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Semimartingale properties of the lower Snell envelope in optimal stopping under model uncertainty

Erick Treviño Aguilar

University of Guanajuato

Abstract. Optimal stopping under model uncertainty is a recent topic under research. The classical approach to characterize the solution of optimal stopping is based on the Snell envelope which can be seen as the value process as time runs. The analogous concept under model uncertainty is the so-called lower Snell envelope and in this paper, we investigate its structural properties. We give conditions under which it is a semimartingale with respect to one of the underlying probability measures and show how to identify the finite variation process by a limiting procedure. An example illustrates that without our conditions, the semimartingale property does not hold in general.

1 Introduction

Optimal stopping is a central topic in probability and statistics, its origin can be traced back at least to Wald's sequential approach to statistical testing; see Wald (1945). In a few decades, the problem of optimal stopping found applications in a rich variety of different areas going beyond statistics. The techniques underlying the solution to optimal stopping problems are diverse and include probabilistic potential theory for Markov reward processes and martingale methods for the Snell envelope; see, for example, Peskir and Shiryayev (2006) and El Karoui (1981), respectively.

In its most essential form, the problem of optimal stopping is formulated with respect to a unique probability measure which seen from the point of view of mathematical modeling, corresponds to a description of relevant probabilistic distributions determined by the phenomenon of interest. In many instances however, and including other problems well beyond optimal stopping, there are good reasons for considering families of probability measures. For example, in composed testing in which a decision must be taken in order to classify a distribution as the member of a composed null-hypothesis, one considers a family of distributions representing the null-hypothesis against another family of distributions which represents the alternative, see, for example, Witting (1985). In decision theory, Ellsberg's paradox Ellsberg (1961), highlights how the information (respectively, ambiguity) about the distribution are crucial in understanding human decisions under risk

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and uncertainty. The solution to the paradox consists in extending von Neumann and Morgenstern utilities von Neumann and Morgenstern (1944) to the so-called maxmin preferences axiomatized by Gilboa and Schmeidler (1989). The axiomatic framework of Gilboa and Schmeidler (1989) yields for each preference a family of probability measures under which utilities are quantified and the worst possible outcome is the utility assigned and under which decisions are taken. Another situation where optimal stopping problems are considered together with a family of probability measures comes from mathematical finance in the analysis of American options in incomplete markets.

Following Föllmer and Schied (2004), Section 6.4, let us at this point formulate a problem of optimal stopping in which a family of probability measures is considered. The numerical representation of a preference in the axiomatic framewok of Gilboa and Schmeidler (1989) takes the form

$$\psi(\cdot) := \inf_{Q\in\mathcal{Q}} E_Q[u(\cdot)], \tag{1}$$

with u a concave utility function and $\mathcal{Q}$ a convex class of probability measures. From this point of view, a reward process Y has maximal "expected" utility

$$\sup_{\tau} \psi(Y_\tau), \tag{2}$$

where the supremum is taken over the class of stopping times τ. There is one intuitive interpretation to this problem: The relevant distributions which in particular describe the probabilistic properties of Y are unclear and a family of probability measures $\mathcal{Q}$ is introduced playing the role of 'probabilistic models' potentially describing the distribution of Y and other relevant variables. Then, the decision of when to stop is made on the basis of obtaining the best reward under the worst of the possible choices of a model $Q \in \mathcal{Q}$. The choice of $\mathcal{Q}$ is subjective and may change according to Y or to the individual solving the stopping problem.

With this motivation, we can define a process $Z_t = u(Y_t)$ and consider the stopping problem

$$\sup_{\tau} \inf_{Q\in\mathcal{Q}} E_Q[Z_\tau]. \tag{3}$$

By analogy with classical theory of optimal stopping and its solution through the Snell envelope, we may want to characterize the solution to the *robust* stopping problem (3) through an analogous envelope. The approach is possible if the class of probability measures $\mathcal{Q}$ satisfies a property called *stability under pasting*, also known as *rectangularity*. Under this property, Gilboa and Schmeider utility functionals are *time-consistent*; see Epstein and Schneider (2003). We give the precise definitions below. Under this special property of time consistency, there exists an extension of the Snell envelope which characterizes the solution to the robust stopping problem (3). This novel envelope, the *lower Snell envelope*, is systematically presented by Föllmer and Schied (2004) in discrete time in the context of martingale measures in financial markets and investigated for other classes of probability measures by Riedel (2009). The construction of the lower Snell envelope, and the solution of the stopping problem (3) in continuous time was studied by the author Trevino-Aguilar (2011). We recall its definition and properties in continuous time in Section 2.

In a recent paper, Cheng and Riedel (2013) investigate the robust stopping problem (3) under g-expectations with backward differential stochastic equations techniques. In particular, this setting covers the κ-ambiguity model due to Chen and Epstein (2002). In this last model, the class $\mathcal{Q}$ is the family of probability measures under which a Brownian motion develops a trend having a density bounded by a constant $\kappa > 0$. Note that this corresponds in our initial discussion to the situation in which a signal is observed and it is unclear if the signal is purely noise. Thus, there is ambiguity in relevant distributions. Their solution consists in stopping as soon as the underlying process touches its lower Snell envelope. Moreover, they obtain a structural result which describes the lower Snell envelope as the sum of a process of bounded variation and a stochastic integral with respect to Brownian motion. Their structural result yields in particular the fact that the lower Snell envelope is a semimartingale. We are going to say that a lower Snell envelope has a *uniform decomposition* if it is the difference of a process which is a submartingale with respect to each element $Q \in \mathcal{Q}$ and a non decreasing process. This is less precise than the structural result of Cheng and Riedel (2013) but is suitable for our purposes here.

The goal of this paper is to study these two questions: Is the lower Snell envelope always a semimartingale?, if so, does it admit a uniform decomposition? A very simple example suggests that in general both questions have a negative answer. Indeed, we present an extreme example of a class $\mathcal{Q}$ under which the lower Snell envelope of a Brownian motion plus a deterministic function may fail to be a semimartingale and the lower Snell envelope of Brownian motion is a semimartingale but fails to have a uniform decomposition.

Thus, only if we introduce further properties we can expect positive answers to our two questions. This is the case in the structural result of Cheng and Riedel (2013) in which weak-compactness is crucial. What we do here is to introduce conditions on the side of the underlying process instead of the class $\mathcal{Q}$. We prove that the lower Snell envelope is a semimartingale if the underlying process is itself a semimartingale and the lower Snell envelope has a uniform decomposition if the underlying process has also this form.

The lower Snell envelope of a process Z can be characterized as the minimal supermartingale, in the generalized sense of Definition 2.2 below, dominating its underlying stochastic process Z. This is a key result that has been proved in the framework of g-expectations by Cheng and Riedel (2013). We extend their result to our setting. We will show how this property allows to identify the non decreasing process in uniform decompositions by a limiting procedure analogous to the Doob–Meyer decomposition and in which time consistency is crucial.

