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Complete Bernstein functions and subordinators with nested ranges. A note on a paper by P. Marchal*

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Abstract

Let $\alpha : [0,1] \to [0,1]$ be a measurable function. It was proved by P. Marchal [2] that the function

$$\phi^{(\alpha)}(\lambda) := \exp\left[\int_0^1 \frac{\lambda - 1}{1 + (\lambda - 1)x}\,\alpha(x)\,\mathrm{d}x\right], \quad \lambda > 0$$

is a special Bernstein function. Marchal used this to construct, on a single probability space, a family of regenerative sets $\mathcal{R}^{(\alpha)}$ such that $\mathcal{R}^{(\alpha)} \stackrel{\text{law}}{=} \overline{\{S_t^{(\alpha)} : t \geq 0\}}$ ($S^{(\alpha)}$ is the subordinator with Laplace exponent $\phi^{(\alpha)}$) and $\mathcal{R}^{(\alpha)} \subset \mathcal{R}^{(\beta)}$ whenever $\alpha \leq \beta$. We give two simple proofs showing that $\phi^{(\alpha)}$ is a complete Bernstein function and extend Marchal's construction to all complete Bernstein functions.

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For a measurable function $\alpha : [0,1] \to [0,1]$ define

$$\phi^{(\alpha)}(\lambda) := \exp\left[\int_0^1 \frac{\lambda - 1}{1 + (\lambda - 1)x}\,\alpha(x)\,\mathrm{d}x\right], \quad \lambda > 0. \tag{0.1}$$

If $\alpha \equiv \alpha_0$ is a constant function with $\alpha_0 \in (0,1)$, then $\phi^{(\alpha)}$ reduces to the fractional power function $\lambda \mapsto \lambda^{\alpha_0}$. In a recent paper, Marchal [2] proves that for any measurable function $\alpha : [0,1] \to [0,1]$, $\phi^{(\alpha)}$ is a special Bernstein function, and the dual Bernstein function $\lambda/\phi^{(\alpha)}(\lambda)$ is $\phi^{(1-\alpha)}$. As an application, Marchal constructs, on a single probability space, a family of regenerative sets $\mathcal{R}^{(\alpha)}$ such that $\mathcal{R}^{(\alpha)} \stackrel{\text{law}}{=} \overline{\{S_t^{(\alpha)} : t \geq 0\}}$ ($S^{(\alpha)}$ is the subordinator with Laplace exponent $\phi^{(\alpha)}$) and $\mathcal{R}^{(\alpha)} \subset \mathcal{R}^{(\beta)}$ whenever $\alpha \leq \beta$. In this short note, we will go further to show that $\phi^{(\alpha)}$ is a complete Bernstein function for all measurable weights $\alpha : [0,1] \to [0,1]$ and that Marchal's construction holds for all complete Bernstein functions. Independently of us this has been remarked by Alili, Jedidi and Rivero in [1, Example 4.2, p. 730].

Let us first briefly recall some basic facts on Bernstein functions. We use the monograph [3] as our standard reference for Bernstein functions. A function is called

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a Bernstein function, if $f:(0,\infty)\to[0,\infty)$, $f\in C^\infty(0,\infty)$ and $(-1)^{k-1}f^{(k)}\geq 0$ for all $k\in\mathbb{N}$. All Bernstein functions admit a unique Lévy–Khintchine representation

$$f(\lambda)=a+b\lambda+\int_{(0,\infty)}\left(1-\mathrm{e}^{-\lambda x}\right)\nu(\mathrm{d}x), \tag{0.2}$$

where $a,b\geq 0$ and ν is a measure on $(0,\infty)$ satisfying $\int_{(0,\infty)}(x\wedge 1)\,\nu(\mathrm{d}x)<\infty$. A Bernstein function f is said to be a special Bernstein function if $f^*(\lambda):=\lambda/f(\lambda)$ is again a Bernstein function; in this case, f^* is called the dual Bernstein function of f. A Bernstein function f is a complete Bernstein function if its Lévy measure ν in (0.2) has a completely monotone density m (i.e. $m\in C^\infty(0,\infty)$ and $(-1)^k m^{(k)}\geq 0$ for all $k\in\mathbb{N}\cup\{0\}$) w.r.t. Lebesgue measure. We use $\mathcal{BF}$, $\mathcal{SBF}$ and $\mathcal{CBF}$ to denote the collections of all Bernstein functions, special Bernstein functions and complete Bernstein functions, respectively. It is known that

