

Universal large deviations for Kac polynomials

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Abstract

We prove the universality of the large deviations principle for the empirical measures of zeros of random polynomials whose coefficients are i.i.d. random variables possessing a density with respect to the Lebesgue measure on \mathbb{C} , \mathbb{R} or \mathbb{R}^+ , under the assumption that the density does not vanish too fast at zero and decays at least as $\exp -|x|^\rho$, $\rho > 0$, at infinity.

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1 Introduction and statement of the main result

Consider random polynomials of the form:

$$P_n(z) = \sum_{k=0}^n a_k z^k = a_n \prod_{i=1}^n (z - z_i) \quad (1.1)$$

where a_0, \dots, a_n are i.i.d. random variables and z_1, \dots, z_n are the complex zeros of P_n . (Such random polynomials are often referred to as *Kac polynomials*.) There is a rich literature about the behavior of the zeros of P_n and we refer to [TV15] for a nice recent review of the subject. An important aspect of the theory is universality. For example, introduce the empirical measure of zeros:

$$\mu_n := \frac{1}{n} \sum_{k=1}^n \delta_{z_k}.$$

Then, Ibragimov and Zaporozhets in [IZ13] showed that $(\mu_n)_{n \in \mathbb{N}}$ converges weakly to the ν_{S^1} , the uniform measure on the unit circle, if and only if $\mathbb{E}(\log(1 + |a_0|)) < \infty$; that is, the limit μ_n is (modulo technical conditions) universal. Other universal properties

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include rescaled limits for μ_n , see [IZ97], correlation functions for the point process of zeros [TV15], and more.

Our interest in this note is in large deviations for the sequence μ_n in the space $\mathcal{M}_1(\mathbb{C})$ equipped with the topology of weak convergence, which makes it into a Polish space. In case the coefficients (a_i) are i.i.d. standard complex Gaussian random variables, Zeitouni and Zelditch¹ proved in [ZZ10] that the sequence of empirical measures of zeros (which we denote by $\mu_n^{\mathbb{C}}$ for this model) satisfies the large deviations principle (LDP) in $\mathcal{M}_1(\mathbb{C})$ with speed n^2 and good rate function $I_{\mathbb{C}}$ defined by

$$I_{\mathbb{C}}(\mu) = - \iint \left(\log |z - w| - \frac{1}{2} \log(1 + |z|^2) - \frac{1}{2} \log(1 + |w|^2) \right) d\mu(z) d\mu(w) + \sup_{z \in S^1} \int (\log |z - w|^2 - \log(1 + |w|^2)) d\mu(w).$$

When $\int \log(1 + |z|^2) d\mu(z)$ is finite, it simplifies to:

$$I_{\mathbb{C}}(\mu) = - \iint \log |z - w| d\mu(z) d\mu(w) + \sup_{z \in S^1} \int \log |z - w|^2 d\mu(w).$$

This has been extended by Butez [But15] to the case of real-valued i.i.d. standard Gaussians (a_i) : the empirical measure of zeros, denoted $\mu_n^{\mathbb{R}}$ for that model, satisfies the LDP in $\mathcal{M}_1(\mathbb{C})$ with speed n^2 and good rate function $I_{\mathbb{R}}$ defined by

$$I_{\mathbb{R}}(\mu) = \begin{cases} \frac{1}{2} I_{\mathbb{C}}(\mu) & \text{if } \mu \text{ is invariant under } z \mapsto \bar{z} \\ \infty & \text{otherwise.} \end{cases}$$

Finally, when the coefficients (a_i) are i.i.d. standard exponential random variables, Ghosh and Zeitouni proved in [GZ16] that the sequence of empirical measures of zeros, denoted by $\mu_n^{\mathbb{R}^+}$, satisfies the LDP in $\mathcal{M}_1(\mathbb{C})$ with speed n^2 and good rate function $I_{\mathbb{R}^+}$ defined by:

$$I_{\mathbb{R}^+}(\mu) = \begin{cases} \frac{1}{2} I_{\mathbb{C}}(\mu) & \text{if } \mu \in \bar{\mathcal{P}} \\ \infty & \text{otherwise.} \end{cases}$$

Where \mathcal{P} is the set of empirical measures of zeros of polynomials with positive coefficients and $\bar{\mathcal{P}}$ is its closure for the weak topology. (An explicit description of \mathcal{P} is provided in [BE15].)