The paper is organized as follows. In Section 2, we fix notation, introduce the concept of stability and time-consistency and recall properties of the lower Snell envelope which are fundamental to our approach. In Section 3, we show that the lower Snell envelope of a semimartingale is itself a semimartingale and give a sufficient condition under which the lower Snell envelope has a uniform decomposition. In Section 4, we specify a stable class of probability measures under which the lower Snell envelope fails to be a semimartingale. In Section 5, we identify the semimartingale parts of the lower Snell envelope by a limiting procedure analogous to the Doob–Meyer decomposition and that uses only the property of time-consistency.

2 The lower Snell envelope

We start with some notation. Let $\mathsf{T} > 0$ be a positive finite number. We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathbb{F} := \{\mathcal{F}_t\}_{0 \le t \le \mathsf{T}}$. The probability measure $\mathbb{P}$ serves as reference measure, and we assume it is $0 - 1$ on $\mathcal{F}_0$. We assume that the filtration $\mathbb{F}$ satisfies the usual assumptions of right continuity and completeness. By $\mathcal{T}$ we denote the class of $\mathbb{F}$-stopping times with values in the interval $[0, \mathsf{T}]$.

Definition 2.1. Let $\tau \in \mathcal{T}$ be a stopping time and Q_1 and Q_2 be probability measures equivalent to $\mathbb{P}$. The probability measure defined through

$$Q_3(A) := E_{Q_1}\big[Q_2[A \mid \mathcal{F}_\tau]\big], \qquad A \in \mathcal{F}_\mathsf{T}$$

is called *the pasting* of Q_1 and Q_2 in τ.

A family of probability measures $\mathcal{Q}$ is *stable under pasting* or simply *stable* if every $Q \in \mathcal{Q}$ is equivalent to $\mathbb{P}$, and if for each Q_1 and Q_2 in $\mathcal{Q}$ and any stopping time $\tau \in \mathcal{T}$, the pasting of Q_1 and Q_2 in τ is an element of $\mathcal{Q}$.

Definition 2.1 is analogous to the concept studied by Föllmer and Schied (2004). It is related to the concepts of fork-convexity and m-stability; see Delbaen (2006). The family of equivalent martingale measures is stable under pasting, and this property is crucial for the analysis of the upper and lower prices $\pi_{\sup}(\cdot)$ and $\pi_{\inf}(\cdot)$ of American options; see Föllmer and Schied (2004). Another important application of the stability concept appears in the problem of representing time consistent risk measures; see, for example, Föllmer and Penner (2006) for details and references.

From now on, we fix a stable class of probability measures $\mathcal{Q}$. In the introduction, we mentioned the concept of being a supermartingale in a generalized sense. In the next definition, we make it precise. Here and in the sequel, we denote by $L^1(Q)$ the space of random variables which are integrable with respect to probability measure Q.

Definition 2.2. For $\tau \in \mathcal{T}$, we denote by $\mathbf{E}^{\downarrow}[\cdot \mid \mathcal{F}_\tau] : \bigcap_{Q \in \mathcal{Q}} L^1(Q) \to \mathcal{F}_\tau$ the operator

$$\mathbf{E}^{\downarrow}[\cdot \mid \mathcal{F}_\tau] := \operatorname*{ess\,inf}_{Q \in \mathcal{Q}} E_Q[\cdot \mid \mathcal{F}_\tau]. \tag{4}$$

An $\mathbb{F}$-adapted process $\{M_t\}_{0 \le t \le \mathsf{T}}$ is a $\mathbf{E}^{\downarrow}$-supermartingale (respectively, a $\mathbf{E}^{\downarrow}$-martingale) if for each pair of stopping times $\tau, \theta \in \mathcal{T}$ with $\mathbb{P}(\tau \ge \theta) = 1$ we have

1. $E_Q[|M_\tau|] < \infty$, for $Q \in \mathcal{Q}$,
2. $\mathbf{E}^{\downarrow}[M_\tau \mid \mathcal{F}_\theta] \le M_\theta$ (respectively $=$).

Definition 2.2 is specific to our setting here but it has been considered by other authors; see, for example, Coquet et al. (2002) and Cheng and Riedel (2013).

Remark 2.1. The "upper version" of our operator $\mathbf{E}^{\downarrow}$ is defined by

$$\mathbf{E}^{\uparrow}[\cdot \mid \mathcal{F}_\tau] := \operatorname*{ess\,sup}_{Q \in \mathcal{Q}} E_Q[\cdot \mid \mathcal{F}_\tau].$$

When $\mathcal{Q}$ is a class of martingale measures, then the corresponding concept of $\mathbf{E}^{\uparrow}$-supermartingale is crucial in the celebrated optional decomposition theorem; see Föllmer and Kabanov (1998) and its references. This is a general result on the structure of $\mathbf{E}^{\uparrow}$-supermartingales and motivates to study analogous properties for the lower non-linear expectation $\mathbf{E}^{\downarrow}$. In particular, to study semimartingale properties.

The stability property of the class $\mathcal{Q}$ translates into a "time-consistency property":

$$\mathbf{E}^{\downarrow}\big[\mathbf{E}^{\downarrow}[\cdot \mid \mathcal{F}_s] \mid \mathcal{F}_t\big] = \mathbf{E}^{\downarrow}[\cdot \mid \mathcal{F}_t], \qquad s \ge t, \tag{5}$$

which is crucial to all of our results here.

In this section, we recall some basic properties of the lower Snell envelope and the solution of the robust stopping problem (3). Recall $\mathcal{Q}$ denotes a stable class of probability measures. We take a stochastic process H that satisfies the following assumption.

Assumption 2.1. *We assume that the process* H *is a càdlàg positive* $\mathbb{F}$*-adapted process which is of class*(D) *with respect to each* $Q \in \mathcal{Q}$*, that is,*

$$\lim_{x \to \infty} \sup_{\tau \in \mathcal{T}} E_Q[\mathsf{H}_\tau ; \mathsf{H}_\tau \ge x] = 0.$$

In particular

$$\sup_{\tau \in \mathcal{T}} E_Q[\mathsf{H}_\tau] < \infty. \tag{6}$$

The stochastic process H *is upper semicontinuous in expectation from the left with respect to each probability measure* $Q \in \mathcal{Q}$*. That is, for any stopping time* θ *of the* *filtration* $\mathbb{F}$ *and an increasing sequence of stopping times* $\{\theta_i\}_{i\in\mathbb{N}}$ *converging to* θ, *we have*

$$\limsup_{i\to\infty} E_Q[\mathsf{H}_{\theta_i}] \le E_Q[\mathsf{H}_\theta]. \tag{7}$$

For a stopping time $\rho \in \mathcal{T}$, we define $\mathcal{T}[\rho, \mathsf{T}] := \{\tau \in \mathcal{T} \mid \mathbb{P}(\tau \ge \rho) = 1\}$.

