

STOCHASTIC INTEGRATION WITH RESPECT TO CYLINDRICAL LÉVY PROCESSES

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A cylindrical Lévy process does not enjoy a cylindrical version of the semimartingale decomposition which results in the need to develop a completely novel approach to stochastic integration. In this work, we introduce a stochastic integral for random integrands with respect to cylindrical Lévy processes in Hilbert spaces. The space of admissible integrands consists of càglàd, adapted stochastic processes with values in the space of Hilbert–Schmidt operators. Neither the integrands nor the integrator is required to satisfy any moment or boundedness condition. The integral process is characterised as an adapted, Hilbert space valued semimartingale with càdlàg trajectories.

1. Introduction. Cylindrical Brownian motion is the most prominent model of the driving noise for stochastic partial differential equations. The attribute *cylindrical* refers here to the fact that cylindrical Brownian motions are not classical stochastic processes attaining values in the underlying space but are generalised objects whose probabilistic distributions are described by a cylindrical, that is, a finitely additive, measure. The reasons for the choice of cylindrical but not classical Brownian motion can be found in the facts that there does not exist a classical Brownian motion with independent components, that is, a standard Brownian motion, in an infinite dimensional Hilbert space, and that cylindrical processes enable a very flexible modelling of random noise in time and space.

The concept of cylindrical Brownian motion is naturally extended to *cylindrical Lévy processes* in one of the authors' work [1] with Applebaum. Some specific examples and their constructions of cylindrical Lévy processes are presented in the work [26] by Riedle. Linear and semi-linear stochastic partial differential equations perturbed by an additive noise which is modelled by various but specific examples of cylindrical Lévy processes can be found, for example, in the works Brzeźniak and Zabczyk [3], Peszat and Zabczyk [21], and Priola and Zabczyk [22]. However, modelling an arbitrary perturbation of a general stochastic partial differential equations beyond the purely additive case requires a theory of stochastic integration of random integrands with respect to cylindrical Lévy processes.

Received December 2015; revised October 2016.

¹Supported by the Polish NCN Grant 2012/07/B/ST1/03508.

²Supported by the EPSRC Grant EP/I036990/1.

MSC2010 subject classifications. Primary 60H05; secondary 60B11, 60G20, 28C20.

Key words and phrases. Cylindrical Lévy processes, stochastic integration, decoupled tangent sequence, cylindrical Brownian motion, random measures.

Stochastic integration with respect to cylindrical Brownian processes is developed, for example, in Daletskij [4], followed by the articles Gaveau [6], Lepingle and Ouvrard [15] and many others. Surprisingly, stochastic integration with respect to other cylindrical processes than cylindrical Brownian motion is much less considered. In fact, only with respect to cylindrical martingales a stochastic integration theory is developed which originates either from an approach by Métivier and Pellaumail in [16] and [17] or from Mikulevičius and Rozovskiĭ in [18] and [19]. The construction by Métivier and Pellaumail is based on Doléans measures whereas the construction by Mikulevičius and Rozovskiĭ uses a family of reproducing kernel Hilbert spaces. Thus, both constructions heavily rely on the assumed existence of finite weak second moments. In Métivier and Pellaumail [16], the construction is extended to cylindrical local martingales. For the special case of a cylindrical Lévy process with finite weak second moments, one can follow a classical Itô approach to define the stochastic integral for random integrands; see Riedle [25]. Another approach to stochastic integration in infinite dimensional spaces is introduced in [13] by Kurtz and Protter for a large class of integrators, so-called $\#$ -semimartingales, under the additional assumption that they are “good integrators”. Although cylindrical Lévy processes belong to the class of $\#$ -semimartingales, it is not clear if they are good integrators in the sense of [13].

The classical approach to stochastic integration with respect to genuine Lévy processes or semimartingales exploits their decomposition into a (local) martingale and a process with trajectories of bounded variation. Alternatively, one can develop an integration theory by starting with “good integrators”. In this case, one concludes from the Bichteler–Dellacherie theorem that good integrators are semimartingales. Vice versa, semimartingales are verified as good integrators by exploiting their decomposition; see, for example, the monograph [23] by Protter. However, any approach based on a semimartingale decomposition fails for cylindrical Lévy processes although they are in the class of *cylindrical semimartingales*. This is due to the conceptual mismatch that a cylindrical semimartingale cannot be decomposed into the sum of a cylindrical local martingale and another cylindrical process; see Remark 2.2. Consequently, our work requires a novel approach to stochastic integration without decomposing the integrator.

To explain our approach in more detail, let $(Y(t) : t \in [0, T])$ be a classical Lévy process in a Hilbert space U with inner product $\langle \cdot, \cdot \rangle$. A simple integrand $(\Psi(t) : t \in [0, T])$ is of the form $\Psi = \mathbb{1}_{(a,b]} \otimes \Phi$ where $0 \leq a \leq b \leq T$ and Φ is a random variable with values in the space of Hilbert–Schmidt operators from U to another Hilbert space V . Each sensible definition of stochastic integration leads to

$$(1) \quad \left\langle \int_0^T \Psi(s) dY(s), v \right\rangle = \langle \Phi(Y(b) - Y(a)), v \rangle = \langle Y(b) - Y(a), \Phi^* v \rangle$$

for every $v \in V$, where Φ^* denotes the adjoint operator. A cylindrical process, like the cylindrical Lévy process, is a family $(L(t) : t \in [0, T])$ of linear and bounded operators $L(t)$ from U to the space of equivalence classes of real valued random

variables. If we substitute Y by the cylindrical Lévy process L in (1), then the inner product on the right-hand side in (1) corresponds to the application of the linear operator $L(b) - L(a)$ to the other argument of the inner product such that we arrive at

$$(2) \quad \left\langle \int_0^T \Psi(s) dL(s), v \right\rangle = (L(b) - L(a))(\Phi^* v).$$

However, a technical and a conceptual problem arise in (2):

(1) the linear operator $L(b) - L(a)$, mapping to the space of equivalence classes of random variables, is applied to a random argument which results in an ambiguity;

(2) in order to obtain a V -valued stochastic integral such as $\Phi(Y(b) - Y(a))$ in (1), there must exist a V -valued random variable J satisfying

$$(L(b) - L(a))(\Phi^* v) = \langle J, v \rangle \quad \text{for all } v \in V.$$

We call the approach for solving the Problems (1) and (2) the *radonification of the increments* and present it in Section 4.

However, a much more complicated problem is to extend the class of admissible integrands to a larger space rather than only simple integrands. Denote by $\mathcal{H}_0(U, V)$ the space of linear combinations of simple integrands of the form as Ψ above. Then, by means of the radonification of the increments one can define an integral operator

$$(3) \quad I: \mathcal{H}_0(U, V) \rightarrow L_p^0(\Omega; V),$$

where $L_p^0(\Omega; V)$ denotes the space of equivalence classes of V -valued random variables. In the classical setting, the integrator Y is decomposed into a martingale M and a bounded variation process A , resulting in integral operators I_A and I_M with $I = I_A + I_M$. It is straightforward to extend the domain of the integral operator I_A . The integral operator I_M turns out to map to the Hilbert space $L_p^2(\Omega; V)$ of equivalence classes of V -valued random variables with finite second moments. Martingale properties and the nice Hilbert space topology of $L_p^2(\Omega; V)$ allow to conclude the continuity of I_M , and thus to extend its domain. However, as mentioned above, the cylindrical Lévy process L does not enjoy an analogous decomposition, and thus we must work with the integrator operator (3) *in a single entity* to solve:

(3) if a sequence $(\Psi_n)_{n \in \mathbb{N}}$ of simple processes in $\mathcal{H}_0(U, V)$ converges to a stochastic process Ψ in a larger space in some sense then $I(\Psi_n)$ converges to a random variable in $L_p^0(\Omega; V)$.

Dealing with problem (3) means in particular that, instead of exploiting the Itô isomorphism to the Hilbert space $L_p^2(\Omega; V)$, one must establish convergence in the much less amenable topology in $L_p^0(\Omega; V)$, that is, convergence in probability.

We solve Problem (3) in Section 5. Here, the main step is establishing tightness of the set $\{I(\Psi_n) : n \in \mathbb{N}\}$ of Hilbert space valued random variables. We prove tightness of this set by exploiting the result that tightness of the sum of the *decoupled tangent sequence* implies tightness of the original sum. This result originates from one of the authors' work [8], and we will introduce and prove a modified version of this result in Section 3. Although this result was originally introduced with a completely different aim, it seems to be tailor-made for the considerations of our current work.

Concerning the class of admissible integrands, in this work we restrict ourselves to the space of càglàd, adapted stochastic processes with values in the space of Hilbert–Schmidt operators. In finite dimensions, introducing stochastic integration for the larger class of predictable integrands can be achieved in two steps: in the first step, the stochastic integral is introduced for integrands with càglàd paths, and in the second step, the definition is extended to predictable integrands; see, for example, the monograph [23] by Protter. In this sense, our work can be seen as completing the first step for stochastic integrals with respect to cylindrical Lévy processes. In finite dimensions, the second step is typically achieved by arguments based on the semimartingale decomposition of the integrator. However, as cylindrical Lévy processes do not enjoy a corresponding decomposition, enlarging the space of admissible integrands to predictable processes will require other novel tools different from those in this work.

2. Preliminaries. Let U and V be separable Hilbert spaces with inner products $\langle \cdot, \cdot \rangle$ and corresponding norms $\|\cdot\|$. The dual spaces are identified by the original Hilbert spaces. The unit ball is denoted by $B_V := \{v \in V : \|v\| \leq 1\}$. Throughout the paper, $\{e_k\}_{k \in \mathbb{N}}$ and $\{f_k\}_{k \in \mathbb{N}}$ denote some orthonormal basis of U and V , respectively.

The space of linear and bounded operators is denoted by $\mathcal{L}(U, V)$ and it is equipped with the operator norm $\|\cdot\|_{U \rightarrow V}$. The subspace of *Hilbert–Schmidt operators* is denoted by $\mathcal{L}_2(U, V)$ and is equipped with the norm

$$\|\varphi\|_{\mathcal{L}_2}^2 := \sum_{k=1}^{\infty} \|\varphi e_k\|^2.$$

A simple argument using the standard characterisation of compact sets in Hilbert spaces show that a set $K \subseteq \mathcal{L}_2(U, V)$ is compact if and only if it is bounded, closed and obeys

$$(4) \quad \lim_{N \rightarrow \infty} \sup_{\varphi \in K} \sum_{k=N+1}^{\infty} \|\varphi e_k\|^2 = 0.$$

The space of $\mathcal{L}_2(U, V)$ -valued càglàd (continue à gauche, limite à droite) functions is denoted by

$$D_-([0, T]; \mathcal{L}_2(U, V)) \\ := \{\psi : [0, T] \rightarrow \mathcal{L}_2(U, V) : \text{left-continuous with right-limits}\}.$$

This space can be metrizable via the Skorokhod metric

$$(5) \quad d_J(\varphi, \psi) := \inf_{j \in \Lambda} \left(\sup_{t \in [0, T]} \|\varphi(t) - \psi \circ j(t)\|_{\mathcal{L}_2} \vee \sup_{t \in [0, T]} |t - j(t)| \right),$$

where the infimum is over the set Λ of all strictly increasing, continuous bijections $j: [0, T] \rightarrow [0, T]$.

The Borel σ -algebra in U is denoted by $\mathcal{B}(U)$ and the space of Borel measures on $\mathcal{B}(U)$ is denoted by $\mathcal{M}(U)$. The space of Borel probability measures is denoted by $\mathcal{M}_1(U)$ and it is equipped with the Prokhorov metric

$$d_P(\mu, \nu) := \inf\{\varepsilon > 0 : \mu(B) \leq \nu(B_\varepsilon) + \varepsilon \text{ and } \nu(B) \leq \mu(B_\varepsilon) + \varepsilon \\ \text{for closed } B \in \mathcal{B}(U)\},$$

where $B_\varepsilon := \{u \in U : \inf\{\|u - b\| : b \in B\} < \varepsilon\}$. Convergence in the Prokhorov metric is equivalent to weak convergence of probability measures.

Let (Ω, \mathcal{A}, P) be a probability space. The space of equivalence classes of measurable functions $X: \Omega \rightarrow U$ is denoted by $L^0_P(\Omega; U)$. If \mathcal{G} is a sub- σ -algebra of \mathcal{A} , we write $L^0_P(\Omega, \mathcal{G}; U)$ for the space of equivalence classes of \mathcal{G} -measurable functions. By defining the function

$$(6) \quad p: L^0_P(\Omega; U) \rightarrow [0, 1], \quad p(X) = E[1 \wedge \|X\|^2],$$

the space $L^0_P(\Omega; U)$ becomes an F -space under the metric $d(X, Y) := p(X - Y)$.

Let S be a subset of U . For every elements $u_1, \dots, u_n \in S, n \in \mathbb{N}$ and $B \in \mathcal{B}(\mathbb{R}^n)$ define

$$C(u_1, \dots, u_n; B) := \{u \in U : (\langle u, u_1 \rangle, \dots, \langle u, u_n \rangle) \in B\}.$$

These sets are called *cylindrical sets with respect to S* and they form an algebra $\mathcal{Z}(U, S)$. The generated σ -algebra is denoted by $\hat{\mathcal{Z}}(U, S)$ and it is called the *cylindrical σ -algebra with respect to S* . If $S = U$, we write $\mathcal{Z}(U) := \mathcal{Z}(U, S)$ and $\hat{\mathcal{Z}}(U) := \hat{\mathcal{Z}}(U, S)$.

A function $\eta: \mathcal{Z}(U) \rightarrow [0, \infty]$ is called a *cylindrical measure on $\mathcal{Z}(U)$* if for each finite subset $S \subseteq U$ the restriction of η to the σ -algebra $\hat{\mathcal{Z}}(U, S)$ is a measure. A cylindrical measure η is called finite if $\eta(U) < \infty$ and a cylindrical probability measure if $\eta(U) = 1$. The characteristic function of a finite cylindrical measure η is defined by

$$\chi_\eta: U \rightarrow \mathbb{C}, \quad \chi_\eta(u) = \int_U e^{i\langle u, h \rangle} \eta(dh).$$

Note that this integral is well defined as the integrand is measurable with respect to $\hat{\mathcal{Z}}(U, \{u\})$ for each $u \in U$. The cylindrical measure η is called *continuous* if χ_η is continuous.

