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ASYMPTOTICS FOR 2*D* CRITICAL FIRST PASSAGE PERCOLATION

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We consider first passage percolation on $\mathbb{Z}^2$ with i.i.d. weights, whose distribution function satisfies $F(0) = p_c = 1/2$. This is sometimes known as the "critical case" because large clusters of zero-weight edges force passage times to grow at most logarithmically, giving zero time constant. Denote $T(\mathbf{0}, \partial B(n))$ as the passage time from the origin to the boundary of the box $[-n, n] \times [-n, n]$. We characterize the limit behavior of $T(\mathbf{0}, \partial B(n))$ by conditions on the distribution function F. We also give exact conditions under which $T(\mathbf{0}, \partial B(n))$ will have uniformly bounded mean or variance. These results answer several questions of Kesten and Zhang from the 1990s and, in particular, disprove a conjecture of Zhang from 1999. In the case when both the mean and the variance go to infinity as $n \to \infty$, we prove a CLT under a minimal moment assumption. The main tool involves a new relation between first passage percolation and invasion percolation: up to a constant factor, the passage time in critical first passage percolation has the same first-order behavior as the passage time of an optimal path constrained to lie in an embedded invasion cluster.

1. Introduction.

1.1. *The model.* Consider the integer lattice $\mathbb{Z}^d$ and denote by $\mathcal{E}^d$ the set of nearest-neighbor edges. Given a distribution function F with $F(0^-) = 0$, let $(t_e : e \in \mathcal{E}^d)$ be a family of i.i.d. random variables (edge-weights) with common distribution function F. In first passage percolation, we study the random pseudometric on $\mathbb{Z}^d$ induced by these edge-weights.

The model is defined as follows. For $x, y \in \mathbb{Z}^d$, a (vertex self-avoiding) path from x to y is a sequence $(v_0, e_1, v_1, \ldots, e_n, v_n)$, where the v_i's, $i = 1, \ldots, n-1$, are distinct vertices in $\mathbb{Z}^d$ which are different from x or y, and $v_0 = x$, $v_n = y$; e_i is an edge in $\mathcal{E}^d$ which connects v_{i-1} and v_i. If $x = y$, the path is called a (vertex self-avoiding) circuit. For a path γ, we define the passage time of γ to be $T(\gamma) = \sum_{e \in \gamma} t_e$. For any $A, B \subset \mathbb{Z}^d$, we define the *first passage time* from A to B

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by

$$T(A, B) = \inf\{T(\gamma) : \gamma \text{ is a path from a vertex in } A \text{ to a vertex in } B\}.$$

For $A = \{x\}$, write $T(x, B)$ for $T(\{x\}, B)$ and similarly for B. A geodesic is a path γ from A to B such that $T(\gamma) = T(A, B)$.

From the sub-additive ergodic theorem, if $\mathbb{E}T(x, y) < \infty$ for all x, y then there exists a constant μ, called the *time constant*, such that

$$\lim_{n\to\infty} \frac{T(\mathbf{0}, n\mathbf{e}_1)}{n} = \mu \qquad \text{almost surely and in } L^1,$$

where $\mathbf{e}_1 = (1, 0, \dots, 0)$. It was shown by Kesten [14], Theorem 6.1, that

$$\mu = 0 \quad \text{if and only if} \quad F(0) \geq p_c, \tag{1.1}$$

where p_c is the critical probability for Bernoulli bond percolation on $\mathbb{Z}^d$. Therefore, the time constant does not provide much information if $F(0) \geq p_c$.

In [23], equation (3), Y. Zhang introduced the following random variable:

$$\rho(F) = \lim_{n\to\infty} T\big(\mathbf{0}, \partial B(n)\big),$$

where $B(n) = \{x \in \mathbb{Z}^2 : \|x\|_\infty \leq n\}$, $\partial B(n) = \{x \in \mathbb{Z}^2 : \|x\|_\infty = n\}$, and $\|\cdot\|_\infty$ is the sup-norm. By monotonicity, $\rho(F)$ exists almost surely. Note that $\rho(F) = \inf\{T(\gamma) : \gamma \text{ is an infinite path starting from } \mathbf{0}\}$. So if $F(0) > p_c$, then one immediately has $\rho(F) < \infty$ almost surely. Furthermore, it was shown in [23], page 254, that if $F(0) > p_c$ and t_e has all moments, then for any $m \in \mathbb{N}$, one has $\mathbb{E}\rho^m(F) < \infty$. Also, it is easy to see that if $F(0) < p_c$, then $\rho(F) = \infty$ almost surely. Then a natural question arises: how about $F(0) = p_c$?

In [24], Zhang proved that for $d = 2$, it is possible to have $\rho(F) < \infty$ or $\rho(F) = \infty$ almost surely when $F(0) = p_c$ [note that by the Kolmogorov zero-one law, either $\rho(F) < \infty$ almost surely or $\rho(F) = \infty$ almost surely]. Specifically, he introduced the following two distributions. For $a > 0$, set

$$F_a(x) = \begin{cases} 1, & \text{if } x^a > 1 - p_c, \\ x^a + p_c, & \text{if } 0 \leq x^a \leq 1 - p_c, \\ 0, & \text{if } x < 0, \end{cases}$$

and for $b > 0$, set

$$G_b(x) = \begin{cases} 1, & \text{if } \exp(-1/x^b) > 1 - p_c, \\ \exp(-1/x^b) + p_c, & \text{if } 0 \leq \exp(-1/x^b) \leq 1 - p_c, \\ 0, & \text{if } x < 0. \end{cases}$$

Zhang showed in [24], Theorem 8.1.1, that if a is small then $\rho(F_a) < \infty$ almost surely. He also made the following conjecture (see [24], page 146).

CONJECTURE 1.1 (Zhang). One has $\sup\{a > 0 : \rho(F_a) < \infty\} < \infty$.

Moreover, Zhang showed in [24], Theorem 8.1.3, that if $b > 1$, then $\rho(G_b) = \infty$ almost surely.

The critical case of first passage percolation is quite different from the standard one and requires different techniques. For example, the model is expected to retain rotational invariance in the limit [22], whereas the usual first passage model has lattice dependent and distribution dependent asymptotics. For this reason, analysis of the critical case relies on detailed estimates from critical and near-critical percolation (for instance, see [11, 20, 21]). The main new insight of our work is that the behavior of passage times is closely related to a "greedy" growth algorithm called invasion percolation, and that optimal paths constrained to lie in the invasion cluster have the correct first-order growth. This relation allows us to derive necessary and sufficient conditions on the edge-weight distribution to have diverging mean or variance for passage times (Theorems 1.2 and 1.5), and these results can be seen as finer versions of Kesten's condition (1.1) for $\mu = 0$. Furthermore, we can derive a type of universality: whenever the passage-time variance diverges, one has Gaussian fluctuations (see Theorem 1.6).

Constants in this paper may depend on the distribution function F and other fixed parameters such as η, r and λ. However, constants do not depend on k or n. We use $C_1, C_2, \dots$ to denote temporary constants whose meaning may vary, while we use notation like $K_{3.1}$ to denote the permanent constants. For example, $K_{3.1}$ denotes the constant in Lemma 3.1.

1.2. *Main results.* In this paper, we will give an exact criterion for $\rho(F) < \infty$ (see Corollary 1.3 below) and consequently provide a negative answer to Conjecture 1.1. Furthermore, we will derive limit theorems for the sequence $(T(\mathbf{0}, \partial B(n)))_{n\ge 1}$. From now on, suppose that $d = 2$ and that $F(0) = p_c$. Furthermore, define $F^{-1}(t) = \inf\{x : F(x) \ge t\}$ for $t > 0$ and

$$\eta_0 := \sup\{\eta \ge 0 : \mathbb{E}[t_e^{\eta/4}] < \infty\}. \tag{1.2}$$

The $1/4$ in the exponent comes from the fact [5], Lemma 3.1, that $\mathbb{E}T(x, y)^\alpha < \infty$ for all x, y and some $\alpha > 0$ if and only if $\mathbb{E}Y^\alpha < \infty$, where Y is the minimum of four i.i.d. variables with distribution function F. However, this last expectation is finite if $\mathbb{E}t_e^\beta < \infty$ for some $\beta > \alpha/4$. Using this, one can show that $\eta_0 > \eta_1$ is equivalent to $\mathbb{E}T(x, y)^{\eta_2} < \infty$ for all x, y and some $\eta_2 > \eta_1$.

1.2.1. *Behavior of the mean.* We begin with bounds on $\mathbb{E}T(\mathbf{0}, \partial B(n))$.

THEOREM 1.2. (i) *Assuming $\eta_0 > 1$, there is $C_1 = C_1(F) > 0$ such that*

$$\mathbb{E}T(\mathbf{0}, \partial B(2^n)) \le C_1 \sum_{k=2}^{n} F^{-1}(p_c + 2^{-k}) \qquad \textit{for } n \ge 2.$$

(ii) *There exists* $C_2 = C_2(F) > 0$ *such that*

$$\mathbb{E}T(\mathbf{0}, \partial B(2^n)) \geq C_2 \sum_{k=2}^{n} F^{-1}(p_c + 2^{-k}) \qquad \textit{for } n \geq 2.$$

REMARK 1. The moment condition in Theorem 1.2 is nearly optimal since, if $\mathbb{E}Y = \infty$ then, by bounding $T(\mathbf{0}, \partial B(2^n))$ below by the minimum of the 4 edge-weights incident to $\mathbf{0}$, one has $\mathbb{E}T(\mathbf{0}, \partial B(2^n)) = \infty$ for $n \geq 0$.

REMARK 2. The above theorem concerns the passage time from the point $\mathbf{0}$ to the set $\partial B(n)$. In Section 5.4, we derive asymptotics for point-to-point passage times $\mathbb{E}T(\mathbf{0}, x)$ for $x \in \mathbb{Z}^2$.

As a corollary, we have an exact criterion for finiteness of $\rho(F)$.

COROLLARY 1.3. *For any* F, *one has* $\rho(F) < \infty$ *almost surely if and only if* $\sum_{n=2}^{\infty} F^{-1}(p_c + 2^{-n}) < \infty$.

As an example (and to clarify the condition), if the right derivative of F at 0 exists and is positive (or infinite), then $\rho(F) < \infty$. This is not necessary, however, as many distributions with $\rho(F) < \infty$ have right derivative 0 at 0 (e.g., F_a with $a > 1$). We will now apply the above results to Zhang's distributions F_a and G_b. The proof follows by a direct computation and the previous corollary.

COROLLARY 1.4. *The following statements hold*:

1. $\rho(F_a) < \infty$ *almost surely for any* $a > 0$, *and so* $\sup\{a > 0 : \rho(F_a) < \infty\} = \infty$. *In particular, Conjecture* 1.1 *is false*.
2. $\rho(G_b) = \infty$ *almost surely if and only if* $b \geq 1$.

REMARK 3. Zhang asked in [24], page 145, if, under the assumption $\mathbb{E}t_e^m < \infty$ for all $m \in \mathbb{N}$, does $\rho(F) < \infty$ almost surely imply that $\mathbb{E}\rho(F) < \infty$? The answer is yes by combining Theorem 1.2 and Corollary 1.3.

1.2.2. *Behavior of the variance and limit theorems*. Now we consider $\mathrm{Var}(T(\mathbf{0}, \partial B(2^n)))$.

THEOREM 1.5. *Assume that* $\eta_0 > 2$:

(i) *There exists* $C_3 = C_3(F) > 0$ *such that*

$$\mathrm{Var}(T(\mathbf{0}, \partial B(2^n))) \leq C_3 \sum_{k=2}^{n} [F^{-1}(p_c + 2^{-k})]^2 \qquad \textit{for } n \geq 2.$$

(ii) *There exists* $C_4 = C_4(F) > 0$ *such that*

$$\mathrm{Var}(T(\mathbf{0}, \partial B(2^n))) \geq C_4 \sum_{k=2}^{n} [F^{-1}(p_c + 2^{-k})]^2 \qquad \text{for } n \geq 2.$$

By Corollary 1.3, when $\sum_{k=2}^{\infty} F^{-1}(p_c + 2^{-k}) = \infty$ we have $T(\mathbf{0}, \partial B(n)) \xrightarrow{\text{a.s.}} \infty$ as $n \to \infty$. The next theorem gives more information about the limit of $T(\mathbf{0}, \partial B(n))$ in this case.

THEOREM 1.6. *Suppose* $\sum_{k=2}^{\infty} F^{-1}(p_c + 2^{-k}) = \infty$ *and* $\eta_0 > 2$.

(i) *If* $\sum_{k=2}^{\infty} [F^{-1}(p_c + 2^{-k})]^2 < \infty$, *then there is a random variable* Z *with* $\mathbb{E}Z = 0$ *and* $\mathbb{E}Z^2 < \infty$ *such that as* $n \to \infty$

$$T(\mathbf{0}, \partial B(n)) - \mathbb{E}T(\mathbf{0}, \partial B(n)) \to Z \qquad \text{a.s. and in } L^2.$$

(ii) *If* $\sum_{k=2}^{\infty} [F^{-1}(p_c + 2^{-k})]^2 = \infty$, *then as* $n \to \infty$

$$\frac{T(\mathbf{0}, \partial B(n)) - \mathbb{E}T(\mathbf{0}, \partial B(n))}{[\mathrm{Var}(T(\mathbf{0}, \partial B(n)))]^{1/2}} \overset{\text{d}}{\Longrightarrow} N(0, 1).$$

REMARK 4. As in the case of Theorem 1.2, in Section 5.4, we derive versions of the variance asymptotics and limit theorems for point-to-point passage times $T(\mathbf{0}, x)$ for $x \in \mathbb{Z}^2$. See Corollaries 5.9 and 5.10.