The *lower Snell envelope* is the “value process” of the robust stopping problem (3). Thus, it is a stochastic process such that for any $t \in [0, \mathsf{T}]$ is equal to

$$\operatorname*{ess\,sup}_{\tau\in\mathcal{T}[t,\mathsf{T}]} \mathbf{E}^{\downarrow}[\mathsf{H}_\tau \mid \mathcal{F}_t], \qquad \mathbb{P}\text{-a.s.} \tag{8}$$

We continue with a central concept.

Definition 2.3. An $\mathbb{F}$-adapted process $\{S_t\}_{0\le t\le \mathsf{T}}$ is a $\mathcal{Q}$-submartingale if for each $Q \in \mathcal{Q}$ and $t, s \in [0, \mathsf{T}]$ with $s \ge t$

1. $E_Q[|S_s|] < \infty$, for each $Q \in \mathcal{Q}$,
2. $E_Q[S_s \mid \mathcal{F}_t] \ge S_t$.

Now we can make more precise the notion of a uniform decomposition first mentioned in the introduction. A lower Snell envelope $U^{\downarrow}$ has a uniform decomposition if there exists a $\mathcal{Q}$-submartingale S and a nondecreasing process A such that $U^{\downarrow} = S - A$.

The next theorem consists of two parts. In the first, an optimal stopping time under model ambiguity is characterized and in the second, a result clarifying the “local” structure of the lower Snell envelope. Both parts are going to be fundamental in the following sections. For $Q \in \mathcal{Q}$, we denote by U^Q the Snell envelope of H with respect to Q. Recall that it is an $\mathbb{F}$-adapted Q-supermartingale dominating H with the property

$$U_t^Q = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}[t,\mathsf{T}]} E_Q[\mathsf{H}_\tau \mid \mathcal{F}_t], \qquad Q\text{-a.s., for } t \in [0, \mathsf{T}].$$

Theorem 2.1. *Take* $Q \in \mathcal{Q}$ *and let* U^Q *be the Snell envelope of* H *with respect to* Q. *For any stopping time* $\rho \in \mathcal{T}$ *let*

$$\tau_\rho^Q := \inf\{s \ge \rho \mid \mathsf{H}_s \ge U_s^Q\}. \tag{9}$$

Then, *the random time*

$$\tau_\rho^{\downarrow} := \operatorname*{ess\,inf}\{\tau_\rho^Q \mid Q \in \mathcal{Q}\}, \tag{10}$$

is a stopping time and under the conditions of Assumption 2.1 it is optimal in the following sense

$$\operatorname*{ess\,sup}_{\tau\in\mathcal{T}[\rho,\mathsf{T}]} \operatorname*{ess\,inf}_{Q\in\mathcal{Q}} E_Q[\mathsf{H}_\tau \mid \mathcal{F}_\rho] = \operatorname*{ess\,inf}_{Q\in\mathcal{Q}} E_Q[\mathsf{H}_{\tau_\rho^{\downarrow}} \mid \mathcal{F}_\rho]. \tag{11}$$

The lower Snell envelope is a $\mathcal{Q}$-*submartingale in stochastic intervals of the form* $[\rho, \tau_\rho^{\downarrow}]$.

See Trevino-Aguilar (2008), Theorem 5.6 for the proof.

An important consequence of Theorem 2.1 is the minimax identity in the next result. It is going to be used in Theorem 2.2 below.

Corollary 2.1. *Assume the conditions of Theorem 2.1 hold true. Then, for $\rho \in \mathcal{T}$ the following minimax identity holds true*

$$\operatorname*{ess\,sup}_{\tau \in \mathcal{T}[\rho,\mathsf{T}]} \operatorname*{ess\,inf}_{Q \in \mathcal{Q}} E_Q[\mathsf{H}_\tau \mid \mathcal{F}_\rho] = \operatorname*{ess\,inf}_{Q \in \mathcal{Q}} \operatorname*{ess\,sup}_{\tau \in \mathcal{T}[\rho,\mathsf{T}]} E_Q[\mathsf{H}_\tau \mid \mathcal{F}_\rho]. \tag{12}$$

Remark 2.2. The stability of the class $\mathcal{Q}$ is crucial in Corollary 2.1. It also allows to construct an optional right-continuous stochastic process $U^\downarrow := \{U_t^\downarrow\}_{0 \le t \le \mathsf{T}}$ such that for any stopping time $\tau \in \mathcal{T}$

$$U_\tau^\downarrow = \operatorname*{ess\,inf}_{Q \in \mathcal{Q}} \operatorname*{ess\,sup}_{\rho \in \mathcal{T}[\tau,\mathsf{T}]} E_Q[\mathsf{H}_\rho \mid \mathcal{F}_\tau], \qquad \mathbb{P}\text{-a.s.},$$

see Trevino-Aguilar (2011), Theorem 2.4. Thus, the stochastic process $U^\downarrow$ is a regular version of the lower Snell envelope.

Remark 2.3. Note that an $\mathbf{E}^\downarrow$-martingale is a Q-submartingale for each Q in the class $\mathcal{Q}$ which in combination with the structural result of Cheng and Riedel (2013) motivates our definition of uniform decomposition.

The proof of Theorem 5.6 in Trevino-Aguilar (2008) establishes that the lower Snell envelope is a $\mathbf{E}^\downarrow$-martingale in stochastic intervals of the form $[\rho, \tau_\rho^\downarrow]$. We are not making use of this concept here and instead focus on the $\mathcal{Q}$-submartingale property of the lower Snell envelope as stated in Theorem 2.1.

The following result will underline the general structure of lower Snell envelopes. It has been established for g-expectations by Cheng and Riedel (2013), Theorem 3.1. We extend their result to our setting.

Theorem 2.2. *Let H satisfy the conditions of Theorem 2.1. Then, its lower Snell envelope is an $\mathbf{E}^\downarrow$-supermartingale.*

Proof. For $t, s \in [0, \mathsf{T}]$ with $t \ge s$

$$\mathbf{E}^\downarrow[U_t^\downarrow \mid \mathcal{F}_s] = \mathbf{E}^\downarrow[\mathbf{E}^\downarrow[\mathsf{H}_{\tau_t^\downarrow} \mid \mathcal{F}_t] \mid \mathcal{F}_s] = \mathbf{E}^\downarrow[\mathsf{H}_{\tau_t^\downarrow} \mid \mathcal{F}_s] \le U_s^\downarrow.$$

The first identity holds true due to equation (11). In the second identity, we have used the stability property of the family $\mathcal{Q}$ in the form of the time consistency property (5) of the non linear expectation $\mathbf{E}^\downarrow$. In the last inequality, we have used the minimax identity (12). □

A simple corollary of Theorem 2.2 is the following.

Corollary 2.2. *Assume the conditions of Theorem 2.1 hold true. Then, the lower Snell envelope is the minimal $\mathbf{E}^{\downarrow}$-supermartingale dominating* H.