A *cylindrical random variable in U* is a linear and continuous mapping

$$Z: U \rightarrow L^0_P(\Omega; \mathbb{R}).$$

If $C = C(u_1, \dots, u_n; B)$ is a cylindrical set for $u_1, \dots, u_n \in U$ and $B \in \mathcal{B}(\mathbb{R}^n)$, we obtain a cylindrical probability measure η by the definition

$$\eta(C) := P((Zu_1, \dots, Zu_n) \in B).$$

The mapping η is called the *cylindrical distribution of Z* . The characteristic function of a cylindrical random variable Z is defined by

$$\varphi_Z : U \rightarrow \mathbb{C}, \quad \varphi_Z(u) = E[\exp(iZu)],$$

and the characteristic function of Z and its cylindrical distribution η coincide. For a function $\varphi \in \mathcal{L}(U, V)$, one can define a cylindrical random variable in V by

$$Z_\varphi : V \rightarrow L^0_P(\Omega; \mathbb{R}), \quad Z_\varphi v = Z(\varphi^* v).$$

In general, Z_φ is only a cylindrical random variable but if φ is a Hilbert–Schmidt operator then there exists a V -valued random variable $\varphi(Z) : \Omega \rightarrow V$ satisfying

$$(7) \quad Z(\varphi^* v) = \langle \varphi(Z), v \rangle \quad \text{for all } v \in V;$$

see [27], Theorem VI.5.2.

A family $(Z(t) : t \geq 0)$ of cylindrical random variables $Z(t)$ in U is called a *cylindrical process in U* . In our work [1], we extended the concept of cylindrical Brownian motion to cylindrical Lévy processes.

DEFINITION 2.1. A cylindrical process $(L(t) : t \geq 0)$ in U is called a *cylindrical Lévy process* if for each $n \in \mathbb{N}$ and any $u_1, \dots, u_n \in U$ we have that

$$((L(t)u_1, \dots, L(t)u_n) : t \geq 0)$$

is a Lévy process in \mathbb{R}^n .

The characteristic function of $L(t)$ is studied in detail in our work [24]. It turns out that the characteristic function of $L(t)$ for each $t \geq 0$ is of the form

$$\varphi_{L(t)} : U \rightarrow \mathbb{C}, \quad \varphi_{L(t)}(u) = \exp(tS(u)),$$

where $S : U \rightarrow \mathbb{C}$ is called the *cylindrical symbol of L* and is of the form

$$(8) \quad S(u) = ia(u) - \frac{1}{2}\langle Qu, u \rangle + \int_U (e^{i\langle u, h \rangle} - 1 - i\langle u, h \rangle \mathbb{1}_{B_{\mathbb{R}}(\langle u, h \rangle)}) \nu(dh).$$

Here, $a : U \rightarrow \mathbb{R}$ is a continuous mapping with $a(0) = 0$, $Q : U \rightarrow \mathbb{R}$ is a positive and symmetric operator and ν is a cylindrical measure on $\mathcal{Z}(U)$ satisfying

$$\int_U (\langle u, h \rangle^2 \wedge 1) \nu(dh) < \infty \quad \text{for all } u \in U.$$

Since $L(t) : U \rightarrow L^0_P(\Omega; \mathbb{R})$ is continuous, it follows that the characteristic function $\varphi_{L(1)} : U \rightarrow \mathbb{C}$ is continuous, and thus the symbol $S : U \rightarrow \mathbb{C}$ is continuous. According to Lemma 3.2 in [26], the cylindrical symbol S maps bounded sets to bounded sets.

REMARK 2.2. It follows from the Lévy–Itô decomposition in \mathbb{R} that for each $u \in U$ and $t \geq 0$ a cylindrical Lévy process L with Lévy symbol (8) can be decomposed into

$$L(t)u = a(u)t + W(t)u + \int_{|\beta| \leq 1} \beta \tilde{N}_u(t, d\beta) + \int_{|\beta| > 1} \beta N_u(t, d\beta),$$

where W is a cylindrical Wiener process in U with covariance operator Q and

$$N_u(t, B) := \sum_{0 \leq s \leq t} \mathbb{1}_B((L(s)u - L(s-))u) \quad \text{for all } t \in [0, T], B \in \mathcal{B}(\mathbb{R} \setminus \{0\}),$$

and \tilde{N}_u is the compensated Poisson random measure defined by $\tilde{N}_u(t, B) := N_u(t, B) - t(v \circ \langle \cdot, u \rangle^{-1})(B)$; see [1], Th. 3.9. If L does not have finite weak second moments, that is, $E[|L(1)u|^2] = \infty$, then the (local) martingale part in the semimartingale decomposition of $(L(t)u : t \in [0, T])$ is given by the sum $W(t)u + R(t)(u)$ where

$$R(t)(u) := \int_{|\beta| \leq 1} \beta \tilde{N}_u(t, d\beta).$$

As the truncation function $\beta \mapsto \mathbb{1}_B(\beta)$ is not linear, the mapping $u \mapsto R(t)(u)$ is not linear neither. Thus, $(L(t)u : t \in [0, T])$ enjoys a semimartingale decomposition for fixed $u \in U$, but the martingale and bounded variation parts are not linear in u in general.

We equip the probability space (Ω, \mathcal{A}, P) with the filtration generated by L and defined by

$$\mathcal{F}_t := \sigma(\{L(s)u : u \in U, s \in [0, t]\}) \quad \text{for all } t \geq 0.$$

For a filtration $\mathcal{G} := \{\mathcal{G}_t\}_{t \in I}$ where $I \subseteq [0, \infty)$ is an arbitrary index set, we define

$$\Upsilon(\mathcal{G}) := \{\tau : \Omega \rightarrow I : \text{is stopping time for } \mathcal{G}\}.$$

3. Tightness by decoupling. In the later part of this work, the main argument on extending the definition of the stochastic integral from simple integrands to a much larger class of integrands is based on establishing tightness of the set of stochastic integrals for a sequence of simple integrands. This will be established by the following result which provides a handy criterion for the tightness of a set of sums of random variables in a Hilbert space. The theorem is a modification of a result by Jakubowski in [8], and which is also published in a more general setting in [11].

THEOREM 3.1. For each $n \in \mathbb{N}$, let $\{X_{n,k} : k \in \mathbb{N}\}$ be a sequence of V -valued random variables adapted to a filtration $F_n := \{\mathcal{F}_{n,k} : k \in \mathbb{N}_0\}$. Define for each $k, n \in \mathbb{N}$ a version of the regular conditional distribution

$$P_{n,k} : \mathcal{B}(V) \times \Omega \rightarrow [0, 1], \quad P_{n,k}(B, \omega) = P(X_{n,k} \in B | \mathcal{F}_{n,k-1})(\omega).$$

If there exists a sequence $\{\sigma_n : n \in \mathbb{N}\}$ of finite stopping times $\sigma_n \in \Upsilon(F_n)$ such that $\{P_{n,1} * \dots * P_{n,\tau} : \tau \in \Upsilon(F_n), 1 \leq \tau \leq \sigma_n, n \in \mathbb{N}\}$ is tight, then

$$(9) \quad \{X_{n,1} + \dots + X_{n,\tau} : \tau \in \Upsilon(F_n), 1 \leq \tau \leq \sigma_n, n \in \mathbb{N}\}$$

is tight.

Theorem 3.1 provides a method for establishing tightness of the random field (9). We call this method *tightness by decoupling* for the following reason: for the given sequences $\{X_{n,k} : k \in \mathbb{N}\}$ there exist sequences $\{X_{n,k}^* : k \in \mathbb{N}\}$ of random variables $X_{n,k}^*$ on a larger probability space $(\Omega^*, \mathcal{A}^*, P^*)$ satisfying:

- (i) for every $n \in \mathbb{N}$ the sequence $\{X_{n,k}^* : k \in \mathbb{N}\}$ is conditionally independent over a σ -algebra $\mathcal{F}^* \subseteq \mathcal{A}^*$;
- (ii) $P^*(X_{n,k}^* \in B | \mathcal{F}^*) = P(X_{n,k} \in B | \mathcal{F}_{n,k-1})$ for all $B \in \mathcal{B}(V)$ and $k, n \in \mathbb{N}$.

The sequence $\{X_{n,k}^* : k \in \mathbb{N}\}$ is called the *decoupled tangent sequence*; see Chapter 6 in [5] or [14]. By defining

$$S_n(\sigma_n) := \begin{cases} 0 & \text{if } \sigma_n = 0, \\ \sum_{k=1}^{\sigma_n} X_{n,k} & \text{else,} \end{cases} \quad S_n^*(\sigma_n) := \begin{cases} 0 & \text{if } \sigma_n = 0, \\ \sum_{k=1}^{\sigma_n} X_{n,k}^* & \text{else,} \end{cases}$$

one can conclude from Theorem 3.1 that if $\{S_n^*(\sigma_n) : n \in \mathbb{N}\}$ is tight then $\{S_n(\sigma_n) : n \in \mathbb{N}\}$ is also tight.

Applying Theorem 3.1 in the one-dimensional case yields another result, the *principle of conditioning*, which we also use in this work. The original proof can be found in [2] and [7], and further extensions to Hilbert spaces in [10].

THEOREM 3.2. For each $n \in \mathbb{N}$, let $\{X_{n,k} : k \in \mathbb{N}\}$ be a sequence of real valued random variables adapted to a filtration $F_n := \{\mathcal{F}_{n,k} : k \in \mathbb{N}_0\}$ and $\sigma_n : \Omega \rightarrow \mathbb{N}$ be a stopping time for $\{\mathcal{F}_{n,k} : k \in \mathbb{N}\}$. Define for each $k, n \in \mathbb{N}$

$$\Delta_{n,k} : \mathbb{R} \times \Omega \rightarrow \mathbb{C}, \quad \Delta_{n,k}(\beta, \omega) = E[e^{i\beta X_{n,k}} | \mathcal{F}_{n,k-1}](\omega).$$

If for each $\beta \in \mathbb{R}$, there exists a deterministic constant $c(\beta) \neq 0$ such that

$$\lim_{n \rightarrow \infty} \prod_{k=1}^{\sigma_n} \Delta_{n,k}(\beta, \cdot) = c(\beta) \quad \text{in probability,}$$

then it follows that

$$\lim_{n \rightarrow \infty} E[e^{i\beta(X_{n,1} + \dots + X_{n,\sigma_n})}] = c(\beta).$$

For the proof of Theorem 3.2, we refer to the literature. For the following proof of Theorem 3.1, we introduce a few notation and objects. A *random measure* is a measurable mapping $M : \Omega \rightarrow \mathcal{M}(V)$, where measurability is with respect to

the Borel σ -algebra corresponding to the weak topology on $M(V)$. The mapping M is called a *random probability measure*, if it maps to the space $\mathcal{M}_1(V)$ of Borel probability measures on $\mathcal{B}(V)$. A random measure M is called *integrable*, if $E[M(V)] < \infty$. In this case, $E[M](B) := E[M(B)]$ for all $B \in \mathcal{B}(V)$ defines an element in $\mathcal{M}(V)$. By starting with simple functions and passing to the limit, one shows for bounded, measurable functions $\varphi: V \rightarrow \mathbb{R}$ that

$$(10) \quad E \left[\int_V \varphi(u) M(du) \right] = \int_V \varphi(u) E[M](du).$$

The characteristic function of a random probability measure M is defined by

$$\chi_M: V \times \Omega \rightarrow \mathbb{C}, \quad \chi_M(v) = \int_V e^{i\langle v, h \rangle} M(dh).$$

For an integrable random measure M , it follows from (10) that

$$(11) \quad E[\chi_M(v)] = \chi_{E[M]}(v) \quad \text{for all } v \in V.$$

A set $\{M_i : i \in I\}$ of random measures is called *tight*, if for each $\varepsilon_1 > 0$ we have:

$$(12) \quad \begin{aligned} &\text{there exist for all } i \in I \text{ a set } A_i \in \mathcal{A} \text{ with } P(A_i) > 1 - \varepsilon_1 \text{ obeying:} \\ &\{M_i(\cdot, \omega) : \omega \in A_i, i \in I\} \text{ is relatively compact in } \mathcal{M}(V). \end{aligned}$$

It follows from Prokhorov’s theorem that the last line is equivalent to

$$(13) \quad \sup_{i \in I} \sup_{\omega \in A_i} M_i(V, \omega) < \infty;$$

for each $\varepsilon_2 > 0$ there exists a compact set $K \subseteq V$ such that

$$(14) \quad \sup_{i \in I} \sup_{\omega \in A_i} M_i(K^c, \omega) \leq \varepsilon_2.$$

If $\{M_i : i \in I\}$ is a family of random probability measures, then it is tight if and only if $\{E[M_i] : i \in I\}$ is tight in $\mathcal{M}(V)$.

