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Borel liftings of graph limits

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Abstract

The cut pseudo-metric on the space of graph limits induces an equivalence relation. The quotient space obtained by collapsing each equivalence class to a point is a metric space with appealing analytic properties. We show the equivalence relation admits a Borel lifting: There exists a Borel-measurable mapping that maps each equivalence class to one of its elements. The result yields a general framework for proving measurability properties on the space of graph limits. We give several examples, including Borel-measurability of the set of isomorphism classes of random-free graphons.

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The present note resolves a measurability question arising in the theory of graph limits. Graph limits have recently found applications in several fields, including extremal combinatorics and property testing [11], probability theory [6], statistics [1], and statistical physics [4]. The theory of these limits revolves around two types of objects: Certain measurable functions, which can be thought of as representations of limits of graph sequences, and isomorphism classes of such functions. It is well known that one can pass in a measurable way from functions to isomorphism classes. It has been conjectured that the converse is also true: There is a measurable mapping that takes each isomorphism class to a representative function. We show that this is indeed the case.

1 Result

Let $(\Omega, \mathcal{B}(\Omega), P)$ be an atomless Borel probability space and $L_1(\Omega^2)$ the Banach space of integrable functions on $\Omega \times \Omega$, equipped with the L_1-metric d_1. Let $\mathbf{W} \subset L_1(\Omega^2)$ be the subspace of symmetric integrable functions $\Omega^2 \to [0,1]$. Define a pseudo-norm on $\mathbf{W}$ by

$$\|w\|_\square := \sup_{S,T \in \mathcal{B}(\Omega)} \int_{S\times T} w(s,t) dP(s) dP(t) \;. \tag{1.1}$$

Following [3], we use $\|\,.\,\|_\square$ to define a pseudo-metric on $\mathbf{W}$ as

$$\delta_\square(w,w') := \inf_\psi \|w^\psi - w'\|_\square \qquad \text{where} \qquad w^\psi(x,y) := w(\psi(x), \psi(y)) \;. \tag{1.2}$$

The infimum is taken over all invertible measure-preserving transformations of Ω, i.e. all invertible measurable mappings $\psi : \Omega \to \Omega$ satisfying $\psi P = P$. The pseudo-metric induces an equivalence relation on $\mathbf{W}$, given by $w \equiv w' :\Leftrightarrow \delta_\square(w,w') = 0$. The relation

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$w \equiv w'$ is also known as *weak isomorphism* of w and w' [11]. Denote the equivalence class of $w \in \mathbf{W}$ by $[w]_\square$, and the quotient space of all equivalence classes by $\widehat{\mathbf{W}}$. On the quotient space, $\delta_\square$ is a metric, and the metric space $(\mathbf{W}, \delta_\square)$ is compact [13]. For each $\widehat{w} \in \widehat{\mathbf{W}}$, we write $[\widehat{w}]_\square \subset \mathbf{W}$ for the corresponding equivalence class of elements of $\mathbf{W}$.

Theorem 1.1 below shows that weak isomorphism admits a Borel lifting, i.e. there exists a Borel-measurable mapping $\xi : (\widehat{\mathbf{W}}, \delta_\square) \to (\mathbf{W}, d_1)$ such that

$$\xi(\widehat{w}) \in [\widehat{w}]_\square \qquad\qquad \text{for all } \widehat{w} \in \widehat{\mathbf{W}} \, . \tag{1.3}$$

The lifting is not unique. More precisely:

Theorem 1.1. *There is a sequence (ξ_n) of measurable mappings $\xi_n \colon (\widehat{\mathbf{W}}, \delta_\square) \to (\mathbf{W}, d_1)$ such that, for every $\widehat{w} \in \widehat{\mathbf{W}}$, the set $\{\xi_n(\widehat{w}) \,|\, n \in \mathbb{N}\}$ is a dense subset of $[\widehat{w}]_\square$.*

2 Applications

Every graphon w defines a probability distribution P_w on infinite random graphs [11]. The parametrization of these measures by graphons is not unique, since $P_w = P_{w'}$ whenever w and w' are weakly isomorphic. Clearly, a unique parametrization can be obtained by substituting graphons by equivalence classes. One implication of Theorem 1.1 is that this parametrization, i.e. the mapping $\widehat{w} \mapsto P_{\widehat{w}}$, is measurable.

