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CENTRAL LIMIT THEOREM FOR RANDOM WALKS IN DOUBLY STOCHASTIC RANDOM ENVIRONMENT: $\mathscr{H}_{-1}$ SUFFICES

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We prove a central limit theorem under diffusive scaling for the displacement of a random walk on $\mathbb{Z}^d$ in stationary and ergodic doubly stochastic random environment, under the $\mathscr{H}_{-1}$-condition imposed on the drift field. The condition is equivalent to assuming that the stream tensor of the drift field be stationary and square integrable. This improves the best existing result [*Fluctuations in Markov Processes—Time Symmetry and Martingale Approximation* (2012) Springer], where it is assumed that the stream tensor is in $\mathscr{L}^{\max\{2+\delta,d\}}$, with $\delta > 0$. Our proof relies on an extension of the *relaxed sector condition* of [*Bull. Inst. Math. Acad. Sin. (N.S.)* **7** (2012) 463–476], and is technically rather simpler than existing earlier proofs of similar results by Oelschläger [*Ann. Probab.* **16** (1988) 1084–1126] and Komorowski, Landim and Olla [*Fluctuations in Markov Processes—Time Symmetry and Martingale Approximation* (2012) Springer].

1. Introduction: Setup and main result. Since its appearance in the probability and physics literature in the mid-seventies, the general topics of *random walks/diffusions in random environment* became the most complex and robust area of research. For a general overview of the subject and its historical development we refer the reader to the surveys Kozlov [14], Zeitouni [30], Biskup [4] or Kumagai [16], written at various stages of this rich story. The main problem considered in our paper is that of diffusive limit in the doubly stochastic (and hence, a priori stationary) case.

1.1. *The random walk and the $\mathscr{H}_{-1}$-condition.* Let $(\Omega, \mathscr{F}, \pi, \tau_z : z \in \mathbb{Z}^d)$ be a probability space with an ergodic $\mathbb{Z}^d$-action. Denote by $\mathscr{E}_+ := \{e_1, \dots, e_d : e_i \in \mathbb{Z}^d, e_i \cdot e_j = \delta_{i,j}\}$ the standard generating basis in $\mathbb{Z}^d$ and let $\mathscr{E} := \{\pm e_j : e_j \in \mathscr{E}_+\} = \{k \in \mathbb{Z}^d : |k| = 1\}$ be the set of possible steps of a nearest-neighbour walk on $\mathbb{Z}^d$. Assume that bounded measurable functions $p_k : \Omega \to [0, s^*]$, $k \in \mathscr{E}$ are

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given (s^* denotes the common bound), and assume the p_k satisfy *bistochasticity*, by which we mean the following property:

$$\sum_{k\in\mathscr{E}} p_k(\omega) = \sum_{k\in\mathscr{E}} p_{-k}(\tau_k\omega). \tag{1}$$

Lift these functions to the lattice $\mathbb{Z}^d$ by defining

$$P_k(x) = P_k(\omega, x) := p_k(\tau_x\omega). \tag{2}$$

Given these, define the continuous time nearest neighbour random walk $X(t)$ as a continuous time Markov chain on $\mathbb{Z}^d$, with $X(0) = 0$ and conditional jump rates

$$\mathbf{P}_\omega\big(X(t+dt) = x + k \mid X(t) = x\big) = P_k(\omega, x)\,dt + o(dt), \tag{3}$$

where the subscript ω denotes that the random walk $X(t)$ is a continuous time Markov chain on $\mathbb{Z}^d$ *conditionally*, with $\omega \in \Omega$ sampled according to π. Note that (1) is equivalent to

$$\sum_{k\in\mathscr{E}} P_k(\omega, x) = \sum_{k\in\mathscr{E}} P_{-k}(\omega, x + k),$$

which is exactly bistochasticity of the random walk defined in (3) above. Since the p_k are bounded, so will be the total jump rate of the walk

$$p(\omega) := \sum_{k\in\mathscr{E}} p_k(\omega) \le 2ds^*.$$

Thus, there is no difference between the long time asymptotics of this walk and the discrete time (possibly lazy) walk $n \mapsto X_n \in \mathbb{Z}^d$ with jump probabilities

$$\mathbf{P}_\omega(X_{n+1} = y \mid X_n = x) = \begin{cases} \big(2ds^*\big)^{-1} P_k(\omega, x) & \text{if } y - x = k \in \mathscr{E}, \\ 1 - \big(2ds^*\big)^{-1} \displaystyle\sum_{l\in\mathscr{E}} P_l(\omega, x) & \text{if } y - x = 0, \\ 0 & \text{if } y - x \notin \mathscr{E} \cup \{0\}. \end{cases}$$

We speak about continuous time walk only for reasons of convenience, in order to easily quote facts and results from Kipnis–Varadhan theory of CLT for additive functionals of Markov processes, without tedious reformulations.

We formulate our problem and prove our main result in the context of nearest neighbour walks. This is only for convenience reason. The main result of this paper holds true for finite range bistochastic RWREs under the appropriate conditions. For more details on this, see the remark after Theorem 1, further down in the paper.

We will use the notation $\mathbf{P}_\omega(\cdot)$, $\mathbf{E}_\omega(\cdot)$ and $\mathbf{Var}_\omega(\cdot)$ for *quenched* probability, expectation and variance. That is: probability, expectation, and variance with respect to the distribution of the random walk $X(t)$, *conditionally, with given fixed environment* ω. The notation $\mathbf{P}(\cdot) := \int_\Omega \mathbf{P}_\omega(\cdot)\,d\pi(\omega)$, $\mathbf{E}(\cdot) := \int_\Omega \mathbf{E}_\omega(\cdot)\,d\pi(\omega)$ and $\mathbf{Var}(\cdot) := \int_\Omega \mathbf{Var}_\omega(\cdot)\,d\pi(\omega) + \int_\Omega \mathbf{E}_\omega(\cdot)^2\,d\pi(\omega) - \mathbf{E}(\cdot)^2$ will be reserved for *annealed* probability, expectation and variance, that is, probability, expectation and

variance with respect to the random walk trajectory $X(t)$ *and* the environment ω, sampled according to the distribution π.

It is well known (and easy to check see; e.g., [14]) that due to double stochasticity (1) the annealed set-up is stationary and ergodic in time: the process of *the environment as seen from the position of the random walker* (to be formally defined soon) is stationary and ergodic in time under the probability measure π and consequently the random walk $t \mapsto X(t)$ will have stationary and ergodic annealed increments.

Next, we define, for $k \in \mathscr{E}$, $v_k : \Omega \to [-s^*, s^*]$, $s_k : \Omega \to [0, s^*]$, and $\psi, \varphi : \Omega \to \mathbb{R}^d$,

$$v_k(\omega) := \frac{p_k(\omega) - p_{-k}(\tau_k\omega)}{2}, \qquad \varphi(\omega) := \sum_{k\in\mathscr{E}} k v_k(\omega), \tag{4}$$

$$s_k(\omega) := \frac{p_k(\omega) + p_{-k}(\tau_k\omega)}{2}, \qquad \psi(\omega) := \sum_{k\in\mathscr{E}} k s_k(\omega). \tag{5}$$

Their corresponding lifting to $\mathbb{Z}^d$ are

$$\begin{aligned} V_k(x) = V_k(\omega, x) &:= v_k(\tau_x\omega), \qquad \Phi(x) = \Phi(\omega, x) := \varphi(\tau_x\omega), \\ S_k(x) = S_k(\omega, x) &:= s_k(\tau_x\omega), \qquad \Psi(x) = \Psi(\omega, x) := \psi(\tau_x\omega). \end{aligned} \tag{6}$$

Note that

$$\begin{aligned} &-s^* \le v_k(\omega) \le s^*, \qquad 0 \le s_k(\omega) \le s^*, \\ &\bigl|\varphi(\omega)\bigr| \le 2\sqrt{d}s^*, \qquad \bigl|\psi(\omega)\bigr| \le \sqrt{d}s^*, \qquad \text{a.s.} \end{aligned}$$

The local *quenched* drift of the random walk is

$$\mathbf{E}_\omega\bigl(dX(t) \mid X(s) : 0 \le s \le t\bigr) = \bigl(\Psi\bigl(\omega, X(t)\bigr) + \Phi\bigl(\omega, X(t)\bigr)\bigr)\,dt + o(dt).$$

Note that from (1) and the definitions (4), (5) it follows that for π-almost all $\omega \in \Omega$:

$$v_k(\omega) = -v_{-k}(\tau_k\omega), \qquad \sum_{k\in\mathscr{E}} v_k(\omega) = 0, \tag{7}$$

$$s_k(\omega) = s_{-k}(\tau_k\omega), \qquad \sum_{k\in\mathscr{E}} s_k(\omega) =: s(\omega). \tag{8}$$

Equation (7) means that $V_k : \mathbb{Z}^d \to [-s^*, s^*]$ is π-almost surely a bounded and *sourceless flow* on $\mathbb{Z}^d$, or equivalently, $\Phi : \mathbb{Z}^d \to \mathbb{R}^d$ is a bounded *divergence-free* vector field on $\mathbb{Z}^d$. On the other hand, (8) implies that

$$\begin{aligned} \psi_i(\omega) &= s_{e_i}(\omega) - s_{e_i}(\tau_{-e_i}\omega), \\ \Psi_i(\omega, x) &= S_{e_i}(\omega, x) - S_{e_i}(\omega, x - e_i). \end{aligned} \tag{9}$$

That is, the vector field $\Psi : \mathbb{Z}^d \to \mathbb{R}^d$ is component-wise a directional derivative. It follows in particular that

$$\mathbf{E}(\Psi) = 0. \tag{10}$$

We assume that a similar condition holds for the drift field Φ, too:

$$\mathbf{E}(\Phi) = \sum_{k \in \mathscr{E}} k \int_\Omega v_k(\omega)\, d\pi(\omega) = 0, \tag{11}$$

which due to (7), in the nearest neighbour set-up, is obviously the same as assuming that for $k \in \mathscr{E}$

$$\int_\Omega v_k(\omega)\, d\pi(\omega) = 0. \tag{12}$$

From (10) and (11), it follows that in the *annealed* mean drift of the walk is nil:

$$\mathbf{E}\big(X(t)\big) = \int_\Omega \mathbf{E}_\omega\big(X(t)\big)\, d\pi(\omega) = 0.$$

Under these conditions, the law of large numbers

$$\lim_{t\to\infty} t^{-1} X(t) = 0, \qquad \text{a.s.} \tag{13}$$

follows directly from the ergodic theorem.

Our next assumption is an ellipticity condition for the symmetric part of the jump rates: there exists another constant $s_* \in (0, s^*]$ such that for π-almost all $\omega \in \Omega$ and all $k \in \mathscr{E}$

$$s_k(\omega) \geq s_*, \qquad \pi\text{-a.s.} \tag{14}$$

Note that no ellipticity condition is imposed on the jump probabilities $(p_k)_{k\in\mathscr{E}}$: it may happen that $p_k = 0$ with positive π-probability. Using a time change, we may assume $s_* = 1$, and we will occasionally make this assumption for simplicity.

Regarding fluctuations around the law of large numbers (13), we will soon prove that under the ellipticity condition (14) a *diffusive lower bound* holds: for any fixed vector $v \in \mathbb{R}^d$

$$\varliminf_{t\to\infty} t^{-1} \mathbf{E}\big(\big(v \cdot X(t)\big)^2\big) > 0. \tag{15}$$

Explicit lower bound will be provided in Proposition 1 below.

A *diffusive upper bound* also holds under a subtle condition on the covariances of the drift field $\Phi : \mathbb{Z}^d \to \mathbb{R}^d$. Denote

$$\begin{aligned} C_{ij}(x) &:= \mathbf{Cov}\big(\Phi_i(0), \Phi_j(x)\big) = \int_\Omega \varphi_i(\omega)\varphi_j(\tau_x\omega)\, d\pi(\omega), \qquad x \in \mathbb{Z}^d, \\ D_{ij}(x) &:= \mathbf{Cov}\big(\Psi_i(0), \Psi_j(x)\big) = \int_\Omega \psi_i(\omega)\psi_j(\tau_x\omega)\, d\pi(\omega), \qquad x \in \mathbb{Z}^d, \end{aligned} \tag{16}$$

$$\begin{aligned}\widehat{C}_{ij}(p) &:= \sum_{x\in\mathbb{Z}^d} e^{\sqrt{-1}x\cdot p} C_{ij}(x), \qquad p\in[-\pi,\pi)^d,\\ \widehat{D}_{ij}(p) &:= \sum_{x\in\mathbb{Z}^d} e^{\sqrt{-1}x\cdot p} D_{ij}(x), \qquad p\in[-\pi,\pi)^d.\end{aligned} \tag{17}$$

The Fourier transform is meant as a distribution on $[-\pi,\pi)^d$. More precisely, by Herglotz's theorem, $\widehat{C}$ and $\widehat{D}$ are positive definite $d\times d$ matrix-valued *measures* on $[-\pi,\pi)^d$. Hence, (12) is equivalent to $\widehat{C}_{ij}(\{0\})=0$, for all $i,j=1,\dots,d$.

The fact that Ψ is a spatial derivative of an L^2 function (9) implies that

$$\int_{[-\pi,\pi)^d}\left(\sum_{j=1}^d(1-\cos p_j)\right)^{-1}\sum_{i=1}^d \widehat{D}_{ii}(p)\,dp<\infty. \tag{18}$$

A similar *infrared bound* imposed on the covariances of the field $x\mapsto\Phi(x)$ is the notorious $\mathscr{H}_{-1}$*-condition* referred to in the title of this paper.

$\mathscr{H}_{-1}$*-condition* (first formulation): We assume

$$\int_{[-\pi,\pi)^d}\left(\sum_{j=1}^d(1-\cos p_j)\right)^{-1}\sum_{i=1}^d \widehat{C}_{ii}(p)\,dp<\infty. \tag{19}$$

For later use, we define the positive definite and bounded $d\times d$ matrices

$$\widetilde{C}_{ij} := \int_{[-\pi,\pi)^d}\left(\sum_{j=1}^d(1-\cos p_j)\right)^{-1}\widehat{C}_{ij}(p)\,dp<\infty, \tag{20}$$

$$\widetilde{D}_{ij} := \int_{[-\pi,\pi)^d}\left(\sum_{j=1}^d(1-\cos p_j)\right)^{-1}\widehat{D}_{ij}(p)\,dp<\infty. \tag{21}$$

The probabilistic content of the infrared bounds (18) and (19) is the following. Let $t\mapsto S(t)$ be a continuous time simple symmetric random walk on $\mathbb{Z}^d$ with jump rate 1, fully independent of the random fields $x\mapsto(\Phi(x),\Psi(x))$. Then (18) and (19) are in turn equivalent to

$$\lim_{T\to\infty} T^{-1}\mathbf{E}\left(\left|\int_0^T \Psi\big(S(t)\big)\,dt\right|^2\right)<\infty,$$

and

$\mathscr{H}_{-1}$*-condition* (second formulation):

$$\lim_{T\to\infty} T^{-1}\mathbf{E}\left(\left|\int_0^T \Phi\big(S(t)\big)\,dt\right|^2\right)<\infty. \tag{22}$$

The expectations in the last two expressions are taken over the random walk $t\mapsto S(t)$ *and* the random scenery $x\mapsto(\Phi(x),\Psi(x))$. We omit the straightforward proof of these equivalences. Two more equivalent formulations of the $\mathscr{H}_{-1}$-condition (19)/(22) will appear later in the paper.

The infrared bounds (18) and (19) imply a diffusive upper bound: for any fixed vector $v \in \mathbb{R}^d$

$$\overline{\lim_{t\to\infty}}\, t^{-1}\mathbf{E}\big((v\cdot X(t))^2\big) < \infty. \tag{23}$$

An explicit upper bound will be provided in Proposition 1 below.