Proof. Let X be an $\mathbb{F}$-adapted $\mathbf{E}^{\downarrow}$-supermartingale dominating H. Then,

$$\begin{aligned} X_t &\geq \operatorname*{ess\,sup}_{\tau\in\mathcal{T}[t,\mathsf{T}]} \mathbf{E}^{\downarrow}[X_\tau \mid \mathcal{F}_t] \\ &\geq \operatorname*{ess\,sup}_{\tau\in\mathcal{T}[t,\mathsf{T}]} \mathbf{E}^{\downarrow}[\mathsf{H}_\tau \mid \mathcal{F}_t]. \end{aligned}$$

The term in the second inequality is precisely the lower Snell envelope as defined in equation (8) and it is an $\mathbf{E}^{\downarrow}$-supermartingale, due to Theorem 2.2. Thus, $X_t \geq U_t^{\downarrow}$. □

The next simple result will have interesting consequences, see Theorem 3.1 below.

Lemma 2.1. *The lower Snell envelope satisfies* $U^{\downarrow}_{\tau_t^{\downarrow}} = \mathsf{H}_{\tau_t^{\downarrow}}$, $\mathbb{P}$-*a.s.*

Proof. There exists a sequence of probability measures $\{Q_n\}_{n\in\mathbb{N}} \subset \mathcal{Q}$ such that the sequence of stopping times $\{\tau_t^{Q_n}\}_{n\in\mathbb{N}}$, with $\tau_t^{Q_n}$ as defined in (9), decreases to $\tau_t^{\downarrow}$; see Trevino-Aguilar (2008), Theorem 5.6, first part. In particular, the stopping time $\tau_t^{Q_n}$ clearly has the following property:

$$\mathsf{H}_{\tau_t^{Q_n}} = U^{Q_n}_{\tau_t^{Q_n}}.$$

Therefore,

$$\limsup_{n\to\infty} \mathsf{H}_{\tau_t^{Q_n}} = \limsup_{n\to\infty} U^{Q_n}_{\tau_t^{Q_n}} \geq \limsup_{n\to\infty} U^{\downarrow}_{\tau_t^{Q_n}}.$$

The limits lim sup exist, due to the right continuity of $U^{\downarrow}$ and H since the sequence of stopping times is non increasing. As a consequence, we obtain the inequality

$$\mathsf{H}_{\tau_t^{\downarrow}} \geq U^{\downarrow}_{\tau_t^{\downarrow}}.$$

The inequality in the other direction is a consequence to the definition of $U^{\downarrow}$. □

Remark 2.4. A natural question is if the stopping time $\tau^{\downarrow}$ is the first time that H touches its lower Snell envelope. An example in discrete time, due to Acciaio and Svindland (2014), suggests that in general, it is not.

3 The lower Snell envelope for a semimartingale process

There is a disjunctive for the lower Snell envelope formulated in terms of the family of stopping times $\{\tau_t^{\downarrow}\}_{0\le t\le \mathsf{T}}$. The Lemma 2.1 yields the set inclusion

$$\{t=\tau_t^{\downarrow}\}\subset\{U_t^{\downarrow}=\mathsf{H}_t\}. \tag{13}$$

In the stochastic set $\{\tau_t^{\downarrow}>t\}$, the lower Snell envelope is "locally" a $\mathcal{Q}$-submartingale; see Theorem 2.1, last part. In the complementary set $\{\tau_t^{\downarrow}=t\}$, the set inclusion (13) suggests that $U_t^{\downarrow}$ "looks like" the underlying process H. We will make this intuition more precise in this and the next section in such a way that we obtain "positive" answers to both questions in the introduction as well as examples in the "negative" direction.

We consider a general stable class of probability measures, but we impose conditions on the side of the underlying process H. In Theorem 3.1 below, we show that the lower Snell envelope of a semimartingale is again a semimartingale. In Theorem 3.2 below, we show that the lower Snell envelope has a uniform decomposition if H itself is of this form. Mainly as an illustration of the results, we show in Theorem 3.1, in the context of mathematical finance, that the payoff process of a put option satisfies the condition of Theorem 3.2 with respect to the class of martingale measures $\mathcal{M}$. Therefore, its lower Snell envelope has a uniform decomposition.

Assumption 3.1. *There exists a probability measure $Q\in\mathcal{Q}$ such that H is of the form*

$$\mathsf{H}_t=\mathsf{H}_0+S_t^Q+L_t^Q-N_t^Q,$$

for S^Q a Q-submartingale and L^Q, N^Q càdlàg non decreasing processes with $S_0^Q=L_0^Q=N_0^Q=0$, and $E_Q[N_\mathsf{T}^Q]<\infty$.

Theorem 3.1. *Suppose the Assumptions 2.1 and 3.1 holds true. Define $V^Q:=U^{\downarrow}+N^Q$. Let τ_1,τ_2 be two stopping times with $0\le\tau_1\le\tau_2\le\mathsf{T}$. Then*

$$E_Q[V_{\tau_1}^Q]\le E_Q[V_{\tau_2}^Q].$$

Thus, V^Q is a Q-submartingale.

Proof. For $\delta>0$, we set

$$\theta_\delta^{(1)}:=(\tau_1+\delta)\wedge\tau_2,$$
$$\theta_\delta^{(2)}:=\tau^{\downarrow}_{\theta_\delta^{(1)}}\wedge\tau_2.$$

Now for $i > 2$ we define recursively

$$\theta_\delta^{(i)} := \big(\theta_\delta^{(i-1)} + \delta\big) \wedge \tau_2, \qquad \text{for } i \text{ odd},$$
$$\theta_\delta^{(i+1)} := \tau^{\downarrow}_{\theta_\delta^{(i)}} \wedge \tau_2, \qquad \text{for } i+1 \text{ even}.$$

1. For $N > \frac{\mathsf{T}}{\delta}$ we have

$$\sum_{i=1}^{N} E_Q[V^Q_{\theta_\delta^{(i+1)}} - V^Q_{\theta_\delta^{(i)}}] = \sum_{i=1}^{\infty} E_Q[V^Q_{\theta_\delta^{(i+1)}} - V^Q_{\theta_\delta^{(i)}}].$$

The series is equal to

$$E_Q[V^Q_{\tau_2} - V^Q_{\theta_\delta^{(1)}}].$$

Moreover

$$\lim_{\delta \searrow 0} E_Q[V^Q_{\theta_\delta^{(1)}}] = E_Q[V^Q_{\tau_1}],$$

since V^Q is of $class(D)$ with respect to Q. As a consequence

$$\lim_{\delta \searrow 0} \sum_{i=1}^{\infty} E_Q[V^Q_{\theta_\delta^{(i+1)}} - V^Q_{\theta_\delta^{(i)}}] = E_Q[V^Q_{\tau_2} - V^Q_{\tau_1}]. \tag{14}$$

2. For $i+1$ even, we have

$$E_Q[V^Q_{\theta_\delta^{(i+1)}} - V^Q_{\theta_\delta^{(i)}}] \geq 0, \tag{15}$$

due to the $\mathcal{Q}$-submartingale property of Theorem 2.1.