Apart for the models described above, to our knowledge no other LDPs for the empirical measure of zeros of Kac polynomials appear in the literature. (In a different direction, Zelditch [Zel13] extended the results of [ZZ10] to the case of Riemann surfaces, and Feng and Zelditch [FZ11] studied some cases of non-i.i.d. coefficients in the context of more general $P(\phi)_2$ random polynomials.)

Our main result concerns the universality of the above large deviation principles, under mild technical conditions.

Theorem 1.1. *Let E be \mathbb{C} , \mathbb{R} or \mathbb{R}^+ . Let a_0, \dots, a_n be i.i.d. random variables with a density g with respect to the Lebesgue measure ℓ_E on E . Assume that:*

1. *There exist $\rho > 0$, $r > 0$ and $R > 0$ such that*

$$\forall z \in \mathbb{C}, \quad g(z) \leq \exp(-r|z|^\rho + R) \tag{1.2}$$

¹In fact, [ZZ10] work in $\mathbb{C}P^1$ and consider more general ensembles of holomorphic polynomial with Gaussian coefficients, but it is not hard to check that their result, when specialized to Kac polynomials with complex Gaussian coefficients, is equivalent to the one here; this is implicitly stated in [ZZ10] and explicitly checked in [But15].

2. There exists $\delta > 0$ such that for all $\lambda > 0$:

$$\int 1_{|x| \leq \delta} \frac{1}{g(x)^\lambda} d\ell_E(x) < \infty \tag{1.3}$$

Then the sequence of empirical measures $(\mu_n)_{n \in \mathbb{N}}$ satisfies a large deviations principle in $\mathcal{M}_1(\mathbb{C})$ with speed n^2 and rate function I_E .

The second assumption in Theorem 1.1 means that either the density g does not vanish around zero or, if it vanishes, g is greater than any $|x|^a$ in a neighborhood of zero.

Before describing the (simple) ideas behind the proof of Theorem 1.1, we explain some of the background and why we find the theorem somewhat surprising. The proof of the LDPs in the Gaussian and Exponential cases is based on an explicit expression for the joint distributions of zeros, that we review below. Given that expression, the proofs of the LDP follow (with some detours) a track well explored in the case of the empirical measure of eigenvalues of random matrices. For the latter, large deviations have been extensively studied, initially by Ben Arous and Guionnet [BAG97], Ben Arous and Zeitouni [BAZ98] and Hiai and Petz [HP00]. Recently, the large deviations for the empirical measure were proved for Wigner matrices with entries possessing heavier-than-Gaussian tails by Bordenave and Caputo [BC14], with a rate function depending on the tail of the entries. Very similar results were obtained by Groux [Gro15] for Wishart matrices. In particular, it follows from these results that in the random matrix setup, the rate function is known to be not universal; this is in sharp contrast with Theorem 1.1.

As mentioned above, the LDP for Kac polynomials in the Gaussian and exponential coefficients cases begin with an explicit expression for the density of zeros, which we now explain. We concentrate first on the case of complex Gaussian coefficients. Note that the second equality in (1.1) gives an $n!$ -to-1 map between (a_n, z_1, \dots, z_n) and (a_0, \dots, a_n) . A classical computation of the Jacobian followed by integration over a_n , see e.g. [BBL92], [But15],[FH99], [ZZ10], shows that the distribution of (z_1, \dots, z_n) possesses a density with respect to the Lebesgue measure $d\ell_{\mathbb{C}^n}$ on \mathbb{C}^n given by:

$$\frac{1}{\pi^n} \frac{\prod_{i < j} |z_i - z_j|^2}{\left(\int \prod_{k=1}^n |z - z_k|^2 d\nu_{S^1}(z) \right)^{n+1}} = \frac{1}{\pi^n} \frac{\prod_{i < j} |z_i - z_j|^2}{\|\tilde{a}\|_2^{2n+2}} \tag{1.4}$$

where $\tilde{a} = (a_0/a_n, \dots, a_{n-1}/a_n, 1)$ is a continuous function of (z_1, \dots, z_n) given explicitly by Vieta's formula.