Now, (15) and (23) jointly *suggest* that the central limit theorem

$$t^{-1/2}X(t) \quad \Rightarrow \quad \mathcal{N}\big(0,\sigma^2\big) \tag{24}$$

should hold with some nondegenerate $d\times d$ covariance matrix σ^2. Attempts to prove the CLT (24) under the minimal conditions of bistochasticity (1), ellipticity (14), no drift (12) and $\mathscr{H}_{-1}$ (19) have a notorious history. In Kozlov [14], a similar CLT is announced under the somewhat restrictive condition that the random field $x\mapsto P(x)$ in (2) be finitely dependent. However, as pointed out in Komorowski and Olla [12] the proof in [14] is incomplete. In the same paper [12], the CLT (24) is stated, but as pointed out in [10] this proof is yet again defective. Finally, in [10] a complete proof is given, however, with more restrictive conditions: instead of the $\mathscr{H}_{-1}$-condition (19) a rather stronger integrability condition on the field $x\mapsto\Phi(x)$ is assumed. See the comments in Section 6. More detailed historical comments on this story can be found in the notes after Chapter 3 of [10]. Our main result in the present paper is a complete proof of the CLT (24), under the conditions listed above.

1.2. *Central limit theorem for the random walk.* We define the *environment process*, as seen from the random walker:

$$\eta(t) := \tau_{X(t)}\omega$$

This is a pure jump process on Ω with bounded total jump rates. So, its construction does not pose any technical difficulty. As already mentioned, it is well known (and easy to check, see; e.g., Kozlov [14]) that due to condition (1) the probability measure π is stationary and ergodic for the Markov process $t\mapsto\eta(t)$. We will denote by $(\mathscr{F}_t)_{t\geq 0}$ the filtration generated by this process:

$$\mathscr{F}_t := \sigma\big(\eta(s) : 0\leq s\leq t\big).$$

It is most natural to decompose $X(t)$ as

$$\begin{aligned} X(t) &= \left\{X(t) - \int_0^t \big(\psi\big(\eta(s)\big)+\varphi\big(\eta(s)\big)\big)\,ds\right\} + \int_0^t \big(\psi\big(\eta(s)\big)+\varphi\big(\eta(s)\big)\big)\,ds \\ &=: M(t)+I(t). \end{aligned} \tag{25}$$

In this decomposition, the first term is clearly a square integrable $(\mathscr{F}_t)$-martingale with stationary and ergodic increments and conditional covariances (or, quadratic variation)

$$\mathbf{E}\big(dM_i(t)dM_j(t)\mid\mathscr{F}_t\big) = \delta_{i,j}\big(p_{e_i}\big(\eta(t)\big)+p_{-e_i}\big(\eta(t)\big)\big)\,dt. \tag{26}$$

Thus, due to the martingale CLT (see, e.g., [7])

$$t^{-1/2}M(t) \quad \Rightarrow \quad \mathcal{N}\big(0,\sigma_M^2\big),$$

where

$$\big(\sigma_M^2\big)_{ij} = 2\delta_{i,j}\int_\Omega s_{e_i}(\omega)\,d\pi(\omega). \tag{27}$$

The difficulty is caused by the compensator integral term $I(t)$.

The following proposition quantifies assertions (15) and (23).

PROPOSITION 1. *Let $t \mapsto X(t)$ be a random walk in doubly stochastic (1) random environment with no drift (12). Then the ellipticity (14) and $\mathscr{H}_{-1}$ (19) conditions imply the following diffusive lower and upper bounds*: *For any vector $v \in \mathbb{R}^d$,*

$$\begin{aligned} 2s_*|v|^2 &\le \varlimsup_{t\to\infty} t^{-1}\mathbf{E}\big(\big(v\cdot X(t)\big)^2\big) \\ &\le 6s^*|v|^2 + \frac{24}{s_*}\sum_{i,j=1}^d (\widetilde{C}_{ij}+\widetilde{D}_{ij})v_i v_j, \end{aligned} \tag{28}$$

where $\widetilde{C}_{ij}$ and $\widetilde{D}_{ij}$ are the matrices defined in (20) and (21).

The proof of Proposition 1 is postponed to the next section. Note that the ellipticity condition (14) is relevant in both (lower and upper) bounds, while the $\mathscr{H}_{-1}$-condition (19) is relevant for the upper bound only.

Let us formally state the main result of the present paper.

THEOREM 1. *Let $t \mapsto X(t)$ be a nearest neighbour random walk (3) in random environment, which is bistochastic (1), has no drift (12) and is elliptic (14). If in addition the $\mathscr{H}_{-1}$-condition (19) holds, then*:

(i) *The asymptotic covariance matrix*

$$\big(\sigma^2\big)_{ij} := \lim_{t\to\infty} t^{-1}\mathbf{E}\big(X_i(t)X_j(t)\big)$$

exists, and it is finite and nondegenerate

$$2s_*I \le \sigma^2 \le 6s^*I_d + 24s_*^{-1}(\widetilde{C}+\widetilde{D}), \tag{29}$$

where I is the $d\times d$ unit matrix and $\widetilde{C}$, $\widetilde{D}$ are the matrices defined in (20), (21).

(ii) *Moreover, for any $m\in\mathbb{N}$, $t_1,\dots,t_m\in\mathbb{R}_+$ and any continuous and bounded test function $F:\mathbb{R}^{md}\to\mathbb{R}$*

$$\lim_{T\to\infty}\int_\Omega \left|\mathbf{E}_\omega\left(F\left(\frac{X(Tt_1)}{\sqrt{T}},\dots,\frac{X(Tt_m)}{\sqrt{T}}\right)\right)\right.$$
$$\left. -\,\mathbf{E}\big(F\big(W(t_1),\dots,W(t_m)\big)\big)\right| d\pi(\omega) = 0,$$

where $t \mapsto W(t) \in \mathbb{R}^d$ *is a Brownian motion with*

$$\mathbf{E}\big(W_i(t)\big) = 0, \qquad \mathbf{E}\big(W_i(s)W_j(t)\big) = \min\{s,t\}\big(\sigma^2\big)_{ij}.$$

REMARK (Remark on the jump range of the walk). Throughout the paper, we speak about nearest neighbour random walk with jump range $\mathscr{E}$. However, we could consider a more general setup, with jump range $\mathscr{U} \subset \mathbb{Z}^d$, with the assumptions that (i) $|\mathscr{U}| < \infty$; (ii) the jump rates are bounded: $p_k(\omega) \le s^*$ almost surely for $k \in \mathscr{U}$; (iii) the ellipticity condition (14) holds for a subset $\mathscr{U}' \subset \mathscr{U}$ which generates $\mathbb{Z}^d$. Under these more general assumptions, Theorem 1 remains still valid. The proof remains essentially the same apart of notational changes.

It is worth noting here that (unlike in the self-adjoint/reversible cases) the $\mathscr{H}_{-1}$-condition is certainly stronger than assuming just finiteness of the asymptotic variance of the walk, (23). So $\mathscr{H}_{-1}$ seems to be a sufficient but by no means necessary condition for the CLT to hold. The following question arises very naturally.

QUESTION. *Let X be a stationary, ergodic random walk in a bistochastic random environment, and assume $\mathbf{E}(|X(t)|^2) \le Ct$. Does it follow that X satisfies a central limit theorem?*

Structure of the paper. The proof of this theorem is the content of Sections 2–4. Section 2 contains Hilbert space generalities and most of the notation. Section 3 describes and slightly extends the relaxed sector condition of [8] on which we rely. Proofs of the extensions are given in an Appendix (the proofs are similar to those of [8], but the statements are stronger). In Section 4, we check the conditions of the relaxed sector condition for the concrete case. Remarks, comments (historical and other) and concrete examples are postponed to Sections 5–7.

Let us remark that assuming that s_k is constant for all $k \in \mathscr{E}$, in other words that the walk is *divergence-free*, removes a number of technical difficulties in the proof. Readers who prefer to see the easier version can see it in the first arxiv version of this paper [15].

2. In the Hilbert space $\mathscr{L}^2(\Omega, \pi)$.

2.1. *Spaces and operators*. It is most natural to put ourselves into the Hilbert space over $\mathbb{C}$:

$$\mathscr{H} := \left\{ f \in \mathscr{L}^2(\Omega, \pi) : \int_\Omega f \, d\pi = 0 \right\}.$$

We denote by T_x, $x \in \mathbb{Z}^d$, the spatial shift operators

$$T_x f(\omega) := f(\tau_x \omega),$$

and note that they are unitary:

$$T_x^* = T_{-x} = T_x^{-1}. \tag{30}$$

The $\mathscr{L}^2$-gradients ∇_k, $k \in \mathscr{E}$, respectively, $\mathscr{L}^2$-Laplacian Δ, are

$$\begin{aligned} &\nabla_k := T_k - I, \qquad \nabla_k^* = \nabla_{-k}, \qquad \|\nabla_k\| \le 2, \\ &\Delta := \sum_{l\in\mathscr{E}} \nabla_l = -\frac{1}{2}\sum_{l\in\mathscr{E}} \nabla_l \nabla_{-l}, \qquad \Delta^* = \Delta \le 0, \qquad \|\Delta\| \le 4d. \end{aligned} \tag{31}$$

We remark that the norm inequalities above are in fact equalities in any nondegenerate case, but we will not need this fact.

Due to ergodicity of the $\mathbb{Z}^d$-action $(\Omega, \mathscr{F}, \pi, \tau_z : z \in \mathbb{Z}^d)$,

$$\operatorname{Ker}(\Delta) = \{0\}. \tag{32}$$

Indeed, $\Delta f = 0$ implies that $0 = \langle f, \Delta f\rangle = -\frac{1}{2}\sum_{k\in\mathscr{E}}\langle \nabla_k f, \nabla_k f\rangle$ and since all terms are non-negative, they must all be 0 and f must be invariant to translations. Ergodicity to $\mathbb{Z}^d$ actions means that f is constant, and since our Hilbert space is that of functions averaging to zero, f must be zero.

We define the bounded positive operator $|\Delta|^{1/2}$ in terms of the spectral theorem (applied to the bounded positive operator $|\Delta| := -\Delta$). Note that due to (32) $\operatorname{Ran}|\Delta|$ is dense in $\mathscr{H}$, and hence so is $\operatorname{Ran}|\Delta|^{1/2}$ which is a superset of it. Hence, it follows that $|\Delta|^{-1/2} := (|\Delta|^{1/2})^{-1}$ is an (unbounded) positive self-adjoint operator with $\operatorname{Dom}|\Delta|^{-1/2} = \operatorname{Ran}|\Delta|^{1/2}$ and $\operatorname{Ran}|\Delta|^{-1/2} = \operatorname{Dom}|\Delta|^{1/2} = \mathscr{H}$. Note that the dense subspace $\operatorname{Dom}|\Delta|^{-1/2} = \operatorname{Ran}|\Delta|^{1/2}$ is invariant under, and the operators $|\Delta|^{1/2}$ and $|\Delta|^{-1/2}$ commute with the translations T_x, $x \in \mathbb{Z}^d$.

We define the *Riesz operators*: for all $k \in \mathscr{E}$,

$$\Gamma_k : \operatorname{Dom}|\Delta|^{-1/2} \to \mathscr{H}, \qquad \Gamma_k = |\Delta|^{-1/2}\nabla_k = \nabla_k|\Delta|^{-1/2}, \tag{33}$$

and note that for any $f \in \operatorname{Dom}|\Delta|^{-1/2}$

$$\|\Gamma_k f\|^2 = \langle |\Delta|^{-1/2} f, \nabla_{-k}\nabla_k |\Delta|^{-1/2} f\rangle \le \langle |\Delta|^{-1/2} f, |\Delta||\Delta|^{-1/2} f\rangle = \|f\|^2.$$

Thus, the operators Γ_k, $k \in \mathscr{E}$, extend as bounded operators to the whole space $\mathscr{H}$. The following properties are easy to check:

$$\Gamma_k^* = \Gamma_{-k}, \qquad \|\Gamma_k\| \le 1, \qquad \frac{1}{2}\sum_{l\in\mathscr{E}} \Gamma_l \Gamma_l^* = I. \tag{34}$$

As before, in fact $\|\Gamma_k\| = 1$ in any nondegenerate case, but we will not need this fact.

A third equivalent formulation of the $\mathscr{H}_{-1}$-condition (19)/(22) is the following:

$\mathscr{H}_{-1}$-condition (third formulation):

$$\varphi_i \in \operatorname{Dom}|\Delta|^{-1/2} = \operatorname{Ran}|\Delta|^{1/2}, \qquad i = 1, \dots, d. \tag{35}$$

In the case of nearest neighbour walks, this is further equivalent to

$$v_k \in \operatorname{Dom}|\Delta|^{-1/2} = \operatorname{Ran}|\Delta|^{1/2}, \qquad k \in \mathscr{E}. \tag{36}$$

LEMMA 1. (i) *Conditions* (35) *and* (22) *are equivalent*.

(ii) *Furthermore, in the case of nearest neighbour walks conditions* (35) *and* (36) *are also equivalent*.

PROOF. (i) Recall that (22) is formulated in terms of continuous time simple random walk S. In operator theory language,

$$\mathbf{E}_\omega\big(\Phi_i\big(S(t)\big)\big) = e^{t\Delta}\varphi_i(\omega). \tag{37}$$

Hence,

$$\frac{1}{t}\mathbf{E}\left(\left|\int_0^t \Phi\big(S(s)\big)\,ds\right|^2\right) \overset{(*)}{=} \sum_{i=1}^d \int_0^t \frac{t-s}{t}\mathbf{E}\big(2\Phi_i(0)\Phi_i\big(S(s)\big)\big)\,ds$$
$$\overset{(37)}{=} \sum_{i=1}^d 2\int_0^t \frac{t-s}{t}\langle\varphi_i, e^{s\Delta}\varphi_i\rangle\,ds,$$

where $(*)$ follows from space stationarity of Φ [recall that S is independent of Φ, so $\Phi_i(S)$ is just some average of some fixed translations of Φ_i]. An application of the spectral theorem for $|\Delta|$ shows that this is bounded in t if and only if all $\varphi_i \in \operatorname{Dom}|\Delta|^{-1/2}$, $i = 1, \dots, d$.

(ii) To conclude from $\varphi \in \operatorname{Dom}|\Delta|^{-1/2}$ that $v \in \operatorname{Dom}|\Delta|^{-1/2}$ we recall that $\varphi_i = (I + T_{-e_i})v_{e_i} = (2I + \nabla_{-e_i})v_{e_i}$. Since $\Gamma_{-e_i} = |\Delta|^{-1/2}\nabla_{-e_i}$ is bounded, we get that $\nabla_{-e_i}v_{e_i} \in \operatorname{Dom}(|\Delta|^{-1/2})$. Rearranging gives

$$\varphi_i - 2v_{e_i} \in \operatorname{Dom}\big(|\Delta|^{-1/2}\big)$$

which shows that $\varphi_i \in \operatorname{Dom}(|\Delta|^{-1/2})$ if and only if so is v_{e_i}. □

REMARK. Note that equivalence of (35) and (36) holds *only* in the case of nearest neighbour jumps. If a larger jump range $\mathscr{U}$ is allowed (see the remark after the formulation of Theorem 1) then (36) is stronger than (35). However, the formulation (36) will not be used in the proof of our main result. It will have a role only in the complementary Section 5, which is not part of the proof. That part could also be reformulated in the context of finite jump rate, relying only on (35) but as the main result does not rely on it we will not bother to do that.

