that $X([0,\infty))$ has compact closure in M and set $L(X) = \bigcap_{t\geqslant 0}\overline{X\big([t,\infty)\big)}$. Let A be an attractor for Φ with basin $\mathcal{W}$. If $X_{t_k} \in \mathcal{W}$ for some sequence $t_k \to \infty$, then $L(X) \subset A$.*

Let $(x^z_t)_{t\geqslant 0}$ be the solution of the SDE

$$dx^z_t = h(t,x^z_t)dB_t + g(x^z_t)dt,\ x^z_0 = z \in \mathbb{R} \tag{3.7}$$

where $(B_t)_{t\geqslant 0}$ is a Brownian motion, $g:\mathbb{R}\to\mathbb{R}$ is a Lipschitz function and $h:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ a continuous function.

The next Theorem gives a sufficient condition on h to ensure that $(x^z_t)_{t\geqslant 0}$ is an almost sure asymptotic pseudotrajectory for the flow induced by the ODE

$$\dot{y} = g(y). \tag{3.8}$$

Theorem 3.3 (Proposition 4.1 in [3])**.** *Assume there exists a non-increasing function $\varepsilon:\mathbb{R}_+\to\mathbb{R}_+$ such that $h^2(t,x)\leqslant\varepsilon(t)$ for all (t,x) and such that*

$$\forall k>0,\ \int_0^\infty \exp(-k/\varepsilon(t))dt < \infty^1. \tag{3.9}$$

Then, for all $z\in\mathbb{R}$, $(x^z_t)_{t\geqslant 0}$ is an almost sure asymptotic pseudotrajectory for the flow induced by (3.8).

Remark 3.4. The same result holds if $(x_t)_{t\geqslant 0}$ solves the SDE

$$dx_t = h(t,x_t)dB_t + \delta(t)h_2(x_t)dt + g(x_t)dt,$$

where h_2 is a bounded function and δ is a random adapted function with $\lim_{t\to\infty}\delta(t)=0$ almost surely.

Proof of Lemma 2.3

The proof is divided into two parts.

Proof of the convergence First we prove that C_t converges almost surely to 1. Recall that

$$dC_t = \sqrt{1-C_t^2}dW_t + [(R_t + \frac{1}{R_t})(1-C_t^2) - \frac{n}{2}C_t]dt. \tag{3.10}$$

Define $\alpha(t) = (\frac{3}{2}t)^{\frac{2}{3}}$ so that $\dot{\alpha}(t) = \alpha^{-\frac{1}{2}}(t)$. Set $Z_t = C_{\alpha(t)}$ and $M_t = W_{\alpha(t)}$. Thus $(M_t)_{t\geqslant 0}$ is a martingale with respect to the filtration $\mathcal{G}_t = \sigma\{W_s \mid 0\leqslant s\leqslant \alpha(t)\}$, whose quadratic variation at time t is $\alpha(t) = \int_0^t (\sqrt{\dot{\alpha}(s)})^2 ds$.

Define $B^{(\alpha)}_t = \int_0^t \frac{dM_s}{\sqrt{\dot{\alpha}(s)}}$, so that $(B^{(\alpha)}_t)_{t\geqslant 0}$ is a Brownian motion. Then

$$Z_t = \int_0^t \sqrt{\dot{\alpha}(s)}\sqrt{1-Z_s^2}dB^{(\alpha)}_s + \int_0^t \frac{R_{\alpha(s)} + \frac{1}{R_{\alpha(s)}}}{\sqrt{\alpha(s)}}(1-Z_s^2)ds - \frac{n}{2}\int_0^t \dot{\alpha}(s)Z_s ds. \tag{3.11}$$

For $y\in[-1,1]$ and $\sigma\geqslant 0$, let $(Y^{\sigma,y}_t)_{t\geqslant\sigma}$ be the solution to the SDE on $[-1,1]$

$$\begin{cases} dY^{\sigma,y}_t = \sqrt{\dot{\alpha}(t)}\sqrt{1-(Y^{\sigma,y}_t)^2}dB^{(\alpha)}_t + [\frac{1}{\sqrt{n}}\big(1-(Y^{\sigma,y}_t)^2\big) - \frac{n}{2}\dot{\alpha}(t)Y^{\sigma,y}_t]dt \\ Y^{\sigma,y}_\sigma = y \end{cases}. \tag{3.12}$$

We divide the proof of the convergence in two steps. In the first one, we prove that for all $y\in[-1,1]$ and $\sigma\geqslant 0$, $Y^{\sigma,y}_t$ converges almost surely to 1; and then prove the convergence of Z_t to 1 in the second one.