In the case of real Gaussian coefficients, the probability of having k real zeros is positive for k having the same parity of n . Following a computation of Zaporozhet in [Zap04], one obtains that the distribution of the roots of P_n is given by:

$$\begin{aligned} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{2^k \Gamma(\frac{n+1}{2})}{k!(n-2k)! \pi^{(n-1)/2}} \frac{\prod_{i < j} |z_i - z_j|}{\left(\int \prod_{i=1}^n |z - z_i|^2 d\nu_{S^1}(z) \right)^{(n+1)/2}} d\ell_{n,k}(z_1, \dots, z_n) \\ = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{2^k \Gamma(\frac{n+1}{2})}{k!(n-2k)! \pi^{(n-1)/2}} \frac{\prod_{i < j} |z_i - z_j|}{\|\tilde{a}\|_2^{n+1}} d\ell_{n,k}(z_1, \dots, z_n) \end{aligned}$$

where

$$d\ell_{n,k}(z_1, \dots, z_n) = d\ell_{\mathbb{R}}(z_1) \dots d\ell_{\mathbb{R}}(z_{n-2k}) d\ell_{\mathbb{C}}(z_{n-k+1}) \dots d\ell_{\mathbb{C}}(z_n). \tag{1.5}$$

Note that the k -th term of the mixture corresponds to the case where P_n has $n - 2k$ real roots.

Finally, in the case of positive exponential real coefficients, the distribution of the vector of the zeros is given by:

$$\begin{aligned} & \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{2^k n!}{k!(n-2k)!} \frac{\prod_{i < j} |z_i - z_j|}{(\prod_{i=1}^n |1 - z_i|)^{(n+1)}} d\ell_{n,k}(z_1, \dots, z_n) \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{2^k n!}{k!(n-2k)!} \frac{\prod_{i < j} |z_i - z_j|}{\|\tilde{a}\|_1^{n+1}} d\ell_{n,k}(z_1, \dots, z_n). \end{aligned}$$

Main idea of the proof of Theorem 1.1. We will prove the universality of the LDP by comparing the distributions of the vectors of the zeros in the different models. Assume that the joint density of (a_0, \dots, a_n) is p and μ_n is the empirical measure associated to this model. Assume one could find two sequences $(b_n)_{n \in \mathbb{N}}$ and $(c_n)_{n \in \mathbb{N}}$ satisfying

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \log b_n = \lim_{n \rightarrow \infty} \frac{1}{n^2} \log c_n = 0 \tag{1.6}$$

and two probability densities on \mathbb{C}^n , F_n and G_n satisfying:

$$\forall (z_1, \dots, z_n) \in \mathbb{C}^n \quad b_n F_n(z_1, \dots, z_n) \leq p(z_1, \dots, z_n) \leq c_n G_n(z_1, \dots, z_n) \tag{1.7}$$

such that, under the distribution given by F_n or G_n , the sequence of empirical measures $(\mu_n^{F_n})_{n \in \mathbb{N}}$ and $(\mu_n^{G_n})_{n \in \mathbb{N}}$ satisfies a LDP in $\mathcal{M}_1(\mathbb{C})$, with speed n^2 and the same rate function I . Then, the sequence $(\mu_n)_{n \in \mathbb{N}}$ satisfies a LDP in $\mathcal{M}_1(\mathbb{C})$ with speed n^2 and rate function I , since for any set A , $\mathbb{P}(\mu_n \in A)$ is an integral with respect to the distribution of the zeros and therefore one can use the bounds (1.7) to obtain the LDP.

In practice, we will obtain (1.7) by noting that if the joint distribution of the coefficients is a function of a norm $\|\cdot\|$ of the vector of the coefficients, the distribution of the zeros is roughly of the form:

$$\frac{\prod_{i < j} |z_i - z_j|^2}{\|\tilde{a}\|^{2n+2}}.$$

If $\|\cdot\|$ can be compared with $\|\cdot\|_2$ with nice constants, then we can relate the density of zeros to the Gaussian cases in the spirit of (1.7). For i.i.d. variables, the first hypothesis of the theorem is used to replace the joint distribution of the coefficients by a function of $\|a\|_\rho$ and then we prove the upper bound for the latter distribution. The second hypothesis means that, for the lower bound, we can replace the joint distribution of a by a $1_{\|a\|_\infty \leq \delta}$ which is also a function of a norm.

We conclude this introduction by stating and proving a technical lemma that will be used in the proof of the LDP lower bound.