Finally, we also define the multiplication operators M_k, N_k, $k \in \mathscr{E}$,

$$M_k f(\omega) := v_k(\omega) f(\omega), \qquad M_k^* = M_k, \qquad \|M_k\| \le s^*, \tag{38}$$

$$N_k f(\omega) := \big(s_k(\omega) - s_*\big) f(\omega), \qquad N_k^* = N_k \ge 0, \qquad \|N_k\| \le s^* \tag{39}$$

[recall that s^* is the overall *upper bound* on p and s_* is the *lower bound* on the symmetric parts s in the ellipticity condition (14)]. It is easy to check that the

following commutation relations hold due to (7) and (8):

$$\begin{aligned}\sum_{l\in\mathscr{E}} M_l\nabla_l &= -\sum_{l\in\mathscr{E}} \nabla_{-l}M_l,\\ \sum_{l\in\mathscr{E}} N_l\nabla_l &= \sum_{l\in\mathscr{E}} \nabla_{-l}N_l = -\frac{1}{2}\sum_{l\in\mathscr{E}} \nabla_{-l}N_l\nabla_l.\end{aligned} \tag{40}$$

The *infinitesimal generator* of the stationary environment process $t \mapsto \eta(t)$, acting on the Hilbert space $\mathscr{L}^2(\Omega,\pi)$ is

$$Lf(\omega) = p_k(\omega)\big(f(\tau_k\omega) - f(\omega)\big),$$

which in terms of the operators introduced above is written as

$$L = -D - T + A, \tag{41}$$

with

$$D := -s_*\Delta, \tag{42}$$

$$T := -\sum_{l\in\mathscr{E}} N_l\nabla_l = \frac{1}{2}\sum_{l\in\mathscr{E}} \nabla_{-l}N_l\nabla_l,$$

$$A := \sum_{l\in\mathscr{E}} M_l\nabla_l = -\sum_{l\in\mathscr{E}} \nabla_{-l}M_l. \tag{43}$$

Note that $D = D^*$, $T = T^*$, $A = -A^*$ and

$$0 \le T \le ds^*s_*^{-1}D. \tag{44}$$

The inequalities are meant in operator sense. The last one follows from

$$D^{-1/2}TD^{-1/2} = \frac{1}{2s_*}\sum_{l\in\mathscr{E}} \Gamma_{-l}N_l\Gamma_l,$$

and hence, due to (34) and (39)

$$\big\|D^{-1/2}TD^{-1/2}\big\| \le \frac{ds^*}{s_*}$$

follows, which implies the upper bound in (44).

2.2. *Proof of Proposition* 1.

PROOF OF THE LOWER BOUND IN (28). We decompose the displacement process $t \to X(t)$ in such a way that the forward-and-backward martingale part will be uncorrelated with the rest. The variance of this forward-and-backward martingale will serve as lower bound for the variance of the displacement. Let

$$u_k(\omega) := \operatorname{sgn}\big(v_k(\omega)\big)\min\big\{\big|v_k(\omega)\big|, s_*\big\}, \qquad w_k(\omega) := \operatorname{sgn}\big(v_k(\omega)\big)\big(\big|v_k(\omega)\big| - s_*\big)_+,$$

$$q_k(\omega) := s_* + u_k(\omega), \qquad r_k(\omega) := \big(s_k(\omega) - s_*\big) + w_k(\omega).$$

Note that the skew symmetry (7) of $v_k(\omega)$ is inherited by $u_k(\omega)$ and $w_k(\omega)$:

$$u_k(\omega) + u_{-k}(\tau_k\omega) = 0, \qquad w_k(\omega) + w_{-k}(\tau_k\omega) = 0. \tag{45}$$

Further on,

$$u_k(\omega) + w_k(\omega) = v_k(\omega), \qquad q_k(\omega) + r_k(\omega) = p_k(\omega),$$
$$q_k(\omega) \geq 0, \qquad r_k(\omega) \geq 0.$$

We further define

$$q(\omega) := \sum_{l\in\mathscr{E}} q_l(\omega) \geq 0, \qquad \widetilde{\varphi}(\omega) := \sum_{l\in\mathscr{E}} l q_l(\omega) \in \mathbb{R}^d,$$
$$r(\omega) := \sum_{l\in\mathscr{E}} r_l(\omega) \geq 0, \qquad \widetilde{\psi}(\omega) := \sum_{l\in\mathscr{E}} l r_l(\omega) \in \mathbb{R}^d,$$

and note that

$$q(\omega) + r(\omega) = p(\omega), \qquad \widetilde{\varphi}(\omega) + \widetilde{\psi}(\omega) = \varphi(\omega) + \psi(\omega).$$

Now let $0 = \theta_0 < \theta_1 < \theta_2 < \cdots$ be the successive jump times of the environment process $t \mapsto \eta(t)$ [or, what is the same, of the random walk $t \mapsto X(t)$]:

$$\theta_0 := 0, \qquad \theta_{n+1} := \inf\bigl\{t > \theta_n : \eta(t) \neq \eta(\theta_n)\bigr\},$$

and define *extra* random variables $\alpha_n \in \{0, 1\}$, $n = 0, 1, 2, \ldots$ with the following conditional law, given the trajectory $t \mapsto \eta(t)$: for $N \in \mathbb{N}$ and $a_n \in \{0, 1\}$, $n = 0, 1, \ldots, N$,

$$\mathbf{P}\bigl(\alpha_n = a_n, n = 0, 1, \ldots, N \mid \eta(t)_{t\geq 0}\bigr) = \prod_{n=0}^{N} \Bigl(\frac{q(\eta(\theta_n))}{p(\eta(\theta_n))}\Bigr)^{a_n} \Bigl(\frac{r(\eta(\theta_n))}{p(\eta(\theta_n))}\Bigr)^{1-a_n}.$$

In plain words, conditionally on the trajectory $t \mapsto \eta(t)$, the random variables α_n, $n = 0, 1, 2, \ldots$, are independent biased coin tosses, with probability of head or tail (1 or 0 respectively) equal to the value of $\frac{q(\eta(t))}{p(\eta(t))}$, respectively, $\frac{r(\eta(t))}{p(\eta(t))}$, in the interval $t \in [\theta_n, \theta_{n+1})$. Now, extend piecewise continuously

$$\alpha(t) := \sum_{n=0}^{\infty} \alpha_n \mathbb{1}_{\{t\in(\theta_n,\theta_{n+1}]\}}.$$

Mind, that $t \mapsto \alpha(t)$ is defined as a *caglad*, not a cadlag process. We decompose the displacement $t \mapsto X(t)$ as follows:

$$X(t) = K(t) + L(t) + J(t),$$

where

$$K(t) := \int_0^t \alpha(s)\,dX(s) - \int_0^t \widetilde{\varphi}(s)\,ds,$$

$$L(t) := \int_0^t \big(1-\alpha(s)\big)\,dX(s) - \int_0^t \widetilde{\psi}(s)\,ds,$$

$$J(t) := \int_0^t \big(\widetilde{\varphi}(s) + \widetilde{\psi}(s)\big)\,ds.$$

Note the following three facts.

(1) $t \mapsto K(t)$ and $t \mapsto L(t)$, being driven by conditionally independent Poisson flows, are *uncorrelated martingales*, with respect to their own joint filtration.

(2) $t \mapsto K(t)$ is forward-and-backward martingale with respect to its own past, respectively, future filtration. This is due to (45) and to the fact that the symmetric part of its jump rates is constant, s_*. Indeed,

$$\begin{aligned}\mathbf{E}\big(K(t+dt) - K(t) \mid \eta_t = \omega\big) &= \sum_{l\in\mathscr{E}} l q_l(\omega) - \widetilde{\varphi}(\omega)\,dt \\ &= \sum_{l\in\mathscr{E}} l u_l(\omega) - \widetilde{\varphi}(\omega)\,dt = 0\,dt,\\ \mathbf{E}\big(K(t) - K(t-dt) \mid \eta_t = \omega\big) &= -\sum_{l\in\mathscr{E}} l q_{-l}(\tau_l\omega) - \widetilde{\varphi}(\omega)\,dt \\ &= -\sum_{l\in\mathscr{E}} l u_{-l}(\tau_l\omega) - \widetilde{\varphi}(\omega)\,dt \\ &= \sum_{l\in\mathscr{E}} l u_l(\omega) - \widetilde{\varphi}(\omega)\,dt = 0\,dt,\end{aligned}$$

and hence the claim.

(3) $t \mapsto J(t)$, being an integral, is forward-and-backward predictable with respect to the same filtrations.

From these three facts, it follows that the process $t \mapsto K(t)$ is *uncorrelated* with $t \mapsto L(t) + J(t)$. Hence, for any vector $v \in \mathbb{R}$

$$\begin{aligned}\mathbf{E}\big((v\cdot X(t))^2\big) &= \mathbf{E}\big((v\cdot K(t))^2\big) + \mathbf{E}\big((v\cdot\big(L(t)+J(t)\big))^2\big)\\ &\geq \mathbf{E}\big((v\cdot K(t))^2\big) = 2s_*|v|^2.\end{aligned}$$

□

PROOF OF THE UPPER BOUND IN (28). We provide upper bounds on the variance of the various terms on the right-hand side of the decomposition $X = M + I$ (25).

As shown in (26)–(27) the variance of the martingale term $M(t)$ on the right-hand side of (25) is computed explicitly: for $v \in \mathbb{R}^d$,

$$\frac{1}{t}\mathbf{E}\big((v\cdot M(t))^2\big) = \sum_{i=1}^d v_i^2 \int_\Omega \big(p_{e_i}(\omega) + p_{-e_i}(\omega)\big)\,d\pi(\omega) \leq 2s^*|v|^2. \tag{46}$$

In order to bound the variance of the integral term $I(t)$ on the right-hand side of (25), we quote Proposition 2.1.1 in Olla [20] (alternatively, Lemma 2.4 in [10] contains the same result with a different constant).

LEMMA 2. *Let $t \mapsto \eta(t)$ be a stationary and ergodic Markov process on the probability space (Ω, π), whose infinitesimal generator acting on $\mathscr{L}^2(\Omega, \pi)$ is L. Let $g \in \mathscr{L}^2(\Omega, \pi)$ such that $\int_\Omega g \, d\pi = 0$. Then*

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \mathbf{E}\Big(\max_{0\le s\le t} \Big| \int_0^s g\big(\eta(u)\big)\, du \Big|^2 \Big) \le 16 \lim_{\lambda\to 0} \big(g, (\lambda I - L - L^*)^{-1} g\big).$$

[Olla denotes the right-hand side by $\|g\|_{-1}$—his definition of $\|g\|_{-1}$, (2.1.2) ibid., is different but it is easy to see that it is equivalent to the above, up to a factor of 2.]

The decomposition (41) of the infinitesimal generator gives that $-L - L^* \ge 2s_*|\Delta|$, and hence by Löwner's theorem (see [5], Theorem 2.6, or [17]) $(-L - L^*)^{-1} \le 1/(2s_*)|\Delta|^{-1}$. It then follows that for any vector $v \in \mathbb{R}^d$

$$\begin{aligned} \overline{\lim_{t\to\infty}} t^{-1} \mathbf{E}\Big(\Big(\int_0^t v \cdot \varphi\big(\eta(s)\big)\, ds \Big)^2 \Big) &\le \frac{8}{s_*} \big((v\cdot\varphi), |\Delta|^{-1} (v\cdot\varphi) \big) \\ &= \frac{8}{s_*} \sum_{i,j=1}^d v_i \widetilde{C}_{ij} v_j, \end{aligned} \tag{47}$$

$$\begin{aligned} \overline{\lim_{t\to\infty}} t^{-1} \mathbf{E}\Big(\Big(\int_0^t v \cdot \psi\big(\eta(s)\big)\, ds \Big)^2 \Big) &\le \frac{8}{s_*} \big((v\cdot\psi), |\Delta|^{-1} (v\cdot\psi) \big) \\ &= \frac{8}{s_*} \sum_{i,j=1}^d v_i \widetilde{D}_{ij} v_j. \end{aligned} \tag{48}$$

From (25), by applying the Cauchy–Schwarz inequality we readily obtain

$$\begin{aligned} \mathbf{E}\big((v\cdot X(t))^2 \big) \le 3\mathbf{E}\big((v\cdot M(t))^2 \big) &+ 3\mathbf{E}\Big(\Big(\int_0^t v\cdot\varphi\big(\eta(s)\big)\, ds \Big)^2 \Big) \\ &+ 3\mathbf{E}\Big(\Big(\int_0^t v\cdot\psi\big(\eta(s)\big)\, ds \Big)^2 \Big). \end{aligned}$$

Finally, the upper bound in (28) follows from here, due to (46), (47) and (48). □

3. Relaxed sector condition. In this section, we recall and slightly extend the *relaxed sector condition* from [8]. This is a functional analytic condition on the operators D, T and A from (41) which ensures that the efficient martingale approximation à la Kipnis–Varadhan of integrals of the type of $I(t)$ in (25) exists.

A clarification is due here. The relaxed sector condition (Theorem RSC1 below) is essentially equivalent to the condition that the range $L\mathscr{H}_{-1}$ of the infinitesimal

generator L be *dense in the $\mathscr{H}_{-1}$-topology of $\mathscr{L}^2(\Omega,\pi)$* [defined by the symmetric part $S := (L + L^*)/2$ of the infinitesimal generator]. This latter one appears in earlier work (see, e.g., Olla [20]). But, to the best of our knowledge it has never been exploited *directly, without stronger sufficient assumptions*. The *strong* and *graded sector conditions* of Varadhan [29], respectively of Sethuraman, Varadhan and Yau [25], are stronger sufficient conditions for this to hold, and applicable in various circumstances. Nevertheless, the equivalent formulation in [8] proved to be a very useful one, applicable in conditions where the graded sector condition does not work. In particular, in the context of the present paper. Let us also stress that the graded sector condition itself gets a very transparent and handy proof through the relaxed sector condition. For more details, see [8].

Since in the present case the infinitesimal generator $L = -D - T + A$ and all operators in the decomposition (41) are *bounded* we recall the result of [8] in a slightly restricted form: we do not have to worry now about domains and cores of the various operators D, T or A. This section will be fairly abstract.

3.1. *Kipnis–Varadhan theory.* Let $(\Omega, \mathscr{F}, \pi)$ be a probability space: the state space of a *stationary and ergodic* pure jump Markov process $t \mapsto \eta(t)$ with bounded jump rates. We put ourselves in the complex Hilbert space $\mathscr{L}^2(\Omega,\pi)$. Denote the *infinitesimal generator* of the semigroup of the process by L. Since the process $\eta(t)$ has bounded jump rates, the infinitesimal generator L is a bounded operator. We denote the *self-adjoint* and *skew-self-adjoint* parts of the generator L by

$$S := -\frac{1}{2}(L + L^*) \geq 0, \qquad A := \frac{1}{2}(L - L^*).$$

We assume that S is itself ergodic, that is,

$$\operatorname{Ker}(S) = \{c\mathbb{1} : c \in \mathbb{C}\},$$

and restrict ourselves to the subspace of codimension 1, orthogonal to the constant functions:

$$\mathscr{H} := \{f \in \mathscr{L}^2(\Omega,\pi) : \langle \mathbb{1}, f\rangle = 0\}.$$

In the sequel, the operators $(\lambda I + S)^{\pm 1/2}$, $\lambda \geq 0$, will play an important role. These are defined in terms of the spectral theorem applied to the self-adjoint and positive operator S. The *unbounded* operator $S^{-1/2}$ is self-adjoint on its domain

$$\begin{aligned}\operatorname{Dom}\bigl(S^{-1/2}\bigr) &= \operatorname{Ran}\bigl(S^{1/2}\bigr)\\ &= \Bigl\{f \in \mathscr{H} : \bigl\|S^{-1/2} f\bigr\|^2 := \lim_{\lambda\to 0}\bigl\|(\lambda I + S)^{-1/2} f\bigr\|^2 < \infty\Bigr\}.\end{aligned}$$

Let $f \in \mathscr{H}$. We ask about CLT/invariance principle for the rescaled process

$$Y_N(t) := \frac{1}{\sqrt{N}} \int_0^{Nt} f\bigl(\eta(s)\bigr)\, ds \tag{49}$$

as $N \to \infty$.

