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Boundedly finite measures: separation and convergence by an algebra of functions*

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Abstract

We prove general results about separation and weak$^{\#}$-convergence of boundedly finite measures on separable metric spaces and Souslin spaces. More precisely, we consider an algebra of bounded real-valued, or more generally a $*$-algebra $\mathcal{F}$ of bounded complex-valued functions and give conditions for it to be separating or weak$^{\#}$-convergence determining for those boundedly finite measures that integrate all functions in $\mathcal{F}$. For separation, it is sufficient if $\mathcal{F}$ separates points, vanishes nowhere, and either consists of only countably many measurable functions, or of arbitrarily many continuous functions. For convergence determining, it is sufficient if $\mathcal{F}$ induces the topology of the underlying space, and every bounded set A admits a function in $\mathcal{F}$ with values bounded away from zero on A.

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Contents

1 Introduction

Boundedly finite measures play an increasingly important role in probability theory. Classical examples are Itô excursion measures, or Lévy measures, which we will come back to in a moment. But they also appear in very recent research as speed measures

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of Brownian motions on $\mathbb{R}$-trees [1, 2] or sampling measures for spatial Fleming-Viot processes [14]. Convergence of metric measure spaces with boundedly finite measures has been analysed with a view towards probabilistic applications in [3].

For the purpose of illustration, we quickly recall the situation for Lévy measures of infinitely divisible random variables. A real-valued random variable X is called *infinitely divisible* if, for any $n \in \mathbb{N}$, we can write $X \stackrel{\mathrm{d}}{=} X_{1,n} + \cdots + X_{n,n}$ for an i.i.d. sequence of random variables $(X_{i,n})_{1 \le i \le n}$. A famous theorem by Lévy and Khintchine states that in that case the Fourier transform of the random variable has a very explicit form:

$$-\log \mathbb{E}\big[\exp(-iuX)\big] = iub + \frac{c}{2}u^2 - \int_{\mathbb{R}\setminus\{0\}} e^{-iuy} - 1 + iuy\mathbb{1}_{|y|\le 1}\, \mu(\mathrm{d}y), \quad u \in \mathbb{R}, \tag{1.1}$$

where $b \in \mathbb{R}$, $c \ge 0$ and μ is a measure on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}\setminus\{0\}} 1 \wedge |y|^2 \,\mu(\mathrm{d}y) < \infty$, see Theorem 8.1 of [26]. Let us look at the measure μ, called *Lévy measure*. The obvious questions are: (a) is the Lévy measure unique?[1] (b) If we have a sequence of infinitely divisible random variables, do their Lévy measures converge and in which sense? Lévy measures are not necessarily finite measures, but are required to be finite on any set which is not close to the origin 0. This motivates to consider measures which are finite on a certain class of sets. Daley and Vere-Jones used Appendix A2.6 in [10] to present a framework for such questions which they call *boundedly finite measures*, because the measures are assumed to be finite on bounded sets. The space of these measures is equipped with weak$^{\#}$-convergence, defined as convergence of integrals over bounded continuous functions with bounded support (see Section 2 for definitions). Some extensions are given in [16] and [20]. Lévy measures fit into this framework if we change the Euclidean metric on $\mathbb{R} \setminus \{0\}$ such that 0 is sent infinitely far away, an idea also used in [4].

How does one prove weak convergence $\mu_n \xrightarrow{\mathrm{w}} \mu$ of probability measures on a topological space X in situations where it is not feasible to show convergence of $\int f \,\mathrm{d}\mu_n$ for *all* bounded continuous $f \in \mathcal{C}_b(X)$ directly? One possibility is to find a class $\mathcal{F} \subseteq \mathcal{C}_b(X)$ of sufficiently "nice" functions, which is still rich enough to be *convergence determining*, i.e.

$$\int f \,\mathrm{d}\mu_n \to \int f \,\mathrm{d}\mu \;\forall f \in \mathcal{F} \quad \Longrightarrow \quad \mu_n \xrightarrow{\mathrm{w}} \mu.$$

This approach has proven to be particularly fruitful if the topology on X itself is defined in terms of a class of functions. A classical example for such a topology is the weak (weak-$*$) topology on (the dual of) a Banach space. A more modern one is the Gromov-weak topology on the space of metric measure spaces, which is induced both by a complete metric and by a class of functions called *polynomials* (see [13]). That polynomials do not only induce the topology but are even convergence determining was shown with some effort in [11]. But it also follows directly from a general result due to Le Cam, as pointed out in [21]. Le Cam's theorem goes back to [19] and states that on a completely regular Hausdorff space X, a set of functions $\mathcal{F} \subseteq \mathcal{C}_b(X)$ is convergence determining for Radon probability measures if it is multiplicatively closed and induces the topology of X. The proof can be found in [15, Proposition 4.1]. A version of Le Cam's theorem for separable metric spaces dropping the "Radon" assumption on the probability measures is given in [7]. This version was used extensively for the construction of a tree-valued pruning process in [22].

Our main goal is to extend Le Cam's result to the case of boundedly finite measures and weak$^{\#}$-convergence and, because convergence determining is sometimes too much to ask for, to obtain (weaker) sufficient conditions for $\mathcal{F}$ to at least *separate* boundedly

[1] this question has an affirmative answer with a neat proof in [5] Theorem 3.7

finite measures. A separating class of functions can also be used to prove weak (or weak$^{\#}$) convergence if tightness is known by other methods. In particular, our results allow to give an answer on the question of uniqueness and convergence of μ in (1.1) within a general framework. More importantly, they will find applications in future work about spaces of metric measure spaces and $\mathbb{R}$-trees such as in the upcoming paper [12] which was a driving motivation for the present article. The results will hopefully also facilitate the analysis of spatial population models on unbounded spaces with infinite total population size such as the one in [14], and of other models appearing in modern areas of probability theory.

The rest of the paper is organized as follows. In Section 2, we give our main results about convergence determining (Theorem 2.3) and separating (Theorem 2.7 and Corollary 2.8) classes of functions for boundedly finite measures. In Section 3, we illustrate in four examples how our results can be applied. There, we consider Lévy measures, excursion theory and mass fragmentations.

2 Separation and convergence of boundedly finite measures

Let (X, d) be a separable metric space, endowed with the Borel σ-field induced by d. By $\mathcal{C}_b(X)$, we denote the set of bounded continuous functions on the metric space (X, d) with values in $\mathbb{C}$. For real-valued functions we write $\mathcal{C}_b(X;\mathbb{R})$. Note that $\mathcal{C}_b(X;\mathbb{R}) \subseteq \mathcal{C}_b(X)$.

Definition 2.1 (Boundedly finite measures and weak$^{\#}$-convergence)**.** *The set of* boundedly finite measures $\mathcal{M}^{\#}(X)$ *on* X *w.r.t.* d *is given as*

$$\mathcal{M}^{\#}(X) = \{\mu \in \mathcal{M}(X) \mid \mu(A) < \infty \text{ for all } d\text{-bounded, measurable } A \subseteq X\}.$$

A sequence $(\mu_n)_{n\in\mathbb{N}}$ *in* $\mathcal{M}^{\#}(X)$ *is said to be* weak$^{\#}$-convergent *to* $\mu \in \mathcal{M}^{\#}(X)$*, denoted by* $\mu_n \xrightarrow{\mathrm{w}^{\#}} \mu$*, if* $\int f \,\mathrm{d}\mu_n \to \int f, \mathrm{d}\mu$ *holds for all* $f \in \mathcal{C}_b(X;\mathbb{R})$ *with* d*-bounded support.*

Remark 2.2 (Weak$^{\#}$-convergence versus vague convergence)**.** If (X, d) is a *Heine-Borel space*, i.e. every closed, bounded set is compact, then $\mathcal{M}^{\#}(X)$ coincides with the set of Radon measures on X, and weak$^{\#}$-convergence with vague convergence. For a general separable metric space, however, $\mathcal{M}^{\#}(X)$ is a subset of the Radon measures and weak$^{\#}$-convergence is a potentially much stronger convergence than vague convergence.