$$\mathcal{CBF}\subsetneqq\mathcal{SBF}\subsetneqq\mathcal{BF},$$

see [3, Propositions 11.16 and 11.17 and Example 11.18]. In contrast to $\mathcal{SBF}$, the class $\mathcal{CBF}$ has well-understood structural properties and many examples of complete Bernstein functions are known, cf. [3, Chapters 6 and 16].

We can now state the main result of this note.

Theorem. For any measurable function $\alpha:[0,1]\to[0,1]$, the function $\phi^{(\alpha)}$ defined by (0.1) is a complete Bernstein function.

Remark 1. Let $c,d\in\mathbb{R}$ with $c<d$. For any measurable function $\alpha:[c,d]\to[0,1]$, it follows from our theorem and a straightforward change of variables that the function

$$\lambda\mapsto\exp\left[\int_c^d\frac{\lambda-1}{(d-c)+(\lambda-1)(x-c)}\,\alpha(x)\,\mathrm{d}x\right],\quad\lambda>0$$

is also a complete Bernstein function.

Remark 2. Our second proof of the theorem shows, in particular, that – up to a multiplicative constant $c>0$ – ***all*** *complete Bernstein functions have a representation of the form* (0.1)*; moreover, the function* $\alpha(x)$ *is uniquely determined by the corresponding Bernstein function and vice versa.*

For this we use the following characterization of complete Bernstein functions, see [3, Theorem 6.17]. We have $f\in\mathcal{CBF}$ if, and only if, there is some $\gamma\in\mathbb{R}$ and a measurable function $\eta:[0,\infty)\to[0,1]$ such that

$$f(\lambda)=\exp\left[\gamma+\int_0^\infty\left(\frac{t}{1+t^2}-\frac{1}{\lambda+t}\right)\eta(t)\,\mathrm{d}t\right],\quad\lambda>0. \tag{0.3}$$

The pair (γ,η) uniquely characterizes $f\in\mathcal{CBF}$ and vice versa.

We will see that there is a one-to-one correspondence $\eta\leftrightarrow\alpha$ given by $\eta(t)=\alpha\left(\frac{1}{1+t}\right)$, $t\in[0,\infty)$, while $\gamma=\gamma(\alpha)$. Since $\phi^{(\alpha)}(1)=1$, this means that any $f\in\mathcal{CBF}$ can be written as $f(1)\times\phi^{(\alpha)}$. At the level of subordinators this amounts to consider the time-changed subordinator $(S_{ct}^{(\alpha)})_{t\geq 0}$, $c=f(1)>0$; obviously, $\{S_{ct}^{(\alpha)}:t\geq 0\}=\{S_t^{(\alpha)}:t\geq 0\}$, i.e. Marchal's Theorem 2 holds for *all* complete Bernstein functions.

1 First proof

Our first proof relies on the fact that $f\in\mathcal{CBF}$ if, and only if, f has an analytic extension onto the open upper complex half-plane $\mathbb{H}^\uparrow:=\{z\in\mathbb{C}:\operatorname{Im}z>0\}$ such that $f:\mathbb{H}^\uparrow\to\mathbb{H}^\uparrow$ and and $f(0+)=\lim_{(0,\infty)\ni\lambda\to 0}f(\lambda)$ exists, see [3, Theorem 6.2].