A nonnegative, symmetric operator $\varphi: V \rightarrow V$ is called an *S-operator* if

$$\text{tr}[\varphi] := \sum_{k=1}^{\infty} \langle \varphi f_k, f_k \rangle < \infty.$$

The space of all S-operators is denoted by $\mathcal{L}_S(V)$. The space $\mathcal{L}_S(V)$ is a subspace of the Banach space of trace class operators, and it is equipped with the Borel σ -algebra of the latter. A set $\{\varphi_i : i \in I\} \subseteq \mathcal{L}_S(V)$ is relatively compact if and only if

$$(15) \quad \sup_{i \in I} \text{tr}[\varphi_i] < \infty;$$

$$(16) \quad \lim_{N \rightarrow \infty} \sup_{i \in I} \sum_{k=N}^{\infty} \langle \varphi_i f_k, f_k \rangle = 0.$$

A set $\{T_i : i \in I\}$ of random variables $T_i : \Omega \rightarrow \mathcal{L}_S(V)$ is *tight* if and only if for each $\varepsilon > 0$ there exist for all $i \in I$ a set $A_i \in \mathcal{A}$ with $P(A_i) > 1 - \varepsilon$ such that

$$(17) \quad \{T_i(\omega) : \omega \in A_i, i \in I\} \quad \text{is relatively compact in } \mathcal{L}_S(V).$$

PROOF OF THEOREM 3.1. For each $\tau \in \Upsilon(F_n)$ with $\tau \geq 1$ and $n \in \mathbb{N}$, define the random probability measure

$$P_n(\tau) : \mathcal{B}(V) \times \Omega \rightarrow [0, 1], \quad P_n(\tau) := P_{n,1} * \dots * P_{n,\tau},$$

and, by denoting $S_n(\tau) := X_{n,1} + \dots + X_{n,\tau}$, the random probability measure

$$Q_n(\tau) : \mathcal{B}(V) \times \Omega \rightarrow [0, 1], \quad Q_n(\tau) = P_n(\tau) * \delta_{-S_n(\tau)},$$

where δ_Y denotes the random Dirac measure in $Y \in L^0(\Omega; V)$. In a first and main step, we show that for each $\varepsilon > 0$ there exists a compact set $\{\varphi_{n,\tau} : \tau \in \Upsilon(F_n), 1 \leq \tau \leq \sigma_n, n \in \mathbb{N}\}$ of deterministic S -operators $\varphi_{n,\tau} \in \mathcal{L}_S(V)$, such that for every $n \in \mathbb{N}$ and each $\tau \in \Upsilon(F_n)$ with $1 \leq \tau \leq \sigma_n$ we have

$$(18) \quad 1 - \text{Re } E[\chi_{Q_n(\tau)}(v)] \leq \langle \varphi_{n,\tau} v, v \rangle + 4\varepsilon \quad \text{for all } v \in V.$$

For this purpose, fix $\varepsilon > 0$ and define the symmetrisation $\tilde{P}_{n,k} := P_{n,k} * P_{n,k}^-$ where $P_{n,k}^-(B, \omega) := P_{n,k}(-B, \omega)$ for all $B \in \mathcal{B}(V)$ and $\omega \in \Omega$. Define for each $\tau \in \Upsilon(F_n)$ with $\tau \geq 1$ and $n \in \mathbb{N}$ the random measure

$$\bar{P}_n(\tau) : \mathcal{B}(V) \times \Omega \rightarrow \mathbb{R}_+, \quad \bar{P}_n(\tau) = \tilde{P}_{n,1} + \dots + \tilde{P}_{n,\tau},$$

and the random S -operator

$$T_n(\tau) : V \times \Omega \rightarrow V, \quad \langle T_n(\tau)v, v \rangle = \int_{B_V} \langle v, h \rangle^2 \bar{P}_n(\tau)(dh).$$

As $\{P_n(\sigma_n) : n \in \mathbb{N}\}$ is tight, it follows that the set $\{\tilde{P}_{n,1} * \dots * \tilde{P}_{n,\sigma_n} : n \in \mathbb{N}\}$ is also tight. Part (a) of Lemma 3.4 implies that $\{T_n(\tau) : \tau \in \Upsilon(F_n), 1 \leq \tau \leq \sigma_n, n \in \mathbb{N}\}$ is a tight set of random S -operators and that $\{\bar{P}_n(\tau)(\cdot \cap B_V^c) : \tau \in \Upsilon(F_n), 1 \leq \tau \leq \sigma_n, n \in \mathbb{N}\}$ is a tight set of random measures. It follows from (13) and from (15), respectively, that there are constants $c_1, c_2 > 0$ such that for each $n \in \mathbb{N}$ we have $P(\bar{P}_n(\tau)(B_V^c) > c_1) \leq \varepsilon$ and $P(\text{tr}[T_n(\tau)] > c_2) \leq \varepsilon$ for all $\tau \in \Upsilon(F_n)$ with $1 \leq \tau \leq \sigma_n$. Define for each $n \in \mathbb{N}$ the stopping times

$$\rho'_n := \inf\{k \in \mathbb{N} : \bar{P}_n(k)(B_V^c) > c_1\}, \quad \rho''_n := \inf\{k \in \mathbb{N} : \text{tr}[T_n(k)] > c_2\}.$$

The stopping time $\rho_n := \rho'_n \wedge \rho''_n$ satisfies for each $n \in \mathbb{N}$ that $P(\rho_n < \tau) \leq 2\varepsilon$ for all $\tau \in \Upsilon(F_n)$ with $1 \leq \tau \leq \sigma_n$. Lemma 3.3 implies for each $v \in V$ that

$$\begin{aligned} 1 - \text{Re } E[\chi_{Q_n(\tau)}(v)] &\leq E[(1 - \text{Re } \chi_{Q_n(\tau)}(v)) \mathbb{1}_{\{\tau \leq \rho_n\}}] + P(\rho_n < \tau) \\ &\leq 1 - \text{Re } E[\chi_{Q_n(\rho_n \wedge \tau)}(v)] + 2\varepsilon \\ &\leq E\left[\sum_{k=1}^{\rho_n \wedge \tau} (1 - \chi_{\tilde{P}_{n,k}}(v))\right] + 2\varepsilon. \end{aligned}$$

The assumed tightness of $\{P_n(\tau) : \tau \in \Upsilon(F_n), 1 \leq \tau \leq \sigma_n, n \in \mathbb{N}\}$ yields tightness of $\{\tilde{P}_{n,1} * \dots * \tilde{P}_{n,\rho_n \wedge \sigma_n} : n \in \mathbb{N}\}$. Moreover, since

$$\overline{P}_n(\rho_n \wedge \sigma_n)(B_V^c) \leq \overline{P}_n(\rho_n - 1)(B_V^c) + \tilde{P}_{n,\rho_n}(B_V^c) \leq c_1 + 1 \quad \text{for all } n \in \mathbb{N},$$

part (c) of Lemma 3.4 guarantees tightness of $\{E[\overline{P}_n(\rho_n \wedge \tau)](\cdot \cap B_V^c) : \tau \in \Upsilon(F_n), 1 \leq \tau \leq \sigma_n, n \in \mathbb{N}\}$. Thus, there exists a constant $d > 1$ such that

$$(19) \quad E[\overline{P}_n(\rho_n \wedge \tau)(\{v \in V : \|v\| > d\})] \leq \varepsilon.$$

Define for each $\tau \in \Upsilon(F_n)$ and $n \in \mathbb{N}$ the random operator

$$R_n(\tau) : V \times \Omega \rightarrow V, \quad \langle R_n(\tau)v, v \rangle = \int_{1 < \|h\| \leq d} \langle v, h \rangle^2 \overline{P}_n(\tau)(dh).$$

Since $1 - \cos \beta \leq 2\beta^2$ for all $\beta \in \mathbb{R}$, we obtain by (19) for all $v \in V$ that

$$\begin{aligned} & E \left[\sum_{k=1}^{\rho_n \wedge \tau} (1 - \chi_{\tilde{P}_{n,k}}(v)) \right] \\ &= E \left[\int_V (1 - \cos(\langle v, h \rangle)) \overline{P}_n(\rho_n \wedge \tau)(dh) \right] \\ &\leq 2E \left[\int_{\|h\| \leq d} \langle v, h \rangle^2 \overline{P}_n(\rho_n \wedge \tau)(dh) \right] + 2E[\overline{P}_n(\rho_n \wedge \tau)(\{h \in V : \|h\| > d\})] \\ &\leq 2E[\langle T_n(\rho_n \wedge \tau)v, v \rangle] + 2E[\langle R_n(\rho_n \wedge \tau)v, v \rangle] + 2\varepsilon \\ &= 2\langle E[T_n(\rho_n \wedge \tau)]v, v \rangle + 2\langle E[R_n(\rho_n \wedge \tau)]v, v \rangle + 2\varepsilon. \end{aligned}$$

In the last line, we applied part (b) of Lemma 3.4, which can be done as the definition of the stopping times ρ'_n and ρ''_n guarantees for all $n \in \mathbb{N}$ that

$$\begin{aligned} \text{tr}[R_n(\rho'_n \wedge \sigma_n)] &\leq \int_{1 < \|h\| \leq d} \|h\|^2 \overline{P}_n(\rho'_n - 1)(dh) + \int_{1 < \|h\| \leq d} \|h\|^2 \tilde{P}_{n,\rho'_n}(dh) \\ &\leq d^2(c_1 + 1), \end{aligned}$$

and analogously

$$\text{tr}[T_n(\rho''_n \wedge \sigma_n)] \leq \text{tr}[T_n(\rho''_n - 1)] + \int_{\|h\| \leq 1} \|h\|^2 \tilde{P}_{n,\rho''_n}(dh) \leq c_2 + 1.$$

Moreover, Lemma 3.4 implies that the sets $\{E[T_n(\rho_n \wedge \tau)] : \tau \in \Upsilon(F_n), 1 \leq \tau \leq \sigma_n, n \in \mathbb{N}\}$ and $\{E[R_n(\rho_n \wedge \tau)] : \tau \in \Upsilon(F_n), 1 \leq \tau \leq \sigma_n, n \in \mathbb{N}\}$ are relatively compact in $\mathcal{L}_S(V)$, which completes the proof of (18).

It follows from (18) by Theorem VI.2.3 in [20] that the set $\{E[Q_n(\tau)] : \tau \in \Upsilon(F_n), 1 \leq \tau \leq \sigma_n, n \in \mathbb{N}\}$, and thus also the set $\{Q_n(\tau) : \tau \in \Upsilon(F_n), 1 \leq \tau \leq \sigma_n, n \in \mathbb{N}\}$ are tight. Since each random probability measure $Q_n(\tau)$ is the convolution of $P_n(\tau)$ and the random Dirac measure $\delta_{-S_n(\tau)}$, and since the set

$\{P_n(\tau) : \tau \in \Upsilon(F_n), 1 \leq \tau \leq \sigma_n, n \in \mathbb{N}\}$ is assumed to be tight, it follows that the set $\{\delta_{-S_n(\tau)} : \tau \in \Upsilon(F_n), 1 \leq \tau \leq \sigma_n, n \in \mathbb{N}\}$ is tight, which completes the proof. \square

The following two results are used in the proof of Theorem 3.1.

LEMMA 3.3. *In the setting of Theorem 3.1, define for some $n \in \mathbb{N}$ and for a stopping time $\tau \in \Upsilon(F_n)$ the random probability measure*

$$Q_n(\tau) : \mathcal{B}(V) \times \Omega \rightarrow [0, 1], \quad Q_n(\tau) = P_{n,1} * \dots * P_{n,\tau} * \delta_{-S_n(\tau)},$$

where $S_n(\tau) := X_{n,1} + \dots + X_{n,\tau}$. Then it follows that

$$1 - \operatorname{Re} E[\chi_{Q_n(\tau)}(v)] \leq E \left[\sum_{k=1}^{\tau} (1 - \chi_{\tilde{P}_{n,k}}(v)) \right] \quad \text{for every } v \in V.$$

PROOF. Fix $v \in V$, $n \in \mathbb{N}$, and define for each $j \in \mathbb{N}$ the $\mathcal{F}_{n,j}$ -measurable random variable

$$X_n(j) := 1 - \chi_{\tilde{P}_{n,j}}(v) - \operatorname{Re} \chi_{Q_n(j-1)}(v) + \operatorname{Re} \chi_{Q_n(j)}(v),$$

where we set $Q_n(0) = \delta_0$. We claim that $Y_n(k) := X_n(1) + \dots + X_n(k)$ defines a submartingale $(Y_n(k) : k \in \mathbb{N})$ with respect to F_n . For each $k \in \mathbb{N}$, we obtain

$$(20) \quad \begin{aligned} E[X_n(k) | \mathcal{F}_{n,k-1}] \\ = 1 - \chi_{\tilde{P}_{n,k}}(v) - \operatorname{Re} \chi_{Q_n(k-1)}(v) + \operatorname{Re} E[\chi_{Q_n(k)}(v) | \mathcal{F}_{n,k-1}]. \end{aligned}$$

Since the random measure $Q_n(j)$ is defined as a convolution its characteristic function obeys

$$\chi_{Q_n(j)}(v) = \chi_{\delta_{-S_n(j)}}(v) \chi_{P_n(j)}(v) = e^{-i\langle v, S_n(j) \rangle} \chi_{P_n(j)}(v) \quad \text{for all } j \in \mathbb{N},$$

where $P_n(k) := P_{n,1} * \dots * P_{n,k}$. Consequently, we arrive at

$$\begin{aligned} E[\chi_{Q_n(k)}(v) | \mathcal{F}_{n,k-1}] &= \chi_{P_n(k)}(v) e^{-i\langle v, S_n(k-1) \rangle} E[e^{-i\langle v, X_{n,k} \rangle} | \mathcal{F}_{n,k-1}] \\ &= \chi_{Q_n(k-1)}(v) \chi_{P_{n,k}}(v) \chi_{P_{n,k}^-}(v) \\ &= \chi_{Q_n(k-1)}(v) \chi_{\tilde{P}_{n,k}}(v). \end{aligned}$$

Applying this equality to (20), we obtain

$$E[X_n(k) | \mathcal{F}_{n,k-1}] = (1 - \operatorname{Re} \chi_{Q_n(k-1)}(v))(1 - \chi_{\tilde{P}_{n,k}}(v)).$$

Since the last line is nonnegative, it follows that $(Y_n(k) : k \in \mathbb{N})$ is a submartingale.

We conclude from (11) that

$$E[\chi_{Q_n(1)}(v)] = E[\chi_{P_{n,1}}(v)] E[e^{-i\langle X_{n,1}, v \rangle}] = \chi_{X_{n,1}}(v) \chi_{-X_{n,1}}(v) = E[\chi_{\tilde{P}_{n,1}}(v)],$$

which yields $E[Y_n(1)] = 0$. Doob’s optional stopping theorem shows for $\tau \in \Upsilon(F_n)$ that $E[Y_n(\tau \wedge N)] \geq E[Y_n(1)] = 0$ for all $N \in \mathbb{N}$, which, by the very definition of $Y_n(k)$, results in

$$\begin{aligned} E \left[\sum_{k=1}^{\tau \wedge N} (1 - \chi_{\tilde{P}_{n,k}}(v)) \right] &\geq E \left[- \sum_{k=1}^{\tau \wedge N} (\operatorname{Re} \chi_{Q_n(j)}(v) - \operatorname{Re} \chi_{Q_n(j-1)}(v)) \right] \\ &= E[1 - \operatorname{Re} \chi_{Q_n(\tau \wedge N)}(v)]. \end{aligned}$$

Applying the result on monotone convergence to the left-hand side and Lebesgue’s theorem of dominated convergence to the right-hand side completes the proof. \square

For the following lemma, note that if $k: V \times V \rightarrow \mathbb{R}$ is a bilinear form then a linear operator $K: V \rightarrow V$ can be defined by $\langle K v_1, v_2 \rangle = k(v_1, v_2)$ for all $v_1, v_2 \in V$.