We denote by R_λ the *resolvent* of the semigroup $s \mapsto e^{sL}$:

$$R_\lambda := \int_0^\infty e^{-\lambda s} e^{sL}\, ds = (\lambda I - L)^{-1}, \qquad \lambda > 0, \tag{50}$$

and given $f \in \mathscr{H}$, we will use the notation

$$u_\lambda := R_\lambda f.$$

The following theorem is a direct extension to general nonreversible setup of the Kipnis–Varadhan theorem [9]. It yields the *efficient martingale approximation* of the additive functional (49). See Tóth [27], or the surveys [20] and [10].

THEOREM KV. *With the notation and assumptions as before, if the following two limits hold in (the norm topology of) $\mathscr{H}$:*

$$\lim_{\lambda\to 0} \lambda^{1/2} u_\lambda = 0, \qquad \lim_{\lambda\to 0} S^{1/2} u_\lambda = v \in \mathscr{H}, \tag{51}$$

then

$$\sigma^2 := 2 \lim_{\lambda\to 0} \langle u_\lambda, f\rangle = 2\|v\|^2 \in [0,\infty),$$

exists, and there also exists a zero mean, $\mathscr{L}^2$-martingale $M(t)$, adapted to the filtration of the Markov process $\eta(t)$, with stationary and ergodic increments and variance

$$\mathbf{E}\big(M(t)^2\big) = \sigma^2 t,$$

such that for $t \in (0,\infty)$

$$\lim_{N\to\infty} \mathbf{E}\left(\left| Y_N(t) - \frac{M(Nt)}{\sqrt{N}} \right|^2\right) = 0.$$

COROLLARY KV. *With the same setup and notation, for any $m \in \mathbb{N}$, $t_1, \dots, t_m \in \mathbb{R}_+$ and $F : \mathbb{R}^m \to \mathbb{R}$ continuous and bounded*

$$\lim_{N\to\infty} \int_\Omega \big|\mathbf{E}_\omega\big(F\big(Y_N(t_1), \dots, Y_N(t_m)\big)\big) - \mathbf{E}\big(F\big(W(t_1), \dots, W(t_m)\big)\big)\big|\, d\pi(\omega) = 0,$$

where $t \mapsto W(t) \in \mathbb{R}$ is a 1-dimensional Brownian motion with variance $\mathbf{E}(W(t)^2) = \sigma^2 t$.

3.2. *Relaxed sector condition.* Let, for $\lambda > 0$,

$$C_\lambda := (\lambda I + S)^{-1/2} A (\lambda I + S)^{-1/2}. \tag{52}$$

These are bounded and skew-self-adjoint.

THEOREM RSC1. *Assume that there exist a dense subspace $\mathscr{C} \subseteq \mathscr{H}$ and an operator $C : \mathscr{C} \to \mathscr{H}$ which is essentially skew-self-adjoint on the core $\mathscr{C}$ and such that for any vector $\psi \in \mathscr{C}$ there exists a sequence $\psi_\lambda \in \mathscr{H}$ such that*

$$\lim_{\lambda\to 0} \|\psi_\lambda - \psi\| = 0 \quad \textit{and} \quad \lim_{\lambda\to 0} \|C_\lambda \psi_\lambda - C\psi\| = 0. \tag{53}$$

Then for any $f \in \mathrm{Dom}(S^{-1/2})$ the limits (51) hold, and thus the martingale approximation and CLT of Theorem KV follow.

REMARKS. 1. The conditions of Theorem RSC1 can be shown to be equivalent to that the sequence of bounded skew-self-adjoint operators C_λ converges in the *strong graph limit* sense to the unbounded skew-self-adjoint operator C; see Lemma 7(ii) below. For various notions of graph limits of operators over Hilbert or Banach spaces, see Chapter VIII of [24], especially Theorem VIII.26 ibid.

2. Theorem RSC1 is a slightly stronger reformulation of Theorem 1 from [8] where the condition (53) was slightly stronger. There it was assumed that for any $\varphi \in \mathscr{C}$, $\lim_{\lambda\to 0} \|C_\lambda \varphi - C\varphi\| = 0$. It turns out that the weaker and more natural condition (53) suffices and this has some importance in our next extension, Theorem RSC2. For sake of completeness, we give the proof of this theorem in the Appendix.

3. The operator C is heuristically some version of $S^{-1/2}AS^{-1/2}$. However, it is not sufficient that a naturally densely defined version of $S^{-1/2}AS^{-1/2}$ is skew-Hermitian. One must show that its closure is actually skew-self-adjoint. The conditions of Theorem RSC1 require to be careful with domains and with point-wise convergence as $\lambda \to 0$, as above.

RSC refers to *relaxed sector condition*: indeed, as shown in [8] this theorem contains the *strong sector condition* of [29] and the *graded sector condition* of [25] as special cases. See the comments at the beginning of Section 3 for the precise relation of RSC to other sector conditions. For comments on history, content and variants of Theorem KV, we refer the reader to the monograph [10]. For some direct consequences of Theorem RSC1, see [8].

Now, we slightly extend the validity of Theorem RSC1. Assume that the symmetric part of the infinitesimal generator decomposes as

$$S = D + T,$$

where $D = D^*$, $T = T^*$ and the "diagonal" part D dominates T in the following sense: there exists $c < \infty$ so that

$$0 \le T \le cD. \tag{54}$$

Further, denote

$$B_\lambda := (\lambda I + D)^{-1/2} A (\lambda I + D)^{-1/2}. \tag{55}$$

The following statement is actually a straightforward consequence of Theorem RSC1.

THEOREM RSC2. *Assume that there exist a dense subspace $\mathscr{B} \subseteq \mathscr{H}$ and an operator $B : \mathscr{B} \to \mathscr{H}$ which is essentially skew-self-adjoint on the core $\mathscr{B}$ and such that for any vector $\varphi \in \mathscr{B}$ there exists a sequence $\varphi_\lambda \in \mathscr{H}$ such that*

$$\lim_{\lambda \to 0} \|\varphi_\lambda - \varphi\| = 0 \quad \text{and} \quad \lim_{\lambda \to 0} \|B_\lambda \varphi_\lambda - B\varphi\| = 0. \tag{56}$$

Then for any $f \in \mathrm{Dom}(D^{-1/2})$ the limits (51) hold, and thus the martingale approximation and CLT of Theorem KV follow.

The proof of Theorem RSC2 is also postponed to the Appendix.

4. The operator $B = D^{-1/2} A D^{-1/2}$ and proof of Theorem 1. We apply Theorem RSC2 to our concrete setup, with the operators D and A defined using (42) and (43), respectively. Recall that without loss of generality we have fixed $s_* = 1$ [see the remark after the ellipticity condition (14)]. Let

$$\mathscr{B} := \mathrm{Dom}\,|\Delta|^{-1/2} = \mathrm{Ran}\,|\Delta|^{1/2},$$

and recall from (33) and (38) the definition of the operators Γ_l and M_l, $l \in \mathscr{E}$. Define the *unbounded* operator $B : \mathscr{B} \to \mathscr{H}$

$$B := -\sum_{l \in \mathscr{E}} \Gamma_{-l} M_l |\Delta|^{-1/2}.$$

(The definition of B uses our assumption that $s_* = 1$, otherwise with our definitions of D and A we would have needed a factor of $1/s_*$ before it.) First, we verify (56), that is, $B_\lambda \to B$ *pointwise* on the core $\mathscr{B}$, where the bounded operator B_λ is expressed by inserting the explicit form of D and A, (42), respectively, (43), into the definition (55) of B_λ:

$$B_\lambda = -\sum_{l \in \mathscr{E}} (\lambda I - \Delta)^{-1/2} \nabla_{-l} M_l (\lambda I - \Delta)^{-1/2}.$$

From the spectral theorem for the commutative C^*-algebra generated by the shift operators T_{e_i}, $i = 1, \dots, d$ (see, e.g., Theorem 1.1.1 on page 2 of [1]), we obtain that $\|(\lambda I - \Delta)^{-1/2} \nabla_l\| \le 1$, $\|(\lambda I - \Delta)^{-1/2} |\Delta|^{1/2}\| \le 1$ and, moreover, for any $\varphi \in \mathscr{H}$,

$$(\lambda I - \Delta)^{-1/2} \nabla_l \varphi \to \Gamma_l \varphi, \qquad (\lambda I - \Delta)^{-1/2} |\Delta|^{1/2} \varphi \to \varphi, \qquad \text{as } \lambda \searrow 0.$$

When $\varphi \in \mathscr{B}$, we get $(\lambda I - \Delta)^{-1/2} \varphi \to |\Delta|^{-1/2} \varphi$ which allows us to write

$$\begin{aligned} B_\lambda \varphi &= -\sum_{l \in \mathscr{E} d} (\lambda I - \Delta)^{-1/2} \nabla_{-l} M_l (\lambda I - \Delta)^{-1/2} \varphi \\ &= -\sum_{l \in \mathscr{E} d} (\lambda I - \Delta)^{-1/2} \nabla_{-l} M_l |\Delta|^{-1/2} \varphi \\ &\quad + O\big(\big\|(\lambda I - \Delta)^{-1/2} \varphi - |\Delta|^{-1/2} \varphi\big\|\big). \end{aligned}$$

Hence, (56) follows readily for any $\varphi \in \mathscr{B}$.

With (56) established, we need to show that B is essentially skew-self-adjoint on $\mathscr{B}$. We start with a light lemma.

LEMMA 3. (i) $B : \mathscr{B} \to \mathscr{H}$ *is skew-Hermitian, that is,* $\langle \varphi, B\psi \rangle = -\langle B\varphi, \psi \rangle$ *for all* $\varphi, \psi \in \mathscr{B}$.

(ii) *The full domain of* B^* *is*

$$\mathscr{B}^* = \left\{ f \in \mathscr{H} : \sum_{l \in \mathscr{E}} M_l \Gamma_l f \in \mathscr{B} \right\} \tag{57}$$

and B^* *acts on* $\mathscr{B}^*$ *by*

$$B^* := -|\Delta|^{-1/2} \sum_{l \in \mathscr{E}} M_l \Gamma_l. \tag{58}$$

REMARK. It is of crucial importance here that $\mathscr{B}^*$ in (57) is the *full* domain of the adjoint operator B^*, that is, the subspace of all f such that the linear functional $g \mapsto \langle f, Bg \rangle$ is bounded on $\mathscr{B}$. It will not be enough for our purposes just to show that $\mathscr{B}^*$ is some core of definition.

PROOF OF LEMMA 3. (i) Let $f, g \in \mathscr{B}$. Then, due to (40)

$$\begin{aligned} \langle f, Bg \rangle &= -\sum_{l \in \mathscr{E}} \langle |\Delta|^{-1/2} f, \nabla_{-l} M_l |\Delta|^{-1/2} g \rangle \\ &\overset{(40)}{=} \sum_{l \in \mathscr{E}} \langle \nabla_{-l} M_l |\Delta|^{-1/2} f, |\Delta|^{-1/2} g \rangle = -\langle Bf, g \rangle, \end{aligned}$$

(ii) Next,

$$\begin{aligned} \mathrm{Dom}(B^*) &= \left\{ f \in \mathscr{H} : (\exists c(f) < \infty)(\forall g \in \mathscr{B}) : \right. \\ &\qquad \left. \left| \left\langle f, \sum_{l \in \mathscr{E}} \Gamma_{-l} M_l |\Delta|^{-1/2} g \right\rangle \right| \le c(f) \|g\| \right\} \\ &= \left\{ f \in \mathscr{H} : (\exists c(f) < \infty)(\forall g \in \mathscr{B}) : \right. \\ &\qquad \left. \left| \left\langle \sum_{l \in \mathscr{E}} M_l \Gamma_l f, |\Delta|^{-1/2} g \right\rangle \right| \le c(f) \|g\| \right\} \\ &= \left\{ f \in \mathscr{H} : \sum_{l \in \mathscr{E}} M_l \Gamma_l f \in \mathscr{B} \right\}, \end{aligned}$$

as claimed. In the last step, we used the fact that $\mathscr{B}$ is the *full domain* of the self-adjoint operator $|\Delta|^{-1/2}$. The action (58) of B^* follows from straightforward manipulations. □

Note that Lemma 3 in particular implies that $\mathscr{B} \subseteq \mathscr{B}^*$, that $B^* : \mathscr{B}^* \to \mathscr{H}$ is in principle an extension of $-\overline{B}$, and hence the operator $B : \mathscr{B} \to \mathscr{H}$ is closable as the adjoint of any operator is automatically closed. We actually ought to prove that

$$B^* = -\overline{B}.$$

We apply von Neumann's criterion (see, e.g., Theorem VIII.3 of Reed and Simon [24]): If for *some* $\alpha > 0$,

$$\operatorname{Ker}\bigl(B^* \pm \alpha I\bigr) = \{0\} \tag{59}$$

then $B^* = -\overline{B}$. For reasons which will become clear very soon, we will choose $\alpha = s^*$. (Actually any $\alpha \geq s^*$ would work equally well.) Thus, (59) is equivalent to showing that the equations

$$\sum_{l \in \mathscr{E}} M_l \Gamma_l \mu + s^* |\Delta|^{1/2} \mu = 0, \tag{60}$$

$$\sum_{l \in \mathscr{E}} M_l \Gamma_l \mu - s^* |\Delta|^{1/2} \mu = 0 \tag{61}$$

admit only the trivial solution $\mu = 0$. We will prove this for (60). The other case is done very similarly.

Note that assuming $\mu \in \mathscr{B}$ the problem becomes fully trivial. Indeed, inserting $\mu = |\Delta|^{1/2} \chi$ in (60) and taking inner product with χ we get

$$\sum_{l \in \mathscr{E}} \langle M_l \nabla_l \chi, \chi \rangle - s^* \langle \Delta \chi, \chi \rangle = 0.$$

The first term is pure imaginary (40) while the second term is real (31), giving that $\langle \Delta \chi, \chi \rangle = 0$ which, due to (32), admits only the trivial solution $\chi = 0$. The point is that μ is not necessarily in $\mathscr{B}$ so $|\Delta|^{-1/2} \mu$ is not necessarily well defined as an element of $\mathscr{H}$. Nevertheless, we are able to define a scalar random field $\Psi : \Omega \times \mathbb{Z}^d \to \mathbb{R}$ of stationary increments (rather than stationary) which can be thought of as the lifting of $|\Delta|^{-1/2} \mu$ to the lattice $\mathbb{Z}^d$.