1.3. *Relations to previous work.* First passage percolation has been studied since its introduction by Hammersley and Welsh [10] in the 1960s, but most work has focused on the noncritical case, where $F(0) < p_c$. There the passage time from $\mathbf{0}$ to a vertex x grows linearly in x, and many results have been proved, including shape theorems, large deviations, concentration inequalities and moment bounds. We refer the reader to the surveys [1, 9]. The supercritical case, where $F(0) > p_c$ is easier to analyze, since there is almost surely an infinite cluster of edges with passage time 0, and so distant vertices need only to travel to the infinite cluster to reach one-another. This produces passage times $T(\mathbf{0}, x)$ that are of order one as $x \to \infty$.

The critical case, where $F(0) = p_c$, is considerably more subtle. It is expected (though only proved in two or high dimensions) that there is no infinite cluster of p_c-open edges (i.e., edges with passage time 0). However, clusters of p_c-open edges occur on all scales, giving, for example, infinite mean size for the p_c-open cluster of the origin. This means that two distant points can be connected by a path which uses mostly zero-weight edges, and this path may be able to find lower and lower edge weights as it moves further into the bulk of the system. Therefore, to characterize passage times, one should understand the balance between the number of edges on each scale with low weights and the number of paths that can access them.

Kesten proved in [14], Theorem 6.1, that the time constant μ is zero in the critical case, implying that $T(\mathbf{0}, x) = o(\|x\|)$ as $x \to \infty$. This result was sharpened by L. Chayes [4], Theorem B, who showed that for any $\delta > 0$, $\lim_{n\to\infty} T(\mathbf{0}, n\mathbf{e}_1)/n^\delta = 0$ almost surely. In [16], Remark 3, Kesten claimed that Chayes's argument can be extended to $T(\mathbf{0}, n\mathbf{e}_1) \le \exp(C\sqrt{\log n})$ for large n almost surely. These results go some way to quantify asymptotics of the passage time in the critical case for general dimension.

More progress has been made in the critical case in $2d$, due to a more developed theory of Bernoulli percolation on planar lattices. It was shown by Chayes, Chayes and Durrett in [2], Theorem 3.3, that if t_e is 0 or 1 with probability $1/2$ then $\mathbb{E}T(\mathbf{0}, n\mathbf{e}_1) \asymp \log n$. In this Bernoulli case, the passage time between $\mathbf{0}$ and x can be represented as the maximum number of disjoint p_c-closed dual circuits separating $\mathbf{0}$ and $n\mathbf{e}_1$, as every p_c-closed edge on a geodesic contributes passage time 1. Recently, Yao [22] showed a law of large numbers on the triangular lattice, using the CLE of Camia and Newman.

Our work was motivated by that of Zhang in 1999, who showed that critical FPP can display "double behavior." That is, he showed that there exist distributions F with $F(0) = p_c$ for which the passage time $T(\mathbf{0}, \partial B(n))$ diverges as $n \to \infty$, and those for which the passage time remains bounded. Intuitively, bounded passage times come from those distributions which have significant mass near zero, so that long paths can find more and more low weights as they move away from $\mathbf{0}$, producing infinite paths with finite passage time. Zhang asked many questions about this case, in particular which distributions have which of the two behaviors. One main point of our work is Corollary 1.3, which gives an exact criterion that this passage time remains bounded if and only if $\sum_k F^{-1}(p_c + 2^{-k}) < \infty$. Our proof involves a new relation to a model called invasion percolation, and it turns out that optimal paths in the invasion cluster have passage time of the same order as geodesics in FPP. (See the next section for more details.) This theorem allows us to answer Zhang's questions in the two-dimensional case.

Our other motivation is the work of Kesten and Zhang in '97. They also considered the critical case in $2d$ and proved central limit theorems for $T(\mathbf{0}, \partial B(n))$ for certain distributions. They showed that if $\mathbb{E}t_e^\delta < \infty$ for some $\delta > 4$, $F(0) = p_c$, and there exists a constant $C_0 > 0$ such that $F(C_0) = p_c$, then the sequence $T(\mathbf{0}, \partial B(n))$ satisfies a Gaussian central limit theorem: there exists a sequence γ_n such that $C_1(\log n)^{1/2} \le \gamma_n \le C_2(\log n)^{1/2}$ and $\gamma_n^{-1}(T(\mathbf{0}, \partial B(n)) - \mathbb{E}T(\mathbf{0}, \partial B(n))) \Rightarrow N(0,1)$. It is important that the condition $F(C_0) = p_c$ gives a positive lower bound for the weight of nonzero edges. Kesten and Zhang do not address any distributions with mass near zero, though they do remark about the double behavior of such distributions.

The second part of our paper, on limit theorems and variance estimates, completes the picture started by Kesten and Zhang. Theorems 1.5 and 1.6(ii) require only $\sum_k (F^{-1}(p_c + 2^{-k}))^2 = \infty$ and a weak moment condition on t_e (lower than that of Kesten and Zhang) to deduce that the variance of $T(\mathbf{0}, \partial B(n))$ diverges and that a Gaussian CLT holds. This result on the CLT shows that in the critical case, no other limiting behavior is possible, in contrast to the subcritical case, where the variance is expected to be of order $n^{2/3}$ with a non-Gaussian limiting distribution (see [12]). Theorem 1.6(i) also addresses the intermediate case, where the mean of $T(\mathbf{0}, \partial B(n))$ diverges but the variance converges. Here, the centered sequence is tight and converges to a nontrivial limit. The limit is unlikely to be explicit since its variance depends heavily on weights of edges near the origin.

2. Setup for the proof. Zhang's proof in [24], Theorem 8.1.1, that $\rho(F_a)$ has all moments used a comparison to a near-critical percolation model introduced in [3] by Chayes, Chayes and Durrett. Their model is a version of an incipient infinite cluster, a term used by physicists to describe large (system-spanning) percolation clusters at criticality. We will, however, need finer asymptotics that are obtained by comparison with a different near-critical model, invasion percolation. Though it has no parameter, it tends on large scales to resemble Bernoulli percolation at criticality. We describe the model of invasion percolation in Section 2.1. We also recall some known facts about Bernoulli percolation in Section 2.2.

We couple the first passage percolation model on $(\mathbb{Z}^2, \mathcal{E}^2)$ with invasion percolation and Bernoulli percolation. To describe the coupling, we consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = [0, 1]^{\mathcal{E}^2}$, $\mathcal{F}$ is the cylinder sigma-field and $\mathbb{P} = \prod_{e\in\mathcal{E}^2} \mu_e$, where each μ_e is an uniform distribution on $[0, 1]$. Write $\omega = (\omega_e)_{e\in\mathcal{E}^2} \in \Omega$. Define the edge weights as $t_e = F^{-1}(\omega_e)$ for $e \in \mathcal{E}^2$.

2.1. *Invasion percolation.* If an edge e has endpoints e_x and e_y, we write $e = \{e_x, e_y\}$. For an arbitrary subgraph $G = (V, E)$ of $(\mathbb{Z}^2, \mathcal{E}^2)$, define the edge boundary ΔG by $\Delta G = \{e \in \mathcal{E}^2 : e \notin E, e_x \in V \text{ or } e_y \in V\}$. Define a sequence of subgraphs $(G_n)_{n=0}^{\infty}$ as follows. Let $G_0 = (\{0\}, \varnothing)$. If $G_i = (V_i, E_i)$ is defined, we let $E_{i+1} = E_i \cup \{e_{i+1}\}$, where e_{i+1} is the edge with $\omega_{e_{i+1}} = \min\{\omega_e : e \in \Delta G_i\}$, and let G_{i+1} be the graph induced by E_{i+1}. The graph $I := \bigcup_{i=0}^{\infty} G_i$ is called the *invasion percolation cluster* (at time infinity).

Invasion percolation is coupled with the first passage percolation model since we have defined $t_e = F^{-1}(\omega_e)$. They can also be coupled with Bernoulli percolation as follows. For each $e \in \mathcal{E}^2$ and $p \in [0, 1]$, we say that e is p-open in ω if $\omega_e \le p$, and otherwise we say that e is p-closed. If there is a p-open path from a vertex set A to a vertex set B then we write that $A \leftrightarrow B$ by a p-open path. The collection of p-open edges has the same distribution as the set of open edges in Bernoulli percolation with parameter p.

We also need the notion of the dual graph. Let $(\mathbb{Z}^2)^* = (1/2, 1/2) + \mathbb{Z}^2$ and $(\mathcal{E}^2)^* = (1/2, 1/2) + \mathcal{E}^2$. For $x \in \mathbb{Z}^2$, we write $x^* = (1/2, 1/2) + x$. For $e \in \mathcal{E}^2$, we denote its endpoints (left respectively right or bottom respectively top) by e_x, $e_y \in \mathbb{Z}^2$. The edge $e^* = \{e_x + (1/2, 1/2), e_y - (1/2, 1/2)\}$ is the *dual edge* to e and its endpoints (bottom respectively top or left respectively right) are written e_x^*

and e_y^*. For $A \subset \mathbb{Z}^2$, A^* is defined as $(1/2, 1/2) + A$. An edge e^* is declared to be p-open in ω when e is, and p-closed otherwise.

The following relations between invasion percolation and Bernoulli percolation are well known. Since they are crucial, and their proofs are short, we add the proofs for the convenience of the reader.

- Almost surely, if $x \in I$ and $y \leftrightarrow x$ by a p_c-open path, then $y \in I$.

PROOF. If y is not in I then we can find $e \in \Delta I$ (on a p_c-open path from x to y) such that e is p_c-open. But then $e \in \Delta G_n$ for all large n. By the definition of the invasion algorithm, this means that for large n, each edge added to the invasion is p_c-open, and from this we can build an infinite p_c-open path. This contradicts the fact that there is almost surely no infinite p_c-open cluster [13], Theorem 1. □

- For $n \geq 0$, let $\hat{p}_n$ be defined as

$$\hat{p}_n = \sup\{\omega_e : e \in I \cap E\big(B(2^n)\big)^c\}, \tag{2.1}$$

where $E(V)$ is the set of edges with both endpoints in V. Then

$$\hat{p}_n > p \Rightarrow A_{n,p} \text{ occurs}, \tag{2.2}$$

where $A_{n,p}$ is the event that there is a p-closed dual circuit around the origin with diameter at least 2^n. Here, the diameter of a set X is $\sup\{\|x - y\|_\infty : x, y \in X\}$.

PROOF. Take $e \in I \cap E(B(2^n))^c$ with $\omega_e > p$. At the moment k that e is added to the invasion cluster, the graph G_k has edge boundary which is ω_e-closed, and so is p-closed. From the edge boundary, we can extract a dual circuit around $\mathbf{0}$ that contains e^*, by [8], Proposition 11.2. This circuit then has diameter at least 2^n. □

2.2. *Correlation length.* A central tool used to study invasion percolation is correlation length [15], equation (1.21). For $m, n \in \mathbb{N}$ and $p \in (p_c, 1]$, let

$$\sigma(n, m, p) = \mathbb{P}\big(\text{there is a } p\text{-open left-right crossing of } [0, n] \times [0, m]\big),$$

where a p-open left-right crossing of $[0, n] \times [0, m]$ means a path γ in $[0, n] \times [0, m]$ with all edges p-open from $\{0\} \times [0, m]$ to $\{n\} \times [0, m]$. For $\varepsilon > 0$ and $p > p_c$, we define

$$L(p, \varepsilon) = \min\{n \geq 1 : \sigma(n, n, p) \geq 1 - \varepsilon\}.$$

$L(p, \varepsilon)$ is called the correlation length. It is known (see [15], equation (1.24)) that there is $\varepsilon_1 > 0$ such that for all $0 < \varepsilon, \varepsilon' \leq \varepsilon_1$, $L(p, \varepsilon)/L(p, \varepsilon')$ is bounded away from 0 and ∞ as $p \downarrow p_c$. We write $L(p) = L(p, \varepsilon_1)$. For $n \geq 1$, define

$$p_n = \min\{p : L(p) \leq n\}. \tag{2.3}$$

We now note the following facts:

- By [11], equation (2.10), there exists $K_{2.4} \in (0, 1)$ such that for all $n \geq 1$,

$$K_{2.4} n \leq L(p_n) \leq n. \tag{2.4}$$

- There exist $C_1, C_2 > 0$ such that for all $m, n \geq 1$, $C_1 |\log \frac{m}{n}| \leq |\log \frac{p_m - p_c}{p_n - p_c}| \leq C_2 |\log \frac{m}{n}|$. This is a consequence of [19], Proposition 34, and a priori estimates on the four-arm exponent. In particular, putting $m = 1$, there exist $\delta_0 > \varepsilon_0 > 0$ such that for $n \geq 2$

$$\frac{1}{n^{\delta_0}} < p_n - p_c < \frac{1}{n^{\varepsilon_0}}. \tag{2.5}$$

  We may and will always assume $\delta_0 > 1$.
- From [15], equation (2.25), and (2.2), there exist $K_{2.6.1}, K_{2.6.2} > 0$ such that for all $p > p_c$ and $n \geq 1$,

$$\mathbb{P}(\hat{p}_n > p) \leq \mathbb{P}(A_{n,p}) \leq K_{2.6.1} \exp\left(-\frac{K_{2.6.2} 2^n}{L(p)}\right). \tag{2.6}$$

- By the RSW theorem (see [8], Section 11.7), there exists $K_{2.7} > 0$ such that for all $k \in \mathbb{N}$,

$$\begin{aligned} &\mathbb{P}\big(\text{there is a } p_{2^k}\text{-closed dual circuit around } \mathbf{0} \text{ in } B(2^k)^* \setminus B(2^{k-1})^*\big) \\ &\quad \geq K_{2.7}. \end{aligned} \tag{2.7}$$

2.3. *Sketch of proofs*. The main tool in our proofs is Lemma 3.1, a moment bound on annulus passage times. We first describe its proof. Consider all paths between $\mathbf{0}$ and $\partial B(2^{n+1})$ which lie in the invasion cluster I and $B(2^{n+1})$. Let γ_n be such a path with minimal passage time. Lemma 3.1 gives an upper bound on the rth moment of the sum of edge weights in γ_n which lie in $B(2^{k+1}) \setminus B(2^k)$ [i.e., $\mathbb{E}T_k^r(\gamma_n)$, where $T_k(\gamma_n)$ is defined in (3.1)].