Step I: Let $y\in[-1,1]$ and assume without loss of generality $\sigma=0$. In order to lighten the notation, we omit the superscripts y and σ in $Y^{\sigma,y}_t$ during this step. We start by

[1] For example $\varepsilon(t) = O(1/(\log(t))^\alpha)$ with $\alpha>1$.

proving that Y_t is an almost sure asymptotic pseudotrajectory for the flow induced by the ODE

$$\dot{x} = \frac{1}{\sqrt{n}}(1-x^2). \tag{3.13}$$

In order to achieve it, we use Theorem 3.3. Since $x \mapsto (1-x^2)$ is Lipschitz continuous on $[-1,1]$ and that $Y_t \in [-1,1]$ for all $t \geqslant 0$, it remains to verify the hypothesis concerning the noise term.

Set

$$\varepsilon(t) = \dot{\alpha}(t) = (\frac{3}{2}t)^{-\frac{1}{3}}. \tag{3.14}$$

It is then obvious that $\varepsilon(t)$ satisfies (3.9). Because $Y_t \in [-1,1]$ for all $t \geqslant 0$, it is clear that the conditions in Remark 3.4 are satisfied. Consequently, by Theorem 3.3, $(Y_t)_t$ is an almost sure asymptotic pseudotrajectory for the flow induced by (3.13).

Because $\{1\}$ is an attractor for the flow induced by (3.13) with basin $]-1,1]$ and that almost surely $Y_t \in]-1,1]$ infinitely often, then

$$\lim_{t\to\infty} Y_t = 1 \text{ a.s} \tag{3.15}$$

by Theorem 3.2.

Step II: Our goal is to prove

$$\mathbb{P}(\lim_{t\to\infty} Z_t = 1) = 1. \tag{3.16}$$

Define the stopping times $\sigma_0 = 0$,

$$\tau_j = \inf\big(t > \sigma_{j-1} \mid \frac{R_{\alpha(t)}}{\sqrt{\alpha(t)}} = \frac{1}{\sqrt{n}}\big), \quad j \geqslant 1 \tag{3.17}$$

and

$$\sigma_j = \inf\big(t > \tau_j \mid \frac{R_{\alpha(t)}}{\sqrt{\alpha(t)}} = \frac{3}{2\sqrt{n}}\big), \quad j \geqslant 1 \tag{3.18}$$

with the convention $\inf \emptyset = +\infty$.

By Lemma 2.2, we have

$$\mathbb{P}\big(\bigcup_{j\geqslant 1}\{\tau_j = \infty\}\big) = 1 \text{ and for all } j \geqslant 1,\ \mathbb{P}\big(\sigma_j < \infty \mid \tau_j < \infty\big) = 1. \tag{3.19}$$

Let us start by estimating $\mathbb{P}(\lim_{t\to\infty} Z_t = 1,\ \tau_{j+1} = \infty \mid \sigma_j < \infty)$. For $s \in [\sigma_j, \tau_{j+1}]$, we have

$$\frac{R_{\alpha(s)} + \frac{1}{R_{\alpha(s)}}}{\sqrt{\alpha(s)}} \geqslant \frac{1}{\sqrt{n}}.$$

So, by Ikeda-Watanabe's comparison result (see Theorem 1.1, Chapter VI in [9]),

$$\mathbb{P}\big(Z_{(t+\sigma_j)\wedge\tau_{j+1}} \geqslant Y^{\sigma_j, Z_{\sigma_j}}_{(t+\sigma_j)\wedge\tau_{j+1}},\ \forall t \geqslant 0 \mid \sigma_j < \infty\big) = 1. \tag{3.20}$$

As a consequence, we have

$$\begin{aligned} \mathbb{P}\big(\lim_{t\to\infty} Z_t = 1,\ \tau_{j+1} = \infty \mid \sigma_j < \infty\big) &\geqslant \mathbb{P}\big(\lim_{t\to\infty} Y^{\sigma_j, Z_{\sigma_j}}_{t+\sigma_j} = 1,\ \tau_{j+1} = \infty \mid \sigma_j < \infty\big) \\ &= \mathbb{P}\big(\tau_{j+1} = \infty \mid \sigma_j < \infty\big). \end{aligned} \tag{3.21}$$

where the last equality follows from Step I.