Let, therefore, μ be a putative solution for (60) and define, for each $k \in \mathscr{E}$,

$$u_k := \Gamma_k \mu. \tag{62}$$

These are vector components and they also satisfy the *gradient condition*: $\forall k, l \in \mathscr{E}$

$$u_k + T_k u_{-k} = 0, \qquad u_k + T_k u_l = u_l + T_l u_k. \tag{63}$$

Note also that

$$\sum_{l \in \mathscr{E}} u_l = |\Delta|^{1/2} \mu.$$

The eigenvalue equation (60) becomes

$$\sum_{l \in \mathscr{E}} v_l u_l + s^* \sum_{l \in \mathscr{E}} u_l = 0. \tag{64}$$

We lift this equation to $\mathbb{Z}^d$. By defining the lattice vector fields $V, U : \Omega \times \mathbb{Z}^d \to \mathbb{R}^d$ as

$$V_k(\omega, x) := v_k(\tau_x \omega), \qquad U_k(\omega, x) := u_k(\tau_x \omega),$$

we obtain the following lifted version of equation (64):

$$\sum_{l \in \mathscr{E}} V_l(\omega, x) U_l(\omega, x) + s^* \sum_{l \in \mathscr{E}} U_l(\omega, x) = 0. \tag{65}$$

Note that U is the $\mathbb{Z}^d$-gradient of a scalar field $\Psi : \Omega \times \mathbb{Z}^d \to \mathbb{R}$, determined uniquely by

$$\Psi(\omega, 0) = 0, \qquad \Psi(\omega, x + k) - \Psi(\omega, x) = U_k(\omega, x). \tag{66}$$

As promised, the scalar field Ψ has stationary increments (or, in the language of ergodic theory: it is a cocycle), that is,

$$\Psi(\omega, y) - \Psi(\omega, x) = \Psi(\tau_x \omega, y - x) - \Psi(\tau_x \omega, 0). \tag{67}$$

Equation (65) gets the form

$$\begin{aligned} &s^* \sum_{l \in \mathscr{E}} \big(\Psi(\omega, x + l) - \Psi(\omega, x)\big) \\ &\quad + \sum_{l \in \mathscr{E}} V_l(\omega, x) \big(\Psi(\omega, x + l) - \Psi(\omega, x)\big) = 0. \end{aligned} \tag{68}$$

Denote the first term by $\operatorname{lap} \Psi$ and the second by $\operatorname{grad} \Psi$ (these are the usual $\mathbb{Z}^d$ Laplacian and gradient, resp.), so the equation becomes

$$s^* \operatorname{lap} \Psi + V \cdot \operatorname{grad} \Psi = 0. \tag{69}$$

We prove that equation (68)/(69) admits $\Psi \equiv 0$ as the only solution satisfying $\mathbf{E}(\Psi(x)) = 0$ for all $x \in \mathbb{Z}^d$. This will be done using *an auxiliary* random walk in random environment which will be denoted by Y. We remark that in the specific case where X is *divergence-free* that is, $s \equiv 1$, or in general when s is constant, we get that Y is the same as X, but in general they differ.

We define the environment for Y on the same probability space Ω as X. The transfer rates p_k^Y, $k \in \mathscr{E}$ are given by

$$p_k^Y(\omega) = s^* + v_k(\omega).$$

In other words, we take from X the anti-symmetric part $v_k = (p_k - p_{-k})/2$ but replace the symmetric part with the constant s^*. The walk Y is also bistochastic, so all results proved so far [in particular, stationarity and ergodicity of Y's environment process $t \mapsto \eta^Y(t) := \tau_{Y(t)} \omega$, and the diffusive lower and upper bounds for $t \mapsto Y(t)$] are in force.

Note that equation (68)/(69) means exactly that for a given $\omega \in \Omega$ fixed (i.e., in the quenched setup) the field $\Psi(\omega, \cdot) : \mathbb{Z}^d \to \mathbb{R}$ is *harmonic* for the random walk $Y(t)$. Thus, the process

$$t \mapsto R(t) := \Psi\big(Y(t)\big) \tag{70}$$

is a martingale [with $R(0) = 0$] in the quenched filtration $\sigma(\omega, Y(s)_{0\le s\le t})_{t\ge 0}$. Hence, $t \mapsto R(t)$ is a martingale in its own filtration $\sigma(R(s)_{0\le s\le t})_{t\ge 0}$, too. We will soon show that $\mathbf{E}(R(t)^2) < \infty$. From stationarity and ergodicity of the environment process $t \mapsto \eta_t^Y$ and (67), it follows that the process $t \mapsto R(t)$ has stationary and ergodic increments with respect to the annealed measure $\mathbf{P}(\cdot) := \int_\Omega \mathbf{P}_\omega(\cdot)\, d\pi(\omega)$. Indeed, let $F(R(\cdot))$ be an arbitrary bounded and measurable functional of the process $t \mapsto R(t)$, $t \ge 0$. Using (67), a straightforward computation shows that

$$\mathbf{E}_\omega\big(F\big(R(t_0 + \cdot) - R(t_0)\big)\big) = \mathbf{E}_\omega\big(\mathbf{E}_{\eta(t_0)}\big(F\big(R(\cdot)\big)\big)\big).$$

Hence, by stationarity and ergodicity of the environment process $t \mapsto \eta(t)$, the claim follows.

Thus, the process $t \mapsto R(t)$ is a martingale [with $R(0) = 0$] with stationary and ergodic increments, in its own filtration $\sigma(R(s)_{0\le s\le t})_{t\ge 0}$, with respect to the annealed measure $\mathbf{P}(\cdot)$.

LEMMA 4. *Let μ be a solution of equation (60), Ψ the harmonic field constructed in (66) and $R(t)$ the martingale defined in (70). Then*

$$\mathbf{E}\big(R(t)^2\big) = 2s^*\|\mu\|^2 t. \tag{71}$$

PROOF. Since $t \mapsto R(t)$ is a martingale with stationary increments [with respect to the annealed measure $\mathbf{P}(\cdot)$], we automatically have $\mathbf{E}(R(t)^2) = \varrho^2 t$ with some $\varrho \ge 0$. We now compute ϱ.

$$\begin{aligned}
\varrho^2 &:= \lim_{t\to 0} \frac{\mathbf{E}(R(t)^2)}{t} \overset{1}{=} \lim_{t\to 0} \int_\Omega \frac{\mathbf{E}_\omega(\Psi(\omega, Y(t))^2)}{t}\, d\pi(\omega) \\
&\overset{2}{=} \int_\Omega \lim_{t\to 0} \frac{\mathbf{E}_\omega(\Psi(\omega, Y(t))^2)}{t}\, d\pi(\omega) \overset{3}{=} \sum_{l\in\mathscr{E}} \int_\Omega \big(s^* + v_l(\omega)\big)\big|u_l(\omega)\big|^2\, d\pi(\omega) \\
&\overset{4}{=} s^* \sum_{l\in\mathscr{E}} \int_\Omega \big|u_l(\omega)\big|^2\, d\pi(\omega) \overset{5}{=} s^* \sum_{l\in\mathscr{E}} \|\Gamma_l \mu\|^2 \overset{6}{=} 2s^*\|\mu\|^2.
\end{aligned}$$

Step 1 is annealed averaging. Step 2 is easily justified by dominated convergence. Step 3 drops out from explicit computation of the conditional variance of one jump. In step 4, we used that due to (7) and (63) $v_{-l}(\omega)|u_{-l}(\omega)|^2 = -v_l(\tau_{-l}\omega)|u_l(\tau_{-l}\omega)|^2$ and translation invariance of the measure π on Ω. In step 5, we use the definition (62) of u_l. Finally, in the last step 6 we used the third identity of (34). □

PROPOSITION 2. *The unique solution of* (60)/(61) *is* $\mu = 0$, *and consequently the operator* B *is essentially skew-self-adjoint on the core* $\mathscr{B}$.

PROOF. Let μ be a solution of the equation (60), Ψ the harmonic field constructed in (66) and $R(t)$ the martingale defined in (70). From the martingale central limit theorem (see, e.g., [7]) and (71) it follows that

$$\frac{R(t)}{\sqrt{t}} \quad \Rightarrow \quad \mathscr{N}\bigl(0, 2s^*\|\mu\|^2\bigr), \qquad \text{as } t \to \infty. \tag{72}$$

On the other hand, we are going to prove that

$$\frac{R(t)}{\sqrt{t}} \xrightarrow{\mathbf{P}} 0, \qquad \text{as } t \to \infty. \tag{73}$$

Jointly, (72) and (73) clearly imply $\mu = 0$, as claimed in the proposition.

The proof of (73) will combine

(A) the (sub)diffusive behaviour of the displacement

$$\overline{\lim_{T\to\infty}}\, T^{-1}\mathbf{E}\bigl(Y(T)^2\bigr) < \infty,$$

which follows from the $\mathscr{H}_{-1}$-condition [see (28)]; and

(B) the fact that the scalar field $x \mapsto \Psi(x)$ having zero mean and stationary increments [cf. (67)], increases *sublinearly* with $|x|$. The sublinearity is the issue here. Since Ψ has stationary, mean zero increments, due to the individual (pointwise) ergodic theorem, it follows that *in any fixed direction* Ψ increases sublinearly almost surely. However, this does not warrant that Ψ increases sublinearly uniformly in $\mathbb{Z}^d$, $d \geq 2$, which is the difficulty we will now tackle.

Let $\delta > 0$ and $K < \infty$. Then

$$\begin{aligned}\mathbf{P}\bigl(|R(t)| > \delta\sqrt{t}\bigr) &\leq \mathbf{P}\bigl(\{|R(t)| > \delta\sqrt{t}\} \cap \{|Y(t)| \leq K\sqrt{t}\}\bigr) \\ &\quad + \mathbf{P}\bigl(|Y(t)| > K\sqrt{t}\bigr).\end{aligned} \tag{74}$$

From (sub)diffusivity (28) and Chebyshev's inequality, it follows directly that

$$\lim_{K\to\infty} \overline{\lim_{t\to\infty}}\, \mathbf{P}\bigl(|Y(t)| > K\sqrt{t}\bigr) = 0. \tag{75}$$

We present two proofs of

$$\lim_{t\to\infty} \mathbf{P}\bigl(\{|R(t)| > \delta\sqrt{t}\} \cap \{|Y(t)| \leq K\sqrt{t}\}\bigr) = 0, \tag{76}$$

with $\delta > 0$ and $K < \infty$ fixed. One with bare hands, valid in $d = 2$ only, and another one valid in any dimension which relies on a heat kernel (upper) bound from Morris and Peres [18].

PROOF OF (76) IN $d = 2$, WITH BARE HANDS. We follow here the approach of [3] where the argument was applied in a different context. In order to keep it

short (as another full proof valid in all dimensions follows), we assume separate ergodicity, that is, $(\Omega, \mathscr{F}, \pi, \tau_{e_i})$ is ergodic for both $i = 1, 2$.

First, note that

$$\mathbf{P}\big(\{|R(t)| > \delta\sqrt{t}\} \cap \{|Y(t)| \le K\sqrt{t}\}\big) \le \mathbf{P}\Big(\max_{|x|<K\sqrt{t}} |\Psi(x)| > \delta\sqrt{t}\Big). \tag{77}$$

Next, since Ψ is harmonic with respect to the random walk $Y(t)$, it obeys the *maximum principle* (this is true for any random walk, no special property of Y is used here). Thus,

$$\max_{|x|_\infty \le L} |\Psi(x)| = \max_{|x|_\infty = L} |\Psi(x)|, \tag{78}$$

where $|x|_\infty := \max\{|x_1|, |x_2|\}$. By spatial stationarity,

$$\begin{aligned}
\max_{|x_1|\le L} |\Psi(x_1, -L) - \Psi(0, -L)| &\sim \max_{|x_1|\le L} |\Psi(x_1, 0)| \\
&\sim \max_{|x_1|\le L} |\Psi(x_1, +L) - \Psi(0, +L)|, \\
\max_{|x_2|\le L} |\Psi(-L, x_2) - \Psi(-L, 0)| &\sim \max_{|x_2|\le L} |\Psi(0, x_2)| \\
&\sim \max_{|x_2|\le L} |\Psi(+L, x_2) - \Psi(+L, 0)|,
\end{aligned} \tag{79}$$

where $\sim$ stands for equality in distribution. Now, note that $\Psi(x_1, 0)$ and $\Psi(0, x_2)$ are Birkhoff sums:

$$\Psi(x_1, 0) = \sum_{j=0}^{x_1-1} u_{e_1}(\tau_{je_1}\omega), \qquad \Psi(0, x_2) = \sum_{j=0}^{x_2-1} u_{e_2}(\tau_{je_2}\omega),$$

where $u_{e_1}(\omega)$ and $u_{e_2}(\omega)$ are zero mean and square integrable [recall the definition of u, (62)]. Hence, by the ergodic theorem,

$$L^{-1} \max\Big\{ \max_{|x_1|\le L} |\Psi(x_1, 0)|, \max_{|x_2|\le L} |\Psi(0, x_2)| \Big\} \to 0, \quad \text{a.s., as } L \to \infty. \tag{80}$$

Putting together (78), (79) and (80), we readily obtain, for any $\varepsilon > 0$,

$$\lim_{L\to\infty} \mathbf{P}\Big(\max_{|x|_\infty \le L} |\Psi(x)| \ge \varepsilon L\Big) = 0. \tag{81}$$

Finally, (76) follows by applying (81) to the right-hand side of (77). □

PROOF OF (76) IN ALL $d \ge 2$. We start with the following uniform upper bound on the (quenched) heat kernel of the walk $Y(t)$.

PROPOSITION 3. *There exists a constant $C = C(d, s^*)$ (depending only on the dimension d and the upper bound s^* on the jump rates) such that for π-almost all $\omega \in \Omega$ and all $t > 0$*

$$\sup_{x \in \mathbb{Z}^d} \mathbf{P}_\omega\big(Y(t) = x\big) \le C t^{-d/2}, \qquad \pi\text{-a.s.} \tag{82}$$

PROOF. This bound (82) follows from Theorem 2 of Morris and Peres [18] through Lemma 5, below, which states essentially the same bound for discrete-time lazy random walks on $\mathbb{Z}^d$ (recall that a random walk is called lazy if there is a lower bound on the probability of the walker staying put at any given point).

LEMMA 5. *Let $V : \mathbb{Z}^d \to [-1, 1]^{\mathscr{E}}$ be a (deterministically given) field such that for all $k \in \mathscr{E}$ and $x \in \mathbb{Z}^d$:*

$$V_k(x) + V_{-k}(x + k) = 0, \qquad \sum_{l \in \mathscr{E}} V_l(x) = 0. \tag{83}$$

Define the discrete-time nearest-neighbour, lazy random walk $n \mapsto Y_n$ on $\mathbb{Z}^d$ with transition probabilities:

$$\begin{aligned} &\mathbf{P}(Y_{n+1} = y \mid Y_n = x) \\ &= p_{x,y} := \begin{cases} \dfrac{1}{2} & \text{if } y = x, \\ \dfrac{1}{4d}\big(1 + V_k(x)\big) & \text{if } y = x + k, k \in \mathscr{E}, \\ 0 & \text{if } |y - x| > 1. \end{cases} \end{aligned} \tag{84}$$

Then there exists a constant $C = C(d)$ depending only on dimension such that for any $x, y \in \mathbb{Z}^d$,

$$\mathbf{P}(Y_n = y \mid Y_0 = x) \le C n^{-d/2}. \tag{85}$$

PROOF. For $A, B \subset \mathbb{Z}^d$, such that $A \cap B = \varnothing$ let

$$Q(A, B) := \sum_{x \in A, y \in B} p_{x,y}.$$

For notational reasons, we extend the definition of $V_k(x)$, $k \in \mathscr{E}$, $x \in \mathbb{Z}^d$, as follows:

$$V_z(x) := \begin{cases} V_k(x) & \text{if } z = k \in \mathscr{E}, \\ 0 & \text{if } z \notin \mathscr{E}. \end{cases}$$

For $S \subset \mathbb{Z}^d$, $|S| < \infty$ let $\partial S := \{(x, y) : x \in S, y \in \mathbb{Z}^d \setminus S, \|x - y\| = 1\}$ and note that by the isoperimetric inequality for $\mathbb{Z}^d$,

$$|\partial S| \ge C |S|^{(d-1)/d}, \tag{86}$$

with some dimension-dependent constant C. (This discrete isoperimteric inequality is a simple corollary of the classic isoperimetric inequality in $\mathbb{R}^d$. See also Theorem V3.1 in [6] for a general discretisation result for isoperimteric inequalities.)