Consider a set $\mathcal{F}$ of measurable, $\mathbb{C}$-valued functions on X and define

$$\mathcal{M}_{\mathcal{F}}(X) = \Big\{\mu \in \mathcal{M}(X) \,\Big|\, \int |f(x)| \,\mu(\mathrm{d}x) < \infty \ \ \forall f \in \mathcal{F}\Big\}, \quad \mathcal{M}^{\#}_{\mathcal{F}}(X) = \mathcal{M}^{\#}(X) \cap \mathcal{M}_{\mathcal{F}}(X). \tag{2.1}$$

Theorem 2.3 (Convergence determining for boundedly finite measures)**.** *Let* (X, d) *be a separable metric space and* $\mathcal{F} \subset \mathcal{C}_b(X)$*. Assume that*

(T.1) $\mathcal{F}$ *is multiplicatively closed and closed under complex conjugation.*

(T.2) $\mathcal{F}$ *induces the topology of* X*.*

(T.3) *For every bounded set* $A \subset X$ *there exists* $f \in \mathcal{F}$ *and* $\delta > 0$ *with* $\inf_{x\in A} |f(x)| > \delta$*.*

Then $\mathcal{F}$ *is weak*$^{\#}$*-convergence determining for measures in* $\mathcal{M}^{\#}_{\mathcal{F}}(X)$*, i.e.*

$$\mu, \mu_n \in \mathcal{M}^{\#}_{\mathcal{F}}(X), \int f \,\mathrm{d}\mu_n \to \int f \,\mathrm{d}\mu \;\forall f \in \mathcal{F} \quad \Longrightarrow \quad \mu_n \xrightarrow{\mathrm{w}^{\#}} \mu.$$

Remark 2.4. 1. (T.1) and (T.2) are classical assumptions for these kind of theorems, see Proposition 4.1 in [15]. Something like (T.3) is necessary to replace the fixed total mass in the weak convergence of probabilities. At least, we have to ensure that $\mathcal{F}$ "vanishes nowhere", because if there was $x \in X$ with $f(x) = 0$ for all $f \in \mathcal{F}$, then $\mathcal{F}$ could not even separate $a \cdot \delta_x$ for different $a \geq 0$. For the purpose of separation of measures, we can do with this weaker requirement S.3 in Theorem 2.7 below. We do not know, however, if it would be enough for Theorem 2.3.

2. For real-valued functions, the part "closed under complex conjugation" is always satisfied.

For the proof, we embed everything in the Hilbert cube H, a technique going back to Urysohn's work on metrisation and also used in [7]. Recall that

$$H = [0,1]^{\mathbb{N}}, \quad \text{with product topology.}$$

Denote the uniform norm on $\mathcal{C}_b(H;\mathbb{R})$ by $\|\cdot\| := \|\cdot\|_\infty$. For $0 < \delta < 1$, we consider the subspace

$$H_\delta := \left\{x = (x_n)_{n\in\mathbb{N}} \in H \;\middle|\; x_1 \geq \delta\right\}$$

and use the following variant of the Stone-Weierstrass theorem.

Definition 2.5 ($\mathcal{P}$, $\mathcal{P}_0$)**.** *Let $\mathcal{P} \subseteq \mathcal{C}_b(H;\mathbb{R})$ be the set of polynomials on H (i.e. functions depending on finitely many coordinates and an algebraic multivariate polynomial in these coordinates). Let $\mathcal{P}_0 := \left\{p \in \mathcal{P} \;\middle|\; p(x) = 0 \;\forall\, x = (x_n)_{n\in\mathbb{N}} \in H \text{ with } x_1 = 0\right\}$.*

Lemma 2.6 (Stone-Weierstrass variant)**.** *Let $g\colon H \to [0,1]$ be continuous with $\operatorname{supp} g \subset H_\delta$ for some $\delta > 0$. Then, for every $\varepsilon > 0$ there exists a polynomial $p_\varepsilon \in \mathcal{P}_0$ such that*

$$|g(x) - p_\varepsilon(x)| \leq \varepsilon x_1 \quad \forall\, x = (x_n)_{n\in\mathbb{N}} \in H.$$

Proof. For $x = (x_n)_{n\in\mathbb{N}} \in H$ define

$$\tilde{g}(x) := \begin{cases} x_1^{-1} g(x) & \text{if } x_1 > 0, \\ 0 & \text{if } x_1 = 0. \end{cases}$$

Then $\tilde{g} \in \mathcal{C}_b(H;\mathbb{R})$ and we can use the Stone-Weierstrass theorem. So, for $\varepsilon > 0$ we find $\tilde{p}_\varepsilon \in \mathcal{P}$ such that

$$|\tilde{g}(x) - \tilde{p}_\varepsilon(x)| < \varepsilon \;\forall x \in H.$$

Define $p_\varepsilon(x) := x_1 \tilde{p}_\varepsilon(x)$, $x \in H$. Then $p_\varepsilon \in \mathcal{P}_0$ and we get for any $a \in (0,1]$ the estimate

$$\sup_{x\in H,\, x_1=a} a^{-1} |g(x) - p_\varepsilon(x)| = \sup_{x\in H,\, x_1=a} |\tilde{g}(y) - \tilde{p}_\varepsilon(y)| < \varepsilon.$$

That is what we needed to show, as the case $x_1 = 0$ is trivial. □

Proof of Theorem 2.3. Step 1. It is enough to consider functions with values in $[0,1]$: First, $\mathcal{F}$ may be replaced by $\mathcal{F}' := \left\{\Re(f), \Im(f) \;\middle|\; f \in \mathcal{F}\right\}$, where $\Re(f)$ and $\Im(f)$ are the real and imaginary parts of f, respectively. Because $\mathcal{F}$ is closed under complex conjugation due to (T.1), the conditions (T.1), (T.2) and (T.3) are also satisfied for $\mathcal{F}'$ instead of $\mathcal{F}$. Thus we may assume $\mathcal{F}$ to consist of real-valued functions. Second, $\mathcal{F}$ may be replaced by $\mathcal{F}' := \{f^2 \mid f \in \mathcal{F}\} \cup \left\{f^2(\|f\|_\infty - f) \;\middle|\; f \in \mathcal{F}\right\}$, which is contained in the vector space generated by $\mathcal{F}$ and easily seen to satisfy the prerequisites of the theorem provided that $\mathcal{F}$ does. Because $\mathcal{F}'$ maps to $\mathbb{R}_+$, we can assume, by normalisation, that the elements of $\mathcal{F}$ map into $[0,1]$.

Step 2. By Assumption (T.2), $\mathcal{F}$ induces the topology of X. Because X is a separable, metric space, it has a countable basis, and thus we can choose a countable subfamily of $\mathcal{F}$ that still induces the topology of X. Indeed, the family of sets of the form $\bigcap_{i=1}^n f_i^{-1}(U_i)$ for $n \in \mathbb{N}$, $f_i \in \mathcal{F}$, $U_i \subseteq [0,1]$ open is a base for the topology, and because X has a countable base, we can select a countable subfamily that is still a base[2]. Therefore there exist $f_1, f_2, \ldots \in \mathcal{F}$ with $0 \le f_m \le 1$, such that $(f_m)_{m\in\mathbb{N}}$ induces the topology of X. Then $\iota\colon X \to H$, $x \mapsto (f_m(x))_{m\in\mathbb{N}}$ is a topological embedding (i.e. a homeomorphism onto its image) of X into H. Identifying X with $\iota(X)$, we assume w.l.o.g. $X \subseteq H$ and f_n to be the (restriction of the) n^{th} coordinate projection. In particular, being an algebra, the linear span of $\mathcal{F}$ contains $\mathcal{P}_0$ (defined in Definition 2.5).