3. For $i+2$ odd, we have

$$V^Q_{\theta_\delta^{(i+2)}} - V^Q_{\theta_\delta^{(i+1)}} = V^Q_{(\tau^{\downarrow}_{\theta_\delta^{(i)}} + \delta) \wedge \tau_2} - V^Q_{\tau^{\downarrow}_{\theta_\delta^{(i)}} \wedge \tau_2}.$$

We are going to show that

$$V^Q_{\theta_\delta^{(i+2)}} - V^Q_{\theta_\delta^{(i+1)}} \geq S^Q_{(\tau^{\downarrow}_{\theta_\delta^{(i)}} + \delta) \wedge \tau_2} - S^Q_{\tau^{\downarrow}_{\theta_\delta^{(i)}} \wedge \tau_2} + L^Q_{(\tau^{\downarrow}_{\theta_\delta^{(i)}} + \delta) \wedge \tau_2} - L^Q_{\tau^{\downarrow}_{\theta_\delta^{(i)}} \wedge \tau_2}. \tag{16}$$

By taking expectation with respect to Q in the inequality (16), we see

$$\begin{aligned} E_Q&[V^Q_{\theta_\delta^{(i+2)}} - V^Q_{\theta_\delta^{(i+1)}}] \\ &\geq E_Q[S^Q_{(\tau^{\downarrow}_{\theta_\delta^{(i)}} + \delta) \wedge \tau_2} - S^Q_{\tau^{\downarrow}_{\theta_\delta^{(i)}} \wedge \tau_2} + L^Q_{(\tau^{\downarrow}_{\theta_\delta^{(i)}} + \delta) \wedge \tau_2} - L^Q_{\tau^{\downarrow}_{\theta_\delta^{(i)}} \wedge \tau_2}] \qquad (17) \\ &\geq 0, \end{aligned}$$

since S^Q is a Q-submartingale and L is a non decreasing process.

We now prove (16). In the event $\{\tau^{\downarrow}_{\theta^{(i)}_\delta} \geq \tau_2\}$ both sides of the inequality (16) are equal to zero.

In the event $\{\tau^{\downarrow}_{\theta^{(i)}_\delta} \leq \tau_2\}$ we have

$$
\begin{aligned}
&V^Q_{\theta^{(i+2)}_\delta} - V^Q_{\theta^{(i+1)}_\delta} \\
&\quad = V^Q_{(\tau^{\downarrow}_{\theta^{(i)}_\delta}+\delta)\wedge\tau_2} - V^Q_{\tau^{\downarrow}_{\theta^{(i)}_\delta}} \\
&\quad = U^{\downarrow}_{(\tau^{\downarrow}_{\theta^{(i)}_\delta}+\delta)\wedge\tau_2} + N^Q_{(\tau^{\downarrow}_{\theta^{(i)}_\delta}+\delta)\wedge\tau_2} - U^{\downarrow}_{\tau^{\downarrow}_{\theta^{(i)}_\delta}} - N^Q_{\tau^{\downarrow}_{\theta^{(i)}_\delta}} \\
&\quad = U^{\downarrow}_{(\tau^{\downarrow}_{\theta^{(i)}_\delta}+\delta)\wedge\tau_2} + N^Q_{(\tau^{\downarrow}_{\theta^{(i)}_\delta}+\delta)\wedge\tau_2} - \mathsf{H}_{\tau^{\downarrow}_{\theta^{(i)}_\delta}} - N^Q_{\tau^{\downarrow}_{\theta^{(i)}_\delta}},
\end{aligned}
$$

where in the last identity we applied the equality $U^{\downarrow}_{\tau^{\downarrow}_{\theta^{(i)}_\delta}} = \mathsf{H}_{\tau^{\downarrow}_{\theta^{(i)}_\delta}}$, proved in Lemma 2.1. Moreover,

$$
\begin{aligned}
&U^{\downarrow}_{(\tau^{\downarrow}_{\theta^{(i)}_\delta}+\delta)\wedge\tau_2} + N^Q_{(\tau^{\downarrow}_{\theta^{(i)}_\delta}+\delta)\wedge\tau_2} - \mathsf{H}_{\tau^{\downarrow}_{\theta^{(i)}_\delta}} - N^Q_{\tau^{\downarrow}_{\theta^{(i)}_\delta}} \\
&\quad \geq \mathsf{H}_{(\tau^{\downarrow}_{\theta^{(i)}_\delta}+\delta)\wedge\tau_2} + N^Q_{(\tau^{\downarrow}_{\theta^{(i)}_\delta}+\delta)\wedge\tau_2} - \mathsf{H}_{\tau^{\downarrow}_{\theta^{(i)}_\delta}} - N^Q_{\tau^{\downarrow}_{\theta^{(i)}_\delta}} \\
&\quad = S^Q_{(\tau^{\downarrow}_{\theta^{(i)}_\delta}+\delta)\wedge\tau_2} - S^Q_{\tau^{\downarrow}_{\theta^{(i)}_\delta}} + L^Q_{(\tau^{\downarrow}_{\theta^{(i)}_\delta}+\delta)\wedge\tau_2} - L^Q_{\tau^{\downarrow}_{\theta^{(i)}_\delta}}.
\end{aligned}
$$

This proves the inequality (16).

4. As a consequence, we obtain

$$E_Q[V^Q_{\tau_2} - V^Q_{\tau_1}] \geq 0,$$

due to (14), (15), and (17). This establishes the theorem.

□

We now establish a variant of Theorem 3.1 under the stronger condition of the next assumption.

Assumption 3.2. *There exists a* Q*-submartingale* $\{S_t\}_{0\leq t\leq \mathsf{T}}$ *with* $S_0 = 0$ *such that* H *is of the form*

$$\mathsf{H}_t = \mathsf{H}_0 + S_t + L_t - N_t,$$

for L, N *càdlàg non decreasing processes with* $L_0 = N_0 = 0$*, and*

$$E_Q[N_{\mathsf{T}}] < \infty,$$

for each $Q \in \mathcal{Q}$.

The following result can be established with analogous arguments in the proof of Theorem 3.1 and we omit the details.

Theorem 3.2. *Assume* H *satisfies the Assumptions* 2.1 *and* 3.2. *Then* $V := U^{\downarrow} + N$ *is a* $\mathcal{Q}$*-submartingale*.

Proof. Analogous to the proof of Theorem 3.1. □

As an illustration of Theorem 3.2, we now prove that the lower Snell envelope of an American put option has a uniform decomposition. We consider the class of martingale measures $\mathcal{M}$ of a non-negative continuous semimartingale X.