Lemma 1.2. *Let E be \mathbb{C} , \mathbb{R} or \mathbb{R}^+ . Let X_0, \dots, X_n be i.i.d. random variables, uniformly distributed on the disk in E of center 0 and radius δ . Assume that there exists $\delta > 0$ such that for all $\lambda > 0$,*

$$c(\lambda) := \int 1_{|x| \leq \delta} \frac{1}{g(x)^\lambda} d\ell_E(x) < \infty. \tag{1.8}$$

Then, for any $K > 0$ and $\varepsilon > 0$ there exists $n_0 = n_0(K, \delta, \varepsilon)$ such that for all $n > n_0$,

$$\int 1_{\{\prod_{k=0}^n g(a_k) \leq e^{-\varepsilon n^2}\}} 1_{\|a\|_\infty < \delta} d\ell_E^{\otimes n}(a_0, \dots, a_n) \leq e^{-Kn^2}. \tag{1.9}$$

Proof of Lemma 1.2. Fix $K > 0$, micking the proof of Chernoff's inequality, we have:

$$\begin{aligned} & \int 1_{\{\prod_{k=0}^n g(a_k) \leq e^{-\varepsilon n^2}\}} 1_{\|a\|_\infty < \delta} d\ell_{E^{n+1}}(a_0, \dots, a_n) \\ &= \int 1_{\{\prod_{k=0}^n g(a_k)^{-\lambda} \geq e^{\lambda \varepsilon n^2}\}} 1_{\|a\|_\infty < \delta} d\ell_{E^{n+1}}(a_0, \dots, a_n) \\ &\leq e^{-\lambda \varepsilon n^2} \int \prod_{k=0}^n [g(a_k)^{-\lambda} 1_{|a_k| < \delta}] d\ell_{E^{n+1}}(a_0, \dots, a_n) \\ &\leq e^{-\lambda \varepsilon n^2} e^{(n+1)c(\lambda)}. \end{aligned}$$

The proof is completed by taking λ large enough so that $\lambda\varepsilon > K$ and then taking n_0 large enough so that $e^{-\lambda\varepsilon n^2} e^{(n+1)c(\lambda)} \leq e^{-Kn^2}$ for all $n > n_0$. \square

2 Proof of Theorem 1.1

The proof of the main theorem is made in two steps: we start by proving the theorem when the coefficients are complex, and then we treat the real and the positive case. The proof of the three cases are very similar, the arguments and ideas are exactly the same.

Proof of Theorem 1.1. Complex coefficients. Recall that the density of the distribution of the random vector of zeros (z_1, \dots, z_n) (taken at random uniform order) with respect to $\ell_{\mathbb{C}^n}$ is given by

$$p(z_1, \dots, z_n) = \frac{1}{n!} \int \prod_{i < j} |z_i - z_j|^2 |a_n|^{2n} g(a_0) \dots g(a_n) d\ell_{\mathbb{C}}(a_n)$$

where the a_j 's are seen as functions of z_1, \dots, z_n and a_n using Vieta's formula. See e.g. [HKPV09, Lemma 1.1.1 p3] for a proof of this classical result.

Upper Bound. Using the inequality (1.2), we obtain:

$$p(z_1, \dots, z_n) \leq \frac{1}{n!} \int \prod_{i < j} |z_i - z_j|^2 |a_n|^{2n} \exp(-r \sum_{k=0}^n |a_k|^\rho) e^{(n+1)R} d\ell_{\mathbb{C}}(a_n)$$

For a vector $b = (b_0, \dots, b_n)$ and $\rho > 0$, set $\|b\|_\rho = (\sum_{i=0}^n |b_i|^\rho)^{1/\rho}$. Then,

$$\int \prod_{i < j} |z_i - z_j|^2 |a_n|^{2n} e^{-r \sum_{k=0}^n |a_k|^\rho} d\ell_{\mathbb{C}}(a_n) = \int \prod_{i < j} |z_i - z_j|^2 |a_n|^{2n} e^{-r |a_n|^\rho \|\tilde{a}\|_\rho^\rho} d\ell_{\mathbb{C}}(a_n)$$

where $\tilde{a} = (a_0/a_n, \dots, a_{n-1}/a_n, 1)$. We note that \tilde{a} only depends on the zeros and not on a_n , so we can compute the last integral using the change of variables $u = a_n \|\tilde{a}\|_\rho$ to obtain:

$$p(z_1, \dots, z_n) \leq \frac{1}{n!} e^{(n+1)R} \frac{\prod_{i < j} |z_i - z_j|^2}{\|\tilde{a}\|_\rho^{2n+2}} \int |u|^{2n} e^{-r|u|^\rho} d\ell_{\mathbb{C}}(u).$$