Step 3. Let $\mu_n, \mu \in \mathcal{M}^{\#}_{\mathcal{F}}(X)$ with

$$\int f \,\mathrm{d}\mu_n \to \int f \,\mathrm{d}\mu \quad \forall\, f \in \mathcal{F}. \tag{2.2}$$

For the claimed weak$^{\#}$-convergence, it is enough to show $\int g \,\mathrm{d}\mu_n \to \int g \,\mathrm{d}\mu$ for all $g\colon X \to [0,1]$ which have d-bounded support and are uniformly continuous (by the Portmanteau theorem, Theorem 2.1 in [20]). Because weak convergence depends on the metric only through the induced topology, we may assume g to be uniformly continuous w.r.t. any other metric on X inducing the same topology as d. To this end, we take any metric on H inducing its topology, and assume that g is uniformly continuous w.r.t. its restriction to X (recall that X is a subspace of H by Step 2). By Assumption (T.3), there is $f \in \mathcal{F}$ and $\delta > 0$ such that for all $x \in \operatorname{supp}(g)$ we have $|f(x)| > \delta$. We may assume w.l.o.g. that $f = f_1$ (if not, we define $f_1' = f$, $f_{m+1}' = f_m$ and observe that ι' defined with these f_m' is still an embedding and g uniformly continuous w.r.t. the restriction of the metric on H). Then $\operatorname{supp}(g) \subseteq H_\delta$. Furthermore, since g is uniformly continuous, it can be extended continuously to the closure of X in H, and by the Tietze extension theorem (e.g. [24, Theorem 35.1]) to a continuous function from H to $[0,1]$ with support in H_δ. We denote the extension again by g. We also identify μ_n and μ with their natural extensions to H.

Step 4. g satisfies the assumptions of Lemma 2.6. For $\varepsilon > 0$, choose $p_\varepsilon \in \mathcal{P}_0 \subseteq \operatorname{span}(\mathcal{F})$ as in the lemma. Because $\mu_n(f_1) \to \mu(f_1)$, we have $M := \sup_{n\in\mathbb{N}} \mu_n(f_1) < \infty$ and obtain for all $n \in \mathbb{N}$

$$\left| \int p_\varepsilon - g \,\mathrm{d}\mu_n \right| \le \int_H \varepsilon x_1 \,\mu_n(\mathrm{d}x) = \varepsilon \int f_1 \,\mathrm{d}\mu_n \le \varepsilon M.$$

Because $\mu_n(p_\varepsilon) \to \mu(p_\varepsilon)$ by (2.2), we conclude for every $\varepsilon > 0$

$$\limsup_{n\to\infty} |\mu_n(g) - \mu(g)| \le \limsup_{n\to\infty} |\mu_n(p_\varepsilon) - \mu(p_\varepsilon)| + |\mu_n(p_\varepsilon - g)| + |\mu(p_\varepsilon - g)| \le 2\varepsilon M.$$

Because ε is arbitrary, the claimed convergence follows. □

It is desirable to have a result which separates two boundedly finite measures but requires less than the previous theorem. While "boundedly finite" is essential for the definition of weak$^{\#}$-convergence, we can drop this assumption for the purpose of separation and work with $\mathcal{M}_{\mathcal{F}}$, the space of measures integrating $\mathcal{F}$ as defined in (2.1), instead of $\mathcal{M}^{\#}_{\mathcal{F}}$. We can also relax the metrisability assumption on the space X, but do need some topological assumption. Recall that a topological space is Hausdorff if any two distinct points can be separated by open sets, and a Hausdorff topological space X is, by definition, a *Souslin space* if there exists a Polish space Y and a continuous surjective map from Y onto X. Note that a Souslin space is separable but need not be metrisable. An example is a separable Banach space in its weak topology, which is clearly Souslin but not metrisable. Conversely, not every separable metrisable space is

[2] One can show this with standard arguments: If $\mathcal{B}, \mathcal{B}'$ are bases, $\mathcal{B}$ countable, $B \in \mathcal{B}$, then $B = \bigcup I$ for some $I \subseteq \mathcal{B}'$, and for every $U \in I$ there is $J_U \subseteq \mathcal{B}$ with $U = \bigcup J_U$. For every $V \in J := \bigcup_{U\in I} J_U$, we select one $U_V \in I$ with $V \in J_{U_V}$. J is a subset of $\mathcal{B}$, hence countable. Because $B = \bigcup J = \bigcup_{V\in J} U_V$, we obtain a countable basis by taking all U_V for all joices of $B \in \mathcal{B}$.

Souslin. In the case of a Souslin space X, and a countable family of functions $\mathcal{F}$, we can drop the topological assumptions on $\mathcal{F}$ from the prerequisites of Theorem 2.3 and still obtain the weaker conclusion that $\mathcal{F}$ is separating for measures in $\mathcal{M}_{\mathcal{F}}$. More precisely, we have

Theorem 2.7 (Separation of boundedly finite measures with measurable functions)**.** *Let X be a Souslin space (for example a Polish space), and $\mathcal{F}$ a* countable *set of bounded, measurable $\mathbb{C}$-valued functions. Assume that*

(**S.1**) *$\mathcal{F}$ is multiplicatively closed and closed under complex conjugation.*

(**S.2**) *$\mathcal{F}$ separates points of X.*

(**S.3**) *$\mathcal{F}$ vanishes nowhere, i.e. for every $x \in X$ there exists an $f_x \in \mathcal{F}$ with $f_x(x) \neq 0$.*

Then $\mathcal{F}$ is separating for measures in $\mathcal{M}_{\mathcal{F}}(X)$, i.e.

$$\mu_1, \mu_2 \in \mathcal{M}_{\mathcal{F}}(X), \int f \,\mathrm{d}\mu_1 = \int f \,\mathrm{d}\mu_2 \;\forall\, f \in \mathcal{F} \quad \Longrightarrow \quad \mu_1 = \mu_2. \tag{2.3}$$

Proof. Assume that $\mu_1, \mu_2 \in \mathcal{M}_{\mathcal{F}}(X)$ are such that $\int f \,\mathrm{d}\mu_1 = \int f \,\mathrm{d}\mu_2$ holds for all $f \in \mathcal{F}$. We have to show $\mu_1 = \mu_2$. Enumerate $\mathcal{F} = \{f_n \mid n \in \mathbb{N}\}$. Using Step 1 from the proof of Theorem 2.3, we may (and do) assume that f_n takes values in $[0,1]$ for all $n \in \mathbb{N}$. We proceed in two steps: first we show that (2.3) holds if we assume that instead of (S.3) the following stronger condition holds,

(**S.3**†) $f_1(x) \neq 0$ for all $x \in X$.

In the second step, we reduce the general case to the one where (S.3†) holds.

Step 1. Assume that (S.3†) holds and define $\iota\colon X \to H$, $x \mapsto \big(f_n(x)\big)_{n\in\mathbb{N}}$. Then ι is measurable and injective by assumption (S.2). Because X is a Souslin space, it is an analytic measurable space ([9, Proposition 8.6.13]) and so is $Y := \iota(X)$ ([9, Corollary 8.6.9]). By [9, Proposition 8.6.2], ι is a Borel isomorphism onto Y, i.e. $\iota^{-1}\colon Y \to X$ is measurable. Therefore,

$$\mu_1 = \mu_2 \iff \mu_1 \circ \iota^{-1} = \mu_2 \circ \iota^{-1}. \tag{2.4}$$

Because of (S.3†), every $x = (x_n)_{n\in\mathbb{N}} \in Y$ satisfies $x_1 \neq 0$. We define the metric

$$r(x,y) := |x_1^{-1} - y_1^{-1}| + \sum_{n\in\mathbb{N}} 2^{-n} |x_n - y_n| \wedge 1, \qquad x, y \in Y,$$

which induces on Y the topology inhereted as a subspace of H. Let $\mathcal{G} := \{f \circ \iota^{-1} \mid f \in \mathcal{F}\}$. We show that (Y, r) and $\mathcal{G}$ satisfy the prerequisites of Theorem 2.3.

$\mathcal{G}$ satisfies (T.1) because $\mathcal{F}$ does by assumption. By construction of ι, $\mathcal{G}$ coincides with the set of restrictions of coordinate projections to Y. Therefore, $\mathcal{G}$ induces the topology of Y, i.e. (T.2) is satisfied. An r-bounded set A in Y satisfies $\delta := \inf_{x\in A} |x_1| > 0$, and (T.3) is satisfied with $f = f_1 \circ \iota^{-1}$. Thus we can apply Theorem 2.3 and obtain that $\mathcal{G}$ is weak$^\#$-convergence determining and a fortiori separating for measures in $\mathcal{M}_{\mathcal{G}}^\#(Y)$.