We denote the local time of X at level K by $\{L_t^K\}_{0\le t\le \mathsf{T}}$.

Remark 3.1. Let X be a non-negative continuous semimartingale and assume that its class of local-martingale measures $\mathcal{M}$ is nonempty. Take $K > 0$ and assume $X_0 > 0$. Then, the payoff process of the American put option $\{(K - X_t)^+\}_{0\le t\le \mathsf{T}}$ satisfies the condition of Assumption 3.2.

Proof. We are going to make use of the put-call parity formula:

$$(K - x)^+ - (x - K)^+ = K - x.$$

The payoff process of the call option $\{(X_t - K)^+\}_{0\le t\le \mathsf{T}}$ can be represented as

$$\begin{aligned}(X_t - K)^+ &= (X_0 - K)^+ + \int_0^t 1_{\{X_s > K\}}\, dX_s + \frac{1}{2} L_t^K \\ &= (X_0 - K)^+ + X_t - X_0 - \int_0^t 1_{\{X_s \le K\}}\, dX_s + \frac{1}{2} L_t^K,\end{aligned}$$

due to Meyer–Itô's formula for convex functions; see, for example, Protter (2005), Theorem 70, page 218. Then

$$\int_0^t 1_{\{X_s \le K\}}\, dX_s \ge (X_0 - K)^+ - X_0 + \frac{1}{2} L_t^K,$$

and the integral in the left-hand side of the inequality is a supermartingale with respect to each $P \in \mathcal{M}$. We then have

$$E_P[L_{\mathsf{T}}^K] \le 2X_0 - 2(X_0 - K)^+.$$

We use the put-call parity formula to write the payoff of a put option in the form

$$(K - X_t)^+ = (X_0 - K)^+ + K - X_0 - \int_0^t 1_{\{X_s \le K\}}\, dX_s + \frac{1}{2} L_t^K.$$

In this representation, it is clear that it satisfies the Assumption 3.2. □

4 A warning example

In this section, we specify a stable family of equivalent probability measures $\mathbb{W}$ under which lower Snell envelopes

- may fail to be semimartingales and
- are supermartingales with respect to each element of $\mathbb{W}$ but do not have a uniform decomposition.

Let B be a Brownian motion defined in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ in this section we assume that $\mathbb{F}$ is the augmented filtration generated by B.

Definition 4.1. The class $\mathbb{W}$ consists of probability measures Q^{γ} determined by density processes satisfying the Dooleans-Dade stochastic equation

$$dZ_t = \gamma_t Z_t \, dB_t, \tag{18}$$

with γ a progressively measurable process satisfying

1. $\gamma_t \leq 0$ for $t \in [0, \mathsf{T}]$, and
2. $E[\int_0^{\mathsf{T}} \gamma_s^2 ds] < \infty$.

The class $\mathbb{W}$ corresponds to a pessimistic view of the future in which the Brownian motion B becomes a supermartingale. Indeed, the stochastic process

$$B_t^{\gamma} := B_t - \int_0^t \gamma_s \, ds, \tag{19}$$

is a Q^{γ} Brownian motion due to Girsanov's transformation theorem.

Remark 4.1. The κ-ambiguity model due to Chen and Epstein (2002) specifies a stable class of probability measures in the following way. One considers probability measures defined by density process described by the Dooleans-Dade exponential (18) with γ satisfying $|\gamma| \leq \kappa$ for κ a nonnegative constant. Our class $\mathbb{W}$ is a extreme case where γ is bounded from above but free from below.

Take a nonnegative continuous deterministic function V defined in the interval $[0, \mathsf{T}]$. In the next result, we show that the lower Snell envelope of the process $\mathcal{H} = B + V$ is equal to $\mathcal{H}$ itself. Note that for a function V of arbitrary p-variation, for $p > 2$, the lower Snell envelope of $\mathcal{H}$ may fail to be a semimartingale.

Proposition 4.1. *The lower Snell envelope of the process $\mathcal{H}$ is equal to $\mathcal{H}$ itself.*

Proof. Take $t \in [0, \mathsf{T})$. Let $\{r_n\}_{n \in \mathbb{N}}$ be a sequence of positive numbers decreasing to t. For $K, n \in \mathbb{N}$ let $\gamma^{n,K}$ be defined by

$$\gamma_s^{n,K} = \begin{cases} 0 & \text{if } s \in [0, t) \\ \frac{-K}{r_n - t} & \text{if } s \in [t, r_n) \\ 0 & \text{if } s \in [r_n, \mathsf{T}]. \end{cases}$$

Let $h^{n,K}$ be the exit time defined by

$$h^{n,K} := \sup\left\{s \in [t, r_n] \mid V_s + \int_t^s \gamma_s^{n,K} ds \geq 0\right\}.$$

For $K > \sup_{t\in[0,\mathsf{T}]} V_t$ sufficiently large, the exit time $h^{n,K}$ is smaller than r_n. It is clearly a deterministic time, and a stopping time.

For a stopping time σ with $\sigma \geq t$, the stopping time $\sigma_n := \sigma \wedge h^{n,K}$ is a stopping time with the property $\sigma_n \leq r_n$ and

$$V_{\sigma_n} + \int_t^{\sigma_n} \gamma_z^{n,K}\, dz \geq V_\sigma + \int_t^{\sigma} \gamma_z^{n,K}\, dz.$$

Let $Q^{\gamma^{n,K}}$ be the probability measure with Dooleans-Dade exponential $\gamma^{n,K}$. Let σ_n^* be the first optimal time of $V + \int_t^{\cdot} \gamma_z^{n,K} dz$ in the interval $[t, \mathsf{T}]$ with respect to Q^{γ^n}. Then, it is clear that σ_n^* is deterministic and dominated from above by r_n due to our previous argument involving the exit time $h^{n,K}$. As a consequence,

$$\begin{aligned}
U_t^{\downarrow} &= \operatorname*{ess\,inf}_{Q\in\mathbb{W}} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}[t,\mathsf{T}]} E_Q[\mathcal{H}_\tau \mid \mathcal{F}_t] \\
&\leq \operatorname*{ess\,sup}_{\tau\in\mathcal{T}[t,\mathsf{T}]} E_{Q^{\gamma^{n,K}}}[\mathcal{H}_\tau \mid \mathcal{F}_t] \\
&= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}[t,\mathsf{T}]} E_{Q^{\gamma^{n,K}}}\left[B_\tau^{\gamma^{n,K}} + \int_t^\tau \gamma_z^{n,K}\, dz + V_\tau \mid \mathcal{F}_t\right] \\
&= B_t^{\gamma^{n,K}} + \operatorname*{ess\,sup}_{\tau\in\mathcal{T}[t,\mathsf{T}]} E_{Q^{\gamma^{n,K}}}\left[\int_t^\tau \gamma_z^{n,K}\, dz + V_\tau \mid \mathcal{F}_t\right] \\
&= B_t + E_{Q^{\gamma^{n,K}}}\left[\int_t^{\sigma_n^*} \gamma_z^{n,K}\, dz + V_{\sigma_n^*} \mid \mathcal{F}_t\right] \\
&\leq B_t + V_{\sigma_n^*}.
\end{aligned}$$

The first inequality is clear. The second identity holds true due to the definition of $\mathcal{H}$ and equation (19) for $B^{\gamma^{n,K}}$. The third identity holds true due to the fact that $B^{\gamma^{n,K}}$ is a $Q^{\gamma^{n,K}}$-Brownian motion. The fourth equality is a consequence to the fact that $\gamma^{n,K}$ is zero in the interval $[0, t]$ and the optimality of σ_n^*. The last inequality holds true since the trend $\int_t^{\sigma_n^*} \gamma_z^{n,K}\, dz$ is negative and $V_{\sigma_n^*}$ is deterministic.