LEMMA 3.4. *For each $n \in \mathbb{N}$, let $\{M_{n,k} : k \in \mathbb{N}\}$ be a sequence of symmetric random probability measures adapted to a filtration $F_n := \{\mathcal{F}_{n,k} : k \in \mathbb{N}_0\}$. For $\tau \in \Upsilon(F_n)$ with $\tau \geq 1$, denote $\bar{M}_n(\tau) := M_{n,1} + \dots + M_{n,\tau}$ and define for some $c > 0$ the random operators*

$$T_n(\tau): V \times \Omega \rightarrow V, \quad \langle T_n(\tau)v_1, v_2 \rangle = \int_{\|h\| \leq c} \langle v_1, h \rangle \langle v_2, h \rangle \bar{M}_n(\tau)(dh),$$

and the random measures

$$N_n(\tau): \mathcal{B}(V) \times \Omega \rightarrow \mathbb{R}_+, \quad N_n(\tau)(B) = \bar{M}_n(\tau)(B \cap \{\|v\| > c\}).$$

If there exists a sequence $\{\sigma_n : n \in \mathbb{N}\}$ of finite stopping times $\sigma_n \in \Upsilon(F_n)$ with $\sigma_n \geq 1$ such that $\{M_{n,1} * \dots * M_{n,\sigma_n} : n \in \mathbb{N}\}$ is tight, then we have:

(a) *the set $\{T_n(\tau) : \tau \in \Upsilon(F_n), 1 \leq \tau \leq \sigma_n, n \in \mathbb{N}\}$ of random S -operators and the set $\{N_n(\tau) : \tau \in \Upsilon(F_n), 1 \leq \tau \leq \sigma_n, n \in \mathbb{N}\}$ of random measures are tight.*

(b) *uniform integrability of $\{\operatorname{tr}[T_n(\sigma_n)] : n \in \mathbb{N}\}$ implies that, for $\tau \in \Upsilon(F_n)$,*

$$E[T_n(\tau)]: V \rightarrow V, \quad \langle E[T_n(\tau)]v, v \rangle = E[\langle T_n(\tau)v, v \rangle]$$

defines a relatively compact set $\{E[T_n(\tau)] : \tau \in \Upsilon(F_n), 1 \leq \tau \leq \sigma_n, n \in \mathbb{N}\}$ of S -operators.

(c) *uniform integrability of $\{N_n(\sigma_n)(V) : n \in \mathbb{N}\}$ implies that $\{E[N_n(\tau)] : \tau \in \Upsilon(F_n), 1 \leq \tau \leq \sigma_n, n \in \mathbb{N}\}$ is relatively compact.*

PROOF. (a) Let R_n denote the infinitely divisible random measure with characteristic function

$$\chi_{R_n}: V \times \Omega \rightarrow \mathbb{C}, \quad \chi_{R_n}(v) = \exp\left(\int_V (e^{i\langle h, v \rangle} - 1) \bar{M}_n(\sigma_n)(dh)\right).$$

The inequality

$$1 - \exp\left(\sum_{k=1}^n (\beta_k - 1)\right) \leq 1 - \prod_{k=1}^n \beta_k \quad \text{for all } \beta_k \in [0, 1], n \in \mathbb{N},$$

yields for every $v \in V$ and $n \in \mathbb{N}$ the estimate

$$1 - E[\chi_{R_n}(v)] = E\left[1 - \exp\left(\sum_{k=1}^{\sigma_n} (\chi_{M_{n,k}}(v) - 1)\right)\right] \leq 1 - E\left[\prod_{k=1}^{\sigma_n} \chi_{M_{n,k}}(v)\right].$$

Tightness of $\{M_{n,1} * \dots * M_{n,\sigma_n} : n \in \mathbb{N}\}$ implies by Theorem VI.2.3 in [20] together with (11) that the set $\{E[R_n] : n \in \mathbb{N}\}$ is tight. It follows that for each $\varepsilon > 0$ there exists for every $n \in \mathbb{N}$ a set $A_n \in \mathcal{A}$ with $P(A_n) > 1 - \varepsilon$ such that the set

$$\{\mathbb{1}_{A_n}(\omega)R_n(\cdot, \omega) + \mathbb{1}_{A_n^c}(\omega)\delta_0(\cdot) : \omega \in \Omega, n \in \mathbb{N}\}$$

of infinitely divisible probability measures is relatively compact. Theorem VI.5.1 in [20] implies that the set $\{T_n(\sigma_n)(\omega) : \omega \in A_n, n \in \mathbb{N}\}$ is compact and the set $\{N_n(\sigma_n)(\cdot, \omega) : \omega \in A_n, n \in \mathbb{N}\}$ is relatively compact. The monotonicity $\langle T_n(\tau)f_k, f_k \rangle \leq \langle T_n(\sigma_n)f_k, f_k \rangle$ for all $k \in \mathbb{N}$ and $N_n(\tau) \leq N_n(\sigma_n)$ for each $\tau \leq \sigma_n$ completes the proof by (13), (14) and (15), (16).

(b) By applying Tonelli’s theorem, we obtain

$$\begin{aligned} \sup_{n \in \mathbb{N}} \text{tr}[E[T_n(\sigma_n)]] &= \sup_{n \in \mathbb{N}} E\left[\sum_{k=1}^{\infty} \langle T_n(\sigma_n)f_k, f_k \rangle\right] \\ (21) \qquad \qquad \qquad &= \sup_{n \in \mathbb{N}} E[\text{tr}[T_n(\sigma_n)]] < \infty. \end{aligned}$$

Let $\varepsilon > 0$ be given and choose $\delta > 0$ such that $P(A) \leq \delta$ for any $A \in \mathcal{A}$ implies $E[\text{tr}[T_n(\sigma_n)]\mathbb{1}_A] \leq \varepsilon$ for all $n \in \mathbb{N}$. From part (a), it follows by (16) that there are $A_n \in \mathcal{A}, n \in \mathbb{N}$, with $P(A_n) > 1 - \delta$ and $N_0 \in \mathbb{N}$ such that

$$\sup_{n \in \mathbb{N}} \sup_{\omega \in A_n} \sum_{k=N_0}^{\infty} \langle T_n(\sigma_n)(\omega)f_k, f_k \rangle \leq \varepsilon.$$

It follows that

$$\begin{aligned} &\sup_{n \in \mathbb{N}} \sum_{k=N_0}^{\infty} \langle E[T_n(\sigma_n)]f_k, f_k \rangle \\ &\leq \sup_{n \in \mathbb{N}} E\left[\mathbb{1}_{A_n} \sum_{k=N_0}^{\infty} \langle T_n(\sigma_n)f_k, f_k \rangle\right] + E[\text{tr}[T_n(\sigma_n)]\mathbb{1}_{A_n^c}] \\ &\leq 2\varepsilon, \end{aligned}$$

which shows that

$$(22) \quad \lim_{N \rightarrow \infty} \sup_{n \in \mathbb{N}} \sum_{k=N}^{\infty} \langle E[T_n(\sigma_n)]f_k, f_k \rangle = 0.$$

Both properties (21) and (22) establish that $\{E[T_n(\sigma_n)] : n \in \mathbb{N}\}$ is relatively compact, which completes the proof by monotonicity $\langle E[T_n(\tau)]f_k, f_k \rangle \leq \langle E[T_n(\sigma_n)]f_k, f_k \rangle$ for all $k \in \mathbb{N}$ for $\tau \leq \sigma_n$ by (15) and (16).

(c) Can be proved as (b). \square

4. Radonification of the increments. In this section, we solve problems (1) and (2) mentioned in the Introduction. Recall the definition $p(X) = E[1 \wedge \|X\|^2]$ for any V -valued random variable X in (6). The following inequality originates from the work [12], but since we only need a special case we give a short proof here.

LEMMA 4.1. *There exists a universal constant $c > 0$ such that for any $\varphi \in \mathcal{L}_2(U, V)$ and any cylindrical random variable $Z : U \rightarrow L_p^0(\Omega; \mathbb{R})$ we have*

$$p(\varphi Z) \leq c \int_U p(Zu)(\gamma \circ (\varphi^*)^{-1})(du),$$

where γ denotes the canonical Gaussian cylindrical measure on V .

PROOF. Let $(\Omega', \mathcal{A}', P')$ be another probability space and let $\Gamma : V \rightarrow L_{p'}^0(\Omega'; \mathbb{R})$ be a cylindrical random variable distributed according to the canonical cylindrical Gaussian distribution γ on V . From the inequality,

$$(1 \wedge |\alpha|)(1 \wedge |\beta|) \leq 1 \wedge |\alpha\beta| \quad \text{for all } \alpha, \beta \in \mathbb{R},$$

it follows for a real-valued, standard normally distributed random variable ξ that

$$(1 \wedge \|v\|^2)E'[1 \wedge |\xi|^2] \leq E'[1 \wedge \|v\|^2|\xi|^2] = E'[1 \wedge |\Gamma v|^2] \quad \text{for all } v \in V.$$

Consequently, by defining $c := (E'[1 \wedge |\xi|^2])^{-1}$ we obtain

$$(1 \wedge \|v\|^2) \leq cE'[1 \wedge |\Gamma v|^2] \quad \text{for all } v \in V.$$

It follows that

$$\begin{aligned} p(\varphi Z) &= E[1 \wedge \|\varphi Z\|^2] \leq cE[E'[1 \wedge |\Gamma(\varphi Z)|^2]] \\ &= cE\left[\int_V (1 \wedge \langle v, \varphi Z \rangle^2)\gamma(dv)\right] \\ &= c \int_V E[1 \wedge |Z(\varphi^*v)|^2]\gamma(dv) \\ &= c \int_U E[1 \wedge |Zu|^2](\gamma \circ (\varphi^*)^{-1})(du), \end{aligned}$$

which completes the proof. \square

Let $(L(t) : t \geq 0)$ be a cylindrical Lévy process in U . We equip the probability space with the filtration generated by L and defined by

$$\mathcal{F}_t := \sigma(\{L(s)u : u \in U, s \in [0, t]\}) \quad \text{for all } t \geq 0.$$

Fix the times $0 \leq s \leq t$. An $\mathcal{L}_2(U, V)$ -valued, \mathcal{F}_s -measurable random variable Φ is called *simple* if it is of the form

$$\Phi(\omega) = \sum_{i=1}^m \mathbb{1}_{A_i}(\omega)\varphi_i \quad \text{for all } \omega \in \Omega,$$

for disjoint sets $A_1, \dots, A_m \in \mathcal{F}_s$ and $\varphi_1, \dots, \varphi_m \in \mathcal{L}_2(U, V)$. The space of all $\mathcal{L}_2(U, V)$ -valued, \mathcal{F}_s -measurable, simple random variables is denoted by $S(\Omega, \mathcal{F}_s; \mathcal{L}_2)$. It follows from (7) that for each $i = 1, \dots, m$ there exists an V -valued random variable $\varphi_i(L(t) - L(s))$ satisfying

$$\langle \varphi_i(L(t) - L(s)), v \rangle = (L(t) - L(s))(\varphi_i^* v) \quad \text{for all } v \in V.$$

Define an \mathcal{F}_s -measurable, V -valued random variable by

$$J(\Phi) := \sum_{i=1}^m \mathbb{1}_{A_i} \varphi_i(L(t) - L(s)).$$

In this situation, we define

$$(L(t) - L(s))(\Phi^* v) := \langle J(\Phi), v \rangle \quad \text{for all } v \in V.$$

The following result enables us to extend this definition of *radonified increments* from simple to arbitrary random variables Φ .

THEOREM 4.2. *Let $0 \leq s \leq t$ be fixed. For each \mathcal{F}_s -measurable, $\mathcal{L}_2(U, V)$ -valued random variable Φ , there exist an V -valued random variable Y and a sequence $\{\Phi_n\}_{n \in \mathbb{N}}$ of simple random variables in $S(\Omega, \mathcal{F}_s; \mathcal{L}_2)$ with $\Phi_n \rightarrow \Phi$ P -a.s. such that*

$$Y = \lim_{n \rightarrow \infty} J(\Phi_n) \quad \text{in probability.}$$

Moreover, the limit Y does not depend on the sequence $\{\Phi_n\}_{n \in \mathbb{N}}$.