We have

$$\begin{aligned} Q(S, S^c) &= \sum_{x\in S, y\in S^c} \frac{1}{4d}\big(1 + V_{y-x}(x)\big) \\ &= \frac{1}{4d}|\partial S| + \frac{1}{4d}\Big(\sum_{x\in S, y\in\mathbb{Z}^d} V_{y-x}(x) - \sum_{x\in S, y\in S} V_{y-x}(x)\Big) \\ &= \frac{1}{4d}|\partial S|, \end{aligned} \tag{87}$$

where the last equality follows from

$$\sum_{x\in S, y\in\mathbb{Z}^d} V_{y-x}(x) = \sum_{x\in S}\sum_{l\in\mathscr{E}} V_l(x) = 0,$$

$$\sum_{x\in S, y\in S} V_{y-x}(x) = \frac{1}{2}\sum_{x\in S, y\in S}\big(V_{y-x}(x) + V_{x-y}(y)\big) = 0,$$

both of which are consequences of (83). Yet another consequence of (83) is that the uniform counting measure on $\mathbb{Z}^d$ is stationary to our walk. Hence, the isoperimetric profile $\Phi(r)$ (in the sense of Morris and Peres [18]) is given by

$$\Phi(r) := \inf_{0<|S|\le r} \frac{Q(S, S^c)}{|S|}.$$

Theorem 2 of [18] (specified to our setup) states that for any $0 < \varepsilon \le 1$, if

$$n > 1 + 4\int_4^{4/\varepsilon} \frac{du}{u\Phi^2(u)} \tag{88}$$

then, for any $x, y \in \mathbb{Z}_d$

$$\mathbf{P}(X_n = y \mid X_0 = x) \le \varepsilon.$$

From (87) and the isoperimetric inequality (86), we have

$$C_1 r^{-1/d} \le \Phi(r) \le C_2 r^{-1/d}, \tag{89}$$

with the constants $0 < C_1 < C_2 < \infty$ depending only on the dimension. Finally, from (88) and (89), we readily get (85). □

In order to obtain (82) from (85), note that $Y(t) = Y_{\nu(t)}$ where Y_n is a discrete time lazy random walk defined in (83) and (84), with $V_k(x) = v_k(\tau_x\omega)/s^*$ and

$t \mapsto \nu(t)$ is a Poisson birth process with intensity s^*t independent of the discrete time walk Y_n. Thus,

$$\begin{aligned}\mathbf{P}_\omega\big(Y(t)=x\big) &= e^{-s^*t/2}\sum_{n=0}^{\infty}\frac{(s^*t/2)^n}{n!}\mathbf{P}_\omega(Y_n=x)\\ &\le e^{-s^*t/2}\left(1+\sum_{n=1}^{\infty}\frac{(s^*t/2)^n}{n!}Cn^{-d/2}\right)\\ &\le C\big(d,s^*\big)t^{-d/2}\end{aligned}$$

This completes the proof of Proposition 3. □

REMARKS. (1) The point in Proposition 3 is that it provides uniform upper bound in any (deterministic) environment which satisfies conditions (83), and thus allows decoupling of the expectation with respect to the walk and with respect to the environment.

(2) In Lemma 5, the "amount of laziness" could be any $\delta \in (0,1)$, with appropriate minor changes in the formulation and proof.

(3) Alternative proofs of Proposition 3 are also valid, using either Nash–Sobolev or Faber–Krahn inequalities; see, for example, Kumagai [16]. These alternative proofs—which we do not present here—are more analytic in flavour. Their advantage is robustness: these proofs are also valid in continuous-space setting (see Section 6 below).

We now return to the proof of (76). By Chebyshev's inequality,

$$\begin{aligned}&\mathbf{P}\big(\{|R(t)|>\delta\sqrt{t}\}\cap\{|Y(t)|\le K\sqrt{t}\}\big)\\ &\quad\le \delta^{-2}t^{-1}\mathbf{E}\big(|R(t)|^2\mathbb{1}_{\{|Y(t)|\le K\sqrt{t}\}}\big).\end{aligned}\tag{90}$$

Since the scalar field Ψ has stationary increments [cf. (67)], and zero mean, we get from the $\mathscr{L}^2$ ergodic theorem that for $k \in \mathscr{E}$

$$\lim_{n\to\infty} n^{-2}\mathbf{E}\big(|\Psi(nk)|^2\big)=0,$$

and, consequently,

$$\lim_{|x|\to\infty} |x|^{-2}\mathbf{E}\big(|\Psi(x)|^2\big)=0.\tag{91}$$

Applying in turn the heat kernel bound (82) of Proposition 3 and the limit (91) on the right- hand side of (90), we obtain

$$t^{-1}\mathbf{E}\big(|R(t)|^2\mathbb{1}_{\{|Y(t)|\le K\sqrt{t}\}}\big)\le Ct^{-d/2-1}\sum_{|x|\le K\sqrt{t}}\mathbf{E}\big(|\Psi(x)|^2\big)\to 0,\qquad \text{as } t\to\infty.$$

Here, the first expectation is both on the random walk $Y(t)$ and on the field ω, while the second is just on the field ω. The point is that with the help of the uniform heat kernel bound of Proposition 3 we can *decouple* the two expectations.

This concludes the proof of (76) in arbitrary dimension. □

We conclude the proof of the Proposition 2 by noting that from (74), (75) and (76) we readily get (73) which, together with (72) implies indeed that $\mu = 0$. So (59) holds with $\alpha = s^*$. We showed that $\mathrm{Ker}(B^* + s^* I) = \{0\}$, the proof that $\mathrm{Ker}(B^* - s^* I) = \{0\}$ is done in the same way with Y defined using $-V$ instead of V. Thus, the operator $B : \mathscr{B} \to \mathscr{H}$ is indeed essentially skew-self-adjoint. □

PROOF OF THEOREM 1. Proposition 2 verifies that the operator B is essentially skew-self-adjoint. The other conditions of Theorem RSC2 are verified on pages 4324–4325. Thus, Theorem RSC2 may be applied and we get that for any $f \in \mathrm{Dom}(|\Delta|^{-1/2})$, the time average $\int_0^N f(\eta(t))$ may be approximated by a Kipnis–Varadhan martingale. The third formulation of the $\mathscr{H}_{-1}$ condition (35) gives that $v_k \in \mathrm{Dom}(|\Delta|^{-1/2})$ while it is always true that $s_k \in \mathrm{Dom}(|\Delta|^{-1/2})$, (18). Applying Theorem RSC2 with $f = v_k + s_k$ for each $k \in \{1, \dots, d\}$ gives that the compensator I from the decomposition $X = M + I$ (25) can be approximated with a Kipnis–Varadhan martingale, which we recall, is a stationary martingale M' which is adapted to the filtration of the environment process η. Hence, $M + M'$ is also a stationary martingale and has a CLT. Proposition 1 gives the bounds (29). □

5. The stream tensor field. The content of this section is not a part of the proof of our main result, but it is an important part of the story and sheds light on the role and limitations of the $\mathscr{H}_{-1}$-condition in this context. We formulate this section in the context of nearest neighbour jumps and part (ii) of Proposition 4 (below) as presented here relies on the equivalence of (35) and (36) which is valid only in the nearest neighbour case. However, we remark that this statement, too, can be easily reformulated for general finite jump rates, but in this case some modifications in the definition of the lattice stream tensor are due and the formulation becomes less transparent. We omit these not particularly instructive details, noting that it is doable with minimum effort.

The following proposition establishes the existence of the stream tensor field and is essentially Helmholtz's theorem. It is the $\mathbb{Z}^d$ lattice counterpart of Proposition 11.1 from [10]. Recall the definition of the field $V : \Omega \times \mathbb{Z}^d \to [-s^*, s^*]^{\mathscr{E}}$ from (6).

PROPOSITION 4. (i) *There exists an antisymmetric tensor field* $H : \Omega \times \mathbb{Z}^d \to \mathbb{R}^{\mathscr{E} \times \mathscr{E}}$ *such that for all* $x \in \mathbb{Z}^d$ *we have* $H_{k,l}(\cdot, x) \in \mathscr{H}$ *and*

$$H_{l,k}(\omega, x) = H_{-k,l}(\omega, x + k) = H_{k,-l}(\omega, x + l) = -H_{k,l}(\omega, x), \tag{92}$$

with stationary increments

$$H(\omega, y) - H(\omega, x) = H(\tau_x\omega, y - x) - H(\tau_x\omega, 0),$$

such that

$$V_k(\omega, x) = \sum_{l\in\mathscr{E}} H_{k,l}(\omega, x). \tag{93}$$

The realization of the tensor field H is uniquely determined by the "pinning down" condition (101) below.

(ii) *The $\mathscr{H}_{-1}$-condition (19) holds if and only if there exist $h_{k,l} \in \mathscr{H}$, $k, l \in \mathscr{E}$, such that*

$$h_{l,k} = T_k h_{-k,l} = T_l h_{k,-l} = -h_{k,l} \tag{94}$$

and

$$v_k(\omega) = \sum_{l\in\mathscr{E}} h_{k,l}(\omega). \tag{95}$$

In this case, the tensor field H can be realized as the stationary lifting of h:

$$H_{k,l}(\omega, x) = h_{k,l}(\tau_x\omega). \tag{96}$$

PROOF. (i) For $k, l, m \in \mathscr{E}$ define

$$g_{m;k,l} := \Gamma_m(\Gamma_l v_k - \Gamma_k v_l),$$

where $\Gamma_l = |\Delta|^{-1/2}\nabla_l$ are the Riesz operators defined in (33), and note that for all $k, l, m, n \in \mathscr{E}$:

$$g_{m;l,k} = T_k g_{m;-k,l} = T_l g_{m;k,-l} = -g_{m;k,l}, \tag{97}$$

$$g_{m;l,k} + T_m g_{n;l,k} = g_{n;l,k} + T_n g_{m;l,k}, \tag{98}$$

$$\sum_{l\in\mathscr{E}} g_{m;k,l} = \nabla_m v_k. \tag{99}$$

Equation (97) means that that keeping the subscript $m \in \mathscr{E}$ fixed, $g_{m;k,l}$ has exactly the symmetries of a $\mathscr{L}^2$-tensor variable indexed by $k, l \in \mathscr{E}$. Equation (98) means that, on the other hand, keeping $k, l \in \mathscr{E}$ fixed, $g_{m;k,l}$ is a $\mathscr{L}^2$-gradient in the subscript $m \in \mathscr{E}$. Finally, (99) means that the $\mathscr{L}^2$-divergence of tensor $g_{m;\cdot,\cdot}$ is actually the $\mathscr{L}^2$-gradient of the vector $v_\cdot$.

Let $G_{m;k,l} : \Omega \times \mathbb{Z}^d \to \mathbb{R}$ be the lifting $G_{m;k,l}(\omega, x) := g_{m;k,l}(\tau_x\omega)$. By (98), for any $k, l \in \mathscr{E}$ fixed $(G_{m;k,l}(\omega, x))_{m\in\mathscr{E}}$ is a lattice gradient. The increments of $H_{k,l}$ are defined by

$$H_{k,l}(\omega, x + m) - H_{k,l}(\omega, x) = G_{m;k,l}(\omega, x), \qquad m \in \mathscr{E}. \tag{100}$$

This is consistent, due to (98).

Next, in order to uniquely determine the tensor field H, we "pin down" its values at $x = 0$. For $e_i, e_j \in \mathscr{E}_+$ choose

$$H_{e_i,e_j}(\omega, 0) = 0, \qquad H_{-e_i,e_j}(\omega, 0) = -g_{-e_i;e_i,e_j}(\omega),$$
$$H_{e_i,-e_j}(\omega, 0) = g_{-e_j;e_i,e_j}(\omega), \tag{101}$$
$$H_{-e_i,-e_j}(\omega, 0) = -g_{-e_i;e_i,e_j}(\omega) + g_{-e_j;e_i,e_j}(\tau_{-e_i}\omega).$$

The tensor field H is fully determined by (100) and (101). Due to (97) and (99), (92), respectively, (93) will hold, indeed.

(ii) We show equivalence with $v_k \in \mathrm{Dom}(|\Delta|^{-1/2})$ (35). First we prove the *only* if part. Assume (35) and let

$$h_{k,l} = \Gamma_l |\Delta|^{-1/2} v_k - \Gamma_k |\Delta|^{-1/2} v_l = |\Delta|^{-1/2}(\Gamma_l v_k - \Gamma_k v_l).$$

Hence, (94) and (95) are readily obtained. Next, we prove the if part. Assume that there exist $h_{k,l} \in \mathscr{H}$ with the symmetries (94) and v_k is realized as in (95). Then we have

$$v_k = \sum_{l\in\mathscr{E}} h_{k,l} = \frac{1}{2}\sum_{l\in\mathscr{E}}(h_{k,l} + h_{k,-l}) = -\frac{1}{2}\sum_{l\in\mathscr{E}} \nabla_l h_{k,-l} = -\frac{1}{2}|\Delta|^{1/2}\sum_{l\in\mathscr{E}} \Gamma_l h_{k,-l},$$

which shows indeed (35). □

$\mathscr{H}_{-1}$*-condition* (fourth formulation): The drift vector field V is realized as the curl of a *stationary and square integrable*, zero mean tensor field H, as shown in (93).

REMARK. If the $\mathscr{H}_{-1}$-condition (19) does not hold, it may still be possible that there exists a *nonsquare integrable* tensor variable $h : \Omega \to \mathbb{R}^{\mathscr{E}\times\mathscr{E}}$ which has the symmetries (94) and with $v : \Omega \to \mathbb{R}^{\mathscr{E}}$ realized as in (95). Then let $H : \Omega \times \mathbb{Z}^d \to \mathbb{R}^{\mathscr{E}\times\mathscr{E}}$ be the stationary lifting (96) and we still get (93) with a *stationary but not square integrable* tensor field. Note that this is not decidable in terms of the covariance matrix (16) or its Fourier transform (17). The question of diffusive (or super-diffusive) asymptotic behaviour of the walk $t \mapsto X(t)$ in these cases is fully open.

In the next proposition—which essentially follows an argument from Kozlov [14]—we give a sufficient condition for the $\mathscr{H}_{-1}$-condition (19) to hold.

PROPOSITION 5. *If $p \mapsto \widehat{C}(p)$ is twice continuously differentiable function in a neighbourhood of $p = 0$, then the $\mathscr{H}_{-1}$-condition* (19) *holds.*

PROOF. For the duration of this proof, we introduce the notation

$$B_{k,l}(x) := \mathbf{E}\big(V_k(0)V_l(x)\big), \qquad \widehat{B}_{k,l}(p) := \sum_{x\in\mathbb{Z}^d} e^{\sqrt{-1}x\cdot p} B_{k,l}(x),$$

with $k, l \in \mathscr{E}$, $x \in \mathbb{Z}^d$, $p \in [-\pi, \pi]^d$. Hence, for $i, j \in \{1, \dots, d\}$,

$$\widehat{C}_{ij}(p) = \widehat{B}_{e_i,e_j}(p) - \widehat{B}_{-e_i,e_j}(p) - \widehat{B}_{e_i,-e_j}(p) + \widehat{B}_{-e_i,-e_j}(p).$$

(The identity is meant in the sense of distributions.)

Note that, due to the first clause in (83),

$$\begin{aligned} \widehat{B}_{k,l}(p) &= -e^{\sqrt{-1}p\cdot k}\widehat{B}_{-k,l}(p) = -e^{-\sqrt{-1}p\cdot l}\widehat{B}_{k,-l}(p) \\ &= e^{\sqrt{-1}p\cdot(k-l)}\widehat{B}_{-k,-l}(p). \end{aligned} \tag{102}$$

Using (102) in the above expression of $C(p)$ in terms of $B(p)$, direct computations yield

$$\widehat{C}_{ij} = \bigl(1 + e^{-\sqrt{-1}p\cdot e_i}\bigr)\bigl(1 + e^{\sqrt{-1}p\cdot e_j}\bigr)\widehat{B}_{e_i,e_j}(p).$$

Thus, the regularity condition imposed on $p \mapsto C(p)$ is equivalent to assuming the same regularity about $p \mapsto \widehat{B}(p)$.