First proof of the main theorem. According to [4, Theorem I.4.3, p. 32], we can pick a sequence of step functions $\{\alpha_n : n \in \mathbb{N}\}$ on $[0,1]$ of the following form

$$\alpha_n = \sum_{i=1}^{n} a_i^{(n)} \mathbb{1}_{\left[t_{i-1}^{(n)}, t_i^{(n)}\right)},$$

where $a_i^{(n)} \in [0,1]$ for all $i \in \{1,\dots,n\}$ and $0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_n^{(n)} = 1$, such that $\alpha_n(x) \to \alpha(x)$ for almost all $x \in [0,1]$ as $n \to \infty$. For $n \in \mathbb{N}$ and $\lambda > 0$, we have

$$\begin{aligned}\phi_n^{(\alpha)}(\lambda) &:= \exp\left[\int_0^1 \frac{\lambda-1}{1+(\lambda-1)x}\,\alpha_n(x)\,\mathrm{d}x\right]\\ &= \exp\left[\sum_{i=1}^{n} a_i^{(n)} \int_{t_{i-1}^{(n)}}^{t_i^{(n)}} \frac{\lambda-1}{1+(\lambda-1)x}\,\mathrm{d}x\right]\\ &= \prod_{i=1}^{n}\left(\frac{1+(\lambda-1)t_i^{(n)}}{1+(\lambda-1)t_{i-1}^{(n)}}\right)^{a_i^{(n)}}.\end{aligned}$$

This representation allows us to extend $\phi_n^{(\alpha)}$ analytically onto the open upper half-plane $\mathbb{H}^{\uparrow}$. Moreover,

$$\lim_{(0,\infty)\ni\lambda\to 0} \phi_n^{(\alpha)}(\lambda) = 0 \quad \text{for all } n \in \mathbb{N},$$

and by the dominated convergence theorem, one has

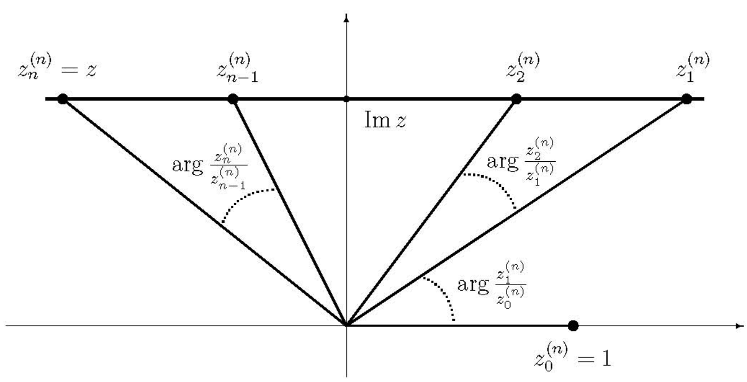
$$\lim_{n\to\infty} \phi_n^{(\alpha)}(\lambda) = \phi^{(\alpha)}(\lambda) \quad \text{for all } \lambda > 0.$$

Let $n \in \mathbb{N}$ and $z \in \mathbb{H}^{\uparrow}$. Note that

$$\begin{aligned}\phi_n^{(\alpha)}(z) &= \prod_{i=1}^{n}\left(\frac{1+(z-1)t_i^{(n)}}{1+(z-1)t_{i-1}^{(n)}}\right)^{a_i^{(n)}}\\ &= \left(t_1^{(n)}\right)^{a_1^{(n)}}\left(z-1+\frac{1}{t_1^{(n)}}\right)^{a_1^{(n)}} \prod_{i=2}^{n}\left(\frac{t_i^{(n)}}{t_{i-1}^{(n)}}\right)^{a_i^{(n)}}\left(\frac{z-1+1/t_i^{(n)}}{z-1+1/t_{i-1}^{(n)}}\right)^{a_i^{(n)}}\\ &= \left(t_1^{(n)}\right)^{a_1^{(n)}}\left(\frac{z_1^{(n)}}{z_0^{(n)}}\right)^{a_1^{(n)}} \prod_{i=2}^{n}\left(\frac{t_i^{(n)}}{t_{i-1}^{(n)}}\right)^{a_i^{(n)}}\left(\frac{z_i^{(n)}}{z_{i-1}^{(n)}}\right)^{a_i^{(n)}},\end{aligned}$$

where

$$z_0^{(n)} := 1, \quad z_i^{(n)} := z-1+\frac{1}{t_i^{(n)}}, \quad i=1,\dots,n.$$