PROOF. Since $\Phi : \Omega \rightarrow \mathcal{L}_2(U, V)$ is strongly \mathcal{F}_s -measurable, there exists a sequence $\{\Phi_n\}_{n \in \mathbb{N}}$ of simple random variables in $S(\Omega, \mathcal{F}_s; \mathcal{L}_2)$ with $\Phi_n \rightarrow \Phi$ P -a.s. It remains to show that $(J(\Phi_n))_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^0_p(\Omega; V)$. By linearity, it is sufficient to show that $\Phi_n \rightarrow 0$ P -a.s. for $n \rightarrow \infty$ implies that $J(\Phi_n) \rightarrow 0$ in probability for $n \rightarrow \infty$. For this purpose, assume that

$$\Phi_n(\omega) = \sum_{i=1}^{m_n} \mathbb{1}_{A_{n,i}}(\omega)\varphi_{n,i} \quad \text{for all } \omega \in \Omega,$$

for disjoint sets $A_{n,1}, \dots, A_{n,m_n} \in \mathcal{F}_s$ and $\varphi_{n,1}, \dots, \varphi_{n,m_n} \in \mathcal{L}_2(U, V)$, $m_n \in \mathbb{N}$ and $\Phi_n \rightarrow 0$ P -a.s. for $n \rightarrow \infty$. Define the cylindrical random variable $Z := L(t) - L(s)$. Independence of Z and \mathcal{F}_s implies that $\varphi_{n,i}Z$ is also independent of \mathcal{F}_s . Using this independence, Lemma 4.1 and (11), we obtain for each $n \in \mathbb{N}$ that

$$\begin{aligned} p(J(\Phi_n)) &= E \left[1 \wedge \left\| \sum_{i=1}^{m_n} \mathbb{1}_{A_{n,i}} \varphi_{n,i} Z \right\|^2 \right] \\ &= E \left[\sum_{i=1}^{m_n} \mathbb{1}_{A_{n,i}} (1 \wedge \|\varphi_{n,i} Z\|^2) \right] \\ &= \int_{\Omega} \sum_{i=1}^{m_n} \mathbb{1}_{A_{n,i}}(\omega) p(\varphi_{n,i} Z) P(d\omega) \\ &\leq c \int_{\Omega} \sum_{i=1}^{m_n} \mathbb{1}_{A_{n,i}}(\omega) \left(\int_U p(Zu) (\gamma \circ (\varphi_{n,i}^*)^{-1})(du) \right) P(d\omega) \\ &= c \int_{\Omega} \left(\int_U p(Zu) M_n(du, \omega) \right) P(d\omega) \\ &= c \int_U p(Zu) E[M_n](du), \end{aligned}$$

where c denotes the constant derived in Lemma 4.1 and M_n is the random probability measure defined by

$$M_n : \mathcal{B}(U) \times \Omega \rightarrow [0, 1], \quad M_n(B, \omega) = \sum_{i=1}^{m_n} \mathbb{1}_{A_{n,i}}(\omega) (\gamma \circ (\varphi_{n,i}^*)^{-1})(B).$$

Recall the definition $E[M_n](B) := E[M_n(B)]$ for all $B \in \mathcal{B}(U)$. For each $n \in \mathbb{N}$ and $\omega \in \Omega$, the measure $M_n(\cdot, \omega)$ is Gaussian with expectation 0 and covariance operator

$$Q_n(\omega) : U \rightarrow U, \quad Q_n(\omega) = \sum_{i=1}^{m_n} \mathbb{1}_{A_{n,i}}(\omega) \varphi_{n,i}^* \varphi_{n,i}.$$

Consequently, we have for all $u \in U$ that

$$(23) \quad E[\varphi_{M_n}(u)] = E[e^{i\langle Q_n u, u \rangle}] = E[e^{i\|\Phi_n u\|_V^2}] \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

Egorov’s theorem implies that for each $\varepsilon > 0$ there exists a set $A \in \mathcal{A}$ with $P(A) > 1 - \varepsilon$ such that

$$\sup_{n \in \mathbb{N}} \sup_{\omega \in A} \text{tr}[Q_n(\omega)] < \infty.$$

As $Q_n(\omega)$ is the covariance operator of the Gaussian measure $M_n(\cdot, \omega)$, it follows that the set $\{M_n(\cdot, \omega) : \omega \in A, n \in \mathbb{N}\}$ is tight. Thus, the set $\{M_n : n \in \mathbb{N}\}$ of random measures is tight, which implies together with (23) that $E[M_n]$ converges weakly to the Dirac measure in 0. Since the function $u \mapsto p(Zu)$ is bounded and continuous, we obtain

$$\int_H p(Zu)E[M_n](du) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Linearity of J guarantees the claimed uniqueness, which completes the proof. \square

Theorem 4.2 enables us to define for each $0 \leq s \leq t$ and \mathcal{F}_s -measurable random variable $\Phi : \Omega \rightarrow \mathcal{L}_2(U, V)$ the V -valued random variable

$$\Phi(L(t) - L(s)) := \lim_{n \rightarrow \infty} J(\Phi_n),$$

where $(\Phi_n)_{n \in \mathbb{N}} \subseteq \mathcal{S}(\Omega, \mathcal{F}_s; \mathcal{L}_2)$ converges to Φ P -a.s. We define then the increments of the cylindrical Lévy process L under the random mapping Φ by

$$(L(t) - L(s))(\Phi^*v) := \langle \Phi(L(t) - L(s)), v \rangle \quad \text{for all } v \in V.$$

We finish this section with calculating the conditional characteristic function of the radonified increments.

LEMMA 4.3. *If $0 \leq s \leq t$ and $\Phi : \Omega \rightarrow \mathcal{L}_2(U, V)$ is an \mathcal{F}_s -measurable random variable, then it follows for each $v \in V$ that*

$$(24) \quad E[\exp(i\langle \Phi(L(t) - L(s)), v \rangle) | \mathcal{F}_s] = \exp((t - s)S(\Phi^*v)), \quad P\text{-a.s.},$$

where $S : U \rightarrow \mathbb{C}$ denotes the cylindrical Lévy symbol of L defined in (8).

PROOF. If Φ is simple, then it is easy to establish the equality claimed in (24). For an arbitrary \mathcal{F}_s -measurable random variable $\Phi : \Omega \rightarrow \mathcal{L}_2(U, V)$, Theorem 4.2 guarantees that there exists a sequence $(\Phi_n)_{n \in \mathbb{N}}$ of simple random variables in $\mathcal{S}(\Omega, \mathcal{F}_s; \mathcal{L}_2)$ satisfying $\Phi_n \rightarrow \Phi$ P -a.s. as $n \rightarrow \infty$ and for all $v \in V$

$$(25) \quad \lim_{n \rightarrow \infty} (L(t) - L(s))(\Phi_n^*v) = (L(t) - L(s))(\Phi^*v) \quad \text{in probability.}$$

On the other hand, as $\Phi_n \rightarrow \Phi$ P -a.s. the continuity of the cylindrical Lévy symbol $S : U \rightarrow \mathbb{C}$ yields for all $v \in V$

$$(26) \quad \lim_{n \rightarrow \infty} \exp((t - s)S(\Phi_n^*v)) = \exp((t - s)S(\Phi^*v)) \quad P\text{-a.s.}$$

Equations (25) and (26) show that the relation (24) can be generalised to arbitrary $\Phi \in L_p^0(\Omega, \mathcal{F}_s; \mathcal{L}_2)$. \square

5. The stochastic integral. We begin the definition of the classical stochastic integral with simple integrands. An $\mathcal{L}_2(U, V)$ -valued, stochastic process $(\Psi(t) : t \in [0, T])$ is called *simple* if it is of the form

$$(27) \quad \Psi(t) = \Phi_0 \mathbb{1}_{\{0\}}(t) + \sum_{j=1}^N \Phi_j \mathbb{1}_{(t_j, t_{j+1}]}(t) \quad \text{for all } t \in [0, T],$$

where $0 = t_1 < \dots < t_{N+1} = T$ is a finite sequence of deterministic times and each $\Phi_j : \Omega \rightarrow \mathcal{L}_2(U, V)$ is an \mathcal{F}_{t_j} -measurable random variable for each $j = 0, \dots, N$. The set of all simple $\mathcal{L}_2(U, V)$ -valued stochastic processes is denoted by $\mathcal{H}_0(U, V)$.

Let $(\Psi(t) : t \in [0, T])$ be a simple process in $\mathcal{H}_0(U, V)$ of the form (27) and $(L(t) : t \geq 0)$ be a cylindrical Lévy process in U . Theorem 4.2 guarantees that for each $j = 1, \dots, N$ and $t \in [0, T]$, there exists the random variable

$$J(\Phi_j)(t) := \Phi_j(L(t \wedge t_{j+1}) - L(t \wedge t_j)) : \Omega \rightarrow V,$$

satisfying

$$(L(t \wedge t_{j+1}) - L(t \wedge t_j))(\Phi_j^* v) = \langle J(\Phi_j)(t), v \rangle \quad \text{for all } v \in V.$$

Thus, we can define a random variable in $L^0_p(\Omega; V)$ for each $t \in [0, T]$ by

$$I(\Psi)(t) : \Omega \rightarrow V, \quad I(\Psi)(t) := J(\Phi_1)(t) + \dots + J(\Phi_N)(t).$$

Obviously, the random variable $I(\Psi)(t)$ obeys

$$\langle I(\Psi)(t), v \rangle = \sum_{j=1}^N \langle J(\Phi_j)(t), v \rangle = \sum_{j=1}^N (L(t \wedge t_{j+1}) - L(t \wedge t_j))(\Phi_j^* v)$$

for all $v \in V$ and $t \in [0, T]$.

In the following, we extend the domain of I to the linear space:

$$\mathcal{H}(U, V) := \{ \Psi : [0, T] \times \Omega \rightarrow \mathcal{L}_2(U, V) : \text{adapted and with càglàd paths} \}.$$

The paths of an element Ψ in $\mathcal{H}(U, V)$ are in the space $D_-([0, T]; \mathcal{L}_2(U, V))$ of $\mathcal{L}_2(U, V)$ -valued functions which are continuous from the left and have limits from the right (càglàd). Recall from Section 2 that this space is equipped with the Skorokhod metric d_J defined in (5). As $\mathcal{L}_2(U, V)$ is separable every $\Psi \in \mathcal{H}(U, V)$ can also be considered as a random variable $\Psi : \Omega \rightarrow D_-([0, T]; \mathcal{L}_2(U, V))$.

The definition of the stochastic integral for arbitrary integrands in the space $\mathcal{H}(U, V)$ is given by the following result. The assumed approximation by simple processes is presented in the subsequent Lemma 5.2.

THEOREM 5.1. *For every sequence $(\Psi_n)_{n \in \mathbb{N}} \subseteq \mathcal{H}_0(U, V)$ which converges to some $\Psi \in \mathcal{H}(U, V)$ in probability in the Skorokhod metric, that is,*

$$\lim_{n \rightarrow \infty} P(d_J(\Psi_n, \Psi) \geq \varepsilon) = 0 \quad \text{for all } \varepsilon > 0,$$

there exists a V -valued, adapted semimartingale $(I(\Psi)(t) : t \in [0, T])$ with càdlàg trajectories obeying for each $t \in [0, T]$ and $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} P(\|I(\Psi_n)(t) - I(\Psi)(t)\| \geq \varepsilon) = 0.$$

The limit $I(\Psi)$ does not depend on the sequence $(\Psi_n)_{n \in \mathbb{N}}$, that is, it is unique up to evanescence.

In this work, we define simple integrands as stochastic processes which equal a random variable Φ_j on deterministic but not random intervals $(t_j, t_{j+1}]$. This guarantees that the radonification of the increments $J(\Phi_j)(t)$ is well defined by the approach in Section 4. If the interval $(t_j, t_{j+1}]$ were random the integrand Φ_j would not be independent of $L(t_{j+1}) - L(t_j)$ and this method could not be applied any more.

Simple integrands defined on random intervals are dense in $\mathcal{H}(U, V)$ with respect to the uniform convergence on $[0, T]$ in probability, that is, the so-called ucp convergence. In our case of deterministic partitions, we have to weaken the topology to the Skorokhod topology.

LEMMA 5.2. *For every $\Psi \in \mathcal{H}(U, V)$, there exists a sequence $(\Psi_n)_{n \in \mathbb{N}}$ of simple stochastic processes $\Psi_n \in \mathcal{H}_0(U, V)$, each defined on a partition $(t_{n,k})_{k=1, \dots, n}$ of the interval $[0, T]$ with $\max_{j=1, \dots, n} |t_{n,j+1} - t_{n,j}| \rightarrow 0$ for $n \rightarrow \infty$ and with $\{\Psi_n(t) : t \in [0, T]\}$ in the closure of $\{\Psi(t) : t \in [0, T]\}$ in V such that*

$$\lim_{n \rightarrow \infty} d_J(\Psi_n(\omega), \Psi(\omega)) = 0 \quad \text{for all } \omega \in \Omega.$$

PROOF. This follows from the construction of the approximating sequence as the discretisation in the analogue result for deterministic, càdlàg functions; see, for example, [20], Lemma VII.6.5. \square

PROPOSITION 5.3. *Let μ be a continuous cylindrical probability measure on $\mathcal{Z}(U)$ and K be a compact set in $\mathcal{L}_2(U, V)$. Then the set $\{\mu \circ \varphi^{-1} : \varphi \in K\}$ is relatively compact in the space $\mathcal{M}_1(V)$ of probability measures on $\mathcal{B}(V)$.*

PROOF. According to [27], Proposition IV.4.2, page 236, there exist a probability space (Ω, \mathcal{A}, P) and a cylindrical random variable $Z : U \rightarrow L^0_P(\Omega; \mathbb{R})$ such that μ is the cylindrical distribution of Z . Then the random variable $\varphi(Z)$ is distributed according to the probability measure $\mu \circ \varphi^{-1}$ for each $\varphi \in K$. By [27], Corollary 1 in I.3.9, p. 52, the set $\{\mu \circ \varphi^{-1} : \varphi \in K\}$ is relatively compact if and only if

$$(28) \quad \lim_{r \rightarrow \infty} \sup_{\varphi \in K} P(\|\varphi(Z)\| \geq r) = 0,$$

$$(29) \quad \lim_{N \rightarrow \infty} \sup_{\varphi \in K} P\left(\sum_{k=N+1}^{\infty} \langle \varphi(Z), f_k \rangle^2 \geq r\right) = 0 \quad \text{for each } r > 0.$$

Recall that $(f_k)_{k \in \mathbb{N}}$ denotes the orthonormal basis in V . For fixed m and N in \mathbb{N} with $N < m$ and for $\varphi \in K$, define the $m - N$ -dimensional random vector $Y := (\langle \varphi(Z), f_{N+1} \rangle, \dots, \langle \varphi(Z), f_m \rangle)$. The characteristic function $\chi_Y: \mathbb{R}^{m-N} \rightarrow \mathbb{C}$ of Y is given for $\beta = (\beta_{N+1}, \dots, \beta_m) \in \mathbb{R}^{m-N}$ by

$$\chi_Y(\beta) = E \left[\exp \left(i \sum_{k=N+1}^m \beta_k \langle \varphi(Z), f_k \rangle \right) \right] = \chi_Z \left(\sum_{k=N+1}^m \varphi^*(\beta_k f_k) \right).$$