Since $(\{\tau_j=\infty\})_{j\geqslant 1}$ is an increasing family of events such that

$$\{\tau_j=\infty\}=\mathcal{N}_j\cup\bigcup_{k=0}^{j-1}\{\tau_{k+1}=\infty,\ \sigma_k<\infty\},$$

where $\mathcal{N}_j$ is an event of probability 0, we obtain from (3.19) and (3.21)

$$\begin{aligned}\mathbb{P}\big(\lim_{t\to\infty}Z_t=1\big) &= \sum_{j\geqslant 0}\mathbb{P}\big(\lim_{t\to\infty}Z_t=1,\ \tau_{j+1}=\infty \text{ and } \sigma_j<\infty\big)\\ &\geqslant \sum_{j\geqslant 0}\mathbb{P}\big(\tau_{j+1}=\infty \text{ and } \sigma_j<\infty\big)\\ &= 1. \end{aligned}\tag{3.22}$$

Consequently, C_t converges almost surely to 1. Therefore, so does

$$\frac{R_t}{t}=\frac{1}{t}\int_0^t C_s ds.\tag{3.23}$$

Proof of the rate of convergence Set $\sigma_0=0$ and define the stopping times

$$\tau_j=\inf\big(t>\sigma_{j-1}\mid C_t=0 \text{ or } \frac{R_t}{t}=\frac{1}{2}\big),\quad j\geqslant 1\tag{3.24}$$

and

$$\sigma_j=\inf\big(t>\tau_j\mid C_t\geqslant\frac{1}{2} \text{ and } \frac{R_t}{t}\geqslant\frac{3}{4}\big),\quad j\geqslant 1\tag{3.25}$$

with the convention $\inf\emptyset=+\infty$. So, by the previous part,

$$\mathbb{P}\big(\bigcup_{j\geqslant 1}\{\tau_j=\infty\}\big)=1 \text{ and for all } j\geqslant 1,\ \mathbb{P}\big(\sigma_j<\infty\mid\tau_j<\infty\big)=1.\tag{3.26}$$

Case $n\geqslant 2$: Set $Z_t=1-C_t$ and define the process $(\vartheta_t)_{t\geqslant 0}$ by $\vartheta_0=2$, $\vartheta_t=Z_t-Z_{\tau_j}+\vartheta_{\tau_j}$ for $t\in[\tau_j,\sigma_j]$ and

$$\vartheta_t=\vartheta_{\sigma_j}-\int_{\sigma_j}^t\sqrt{1-C_s^2}dW_s-\frac{1}{2}\int_{\sigma_j}^t s\vartheta_s ds+\frac{n}{2}(t-\sigma_j)\tag{3.27}$$

for $t\in[\sigma_j,\tau_{j+1}]$. Thanks to (3.10), one can also write Z_t, for $\sigma_j\leqslant t\leqslant\tau_{j+1}$,

$$Z_t=Z_{\sigma_j}-\int_{\sigma_j}^t\sqrt{1-C_s^2}dW_s-\int_{\sigma_j}^t\Big(\big(R_s+\frac{1}{R_s}\big)(1+C_s)+\frac{n}{2}\Big)Z_s ds+\frac{n}{2}(t-\sigma_j).\tag{3.28}$$

Moreover, for such times t, we have

$$\Big(R_t+\frac{1}{R_t}\Big)(1+C_t)\geqslant\frac{t}{2}.$$

Hence, from Ikeda-Watanabe comparison's result

$$\mathbb{P}(Z_t\leqslant\vartheta_t,\ \forall t\geqslant 0)=1.\tag{3.29}$$

Since $1-C_t^2\in[0,1]$, we have by Proposition A.1

$$\mathbb{P}\Big(\vartheta_t=O\big(t^{-1/2}\sqrt{\ln(t)}\big),\tau_{j+1}=\infty\mid\sigma_j<\infty\Big)=\mathbb{P}\Big(\tau_{j+1}=\infty\mid\sigma_j<\infty\Big).\tag{3.30}$$

By the same argumentation as in Step II of the proof of the convergence, one obtains

$$1-C_t=O\big(t^{-1/2}\sqrt{\ln(t)}\big)\ a.s.\tag{3.31}$$

Case $n = 1$: Set $\theta_t = \arccos(C_t) \in [0, \pi]$. Then, as long as $R_t > 0$, θ_t solves

$$d\theta_t = -dW_t - (R_t + \frac{1}{R_t})\sin(\theta_t)dt + dL_t(0) - dL_t(\pi), \tag{3.32}$$

where $(L_t(0))_{t\geqslant 0}$ (resp. $(L_t(\pi))_{t\geqslant 0}$) is a process of finite variation that increases when $\theta_t = 0$ (resp. $\theta_t = \pi$).