One has $T(\gamma_n) \leq T(\mathbf{0}, \partial B(2^n))$, and γ_n is a nicer path than the geodesic for the weights (t_e). Once the invasion has reached $\partial B(2^k)$, all of its further edges are likely to be nearly p_{2^k}-open [i.e., $\hat{p}_k$ from (2.1) is of order p_{2^k}], and so the edges e in γ_n outside of $B(2^k)$ will have $t_e \leq F^{-1}(p_{2^k})$. Bounding p_{2^k} with (2.5), each edge e has $t_e \leq a_k$ [defined in (3.2)]. We only know this behavior of $\hat{p}_k$ with high probability, so we need to decompose the probability space over different values of $\hat{p}_k$ using an idea of A. Járai [11].

This gives $T_k(\gamma_n) \lesssim a_k \#\{e \in \gamma_n \cap (B(2^{k+1}) \setminus B(2^k)) : e \text{ is } p_c\text{-closed}\}$. The reason is that the only edges contributing to $T(\gamma_n)$ are the p_c-closed ones. In Lemma 3.2, we show that each such edge has "4-arms." That is, they have the properties (a) their weight is between p_c and p_{2^k}, (b) they have two disjoint p_{2^k}-open arms to distance 2^{k-1} and (c) they have two disjoint p_c-closed arms to distance 2^{k-1}. All moments of the number of such edges in an annulus were bounded in [7] (see Lemma 3.3 below), so we can conclude.

2.3.1. *Idea of the proof of Theorem* 1.2. The proof of (ii) follows that of Zhang [24], Theorem 8.1.2. The proof of (i) follows immediately from Lemma 3.1. Indeed, to find the upper bound for $\mathbb{E}T(\mathbf{0}, \partial B(2^n))$, we simply use the inequality $T(\mathbf{0}, \partial B(2^n)) \le T(\gamma_n) = \sum_{k=-1}^{n} T_k(\gamma_n)$, where, as above, each $T_k(\gamma_n)$ is the time that γ_n spent in the annulus $B(2^{k+1}) \setminus B(2^k)$. Applying the moment bounds from Lemma 3.1 gives (i).

2.3.2. *Idea of the proof of Theorem* 1.5 *and* 1.6. We follow Kesten–Zhang [17]. Instead of dealing with $\mathrm{Var}(T(\mathbf{0}, \partial B(2^n)))$ directly, we consider $\mathrm{Var}(T(\mathbf{0}, \mathcal{C}_n))$, where $\mathcal{C}_n$ is the innermost p_c-open circuit in $B(2^{m+1}) \setminus B(2^m)$ for $m \ge n$ surrounding $\mathbf{0}$. It can be shown that these two variances are close to each other. The variance bounds for $T(\mathbf{0}, \mathcal{C}_n)$ are stated in Theorem 5.1 and the CLT is stated in Theorem 5.2.

Writing $T(\mathbf{0}, \mathcal{C}_n) - \mathbb{E}T(\mathbf{0}, \mathcal{C}_n)$ as a sum of martingale differences $\Delta_k = \mathbb{E}[T(\mathbf{0}, \mathcal{C}_n) \mid \mathcal{F}_k] - \mathbb{E}[T(\mathbf{0}, \mathcal{C}_n) | \mathcal{F}_{k-1}]$, one has $\mathrm{Var}(T(\mathbf{0}, \mathcal{C}_n)) = \sum_{k=0}^{n} \mathbb{E}\Delta_k^2$. The idea of Kesten–Zhang was to take $\mathcal{F}_k$ generated by the edge-weights on and in the interior of $\mathcal{C}_k$, and they proved an alternate representation for such Δ_k's [see Lemma 5.3(ii)]. With this choice, we can use the bounds in Lemma 3.1 to prove moment bounds on the Δ_k's in Lemma 5.5. We note that by the representation in Lemma 5.3(ii), Δ_k does not depend on n.

Given the above moment bounds, and growth of both the variance and mean of $T(\mathbf{0}, \partial B(n))$, the proof of the CLT for $T(\mathbf{0}, \partial B(n))$ [item (ii) in Theorem 1.6] is similar to the original one of Kesten–Zhang. It consists of verifying the conditions of McLeish's CLT [18]. Because this is standard, we omit the details, and refer the reader to the arXiv version of this paper [6]. For (i), if the variance does not diverge, then by the martingale convergence theorem, $T(\mathbf{0}, \mathcal{C}_n) - \mathbb{E}T(\mathbf{0}, \mathcal{C}_n)$ will converge to some random variable Z. Using a stronger comparison to $T(\mathbf{0}, \partial B(n))$ given in Lemma 5.7 allows us to complete the proof.

3. Moment bounds for annulus times. In this section, we prove the main lemma of the paper, Lemma 3.1, bounding certain annulus passage times $T_k(\gamma_n)$ through the invasion cluster I. Define $\mathcal{E}_{-1} := E(B(1))$ and $\mathcal{E}_n := E(B(2^{n+1})) \setminus E(B(2^n))$ for $n \ge 0$. Note that $|\mathcal{E}_{-1}| = 12$ and $|\mathcal{E}_n| = 24 \cdot 4^n + 4 \cdot 2^n$ for $n \ge 0$. For any path γ, define for $k \ge -1$

$$T_k(\gamma) := \sum_{e \in \gamma \cap \mathcal{E}_k} t_e. \tag{3.1}$$

For $n \ge -1$, let γ_n be a path from $\mathbf{0}$ to $\partial B(2^{n+1})$ such that

$$T(\gamma_n) = \inf\{T(\gamma) : \gamma \text{ is a path from } \mathbf{0} \text{ to } \partial B(2^{n+1}) \text{ and } \gamma \subset B(2^{n+1}) \cap I\}.$$

As with any path from $\mathbf{0}$ to $\partial B(2^{n+1})$, γ_n satisfies $T(\gamma_n) = \sum_{k=-1}^{n} T_k(\gamma_n)$. Recall ε_0 from (2.5). For simplicity of notation, define

$$a_k := F^{-1}(p_c + 2^{-\varepsilon_0 k/2}) \qquad \text{for } k \in \mathbb{N}. \tag{3.2}$$

Note that a_k is only defined when the argument of F^{-1} is strictly less than 1, and this will be guaranteed by the condition $k \geq k_0$ in the lemma below.

The main goal now is to prove the following lemma. The proof is delayed until the end of the section, so we can build up other results needed for it.

LEMMA 3.1. *Recall the definition of* η_0 *from* (1.2) *and suppose* $\eta_0 > 1$.

(i) *For all* $r \in [1, \eta_0)$ *and integers* $k \geq -1$, *we have* $\sup_{n \geq k} \mathbb{E}[T_k^r(\gamma_n)] < \infty$.

(ii) *Given* $r \in [1, \infty)$ *and* $\lambda \in (0, \infty)$, *there are* $k_0 = k_0(r, \lambda, F) > 0$ *and* $K_{3.1} = K_{3.1}(r, \lambda, F) > 0$, *such that for all* n, k *satisfying* $n - 1 \geq k \geq k_0$,

$$\mathbb{E}\big[T_k^r(\gamma_n)\big] \leq K_{3.1}\big(a_k^r + e^{-\lambda k}\big).$$

REMARK 5. To prove Theorem 1.2, it is sufficient to use the above lemma with $r = 1$. Here, we prove it in the general form for future use in Section 5.

For $m_1, m_2 \geq 1$, $p \in (p_c, 1]$, and $e \in \mathcal{E}^2$, let $A_e(m_1, p)$ be the event that

(a) e is connected to $\partial B(e_x, m_1)$ by two vertex disjoint p-open paths,

(b) e^* is connected to $\partial B(e_x, m_1)^*$ by two vertex disjoint p_c-closed dual paths and

(c) $\omega_e \in (p_c, p]$.

Here, $\partial B(e_x, m_1) = e_x + \partial B(m_1)$. Let $N(m_1, m_2, p)$ be the number of edges e in $E(B(2m_2)) \setminus E(B(m_2))$ such that $A_e(m_1, p)$ occurs; that is,

$$N(m_1, m_2, p) = \sum_{e \in E(B(2m_2)) \setminus E(B(m_2))} \mathbb{1}_{A_e(m_1, p)}.$$

LEMMA 3.2. *Let* $\hat{p}_k$ *be as in* (2.1). *For all* $p > p_c$ *and* $1 \leq k \leq n - 1$,

$$T_k(\gamma_n)\mathbb{1}\{\hat{p}_k \leq p\} \leq N\big(2^{k-1}, 2^k, p\big) \cdot F^{-1}(p).$$

PROOF. Suppose $\hat{p}_k \leq p$ for some $p > p_c$. Define, for $n \geq 1$ and $1 \leq k \leq n - 1$, $T'_{k,n} = \#\{e \in \gamma_n \cap \mathcal{E}_k : \omega_e > p_c\}$. Since $\hat{p}_k \leq p$ and $\gamma_n \subset I$, we have $T_k(\gamma_n) \leq T'_{k,n} F^{-1}(p)$. Then it is sufficient to show

$$T'_{k,n} \leq N\big(2^{k-1}, 2^k, p\big). \tag{3.3}$$

Let $e \in \gamma_n \cap \mathcal{E}_k$ be p_c-closed. As $\gamma_n \subset I$ and $\hat{p}_k \leq p$, e is p-open. Note that there exist disjoint paths $\gamma_{n,1}, \gamma_{n,2} \subset \gamma_n$ such that $\gamma_{n,1}$ is a p-open path joining e_x to $\partial B(e_x, 2^{k-1})$ and $\gamma_{n,2}$ is a p-open path joining e_y to $\partial B(e_x, 2^{k-1})$. [This holds because e_x is invaded but $0 \notin B(e_x, 2^{k-1})$.]

For an illustration of the following argument, see Figure 1. If $\gamma_{n,1} \leftrightarrow \gamma_{n,2}$ by a p_c-open path γ' in $B(e_x, 2^{k-1})$, and if we let $u \in \gamma_{n,1}$ and $v \in \gamma_{n,2}$ be such that $u \leftrightarrow v$ via γ', then every vertex in γ' is in I (see the first bulleted fact in

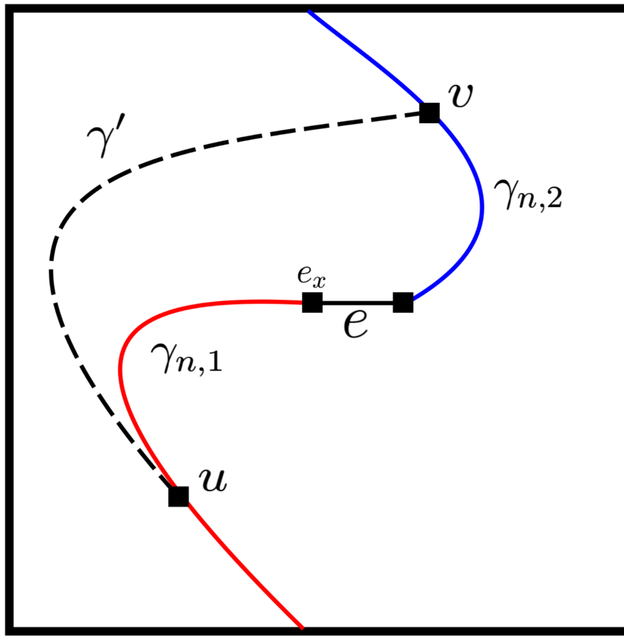

FIG. 1. *Depiction of the proof of Lemma 3.2. The box is $B(e_x, 2^{k-1})$. The path γ' is p_c-open and connects vertices u and v on γ_n, but bypasses e.*

Section 2.1), so $\gamma' \subset I$. Now let γ_n' be the path connecting 0 and u via γ_n, u to v via γ' and v to $\partial B(2^{n+1})$ via γ_n. Then γ_n' is in I and has at least one p_c-closed edge less (namely e) than γ_n. Also each p_c-closed edge of γ_n' is a p_c-closed edge of γ_n, and this implies $T(\gamma_n') < T(\gamma_n)$, contradicting the minimality of γ_n. Hence, $\gamma_{n,1} \not\leftrightarrow \gamma_{n,2}$ by a p_c-open path in $B(e_x, 2^{k-1})$. Note that by duality, exactly one of the following will happen:

1. e_x^* and e_y^* are connected to $\partial B(e_x, 2^{k-1})^*$ by two disjoint p_c-closed dual paths, which are also disjoint from $\gamma_{n,1} \cup \gamma_{n,2} \cup \{e\}$;
2. there is a p_c-open path connecting $\gamma_{n,1}$ and $\gamma_{n,2}$ in $B(e_x, 2^{k-1})$.

So the first event, and thus $A_e(2^{k-1}, p)$, occurs. □

Next we bound the moments of $N(2^{k-1}, 2^k, p)$ using [7], Lemma 5.1.