Let $\varepsilon > 0$ be given. The continuity of the characteristic function $\chi_Z: U \rightarrow \mathbb{C}$ implies that there exists a $\delta > 0$ such that

$$(30) \quad |1 - \chi_Z(u)| \leq \varepsilon \mathbb{1}_{B_\delta}(u) + 2 \mathbb{1}_{B_\delta^c}(u) \leq \varepsilon + 2 \frac{\|u\|^2}{\delta^2} \quad \text{for all } u \in U,$$

where $B_\delta := \{u \in U : \|u\| \leq \delta\}$. By applying [27], Proposition IV.5.2, p. 205, we obtain for every $r > 0$ the inequality

$$(31) \quad \begin{aligned} P \left(\sum_{k=N+1}^m \langle \varphi(Z), f_k \rangle^2 \geq r^2 \right) &= P_Y(\beta \in \mathbb{R}^{m-N} : |\beta| \geq r) \\ &\leq 3 \int_{\mathbb{R}^{m-N}} \left(1 - \chi_Y \left(\frac{\beta}{r} \right) \right) d\gamma_{m-N}(\beta), \end{aligned}$$

where γ_{m-N} denotes the standard normal distribution on $\mathcal{B}(\mathbb{R}^{m-N})$. Inequality (30) implies

$$\begin{aligned} &\int_{\mathbb{R}^{m-N}} \left(1 - \chi_Y \left(\frac{\beta}{r} \right) \right) d\gamma_{m-N}(\beta) \\ &= \int_{\mathbb{R}^{m-N}} \left(1 - \chi_Z \left(\frac{1}{r} \sum_{k=N+1}^m \varphi^*(\beta_k f_k) \right) \right) d\gamma_{m-N}(\beta_{N+1}, \dots, \beta_m) \\ &\leq \int_{\mathbb{R}^{m-N}} \left(\varepsilon + \frac{2}{\delta^2 r^2} \left\| \sum_{k=N+1}^m \varphi^*(\beta_k f_k) \right\|^2 \right) d\gamma_{m-N}(\beta_{N+1}, \dots, \beta_m) \\ &= \varepsilon + \frac{2}{\delta^2 r^2} \sum_{k=N+1}^m \sum_{\ell=N+1}^m \langle \varphi^*(f_k), \varphi^*(f_\ell) \rangle \int_{\mathbb{R}^{m-N}} \beta_k \beta_\ell d\gamma_{m-N}(\beta_{N+1}, \dots, \beta_m) \\ &= \varepsilon + \frac{2}{\delta^2 r^2} \sum_{k=N+1}^m \|\varphi^* f_k\|^2. \end{aligned}$$

By applying the last estimate to inequality (31), we obtain for every $r > 0$

$$(32) \quad \begin{aligned} \sup_{\varphi \in K} P \left(\sum_{k=N+1}^m \langle \varphi(Z), f_k \rangle^2 \geq r \right) &= \sup_{\varphi \in K} \lim_{m \rightarrow \infty} P \left(\sum_{k=N+1}^m \langle \varphi(Z), f_k \rangle^2 \geq r \right) \\ &\leq 3\varepsilon + \frac{6}{\delta^2 r^2} \sup_{\varphi \in K} \sum_{k=N+1}^{\infty} \|\varphi^* f_k\|^2. \end{aligned}$$

Since the mapping $\varphi \mapsto \varphi^*$ is continuous on $\mathcal{L}_2(U, V)$, the set $\{\varphi^* : \varphi \in K\}$ is compact in $\mathcal{L}_2(V, U)$. Thus, Condition (29) follows from (32) by the characterisation (4) of the compact set K . Similarly, we conclude for every $r > 0$ that

$$P(\|\varphi(Z)\| \geq r) = \lim_{m \rightarrow \infty} P\left(\sum_{k=1}^m \langle \varphi(Z), f_k \rangle^2 \geq r^2\right) \leq 3\varepsilon + \frac{6}{\delta^2 r^2} \sup_{\varphi \in K} \|\varphi\|_{\mathcal{L}_2}^2.$$

Thus, Condition (28) follows from boundedness of K , which completes the proof. □

LEMMA 5.4. *If the set $\{\mu_\alpha : \alpha \in J\}$ of infinitely divisible probability measures μ_α on $\mathcal{B}(V)$ for an arbitrary index set J is relatively compact in $\mathcal{M}_1(V)$, then*

$$\{\mu_{\alpha_1}^{*t_1} * \dots * \mu_{\alpha_n}^{*t_n} : \alpha_i \in J, t_i \geq 0, t_1 + \dots + t_n \leq T \text{ for } i = 1, \dots, n, n \in \mathbb{N}\}$$

is also relatively compact in $\mathcal{M}_1(V)$.

PROOF. According to [20], Theorem VI.5.3, p. 187, the set $\{\mu_\alpha : \alpha \in J\}$ of infinitely divisible probability measures μ_α with characteristics $(a_\alpha, Q_\alpha, \nu_\alpha)$ is relatively compact if and only if the following three conditions are satisfied:

- (1) the set $\{a_\alpha : \alpha \in J\} \subseteq V$ is relatively compact;
- (2) the set $\{\nu_\alpha : \alpha \in J\}$ restricted to the complement of any neighborhood of the origin is relatively compact;
- (3) the operators $T_\alpha : V \rightarrow V$ defined by

$$\langle T_\alpha v, v \rangle := \langle Q_\alpha v, v \rangle + \int_{\|h\| \leq 1} \langle v, h \rangle^2 \nu_\alpha(dh)$$

satisfy the conditions

- (i) $\sup_{\alpha \in J} \sum_{k=1}^\infty \langle T_\alpha f_k, f_k \rangle < \infty,$
- (ii) $\lim_{N \rightarrow \infty} \sup_{\alpha \in J} \sum_{k=N}^\infty \langle T_\alpha f_k, f_k \rangle = 0.$

For $\alpha_i \in J, t_i \geq 0$ and $t_1 + \dots + t_n \leq T$, the infinitely divisible probability measure $\mu_{\alpha_1}^{*t_1} * \dots * \mu_{\alpha_n}^{*t_n}$ has the characteristics

$$\left(\sum_{i=1}^n t_i a_{\alpha_i}, \sum_{i=1}^n t_i Q_{\alpha_i}, \sum_{i=1}^n t_i \nu_{\alpha_i} \right).$$

It remains to show that these characteristics satisfy the corresponding Conditions (1)–(3) above.

(1) Define the set

$$D := \left\{ \sum_{i=1}^n t_i a_{\alpha_i} : \alpha_i \in J, t_1, \dots, t_n \geq 0, t_1 + \dots + t_n \leq T, n \in \mathbb{N} \right\}.$$

Since the set $A := \{a_\alpha : \alpha \in J\}$ is relatively compact, it follows by a theorem of S. Mazur that the convex hull $\text{co}(A)$ of A is relatively compact. Since the mapping

$$m : [0, T] \times \overline{\text{co}(A)} \rightarrow V, \quad m(t, v) = tv,$$

is continuous and $D \subseteq m([0, T] \times \overline{\text{co}(A)})$, we can conclude that the set D is relatively compact.

(2) Prokhorov’s theorem guarantees that the set $\{v_\alpha : \alpha \in J\}$ restricted to the complement of any neighborhood of the origin is tight and uniformly bounded in total variation norm. Clearly, the same applies to

$$\left\{ \sum_{i=1}^n t_i v_{\alpha_i} : \alpha_i \in J, t_1, \dots, t_n \geq 0, t_1 + \dots + t_n \leq T, n \in \mathbb{N} \right\},$$

and another application of Prokhorov’s theorem shows that this set restricted to the complement of any neighborhood of the origin is relatively compact.

(3) For every $n, N \in \mathbb{N}$, one obtains

$$\begin{aligned} & \sup_{\substack{t_1 + \dots + t_n \leq T \\ t_1, \dots, t_n \geq 0}} \sup_{\alpha_1, \dots, \alpha_n \in J} \sum_{k=N}^\infty \sum_{i=1}^n \langle t_i T_{\alpha_i} f_k, f_k \rangle \\ & \leq \sup_{\substack{t_1 + \dots + t_n \leq T \\ t_1, \dots, t_n \geq 0}} \sum_{i=1}^n t_i \sup_{\alpha \in J} \sum_{k=N}^\infty \langle T_\alpha f_k, f_k \rangle \\ & \leq T \sup_{\alpha \in J} \sum_{k=N}^\infty \langle T_\alpha f_k, f_k \rangle \rightarrow 0 \quad \text{as } N \rightarrow \infty. \end{aligned}$$

Analogously, we conclude

$$\sup_{\substack{t_1 + \dots + t_n \leq T \\ t_1, \dots, t_n \geq 0}} \sup_{\alpha_1, \dots, \alpha_n \in J} \sum_{k=1}^\infty \sum_{i=1}^n \langle t_i T_{\alpha_i} f_k, f_k \rangle \leq T \sup_{\alpha \in J} \sum_{k=1}^\infty \langle T_\alpha f_k, f_k \rangle < \infty.$$

The proof is complete. \square

For the proof of Theorem 5.1, we now introduce an alternative definition of a stochastic integral $\tilde{I}(\Psi)(t)$ for simple integrands Ψ . Its definition guarantees that the integral processes $(\tilde{I}(\Psi_n)(t) : t \in [0, T])$ for an approximating sequence (Ψ_n) of simple integrands converge uniformly in probability (Proposition 5.6), which guarantees that its limit has càdlàg trajectories. The original stochastic integral

$I(\Psi_n)(t)$ converges only as a V -valued random variable, that is, at each fixed time $t \in [0, T]$; see Theorem 5.1.

For the definition, assume that $\Psi \in \mathcal{H}_0(U, V)$ is of the form

$$\Psi(t) = \Phi_0 \mathbb{1}_{\{0\}}(t) + \sum_{j=1}^N \Phi_j \mathbb{1}_{(t_j, t_{j+1}]}(t) \quad \text{for all } t \in [0, T],$$

where $0 = t_1 < \dots < t_{N+1} = T$ is a finite sequence of deterministic times and each $\Phi_j : \Omega \rightarrow \mathcal{L}_2(U, V)$ is an \mathcal{F}_{t_j} -measurable random variable for each $j = 0, \dots, N$. Then we define a V -valued stochastic process $(\tilde{I}(\Psi)(t) : t \in [0, T])$ by

$$\tilde{I}(\Psi)(t) := \begin{cases} 0 & \text{if } t \in [0, t_2), \\ \sum_{j=1}^k \Phi_j (L(t_{j+1}) - L(t_j)) & \text{if } t \in [t_{k+1}, t_{k+2}), k \in \{1, \dots, N - 1\}, \\ \sum_{j=1}^N \Phi_j (L(t_{j+1}) - L(t_j)) & \text{if } t = T. \end{cases}$$

Obviously, $\tilde{I}(\Psi)$ is an adapted stochastic process in V with càdlàg trajectories.

PROPOSITION 5.5. *For every $n \in \mathbb{N}$, let Ψ_n be a simple stochastic process in $\mathcal{H}_0(U, V)$ defined on a partition $\{t_{n,k}\}_{k=1, \dots, N_n+1}$ and define $\mathcal{G}_n := \{\mathcal{F}_{t_{n,k}} : k = 1, \dots, N_n + 1\}$. If $\{\Psi_n : n \in \mathbb{N}\}$ is tight in $\mathcal{M}_1(D_-([0, T]; \mathcal{L}_2(U, V)))$ then*

$$\{\tilde{I}(\Psi_n)(\tau) : \tau \in \Upsilon(\mathcal{G}_n), n \in \mathbb{N}\}$$

is tight in V , where $\Upsilon(\mathcal{G}_n) = \{\tau : \Omega \rightarrow \{t_{n,1}, \dots, t_{n,N_n+1}\} : \text{is stopping time for } \mathcal{G}_n\}$.

PROOF. Each Ψ_n is of the form

$$\Psi_n(t) = \Phi_{n,0} \mathbb{1}_{\{0\}}(t) + \sum_{j=1}^{N_n} \Phi_{n,j} \mathbb{1}_{(t_{n,j}, t_{n,j+1}]}(t),$$

for $0 = t_{n,1} < \dots < t_{n,N_n+1} = T$ and $\Phi_{n,j} \in L_P^0(\Omega, \mathcal{F}_{t_{n,j}}; \mathcal{L}_2)$ for each $j = 0, \dots, N_n$ and $n \in \mathbb{N}$. Define for each $j = 2, \dots, N_n + 1$ and $n \in \mathbb{N}$ the $\mathcal{F}_{t_{n,j}}$ -measurable random variable

$$X_{n,j} := \Phi_{n,j-1}(L(t_{n,j}) - L(t_{n,j-1})) : \Omega \rightarrow V,$$

and choose a regular conditional distribution

$$P_{n,j} : \mathcal{B}(V) \times \Omega \rightarrow [0, 1], \quad P_{n,j}(B, \omega) = P(X_{n,j} \in B | \mathcal{F}_{t_{n,j-1}})(\omega).$$

Lemma 4.3 guarantees for every $v \in V$ that

$$E[\exp(i\langle X_{n,j}, v \rangle) | \mathcal{F}_{t_{n,j-1}}] = \exp((t_{n,j} - t_{n,j-1})S(\Phi_{n,j-1}^* v)) \quad P\text{-a.s.},$$

where $S: U \rightarrow \mathbb{C}$ denotes the cylindrical Lévy symbol of L . It follows for P -a.a. $\omega \in \Omega$ that

$$(33) \quad P_{n,j}(\cdot, \omega) = (\lambda \circ (\Phi_{n,j-1}(\omega))^{-1})^{*(t_{n,j}-t_{n,j-1})},$$

where λ is the cylindrical distribution of $L(1)$. For $\tau \in \Upsilon(\mathcal{G}_n)$ introduce the notation

$$[\tau](\omega) := \inf\{k \in \{1, \dots, N_n + 1\} : \tau(\omega) = t_{n,k}\}.$$

Define for every stopping time $\tau \in \Upsilon(\mathcal{G}_n)$ with $2 \leq [\tau] \leq N_n + 1$ the random probability measure

$$P_n(\tau): \mathcal{B}(V) \times \Omega \rightarrow [0, 1], \quad P_n(\tau) = P_{n,2} * \dots * P_{n,[\tau]}.$$