If $\mu_1, \mu_2 \in \mathcal{M}_{\mathcal{F}}(X)$, then they integrate f_1, and $\hat{\mu}_i := \mu_i \circ \iota^{-1}$, $i = 1, 2$, are boundedly finite measures on (Y, r). Thus obviously $\hat{\mu}_i \in \mathcal{M}_{\mathcal{G}}^\#(Y)$ and the claim of the theorem follows with (2.4).

Step 2. Now consider the general case, where (S.3†) does not necessarily hold. Define

$$f_0 := \sum_{n\in\mathbb{N}} 2^{-n} \frac{f_n}{1 \vee \int f_n \,\mathrm{d}\mu_1},$$

and let $\mathcal{F}'$ be the set of finite products of elements of $\mathcal{F} \cup \{f_0\}$. Then $\mathcal{F}'$ is a countable set of measurable functions satisfying (S.1), (S.2) and, because $\mathcal{F}$ vanishes nowhere, also (S.3†) (with f_1 replaced by $f_0 \in \mathcal{F}'$). Hence, by Step 1, $\mathcal{F}'$ is separating for measures in $\mathcal{M}_{\mathcal{F}'}(X)$.

According to the monotone convergence theorem, we have

$$\int f_0 \,\mathrm{d}\mu_1 = \int f_0 \,\mathrm{d}\mu_2 \le 1.$$

Because every element of $\mathcal{F}'$ is dominated by an element of $\mathcal{F} \cup \{f_0\}$ (recall that elements of $\mathcal{F}'$ map to $[0,1]$), this implies $\mu_1, \mu_2 \in \mathcal{M}_{\mathcal{F}'}(X)$. Moreover, dominated convergence yields

$$\int g \,\mathrm{d}\mu_1 = \int g \,\mathrm{d}\mu_2 \quad \forall g \in \mathcal{F}',$$

which implies $\mu_1 = \mu_2$ by Step 1. □

In the case of *continuous* functions, we can drop the countability of $\mathcal{F}$.

Corollary 2.8 (Separation of boundedly finite measures with continuous functions)**.** *Let X be a Souslin space (e.g. a Polish space), and $\mathcal{F} \subseteq \mathcal{C}_b(X)$. Assume* (S.1), (S.2)*, and* (S.3) *from Theorem 2.7. Then $\mathcal{F}$ is separating for measures in $\mathcal{M}_{\mathcal{F}}(X)$, i.e.* (2.3) *holds.*

Proof. For $x \in X$, let f_x be as in (S.3). There is an open neighbourhood U_x of x with $f_x(y) \neq 0$ for all $y \in U_x$. Recall that a topological space is called Lindelöf if every open cover has a countable subcover, and every Polish space has this property (because it has a countable base). Because the property is obviously preserved by continuous maps, every Souslin space is Lindelöf as well. Hence $(U_x)_{x \in X}$ has a countable subcover, and there exists a countable subfamily $\mathcal{F}_1$ of $\mathcal{F}$ satisfying (S.3).

Similarly, for $x, y \in X$ let $f_{xy} \in \mathcal{F}$ be such that $f_{xy}(x) \neq f_{xy}(y)$. Then there is an open neighbourhood U_{xy} of (x,y) in X^2 with $f_{xy}(u) \neq f_{xy}(v)$ for all $(u,v) \in U_{xy}$. Because X^2 is also Souslin and hence Lindelöf, we find a countable subfamily $\mathcal{F}_2$ of $\mathcal{F}$ satisfying (S.2). Let $\mathcal{F}'$ be the closure of $\mathcal{F}_1 \cup \mathcal{F}_2$ under multiplication and complex conjugation. Then $\mathcal{F}'$ satisfies the prerequisites of Theorem 2.7 and the conclusion follows. □

3 Examples

3.1 Example 1: Lévy-Khintchine formula on $\mathbb{R}^D$

Let Z be an infinitely divisible random variable with values in $\mathbb{R}^D$ for $D \in \mathbb{N}$. That means for any $n \in \mathbb{N}$ there are i.i.d. random variables $Z_{1,n}, \dots, Z_{n,n}$ such that $Z \overset{\mathrm{d}}{=} Z_{1,n} + \dots + Z_{n,n}$. Consider

$$X := \mathbb{R}^D \setminus \{0\} \quad \text{with metric} \quad d(x,y) := \|x-y\|_\infty + \left| \|x\|_\infty^{-1} - \|y\|_\infty^{-1} \right|, \; x, y \in X.$$

It is well-known that there exist $b \in \mathbb{R}^D$, $C \in S_+(\mathbb{R}^D)$ a symmetric, positive semidefinite matrix, and $\mu \in \mathcal{M}^\#(X)$ with $\int (1 \wedge \|x\|_\infty^2) \,\mu(\mathrm{d}x) < \infty$ such that for all $u \in \mathbb{R}^D$

$$\Psi(u) := \log \mathbb{E}[\exp(iu^t Z)] = iu^t b - \frac{1}{2} u^t C u + \int_{\mathbb{R}^D \setminus \{0\}} \exp(iu^t x) - 1 - iu^t x \mathbb{1}_{|x| \le 1} \,\mu(\mathrm{d}x). \quad (3.1)$$

This formula is called the Lévy-Khintchine formula, see [26, Theorem 8.1]. The function $\mathbb{R}^D \to \mathbb{C}$, $u \mapsto \Psi(u)$ characterizes the distribution of the random vector Z. On a first glance, however, it is not clear why the Lévy triple (b, C, μ) should be unique. Theorem 2.3 allows to give simple verification of that known fact in a general setup.

Proposition 3.1 (Uniqueness and convergence of Lévy measures)**.** (3.1) *determines the Lévy triple* $(b, C, \mu) \in \mathbb{R}^D \times S_+(\mathbb{R}^D) \times \mathcal{M}^\#(X)$ *uniquely. Furthermore, if* Z_n *are infinitely divisible random variables converging in distribution to* Z*, and* (b_n, C_n, μ_n) *is the Lévy triple of* Z_n*, then* $\mu_n \xrightarrow{\mathrm{w}^\#} \mu$*.*

Proof. We start with the unique identification of the law.

Step 1. First note that

$$C_{k,j} = -\lim_{m\to\infty} m^{-2} \left[\Psi(m(\mathrm{e}_k + \mathrm{e}_j)) - \Psi(m\mathrm{e}_k) - \Psi(m\mathrm{e}_j)\right],\ 1 \le k, j \le D,$$

where e_k, $k = 1, \dots, D$ are unit vectors in $\mathbb{R}^D$. Moreover, for $k = 1, \dots, D$:

$$b_k = \lim_{m\to\infty} -\frac{i}{m}\left(\Psi(m\mathrm{e}_k) - \frac{1}{2}m^2 C_{k,k}\right).$$

Hence C and b are unique.

Step 2. Now suppose $\mu_1, \mu_2 \in \mathcal{M}^\#(X)$ both satisfy $\int (1 \wedge \|x\|_\infty^2)\, \mu_i(\mathrm{d}x) < \infty$ and (3.1) with μ replaced by μ_i, $i = 1, 2$.