Another consequence is that, for any function $f : \mathbf{W} \to \mathbb{R}$ that is Borel with respect to the L_1 topology on $\mathbf{W}$, the composite map $\xi \circ f$ is Borel on $\widehat{\mathbf{W}}$. This property has a number of applications. For example, a graphon w is called *random-free* if its range is $\{0, 1\}$. The set of all such graphons is not closed with respect to the metric $\delta_\square$, and their equivalence classes are hence not closed in $\widehat{\mathbf{W}}$ [11]. However:

Corollary 2.1. *The set of equivalence classes of random-free graphons is Borel in $\widehat{\mathbf{W}}$.*

Measurability of composite maps also implies that densities of weighted subgraphs are Borel measurable: Let $M \in \mathbb{R}^{n \times n}$ be a symmetric matrix with non-negative entries and vanishing diagonal, i.e. an edge-weighted, complete graph of size n. Define the density of M in w as

$$t(M, w) := \int \prod_{i<j\leq n} w(x_i, x_j)^{M_{ij}} dP^n \, . \tag{2.1}$$

The mapping $w \mapsto t(M, w)$ on $\mathbf{W}$ depends only on the equivalence class of w, and induces a mapping $\widehat{w} \mapsto t(M, \widehat{w})$ on the quotient space $\widehat{\mathbf{W}}$. The mapping $\widehat{w} \mapsto t(M, \widehat{w})$ is continuous if M is binary, but the same is not true in general.

Corollary 2.2. *For every M, the map $\widehat{w} \mapsto t(M, \widehat{w})$ is Borel on $\widehat{\mathbf{W}}$.*

In particular, for $n = 2$ and any $k \in \mathbb{N}$, the moment $\int_{\Omega^2} w^k(x, y) P(dx) P(dy)$ depends in a measurable way on the equivalence class. These moments completely determine the value distribution of the graphon, i.e. the image probability measure $w(P \otimes P)$ on $[0, 1]$. Hence, the mapping $w \mapsto w(P \otimes P)$ that takes a graphon to its value distribution is Borel with respect to the weak topology of probability measures on $[0, 1]$.

3 Proof

Theorem 1.1 can be stated equivalently by defining a set-valued mapping

$$\phi_\square : \widehat{\mathbf{W}} \to 2^{\mathbf{W}} \qquad \text{with} \qquad \phi_\square(\widehat{w}) := [\widehat{w}]_\square \, . \tag{3.1}$$

We then have to show that there are measurable mappings $\xi_n \colon (\widehat{\mathbf{W}}, \delta_\square) \to (\mathbf{W}, d_1)$ with

$$\overline{\{\xi_n(\widehat{w}) \,|\, n \in \mathbb{N}\}} = \phi_\square(\widehat{w}) \qquad \text{for all } \widehat{w} \in \widehat{\mathbf{W}} \, , \tag{3.2}$$

where $\overline{A}$ denotes the closure of a set A.

Liftings of set-valued maps are a well-studied topic in analysis, and we use a result of Kuratowski and Ryll-Nardzewski [10] on the existence of liftings, and a generalization by Castaing [5] (see e.g. [8], Theorem 12.16, and [9], Theorem 14.4.1, for textbook statements). For our purposes, these results can be summarized as follows:

Theorem 3.1. *Let $\mathbf{X}$ be a measurable space, $\mathbf{Y}$ a Polish space, and $\phi : \mathbf{X} \to 2^{\mathbf{Y}}$ a set-valued mapping. Require $\phi(x)$ to be non-empty and closed for all $x \in \mathbf{X}$, and that*

$$\phi^{-1}(A) := \{x \in \mathbf{X} \,|\, \phi(x) \cap A \neq \emptyset\} \tag{3.3}$$

is a measurable set in $\mathbf{X}$ for each open set A in $\mathbf{Y}$. Then there exists a sequence of measurable mappings $\xi_n : \mathbf{X} \to \mathbf{Y}$ such that $\overline{\{\xi_n(x) \,|\, n \in \mathbb{N}\}} = \phi(x)$ for all $x \in \mathbf{X}$.

For $\phi_\Box$ as defined in (3.1) and any subset $A \subset \mathbf{W}$, the set $\phi_\Box^{-1}(A)$ in (3.3) simply consists of all $\widehat{w} \in \widehat{\mathbf{W}}$ for which A contains at least one element of the equivalence class $[\widehat{w}]_\Box$. If A is in particular an open d_1-ball in $\mathbf{W}$, this set has the following property:

Lemma 3.2. *Denote by $U_\varepsilon(v)$ the open d_1-ball of radius ε centered at $v \in \mathbf{W}$. If $\varepsilon < \delta$,*

$$\overline{\phi_\Box^{-1}(U_\varepsilon(v))} \;\subseteq\; \phi_\Box^{-1}(U_\delta(v)) \tag{3.4}$$

for all $v \in \mathbf{W}$.