Because C_t converges almost surely to 1, then, from the second order Taylor expansion of $\cos(.)$ about 0, an estimate of its rate of convergence is given by the one of θ_t^2 to 0.

Set $\Theta_t = \theta_t^2$. Then, as long as $R_t > 0$, it solves

$$d\Theta_t = -2\sqrt{\Theta_t}dW_t - 2(R_t + \frac{1}{R_t})\sqrt{\Theta_t}\sin(\sqrt{\Theta_t})dt + dt - 2\sqrt{\Theta_t}dL_t(\pi). \tag{3.33}$$

Note that $L_t(\pi)$ increases only when $\Theta_t = \pi^2$.

Following the same methodology as for the case $n \geqslant 2$, define a process $(\Psi_t)_{t\geqslant 0}$ as follows: $\Psi_0 = \pi^2$, $\Psi_t = \Theta_t - \Theta_{\tau_j} + \Psi_{\tau_j}$ if $t \in [\tau_j, \sigma_j]$ and for $t \in [\sigma_j, \tau_{j+1}]$,

$$\Psi_t = \Psi_{\sigma_j} - 2\int_{\sigma_j}^t \sqrt{\Psi_s}dW_s - 2\int_{\sigma_j}^t \frac{s}{\pi}\Psi_s ds + (t - \sigma_j). \tag{3.34}$$

Because for $t \in [\sigma_j, \tau_{j+1}]$, we have

$$(R_t + \frac{1}{R_t}) \geqslant \frac{t}{2} \text{ and } \sqrt{\Theta_t}\sin(\sqrt{\Theta_t}) \geqslant \frac{2}{\pi}\Theta_t,$$

it follows from Ikeda-Watanabe's comparison result

$$\mathbb{P}(\Theta_t \leqslant \Psi_t,\ \forall t \geqslant 0) = 1. \tag{3.35}$$

Since $(\Psi_{t\wedge\tau_{j+1}})_{t\geqslant\sigma_j}$ has the same law as $(Z^2_{t\wedge\tau_{j+1}})_{t\geqslant\sigma_j}$, where $(Z_t)_{t\geqslant\sigma_j}$ is the solution of the SDE

$$dZ_t = dW_t - \frac{t}{\pi}Z_t dt,\ Z_{\sigma_j} = \theta_{\sigma_j}, \tag{3.36}$$

it follows from Proposition A.1

$$\mathbb{P}\Big(\Psi_t = O\big(t^{-1}\ln(t)\big), \tau_{j+1} = \infty \mid \sigma_j < \infty\Big) = \mathbb{P}\Big(\tau_{j+1} = \infty \mid \sigma_j < \infty\Big).$$

Arguing like in Step II of the proof of the convergence, one obtains

$$\Theta_t = O\big(t^{-1}\ln(t)\big)\ a.s.$$

Thus

$$1 - C_t = O\big(t^{-1}\ln(t)\big)\ a.s.$$

Remark 3.5. Following the proof of the rate of convergence from the case $n = 1$, one proves that the rate of convergence to 1 of the solution of the SDE

$$\begin{cases} dC_t^{(n)} = \sqrt{n}\sqrt{1 - (C_t^{(n)})^2}dW_t + [(R_t + \frac{1}{R_t})\big(1 - (C_t^{(n)})^2\big) - \frac{n}{2}C_t^{(n)}]dt \\ C_0^{(n)} = y \end{cases} \tag{3.37}$$

is $O\big(t^{-1}\ln(t)\big)$. Therefore, we conjecture that so does C_t for any $n \geqslant 2$.

4 Conclusion

The motivating model of this work was the real-valued self attracting diffusion

$$dX_t = \nu dW_t + a\int_0^t \sin(X_t - X_s)dsdt, \qquad X_0 = 0$$

with $\nu > 0$ and $a < 0$. Identifying it with $(\cos(X_t), \sin(X_t))$, we had that the almost sure convergence of X_t was an immediate consequence of the more general diffusion on the $n-$dimensional unit sphere $\mathbb{S}^n$

$$dX_t = \nu \circ dW_t(X_t) - a\int_0^t \nabla_{\mathbb{S}^n} V_{X_s}(X_t)dsdt, \qquad X_0 = x \in \mathbb{S}^n$$

with $V_y(x) = \langle x, y\rangle$.