The stopping time σ_n^* has the property that it takes values in the interval $[t, r_n]$ and therefore, the difference $V_{\sigma_n^*} - V_t$ converges uniformly to zero. Then

$$U_t^{\downarrow} \leq \mathcal{H}_t + (V_{\sigma_n^*} - V_t) \to \mathcal{H}_t.$$

This proves that the lower Snell envelope of $\mathcal{H}$ is again $\mathcal{H}$. □

The lower Snell envelope of the Brownian motion B is B itself due to Proposition 4.1. The following result shows that there exists no uniform decomposition.

Proposition 4.2. *There exists no continuous uniform decomposition for the Brownian motion B with respect to $\mathbb{W}$.*

Proof. By way of contradiction, assume $B = Z + V$ where Z is a continuous $\mathbb{W}$-submartingale and V is a continuous process of locally-bounded variation. Let τ be a stopping time such that V is of bounded variation in the interval $[0, \tau]$. For $k > 0$ let $\tau_k := \inf\{s \geq 0 \mid Z_s \geq k\} \wedge \tau$. Take $Q^\gamma \in \mathbb{W}$. Recall our notation of the Q^γ-Brownian motion $B^\gamma = B_t - \int_0^t \gamma_s \, ds$. Then, we must have

$$Z_t - B_t^\gamma = \int_0^t \gamma_s \, ds - V_t.$$

We see that the stopped process $Z_{t\wedge\tau_k} - B^\gamma_{t\wedge\tau_k}$ is a Q^γ-submartingale of bounded variation. It is of *class*(D) and it has a Doob–Meyer decomposition in the interval $[0, \tau_k]$ as the sum $X^\gamma + A^\gamma$, with X^γ a continuous Q^γ-martingale and A^γ a continuous non decreasing process; see Ethier and Kurtz (1986), Corollary 5.2 p.78. Therefore, up to the stopping time τ_k, the process X^γ is a martingale of bounded variation. It is then, a purely discontinuous local martingale; see Yoeurp (1976), Théorème (1-6), p. 443. As a consequence, X^γ is constant, since it is continuous. We then see that the process V satisfies

$$V_{t\wedge\tau_k} = \int_0^{t\wedge\tau_k} \gamma_s \, ds - A^\gamma_{t\wedge\tau_k} - X^\gamma_0.$$

In particular, this implies that for any progressively measurable stochastic process ξ with $-1 \leq \xi \leq 0$

$$\int_{[0,\tau_k]} \xi_{s-} \, dV_s \geq \int_{[0,\tau_k]} \xi_{s-}\gamma_s \, ds.$$

This is impossible since the left hand side of the inequality is dominated from above by the total variation of V in the interval $[0, \tau_k]$, while the supremum over ξ and γ in the right-hand side is infinite. □

5 Identification of the uniform parts

In Theorem 3.2, we gave a condition under which the lower Snell envelope is the difference of a $\mathcal{Q}$-submartingale and a non decreasing process. In this section, we use the $\mathbf{E}^\downarrow$-supermartingale property of the lower Snell envelope, established in Theorem 2.2, in order to identify the nondecreasing part through a limiting procedure defined analogously to the proof of the Doob–Meyer decomposition. Here however, the operator $\mathbf{E}^\downarrow$ is nonlinear and we instead use the time consistency property (5).

We will consider convergence of stochastic processes in the sense of Fatou convergence as in the next definition.

Definition 5.1. Let D be a dense subset of $[0, \mathsf{T}]$. A sequence of processes $\{V^n\}_{n\in\mathbb{N}}$ is Fatou convergent on D to a process V if the sequence is bounded from below by an integrable variable, and if for any $t \in [0, \mathsf{T}]$ we have

$$V_t = \limsup_{s\downarrow t, s\in D} \limsup_{n\to\infty} V_s^n$$
$$= \liminf_{s\downarrow t, s\in D} \liminf_{n\to\infty} V_s^n.$$

This form of convergence has been considered previously by other authors. The main reason to consider this mode of convergence here is the compactness principle in Delbaen and Schachermayer (1999) when total variation is controlled.

Theorem 5.1. *Let U be a nonnegative $\mathbf{E}^{\downarrow}$-supermartingale of class(D) with respect to each element Q of $\mathcal{Q}$. Let Ξ be the class of finite partitions with points in D, a countable dense subset of $[0, \mathsf{T}]$. Let $DM(\Xi)$ be the class of random variables defined by*

$$V_{\mathsf{T}}^{\Pi} = \sum_{i=0}^{n-1} (U_{t_i} - \mathbf{E}^{\downarrow}[U_{t_{i+1}} \mid \mathcal{F}_{t_i}]), \tag{20}$$

for $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = \mathsf{T}\}$ a partition in Ξ.

If the class $DM(\Xi)$ is uniformly integrable with respect to a $Q_0 \in \mathcal{Q}$, then there exists a right-continuous nondecreasing process V such that $Z := U + V$ is a Q_0-submartingale. In this case, for each probability measure Q under which the class $DM(\Xi)$ is uniformly integrable, the process Z is also a submartingale.

Proof.

1. Given a partition $\Pi := \{0 = t_0 < t_1 < \cdots < t_n = \mathsf{T}\}$ we define the non decreasing predictable process V^{Π} by $V_0^{\Pi} = 0$ and for $t > 0$

$$V_t^{\Pi} := \sum_{i=0}^{n-1} (U_{t_i} - \mathbf{E}^{\downarrow}[U_{t_{i+1}} \mid \mathcal{F}_{t_i}]) 1_{[t_{i+1}, \mathsf{T}]}(t).$$

Note that

$$V_{t_{k+1}}^{\Pi} - V_{t_k}^{\Pi} = U_{t_k} - \mathbf{E}^{\downarrow}[U_{t_{k+1}} \mid \mathcal{F}_{t_k}]. \tag{21}$$

2. We are going to prove the equality

$$\mathbf{E}^{\downarrow}[U_{t_l} + V_{t_l}^{\Pi} \mid \mathcal{F}_{t_k}] = U_{t_k} + V_{t_k}^{\Pi}, \qquad \text{for } l \in \{k+1, \ldots, n\}, \tag{22}$$

in which the time consistency property (5) of the operator $\mathbf{E}^{\downarrow}$ is crucial.