Let $\varepsilon > 0$ be given. Since $\{\Psi_n : n \in \mathbb{N}\}$ is tight, there exists a compact set $C \subseteq D_-([0, T]; \mathcal{L}_2(U, V))$ such that $P(\Psi_n \in C) \geq 1 - \varepsilon$ for all $n \in \mathbb{N}$. Proposition 1.6 in [9] guarantees that there exists a compact set $K \subseteq \mathcal{L}_2(U, V)$ such that $\{\Psi_n \in C\} \subseteq \{\Psi_n(t) \in K \text{ for all } t \in [0, T]\}$ for all $n \in \mathbb{N}$. Consequently, the set

$$A_n := \{\Psi_n(t) \in K \text{ for all } t \in [0, T]\} = \{\Phi_{n,j} \in K \text{ for all } j = 0, \dots, N_n\},$$

satisfies $P(A_n) \geq 1 - \varepsilon$ for all $n \in \mathbb{N}$. Denoting $\lambda_\varphi := \lambda \circ \varphi^{-1}$ for every $\varphi \in K$, Proposition 5.3 guarantees that the set $\{\lambda_\varphi : \varphi \in K\}$ of infinitely divisible probability measures λ_φ is relatively compact in $\mathcal{M}_1(V)$. Lemma 5.4 yields that the set

$$\mathfrak{X} := \{\lambda_{\varphi_1}^{*s_1} * \dots * \lambda_{\varphi_n}^{*s_n} : s_1, \dots, s_n \geq 0, s_1 + \dots + s_n \leq T, \varphi_j \in K, n \in \mathbb{N}\}$$

is relatively compact in $\mathcal{M}_1(V)$. Since (33) implies

$$\begin{aligned} & \{P_n(\tau(\omega))(\cdot, \omega) : \tau \in \Upsilon(\mathcal{G}_n), 2 \leq [\tau] \leq N_n + 1, \omega \in A_n, n \in \mathbb{N}\} \\ & \subseteq \{P_n(k)(\cdot, \omega) : k \in \{2, \dots, N_n + 1\}, \omega \in A_n, n \in \mathbb{N}\} \\ & \subseteq \mathfrak{X}, \end{aligned}$$

it follows that the set

$$\{P_n(\tau) : \tau \in \Upsilon(\mathcal{G}_n), 2 \leq [\tau] \leq N_n + 1, n \in \mathbb{N}\}$$

of random probability measures is tight. Theorem 3.1 implies that

$$\{X_{n,2} + \dots + X_{n,[\tau]} : \tau \in \Upsilon(\mathcal{G}_n), 2 \leq [\tau] \leq N_n + 1, n \in \mathbb{N}\}$$

is tight which completes the proof. \square

PROPOSITION 5.6. *For every sequence $(\Psi_n)_{n \in \mathbb{N}} \subseteq \mathcal{H}_0(U, V)$ which converges to some $\Psi \in \mathcal{H}(U, V)$ in probability in the Skorokhod metric, that is,*

$$\lim_{n \rightarrow \infty} P(d_J(\Psi_n, \Psi) \geq \varepsilon) = 0 \quad \text{for all } \varepsilon > 0,$$

there exists a V -valued, adapted stochastic process $(I(\Psi)(t) : t \in [0, T])$ with càdlàg trajectories obeying for each $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} P \left(\sup_{t \in [0, T]} \|\tilde{I}(\Psi_n)(t) - I(\Psi)(t)\| \geq \varepsilon \right) = 0.$$

The limit $I(\Psi)$ does not depend on the sequence $(\Psi_n)_{n \in \mathbb{N}}$, that is, it is unique up to evanescence.

PROOF. It is sufficient to show for an arbitrary $\varepsilon > 0$ that

$$(34) \quad \lim_{m, n \rightarrow \infty} P \left(\sup_{t \in [0, T]} \|\tilde{I}(\Psi_m)(t) - \tilde{I}(\Psi_n)(t)\| > \varepsilon \right) = 0.$$

Recall that the V -valued stochastic process $(\tilde{I}(\Psi_m)(t) : t \in [0, T])$ has càdlàg paths due to its definition. Define for each $m, n \in \mathbb{N}$ the stopping time

$$\tau_{m,n} := \inf\{t > 0 : \|\tilde{I}(\Psi_m)(t) - \tilde{I}(\Psi_n)(t)\| > \varepsilon\} \wedge T,$$

where $\inf\{\emptyset\} = \infty$. By the very definition of $\tau_{m,n}$ it follows that (34) is satisfied if and only if

$$(35) \quad \lim_{m, n \rightarrow \infty} P(\|\tilde{I}(\Psi_m)(\tau_{m,n}) - \tilde{I}(\Psi_n)(\tau_{m,n})\| > \varepsilon) = 0.$$

In order to establish (35), it is according to Lemma 2.4 in [11] sufficient to show that

$$(36) \quad \{\tilde{I}(\Psi_m)(\tau_{m,n}) - \tilde{I}(\Psi_n)(\tau_{m,n}) : m, n \in \mathbb{N}\} \quad \text{is tight in } V;$$

and that for every $v \in V$ we have

$$(37) \quad \lim_{m, n \rightarrow \infty} \langle \tilde{I}(\Psi_m)(\tau_{m,n}) - \tilde{I}(\Psi_n)(\tau_{m,n}), v \rangle = 0 \quad \text{in probability.}$$

By merging the partitions where Ψ_m and Ψ_n are defined on, we obtain for every $m, n \in \mathbb{N}$ and $t \in [0, T]$ the representation

$$\Psi_m(t) - \Psi_n(t) = \Phi_{m,n,0} \mathbb{1}_{\{0\}}(t) + \sum_{j=1}^{N_{m,n}} \Phi_{m,n,j} \mathbb{1}_{(t_{m,n,j}, t_{m,n,j+1}]}(t),$$

where $0 = t_{m,n,1} < \dots < t_{m,n,N_{m,n}+1} = T$ is a finite sequence of deterministic times and $\Phi_{m,n,j} : \Omega \rightarrow \mathcal{L}_2(U, V)$ is an $\mathcal{F}_{t_{m,n,j}}$ -measurable random variable for each $j = 0, \dots, N_{m,n}$.

In order to establish (36), note that for each $m, n \in \mathbb{N}$ the stochastic process $(\tilde{I}(\Psi_m - \Psi_n)(t) : t \in [0, T])$ varies only at points of the partition $\pi_{m,n} := \{t_{m,n,k} : k = 2, \dots, N_{m,n} + 1\}$. Consequently, the stopping time $\tau_{m,n}$ only attains values in $\pi_{m,n}$, and thus it follows $\tau_{m,n} \in \Upsilon(\mathcal{G}_{m,n})$ for $\mathcal{G}_{m,n} := \{\mathcal{F}_{t_{m,n,k}} : k = 1, \dots, N_{m,n} + 1\}$. Consequently, one can apply Proposition 5.5 to conclude (36).

For establishing (37), define for every $j = 2, \dots, N_{m,n} + 1$ the V -valued random variable

$$X_{m,n,j} := \Phi_{m,n,j-1}(L(t_{m,n,j}) - L(t_{m,n,j-1})): \Omega \rightarrow V.$$

Obviously, we have

$$(38) \quad \tilde{I}(\Psi_m)(\tau_{m,n}) - \tilde{I}(\Psi_n)(\tau_{m,n}) = X_{m,n,2} + \dots + X_{m,n,[\tau_{m,n}]},$$

where we use the notation

$$[\tau_{m,n}](\omega) := \inf\{k \in \{2, \dots, N_{m,n} + 1\} : \tau_{m,n}(\omega) = t_{m,n,k}\}.$$

Lemma 4.3 implies for all $\beta \in \mathbb{R}$ and $v \in V$

$$\begin{aligned} F_{m,n,j}(\beta) &:= E[e^{i\beta \langle X_{m,n,j}(t), v \rangle} | \mathcal{F}_{t_{m,n,j-1}}] \\ &= \exp((t_{m,n,j} - t_{m,n,j-1})S(\beta \Phi_{m,n,j-1}^* v)), \quad P\text{-a.s.} \end{aligned}$$

Consequently, we obtain P -a.s. that

$$\begin{aligned} F_{m,n}(\beta) &:= \prod_{j=2}^{[\tau_{m,n}]} F_{m,n,j}(\beta) = \exp\left(\sum_{j=2}^{[\tau_{m,n}]} (t_{m,n,j} - t_{m,n,j-1})S(\beta \Phi_{m,n,j-1}^* v)\right) \\ &= \exp\left(\int_0^{\tau_{m,n}} S(\beta(\Psi_m^*(s) - \Psi_n^*(s))v) ds\right). \end{aligned}$$

In order to show $F_{m,n}(\beta) \rightarrow 1$ in probability for $m, n \rightarrow \infty$ we have to show that each subsequence $(F_{m_k, n_k}(\beta))_{k \in \mathbb{N}}$ has a further subsequence converging to 1 P -a.s. As $d_J(\Psi_{m_k}, \Psi)$ and $d_J(\Psi_{n_k}, \Psi)$ converge to 0 in probability for $k \rightarrow \infty$ there exists subsequences $(d_J(\Psi_{m_{k_\ell}}, \Psi))_{\ell \in \mathbb{N}}$ and $(d_J(\Psi_{n_{k_\ell}}, \Psi))_{\ell \in \mathbb{N}}$ converging to 0 P -a.s. for $\ell \rightarrow \infty$. It follows that there exists a set $\Omega_0 \in \mathcal{A}$ with $P(\Omega_0) = 1$ such that for each $\omega \in \Omega_0$ we have that

$$(39) \quad \sup_{\ell \in \mathbb{N}} \sup_{t \in [0, T]} \|\Psi_{m_{k_\ell}}(t)(\omega) - \Psi_{n_{k_\ell}}(t)(\omega)\|_{\mathcal{L}_2} < \infty,$$

and that there exists a Lebesgue null set $N_\omega \subseteq [0, T]$ depending on ω such that

$$(40) \quad \lim_{\ell \rightarrow \infty} \|\Psi_{m_{k_\ell}}(s)(\omega) - \Psi_{n_{k_\ell}}(s)(\omega)\|_{\mathcal{L}_2} = 0 \quad \text{for all } s \in [0, T] \setminus N_\omega.$$

Lemma 3.2 in [26] guarantees that the cylindrical Lévy symbol S maps bounded sets to bounded sets. Consequently, we can conclude from Lebesgue's theorem of dominated convergence by applying (39) and (40), that

$$\begin{aligned} &\lim_{\ell \rightarrow \infty} \left| \int_0^{\tau_{m_{k_\ell}, n_{k_\ell}}(\omega)} S(\beta(\Psi_{m_{k_\ell}}^*(s)(\omega) - \Psi_{n_{k_\ell}}^*(s)(\omega))v) ds \right| \\ &\leq \lim_{\ell \rightarrow \infty} \int_0^T |S(\beta(\Psi_{m_{k_\ell}}^*(s)(\omega) - \Psi_{n_{k_\ell}}^*(s)(\omega))v)| ds = 0. \end{aligned}$$

Since we considered an arbitrary subsequence, we can conclude that $F_{m,n}(\beta) \rightarrow 1$ in probability for $m, n \rightarrow \infty$ for every $\beta \in \mathbb{R}$. The principle of conditioning, Theorem 3.2, yields

$$\lim_{m,n \rightarrow \infty} E \left[\exp \left(i\beta \sum_{j=2}^{[\tau_{m,n}]} \langle X_{m,n,j}, v \rangle \right) \right] = 1 \quad \text{for every } \beta \in \mathbb{R}.$$

Because of the representation (38), this establishes (37).

Let $(\Psi_n)_{n \in \mathbb{N}}$ and $(\Psi'_n)_{n \in \mathbb{N}}$ be two sequences converging to Ψ in probability in the Skorokhod metric d_J and denote by $I(\Psi)$ and $I'(\Psi)$ the limits of $(\tilde{I}(\Psi_n))_{n \in \mathbb{N}}$ and $(\tilde{I}(\Psi'_n))_{n \in \mathbb{N}}$. As in the proof of (37), we can conclude that

$$\lim_{n \rightarrow \infty} \langle \tilde{I}(\Psi_n - \Psi'_n)(t), v \rangle = 0 \quad \text{in probability for all } v \in V \text{ and } t \in [0, T],$$

which shows $I(\Psi)(t) = I'(\Psi)(t)$ P -a.s. for each $t \in [0, T]$. Since the stochastic processes $I(\Psi)$ and $I'(\Psi)$ have càdlàg paths, it follows that they are indistinguishable. \square

Combining Lemma 5.2 and Proposition 5.6 enables us to define for every $\Psi \in \mathcal{H}(U, V)$

$$I(\Psi) := \lim_{n \rightarrow \infty} \tilde{I}(\Psi_n),$$

where $(\Psi_n)_{n \in \mathbb{N}}$ is an approximating sequence of simple processes in $\mathcal{H}_0(U, V)$ and the limit is in probability in the uniform norm as stated in Proposition 5.6.