For $u \in \mathbb{R}^D$, define $F_u, \psi_u, G_u \colon \mathbb{R}^D \to \mathbb{C}$ by

$$F_u(x) := \exp(iu^t x) - 1, \qquad \psi_u(x) := iu^t x \mathbb{1}_{|x|\le 1}, \qquad G_u(x) := F_u(x) - \psi_u(x) \tag{3.2}$$

for $x \in \mathbb{R}^D$. Consider the following two classes of functions, where span denotes the linear span:

$$\mathcal{G} := \mathrm{span}\{G_u \mid u \in \mathbb{R}^D\}, \qquad \mathcal{F} := \mathrm{span}\{F_u \cdot F_v \mid u, v \in \mathbb{R}^D\}.$$

Then with (3.1) and the uniqueness of b and C from Step 1, we have $\int G\,\mathrm{d}\mu_1 = \int G\,\mathrm{d}\mu_2$ for all $G \in \mathcal{G}$. Now observe that, using linearity of $u \mapsto \psi_u(x)$ for every $x \in \mathbb{R}^D$,

$$F_u \cdot F_v = F_{u+v} - F_u - F_v = G_{u+v} - G_u - G_v \in \mathcal{G} \qquad \forall u, v \in \mathbb{R}^D.$$

Hence, $\mathcal{F}$ is multiplicatively closed (by the first equality) and $\mathcal{F} \subseteq \mathcal{G}$. In particular, (S.1) holds,

$$\int f\,\mathrm{d}\mu_1 = \int f\,\mathrm{d}\mu_2 \qquad \forall f \in \mathcal{F},$$

and $\mu_1, \mu_2 \in \mathcal{M}^\#_{\mathcal{F}}$, because functions from $\mathcal{G}$ are integrable. Furthermore, $\mathcal{F}$ is contained in $\mathcal{C}_b(X)$ and (S.2) and (S.3) are easily verified. Thus $\mu_1 = \mu_2$ follows from Corollary 2.8.

Now we show the convergence result.

Step 3. First, $\Psi_n(u) = \log \mathbb{E}[\exp(iu^t Z_n)] \to \Psi(u)$ pointwise since $x \mapsto \exp(iu^t x)$ is a bounded continuous function.

Step 4. Recall that for the Lévy-triple (b, C, μ) the linear part b depends on the choice of the compensation function ψ_u, but C and μ do not. So in order to show $\mu_n \xrightarrow{\mathrm{w}^\#} \mu$ we may choose any (admissible) ψ_u we like. Replace ψ_u in (3.2) by $\hat{\psi}_u(x) = iu^t x h(x)$ for a $\mathcal{C}^1$-function $h \colon \mathbb{R}^D \to \mathbb{R}$ with $h(0) = 1$ and compact support. Then the argument from above still works: $\mathcal{F}$ is multiplicatively closed and $\mathcal{F} \subset \mathcal{C}_b(X)$, so (T.1) holds.

Moreover, Assumption (T.2) holds by the fact that $F_u(x_n) \to F_u(x) \Leftrightarrow \exp(iu^t x_n) \to \exp(iu^t x)$ for all $u \in \mathbb{R}^D$. The latter is nothing else than the convergence of the characteristic function of the measures $\delta(x_n)$ to $\delta(x)$. But this implies that $x_n \to x$ in $\mathbb{R}^D$, so (T.2) holds.

Finally, let $A \subset \mathbb{R}^D \setminus \{0\}$ be bounded w.r.t. d. Then there is $\varepsilon > 0$ s.t. $\varepsilon < \inf\{\|x\|_\infty \mid x \in A\} \leq \sup\{\|x\|_\infty \mid x \in A\} < \varepsilon^{-1}$. Consider $u = (u^*, \dots, u^*) \in \mathbb{R}^D$ with $u^* = (\varepsilon\pi)/(2D)$. Then $u^t x \in (\pi\varepsilon^2/(2D), \pi/2)$ for $x \in A$ and moreover:

$$\begin{aligned}\inf_{x\in A} |F_u(x)|^2 &= \inf_{x\in A} \left|e^{iu^t x} - 1\right|^2 \geq \inf_{x\in A} \left|\cos(u^t x) - 1\right|^2 \\ &= \inf_{z\in(\pi\varepsilon^2/(2D),\pi/2)} |\cos(z) - 1|^2 = \big(1 - \cos(\pi\varepsilon^2/(2D))\big)^2 =: \delta^2.\end{aligned}$$

Thus, (T.3) holds and Theorem 2.3 applies to show $\mu_n \xrightarrow{\mathrm{w}^\#} \mu$. □

Remark 3.2. Of course, the previous result is well-known, see [26, Theorem 8.7].

3.2 Example 2: Lévy-Khintchine formula on $\mathcal{M}_f(E)$

Let E be a Polish space and $\mathcal{M}_f(E)$ denote the finite measures on E. Suppose that Z takes values in finite measures on E ($E = \{1, \dots, D\}$ is a special case of Example 1 if we restrict to nonnegative random variables there) and that Z is infinitely divisible. Then, under the assumption that $\mathbb{E}[Z(E)] < \infty$, Theorem 6.1 of [18] states that there exists $b \in \mathcal{M}_f(E)$ and $\mu \in \mathcal{M}^\#(\mathcal{M}_f(E) \setminus \{0\})$ such that

$$L(\phi) := -\log \mathbb{E}[\exp(-\langle\phi, Z\rangle)] = \langle\phi, b\rangle + \int 1 - \exp(-\langle\phi,\nu\rangle)\, \mu(\mathrm{d}\nu), \quad \phi \in \mathcal{C}_b\big(E, [0,\infty)\big), \tag{3.3}$$

where the set $\mathcal{C}_b\big(E, [0,\infty)\big)$ denotes the nonnegative, bounded, continuous functions on E. We use the metric space $(X, d) = (\mathcal{M}_f(E) \setminus \{0\}, d)$ with

$$d(\nu, \nu') = d_{\text{Prohorov}}(\nu, \nu') + |\nu(E)^{-1} - \nu'(E)^{-1}|$$

for the definition of $\mathcal{M}^\#(X)$. The uniqueness of the pair (b, μ) in the Lévy-Khintchine formula in (3.3) can be shown with our methodology.

Proposition 3.3. *The pair* $(b, \mu) \in \mathcal{M}_f(E) \times \mathcal{M}^\#(\mathcal{M}_f(E) \setminus \{0\})$ *in* (3.3) *is unique.*

Proof. Step 1. First, b can be identified via $\langle b, \phi\rangle = \lim_{m\to\infty} \frac{1}{m} L(m\phi)$.

Step 2. To identify μ we want to use Corollary 2.8. Since (X, d) is a Polish space it is also a Souslin space. Define for $\phi \in \mathcal{C}_b(E, [0,\infty))$ the function $F_\phi : X \to [0, 1]$ via

$$F_\phi(\nu) = 1 - \exp(-\langle\phi, \nu\rangle),\ \nu \in X.$$

The linear span of functions of this kind is defined

$$\mathcal{F} := \mathrm{span}\{F_\phi \mid \phi \in \mathcal{C}_b(E, [0,\infty))\}.$$

Then it is easy to see that $\mathcal{F} \subset \mathcal{C}_b(X)$.

Step 3. Now we want to verify the remaining conditions of Corollary 2.8. (S.1) holds since $\mathcal{F}$ are real-valued functions and $F_\phi \cdot F_\psi = F_{\phi+\psi} - F_\phi - F_\psi$ for $\phi, \psi \in \mathcal{C}_b(E, [0,\infty))$. (S.2) and (S.3) trivially hold. So we can apply the corollary and deduce the uniqueness of μ. □

Remark 3.4. The previous proof also works with Theorem 2.3 and thus allows to deduce a result on the convergence of the characteristics for sequences of infinitely divisible random measures.

3.3 Example 3: excursion measure of Brownian motion

Let $P_x \in \mathcal{M}_1\big(\mathcal{C}(\mathbb{R}_+, \mathbb{R})\big)$ be the law of a 1-dimensional Brownian motion started in $x \in \mathbb{R}$ and denote the canonical process by $(B_t)_{t\geq 0}$. Let $T_0 := \inf\{t > 0 : B_t = 0\}$

be the first hitting time of the origin 0. It is a folklore fact that the measure $\mu_n := nP_{1/n}((B_{t\wedge T_0})_{t\geq 0} \in \cdot)$ converges, as $n \to \infty$, to the Itô excursion measure μ_{exc} of the reflected Brownian motion $(|B_t|)_{t\geq 0}$. There are several ways to define μ_{exc}. We use the characterisation given in Theorem XII.4.2 of [25] (where $\mu_{\mathrm{exc}} = 2n_+$ for n_+ used in [25]) as definition.

