STOCHASTIC DE GIORGI ITERATION AND REGULARITY OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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Under general conditions, we devise a stochastic version of De Giorgi iteration scheme for semilinear stochastic parabolic partial differential equation of the form

$$\partial_t u = \text{div}(A\nabla u) + f(t, x, u) + g_i(t, x, u)\dot{w}_t^i$$

with progressively measurable diffusion coefficients. We use the scheme to show that the solution of the equation is almost surely Hölder continuous in both space and time variables.

1. Introduction. Stochastic partial differential equations (SPDEs) arise in many pure and applied sciences. Regularity of solutions is of central importance for theoretical development as well as for numerical simulation. For linear equations with constant diffusion coefficients, the $W^{n,2}$-theory has been well developed (see Pardoux [14] and Rozovskii [16]), and a more general $W^{k,p}$-theory has been established by Krylov [8]. Such equations can also be studied from a semigroup point of view (Brzèzniak, van Neerven, Veraar and Weis [1] and Da Prato and Zabczyk [5]). Results concerning nonlinear equations can be found in Debussche, De Moor and Hofmanova [6] and Pardoux [13]. In particular, many examples of semilinear SPDEs with measurable coefficients can be found in the survey monograph edited by Carmona and Rozovskii [4]. Although an obviously important question in applications, regularity of solutions of semilinear SPDEs with random diffusion coefficients does not seem to have been adequately addressed in the literature.

In this paper, we consider the following type of semilinear SPDEs on $\mathbb{R}^n$:

$$\partial_t u = \text{div}(A\nabla u) + f(t, x, u) + g_i(t, x, u)\dot{w}_t^i, \tag{1.1}$$

where $\{w^i\}$ is a sequence of independent standard Brownian motions on a filtered probability space $(\Omega, \mathscr{F}_*, \mathbb{P})$, the diffusion coefficients A are $\mathscr{F}_* = \{\mathscr{F}_t\}$-progressively measurable, and $g = \{g_i\}$ is an ℓ^2-valued function such that for each fixed x and a progressively measurable process h, the process $g(t, x, h_t)$ is also progressively measurable. We will show that almost surely a stochastically strong solution with L^2-initial data is Hölder continuous in both space and (strictly positive) time variables and its Hölder norm has finite moments of all orders.

The basic assumptions on the SPDE (1.1) are as follows:

(1) Uniform ellipticity: $A(t, x; \omega)$ is $\mathscr{F}_*$-progressively measurable and uniformly elliptic, that is, there is a positive constant λ such that

$$\lambda I \leq A(t, x; \omega) \leq \lambda^{-1} I \qquad \forall (t, x, \omega) \in \mathbb{R}_+ \times \mathbb{R}^n \times \Omega.$$

(2) Linear growth: there exist a nonnegative function $K \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and a positive constant Λ such that

$$(1.2) \quad \left| f(t, x, u) \right| + \left| g(t, x, u) \right|_{\ell^2} \leq K(x) + \Lambda |u| \qquad \forall (t, x, u) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}.$$

We emphasize that no further conditions concerning the continuity A, f or g are imposed. A stochastic process $u = u(t, x; \omega)$ is said to be a (stochastically strong) solution of (1.1) if it is an almost surely continuous L^2 process belonging to the space $L^2(\Omega \times \mathbb{R}^+, \mathscr{P}, W^{1,2}(\mathbb{R}^n))$ and satisfies the SPDE (1.1) in the sense that

$$
\begin{aligned}
(1.3) \quad \langle u(t), \varphi \rangle = {} & \langle u(0), \varphi \rangle - \int_0^t \langle A \nabla u(s), \nabla \varphi \rangle \, ds + \int_0^t \langle f(u(s)), \varphi \rangle \, ds \\
& + \int_0^t \langle g_i(u(s)), \varphi \rangle \, dw_s^i
\end{aligned}
$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$. Here, $\mathscr{P}$ is the completion of the progressively measurable σ-algebra on $\Omega \times \mathbb{R}_+$ under the product measure $\mathbb{P}(d\omega) \times dt$, and $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $L^2(\mathbb{R}^n)$. The main result of the current work is the following moment estimate.