PROPOSITION 5.7. *If Ψ is in $\mathcal{H}(U, V)$, then the stochastic process $(I(\Psi)(t) : t \in [0, T])$ is a semimartingale.*

PROOF. Denote by $\mathcal{E}(V, V)$ the space of all adapted, càglàd, simple processes with values in $\mathcal{L}(V, V)$ which are bounded by 1, that is each $\Theta \in \mathcal{E}(V, V)$ is of the form

$$(41) \quad \Theta(t) = \Gamma_0 \mathbb{1}_{\{0\}}(t) + \sum_{k=1}^N \Gamma_k \mathbb{1}_{(s_k, s_{k+1}]}(t) \quad \text{for all } t \in [0, T],$$

where $0 = s_1 < \dots < s_{N+1} = T$ is a finite sequence of deterministic times and each $\Gamma_k : \Omega \rightarrow \mathcal{L}(V, V)$ is an \mathcal{F}_{s_k} -measurable random variable with $\|\Gamma_k(\omega)\|_{V \rightarrow V} \leq 1$ for all $\omega \in \Omega$ and for each $k = 0, \dots, N$. The elementary integral is then defined by

$$\int_0^T \Theta(s) I(\Psi)(ds) = \sum_{k=1}^N \Gamma_k (I(\Psi)(s_{k+1}) - I(\Psi)(s_k)).$$

Theorem 2.1 in [12] shows that $I(\Psi)$ is a semimartingale if and only

$$(42) \quad \left\{ \left\| \int_0^T \Theta(s) I(\Psi)(ds) \right\| : \Theta \in \mathcal{E}(V, V) \right\} \quad \text{is stochastically bounded.}$$

Lemma 5.2 guarantees that there exists a sequence $(\Psi_n)_{n \in \mathbb{N}}$ of simple processes in $\mathcal{H}_0(U, V)$ converging to Ψ in probability in the Skorokhod metric. As Proposition 5.6 implies that $I(\Psi_n)(s)$ converges to $I(\Psi)(s)$ in probability for all $s \in [0, T]$, it follows that (42) is established by showing

$$(43) \quad \left\{ \left\| \int_0^T \Theta(s) \tilde{I}(\Psi_n)(ds) \right\| : \Theta \in \mathcal{E}(V, V), n \in \mathbb{N} \right\}$$

is stochastically bounded. Each Ψ_n is of the form

$$(44) \quad \Psi_n(t) = \Phi_{n,0} \mathbb{1}_{\{0\}}(t) + \sum_{\ell=1}^{N_n} \Phi_{n,\ell} \mathbb{1}_{(t_{n,\ell}, t_{n,\ell+1}]}(t) \quad \text{for all } t \in [0, T],$$

for $0 = t_{n,1} < \dots < t_{n,N_n+1} = T$ and $\Phi_{n,\ell} \in L^0_P(\Omega, \mathcal{F}_{t_{n,\ell}}; \mathcal{L}_2)$ for each $\ell = 1, \dots, N_n$ and $n \in \mathbb{N}$. For given $\Theta \in \mathcal{E}(V, V)$ of the form (41) and $\Psi_n \in \mathcal{H}_0(U, V)$ of the form (44), we can assume, by possibly enlarging the partition $(t_{n,\ell})_{\ell=1, \dots, N_n+1}$, that for every $k \in \{1, \dots, N+1\}$ there exists $\ell_k \in \{1, \dots, N_n+1\}$ such that $s_k = t_{n,\ell_k}$. It follows that

$$\begin{aligned} \int_0^T \Theta(s) \tilde{I}(\Psi_n)(ds) &= \sum_{k=1}^N \Gamma_k (\tilde{I}(\Psi_n)(s_{k+1}) - \tilde{I}(\Psi_n)(s_k)) \\ &= \sum_{k=1}^N \Gamma_k \left(\sum_{\ell=\ell_k}^{\ell_{k+1}-1} \Phi_{n,\ell} (L(t_{n,\ell+1}) - L(t_{n,\ell})) \right) \\ &= \sum_{\ell=1}^{N_n} (\tilde{\Gamma}_\ell \circ \Phi_{n,\ell})(L(t_{n,\ell+1}) - L(t_{n,\ell})), \end{aligned}$$

where we use the definition

$$\tilde{\Gamma}_\ell := \Gamma_k \quad \text{for every } \ell \in \{\ell_k, \dots, \ell_{k+1} - 1\}.$$

Note, that $\tilde{\Gamma}_\ell$ is $\mathcal{F}_{t_{n,\ell}}$ -measurable for all $\ell = 1, \dots, N_n$. Define for each $n \in \mathbb{N}$ and $\ell = 2, \dots, N_n + 1$ the $\mathcal{F}_{t_{n,\ell}}$ -measurable random variable

$$X_{n,\ell}^\ominus := (\tilde{\Gamma}_{\ell-1} \circ \Phi_{n,\ell-1})(L(t_{n,\ell}) - L(t_{n,\ell-1})): \Omega \rightarrow V.$$

Obviously, we have

$$\int_0^T \Theta(s) \tilde{I}(\Psi_n)(ds) = X_{n,2}^\ominus + \dots + X_{n,N_n+1}^\ominus.$$

Choose a regular conditional distribution

$$P_{n,\ell}^\Theta : \mathcal{B}(V) \times \Omega \rightarrow [0, 1], \quad P_{n,\ell}^\Theta(B, \omega) = P(X_{n,\ell}^\Theta \in B | \mathcal{F}_{t_{n,\ell-1}})(\omega).$$

Lemma 4.3 guarantees for every $v \in V$ that P -a.s.

$$E[\exp(i\langle X_{n,\ell}^\Theta, v \rangle) | \mathcal{F}_{t_{n,\ell-1}}] = \exp((t_{n,\ell} - t_{n,\ell-1})S((\tilde{\Gamma}_{\ell-1} \circ \Phi_{n,\ell-1})^*v)),$$

where $S : U \rightarrow \mathbb{C}$ denotes the cylindrical Lévy symbol of L . It follows for P -a.a. $\omega \in \Omega$

$$(45) \quad P_{n,\ell}^\Theta(\cdot, \omega) = (\lambda \circ (\tilde{\Gamma}_{\ell-1}(\omega) \circ \Phi_{n,\ell-1}(\omega))^{-1})^{*(t_{n,\ell} - t_{n,\ell-1})},$$

where λ is the cylindrical distribution of $L(1)$. Define for every $k \in \{2, \dots, N_n + 1\}$ the random probability measure

$$P_n^\Theta(k) : \mathcal{B}(V) \times \Omega \rightarrow [0, 1], \quad P_n^\Theta(k) = P_{n,2}^\Theta * \dots * P_{n,k}^\Theta.$$

Let $\varepsilon > 0$ be given. Since $\{\Psi_n : n \in \mathbb{N}\}$ is tight, we can conclude as in the proof of Proposition 5.5 by using Proposition 1.6 in [9] that there exists a compact set $K \subseteq \mathcal{L}_2(U, V)$ such that the sets

$$A_n := \{\Phi_{n,\ell} \in K \text{ for all } \ell = 1, \dots, N_n\},$$

satisfy $P(A_n) \geq 1 - \varepsilon$ for all $n \in \mathbb{N}$. The ideal property of $\mathcal{L}_2(U, V)$ guarantees that the set

$$K_\varepsilon := \{\vartheta \circ \psi : \psi \in K, \vartheta \in \mathcal{L}(V, V) \text{ with } \|\vartheta\|_{V \rightarrow V} \leq 1\}$$

is a subset of $\mathcal{L}_2(U, V)$. Moreover, as K is compact it follows that K_ε is bounded and satisfies (4), and thus the closure \overline{K}_ε is a compact set in $\mathcal{L}_2(U, V)$. Denoting $\lambda_\sigma := \lambda \circ \sigma^{-1}$ for every $\sigma \in \overline{K}_\varepsilon$, Proposition 5.3 guarantees that the set $\{\lambda_\sigma : \sigma \in \overline{K}_\varepsilon\}$ of infinitely divisible probability measures λ_σ is relatively compact. Lemma 5.4 yields that the set

$$\begin{aligned} \mathfrak{X} := & \{\lambda_{\sigma_1}^{*s_1} * \dots * \lambda_{\sigma_n}^{*s_n} : s_j \geq 0, s_1 + \dots + s_n \leq T, \\ & \sigma_j \in \overline{K}_\varepsilon, j = 1, \dots, n, n \in \mathbb{N}\} \end{aligned}$$

is relatively compact. Since (45) implies

$$\{P_n^\Theta(k)(\cdot, \omega) : k \in \{2, \dots, N_n + 1\}, \omega \in A_n, \Theta \in \mathcal{E}(V, V), n \in \mathbb{N}\} \subseteq \mathfrak{X},$$

it follows that the set

$$\{P_n^\Theta(k) : k \in \{2, \dots, N_n + 1\}, \Theta \in \mathcal{E}(V, V), n \in \mathbb{N}\}$$

of random probability measures is tight. Theorem 3.1 implies that

$$\{X_{n,2}^\Theta + \dots + X_{n,k}^\Theta : k \in \{2, \dots, N_n + 1\}, \Theta \in \mathcal{E}(V, V), n \in \mathbb{N}\}$$

is tight which establishes that the set (43) is stochastically bounded. \square

PROOF OF THEOREM 5.1. Let $(\Psi_n)_{n \in \mathbb{N}}$ be the sequence of simple processes in $\mathcal{H}_0(U, V)$ converging to Ψ in probability in the Skorokhod metric, which exists due to Lemma 5.2. In particular, $\{\Psi_n(t) : t \in [0, T]\}$ is in the closure of $\{\Psi(t) : t \in [0, T]\}$ and the partition $(t_{n,j})_{j=1, \dots, N_n+1}$ obeys

$$(46) \quad \lim_{n \rightarrow \infty} \sup_{j=1, \dots, N_n} |t_{n,j+1} - t_{n,j}| = 0.$$

Propositions 5.6 and 5.7 guarantee the existence of the adapted semimartingale $(I(\Psi)(t) : t \in [0, T])$ in V obeying

$$\sup_{t \in [0, T]} \|\tilde{I}(\Psi_n)(t) - I(\Psi)(t)\| \rightarrow 0 \quad \text{in probability.}$$

It remains to show that for each $t \in [0, T]$ the V -valued random variable

$$\Delta_n(t) := I(\Psi_n)(t) - \tilde{I}(\Psi_n)(t) : \Omega \rightarrow V,$$

converges to 0 in probability for $n \rightarrow \infty$. In order to show this, fix some $t \in (0, T)$ and denote by k_n the element in $\{1, \dots, N_n\}$ such that $t \in (t_{n,k_n}, t_{n,k_n+1}]$ and by Φ_{n,k_n} the $\mathcal{L}_2(U, V)$ -valued random variable satisfying $\Psi_n(t) = \Phi_{n,k_n}$ for all $n \in \mathbb{N}$. Thus, we obtain $\Delta_n(t) = \Phi_{n,k_n}(L(t) - L(t_{n,k_n}))$ for all $n \in \mathbb{N}$.

Choose a regular conditional distribution

$$P_n : \mathcal{B}(V) \times \Omega \rightarrow [0, 1], \quad P_n(B, \omega) = P(\Delta_n \in B | \mathcal{F}_{t_{n,k_n}})(\omega).$$

As in the proof of Proposition 5.5, it follows that $\{P_n : n \in \mathbb{N}\}$ is tight. By taking expectation, we obtain that $\{\Delta_n(t) : n \in \mathbb{N}\}$ is tight.

Furthermore, Lemma 4.3 implies for every $\beta \in \mathbb{R}$ and $v \in V$ that P -a.s.

$$(47) \quad E[\exp(i\beta \langle \Delta_n(t), v \rangle) | \mathcal{F}_{n,k_n}] = \exp((t - t_{n,k_n})S(\beta \Phi_{n,k_n}^* v)),$$

where $S : U \rightarrow \mathbb{C}$ denotes the cylindrical Lévy symbol of L ; see (8). The set $\{\Phi_{n,k_n}^*(\omega)v : n \in \mathbb{N}\}$ is uniformly bounded as $\Phi_{n,k_n}(\omega)$ is in the closure of $\{\Psi(t)(\omega) : t \in [0, T]\}$ for every $\omega \in \Omega$ and the closure of the latter is compact by Proposition 1.1 in [9]. As S maps bounded sets to bounded sets by Lemma 3.2 in [26], we conclude from (46) and (47) that

$$(48) \quad \lim_{n \rightarrow \infty} E[\exp(i\beta \langle \Delta_n(t), v \rangle) | \mathcal{F}_{n,k_n}] = 1 \quad P\text{-a.s. for every } \beta \in \mathbb{R}.$$

Taking expectation yields $\langle \Delta_n(t), v \rangle \rightarrow 0$ in probability for $n \rightarrow \infty$ for all $v \in V$. Together with tightness of $\{\Delta_n(t) : n \in \mathbb{N}\}$ it follows from Lemma 2.4 in [11] that $\Delta_n(t) \rightarrow 0$ for $n \rightarrow \infty$ in probability, which yields

$$(49) \quad \lim_{n \rightarrow \infty} P(\|I(\Psi_n)(t) - I(\Psi)(t)\| \geq \varepsilon) = 0 \quad \text{for every } \varepsilon > 0.$$

It remains to show that (49) holds true for each sequence $(\Psi'_n)_{n \in \mathbb{N}}$ in $\mathcal{H}_0(U, V)$ converging to Ψ in probability in the Skorokhod metric d_J ; that is, we have to establish for every $t \in [0, T]$ that

$$(50) \quad \lim_{n \rightarrow \infty} P(\|I(\Psi'_n)(t) - I(\Psi)(t)\| \geq \varepsilon) = 0 \quad \text{for every } \varepsilon > 0.$$

For this purpose, define for each $t \in [0, T]$ and $n \in \mathbb{N}$ the V -valued random variable

$$\Delta'_n(t) := I(\Psi_n)(t) - I(\Psi'_n)(t): \Omega \rightarrow V,$$

where $(\Psi_n)_{n \in \mathbb{N}}$ denotes the sequence from above. Because of (49), we can establish (50) by showing

$$(51) \quad \lim_{n \rightarrow \infty} P(\|\Delta'_n(t)\| \geq \varepsilon) = 0 \quad \text{for all } \varepsilon > 0.$$

In order to establish (51), it is according to Lemma 2.4 in [11] sufficient to show that

$$(52) \quad \{I(\Psi_n)(t) - I(\Psi'_n)(t) : n \in \mathbb{N}\} \quad \text{is tight in } V;$$

and that for every $v \in V$ we have

$$(53) \quad \lim_{n \rightarrow \infty} \langle I(\Psi_n)(t) - I(\Psi'_n)(t), v \rangle = 0 \quad \text{in probability.}$$

By merging the partitions where Ψ_n and Ψ'_n are defined, we obtain for every $n \in \mathbb{N}$ the representation

$$\Psi_n(t) - \Psi'_n(t) = \Phi_{n,0} \mathbb{1}_{\{0\}}(t) + \sum_{j=1}^{N_n} \Phi_{n,j} \mathbb{1}_{(t_{n,j}, t_{n,j+1}]}(t) \quad \text{for all } t \in [0, T],$$

where $0 = t_{n,1} < \dots < t_{n,N_n+1} = T$ is a finite sequence of deterministic times and $\Phi_{n,j}: \Omega \rightarrow \mathcal{L}_2(U, V)$ is an $\mathcal{F}_{t_{n,j}}$ -measurable random variable for each $j = 0, \dots, N_n$. For a fixed $t \in (0, T]$, we can assume that for every $n \in \mathbb{N}$ there exists $k_n \in \{2, \dots, N_n + 1\}$ such that $t = t_{n,k_n}$. Now we can prove (52) and (53) as (36) and (37) in the proof of Proposition 5.6. \square

Acknowledgements. The authors thank the referees for some helpful and insightful comments.

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