Definition 3.5 (Brownian excursion measure μ_{exc})**.** *Let $X' := \mathcal{C}(\mathbb{R}_+;\mathbb{R}_+)$ be equipped with the topology of uniform convergence on compacta. For $r > 0$, let $\nu_r \in \mathcal{M}_1(X')$ be the law of a 3-dimensional Bessel bridge of length r. Define*

$$\mu_{exc} := \int_{\mathbb{R}_+} \nu_r\, \kappa(r)\,\mathrm{d}r \qquad \textit{for} \qquad \kappa(r) := (2\pi r^3)^{-1/2}.$$

Then the σ-finite measure μ_{exc} on X' is called Brownian excursion measure.

Since μ_{exc} is obviously not a finite measure, we have to be more precise about what we mean by convergence of μ_n to μ_{exc}. In Theorem 1 of [17] it is shown (for a more general class of diffusions) that $\int F\,\mathrm{d}\mu_n \to \int F\,\mathrm{d}\mu_{\mathrm{exc}}$ holds for every $F \in \mathcal{C}_b(X')$ with the property that there is an $\varepsilon > 0$ with $F(e) = 0$ whenever $\|e\|_\infty < \varepsilon$. This looks very much like weak$^\#$-convergence on $X' \setminus \{0\}$, where 0 denotes the zero function and is sent infinitely far away.[3] This is, however, not precisely the case, because the map $e \mapsto \|e\|_\infty$ is not continuous w.r.t. uniform convergence on compacta.

In this subsection we give a setup, where we can apply Theorem 2.3 to obtain a weak$^\#$-convergence $\mu_n \xrightarrow{\mathrm{w}^\#} \mu_{\mathrm{exc}}$. To this end, we have to modify the topology on (a subspace of) X' in two ways. First, we weaken uniform convergence on compacta to convergence in Lebesgue measure, because the latter is induced by “nice” functions and therefore much easier to handle in our framework. This, of course, substantially weakens our result, so that it does not imply the one in [17]. Second, we strengthen the topology (and therefore our result) a bit by additionally requiring convergence of excursion lengths for the convergence of excursions. This allows us to send the zero function infinitely far away by a continuous function, and the result in [17] does not directly include ours.

Definition 3.6 (Our excursion space)**.** *Define the excursion length $\zeta\colon X' \to [0,\infty]$ by*

$$\zeta(e) = \sup\{t > 0 \mid e(t) \neq 0\} \cup \{0\},$$

the space of excursions $X := \zeta^{-1}\big((0,\infty)\big)$, and the metric

$$d(e,\hat{e}) = \Big(\int_0^\infty |e(t) - \hat{e}(t)| \wedge 1\,\mathrm{d}t\Big) \wedge 1 + \big|\zeta(e)^{-1} - \zeta(\hat{e})^{-1}\big|$$

on X.

The topology induced by d on X is the Meyer-Zheng topology (or pseudo-path topology) introduced in [23] plus convergence of excursion lengths as we show in Lemma 3.8.

Definition 3.7 (Meyer-Zheng topology)**.** *Let λ be the probability measure on $\mathbb{R}_+$ with Lebesgue-density $t \mapsto e^{-t}$, and $e_n, e\colon \mathbb{R}_+ \to \mathbb{R}_+$ measurable. Then e_n is said to converge to e in Meyer-Zheng topology if the image measures of λ under e_n converge weakly to the one under e.*

Lemma 3.8 (The topology induced by d)**.** *Let $e_n, e \in X$. Then the following are equivalent:*

[3] Given a metric space (X,d) and $x \in X$, “sending x infinitely far away” is a figure of speech for considering $X \setminus \{x\}$ with a metric d' topologically equivalent to d, but making every sequence that d-converges to x leave every d'-ball. A possible choice is $d'(y,z) = d(y,z) + |d(x,y)^{-1} - d(x,z)^{-1}|$. Formally, $d'(x,y) = \infty$ for all $y \in X \setminus \{x\}$.

1. *$e_n \to e$ with respect to d.*
2. *e_n converges to e in Lebesgue-measure, and $\zeta(e_n) \to \zeta(e)$.*
3. *e_n converges to e in Meyer-Zheng topology, and $\zeta(e_n) \to \zeta(e)$.*

In particular, (X, d) is a separable metric space.

Proof. For the "in particular" note that separability of convergence in Lebesgue-measure is well-known and carries over to d-convergence by the first equivalence.
1⇔2: Since $\zeta(e) \in (0, \infty)$ for all $e \in X$, we have that $\left|\zeta(e)^{-1} - \zeta(\hat{e})^{-1}\right| \to 0$ is equivalent to $\zeta(e_n) \to \zeta(e)$. It is well-known that $d_{\mathrm{Leb}}(e, \hat{e}) = \int |e(t) - \hat{e}(t)| \wedge 1 \,\mathrm{d}t$ induces convergence in Lebesgue-measure (e.g. [8, Exercise 4.7.61]), so the same is true for $d_{\mathrm{Leb}} \wedge 1$.
3⇔2: In [23, Lemma 1] it is shown that Meyer-Zheng topology coincides with convergence in λ-measure. Now $\zeta(e_n) \to \zeta(e)$ implies $M := \sup_{n\in\mathbb{N}} \zeta(e_n) < \infty$, and λ is equivalent to Lebesgue-measure on $[0, M]$. □

Theorem 3.9 (Brownian excursion measure)**.** *Let (X, d) be the excursion space introduced in Definition 3.6, μ_{exc} the Brownian excursion measure, and $\widehat{B} = (\widehat{B}_t)_{t\geq 0}$ Brownian motion killed in 0. In $\mathcal{M}^{\#}(X)$,*

$$\mu_n := nP_{1/n}(\widehat{B} \in \cdot) \xrightarrow{\mathrm{w}^{\#}} \mu_{\text{exc}} \quad \text{as } n \to \infty.$$

In order to use Theorem 2.3 to prove Theorem 3.9, we need a set $\mathcal{F}$ of continuous functions on X satisfying (T.1) – (T.3). To this end, define for $f \in L^1(\mathbb{R}_+)$ and $g \in \mathcal{C}_b(\mathbb{R}_+)$

$$F_{f,g}(e) = \int_0^\infty f(t)g(e(t))\,\mathrm{d}t.$$

Denote by $\mathcal{C}_c$ the continuous functions with compact support and define a set of bounded functions on X by

$$\mathcal{F}' = \left\{F_{f,g} \mid f \in \mathcal{C}_c(\mathbb{R}_+),\, x \mapsto (1 \wedge x)^{-1}g(x) \in \mathcal{C}_b(\mathbb{R}_+)\right\} \cup \{h \circ \zeta \mid h \in \mathcal{C}_b(\mathbb{R}_+)\}.$$

Definition 3.10 ($\mathcal{F}$)**.** *Let $\mathcal{F}$ be the multiplicative closure of $\mathcal{F}'$.*

Lemma 3.11. *$\mathcal{F}$ is weak$^{\#}$-convergence determining for measures in $\mathcal{M}^{\#}_{\mathcal{F}}(X)$.*

Proof. $\mathcal{F}$ obviously satisfies (T.1) and (T.3). Indeed, $\mathcal{F}$ is multiplicatively closed by definition, and $A \subseteq X$ is d-bounded if and only if $\inf_{e\in A} \zeta(e) > 0$. Once we have shown (T.2), i.e. that $\mathcal{F}$ induces the same topology as d, the claim follows from Theorem 2.3. To this end, note that $\zeta(e_n) \to \zeta(e)$ is necessary for both d- and $\mathcal{F}$-convergence, and recall that under this condition, by Lemma 3.8, d-convergence is equivalent to

$$\int_0^\infty \varphi(t, e_n(t))\,\lambda(\mathrm{d}t) \to \int_0^\infty \varphi(t, e(t))\,\lambda(\mathrm{d}t) \qquad \forall \varphi \in \mathcal{C}_b(\mathbb{R}_+^2). \tag{3.4}$$