It would now be interesting to study the self interacting diffusion

$$dX_t = \nu dW_t + \sum_{k=1}^n k a_k \int_0^t \sin(k(X_t - X_s))dsdt,$$

where the coefficient $a_k \neq 0$ are such that $\sum_{k=1}^n k^2 a_k < 0$.

Because $\sum_{k=1}^n k^2 a_k = (\sum_{k=1}^n k a_k \sin(k.))'(0)$ and that it has to play a more and more important role if $(X_t)_t$ localizes, it sounds reasonable to formulate the following conjecture:

Conjecture 4.1. *Let $(X_t)_{t\geqslant 0}$ be the solution of the self interacting diffusion*

$$dX_t = \nu dW_t + \sum_{k=1}^n k a_k \int_0^t \sin(k(X_t - X_s))dsdt,\ X_0 = x.$$

If $\sum_{k=1}^n k^2 a_k < 0$ (resp. $\sum_{k=1}^n k^2 a_k > 0$), then X_t converges almost-surely (resp. $\limsup_t X_t > \liminf_t X_t$).

A Almost sure convergence for a time-inhomogeneous linear SDE

In this paper, we needed to use the rate of convergence to 0 for the solution of the SDE

$$dX_t = g_t dW_t + \mu_t dt - (1+\alpha)\lambda t^\alpha X_t dt, \tag{A.1}$$

when $t \mapsto \mu_t$ is a deterministic constant, $\alpha = 1$ and $(g_t)_{t\geqslant 0}$ is an adapted process bounded by 1. Here, $(W_t)_{t\geqslant 0}$ stands for a real Brownian motion and $\lambda > 0$.

Proposition A.1. *Let X_t be the solution of (A.1) with initial condition $X_0 = x$. Assume that $(g_t)_{t\geqslant 0}$ and $(\mu_t)_{t\geqslant 0}$ are adapted processes bounded by some deterministic constant K and let $\alpha \geqslant 0$. Then*

$$X_t = O\Big(t^{-\alpha/2}\sqrt{\log(t)}\Big)\ a.s.$$

Proof. The solution of Equation (A.1) with initial condition $X_0 = x$ is

$$\begin{aligned} X_t &= e^{-\lambda t^{1+\alpha}}\Big(x + \int_0^t e^{\lambda s^{1+\alpha}} g_s dW_s + \int_0^t e^{\lambda s^{1+\alpha}}\mu_s ds\Big) \\ &= e^{-\lambda t^{1+\alpha}}\Big(x + M_t + \int_0^t e^{\lambda s^{1+\alpha}}\mu_s ds\Big). \end{aligned} \tag{A.2}$$

Since

$$e^{-\lambda t^{1+\alpha}}\Big|\int_0^t e^{\lambda s^{1+\alpha}}\mu_s ds\Big| \leqslant K e^{-\lambda t^{1+\alpha}}\int_0^t e^{\lambda s^{1+\alpha}} ds = O\Big(t^{-\alpha}\Big), \tag{A.3}$$

then, on the event $\{\langle M\rangle_\infty := \int_0^\infty e^{2\lambda s^{1+\alpha}} g_s^2 ds < \infty\}$, the result is immediate because in that case, M_t converges almost surely. In the sequel, we assume that we are on the event $\{\langle M\rangle_\infty = \infty\}$.

By the Dubins-Schwarz Theorem (see Theorem 4.6 in [10], Chapter 3) with the law of Iterated Logarithm for Brownian motion (see Theorem 9.23 in [10], Chapter 2), we have

$$M_t = O\Big(\sqrt{\langle M\rangle_t \log\log(\langle M\rangle_t)}\Big). \tag{A.4}$$

By the hypothesis on g_t, we have

$$\langle M\rangle_t = \int_0^t e^{2\lambda s^{1+\alpha}} g_s^2 ds \leqslant K^2 \int_0^t e^{2\lambda s^{1+\alpha}} ds.$$

Therefore,

$$\langle M\rangle_t = O\Big(e^{2\lambda t^{1+\alpha}} t^{-\alpha}\Big). \tag{A.5}$$

The desired result follows from (A.2)–(A.5). $\square$

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