THEOREM 1.1. *Let u be a* (*stochastically strong*) *solution of the SPDE* (1.1) *with* (*nonrandom*) *initial data $u(0) = u_0$. Then for every $p > 0$ there is a constant $C = C(n, \lambda, \Lambda, T, p)$ such that*

$$
\begin{aligned}
\mathbb{E} \int_0^{2T} & \|u(t)\|_{L^2(\mathbb{R}^n)}^p \, dt + \mathbb{E} \|u\|_{L^\infty([T,2T] \times \mathbb{R}^n)}^p \\
& \leq C \big(\|u_0\|_{L^2(\mathbb{R}^n)} + \|K\|_{L^2(\mathbb{R}^n)} + \|K\|_{L^\infty(\mathbb{R}^n)} \big)^p.
\end{aligned}
$$

Using this moment estimate and following a suggestion from Professor Nicolai Krylov and the approaches used in Debussche, De Moor and Hofmanova [6], we will prove the following regularity statement for the solution.

THEOREM 1.2. *Let u be a solution of the SPDE* (1.1) *with a* (*deterministic*) *initial condition $u(0) = u_0 \in L^2(\mathbb{R}^n)$. Then there exists a positive exponent $\alpha = \alpha(n, \lambda, \Lambda)$ such that for all $T > 0$ the solution $u \in C^\alpha([T, 2T] \times \mathbb{R}^n)$ almost surely. Furthermore, for every $p > 0$, there is a constant $C = C(n, \lambda, \Lambda, T, p)$ such that*

$$\mathbb{E} \|u\|_{C^\alpha([T,2T] \times \mathbb{R}^n)}^p \leq C \big(\|u_0\|_{L^2(\mathbb{R}^n)} + \|K\|_{L^2(\mathbb{R}^n)} + \|K\|_{L^\infty(\mathbb{R}^n)} \big)^p.$$

REMARK 1.3. In general, (1.1) may not admit a stochastically strong solution if the coefficients are merely progressively measurable. We have assumed the existence of such strong solution in our settings.

However, under some general conditions a weak solution exists. This weak solution, under some additional assumptions, can be understood as a strong solution on another probability space therefore our result can be applied; see Carmona and Rozovskii [4] or Viot [17] for a detailed exposition.

The novelty of our result is that we do not impose any assumptions on the smoothness of A, f or g. Indeed, if A and g have some continuity, for example, Dini continuity, then the above result follows directly from Krylov [8, 9]. The approach we adopted in this work is quite different from the usual ones in the study of SPDEs. Largely motivated by the recent work of Glatt-Holtz, Šverák and Vicol [7] and Krylov [10], rather than relying on abstract or explicit estimates of the solution kernel, we analyze the energy of the solution by a combination of PDE techniques and stochastic analysis. Indeed, our work can be viewed as a stochastic version of De Giorgi–Nash–Moser theory. As such our flexible method is potentially applicable to other types of nonlinear SPDEs.

The paper is organized as follows. In Section 2, we present a stochastic modification of De Giorgi's iteration. In Section 3, we prove the decay of the tail probability of the solution. The main results, Theorem 1.1 and Theorem 1.2 stated above are proved in the last section.

2. Stochastic De Giorgi iteration.

De Giorgi's iteration is a classical method for studying elliptic and parabolic equations with measurable coefficients. In this section, we develop a stochastic extension of this method appropriate for the type of SPDEs under investigation. See Cafarelli and Vasseur [2, 3] for an exposition of the classical theory without random perturbation.

Throughout the paper, an L^p-norm without specifying a domain is implicitly assumed to be taken on $\mathbb{R}^n$; thus $\|K\|_p = \|K\|_{L^p(\mathbb{R}^n)}$. For a time interval $I \subset \mathbb{R}^+$, we define the norm

$$\|g\|_{p_1, p_2, I} := \|u\|_{L^{p_1}(I, L^{p_2}(\mathbb{R}^n))} = \left(\int_I \|g\|_{p_2}^{p_1} \, dt \right)^{1/p_1}.$$

The norm most relevant for this paper is $\|\cdot\|_{4,2,I}$.