LEMMA 3.3. *There exists $K_{3.3} > 0$ such that for all $p > p_c$, $L(p) < m_1 \le m_2$ and integers $t \ge 1$*

$$\mathbb{E}[N^t(m_1, m_2, p)] \le \mathbb{E}[N^t(L(p), m_2, p)] \le t!\left(\frac{K_{3.3} m_2}{L(p)}\right)^{2t}.$$

PROOF. The first inequality immediately follows from the definition of $N(m_1, m_2, p)$. In [7], Lemma 5.1, it was shown that there exists $C_1 > 0$ such that if $p > p_c$, $m' \le L(p)$ and $m' \le m_2$, then for all integers $t \ge 0$,

$$\mathbb{E}\big[N^t\big(m', m_2, p\big)\big] \le t!\bigg(C_1 \frac{m_2}{m'}\bigg)^{2t}.$$

Taking $m' = L(p)$, completes the proof. □

The next lemma controls moments of $T_k(\gamma_n)$ when $\hat{p}_k$ is large. Define

$$\hat{t}_k := F^{-1}(\hat{p}_k). \tag{3.4}$$

LEMMA 3.4. *Suppose $\mathbb{E}[t_e^\eta] < \infty$ for some $\eta > 0$. Define $c_{-1} = 4$ and $c_k := 2^{k+1} + 4$ for $k \ge 0$. For all integers $k \ge -1$ and $r \in (0, c_k\eta)$, one has $\mathbb{E}[\hat{t}_k^r] < \infty$. In particular, for any fixed $r > 0$, there exists $K_{3.4} = K_{3.4}(r, \eta, F)$ such that for all integers $k > \log(r/\eta)/\log 2$, we have $\mathbb{E}[\hat{t}_k^r] \le K_{3.4}$.*

PROOF. Note that $t \ge F^{-1}(F(t))$ for all $t \ge 0$. Then we have

$$\mathbb{P}(\hat{t}_k > t) \le \mathbb{P}\big(\hat{t}_k > F^{-1}\big(F(t)\big)\big) \le \mathbb{P}\big(\hat{p}_k \ge F(t)\big). \tag{3.5}$$

To bound the tail probability of $\hat{t}_k$, we need to bound $\mathbb{P}(\hat{p}_k > p)$ when p is near one. By (2.2), for any $k \ge -1$, $\hat{p}_k > p$ implies that there exists a p-closed *dual* circuit around the origin with diameter at least $\lfloor 2^k + 1\rfloor$. Such a dual circuit has length at least $2\lfloor 2^k + 1\rfloor + 2 = c_k$, for $k \ge -1$. For any even $m \ge 4$, observe that since dual circuits around the origin with length m must intersect the line $\{(x, 0) : x \in (-1, m/2-1)\}$, the total number of such circuits is bounded by $\frac{m}{2} \cdot 3^m$. Each of these dual circuits is p-closed with probability $(1 - p)^m$. Therefore, when $p \in [5/6, 1)$ we have

$$\mathbb{P}(\hat{p}_k \ge p) \le \sum_{m=c_k}^{\infty} \frac{m3^m}{2} \cdot (1-p)^m \le \sum_{m=c_k}^{\infty} \frac{m}{2^{(1-\alpha)m}}\big(3(1-p)\big)^{\alpha m}, \tag{3.6}$$

where the second inequality uses $3(1-p) \le 1/2$ and the value of $\alpha \in (0, 1)$ will be specified later. Define $C_1 = C_1(\alpha) := \max_{m\ge 4}\{m2^{-(1-\alpha)m}\}/(1 - 2^{-\alpha})$ and $C_2 := (3\mathbb{E}t_e^\eta)^{1/\eta}$. Combining (3.5) and (3.6), when $t \ge C_3 := F^{-1}(5/6)/C_2$ we have $F(C_2t) \ge 5/6$ and

$$\mathbb{P}(\hat{t}_k > C_2t) \le \mathbb{P}\big(\hat{p}_k \ge F(C_2t)\big) \le C_1\big(3\mathbb{P}(t_e > C_2t)\big)^{c_k\alpha} \le C_1\bigg(\frac{3\mathbb{E}[t_e^\eta]}{(C_2t)^\eta}\bigg)^{c_k\alpha}$$
$$= \frac{C_1}{t^{c_k\alpha\eta}}.$$

The second inequality above follows from (3.6) [with $p = F(C_2t)$], and also uses the definition of C_1 and the fact that $1 - F(C_2t) = \mathbb{P}(t_e > F(C_2t))$. Since $r < c_k\eta$, taking $\alpha = \alpha_k := \frac{c_k\eta + r}{2c_k\eta}$ we have

$$\begin{aligned}\mathbb{E}\biggl[\biggl(\frac{\hat{t}_k}{C_2}\biggr)^r\biggr] &= \int_0^\infty rt^{r-1}\mathbb{P}(\hat{t}_k \geq C_2 t)\,dt \\ &\leq \int_0^{1\vee C_3} rt^{r-1}\,dt + \int_{1\vee C_3}^\infty rt^{r-1}\cdot\frac{C_1(\alpha_k)}{t^{c_k\alpha_k\eta}}\,dt \\ &\leq (1\vee C_3)^r + \frac{C_1(\alpha_k)r}{c_k\alpha_k\eta - r}.\end{aligned}\tag{3.7}$$

In the last inequality, we have used that $(1 \vee C_3)^{r - c_k\alpha_k\eta} \leq 1$. Therefore, using the relation $c_k\alpha_k\eta - r = (c_k\eta - r)/2$, we have

$$\mathbb{E}[\hat{t}_k^r] \leq \bigl(C_2 \vee F^{-1}(5/6)\bigr)^r + \frac{2rC_1(\alpha_k)C_2^r}{c_k\eta - r} < \infty.$$

Next, when $r > 0$ and $k > \log(r/\eta)/\log 2$, taking $\alpha := 1/2$ in the above proof, we have $c_k\alpha\eta - r \geq 2^k\eta - r \geq 2^{k_1}\eta - r > 0$ where $k_1 := \lfloor\log(r/\eta)/\log 2\rfloor + 1$. Then, using (3.7) again (which also holds in the current situation since we still have $c_k\alpha_k\eta - r > 0$),

$$\mathbb{E}[\hat{t}_k^r] \leq C_2^r + \bigl(F^{-1}(5/6)\bigr)^r + \frac{2rC_1(1/2)C_2^r}{2^{k_1}\eta - r},$$

which gives the expression of $K_{3.4}$. □

PROOF OF LEMMA 3.1. First we prove part (i). Recall $\hat{t}_k$ from (3.4). Since $T_k(\gamma_n) \leq |\mathcal{E}_k|\hat{t}_k$ and $|\mathcal{E}_k| \leq 48\cdot 4^k$ for $k \geq -1$, we have

$$\mathbb{E}\bigl[T_k^r(\gamma_n)\bigr] \leq \mathbb{E}\bigl[\bigl(|\mathcal{E}_k|\hat{t}_k\bigr)^r\bigr] \leq \bigl(48\cdot 4^k\bigr)^r\mathbb{E}\bigl[\hat{t}_k^r\bigr].\tag{3.8}$$

For any $r < \eta_0$ and $\eta \in (r, \eta_0)$, one has $\mathbb{E}[t_e^{\eta/4}] < \infty$. Recall c_k in Lemma 3.4. For $k \geq -1$, $c_k\eta \geq \eta > r$, so by Lemma 3.4, $\mathbb{E}[\hat{t}_k^r] < \infty$ and (i) is proved.

Next we prove (ii). The constants ε_0, δ_0 are from (2.5). We will perform a decomposition for $\hat{p}_k$ introduced by Járai ([11], page 319) using iterated logarithms. Its main purpose is to allow to obtain the term a_k^r in the statement of the lemma without any logarithmic prefactors, which may arise if the decomposition were only made using two intervals for the value of $\hat{p}_k$. Define $\log^{(0)} k = k$ and $\log^{(j)} k = \log(\log^{(j-1)} k)$ for $j \geq 1$ such that it is well defined. For $k > 10$, let

$$\log^* k = \min\bigl\{j > 0 : \log^{(j)} k \text{ is well defined and } \log^{(j)} k \leq 10\bigr\}.$$

Let $r \in [1, \infty)$ and $\lambda \in (0, \infty)$ be given. Denote for $j = 0, 1, 2, \dots, \log^* k$,

$$q_k(j) := p_{\lfloor 2^k/(C_1 \log^{(j)} k)\rfloor},$$

where C_1 is so large that

$$C_1 > 2/\log 10, \tag{3.9}$$

$$2r\log 2 - K_{2.6.2}C_1/2 < -\lambda, \tag{3.10}$$

$$\lceil 2r\rceil - K_{2.6.2}C_1/2 < -1. \tag{3.11}$$

Given C_1, let $k_0 > 10$ be the smallest integer such that for all $k \ge k_0$,

$$2^{k/2-1} > C_1 k, \qquad p_c + 2^{-\varepsilon_0 k/2} < 1, \quad \text{and} \quad k > \frac{\log r}{\log 2} + 3. \tag{3.12}$$

The reason for the above choices will be clear as the proof proceeds. We assume $k \ge k_0$ for the rest of the proof. By Lemma 3.4, the third condition in (3.12) gives $\mathbb{E}[\hat{t}_k^{2r}] \le K_{3.4}(2r, 1/4, F)$, and with (3.8) we have for all $k \ge k_0$,

$$\mathbb{E}\big[T_k^{2r}(\gamma_n)\big] \le \big(48\cdot 4^k\big)^{2r} K_{3.4}. \tag{3.13}$$

Note $q_k(\log^* k) < \cdots < q_k(1)$ are well defined if $2^k > C_1 k$. We write

$$\begin{aligned}\mathbb{E}\big[T_k^r(\gamma_n)\big] &= \mathbb{E}\big[T_k^r(\gamma_n)\mathbb{1}\{\hat{p}_k > q_k(0)\}\big]\\ &\quad + \sum_{j=0}^{\log^* k-1} \mathbb{E}\big[T_k^r(\gamma_n)\mathbb{1}\{q_k(j+1) < \hat{p}_k \le q_k(j)\}\big]\\ &\quad + \mathbb{E}\big[T_k^r(\gamma_n)\mathbb{1}\{\hat{p}_k \le q_k(\log^* k)\}\big].\end{aligned} \tag{3.14}$$

By (2.4) and the fact that $C_1 > 2/\log 10$, for $j = 0, 1, \ldots, \log^* k$ and $k \ge k_0$,

$$L\big(q_k(j)\big) \le \left\lfloor \frac{2^k}{C_1 \log^{(j)} k} \right\rfloor \le \frac{2^k}{C_1 \log^{(\log^* k)} k} \le \frac{2^k}{C_1 \log 10} < 2^{k-1}.$$

Then applying Lemma 3.2 and Lemma 3.3, for all $\alpha \ge 1$, $k_0 \le k \le n-1$ and $j = 0, 1, \ldots, \log^* k$,

$$\begin{aligned}\mathbb{E}\big[T_k^\alpha(\gamma_n)\mathbb{1}\{\hat{p}_k \le q_k(j)\}\big] &\le \big[F^{-1}\big(q_k(j)\big)\big]^\alpha \mathbb{E}\big[N^\alpha\big(2^{k-1}, 2^k, q_k(j)\big)\big]\\ &\le \big[F^{-1}\big(q_k(j)\big)\big]^\alpha \cdot \lceil\alpha\rceil! \left(\frac{K_{3.3}2^k}{L(q_k(j))}\right)^{2\lceil\alpha\rceil}.\end{aligned} \tag{3.15}$$

By (3.12), we have for $k \ge k_0$ and $j = 0, \ldots, \log^* k$,

$$\left\lfloor \frac{2^k}{C_1 \log^{(j)} k} \right\rfloor \ge \frac{2^{k-1}}{C_1 \log^{(j)} k} \ge \frac{2^{k-1}}{C_1 k} > 2^{k/2}. \tag{3.16}$$

Then by (2.5), we have

$$q_k(j) \le p_c + \left\lfloor \frac{2^k}{C_1 \log^{(j)} k} \right\rfloor^{-\varepsilon_0} \le p_c + 2^{-k\varepsilon_0/2} < 1. \tag{3.17}$$

Applying (2.4) and (3.17) in (3.15) and recalling the definition of a_k in (3.2), we have for $k_0 \le k \le n-1$ and $j = 0, \dots, \log^* k$ with $\alpha \ge 1$,

$$\mathbb{E}\big[T_k^\alpha(\gamma_n)\mathbb{1}\{\hat{p}_k \le q_k(j)\}\big] \le \lceil\alpha\rceil!\big(C_2 \log^{(j)} k\big)^{2\lceil\alpha\rceil} a_k^\alpha, \tag{3.18}$$

where $C_2 := 2K_{3.3}C_1/K_{2.4}$. We bound the sum in (3.14), starting with the last term. Applying (3.18) with $\alpha = r$ and $j = \log^* k$, one has for $k_0 \le k \le n-1$ and $r \ge 1$,

$$\mathbb{E}\big[T_k^r(\gamma_n)\mathbb{1}\{\hat{p}_k \le q_k(\log^* k)\}\big] \le \lceil r\rceil!(10C_2)^{2\lceil r\rceil} a_k^r. \tag{3.19}$$

For the first term in (3.14), applying the Cauchy–Schwarz inequality, (3.13) and (2.6), for $k_0 \le k \le n-1$,

$$\begin{aligned}\mathbb{E}\big[T_k^r(\gamma_n)\mathbb{1}\{\hat{p}_k > q_k(0)\}\big] &\le \mathbb{E}\big[T_k^{2r}(\gamma_n)\big]^{1/2}\big[\mathbb{P}\big(\hat{p}_k > q_k(0)\big)\big]^{1/2} \\ &\le \big((48\cdot 4^k)^{2r} K_{3.4}\big)^{1/2} \cdot K_{2.6.1}^{1/2}\exp(-K_{2.6.2}C_1 k/2) \\ &= 48^r (K_{3.4}K_{2.6.1})^{1/2}\exp(2rk\log 2 - K_{2.6.2}C_1 k/2).\end{aligned} \tag{3.20}$$