The set of $\varphi_{f,g}$ of the form $\varphi_{f,g}(t, x) = f(t)g(x)e^t$ with $f \in \mathcal{C}_c(\mathbb{R}_+)$, $g \in \mathcal{C}_b(\mathbb{R}_+)$ and $x \mapsto x^{-1}g(x) \in \mathcal{C}_b(\mathbb{R}_+)$ is a multiplicatively closed subset of $\mathcal{C}_b(\mathbb{R}_+^2)$, and induces the Euclidean topology on $\mathbb{R}_+^2$. Thus, by the classical Le Cam theorem, it is convergence determining, and (3.4) is equivalent to

$$F_{f,g}(e_n) = \int_0^\infty \varphi_{f,g}(t, e_n(t))\,\lambda(\mathrm{d}t) \to F_{f,g}(e) \qquad \forall F_{f,g} \in \mathcal{F}',$$

which implies the claim. □

Proof of Theorem 3.9. In view of Lemma 3.11 it is sufficient to show

$$\lim_{\varepsilon\to 0} \tfrac{1}{\varepsilon}\mathbb{E}_\varepsilon\big[F(\widehat{B})\big] = \int F \,\mathrm{d}\mu_{\mathrm{exc}} \qquad \forall F\in\mathcal{F}. \tag{3.5}$$

Fix $F\in\mathcal{F}$. There is $h\in\mathcal{C}_b(\mathbb{R}_+)$, $n\in\mathbb{N}_0$, $f_i\in L^1\cap\mathcal{C}_c(\mathbb{R}_+)$, $x\mapsto(1\wedge x)^{-1}g_i(x)\in\mathcal{C}_b(\mathbb{R}_+)$ for $1\le i\le n$, such that

$$F(e) = h\circ\zeta(e)\cdot\prod_{i=1}^n F_{f_i,g_i}(e) = \int_{\mathbb{R}_+^n} f(\underline{t})h(\zeta(e))g\big(e_1(t_1),\dots,e_n(t_n)\big)\,\mathrm{d}\underline{t},$$

where we set $\underline{t}=(t_1,\dots,t_n)$, $f(\underline{t})=\prod_{i=1}^n f_i(t_i)$ and $g(\underline{t})=\prod_{i=1}^n g_i(t_i)$. We abbreviate $\bar{t}:=\max\{t_1,\dots,t_n\}$. Using that $\zeta(e)=r$ holds ν_r-a.s., and $g_i(e(t_i))=0$ whenever $t_i>\zeta(e)$, we obtain

$$\begin{aligned}\int F\,\mathrm{d}\mu_{\mathrm{exc}} &= \int_0^\infty \kappa(r)h(r)\int\prod_{i=1}^n F_{f_i,g_i}\,\mathrm{d}\nu_r\,\mathrm{d}r \qquad (3.6)\\ &= \int_{\mathbb{R}_+^n} f(\underline{t})\int_0^\infty h(\bar{t}+r)\kappa(\bar{t}+r)\int g\big(e_1(t_1),\dots,e_n(t_n)\big)\,\nu_{\bar{t}+r}(\mathrm{d}e)\,\mathrm{d}r\,\mathrm{d}\underline{t}\\ &= \int_{\mathbb{R}_+^n} f(\underline{t})\int_0^\infty h(\bar{t}+r)\mathbb{E}_0\Big[\frac{g(\rho_{t_1},\dots,\rho_{t_n})}{\rho_{\bar{t}}}\ell^{\rho_{\bar{t}}}(r)\Big]\,\mathrm{d}r\,\mathrm{d}\underline{t},\end{aligned}$$

where $\rho=(\rho_t)_{t\ge0}$ denotes a 3-dimensional Bessel process, ℓ^α the density of a Lévy distribution with scale parameter α, and we have used the relation of densities of the Bessel bridge and process (as obtained, e.g., in [25, XI.§3]).

On the other hand, recall that $T_0=\zeta(\widehat{B})$ is the hitting time of 0 of $\widehat{B}$ and observe

$$\tfrac{1}{\varepsilon}\mathbb{E}_\varepsilon\big[F(\widehat{B})\big] = \int f(\underline{t})\mathbb{E}_\varepsilon\big[h(T_0)\tfrac{1}{\varepsilon}\prod_{i=1}^n g_i(\widehat{B}_{t_i})\big]\,\mathrm{d}\underline{t}. \tag{3.7}$$

For fixed $\underline{t}$, reordering if necessary, we assume for notational convenience that $t_1\le\dots\le t_n=\bar{t}$ and set $t_0=0$. Then, using the Markov property at t_n in the first step,

$$\begin{aligned}\mathbb{E}_\varepsilon\Big[h(T_0)\tfrac{1}{\varepsilon}\prod_{i=1}^n g_i(\widehat{B}_{t_i})\Big] &= \mathbb{E}_\varepsilon\Big[\mathbb{E}_{\widehat{B}_{t_n}}\big[h(t_n+T_0)\big]\frac{1}{\widehat{B}_{t_n}}\prod_{i=1}^n\frac{\widehat{B}_{t_i}}{\widehat{B}_{t_{i-1}}}g_i(\widehat{B}_{t_i})\Big]\\ &= \mathbb{E}_\varepsilon\Big[\mathbb{E}_{\rho_{t_n}}\big[h(t_n+T_0)\big]\frac{1}{\rho_{t_n}}\prod_{i=1}^n g_i(\rho_{t_i})\Big], \qquad (3.8)\end{aligned}$$

where we have used that the sub-Markovian semigroup $(Q_t)_{t\ge0}$ of the killed Brownian motion $\widehat{B}$ and the Markovian semigroup $(H_t)_{t\ge0}$ of the Bessel process ρ are related by

$$H_t(x,\mathrm{d}y) = \begin{cases} x^{-1}Q_t(x,\mathrm{d}y)y & \text{for } x>0,\\ 2\kappa(t)y^2\exp(-y^2/(2t))\,\mathrm{d}y & \text{for } x=0.\end{cases}$$

Because $\widehat{B}$ is a Feller-process and $h\in\mathcal{C}_b$, the function $x\mapsto\mathbb{E}_x\big[h(t_n+T_0)\big]$ is also a bounded continuous function. Because also $x\mapsto x^{-1}g_n(x)$ is bounded and continuous by assumption, we see that the term inside the outer expectation in (3.8) is a bounded continuous function in $\rho_{t_1},\dots,\rho_{t_n}$. Using that ρ is a Feller process, we obtain from (3.7) and (3.8)

$$\lim_{\varepsilon\to0}\tfrac{1}{\varepsilon}\mathbb{E}_\varepsilon\big[F(\widehat{B})\big] = \int f(\underline{t})\mathbb{E}_0\Big[\mathbb{E}_{\rho_{\bar{t}}}\big[h(\bar{t}+T_0)\big]\frac{g(\rho_{t_1},\dots,\rho_{t_n})}{\rho_{\bar{t}}}\Big]\,\mathrm{d}\underline{t}. \tag{3.9}$$

The hitting time T_0 of 0 under P_x is known to be Lévy-distributed with scale parameter x, hence

$$\mathbb{E}_{\rho_{\bar{t}}}\big[h(\bar{t}+T_0)\big] = \int_0^\infty h(\bar{t}+r)\ell^{\rho_{\bar{t}}}(r)\,\mathrm{d}r. \tag{3.10}$$

Inserting (3.10) into (3.9), and applying (3.6), we obtain the claimed convergence (3.5). □

3.4 Example 4: mass fragmentations

Consider the set

$$S^{\downarrow} = \Big\{\underline{s} = (s_1, s_2, \dots) \in [0,1]^{\mathbb{N}} \;\Big|\; \sum_{i\geq 1} s_i \leq 1,\; s_1 \geq s_2 \geq s_3 \geq \dots \geq 0\Big\}$$

of decreasing sequences $(s_1, s_2, \dots)$ with sum less than 1. This set is used to model (mass) fragmentation processes. A perfect introduction to the topic is given in Bertoin's book [6]. In his Section 2.1, the set $S^{\downarrow}$ is introduced and endowed with the topology of pointwise convergence. Our goal is to present two sets of real-valued convergence-determining functions on $S^{\downarrow}$, namely

$$\mathcal{G}_1 = \Big\{S^{\downarrow} \ni \underline{s} = (s_1, s_2, \dots) \mapsto G_p(\underline{s}) = \sum_{i\geq 1} s_i^p \;\Big|\; p \in \mathbb{N}\Big\},$$
$$\mathcal{G}_2 = \Big\{S^{\downarrow} \ni \underline{s} = (s_1, s_2, \dots) \mapsto H_\alpha(\underline{s}) = \sum_{i\geq 1} (1 - e^{-\alpha s_i}) \;\Big|\; \alpha > 0\Big\}.$$