Let $I_k = [(1 - 2^{-k})T, 2T]$, a sequence of time intervals shrinking from $[0, 2T]$ to $[T, 2T]$. For each $a \geq 1$, write $u_{k,a} = (u - a(1 - 2^{-k}))^+$ and let

$$U_{k,a} := \|u_{k,a}\|_{4,2,I_k}^2 = \sqrt{\int_{I_k} \|u_{k,a}\|_2^4 \, dt}$$

be the energy of u on $I_k \times \mathbb{R}^n$ above level $a(1 - 2^{-k})$.

For simplicity, we will denote $f(t, x, u)$ and $g(t, x, u)$ by $f(u)$ and $g(u)$, respectively. We have the following iterative inequality.

PROPOSITION 2.1.   *Assume that the function $K(x)$ in the linear growth condition (1.2) satisfies $\|K\|_\infty \leq 1$. Then for $n \geq 3$, there exists a constant $C = C(n, \lambda, \Lambda, T)$ such that*

$$(2.1) \qquad U_{k,a} \leq \frac{C^k}{a^{2/(n+1)}} (U_{k-1,a} + X^*_{k-1,a}) U_{k-1,a}^{1/(n+1)},$$

where

$$(2.2) \qquad X^*_{k-1,a} = \sup_{(1-2^{-k})T \leq s \leq t \leq 2T} \int_s^t \langle g_i(u(\tau)), u_{k,a}(\tau) \rangle \, dw_\tau^i.$$

PROOF.   Hölder's inequality with the conjugate exponents $(n+1)/n$ and $n+1$ gives

$$(2.3) \qquad \|u_{k,a}(t)\|_2^2 \leq \|u_{k,a}(t)\|_{2(n+1)/n}^2 \cdot |\{u_{k,a}(t) > 0\}|^{1/(n+1)}.$$

Using Chebyshev's inequality, we have

$$|\{u_{k,a}(t) > 0\}| = |\{u_{k-1,a}(t) > 2^{-k}a\}| \leq \left(\frac{2^k}{a}\right)^2 \|u_{k-1,a}(t)\|_2^2.$$

Squaring (2.3) and integrating with respect to t on I_k, we have

$$U_{k,a}^2 \leq \left(\frac{2^k}{a}\right)^{4/(n+1)} \int_{I_k} \|u_{k,a}(t)\|_{2(n+1)/n}^4 \|u_{k-1,a}(t)\|_2^{4/(n+1)} \, dt.$$

Applying Hölder's inequality again with the same conjugate exponents we obtain

$$(2.4) \qquad \begin{aligned} U_{k,a} &\leq \left(\frac{2^k}{a}\right)^{2/(n+1)} \left(\int_{I_k} \|u_{k,a}(t)\|_{2(n+1)/n}^{4(n+1)/n} \, dt\right)^{n/2(n+1)} \\ &\quad \times \left(\int_{I_k} \|u_{k-1,a}(t)\|_2^4 \, dt\right)^{1/2(n+1)}. \end{aligned}$$

The third factor on the right-hand side can be estimated by $U_{k-1,a}^{1/(n+1)}$. The second factor is exactly $\|u_{k,a}\|_{4(n+1)/n, 2(n+1)/n, I_k}^2$. We claim that

$$(2.5) \qquad \|u_{k,a}\|_{4(n+1)/n, 2(n+1)/n, I_k}^2 \leq \sup_{t \in I_k} \|u_{k,a}(t)\|_2^2 + \int_{I_k} \|u_{k,a}(t)\|_{2n/(n-2)}^2 \, dt.$$