Finally, using the classical inequalities on \mathbb{C}^{n+1} , which are a consequence of the Holder inequality:

$$\begin{aligned} \text{if } \rho > 2, \quad \|\cdot\|_\rho &\geq \frac{1}{n^{1/2-1/\rho}} \|\cdot\|_2 \\ \text{if } \rho \leq 2, \quad \|\cdot\|_\rho &\geq \|\cdot\|_2 \end{aligned}$$

we obtain that there exists a sequence $(\gamma_n)_{n \in \mathbb{N}}$ such that, for any $\rho > 0$,

$$\|\cdot\|_\rho \geq \gamma_n \|\cdot\|_2 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \gamma_n = 0. \tag{2.1}$$

Using this inequality we get

$$\begin{aligned} p(z_1, \dots, z_n) &\leq e^{(n+1)R} \frac{\prod_{i < j} |z_i - z_j|^2}{\|\tilde{a}\|_2^{2n+2}} \frac{1}{\gamma_n^{2n+2}} \int |u|^{2n} e^{-r|u|^\rho} d\ell_{\mathbb{C}}(u) \\ &\leq \frac{1}{n!} e^{(n+1)R} \frac{1}{\pi^n} \frac{\prod_{i < j} |z_i - z_j|^2}{\|\tilde{a}\|_2^{2n+2}} \times \frac{\pi^n}{\gamma_n^{2n+2}} \int |u|^{2n} e^{-r|u|^\rho} d\ell_{\mathbb{C}}(u). \end{aligned}$$

The first term of the product is the distribution $\mu_n^{\mathbb{C}}$, see (1.4), and thanks to (2.1) we have

$$c_n := e^{(n+1)R} \frac{\pi^n}{n! \gamma_n^{2n+2}} \int |u|^{2n} e^{-r|u|^{\rho}} d\ell_{\mathbb{C}}(u) = e^{O(n \log n)}.$$

Let $A \subset \mathcal{M}_1(\mathbb{C})$ be a Borel set. Then,

$$\begin{aligned} \frac{1}{n^2} \log \mathbb{P}(\mu_n \in A) &= \frac{1}{n^2} \log \int 1_{\mu_n \in AP}(z_1, \dots, z_n) d\ell_{\mathbb{C}^n}(z_1, \dots, z_n) \\ &\leq \frac{1}{n^2} \log \int \frac{1}{\pi^n} \frac{1_{\mu_n \in A} \prod_{i < j} |z_i - z_j|^2}{\left(\int \prod_{k=1}^n |z - z_k|^2 d\nu_{S^1}(z)\right)^{n+1}} d\ell_{\mathbb{C}^n}(z_1, \dots, z_n) + \frac{\log c_n}{n^2} \\ &= \frac{1}{n^2} \log \mathbb{P}(\mu_n^{\mathbb{C}} \in A) + \frac{\log c_n}{n^2}. \end{aligned}$$

Therefore, using the LDP upper bound for $\mu_n^{\mathbb{C}}$, we complete the proof of the upper bound by noting that

$$\limsup_{n \rightarrow \infty} \frac{1}{n^2} \log \mathbb{P}(\mu_n \in A) \leq - \inf_{\text{clo} A} I_{\mathbb{C}}.$$

Lower Bound. First, we show that the technical lemma allows us to reduce the problem to the proof of the lower bound for i.i.d. (a_i) , with uniform distribution on the disk $D(0, \delta)$. Let $A \subset \mathcal{M}_1(\mathbb{C})$ be a Borel set with $\inf_{\text{int} A} I_{\mathbb{C}} < \infty$, fix $K > \inf_{\text{int} A} I_{\mathbb{C}}$ and $\varepsilon > 0$ then, thanks to Lemma 1.2 there exists n_0 such that for any $n > n_0$:

$$\begin{aligned} \mathbb{P}(\mu_n \in A) &= \int 1_{\mu_n \in A} \prod_{k=0}^n g(a_k) d\ell_{\mathbb{C}}(a_0) \dots d\ell_{\mathbb{C}}(a_n) \\ &\geq \int 1_{\{\prod_{k=0}^n g(a_i) \geq e^{-\varepsilon n^2}\}} 1_{\mu_n \in A} 1_{\|a\|_{\infty} < \delta} \prod_{k=0}^n g(a_k) d\ell_{\mathbb{C}}(a_0) \dots d\ell_{\mathbb{C}}(a_n) \\ &\geq e^{-\varepsilon n^2} \int 1_{\{\prod_{k=0}^n g(a_i) \geq e^{-\varepsilon n^2}\}} 1_{\mu_n \in A} 1_{\|a\|_{\infty} < \delta} d\ell_{\mathbb{C}}(a_0) \dots d\ell_{\mathbb{C}}(a_n) \\ &\geq e^{-\varepsilon n^2} \int 1_{\mu_n \in A} 1_{\|a\|_{\infty} < \delta} d\ell_{\mathbb{C}}(a_0) \dots d\ell_{\mathbb{C}}(a_n) - e^{-(K+\varepsilon)n^2}. \end{aligned} \tag{2.2}$$

The integral $\int 1_{\mu_n \in A} 1_{\|a\|_{\infty} < \delta} d\ell_{\mathbb{C}}(a_0) \dots d\ell_{\mathbb{C}}(a_n)$ is, up to a normalizing factor $(\pi\delta^2)^{n+1}$ which is of order $e^{O(n)}$, the probability that the empirical measure of the zeros of a random polynomial with i.i.d. uniform coefficients on the disk $D(0, \delta)$ belongs to A .

Now we deal with this integral using the same techniques used for the upper bound:

$$\begin{aligned} &\int 1_{\mu_n \in A} 1_{\|a\|_{\infty} < \delta} d\ell_{\mathbb{C}}(a_0) \dots d\ell_{\mathbb{C}}(a_n) \\ &= \int 1_{\mu_n \in A} \frac{1}{n!} \prod_{i < j} |z_i - z_j|^2 \int 1_{\|a_n\|_{\infty} < \delta} |a_n|^{2n} d\ell_{\mathbb{C}}(a_n) d\ell_{\mathbb{C}^n}(z_1, \dots, z_n) \\ &= \int 1_{\mu_n \in A} \frac{1}{n!} \frac{\prod_{i < j} |z_i - z_j|^2}{\|\tilde{a}\|_{\infty}^{2n+2}} d\ell_{\mathbb{C}^n}(z_1, \dots, z_n) \int |u|^{2n} 1_{|u| < \delta} d\ell_{\mathbb{C}}(u) \\ &= \frac{1}{\pi^n} \int 1_{\mu_n \in A} \frac{\prod_{i < j} |z_i - z_j|^2}{\|\tilde{a}\|_2^{2n+2}} d\ell_{\mathbb{C}^n}(z_1, \dots, z_n) \frac{\pi^n}{n!} \int |u|^{2n} 1_{|u| < \delta} d\ell_{\mathbb{C}}(u) \\ &= \mathbb{P}(\mu_n^{\mathbb{C}} \in A) \frac{\pi^n}{n!} \int |u|^{2n} 1_{|u| < \delta} d\ell_{\mathbb{C}}(u). \end{aligned}$$

Here we used the change of variables $u = \|\tilde{a}\|_{\infty} a_n$, using the fact that $\|\tilde{a}\|_{\infty}$ does not depend on a_n and the inequality $\|\cdot\|_{\infty} \leq \|\cdot\|_2$ in \mathbb{C}^{n+1} . Since

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \log \left(\frac{\pi^n}{n!} \int |u|^{2n} 1_{|u| < \delta} d\ell_{\mathbb{C}}(u) \right) = 0,$$

we obtain

$$\liminf_{n \rightarrow \infty} \frac{1}{n^2} \log \int 1_{\mu_n \in A} \frac{\prod_{i < j} |z_i - z_j|^2}{n!} \int 1_{\|a_n\| \leq \delta} |a_n|^{2n} d\ell_{\mathbb{C}}(a_n) d\ell_{\mathbb{C}^n}(z_1, \dots, z_n) \geq - \inf_{\text{int} A} I_{\mathbb{C}}.$$

Combined with (2.2) we obtain

$$\liminf_{n \rightarrow \infty} \frac{1}{n^2} \log \mathbb{P}(\mu_n \in A) \geq -\varepsilon - \inf_{\text{int} A} I_{\mathbb{C}}.$$

Taking the limit as ε goes to zero completes the proof of the lower bound.