For the second term in (3.14), applying the Cauchy–Schwarz inequality, (3.18) with $\alpha = 2r$, and (2.6), we have for $j = 0, 1, \dots, \log^* k - 1$ and $k_0 \le k \le n-1$,

$$\begin{aligned}&\mathbb{E}\big[T_k^r(\gamma_n)\mathbb{1}\{q_k(j+1) < \hat{p}_k \le q_k(j)\}\big] \\ &\quad\le \mathbb{E}\big[T_k^{2r}(\gamma_n)\mathbb{1}\{\hat{p}_k \le q_k(j)\}\big]^{1/2}\big[\mathbb{P}\big(\hat{p}_k > q_k(j+1)\big)\big]^{1/2} \\ &\quad\le \big[\lceil 2r\rceil!\big(C_2\log^{(j)} k\big)^{2\lceil 2r\rceil} a_k^{2r}\big]^{1/2} \cdot K_{2.6.1}^{1/2}\exp\big(-K_{2.6.2}C_1 \log^{(j+1)} k/2\big) \\ &\quad= \big(\lceil 2r\rceil!\big)^{1/2} C_2^{\lceil 2r\rceil} K_{2.6.1}^{1/2} a_k^r \big(\log^{(j)} k\big)^{\lceil 2r\rceil - K_{2.6.2}C_1/2}.\end{aligned} \tag{3.21}$$

Then combining (3.20), (3.21), (3.19) and using the definition of C_1 in (3.10) and (3.11), there are $C_3, C_4, C_5 > 0$ such that for $k_0 \le k \le n-1$,

$$\mathbb{E}\big[T_k^r(\gamma_n)\big] \le C_3 e^{-\lambda k} + C_4 a_k^r \sum_{j=0}^{\log^* k - 1}\big(\log^{(j)} k\big)^{-1} + C_5 a_k^r.$$

Note that C_1 was chosen initially to depend on r so that exponent of $\log^{(j)} k$ in this inequality can be taken to be -1. (A similar choice appears in [7], Theorem 1.3.) This forces the other constants to depend on r, but the important point is that none of them depend on n or k. Using [21], equation (2.16), which says $\sum_{j=0}^{\log^* k}(\log^{(j)} k)^{-1}$ is uniformly bounded in k, we complete the proof of Lemma 3.1. □

4. Study of the mean. In this section, we give the proof of Theorem 1.2. We prove Corollary 1.3 in Section 4.2.

4.1. *Proof of Theorem* 1.2. First we prove an elementary lemma.

LEMMA 4.1. *Let* $f(t)$, $t \in [0,\infty)$, *be a positive nonincreasing function. Fix* $\delta > \varepsilon > 0$ *and integers* $k_1, k_2 \geq 1$. *Then there exist constants* $C_1, C_2 \in (0,\infty)$ *such that for all* $n \geq \max\{k_1, k_2\}$,

$$C_1 \sum_{k=k_2}^{n} f(\varepsilon k) \leq \sum_{k=k_1}^{n} f(\delta k) \leq C_2 \sum_{k=k_2}^{n} f(\varepsilon k).$$

PROOF. It suffices to take $k_1 = k_2 = 1$. If $\delta k \leq \varepsilon k' < \delta(k+1)$, then $f(\varepsilon k') \leq f(\delta k)$, and for each k there are at most $\lceil \delta/\varepsilon \rceil$ such integers k'. Therefore, $\sum_{k=1}^{n} f(\varepsilon k) \leq \lceil \delta/\varepsilon \rceil (f(\varepsilon) + \sum_{k=1}^{n} f(\delta k))$. As $\varepsilon < \delta$, we obtain

$$1 \leq \frac{\sum_{k=1}^{n} f(\varepsilon k)}{\sum_{k=1}^{n} f(\delta k)} \leq \frac{\lceil \delta/\varepsilon \rceil (f(\varepsilon) + \sum_{k=1}^{n} f(\delta k))}{\sum_{k=1}^{n} f(\delta k)}. \qquad \square$$

PROOF OF THEOREM 1.2. For the upper bound, note that for $n \geq -1$, $\mathbb{E}T(\mathbf{0}, \partial B(2^n)) \leq \sum_{k=-1}^{n-1} \mathbb{E}T_k(\gamma_n)$. Take k_0 as in Lemma 3.1 and apply this lemma with $r = 1$. We obtain $\mathbb{E}T(\mathbf{0}, \partial B(2^n)) < \infty$ for all $n \geq -1$ and in particular for $n \geq k_0 + 1$, $\mathbb{E}T(\mathbf{0}, \partial B(2^n))$ is bounded by

$$\sum_{k=-1}^{k_0-1} \mathbb{E}T_k(\gamma_n) + K_{3.1} \sum_{k=k_0}^{n-1} [a_k + e^{-k}] \leq C_1 \sum_{k=2}^{n} F^{-1}(p_c + 2^{-k}).$$

The last inequality uses Lemma 4.1 and $F^{-1}(p_c + 1/4) > 0$. This proves (i).

The proof of the lower bound is similar to that of [24], Theorem 8.1.2. By (2.5), crossing a p_{2^k}-closed dual circuit costs time at least $F^{-1}(p_c + 2^{-k\delta_0})$. If A_k is the event that there is a p_{2^k}-closed dual circuit around $\mathbf{0}$ in $B(2^k)^* \setminus B(2^{k-1})^*$, then by (2.7), $\mathbb{E}T(\mathbf{0}, \partial B(2^n))$ is bounded below by

$$\sum_{k=1}^{n} \mathbb{P}(A_k) \cdot F^{-1}(p_c + 2^{-k\delta_0}) \geq \sum_{k=1}^{n} K_{2.7} F^{-1}(p_c + 2^{-k\delta_0}).$$

Applying Lemma 4.1 completes the proof of (ii). $\square$

4.2. *Proof of Corollary* 1.3. To prove Corollary 1.3, we need the following definition from [24], page 146. Given two distribution functions G and H, we say that $G \preceq H$ if there exists $\xi > 0$ such that $G(x) \leq H(x)$ for all $0 \leq x \leq \xi$. By [24], Theorem 8.1.4, if $\rho(G) < \infty$ almost surely and if $G \preceq H$, then $\rho(H) < \infty$ almost surely. (This is, in fact, provable in general dimensions, though Zhang only gave a proof for $d = 2$.)

PROOF OF COROLLARY 1.3. Suppose that $\sum_{n=2}^{\infty} F^{-1}(p_c + 2^{-n}) < \infty$. Let $\xi > 0$, and let $\tilde{F}$ be a distribution function such that $\tilde{F} = F$ on $[0,\xi]$ and $\tilde{F}(x_0) = 1$

for some x_0. Note that we still have $\sum_{n=2}^{\infty} \tilde{F}^{-1}(p_c + 2^{-n}) < \infty$ and $\tilde{F}$ has all moments. By Theorem 1.2(i), $\rho(\tilde{F}) < \infty$ almost surely. Since $\tilde{F} \preceq F$, we have $\rho(F) < \infty$ almost surely.

Now suppose $\sum_{n=2}^{\infty} F^{-1}(p_c + 2^{-n}) = \infty$, so that $\sum_k F^{-1}(p_c + 2^{-\delta_0 k}) = \infty$ for δ_0 from (2.5). For $k \geq 1$, write A_k for the event that there is a p_{2^k}-closed dual circuit around $\mathbf{0}$ in $B(2^k)^* \setminus B(2^{k-1})^*$ and $b_k := F^{-1}(p_c + 2^{-k\delta_0})$. For $n \geq 1$, define $S_n = \sum_{k=1}^n b_k \mathbb{1}_{A_k}$ and compute by Cauchy–Schwarz, (2.7) and independence of the A_k's:

$$\mathbb{E}S_n^2 \leq \sum_{j,k=1}^n b_j b_k = \left(\sum_{k=1}^n b_k\right)^2 \leq \frac{1}{K_{2.7}^2}(\mathbb{E}S_n)^2.$$

By the Paley–Zygmund inequality (second moment method), we can find $D > 0$ such that for all $n \geq 1$, $\mathbb{P}(S_n \geq D\mathbb{E}S_n) > D$. Since $\rho(F) \geq S(n)$ for all $n \geq 1$ and $\mathbb{E}S_n \to \infty$ as $n \to \infty$, we get $\mathbb{P}(\rho(F) = \infty) > 0$. Finally, since $\mathbb{P}(\rho(F) = \infty) \in \{0, 1\}$, this completes the proof. $\square$

5. Study of the variance. Here we prove Theorems 1.5 and 1.6 using a martingale introduced in [17]. We start with some definitions.

Define $\mathrm{Ann}(n) = B(2^{n+1}) \setminus B(2^n)$, for $n \geq 0$ and $\mathrm{Ann}(-1) = B(1)$. For a vertex self-avoiding circuit $\mathcal{C}$ in $\mathbb{Z}^2$, write $\bar{\mathcal{C}}$ for the graph induced by all the vertices in $\mathbb{Z}^2$ that are either on or in the interior of $\mathcal{C}$. Define for $n \geq -1$

$$m(n) := \inf\{k \geq n : \text{there is a } p_c\text{-open circuit in } \mathrm{Ann}(k) \text{ around } \mathbf{0}\}. \tag{5.1}$$

Note that $m(n) \geq n$. We write $m(n) = m(n, \omega)$ to emphasize the underlying weights $\omega \in \Omega$. Put

$$\mathcal{C}_n := \text{the innermost } p_c\text{-open circuit } \mathcal{C} \subset \mathrm{Ann}(m(n)) \text{ around } \mathbf{0} \tag{5.2}$$

and

$$\mathcal{F}_n := \text{sigma-field generated by } \mathcal{C}_n \text{ and } \{\omega_e : e \in \bar{\mathcal{C}}_n\}. \tag{5.3}$$

By definition, we have $\mathcal{C}_n(\omega) = \mathcal{C}_{m(n,\omega)}(\omega)$. For $n < n'$, we have $m(n) \leq m(n')$, thus $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ forms a filtration. Denote $\mathcal{F}_{-1} = \{\varnothing, \Omega\}$ and $\mathcal{C}_{-1} = \{\mathbf{0}\}$. Instead of $T(\mathbf{0}, \partial B(2^n))$, we first try to study $T(\mathbf{0}, \mathcal{C}_n)$. Write $T(\mathbf{0}, \mathcal{C}_n) - \mathbb{E}T(\mathbf{0}, \mathcal{C}_n) = \sum_{k=0}^n (\mathbb{E}[T(\mathbf{0}, \mathcal{C}_n) \mid \mathcal{F}_k] - \mathbb{E}[T(\mathbf{0}, \mathcal{C}_n) \mid \mathcal{F}_{k-1}]) =: \sum_{k=0}^n \Delta_k$. Then $\{\Delta_k\}_{0\leq k\leq n}$ is an $\mathcal{F}_k$-martingale increment sequence. Thus,

$$\mathrm{Var}(T(\mathbf{0}, \mathcal{C}_n)) = \sum_{k=0}^n \mathbb{E}[\Delta_k^2]. \tag{5.4}$$

The following are the results for $T(\mathbf{0}, \mathcal{C}_n)$ corresponding to those in Theorems 1.5 and 1.6.

THEOREM 5.1. *Let η_0 be as defined in (1.2):*

(i) *If* $\eta_0 > 2$, *then there exists* $C_1 > 0$ *such that for* $n \geq 2$,

$$\mathrm{Var}(T(\mathbf{0}, \mathcal{C}_n)) \leq C_1 \sum_{k=2}^{n} [F^{-1}(p_c + 2^{-k})]^2.$$

(ii) *There exists* $C_2 > 0$ *such that for* $n \geq 2$,

$$\mathrm{Var}(T(\mathbf{0}, \mathcal{C}_n)) \geq C_2 \sum_{k=2}^{n} [F^{-1}(p_c + 2^{-k})]^2.$$

THEOREM 5.2. *Assume that* $\eta_0 > 2$. *Further assume* $\sum_{k=1}^{\infty} [F^{-1}(p_c + 2^{-k})]^2 = \infty$. *Then as* $n \to \infty$,

$$\frac{T(\mathbf{0}, \mathcal{C}_n) - \mathbb{E}T(\mathbf{0}, \mathcal{C}_n)}{(\mathrm{Var}(T(\mathbf{0}, \mathcal{C}_n)))^{1/2}} = \frac{\sum_{k=0}^{n} \Delta_k}{(\sum_{k=0}^{n} \mathbb{E}[\Delta_k^2])^{1/2}} \xrightarrow{\mathrm{d}} N(0, 1).$$

We prove Theorem 5.1 in Section 5.1. In Section 5.2, we prove the CLT in Theorem 5.2. In Section 5.3, we control the difference between $T(\mathbf{0}, \mathcal{C}_q)$ and $T(\mathbf{0}, \partial B(n))$ for $2^{q-1} \leq n \leq 2^q - 1$ and prove Theorems 1.5 and 1.6.

5.1. *Proof of Theorem* 5.1. Due to (5.4), we study bounds on the moments of Δ_k. An important ingredient is a formula for Δ_k from [17], Lemma 2, and we state it as part (ii) in the following lemma. Denote $(\Omega', \mathcal{F}', \mathbb{P}')$ as another copy of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathbb{E}'$ denote the expectation with respect to $\mathbb{P}'$, and ω' denote a sample point in Ω'. Denote $m(n, \omega)$, $\mathcal{C}_k(\omega)$ and $T(\cdot, \cdot)(\omega)$ for the quantities defined as in the previous sections, but with explicit dependence on ω. Define $\ell(n, \omega, \omega') := m(m(n, \omega) + 1, \omega')$. We need the following results, which are [17], Lemma 3 and [17], Lemma 2. The first result is older than [17] and is standard.