To prove this inequality, we use the $L_t^p L_x^q$ interpolation inequality

$$\|u\|_{r_1, r_2, I} \leq \|u\|_{p_1, p_2, I}^\gamma \|u\|_{q_1, q_2, I}^{1-\gamma}$$

with

$$\frac{1}{r_1} = \frac{\gamma}{p_1} + \frac{1-\gamma}{q_1}, \qquad \frac{1}{r_2} = \frac{\gamma}{p_2} + \frac{1-\gamma}{q_2}.$$

Using this inequality with the parameters

$$r_1 = \frac{4(n+1)}{n}, \qquad r_2 = \frac{2(n+1)}{n}, \qquad p_1 = \infty, \qquad q_1 = p_2 = 2,$$

$$q_2 = \frac{2n}{n-2}, \qquad \gamma = \frac{n+2}{2(n+1)}$$

followed by the elementary inequality

$$ab \le \frac{a^p}{p} + \frac{b^q}{q} \le a^p + b^q$$

with $p = 2(n+1)/(n+2)$ and $q = 2(n+1)/n$ we obtain (2.5) immediately.

Applying the Sobolev inequality on $\mathbb{R}^n$ to the second term on the right-hand side of (2.5) and then substituting the result in (2.4), we obtain

$$(2.6) \qquad U_{k,a} \le C\left(\frac{2^k}{a}\right)^{2/(n+1)} \left[\sup_{t \in I_k} \|u_{k,a}(t)\|_2^2 + \int_{I_k} \|\nabla u_{k,a}(t)\|_2^2 \, dt\right] U_{k-1,a}^{1/(n+1)}.$$

We now come to the key step of the proof, namely using Itô's formula to bound the terms involving the supremum over I_k and the gradient of u. The function $h_r(u) = |(u-r)^+|^2$ is piecewise smooth with continuous derivative and its second derivative has a single point of discontinuity (a jump) at $u = r$. The quadratic variation process of the martingale part of the process $u(t)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$. Thus, formally applying Itô's formula and the SPDE (1.1) to the composition $h_{a_k}(u(t)) = |u_{k,a}(t)|^2$ we have

$$\begin{aligned}
(2.7) \quad d\|u_{k,a}(t)\|_2^2 = {}& -2\langle \nabla u_{k,a}(t), A \nabla u_{k,a}(t)\rangle dt + 2\langle g_i(u), u_{k,a}(t)\rangle dw_t^i \\
& + \left[\int_{\mathbb{R}^n} \{|g(u(t))|^2 + 2u_{k,a}(t) f(u(t))\} 1_{\{u_{k,a}(t)>0\}} \, dx\right] dt.
\end{aligned}$$

The validity of the above application of Itô's formula can be fully justified; see Remark 2.3 below.

We now apply the uniform ellipticity assumption to the first term on the right-hand side of (2.7). For the third term, we observe that if $u_{k,a} > 0$, then the inequalities $1 \le a \le 2^k u_{k-1,a}$ and $0 < u \le u_{k-1,a} + a \le (1+2^k)u_{k-1,a}$ hold. By the linear growth condition (2) on f and g, the fact $u_{k,a} \le u_{k-1,a}$ and the assumption $\|K\|_\infty \le 1$, this term is bounded by $C^k \|u_{k-1,a}\|_2^2 \, dt$ for some C. Now, integrating (2.7) from t_0 to t with $t_0 \in I_{k-1} \setminus I_k$ and $t \in I_k$ gives

$$\|u_{k,a}(t)\|_2^2 + \lambda \int_{t_0}^t \|\nabla u_{k,a}(s)\|_2^2 \, ds$$

$$\le \|u_{k,a}(t_0)\|_2^2 + C^k U_{k-1,a} + \int_{t_0}^t \langle g_i(u(s)), u_{k,a}(s)\rangle dw_s^i.$$