Real coefficients. Let E be \mathbb{R} or \mathbb{R}^+ . The proof for real coefficients is essentially the same as for complex coefficients, except that the distribution of the roots is a mixture of measures instead of an absolutely continuous measure. We will apply the same ideas to each term of the mixture to obtain the upper and lower bound. If the coefficients a_k 's are i.i.d. random variables with density g with respect to the Lebesgue measure on E , then the distribution of the vector (z_1, \dots, z_n, a_n) is given by:

$$\begin{aligned} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{2^k}{k!(n-2k)!} |a_n|^n \prod_{i < j} |z_i - z_j| \prod_{k=0}^n g(a_k) d\ell_E(a_n) d\ell_{n,k}(z_1, \dots, z_n) \\ = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{2^k}{k!(n-2k)!} p_{n,k}(z_1, \dots, z_n, a_n) d\ell_E(a_n) d\ell_{n,k}(z_1, \dots, z_n). \end{aligned}$$

Using exactly the same reasoning as in the complex case, we define θ_n^E as:

$$\theta_n^E = \begin{cases} \frac{\pi^{(n-1)/2}}{\Gamma(\frac{n+1}{2})} & \text{if } E = \mathbb{R} \\ \frac{1}{n!} & \text{if } E = \mathbb{R}^+ \end{cases}$$

and we notice that

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \log \theta_n^E = 0.$$

We obtain that for any k :

$$\int p_{n,k}(z_1, \dots, z_n) d\ell_E(a_n) \leq \frac{1}{\theta_n^E} \frac{\prod_{i < j} |z_i - z_j|}{(\int \prod_{i=1}^n |z - z_i|^2 d\nu_{S^1})^{(n+1)/2}} \theta_n^E.$$

This inequality implies that, for any Borel set $A \in \mathcal{M}_1(\mathbb{C})$:

$$\begin{aligned} \mathbb{P}(\mu_n \in A) &= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{2^k}{k!(n-2k)!} \int 1_{\mu_n \in A} \int p_{n,k}(z_1, \dots, z_n) d\ell_E(a_n) d\ell_{n,k}(z_1, \dots, z_n) \\ &\leq \theta_n^E \mathbb{P}(\mu_n^E \in A). \end{aligned}$$

Using the large deviations principle for $(\mu_n^E)_{n \in \mathbb{N}}$ ends the proof of the upper bound.

The proof of the lower bound is very similar to the complex case, we use the technical lemma to deal with i.i.d. uniform random variables on the disk $D(0, \delta)$.

$$\begin{aligned} \mathbb{P}(\mu_n \in A) &= \int 1_{\mu_n \in A} \prod_{k=0}^n g(a_k) d\ell_E(a_0) \dots d\ell_E(a_n) \\ &\geq \int 1_{\{\prod_{k=0}^n g(a_i) \geq e^{-\varepsilon n^2}\}} 1_{\mu_n \in A} 1_{\|a\|_\infty < \delta} \prod_{k=0}^n g(a_k) d\ell_E(a_0) \dots d\ell_E(a_n) \\ &\geq e^{-\varepsilon n^2} \int 1_{\mu_n \in A} 1_{\|a\|_\infty < \delta} d\ell_E(a_0) \dots d\ell_E(a_n) - e^{-(K+\varepsilon)n^2}. \end{aligned} \tag{2.3}$$

We transform this integral in order to compare it to one of the known cases.

$$\begin{aligned} & \int 1_{\mu_n \in A} 1_{\|a\|_\infty < \delta} d\ell_E(a_0) \dots d\ell_E(a_n) \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{2^k}{k!(n-2k)!} \int 1_{\mu_n \in A} 1_{\|a\|_\infty < \delta} |a_n|^n \prod_{i < j} |z_i - z_j| d\ell_E(a_n) d\ell_{n,k}(z_1, \dots, z_n) \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{2^k}{k!(n-2k)!} \int |u|^{n+1} 1_{|u| < \delta} d\ell_E(u) \int 1_{\mu_n \in A} \frac{\prod_{i < j} |z_i - z_j|}{\|\tilde{a}\|_\infty^{n+1}} d\ell_{n,k}(z_1, \dots, z_n). \end{aligned}$$