It is easy to see that $\operatorname{Im} z_i^{(n)} = \operatorname{Im} z$ for all $i \in \{1, \dots, n\}$, and $\operatorname{Re} z_{i-1}^{(n)} > \operatorname{Re} z_i^{(n)}$ for all $i \in \{2, \dots, n\}$; see the figure on the previous page.
Then we find that

$$\arg \frac{z_i^{(n)}}{z_{i-1}^{(n)}} > 0 \text{ for every } i \in \{1, \dots, n\}, \text{ and } \sum_{i=1}^n \arg \frac{z_i^{(n)}}{z_{i-1}^{(n)}} = \arg z_n^{(n)} = \arg z \in (0, \pi). \quad (1.1)$$

If $a_i^{(n)} = 0$ for all $i \in \{1, \dots, n\}$, then $\phi_n^{(\alpha)} \equiv 1$, which is obviously of class $\mathcal{CBF}$; otherwise, we obtain from (1.1) and $a_i^{(n)} \in [0, 1]$ that

$$\arg \phi_n^{(\alpha)}(z) = \sum_{i=1}^n a_i^{(n)} \arg \frac{z_i^{(n)}}{z_{i-1}^{(n)}} \in (0, \pi),$$

which implies that $\phi_n^{(\alpha)}$ preserves the open upper half-plane, and so $\phi_n^{(\alpha)} \in \mathcal{CBF}$. Thus, we conclude that $\phi_n^{(\alpha)} \in \mathcal{CBF}$ for all $n \in \mathbb{N}$. Since $\mathcal{CBF}$ is closed under pointwise limits, cf. [3, Corollary 7.6 (ii)], it follows that $\phi^{(\alpha)} \in \mathcal{CBF}$. □

2 Second proof

Our second proof is based on the characterization (0.3) of complete Bernstein functions mentioned in Remark 2. Alili, Jedidi and Rivero have discovered the same argument, independently of us in [1, Example 4.2, p. 730]. Since their proof appears in a different context and contains a small mistake, we provide the short proof for the readers' convenience. We are grateful to an anonymous referee pointing out the reference [1] and we acknowledge their priority for this argument.

Second proof of the main theorem. Observe that

$$\begin{aligned}\log \phi^{(\alpha)}(\lambda) &= \int_0^1 \frac{\lambda - 1}{1 + (\lambda - 1)x} \, \alpha(x) \, \mathrm{d}x \\ &= \int_0^1 \frac{1}{x^2} \left(x - \frac{1}{(1-x)/x + \lambda} \right) \alpha(x) \, \mathrm{d}x, \quad \lambda > 0.\end{aligned}$$

Changing variables according to $t = (1-x)/x$ yields that for $\lambda > 0$

$$\begin{aligned}\log \phi^{(\alpha)}(\lambda) &= \int_0^\infty \left(\frac{1}{1+t} - \frac{1}{\lambda + t} \right) \alpha\left(\frac{1}{1+t} \right) \mathrm{d}t \\ &= \int_0^\infty \left(\frac{1}{1+t} - \frac{t}{1+t^2} \right) \alpha\left(\frac{1}{1+t} \right) \mathrm{d}t \\ &\quad + \int_0^\infty \left(\frac{t}{1+t^2} - \frac{1}{\lambda+t} \right) \alpha\left(\frac{1}{1+t} \right) \mathrm{d}t.\end{aligned}$$

Since

$$\left| \frac{t}{1+t^2} - \frac{1}{\lambda+t} \right| = \left| \frac{\lambda t - 1}{\lambda + t} \right| \frac{1}{1+t^2} \in L^1((0,\infty); \mathrm{d}t) \quad \text{for all } \lambda > 0,$$

we know that both integrals appearing in the above representation of $\log \phi^{(\alpha)}(\lambda)$ are finite. This shows that $\phi^{(\alpha)}$ is a complete Bernstein function of the form (0.3) with parameters

$$\gamma := \int_0^\infty \left(\frac{1}{1+t} - \frac{t}{1+t^2} \right) \alpha\left(\frac{1}{1+t} \right) \mathrm{d}t \quad \text{and} \quad t \mapsto \eta(t) := \alpha\left(\frac{1}{1+t} \right). \qquad □$$

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