Taking the supremum over $t \in I_k$, we have for some constant C depending only on n, λ and Λ,

$$
\text{(2.8)} \qquad
\begin{aligned}
\sup_{t \in I_k} &\|u_{k,a}(t)\|_2^2 + \int_{t_0}^2 \|\nabla u_{k,a}(s)\|_2^2 \, ds \\
&\leq C \|u_{k,a}(t_0)\|_2^2 + C^k U_{k-1,a} + C X_{k-1,a}^*
\end{aligned}
$$

with $X_{k-1,a}^*$ as defined in (2.2). Noting the fact that $u_{k,a} \leq u_{k-1,a}$, we can find a $t_0 \in I_{k-1} \setminus I_k$ by the mean value theorem such that

$$
\text{(2.9)} \qquad
\|u_{k,a}(t_0)\|_2^2 = \frac{1}{|I_{k-1} \setminus I_k|} \int_{I_{k-1} \setminus I_k} \|u_{k,a}(t)\|_2^2 \, dt \leq 2^k T^{-1} U_{k-1,a}.
$$

Combining (2.6), (2.8) and (2.9), we obtain the desired iterative inequality (2.1). □

REMARK 2.2. In the cases $n = 1$ or 2, the proof in this section shows that for any $\mu \in (0, 1/3)$, there is a constant $C = C(n, \lambda, \Lambda, T, \mu)$ such that

$$
U_{k,a} \leq \frac{C^k}{a^{2\mu}} \big(U_{k-1,a} + X_{k-1,a}^*\big) U_{k-1,a}^\mu.
$$

This is sufficient for estimating the tail probability of $\|u\|_\infty$ in the next section, for all we need is that the factor $U_{k-1,a}$ carries an exponent strictly greater than 0.

REMARK 2.3. For the justification of the Itô expansion in (2.7), we use a sequence φ_ε of smooth approximations of the function $h_r(u) = |(u-r)^+|^2$. In the definition (1.3) of a solution, we use an approximation of the identity ζ_δ as the test function. The desired expansion is obtained by letting $\delta \to 0$ and then $\varepsilon \to 0$. The details of these passing to the limit are very similar to those in Krylov [10].

3. Estimate of the tail probability. In the context of the stochastic De Giorgi iteration, controlling the size of $\|u^+\|_{\infty,[T,2T] \times \mathbb{R}^n}$ means estimating the decay of the tail probability $\mathbb{P}\{\|u\|_{\infty,[T,2T] \times \mathbb{R}^n} \geq a\}$. In order to use the iterative inequality in Propostion 2.1 for this purpose, we need to show that $X_{k-1,a}^*$ is comparable with $U_{k-1,a}$. This is accomplished in Lemma 3.2 below, whose proof depends on the following simple result from stochastic analysis (see Norris [12], page 123).

LEMMA 3.1. *Suppose that $\{M_t\}$ is a continuous local martingale. Then we have*

$$
\mathbb{P}\Big\{ \sup_{0 \leq s \leq t \leq S} (M_t - M_s) \geq a, \langle M \rangle_S \leq b \Big\} \leq 2 e^{-a^2/4b}.
$$

PROOF. According to the Dambis, Dubins–Schwarz theorem (see Revuz and Yor [15], Chapter V, Section 1, Theorem 1.6), there is a Brownian motion B such that $M_t - M_0 = B_{\langle M \rangle_t}$, hence the event in the statement implies the event $\{\sup_{0 \le t \le b} B_t \ge a/2\}$ or $\{\inf_{0 \le t \le b} B_t \le -a/2\}$. Since $\sup_{0 \le t \le b} B_t$ has the same distribution as $\sqrt{b}|B_1|$ by the reflection principle, we obtain the inequality from the explicit density function of a standard Gaussian random variable. □

Consider the continuous martingale

$$X_t := \int_0^t \langle g_i(u(s)), u_{k+1,a}(s) \rangle \, dw_s^i$$

and recall from (2.2) that $X_{k,a}^* = \sup_{(1-2^{-k-1})T \le s \le t \le 2T}(X_t - X_s)$.