If $E = \mathbb{R}$, we use the inequality $\|\cdot\|_\infty \leq \|\cdot\|_2$ on \mathbb{R}^{n+1} to obtain

$$\begin{aligned} & \int 1_{\mu_n \in A} 1_{\|a\|_\infty < \delta} d\ell_{\mathbb{R}}(a_0) \dots d\ell_{\mathbb{R}}(a_n) \\ & \geq \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{2^k}{k!(n-2k)!} \int |u|^{n+1} 1_{|u| < \delta} d\ell_{\mathbb{R}}(u) \int 1_{\mu_n \in A} \frac{\prod_{i < j} |z_i - z_j|}{\|\tilde{a}\|_2^{n+1}} d\ell_{n,k}(z_1, \dots, z_n). \end{aligned}$$

If $E = \mathbb{R}^+$, we use the inequality $\|\cdot\|_\infty \leq \|\cdot\|_1$ on \mathbb{R}^{n+1} to obtain

$$\begin{aligned} & \int 1_{\mu_n \in A} 1_{\|a\|_\infty < \delta} d\ell_{\mathbb{R}^+}(a_0) \dots d\ell_{\mathbb{R}^+}(a_n) \\ & \geq \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{2^k}{k!(n-2k)!} \int |u|^{n+1} 1_{|u| < \delta} d\ell_{\mathbb{R}^+}(u) \int 1_{\mu_n \in A} \frac{\prod_{i < j} |z_i - z_j|}{\|\tilde{a}\|_1^{n+1}} d\ell_{n,k}(z_1, \dots, z_n). \end{aligned}$$

Note that the only difference between the case \mathbb{R} and the case \mathbb{R}^+ is the reference norm employed. Using (2.3) with the last two inequalities, we obtain that for any $\varepsilon > 0$ fixed, we have:

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{n^2} \log \mathbb{P}(\mu_n \in A) & \geq \liminf_{n \rightarrow \infty} \frac{1}{n^2} \log \mathbb{P}(\mu_n^E \in A) + \lim_{n \rightarrow \infty} \frac{1}{n^2} \log \theta_n^E - \varepsilon \\ & \geq - \inf_{\text{int}A} I_E - \varepsilon. \end{aligned}$$

Taking the limit as ε goes to zero ends the proof of the large deviations lower bound for the real and positive cases. □

3 Concluding remarks and an open problem

We focused in this note on Kac polynomials but we could as well study the universality of the large deviations for the zeros of

$$P_n = \sum_{k=0}^n a_k R_k$$

where the R_k 's are orthogonal polynomials satisfying the assumptions of regularity given in [ZZ10] and [But15]. In this case, the distribution of the zeros can be computed ([But15, Theorem 5.1]) and, under the same hypotheses as in Theorem 1.1, the same large deviations principle as for Gaussian coefficients holds. Similar ideas apply to certain non i.i.d. models such as the $P(\phi)_2$ model of [FZ11].

A significant limitation of our approach is the use of the assumption (1.3) in Theorem 1.1. While it is possible that it can be relaxed, we note that some assumption of this type is necessary for the universality result. Indeed, if the support of the distribution of the coefficients is inside an annulus, it follows from Jensen's formula, see [HN08], that μ_n converges deterministically towards ν_{S^1} . Hence, no non-trivial LDP can hold in this case. An interesting test case is the case where the i.i.d. coefficients possess the

density $|z|^\alpha 1_{|z|<\delta}$ for some $\alpha > 0$ and $\delta > 0$. In that case, the distribution of the zeros (z_1, \dots, z_n) is absolutely continuous with respect to the Lebesgue measure on \mathbb{C}^n with density proportional to:

$$\prod_{i<j} |z_i - z_j|^2 \int |a_n|^{2n} \prod |a_i|^{\alpha} 1_{\|a\|_\infty < \delta} d\ell_{\mathbb{C}}(a_n) = \frac{\prod_{i<j} |z_i - z_j|^2}{\|\tilde{a}\|_\infty^{2n+2+n\alpha}} \prod_{k=0}^n \frac{|a_k|}{|a_n|}.$$

If we are able to prove that the term $\prod_{k=0}^n \frac{|a_k|}{|a_n|}$ does not contribute to the large deviations, then a LDP at speed n^2 would hold with rate function

$$I_\alpha(\mu) = - \iint \log |z - w| d\mu(z) d\mu(w) + (2 + \alpha) \sup_{z \in S^1} \int \log |z - w| d\mu(w).$$

In particular, we do not expect universality in that case. We have not been able to carry out the analysis of this setup.

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