LEMMA 5.3 (Kesten and Zhang [17]). (i) *There exists* $K_{5.3} > 0$ *such that for all integers* $k, t \geq 1$,

$$\mathbb{P}(m(k) \geq k + t) \leq \exp(-K_{5.3} t).$$

(ii) *For all* $k \geq 0$, Δ_k *does not depend on* n. *Precisely*,

$$\Delta_k(\omega) = T(\mathcal{C}_{k-1}(\omega), \mathcal{C}_k(\omega)) + \mathbb{E}'[T(\mathcal{C}_k(\omega), \mathcal{C}_{\ell(k,\omega,\omega')}(\omega'))(\omega')] - \mathbb{E}'[T(\mathcal{C}_{k-1}(\omega), \mathcal{C}_{\ell(k,\omega,\omega')}(\omega'))(\omega')].$$

We will write $T(\cdot, \cdot)$ instead of $T(\cdot, \cdot)(\omega)$ or $T(\cdot, \cdot)(\omega')$ when the meaning is clear from the context. The following lemma is a consequence of Lemma 3.1. Recall the definition of a_k in (3.2).

LEMMA 5.4. *Assume* $\eta_0 > 1$.

(i) *For any* $r \in [1,\infty)$ *and* $\lambda \in (0,\infty)$, *there exist* $k_0 = k_0(r,\lambda,F) > 0$ *and* $K_{5.4} = K_{5.4}(r) = K_{5.4}(r,\lambda,F) > 0$ *such that for all* $k \geq k_0$ *and* $\ell \geq 1$,

$$\mathbb{E}\big[T^r\big(\partial B\big(2^k\big),\partial B\big(2^{k+\ell}\big)\big)\big] \leq K_{5.4}\ell^r\big(a_k^r + e^{-\lambda k}\big).$$

(ii) *For any* $r \in [1,\eta_0)$, *there exists a constant* $K_{5.4} = K_{5.4}(r) = K_{5.4}(r,F) > 0$ *such that for all* $k \geq -1$ *and* $\ell \geq 1$,

$$\mathbb{E}\big[T^r\big(\partial B\big(2^k\big),\partial B\big(2^{k+\ell}\big)\big)\big] \leq K_{5.4}\ell^r.$$

PROOF. Take $n := k + \ell + 1$. Since $\gamma_n \cap (B(2^{k+\ell}) \setminus B(2^k))$ provides a specific path connecting the inner and outer boundaries of the annulus, we have $T(\partial B(2^k), \partial B(2^{k+\ell})) \leq \sum_{i=k}^{k+\ell-1} T_i(\gamma_n)$. Applying Jensen's inequality and Lemma 3.1 for $k \geq k_0$, $\mathbb{E}[T^r(\partial B(2^k), \partial B(2^{k+\ell}))]$ is bounded by

$$\ell^{r-1}\left(\sum_{i=k}^{k+\ell-1}\mathbb{E}\big[T_i^r(\gamma_n)\big]\right) \leq \ell^{r-1}\cdot K_{3.1}\sum_{i=k}^{k+\ell-1}\big(a_i^r + e^{-\lambda i}\big) \leq K_{3.1}\ell^r\big(a_k^r + e^{-\lambda k}\big).$$

This proves (i). To prove (ii), if $r \in [1,\eta_0)$, by Lemma 3.1 [parts (i) and (ii) combined], for all $n \geq k \geq -1$ we have $\mathbb{E}[T_k^r(\gamma_n)] < C_1$ for some constant $C_1 = C_1(r,F) > 0$. Using this fact in the above bound proves (ii). □

The above lemma implies bounds on moments of the Δ_k's.

LEMMA 5.5. *Assume* $\eta_0 > 1$:

(i) *For any* $r \in [1,\infty)$ *and* $\lambda \in (0,\infty)$, *there exist* $K_{5.5} = K_{5.5}(r) = K_{5.5}(r,\lambda,F) > 0$ *and* $k_0 = k_0(r,\lambda,F) > 0$ *such that for all* $k \geq k_0 + 1$,

$$\mathbb{E}\big[|\Delta_k|^r\big] \leq K_{5.5}\cdot\big(a_{k-1}^r + e^{-\lambda k}\big).$$

(ii) *For any* $r \in [1,\eta_0)$ *and* $k \geq 0$, *we have* $\mathbb{E}[|\Delta_k|^r] < \infty$.

PROOF. Using $T(\mathcal{C}_k(\omega), \mathcal{C}_{\ell(k,\omega,\omega')}(\omega')) \leq T(\mathcal{C}_{k-1}(\omega), \mathcal{C}_{\ell(k,\omega,\omega')}(\omega'))$ and Lemma 5.3(ii), we have $|\Delta_k(\omega)| \leq T(\mathcal{C}_{k-1}(\omega), \mathcal{C}_k(\omega)) + \mathbb{E}'[T(\mathcal{C}_{k-1}(\omega), \mathcal{C}_{\ell(k,\omega,\omega')}(\omega'))]$. By Jensen's inequality,

$$\begin{aligned}\frac{1}{2^{r-1}}\mathbb{E}\big|\Delta_k(\omega)\big|^r &\leq \mathbb{E}\big[T^r\big(\mathcal{C}_{k-1}(\omega),\mathcal{C}_k(\omega)\big)\big]\\ &\quad + \mathbb{E}\big[\big(\mathbb{E}'\big[T\big(\mathcal{C}_{k-1}(\omega),\mathcal{C}_{\ell(k,\omega,\omega')}\big(\omega'\big)\big)\big]\big)^r\big].\end{aligned} \tag{5.5}$$

First we give an upper bound for the second term. Recall $k_0(r,\lambda,F)$ from Lemma 5.5. Fix $\omega \in \Omega$, and estimate for $k \geq k_0(2,\lambda,F) + 1$,

$$\begin{aligned}&\mathbb{E}'\big[T\big(\mathcal{C}_{k-1}(\omega),\mathcal{C}_{\ell(k,\omega,\omega')}\big(\omega'\big)\big)\big]\\ &\quad = \sum_{t=0}^{\infty}\mathbb{E}'\big[T\big(\mathcal{C}_{k-1}(\omega),\mathcal{C}_{\ell(k,\omega,\omega')}\big(\omega'\big)\big)\mathbb{1}_{\{\ell(k,\omega,\omega')-m(k,\omega)-1=t\}}\big]\end{aligned} \tag{5.6}$$

$$\le \sum_{t=0}^{\infty} \mathbb{E}'\big[T\big(\partial B\big(2^{k-1}\big), \partial B\big(2^{m(k,\omega)+2+t}\big)\big)\mathbb{1}_{\{\ell(k,\omega,\omega')-m(k,\omega)-1=t\}}\big]$$

$$\le \sum_{t=0}^{\infty} \mathbb{E}'\big[T^2\big(\partial B\big(2^{k-1}\big), \partial B\big(2^{m(k,\omega)+2+t}\big)\big)\big]^{1/2}$$

$$\times \mathbb{P}'\big(\ell\big(k,\omega,\omega'\big) - m(k,\omega) - 1 = t\big)^{1/2}$$

$$\le \sum_{t=0}^{\infty} \big(K_{5.4}(2)\big)^{1/2}\big(a_{k-1}^2 + e^{-\lambda(k-1)}\big)^{1/2}\big(m(k,\omega) - k + t + 3\big)e^{-K_{5.3}t/2}$$

$$\le C_1\big(m(k,\omega) - k + 1\big)\big(a_{k-1} + e^{-\lambda(k-1)/2}\big),$$

where the fourth line uses the Cauchy–Schwarz inequality, the sixth line uses Lemma 5.4 with $r = 2$ and Lemma 5.3(i), and in the fifth line $C_1 := K_{5.4}(2)^{1/2} \sum_{t=0}^{\infty}(t+2)e^{-K_{5.3}t/2}$. Therefore,

$$\begin{aligned} &\mathbb{E}\big[\big(\mathbb{E}'\big[T\big(\mathcal{C}_{k-1}(\omega), \mathcal{C}_{\ell(k,\omega,\omega')}\big(\omega'\big)\big)\big]\big)^r\big] \\ &\quad \le C_1^r\big(a_{k-1} + e^{-\lambda(k-1)/2}\big)^r \mathbb{E}\big[\big(m(k,\omega) - k + 1\big)^r\big] \\ &\quad \le C_1^r \mathbb{E}\big[\big(m(k,\omega) - k + 1\big)^r\big] \cdot 2^{r-1}\big(a_{k-1}^r + e^{-\lambda r(k-1)/2}\big). \end{aligned} \tag{5.7}$$

By Lemma 5.3(i) $\mathbb{E}[(m(k,\omega) - k + 1)^r] < \infty$ uniformly in k, so this bounds the second term in (5.5). To bound the first term in (5.5), similar to (5.6), applying the Cauchy–Schwarz inequality, we have for $k \ge k_0(2r, \lambda, F) + 1$,

$$\begin{aligned} &\mathbb{E}\big[T^r\big(\mathcal{C}_{k-1}(\omega), \mathcal{C}_k(\omega)\big)\big] \\ &\quad \le \sum_{t=0}^{\infty} \mathbb{E}\big[T^{2r}\big(\partial B\big(2^{k-1}\big), \partial B\big(2^{k+t+1}\big)\big)\big]^{1/2} \mathbb{P}\big(m(k) - k = t\big)^{1/2} \\ &\quad \le \sum_{t=0}^{\infty} \big[K_{5.4}(2r)\big]^{1/2}\big(a_{k-1}^{2r} + e^{-\lambda(k-1)}\big)^{1/2}(t+2)^r \cdot e^{-K_{5.3}t/2} \\ &\quad \le \big(a_{k-1}^r + e^{-\lambda(k-1)/2}\big)\left(\big[K_{5.4}(2r)\big]^{1/2} \sum_{t=0}^{\infty}(t+2)^r e^{-K_{5.3}t/2}\right). \end{aligned} \tag{5.8}$$

Combining (5.5), (5.7) and (5.8) completes the proof of Lemma 5.5(i). The proof of part (ii) can be done in the same way, using Lemma 5.4(ii). □

The next lemma gives a lower bound for $\mathbb{E}[\Delta_k^2]$.

LEMMA 5.6. *There is $K_{5.6} > 0$ such that for all integers $k \ge 2$,*

$$\mathbb{E}\big[\Delta_k^2\big] \ge K_{5.6}\big[F^{-1}\big(p_c + 2^{-\delta_0 k}\big)\big]^2,$$

where δ_0 is from (2.5).

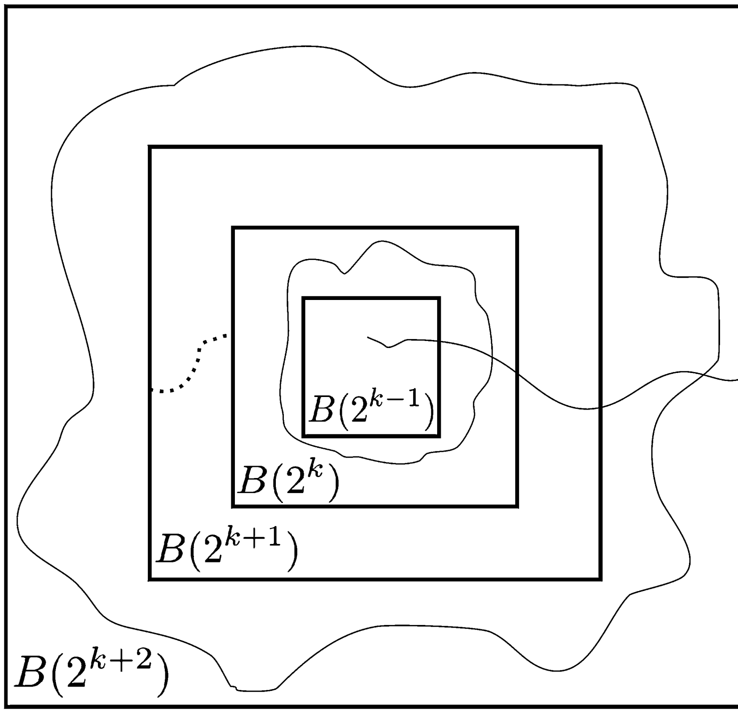

FIG. 2. *The events* (1)–(4) *in the proof of Lemma* 5.6. *The* p_c*-open crossing on the right connects the two* p_c*-open circuits around the origin, but the* p_c*-closed path on the left* (*shown as a dotted curve*) *blocks the existence of a* p_c*-open circuit around* **0** *in* $B(2^{k+1}) \setminus B(2^k)$.

PROOF. Recall the expression of Δ_k in Lemma 5.3(ii) and the filtration $\mathcal{F}_k$ in (5.3). The goal of the proof is to construct an event $E \in \mathcal{F}_k$ with $\mathbb{P}(E) > 0$ uniformly in n, k such that for $\omega \in E$,

$$T(\mathcal{C}_{k-1}(\omega), \mathcal{C}_k(\omega))(\omega) = 0, \tag{5.9}$$

$$\begin{aligned} &\mathbb{E}'[T(\mathcal{C}_{k-1}(\omega), \mathcal{C}_{\ell(k,\omega,\omega')}(\omega'))] - \mathbb{E}'[T(\mathcal{C}_k(\omega), \mathcal{C}_{\ell(k,\omega,\omega')}(\omega'))] \\ &\quad > C_2 F^{-1}(p_c + 2^{-\delta_0 k}), \end{aligned} \tag{5.10}$$

where $C_2 > 0$ is a constant. Let $\tilde{E}$ be the intersection of the following events (see Figure 2):

(1) there exists a p_c-open circuit around **0** in $B(2^k) \setminus B(2^{k-1})$,
(2) there exists a p_c-open circuit around **0** in $B(2^{k+2}) \setminus B(2^{k+1})$,
(3) there exists a p_c-open left-right crossing of $[0, 2^{k+2}] \times [-2^{k-1}, 2^{k-1}]$, and
(4) there exists a *dual* p_c-closed left-right crossing of $[-2^{k+1}, -2^k]^* \times [-2^k, 2^k]^*$.