Theorem 3.12. *The sets $\mathcal{G}_1$ and $\mathcal{G}_2$ are convergence-determining on $S^{\downarrow}$ in the sense that for $\underline{s}(n), \underline{s} \in S^{\downarrow}$, $n \in \mathbb{N}$ the following holds for $j = 1, 2$:*

$$G(\underline{s}(n)) \xrightarrow{n\to\infty} G(\underline{s})\ \forall G \in \mathcal{G}_j \quad \Longrightarrow \quad \underline{s}(n) \xrightarrow{n\to\infty} \underline{s}.$$

Before we give the proof, we relate the set $S^{\downarrow}$ to boundedly finite measures. Therefore, let

$$X = (0,1] \text{ with metric } d(x,y) = \left|x^{-1} - y^{-1}\right|.$$

Consider the mapping

$$\Phi : \begin{cases} S^{\downarrow} & \to \mathcal{M}^{\#}(X), \\ (s_1, s_2 \dots) & \mapsto \sum_{i\geq 1} \delta_{s_i}. \end{cases}$$

Lemma 3.13. *The mapping Φ is a homeomorphism from $S^{\downarrow}$ to $\Phi(S^{\downarrow})$.*

Proof. Step 1. The mapping Φ is injective: For $\underline{s} \in S^{\downarrow}$, all $s_i, i \geq 1$ can be easily reconstructed.

Step 2. The mapping Φ is continuous: Let $\underline{s}(n), \underline{s} \in S^{\downarrow}$ with $\underline{s}(n) \to \underline{s}$. If A is a bounded set in X w.r.t. d with $\Phi(\underline{s})(\partial A) = 0$, then it is clear that $\Phi(\underline{s}(n))(A) \to \Phi(\underline{s})(A)$ and we can use Proposition A.2.6.II (d) in [10].

Step 3. The mapping $\Phi^{-1}|_{\Phi(S^{\downarrow})}$ is continuous: Suppose $\Phi(\underline{s}(n)) \xrightarrow{\mathrm{w}^{\#}} \Phi(\underline{s})$. Fix $z \in (0,1)$ with $z \notin \{\underline{s}_i \mid i \in \mathbb{N}\}$. Then we have $\Phi(\underline{s}(n))|_{[z,1]} \xrightarrow{w} \Phi(\underline{s})|_{[z,1]}$. But this implies convergence of $\underline{s}(n)$ to $\underline{s}$ on those coordinates that lie in $[z,1]$. Since we may choose z arbitrarily small, this suffices. □

Proof of Theorem 3.12. We only provide the proof for $\mathcal{G}_1$, since the other proof is similar.

Step 1. Note that we can write

$$G_p(\underline{s}) = \int x^p\,\Phi(\underline{s})(\mathrm{d}x),\; p \in \mathbb{N}.$$

Thus, $G(\underline{s}(n)) \to G(\underline{s})\, \forall G \in \mathcal{G}_1$ can be written as

$$\int x^p\, \Phi(\underline{s}(n))(\mathrm{d}x) \to \int x^p\, \Phi(\underline{s})(\mathrm{d}x),\ p \in \mathbb{N}.$$

We use Theorem 2.3 to show that $\Phi(\underline{s}(n)) \xrightarrow{\mathrm{w}^\#} \Phi(\underline{s})$ and this suffices for $\underline{s}(n) \to \underline{s}$ by Lemma 3.13.

Step 2. Of course (X, d) as before Lemma 3.13 is a separable metric space. Moreover, the class $\mathcal{F} = \mathrm{span}\{x \mapsto x^p \mid p \in \mathbb{N}\} \subset \mathcal{C}_b(X)$ of polynomials without constant term on X satisfies the prerequisites of Theorem 2.3, which is easy to check. Finally, $\Phi(\underline{s}) \in \mathcal{M}^\#_{\mathcal{F}}(X)$ for all $\underline{s} \in S^\downarrow$ since for all $p \in \mathbb{N}$:

$$\int x^p\, \Phi(\underline{s})(\mathrm{d}x) \le \int x\, \Phi(\underline{s})(\mathrm{d}x) = \sum_{i \ge 1} s_i \le 1 < \infty.$$

Thus, Theorem 2.3 applies and yields the claim. □

Remark 3.14. Note that $\mathcal{G}_1 \not\subset \mathcal{C}(S^\downarrow)$: for $\underline{s}_i(n) = n^{-1}\mathbb{1}_{1 \le i \le n}$, $i \in \mathbb{N}$, we have $\underline{s}(n) \to \underline{0}$, but $G_1(\underline{s}(n)) = 1 \not\to 0 = G_1(\underline{0})$.

The stronger statement than Theorem 3.12 including the continuity holds on the subset of decreasing sequences summing up to 1.

Corollary 3.15. *On the subset $S^\downarrow_1 = \{\underline{s} \in S^\downarrow \mid \sum_{i \ge 1} s_i = 1\}$, the set of functions $\mathcal{G}_1$ generates the topology of pointwise convergence.*

Proof. First, the subset is a Polish space with the relative topology since it is a G_δ-subset of $S^\downarrow$.

Let $\underline{s}(n), \underline{s} \in S^\downarrow_1$, $n \in \mathbb{N}$.

Step 1. First suppose that $\underline{s}(n) \to \underline{s}$ as $n \to \infty$. We can use an approximation argument to get $G(\underline{s}(n)) \to G(\underline{s})$ for all $G \in \mathcal{G}_1$: Let $\varepsilon > 0$. Fix $z > 0$ s.t. $\sum_{i \ge 1} s_i \mathbb{1}_{s_i \le z} < \varepsilon$ and $s_i \neq z$ for all $i \ge 1$. By Proposition A.2.6.II (d) from [10] and our Proposition 3.13, we may choose n so large that $\Phi(\underline{s}(n))|_{[z,1]}$ and $\Phi(\underline{s})|_{[z,1]}$ are very close in the following sense:

$$\sup_{q \in \{1,p\}} \Big| \sum_{i \ge 1} s_i(n)^q\, \mathbb{1}_{s_i(n) > z} - \sum_{i \ge 1} s_i^q\, \mathbb{1}_{s_i(n) > z} \Big| = A(\varepsilon) < \varepsilon.$$

Use that in the following equation

$$\begin{aligned} G_p(\underline{s}(n)) &= \sum_{i \ge 1} s_i(n)^p\, \mathbb{1}_{s_i(n) > z} + \sum_{i \ge 1} s_i(n)^p\, \mathbb{1}_{s_i(n) \le z} \\ &= A(\varepsilon) + \sum_{i \ge 1} s_i^p\, \mathbb{1}_{s_i > z} + \sum_{i \ge 1} s_i(n)^p\, \mathbb{1}_{s_i(n) \le z}. \end{aligned}$$

Once we establish that the last term is small, we see that $G_p(\underline{s}(n)) \to G_p(\underline{s})$. But the last term is bounded by:

$$\begin{aligned} \sum_{i \ge 1} s_i(n)^p\, \mathbb{1}_{s_i(n) \le z} &\le \sum_{i \ge 1} s_i(n) \mathbb{1}_{s_i(n) \le z} \\ &= 1 - \sum_{i \ge 1} s_i(n) \mathbb{1}_{s_i(n) > z} \le 1 - (1 - \varepsilon) = \varepsilon. \end{aligned}$$

Note that it was crucial to know that $\sum_{i \ge 1} s_i = 1$ in the last proof.

Step 2. The converse direction was established in Theorem 3.12. □

Remark 3.16. An application in a similar spirit is given in [12], where X is the set of ultrametric measure spaces with diameter in $[0, 2h)$ for certain $h > 0$. Any ultrametric measure space with diameter in $[0, 2h]$ can be written as a boundedly finite measure on

X (in a unique way similar to a prime factorization). This relation can be used for a result analogous to Corollary 3.15.

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