Proof of Lemma 3.2. Let $(\widehat{w}_1, \widehat{w}_2, \ldots)$ be a sequence in $\phi_\Box^{-1}(U_\varepsilon(v))$ with $\widehat{w}_i \xrightarrow{\delta_\Box} \widehat{w}$. We have to show that $\widehat{w} \in \phi_\Box^{-1}(U_\delta(v))$. By definition of $\phi_\Box^{-1}$, the sets $\phi_\Box(\widehat{w}_i) \cap U_\varepsilon(v)$ are non-empty. Suppose (w_i) is a sequence with $w_i \in \phi_\Box(\widehat{w}_i) \cap U_\varepsilon(v)$ for each $i \in \mathbb{N}$. By Lemma 2.11 of [14], convergence of $(\widehat{w}_i)$ to $\widehat{w}$ then implies

$$\varepsilon \geq \liminf \delta_1(w_i, v) \geq \delta_1(w, v) = \inf_\psi d_1(w^\psi, v) \tag{3.5}$$

for any $w \in \phi_\Box(\widehat{w})$. Since $\varepsilon < \delta$, there is hence a measure-preserving transformation ψ such that $d_1(w^\psi, v) < \delta$, that is, $w^\psi \in U_\delta(v)$. Because w^ψ and w are weakly isomorphic, we also have $w^\psi \in \phi_\Box(\widehat{w})$, and therefore

$$\widehat{w} \;\in\; \phi_\Box^{-1}(\phi_\Box(\widehat{w}) \cap U_\delta(v)) \;\subset\; \phi_\Box^{-1}(U_\delta(v)) \,. \tag{3.6}$$

□

Proof of Theorem 1.1. The space (W, d_1) is a closed subspace of the separable Banach space $L_1(\Omega^2)$, and hence Polish. The sets $\phi_\Box(\widehat{w})$ are non-empty, by definition of the space $\widehat{\mathbf{W}}$ as a quotient. We will show that, additionally:

i. The sets $\phi_\Box(\widehat{w})$ are closed.
ii. For all open sets A in $\mathbf{W}$, the set $\phi_\Box^{-1}(A)$ is Borel in $\widehat{\mathbf{W}}$.

The mapping $\phi_\Box$ therefore satisfies the hypothesis of Theorem 3.1, and Theorem 1.1 follows.

(i) Denote by $t_F : \mathbf{W} \to [0,1]$ the homomorphism density indexed by a finite graph F [12]. Two elements of $\mathbf{W}$ are weakly isomorphic if and only if their homomorphism densities coincide for all finite graphs F [2, 7]. Let $\widehat{w} \in \widehat{\mathbf{W}}$, and let $(w_1, w_2, \ldots)$ be a sequence in the set $\phi(\widehat{w})$ with limit w in $(\mathbf{W}, d_1)$. The homomorphism densities are $\delta_\Box$-continuous and hence d_1-continuous. Therefore,

$$\lim t_F(w_i) = t_F(w) \qquad \text{for all } F \,, \tag{3.7}$$

and since the w_i are weakly isomorphic, $t_F(w_i) = t_F(w)$ for all i and all F. Thus, $w \in \phi(\widehat{w})$, and the set is closed.

(ii) Let $U_\delta(v)$ denote the open ball of radius δ centered at $v \in \mathbf{W}$. Since W is Polish, the open balls form a base of the topology, and it is sufficient to consider sets of the form $A = U_\delta(v)$. Let $\delta_i \in \mathbb{R}_+$ be an increasing sequence $\delta_i \to \delta$. Then, by Lemma 3.2,

$$\phi_\square^{-1}(U_\delta(v)) \;=\; \bigcup_i \phi_\square^{-1}(U_{\delta_i}(v)) \;\subseteq\; \bigcup_i \overline{\phi_\square^{-1}(U_{\delta_i}(v))} \overset{(3.4)}{\subseteq} \bigcup_i \phi_\square^{-1}(U_\delta(v)) \;=\; \phi_\square^{-1}(U_\delta(v))\;.$$

In particular, $\phi_\square^{-1}(U_\delta(v))$ is a countable union of the closed sets $\overline{\phi_\square^{-1}(U_{\delta_i}(v))}$, and hence Borel. ☐

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