For $l \in \{k+1, \ldots, n\}$, we define

$$f^l := U_{t_l} + V_{t_l}^{\Pi},$$

and recursively

$$f^j := \mathbf{E}^{\downarrow}[f^{j+1} \mid \mathcal{F}_{t_j}], \qquad \text{for } j = l-1, l-2, \ldots, k.$$

Time consistency and mathematical induction implies that

$$\mathbf{E}^{\downarrow}[f^l \mid \mathcal{F}_{t_j}] = f^j, \qquad \text{for } j = l-1, l-2, \ldots, k. \tag{23}$$

Let us now compute f^{l-1}

$$\begin{aligned} f^{l-1} &= \mathbf{E}^{\downarrow}[f^l \mid \mathcal{F}_{t_{l-1}}] \\ &= \mathbf{E}^{\downarrow}[U_{t_l} \mid \mathcal{F}_{t_{l-1}}] + V^{\Pi}_{t_l} \\ &= \mathbf{E}^{\downarrow}[U_{t_l} \mid \mathcal{F}_{t_{l-1}}] + V^{\Pi}_{t_{l-1}} + \left(U_{t_{l-1}} - \mathbf{E}^{\downarrow}[U_{t_l} \mid \mathcal{F}_{t_{l-1}}]\right) \\ &= V^{\Pi}_{t_{l-1}} + U_{t_{l-1}}. \end{aligned}$$

The second equality holds true since $V^{\Pi}_{t_l}$ is $\mathcal{F}_{t_{l-1}}$-measurable. By mathematical induction, we get

$$f^{j-1} = V^{\Pi}_{t_{j-1}} + U_{t_{j-1}}, \qquad \text{for } j = l-1, l-2, \ldots, k. \tag{24}$$

Equations (23) and (24) clearly imply the equality (22).

3. Take a sequence of partitions $\{\Pi_n\}_{n\in\mathbb{N}}$ increasingly exhausting the points on D and mesh converging to zero. There exists a right continuous process V and a sequence of convex combinations $\{V^n\}_{n\in\mathbb{N}}$ with

$$V^n = \sum_{j=n}^{\infty} \lambda^n_j V^{\Pi_j},$$

such that V^n Fatou-converges to V on D, due to Lemma 2.2 of Delbaen and Schachermayer (1999), since the sequence is uniformly bounded in variation with respect to a $Q_0 \in \mathcal{Q}$, due to our condition (20). The total variation of V is Q_0-integrable, again due to Lemma 2.2 of Delbaen and Schachermayer (1999). As a consequence, V_{T} is a Q_0-integrable random variable.

For $Q \in \mathcal{Q}$ under which the family $DM(\Xi)$ is uniformly integrable, we are going to prove that $U + V$ is a Q-submartingale, by making use of the equality (22). It will require several steps.

4. In this step, we are going to show that for $s, t \in \Pi_n$ with $s > t$

$$E_Q[U_s + V^n_s \mid \mathcal{F}_t] \geq U_t + V^n_t. \tag{25}$$

Let

$$W^{n,m}_s := \sum_{j=n}^{n+m} \lambda^n_j V^{\Pi_j}_s.$$

Note that $W_s^{n,m}$ converges in $L^1(Q)$ to V_s^n, due to monotone convergence. Hence, there exists a sequence m_k such that

$$E_Q[V_s^n \mid \mathcal{F}_t] = \lim_{k\to\infty} E_Q[W_s^{n,m_k} \mid \mathcal{F}_t], \qquad \mathbb{P}\text{-a.s.}$$

Now we show the inequality (25)

$$\begin{aligned}
&E_Q[U_s + V_s^n \mid \mathcal{F}_t] \\
&\quad = \lim_{k\to\infty} \{E_Q[U_s + W_s^{n,m_k} \mid \mathcal{F}_t]\} \\
&\quad = \lim_{k\to\infty} \left\{ E_Q\left[\sum_{j=n}^{n+m_k} \lambda_j^n (U_s + V_s^{\Pi_j}) \mid \mathcal{F}_t\right] + E_Q[U_s \mid \mathcal{F}_t] \sum_{j=n+m_k+1}^{\infty} \lambda_j^n \right\} \\
&\quad = \lim_{k\to\infty} E_Q\left[\sum_{j=n}^{n+m_k} \lambda_j^n (U_s + V_s^{\Pi_j}) \mid \mathcal{F}_t\right] \geq U_t + V_t^n.
\end{aligned}$$

The last inequality follows from the equality (22).

5. Take $s, t \in [0, \mathsf{T}]$ with $s > t$. Take a sequence $\{t_j\}_{j\in\mathbb{N}} \subset D \cap (t, s)$ decreasing to t. Let $\{s_j\}_{j\in\mathbb{N}} \subset D \cap (s, \mathsf{T})$ be a sequence decreasing to s. For $A \in \mathcal{F}_t$ we have

$$\begin{aligned}
E_Q[(U_{s_j} + V_{s_j})1_A] &= E_Q\Big[\Big(U_{s_j} + \limsup_{z\downarrow s_j, z\in D} \limsup_{n\to\infty} V_z^n\Big)1_A\Big] \\
&\geq \limsup_{z\downarrow s_j, z\in D} \limsup_{n\to\infty} E_Q[(U_z + V_z^n)1_A] \qquad (26) \\
&\geq \limsup_{n\to\infty} E_Q[(U_{t_j} + V_{t_j}^n)1_A]. \qquad (27)
\end{aligned}$$

The inequality (26) holds true due to our assumption of uniform integrability. The second inequality (27) follows by the inequality (25). As a consequence

$$\begin{aligned}
E_Q[(U_{s_j} + V_{s_j})1_A] &\geq \liminf_{t_j\downarrow t} \liminf_{n\to\infty} E_Q[(U_{t_j} + V_{t_j}^n)1_A] \\
&\geq E_Q[(U_t + V_t)1_A]. \qquad (28)
\end{aligned}$$

6. Now

$$\begin{aligned}
E_Q[(U_s + V_s)1_A] &= \lim_{j\to\infty} E_Q[(U_{s_j} + V_{s_j})1_A] \qquad (29) \\
&\geq E_Q[(U_t + V_t)1_A], \qquad (30)
\end{aligned}$$

where (29) follows by uniform integrability and right continuity. The inequality (30) follows by equation (28).

This proves that $U + V$ is a Q-submartingale. □

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Department of Economics and Finance
University of Guanajuato
DCEA—Campus Marfil
Fraccionamiento I, El Establo
Guanajuato GTO 36250
Mexico
E-mail: erick.trevino@ugto.org