Next, due to the second clause of (83),

$$\sum_{k\in\mathscr{E}} \widehat{B}_{k,l}(p) = \sum_{l\in\mathscr{E}} \widehat{B}_{k,l}(p) = 0, \tag{103}$$

and, from (102) and (103) again by direct computations we obtain

$$\sum_{k,l\in\mathscr{E}} \bigl(1 - e^{-\sqrt{-1}p\cdot k}\bigr)\bigl(1 - e^{\sqrt{-1}p\cdot l}\bigr)\widehat{B}_{k,l}(p) \equiv 0. \tag{104}$$

At $p = 0$, we apply $\partial^2/\partial p_i \partial p_j$ to (104) and get

$$\widehat{C}_{ij}(0) = \sum_{k,l\in\mathscr{E}} k_i l_j \widehat{B}_{k,l}(0) = 0, \qquad i, j = 1, \dots, d. \tag{105}$$

Since $\widehat{C}_{j,i}(p) = \widehat{C}_{ij}(-p) = \overline{\widehat{C}_{ij}(p)}$ and $p \mapsto \widehat{C}(p)$ is assumed to be twice continuously differentiable at $p = 0$, from (105) it follows that

$$\widehat{C}(p) = \mathscr{O}\bigl(|p|^2\bigr), \qquad \text{as } |p| \to 0,$$

which implies (19). □

In particular, it follows that sufficiently fast decay of correlations of the divergence-free drift field $V(x)$ implies the $\mathscr{H}_{-1}$-condition (19). Note that the divergence-free condition (7) is crucial in this argument.

6. Historical remarks. There exist a fair number of important earlier results to which we should compare Theorem 1.

(1) In Kozlov [14], Theorem II.3.3 claims the same result under the supplementary restrictive condition that the random field of jump probabilities $x \mapsto P(x)$ in (2) be *finitely dependent*. However, as pointed out by Komorowski and Olla [11], the proof is incomplete there. Also, the condition of finite dependence of the field of jump probabilities is a very serious restriction.

(2) In Komorowski and Olla [12], Theorem 2.2, essentially the same result is announced as above. However, as noted in Section 3.6 of [10] this proof is yet again incomplete.

(3) To our knowledge, the best fully proved result is Theorem 3.6 of [10] where the same result is proved under the condition that the stream tensor field $x \mapsto H(x)$ of Proposition 4 be stationary and in $\mathscr{L}^{\max\{2+\delta,d\}}$, $\delta > 0$, rather than $\mathscr{L}^2$. Note that the conditions of our theorem only request that the tensor field $x \mapsto H$ be square integrable. The proof of Theorem 3.6 in [10] is very technical; see Sections 3.4 and 3.5 of the monograph.

(4) The special case when the tensor field H is actually in $\mathscr{L}^\infty$ is fundamentally simpler. In this case, the so-called *strong sector condition* of Varadhan [29] applies directly. This was noticed in [12]. See also Section 3.3 of [10] and Section 7 below.

(5) Examine the following *diffusion* problem is as follows. Let $t \mapsto X(t) \in \mathbb{R}^d$ be the strong solution of the SDE

$$dX(t) = dB(t) + \Phi\big(X(t)\big)\,dt, \tag{106}$$

where $B(t)$ is standard d-dimensional Brownian motion and $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ is a stationary and ergodic (under space-shifts) vector field on $\mathbb{R}^d$ which has zero mean

$$\mathbf{E}\big(\Phi(x)\big) = 0,$$

and is almost surely *divergence-free*:

$$\operatorname{div}\Phi \equiv 0, \qquad \text{a.s.} \tag{107}$$

It is analogous to the discrete-space problem studied in this paper in the case that s_k is constant for all $k \in \mathscr{E}$. In this case, the $\mathscr{H}_{-1}$-condition is

$$\sum_{i=1}^{d} \int_{\mathbb{R}^d} |p|^{-2} \widehat{C}_{ii}(p)\,dp < \infty, \tag{108}$$

where

$$\widehat{C}_{ij}(p) := \int_{\mathbb{R}^d} \mathbf{E}\big(\Phi_i(0)\Phi_j(x)\big) e^{\sqrt{-1}\,p\cdot x}\,dx, \qquad p \in \mathbb{R}^d.$$

It is a fact that, similar to the $\mathbb{Z}^d$ lattice case, under minimally restrictive regularity conditions, a stationary and square integrable divergence-free drift field $x \mapsto \Phi(x)$ on $\mathbb{R}^d$ can be written as the curl of an antisymmetric stream tensor field with stationary increments $H : \mathbb{R}^d \to \mathbb{R}^{d\times d}$:

$$\Phi_i(x) = \sum_{j=1}^{d} \frac{\partial H_{ji}}{\partial x_j}(x).$$

This is essentially Helmholtz's theorem. See Proposition 11.1 of [10], which is the continuous-space analogue of Proposition 4 of Section 5 above. As shown in

[10], the $\mathscr{H}_{-1}$-condition (108) is equivalent with the fact that the stream tensor H is *stationary* (not just of stationary increments) *and square integrable*. The case of bounded H was first considered in Papanicolaou and Varadhan [22]. This paper is historically the first instant where the problem of diffusion in stationary divergence-free drift field was considered with mathematical rigour. Homogenization and central limit theorem for the diffusion (106), (107) in *bounded* stream field, $H \in \mathscr{L}^\infty$, was first proven in Osada [21]. Today the strongest result in the continuous space-time setup is due to Oelschläger [19] where homogenization and CLT for the displacement is proved for square-integrable stationary stream tensor field, $H \in \mathscr{L}^2$. Oelschläger's proof consists in truncating the stream tensor and bounding the error. If the stream tensor field is stationary *Gaussian* then—as noted by Komorowski and Olla [13]—the *graded sector condition* of [25] can be applied. See also Chapters 10 and 11 of [10] for all existing results on the diffusion model (106), (107).

(6) Attempts to apply Oelschläger's method in the discrete ($\mathbb{Z}^d$ rather than $\mathbb{R}^d$) setting run into enormous technical difficulties (see Chapter 3 of [10]) and seemingly this approach cannot be fully accomplished beyond the overly restrictive condition $H \in \mathscr{L}^{\max\{2+\delta,d\}}$. The main result of this paper, Theorem 1 fills this gap between the restrictive condition $H \in \mathscr{L}^{\max\{2+\delta,d\}}$ of Theorem 3.6 in [10] and the minimal restriction $H \in \mathscr{L}^2$. The content of our Theorem 1 is the discrete $\mathbb{Z}^d$-counterpart of Theorem 1 in Oelschläger [19]. We also stress that our proof is conceptually and technically much simpler that of Theorem 3.6 in [10] or Theorem 1 in [19]. The continuous space-time diffusion model—under the same regularity conditions as those of Oelschläger [19] can be treated in a very similar way reproducing this way Theorem 1 of [19] in a conceptually and technically simpler way. In order to keep this paper relatively short and transparent, those details will be presented elsewhere.

(7) There exist results on *super-diffusive* behaviour of the random walk in doubly stochastic random environment (3), (1) or diffusion in divergence-free random drift field (106), (107), when the $\mathscr{H}_{-1}$-condition (19) fails to hold. In Komorowski and Olla [11] and Tóth and Valkó [28], the diffusion model (106), (107) is considered when the drift field Φ is Gaussian and the stream tensor field H is *genuinely* delocalized: of stationary increment but not stationary. Super-diffusive bounds are proved.

7. Examples. Before formulating concrete examples let us spend a few words about the physical motivation and phenomenology of the problem considered. The continuous case discussed in the previous section, diffusion in divergence-free drift field [cf. (106)–(107)] may model the drifting of a suspended particle in stationary turbulent incompressible flow. Very similarly, the lattice counterpart (3) with jump rates satisfying (1) describe a random walk whose local drift is driven by a stationary source- and sink-free flow. The interest in the asymptotic description of this kind of displacement dates back to the discovery of turbulence. However,

divergence-free environments appear in many other natural contexts, too; see, for example, [10], Chapter 11, or a surprising recent application to group theory by Bartholdi and Erschler [2].

A phenomenological picture of these walks can be formulated in terms of randomly oriented cycles. Imagine that a translation invariant random "soup of cycles"—that is, a Poisson point process of oriented cycles—is placed on the lattice, and the walker is drifted along by these whirls. Now, local small cycles contribute to the diffusive behaviour. But occasionally very large cycles may cause on the long time scale faster-than-diffusive transport. Actually, this happens: in Komorowski and Olla [11] and Tóth and Valkó [28] anomalous *superdiffusive* behaviour is proved in particular cases when the $\mathscr{H}_{-1}$-bound (108) does not hold. Our result establishes that on the other hand, the $\mathscr{H}_{-1}$-bound (19) ensures not only boundedness of the diffusivity but also normal behaviour under diffusive scaling.

And now, to some examples:

(1) *Stationary and bounded stream field*: When there exists a *bounded* tensor valued variable $h : \Omega \to \mathbb{R}^{\mathscr{E}\times\mathscr{E}}$ with the symmetries (94) and such that (95) holds we define the multiplication operators $M_{k,l}$, $k, l \in \mathscr{E}$, acting on $f \in \mathscr{H}$:

$$M_{k,l} f(\omega) := h_{k,l}(\omega) f(\omega). \tag{109}$$

These are bounded self-adjoint operators and they inherit the symmetries of h [recall the shift operators T_k, $k \in \mathscr{E}$ from (30)]:

$$\begin{aligned} &M_{l,k} = T_k M_{-k,l} T_{-k} = T_l M_{k,-l} T_{-l} = -M_{k,l}, \\ &\sum_{l\in\mathscr{E}} M_{k,l} = M_k. \end{aligned} \tag{110}$$

As an alternative to (43), using (110), the skew-self-adjoint part of the infinitesimal generator is expressed as

$$A = \sum_{k,l\in\mathscr{E}} \nabla_{-k} M_{k,l} \nabla_l. \tag{111}$$

In [12] and [10], this form of the operator A is used. The operators $M_{k,l}$ are bounded and so is the operator

$$B := |\Delta|^{-1/2} A |\Delta|^{-1/2} = \sum_{k,l\in\mathscr{E}} \Gamma_{-k} M_{k,l} \Gamma_l \tag{112}$$

which plays a key role in our proof. Due to boundedness of B, the *strong sector condition* is valid in these cases and the central limit theorem for the displacement readily follows. See [12] and Section 3.3 of [10].

Finitely dependent constructions of this type appear in Kozlov [14]. The so-called *cyclic walks* analysed in [12] and in Section 3.3 of [10] are also of this nature.

When the tensor variables $h : \Omega \to \mathbb{R}^{\mathscr{E}\times\mathscr{E}}$ in (94) are in $\mathscr{L}^2 \setminus \mathscr{L}^\infty$, the multiplication operators $M_{k,l}$ defined in (109) are *unbounded*, the representation (111) of the skew-self-adjoint part of the infinitesimal generator and the operator B defined in (112) become just *formal*. Nevertheless, Theorem 1 in Oelschläger [19] and Theorem 3.6 in [10] are proved by controlling approximations of $h_{k,l}$ and the unbounded operators $M_{k,l}$ by truncations at high levels.

(2) *Stationary, square integrable but unbounded stream field*: We let, in arbitrary dimension d, $\Psi : \mathbb{Z}^d + (1/2, \dots, 1/2) \to \mathbb{Z}$ be a stationary, scalar, Lipschitz field with Lipschitz constant 1. As shown in Peled [23], such fields exist in sufficiently high dimension. Define $H : \mathbb{Z}^d \to \mathbb{R}^{\mathscr{E}_2\times\mathscr{E}_2}$ by

$$H_{e_i,e_j}(x) := \frac{1}{d}\Psi\big(x + (e_i + e_j)/2\big), \qquad x \in \mathbb{Z}^d, 1 \le i < j \le d,$$

and extend to $(H_{k,l}(x))_{k,l\in\mathscr{E}}$ by the symmetries (92). The tensor field $H : \mathbb{Z}^d \to \mathbb{R}^{\mathscr{E}_2\times\mathscr{E}_2}$ defined this way will be stationary and $\mathscr{L}^2$, but not necessary in $\mathscr{L}^\infty$—the uniform graph homomorphism of Peled [23], for example, is not bounded. Nevertheless, V is bounded by 1, as it should, since $|H_{k,l}(x) + H_{-k,l}(x)| = |H_{k,l}(x) - H_{k,l}(x-k)| \le \frac{1}{d}$ and V is a sum of d such terms.

(3) *Randomly oriented Manhattan lattice*: Let $u_i : \mathbb{Z}^{d-1} \to \{-1, +1\}$, $i = 1, \dots, d$, be translation invariant and ergodic, zero mean random fields, which are independent between them. Denote their covariances

$$c_i(y) := \mathbf{E}\big(u_i(0), u_i(y)\big), \qquad y \in \mathbb{Z}^{d-1},$$
$$\hat{c}_i(p) := \sum_{y\in\mathbb{Z}^{d-1}} e^{\sqrt{-1}p\cdot y} c_i(y), \qquad p \in [-\pi, \pi)^{d-1}.$$

Define now the lattice vector field

$$V_{\pm e_i}(x) := \pm u_i(x_1, \dots, x_{i-1}, \cancel{x_i}, x_{i+1}, \dots, x_d).$$

Then the random vector field V will satisfy all conditions in (83) and $t \mapsto X(t)$ will actually be a random walk on the lattice $\mathbb{Z}^d$ whose line-paths parallel to the coordinate axes are randomly oriented in a shift-invariant and ergodic way. This oriented graph is called the *randomly oriented Manhattan lattice*. The covariances C and $\widehat{C}$ defined in (16), respectively, (17) will be

$$C_{ij}(x) = \delta_{i,j} c_i(x_1, \dots, x_{i-1}, \cancel{x_i}, x_{i+1}, \dots, x_d),$$
$$\widehat{C}_{ij}(p) = \delta_{i,j}\delta(p_i)\hat{c}_i(p_1, \dots, p_{i-1}, \cancel{p_i}, p_{i+1}, \dots, p_d).$$

The $\mathscr{H}_{-1}$-condition (19) is in this case

$$\sum_{i=1}^{d} \int_{[-\pi,\pi]^{d-1}} \widehat{D}(q)^{-1}\hat{c}_i(q)\, dq < \infty. \tag{113}$$

In the particular case when the random variables $u_i(y)$, $i \in \{1, \dots, d\}$, $y \in \mathbb{Z}^{d-1}$, are *independent fair coin-tosses*, $\hat{c}_i(q) \equiv 1$. In this case, for $d = 2, 3$ the $\mathscr{H}_{-1}$-condition (113) fails to hold, the tensor field H is *genuinely* of stationary increments. In these cases, super-diffusivity of the walk $t \mapsto X(t)$ can be proved with the method of Tarrès, Tóth and Valkó [26] (in the $2d$ case), respectively, of Tóth and Valkó [28] (in the $3d$ case). In dimensions $d \geq 4$, the $\mathscr{H}_{-1}$-condition (113) [and thus (19)] holds and the central limit theorem for the displacement follows from our Theorem 1.

APPENDIX: PROOF OF THEOREM RSC1 AND THEOREM RSC2

PROOF OF THEOREM RSC1. Since the operators C_λ, $\lambda > 0$, defined in (52) are a priori and the operator C is by assumption skew-self-adjoint, we can define the following bounded operators (actually contractions):

$$\begin{aligned} &K_\lambda := (I - C_\lambda)^{-1}, \qquad \|K_\lambda\| \leq 1, \qquad \lambda > 0, \\ &K := (I - C)^{-1}, \qquad \|K\| \leq 1. \end{aligned} \tag{114}$$

Hence, we can write the resolvent $R_\lambda = (\lambda I - L)^{-1}$ (50) as

$$R_\lambda = (\lambda + S)^{-1/2} K_\lambda (\lambda + S)^{-1/2}. \tag{115}$$

LEMMA 6. *Assume that the sequence of bounded operators K_λ converges to K in the strong operator topology*:

$$K_\lambda \overset{\text{st.op.top.}}{\longrightarrow} K \qquad \text{as } \lambda \to 0. \tag{116}$$

Then for any $f \in \mathrm{Dom}(S^{-1/2}) = \mathrm{Ran}(S^{1/2})$, the limits in (51) hold.