By the RSW theorem ([8], Section 11.7), each of the above events has probability bounded from below for all $k \geq 1$. The events (1), (2) and (3) are all nonincreasing, and they are jointly independent from (4). Therefore, applying independence and the FKG inequality, there exists a constant $C_3 > 0$ such that $\mathbb{P}(\tilde{E}) \geq C_3$ for all $k \geq 1$. Now consider a new event (3′): There exists a p_c-open left-right crossing of $\bar{\mathcal{C}}_k \cap ([0, 2^{k+2}] \times [-2^{k-1}, 2^{k-1}])$. Define the event E to be the intersection of the events (1), (2), (4) and (3′). Then $E \in \mathcal{F}_k$, $\tilde{E} \subset E$ and, therefore, $\mathbb{P}(E) \geq \mathbb{P}(\tilde{E}) \geq C_3 > 0$. By definition, we have $T(\mathcal{C}_{k-1}(\omega), \mathcal{C}_k(\omega))(\omega) = 0$ for $\omega \in E$, so (5.9) holds. To see (5.10), recall the definition of p_n in (2.3). Let $E' \subset \Omega'$ be the event

$$E' = \{\text{There is a } p_{2^k}\text{-closed dual circuit around } \mathbf{0} \text{ in } B(2^{k+1})^* \setminus B(2^k)^*\}.$$

From (2.7), let $C_2 > 0$ be such that $\mathbb{P}'(E') > C_2$ for all k. When $\omega \in E$ and $\omega' \in E'$, since every path between $\mathcal{C}_{k-1}(\omega)$ and $\mathcal{C}_{\ell(k,\omega,\omega')}(\omega')$ must cross the p_{2^k}-closed dual circuit defined in E' and then cross $\mathcal{C}_k(\omega)$, we have for $k \geq 2$

$$T(\mathcal{C}_{k-1}(\omega), \mathcal{C}_{\ell(k,\omega,\omega')}(\omega'))(\omega') - T(\mathcal{C}_k(\omega), \mathcal{C}_{\ell(k,\omega,\omega')}(\omega'))(\omega') \geq F^{-1}(p_{2^k}),$$

which by (2.5) is bounded below by $F^{-1}(p_c + 2^{-\delta_0 k})$. This proves (5.10) and therefore we have $\mathbb{P}(\Delta_k < -C_2 F^{-1}(p_c + 2^{-\delta_0 k})) \geq \mathbb{P}(E) \geq C_3$, completing the proof of Lemma 5.6. □

PROOF OF THEOREM 5.1. First we prove (i). Lemma 5.5(i) with $r = 2$ and $\lambda = 1$ implies that there exists $k_0 \geq 1$ such that for all $k \geq k_0 + 1$, we have $\mathbb{E}[\Delta_k^2] \leq K_{5.5}(a_{k-1}^2 + e^{-k})$. For $k \leq k_0$, we will use the general fact that $\mathbb{E}[\Delta_k^2] < C_1$ for some $C_1 > 0$. Therefore, for $k \geq k_0 + 1$,

$$\begin{aligned}\operatorname{Var}(T(\mathbf{0}, \mathcal{C}_n)) &= \sum_{k=0}^{k_0} \mathbb{E}[\Delta_k^2] + \sum_{k=k_0+1}^{n} \mathbb{E}[\Delta_k^2] \leq (k_0+1)C_1 + K_{5.5} \sum_{k=k_0}^{n-1} (a_k^2 + e^{-k}) \\ &\leq C_2 + K_{5.5} \sum_{k=k_0}^{n-1} a_k^2,\end{aligned}$$

where $C_2 > 0$. Using Lemma 4.1 with $f(t) := (F^{-1}(p_c + 2^{-t}) \wedge a_{k_0})^2$, $t \geq 0$, completes the proof of (i).

By Lemma 5.6, $\operatorname{Var}(T(\mathbf{0}, \mathcal{C}_n)) = \sum_{k=0}^{n} \mathbb{E}[\Delta_k^2] \geq K_{5.6} \sum_{k=2}^{n} [F^{-1}(p_c + 2^{-\delta_0 k})]^2$ for $n \geq 2$. Applying Lemma 4.1 again completes the proof of (ii). □

5.2. *Proof of Theorem* 5.2.

PROOF OF THEOREM 5.2. By Lemma 5.5, there exist $k_1, C_3, C_4 > 0$ such that

$$\mathbb{E}[\Delta_k^2] \leq C_3 \qquad \text{for all } k \geq 0, \tag{5.11}$$

$$\mathbb{E}[|\Delta_k|^r] \leq C_4(a_{k-1}^r + e^{-k}) \qquad \text{for all } k \geq k_1, r \in \{2, 3, 6\}. \tag{5.12}$$

Here, the choice of $r \in \{2, 3, 6\}$ is sufficient for proving the CLT. Though the constants C_4 and k_1 may depend on r, this will not be an issue since we only consider finitely many different values of r.

By Theorem 1.5(ii) and the assumption $\sum_{k=2}^{\infty}(F^{-1}(p_c + 2^{-k}))^2 = \infty$, we have $\mathrm{Var}(T(\mathbf{0}, \mathcal{C}_n)) \to \infty$ as $n \to \infty$. By (5.11), we have $\sum_{k=0}^{k_1-1} \mathbb{E}[\Delta_k^2] \le C_3 k_1$, thus we can throw away the first k_1 terms and it is sufficient to prove

$$\frac{\sum_{k=k_1}^{n} \Delta_k}{(\sum_{k=k_1}^{n} \mathbb{E}[\Delta_k^2])^{1/2}} \xRightarrow{\mathrm{d}} N(0, 1).$$

This can be proved in a similar way as in [17]. The key tool is a martingale CLT from McLeish [18], Theorem 2.3. The moment bounds in (5.11) and (5.12) for $r \in \{2, 3, 6\}$ are sufficient to verify its hypotheses. For a full proof, see the arXiv version of this paper [6]. $\square$

5.3. *Proofs of Theorems 1.5 and 1.6.* For any $n \ge 1$, let $q \in \mathbb{Z}$ satisfy $2^{q-1} \le n < 2^q$. The next lemma bounds $|T(\mathbf{0}, \partial B(n)) - T(\mathbf{0}, \mathcal{C}_q)|$.

LEMMA 5.7. *Recall η_0 from (1.2). Assume $\eta_0 > 1$:*

(i) *For any $r \in [1, \eta_0)$, there is $C_0 > 0$ such that for all $n \ge 1$ and $q \ge 1$ such that $2^{q-1} \le n < 2^q$*

$$\mathbb{E}\big[\big|T\big(\mathbf{0}, \partial B(n)\big) - T(\mathbf{0}, \mathcal{C}_q)\big|^r\big] < C_0.$$

(ii) *Assume that $\sum_k a_k^{\eta_1} < \infty$ for some $\eta_1 \in [1, \eta_0)$. Then*

$$\sum_{q=0}^{\infty} \sup_{2^{q-1} \le n < 2^q} \mathbb{E}\big[\big|T\big(\mathbf{0}, \partial B(n)\big) - T(\mathbf{0}, \mathcal{C}_q)\big|^{\eta_1}\big] < \infty.$$

PROOF. We first prove (i). Observe that for $1 \le \ell \le q$, on the event $\{m(q - \ell) \ge q - 1 > m(q - \ell - 1)\}$, $\partial B(n)$ is sandwiched between $\mathcal{C}_{q-\ell-1}$ and $\mathcal{C}_q$. Furthermore, for integers $1 \le \ell \le q$ and $t \ge 0$, restricted to the event $\{m(q - \ell) \ge q - 1 > m(q - \ell - 1)\} \cap \{m(q) = q + t\}$, we have

$$\big|T\big(\mathbf{0}, \partial B(n)\big) - T(\mathbf{0}, \mathcal{C}_q)\big| \le T\big(\partial B\big(2^{q-\ell-1}\big), \partial B\big(2^{q+t+1}\big)\big). \tag{5.13}$$

Then define the events $A_\ell := \{m(q - \ell) \ge q - 1 > m(q - \ell - 1)\}$, for $1 \le \ell \le q$, and $B_t := \{m(q) = q + t\}$, for $t \ge 0$. Using (5.13) and the fact that $\bigcup_{1 \le \ell \le q} \bigcup_{t \ge 0}(A_\ell \cap B_t)$ cover the whole probability space Ω, we have

$$\begin{aligned} &\mathbb{E}\big[\big|T\big(\mathbf{0}, \partial B(n)\big) - T(\mathbf{0}, \mathcal{C}_q)\big|^r\big] \\ &\quad \le \sum_{\ell=1}^{q} \sum_{t=0}^{\infty} \mathbb{E}\big[T^r\big(\partial B\big(2^{q-\ell-1}\big), \partial B\big(2^{q+t+1}\big)\big) \mathbb{1}_{A_\ell} \mathbb{1}_{B_t}\big] \qquad (5.14) \\ &\quad \le \sum_{\ell=1}^{q} \sum_{t=0}^{\infty} \mathbb{E}\big[T^\eta\big(\partial B\big(2^{q-\ell-1}\big), \partial B\big(2^{q+t+1}\big)\big)\big]^{\frac{r}{\eta}} \mathbb{P}(A_\ell)^{\frac{\eta-r}{2\eta}} \mathbb{P}(B_t)^{\frac{\eta-r}{2\eta}}, \end{aligned}$$

where the last line uses Hölder's inequality with $\eta \in (r, \eta_0)$. Recall k_0 from Lemma 3.1. Define $b_k := a_k + e^{-k}$ for $k \geq k_0$ and $b_k := b_{k_0}$ for $-1 \leq k < k_0$. By Lemma 5.4, there is $C_1 > 0$ such that for all integers $k \geq -1$ and $r \geq 0$

$$\mathbb{E}\big[T^{\eta}\big(\partial B(2^k), \partial B(2^{k+r})\big)\big] \leq (C_1 r b_k)^{\eta}. \tag{5.15}$$

By Lemma 5.3(i), there exists $C_2 > 0$ such that for $1 \leq \ell \leq q$ and $t \geq 0$

$$\mathbb{P}(A_\ell)^{\frac{\eta-1}{2\eta}} \mathbb{P}(B_t)^{\frac{\eta-1}{2\eta}} \leq e^{-C_2(\ell-1)} e^{-C_2 t}. \tag{5.16}$$

Combining (5.14), (5.15) and (5.16), $\mathbb{E}[|T(\mathbf{0}, \partial B(n)) - T(\mathbf{0}, \mathcal{C}_q)|^r]$ is bounded by

$$\sum_{\ell=1}^{q} \sum_{t=0}^{\infty} C_1^r (t + \ell + 2)^r b_{q-\ell-1}^r e^{-C_2(\ell-1)} e^{-C_2 t} = \sum_{\ell=1}^{q} b_{q-\ell-1}^r c_\ell,$$

where $c_\ell := e^{-C_2(\ell-1)} \sum_{t=0}^{\infty} C_1^r (t + \ell + 2) e^{-C_2 t}$ for $\ell \geq 1$. Write $b_k := 0$ for $k \leq -2$ and $c_\ell := 0$ for $\ell \leq -1$. Define $\tilde{b} := (b_k^r : k \in \mathbb{Z})$ and $\tilde{c} := (c_k : k \in \mathbb{Z})$. Then the above bound can be written as $(\tilde{b} * \tilde{c})_{q-1}$, where $\tilde{b} * \tilde{c}$ is the convolution of $\tilde{b}$ and $\tilde{c}$. Note that $\|\tilde{b}\|_\infty < \infty$ and $\|\tilde{c}\|_1 < \infty$. Then (i) follows from Young's inequality, which says $\|\tilde{b} * \tilde{c}\|_\infty \leq \|\tilde{b}\|_\infty \|\tilde{c}\|_1$.

Next we prove (ii). Replacing r with η_1 in the above argument, we have $\mathbb{E}[|T(\mathbf{0}, \partial B(n)) - T(\mathbf{0}, \mathcal{C}_q)|^{\eta_1}] \leq (\tilde{b} * \tilde{c})_{q-1}$. Therefore, by Young's inequality,

$$\sum_{q=0}^{\infty} \sup_{n: 2^{q-1} \leq n < 2^q} \mathbb{E}\big[\big|T\big(\mathbf{0}, \partial B(n)\big) - T(\mathbf{0}, \mathcal{C}_q)\big|^{\eta_1}\big] \leq \|\tilde{b} * \tilde{c}\|_1 \leq \|\tilde{b}\|_1 \|\tilde{c}\|_1.$$

The assumption $\sum_{k=k_0}^{\infty} a_k^{\eta_1} < \infty$ implies $\|\tilde{b}\|_1 < \infty$. Thus, the proof of (ii) is completed. $\square$

We now give the main results about $T(\mathbf{0}, \partial B(n))$, beginning with the variance bound.

PROOF OF THEOREM 1.5. For simplicity, denote $s_q := \sum_{k=2}^{q} [F^{-1}(p_c + 2^{-k})]^2$. For $n \geq 2$, let $q \geq 2$ be the integer such that $2^{q-1} \leq n < 2^q - 1$. Denote $X_n := T(\mathbf{0}, \partial B(n)) - \mathbb{E}T(\mathbf{0}, \partial B(n))$ and $Y_n := T(\mathbf{0}, \mathcal{C}_q) - \mathbb{E}T(\mathbf{0}, \mathcal{C}_q)$. Since $\eta_0 > 2$, we may apply Lemma 5.7(i) with $r = 2$, there exists a constant $C_0 > 0$ such that for all $n \geq 2$

$$\|X_n - Y_n\|_2 = \mathbb{E}\big[|X_n - Y_n|^2\big]^{1/2} \leq C_0. \tag{5.17}$$

By Theorem 5.1, there exist $C_1, C_2 > 0$ such that for all $n \geq 2$, $C_1 \sqrt{s_q} \leq \|Y_n\|_2 \leq C_2 \sqrt{s_q}$. Combining the above two bounds and the triangle inequality, we have $((C_1 \sqrt{s_q} - C_0) \vee 0)^2 \leq \mathbb{E}[X_n^2] \leq (C_2 \sqrt{s_q} + C_0)^2$. This suffices to prove the upper bound.

For the lower bound, $C_1\sqrt{s_q} - C_0$ may be negative for small n, so one needs $\operatorname{Var} T(\mathbf{0}, \partial B(n)) > 0$ uniformly in $n \geq 1$. Because this is standard (see [16], equation (4.7)), we omit the proof. $\square$

PROOF OF THEOREM 1.6. First we prove (i). Suppose $\sum_{k=2}^{\infty}[F^{-1}(p_c + 2^{-k})]^2 < \infty$. Then $\sum_{k=k_0}^{\infty} a_k^2 < \infty$, where k_0 is defined in Lemma 3.1. Also note that $T(\mathbf{0}, \mathcal{C}_q) - \mathbb{E}T(\mathbf{0}, \mathcal{C}_q) = \sum_{k=0}^{q} \Delta_k$ and Δ_k, for $k \geq 0$, does not depend on n or q. By Theorem 5.1(ii), we have $\sum_{k=1}^{\infty} \mathbb{E}\Delta_k^2 < \infty$. Then by the martingale convergence theorem, there exists a random variable Z with $\mathbb{E}Z = 0$ and $\mathbb{E}Z^2 < \infty$ such that as $q \to \infty$

$$T(\mathbf{0}, \mathcal{C}_q) - \mathbb{E}T(\mathbf{0}, \mathcal{C}_q) \to Z \qquad \text{a.s. and in } L^2. \tag{5.18}$$

Applying Lemma 5.7(ii) with $\eta_1 = 2$ and taking $n_q = 2^{q-1}$ or $2^q - 1$, for $q \geq 0$, we have $\sum_{q=0}^{\infty} \mathbb{E}[|T(\mathbf{0}, \partial B(n_q)) - T(\mathbf{0}, \mathcal{C}_q)|^2] < \infty$. Therefore, by Borel–Cantelli and (5.18), as $q \to \infty$,

$$T\big(\mathbf{0}, \partial B(n_q)\big) - \mathbb{E}T\big(\mathbf{0}, \partial B(n_q)\big) \to Z \qquad \text{a.s. and in } L^2. \tag{5.19}$$

Note that for all n, q with $2^{q-1} \leq n < 2^q$, $|T(\mathbf{0}, \partial B(n)) - T(\mathbf{0}, \mathcal{C}_q)|$ is bounded by $\max\{|T(\mathbf{0}, \partial B(2^{q-1})) - T(\mathbf{0}, \mathcal{C}_q)|, |T(\mathbf{0}, \partial B(2^q - 1)) - T(\mathbf{0}, \mathcal{C}_q)|\}$. Combining the above observation and (5.19) completes the proof of Theorem 1.6(i).

Next we prove (ii). Suppose $\sum_{k=2}^{\infty}[F^{-1}(p_c + 2^{-k})]^2 = \infty$. Define $\sigma_n := \operatorname{Var}(T(\mathbf{0}, \mathcal{C}_q))^{1/2}$ where $q \in \mathbb{N}$ is such that $2^{q-1} \leq n < 2^q$. Define $\gamma_n := \sqrt{\operatorname{Var}(T(\mathbf{0}, \partial B(n)))}$. By Theorem 5.1(ii), we have $\lim_{n\to\infty} \sigma_n = \infty$. By Lemma 5.7(i) with $r = 2$, there is $C_0 > 0$ such that for all $n \geq 2$

$$|\sigma_n - \gamma_n| \leq C_0. \tag{5.20}$$

Furthermore, there is $C_1 > 0$ such that for all $n \geq 2$

$$\mathbb{E}\big[\big|T\big(\mathbf{0}, \partial B(n)\big) - T(\mathbf{0}, \mathcal{C}_q)\big|\big] \leq C_1. \tag{5.21}$$

Theorem 1.6(ii) follows from Theorem 5.2, (5.21), (5.20) and the fact that $\lim_{n\to\infty} \sigma_n = \infty$. $\square$

5.4. *Limit theorems for point-to-point times.* In this section, we extend results from the last section to point-to-point passage times.

COROLLARY 5.8. (i) *Assuming $\eta_0 > 1$, there exists $C_1 = C_1(F) > 0$ such that*

$$\mathbb{E}T(\mathbf{0}, x) \leq C_1 \sum_{k=2}^{q} F^{-1}\big(p_c + 2^{-k}\big) \qquad \textit{for } x \in B\big(2^{q+1}\big) \setminus B\big(2^q\big) \textit{ and } q \geq 2.$$

(ii) *There exists $C_2 = C_2(F) > 0$ such that*

$$\mathbb{E}T(\mathbf{0}, x) \geq C_2 \sum_{k=2}^{q} F^{-1}\big(p_c + 2^{-k}\big) \qquad \textit{for } x \in B\big(2^{q+1}\big) \setminus B\big(2^q\big) \textit{ and } q \geq 2.$$

COROLLARY 5.9. *Assume that* $\eta_0 > 2$.

(i) *There exists* $C_3 = C_3(F) > 0$ *such that*

$$\operatorname{Var}(T(\mathbf{0},x)) \le C_3 \sum_{k=2}^{q} [F^{-1}(p_c + 2^{-k})]^2 \qquad \text{for } x \in B(2^{q+1}) \setminus B(2^q) \text{ and } q \ge 2.$$

(ii) *There exists* $C_4 = C_4(F) > 0$ *such that*

$$\operatorname{Var}(T(\mathbf{0},x)) \ge C_4 \sum_{k=2}^{q} [F^{-1}(p_c + 2^{-k})]^2 \qquad \text{for } x \in B(2^{q+1}) \setminus B(2^q) \text{ and } q \ge 2.$$

COROLLARY 5.10. *Assume that* $\eta_0 > 2$ *and* $\sum_{k=2}^{\infty} F^{-1}(p_c + 2^{-k}) = \infty$.

(i) *If* $\sum_{k=2}^{\infty}[F^{-1}(p_c+2^{-k})]^2 < \infty$, *then there exists a random variable* $\tilde{Z}$ *with* $\mathbb{E}\tilde{Z} = 0$ *and* $\mathbb{E}\tilde{Z}^2 < \infty$ *such that*

$$T(\mathbf{0},x) - \mathbb{E}T(\mathbf{0},x) \stackrel{\mathrm{d}}{\Longrightarrow} \tilde{Z} \qquad \text{as } \|x\|_\infty \to \infty.$$

$\tilde{Z}$ *has the same distribution as the sum of two independent copies of* Z, *defined in Theorem* 1.6.

(ii) *If* $\sum_{k=2}^{\infty}[F^{-1}(p_c + 2^{-k})]^2 = \infty$, *then*

$$\frac{T(\mathbf{0},x) - \mathbb{E}T(\mathbf{0},x)}{\operatorname{Var}(T(\mathbf{0},x))^{1/2}} \stackrel{\mathrm{d}}{\Longrightarrow} N(0,1) \qquad \text{as } \|x\|_\infty \to \infty.$$

In particular, letting $q = q(x)$ *be the integer such that* $2^q < \|x\|_\infty \le 2^{q+1}$, *we have*

$$\frac{\operatorname{Var}(T(\mathbf{0},x))}{\operatorname{Var}(T(\mathbf{0},\partial B(2^{q(x)})))} \to 2 \qquad \text{as } \|x\|_\infty \to \infty.$$

REMARK 6. In contrast to Theorem 1.6(ii), one only expects convergence in distribution in Corollary 5.10(i), since $T(\mathbf{0},x)$ heavily depends on the edge-weights near the point x, which tends to infinity. As x changes, the edge weights near it only share the same distribution.

Now we describe the construction used in the proof of the above three corollaries. This construction was introduced in [17]. Suppose $x \in B(2^{q+1}) \setminus B(2^q)$. Then the two boxes $B(\mathbf{0}, 2^{q-1})$ and $B(x, 2^{q-1})$ are disjoint and, therefore, $T(\mathbf{0}, \partial B(\mathbf{0}, 2^{q-1}))$ and $T(x, \partial B(x, 2^{q-1}))$ are i.i.d. Define $Y(x) := T(\mathbf{0}, \partial B(\mathbf{0}, 2^{q-1})) + T(x, \partial B(x, 2^{q-1}))$ for $x \in B(2^{q+1}) \setminus B(2^q)$ and $q \ge 2$. Then $T(\mathbf{0}, x) \ge Y(x)$. The statements in the above three corollaries, with $T(\mathbf{0}, x)$ replaced by $Y(x)$, are immediate consequences of Theorems 1.2, 1.5 and 1.6. We only need to control the error between $T(\mathbf{0}, x)$ and $Y(x)$. To bound $T(\mathbf{0}, x)$ from above, recall the definition of $\mathcal{C}_{q+2}$ from (5.2). One can construct a path between $\mathbf{0}$ and x by concatenating a geodesic from $\mathbf{0}$ to $\mathcal{C}_{q+2}$, a p_c-open path along $\mathcal{C}_{q+2}$, and a geodesic from $\mathcal{C}_{q+2}$ to x. Thus, $T(\mathbf{0}, x)$ can be bounded above by $T(\mathbf{0}, \mathcal{C}_{q+2}) + T(x, \mathcal{C}_{q+2})$. This implies

$$
\begin{aligned}
\big|T(\mathbf{0}, x) - Y(x)\big| &\le \big|T(\mathbf{0}, \mathcal{C}_{q+2}) - T\big(\mathbf{0}, \partial B\big(\mathbf{0}, 2^{q-1}\big)\big)\big| \\
&\quad + \big|T(x, \mathcal{C}_{q+2}) - T\big(x, \partial B\big(x, 2^{q-1}\big)\big)\big|.
\end{aligned} \tag{5.22}
$$

The first term in the above bound can be controlled by Lemma 5.7 and the second term can be controlled by the following lemma, which is also analogous to Lemma 5.7.

LEMMA 5.11. *Recall η_0 from (1.2). Assume $\eta_0 > 1$:*

(i) *For any $r \in [1, \eta_0)$, there is $C_0 > 0$ such that for all $q \ge 0$ and $x \in B(2^{q+1}) \setminus B(2^q)$*

$$\mathbb{E}\big[\big|T(x, \mathcal{C}_{q+2}) - T\big(x, \partial B\big(x, 2^{q-1}\big)\big)\big|^r\big] < C_0.$$

(ii) *Assume that $\sum_k a_k^{\eta_1} < \infty$ for some $\eta_1 \in [1, \eta_0)$. Then*

$$\sum_{q=0}^{\infty} \sup_{x \in B(2^{q+1}) \setminus B(2^q)} \mathbb{E}\big[\big|T(x, \mathcal{C}_{q+2}) - T\big(x, \partial B\big(x, 2^{q-1}\big)\big)\big|^{\eta_1}\big] < \infty.$$

The proof of the above lemma is similar to the one of Lemma 5.7 and, therefore, is omitted.

PROOF OF COROLLARY 5.8. Lemma 5.11(i), Lemma 5.7(i) and (5.22) give $C_0 > 0$ with $\mathbb{E}|T(\mathbf{0}, x) - Y(x)| \le C_0$ for all x, proving (i). Combining the lower bound $T(\mathbf{0}, x) \ge Y(x)$ and Theorem 1.2(ii) proves (ii). □

PROOF OF COROLLARY 5.9. When $\eta_0 > 2$, by Lemma 5.11(i), Lemma 5.7(i) and (5.22), there is $C_0 > 0$ such that $\mathbb{E}|T(\mathbf{0}, x) - Y(x)|^2 \le C_0$ for all x. Then the rest of the proof is similar to the proof of Theorem 1.5. □

PROOF OF COROLLARY 5.10. To show (i), since $\sum_k a_k^2 < \infty$ and $\eta_0 > 2$, by Lemma 5.11(ii) we have $\mathbb{E}|T(x, \mathcal{C}_{q+2}) - T(x, \partial B(x, 2^{q-1}))|^2 \to 0$ as $\|x\|_\infty \to \infty$. Then by (5.22), we have $T(\mathbf{0}, x) - Y(x) \to 0$ in L^2 as $\|x\|_\infty \to \infty$. By Theorem 1.6(i) and that $T(\mathbf{0}, \partial B(2^{q-1}))$ and $T(x, \partial B(x, 2^{q-1}))$ are independent, $Y(x) \overset{\mathrm{d}}{\Longrightarrow} Z + Z'$, as $\|x\|_\infty \to \infty$, where Z' is another independent copy of Z as in Theorem 1.6(i). Combining these proves (i). The proof of (ii) is similar to that of Theorem 1.6(ii). □

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