PROOF. From the spectral theorem applied to the positive operator S, it is obvious that, as $\lambda \to 0+$,

$$\begin{aligned} &\|\lambda^{1/2}(\lambda + S)^{-1/2}\| \leq 1, \qquad \lambda^{1/2}(\lambda + S)^{-1/2} \overset{\text{st.op.top.}}{\longrightarrow} 0, \\ &\|S^{1/2}(\lambda + S)^{-1/2}\| \leq 1, \qquad S^{1/2}(\lambda + S)^{-1/2} \overset{\text{st.op.top.}}{\longrightarrow} I. \end{aligned} \tag{117}$$

We can write $f = S^{1/2} g$ with $g \in \mathscr{H}$. Now, using (115), we get

$$\begin{aligned} \lambda^{1/2} u_\lambda &= \lambda^{1/2}(\lambda + S)^{-1/2} K_\lambda (\lambda + S)^{-1/2} S^{1/2} g, \\ S^{1/2} u_\lambda &= S^{1/2}(\lambda + S)^{-1/2} K_\lambda (\lambda + S)^{-1/2} S^{1/2} g. \end{aligned}$$

We get

$$\begin{aligned} S^{1/2}u_\lambda &= S^{1/2}(\lambda+S)^{1/2}K_\lambda(\lambda+S)^{-1/2}S^{1/2}g \\ &\overset{(117)}{=} S^{1/2}(\lambda+S)^{-1/2}K_\lambda\big(g+o(1)\big), \\ \text{By (116,114)} &= S^{1/2}(\lambda+S)^{-1/2}\big(Kg+o(1)\big) \\ &\overset{(117)}{=} Kg+o(1), \end{aligned}$$

where the notation $o(1)$ is for convergence in norm as $\lambda \to 0$. Verifying the other condition of (51) is similar. □

In the next lemma, we formulate a sufficient condition for (116) to hold.

LEMMA 7. *Let C_n, $n \in \mathbb{N}$, and $C = C_\infty$ be densely defined closed (possibly unbounded) operators over the Hilbert space $\mathscr{H}$. Let also $\mathscr{C}_n$ and $\mathscr{C}$ be a cores of definition of the operators C_n and C, respectively. Assume that some (fixed) $\mu \in \mathbb{C}$ is in the intersection of the resolvent set of all operators C_n, $n \le \infty$, and*

$$\sup_{1\le n\le\infty} \big\|(\mu I - C_n)^{-1}\big\| < \infty, \tag{118}$$

and for any $h \in \mathscr{C}$ there exists a sequence $h_n \in \mathscr{C}_n$ such that the following limits hold:

$$\lim_{n\to\infty} \|h_n - h\| = 0 \quad \textit{and} \quad \lim_{n\to\infty} \|C_n h_n - Ch\| = 0. \tag{119}$$

Then (i) *and* (ii) *below hold*:

(i)

$$(\mu I - C_n)^{-1} \overset{\text{st.op.top.}}{\longrightarrow} (\mu I - C)^{-1}. \tag{120}$$

(ii) *The sequence of operators C_n converges* in the strong graph limit *sense to C.*

PROOF. (i) Since $\mathscr{C}$ is a core for the densely defined closed operator C and μ is in the resolvent set of C, the subspace $\widehat{\mathscr{C}} := \{\widehat{h} = (\mu I - C)h : h \in \mathscr{C}\}$ is dense in $\mathscr{H}$. For $\widehat{h} \in \widehat{\mathscr{C}}$, let $h := (\mu I - C)^{-1}\widehat{h} \in \mathscr{C}$ and choose a sequence $h_n \in \mathscr{C}_n$ for which (119) holds. Then

$$\begin{aligned} &(\mu I - C_n)^{-1}\widehat{h} - (\mu I - C)^{-1}\widehat{h} \\ &\quad = \big(\mu(\mu I - C_n)^{-1} - I\big)(h - h_n) + (\mu I - C_n)^{-1}(C_n h_n - Ch), \end{aligned}$$

and hence

$$\begin{aligned} &\big\|(\mu I - C_n)^{-1}\widehat{h} - (\mu I - C)^{-1}\widehat{h}\big\| \\ &\quad \le \big(|\mu|\big\|(\mu I - C_n)^{-1}\big\| + 1\big)\|h - h_n\| + \big\|\mu I - C_n)^{-1}\big\|\|C_n h_n - Ch\| \to 0 \end{aligned}$$

due to (118) and (119). Since this is valid on the *dense* subspace $\widehat{\mathscr{C}} \subset \mathscr{H}$, using again (118), we conclude (120).

(ii) The proof of the "if" part of Theorem VIII. 26 in [24] can be transposed without any essential alteration. □

To complete the proof of Theorem RSC1, first apply Lemma 7(i) to C_λ, $\lambda \to 0+$, defined in (52), C assumed (essentially) skew-self-adjoint, and $\mu = 1$. Note that $\mu = 1$ is indeed in the resolvent set of all these operators and, indeed $\sup_{\lambda>0} \|(I - C_\lambda)^{-1}\| < \infty$ and $\|(I - C)^{-1}\| < \infty$, as required in (118), since the operators C_λ are bounded and skew- self-adjoint and the operator C is assumed to be essentially skew-self-adjoint. From Lemma 7(i), it follows that that (116) holds. Finally, quoting Lemma 6 we complete the proof of Theorem RSC1. □

PROOF OF THEOREM RSC2. From $0 \le T \le cD$ (54), it follows that

$$0 \le D \le S \le (1+c)D \tag{121}$$

Let

$$V_\lambda := (\lambda I + D)^{1/2}(\lambda I + S)^{-1/2}, \qquad V = V_0 := D^{1/2}S^{-1/2}.$$

The operator V is a priori defined on $\mathrm{Dom}(S^{-1/2}) = \mathrm{Ran}(S^{1/2})$, but as we see next, it extends by continuity to a bounded and invertible linear operator defined on the whole space $\mathscr{H}$. Due to (121) the following bounds hold uniformly for $\lambda \ge 0$:

$$\|V_\lambda\| = \|V_\lambda^*\| \le 1, \qquad \|V_\lambda^{-1}\| = \|(V_\lambda^{-1})^*\| \le \sqrt{1+c}.$$

Let us show that bound on $\|V_\lambda\|$, the bound on $\|V_\lambda^{-1}\|$ is similar. We write

$$\begin{aligned}
\|V_\lambda\varphi\|^2 &= \big\langle (\lambda I + D)^{1/2}(\lambda I + S)^{-1/2}\varphi, (\lambda I + D)^{1/2}(\lambda I + S)^{-1/2}\varphi\big\rangle \\
&= \big\langle (\lambda I + S)^{-1/2}\varphi, (\lambda I + D)(\lambda I + S)^{-1/2}\varphi\big\rangle \\
&\le \big\langle (\lambda I + S)^{-1/2}\varphi, (\lambda I + S)(\lambda I + S)^{-1/2}\varphi\big\rangle = \|\varphi\|^2.
\end{aligned}$$

From here, first of all, it follows that

$$\mathrm{Dom}\big(S^{-1/2}\big) = \mathrm{Dom}\big(D^{-1/2}\big),$$

and thus the $\mathscr{H}_{-1}$-conditions $f \in \mathrm{Dom}(S^{-1/2})$, respectively, $f \in \mathrm{Dom}(D^{-1/2})$ in Theorem RSC1, respectively, Theorem RSC2, are actually the same. It is also easy to see that for any $\varphi \in \mathscr{H}$

$$\lim_{\lambda\to 0} V_\lambda\varphi = V\varphi \quad \text{and} \quad \lim_{\lambda\to 0} V_\lambda^{-1}\varphi = V^{-1}\varphi.$$

That is, $V_\lambda \overset{\text{st.op.top.}}{\longrightarrow} V$ and $V_\lambda^{-1} \overset{\text{st.op.top.}}{\longrightarrow} V^{-1}$, as $\lambda \to 0$, where $\overset{\text{st.op.top.}}{\longrightarrow}$ stands for convergence in the strong operator topology.

Next, write the operators C_λ and C from Theorem RSC1, as

$$C_\lambda = V_\lambda^* B_\lambda V_\lambda, \qquad C = V^* B V.$$

Now, from the fact that V_λ and V_λ^{-1} are all bounded, uniformly in $\lambda \geq 0$, it readily follows that: (a) one can use $\mathscr{C} = V^{-1}\mathscr{B}$ as a core for the operator C; (b) C is essentially skew-self-adjoint on $\mathscr{C}$ if so was B on $\mathscr{B}$; and (c) the limit (53) follows from (56) by straightforward manipulations. Indeed, for $\psi \in \mathscr{C}$ define $\varphi := V\psi \in \mathscr{B}$ and let $\varphi_\lambda \in \mathscr{H}$ be such that the limits in (56) hold. Define $\psi_\lambda := V_\lambda^{-1}\varphi_\lambda$. Then the limits in (53) clearly hold:

$$\begin{aligned}
\|\psi_\lambda - \psi\| &= \|V_\lambda^{-1}\varphi_\lambda - V^{-1}\varphi\| \\
&\leq \|V_\lambda^{-1}\|\|\varphi_\lambda - \varphi\| + \|V_\lambda^{-1}\varphi - V^{-1}\varphi\| \to 0, \\
\|C_\lambda\psi_\lambda - C\psi\| &= \|V_\lambda^* B_\lambda \varphi_\lambda - V^* B\varphi\| \\
&\leq \|V_\lambda^*\|\|B_\lambda\varphi_\lambda - B\varphi\| + \|V_\lambda^* B\varphi - V^* B\varphi\| \to 0. \qquad \square
\end{aligned}$$

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REFERENCES

[1] ARVESON, W. (1976). *An Invitation to C^*-Algebras*. *Graduate Texts in Mathematics* **39**. Springer, New York. MR0512360

[2] BARTHOLDI, L. and ERSCHLER, A. Poisson-Fürstenberg boundary and growth of groups. Preprint. Available at arXiv:1107.5499.

[3] BERGER, N. and BISKUP, M. (2007). Quenched invariance principle for simple random walk on percolation clusters. *Probab. Theory Related Fields* **137** 83–120. MR2278453

[4] BISKUP, M. (2011). Recent progress on the random conductance model. *Probab. Surv.* **8** 294–373.

[5] CARLEN, E. (2010). Trace inequalities and quantum entropy: An introductory course. In *Entropy and the Quantum* (R. Sims and D. Ueltschi, eds.). *Contemp. Math.* **529** 73–140. Amer. Math. Soc., Providence, RI.

[6] CHAVEL, I. (2001). *Isoperimetric Inequalities. Differential Geometric and Analytic Perspectives*. *Cambridge Tracts in Mathematics* **145**. Cambridge Univ. Press, Cambridge.

[7] HELLAND, I. S. (1982). Central limit theorems for martingales with discrete or continuous time. *Scand. J. Stat.* **9** 79–94. MR0668684

[8] HORVÁTH, I., TÓTH, B. and VETŐ, B. (2012). Relaxed sector condition. *Bull. Inst. Math. Acad. Sin. (N.S.)* **7** 463–476. MR3077467

[9] KIPNIS, C. and VARADHAN, S. R. S. (1986). Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions. *Comm. Math. Phys.* **104** 1–19. MR0834478

[10] KOMOROWSKI, T., LANDIM, C. and OLLA, S. (2012). *Fluctuations in Markov Processes—Time Symmetry and Martingale Approximation*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **345**. Springer, Heidelberg. MR2952852

[11] KOMOROWSKI, T. and OLLA, S. (2002). On the superdiffusive behavior of passive tracer with a Gaussian drift. *J. Stat. Phys.* **108** 647–668. MR1914190

[12] KOMOROWSKI, T. and OLLA, S. (2003). A note on the central limit theorem for two-fold stochastic random walks in a random environment. *Bull. Pol. Acad. Sci. Math.* **51** 217–232. MR1990811

[13] KOMOROWSKI, T. and OLLA, S. (2003). On the sector condition and homogenization of diffusions with a Gaussian drift. *J. Funct. Anal.* **197** 179–211.

[14] KOZLOV, S. M. (1985). The averaging method and walks in inhomogeneous environments. *Uspekhi Mat. Nauk* **40** 61–120, 238. English version: *Russian Math. Surveys* **40** 73–145 (1985). MR0786087

[15] KOZMA, G. and TÓTH, B. Central limit theorem for random walks in divergence-free random drift field: $\mathscr{H}_{-1}$ suffices. First arXiv version. Available at arXiv:1411.4171v1.

[16] KUMAGAI, T. (2014). *Random Walks on Disordered Media and Their Scaling Limits*. *Lecture Notes in Math.* **2101**. Springer, New York. MR3156983

[17] LÖWNER, K. (1934). Über monotone Matrixfunktionen. *Math. Z.* **38** 177–216. MR1545446

[18] MORRIS, B. and PERES, Y. (2005). Evolving sets, mixing and heat kernel bounds. *Probab. Theory Related Fields* **133** 245–266.

[19] OELSCHLÄGER, K. (1988). Homogenization of a diffusion process in a divergence-free random field. *Ann. Probab.* **16** 1084–1126. MR0942757

[20] OLLA, S. (2001). Central limit theorems for tagged particles and for diffusions in random environment. In *Milieux Aléatoires* (F. Comets and É. Pardoux, eds.). *Panor. Synthèses* **12** 75–100. Soc. Math. France, Paris. MR2226846

[21] OSADA, H. (1983). Homogenization of diffusion processes with random stationary coefficients. In *Probability Theory and Mathematical Statistics* (*Tbilisi*, 1982). *Lecture Notes in Math.* **1021** 507–517. Springer, Berlin. MR0736016

[22] PAPANICOLAOU, G. C. and VARADHAN, S. R. S. (1981). Boundary value problems with rapidly oscillating random coefficients. In *Random Fields*, *Vol. I, II* (*Esztergom*, 1979) (J. Fritz, D. Szász and J. L. Lebowitz, eds.). *Colloquia Mathematica Societatis János Bolyai* **27** 835–873. North-Holland, Amsterdam. MR0712714

[23] PELED, R. (2017). High-dimensional Lipschitz functions are typically flat. *Ann. Probab.* **45** 1351–1447. MR3650405

[24] REED, M. and SIMON, B. (1975). *Methods of Modern Mathematical Physics*. *Vols*. 1, 2. Academic Press, New York.

[25] SETHURAMAN, S., VARADHAN, S. R. S. and YAU, H.-T. (2000). Diffusive limit of a tagged particle in asymmetric simple exclusion processes. *Comm. Pure Appl. Math.* **53** 972–1006. MR1755948

[26] TARRÈS, P., TÓTH, B. and VALKÓ, B. (2012). Diffusivity bounds for 1d Brownian polymers. *Ann. Probab.* **40** 695–713.

[27] TÓTH, B. (1986). Persistent random walks in random environment. *Probab. Theory Related Fields* **71** 615–625. MR0833271

[28] TÓTH, B. and VALKÓ, B. (2012). Superdiffusive bounds on self-repellent Brownian polymers and diffusion in the curl of the Gaussian free field in $d = 2$. *J. Stat. Phys.* **147** 113–131.

[29] VARADHAN, S. R. S. (1995). Self-diffusion of a tagged particle in equilibrium for asymmetric mean zero random walk with simple exclusion. *Ann. Inst. Henri Poincaré Probab. Stat.* **31** 273–285. MR1340041

[30] ZEITOUNI, O. (2004). Lecture notes on random walks in random environment. In *Lectures on Probability Theory and Statistics—Saint-Flour* 2001 (J. Picard, ed.). *Lecture Notes in Math*. **1837**. Springer, Berlin.

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