LEMMA 3.2. *Assume that $\|K\|_\infty \le 1$. There exists a constant $C = C(n, \lambda, \Lambda)$ such that for all positive α and β,*

$$\mathbb{P}\{X_{k,a}^* \ge \alpha\beta, U_{k,a} \le \beta\} \le C e^{-\alpha^2/C^k}.$$

PROOF. Let $T_k = (1 - 2^{-k-1})T$ for simplicity. If we can show that there is a constant C such that

(3.1) $$\langle X \rangle_{2T} - \langle X \rangle_{T_k} \le C^k U_{k,a}^2,$$

then

$$\{X_{k,a}^* \ge \alpha\beta, U_{k,a} \le \beta\} \subset \left\{ \sup_{T_k \le s \le t \le 2T}(X_t - X_s) \ge \alpha\beta, \langle X \rangle_{2T} - \langle X \rangle_{T_k} \le C^k \beta^2 \right\}$$

and the desired estimate follows immediately from Lemma 3.1. To prove (3.1), we start with

$$\langle X \rangle_{2T} - \langle X \rangle_{T_k} = \sum_{i \in \mathbb{N}} \int_{I_k} \langle g_i(u), u_{k+1,a} \rangle^2 \, ds,$$

which follows from the definition of X_t. We observe that if $u_{k+1,a} > 0$, then the inequalities $1 \le a \le 2^{k+1} u_{k,a}$ and $0 < u \le u_{k,a} + a \le (1 + 2^{k+1})u_{k,a}$ hold. By Minkowski's inequality (integral form), the linear growth condition (2) on f and g and the fact $u_{k+1,a} \le u_{k,a}$ we have

$$\sum_{i \in \mathbb{N}} \left(\int_{\mathbb{R}^n} g_i(u)u_{k+1,a} \, dx \right)^2 \le \left(\int_{\mathbb{R}^n} |g(u)|u_{k+1,a} \, dx \right)^2 \le C^k \left(\int_{\mathbb{R}^n} u_{k,a}^2 \, dx \right)^2.$$

Integrating over the interval I_k, we obtain the desired inequality (3.1). □

Armed with the iterative inequality (2.1) and the comparison result Lemma 3.2, we are in a position to control the size of $\|u^+\|_{\infty, [T, 2T] \times \mathbb{R}^n}$ by estimating its tail probability. Without loss of generality, we will only work with the case $T = 1$. It is important that the constant M_0 in the following proposition is independent of a.

PROPOSITION 3.3. *Assume that $\|K\|_\infty \leq 1$. There exists a constant $M_0 = M_0(n, \lambda, \Lambda)$ such that for all $a \geq 1$ and $M > M_0$,*

$$\mathbb{P}\{\|u^+\|_{\infty,[1,2]\times\mathbb{R}^n} > a, M\|u^+\|_{4,2,[0,2]} \leq a\} \leq e^{-M^\delta}.$$

Here, $\delta = 1/(n+1)$ when $n \geq 3$ and δ can be any value from $(0, 1/3)$ when $n = 1$ or 2.

PROOF. As in the classical theory, we start with the observation that $\{\|u^+\|_{\infty,[1,2]\times\mathbb{R}^n} > a\} \subset G_a^c$, where $G_a = \{\lim_{k\to\infty} U_{k,a} = 0\}$. Consider the events $\mathcal{E}_k = \{U_{k,a} \leq (a/M)^2 \gamma^k\}$ for a constant $\gamma < 1$ to be determined later. Since $\|u\|_{4,2,[0,2]} = \sqrt{U_{0,a}}$, it suffices to prove

$$\mathbb{P}\{G_a^c \cap \mathcal{E}_0\} \leq e^{-M^\delta}.$$

It is clear that

$$G_a^c \subset \bigcup_{k\geq 0} \mathcal{E}_k^c \subset \mathcal{E}_0^c \cup \left[\bigcup_{k\geq 1}(\mathcal{E}_k^c \cap \mathcal{E}_{k-1})\right],$$

which implies

(3.2) $$\mathbb{P}\{G_a^c \cap \mathcal{E}_0\} \leq \sum_{k\geq 1}\mathbb{P}\{\mathcal{E}_k^c \cap \mathcal{E}_{k-1}\}.$$

We estimate the probability $\mathbb{P}\{\mathcal{E}_k^c \cap \mathcal{E}_{k-1}\}$. We take $\alpha = (2C)^{k/2}M^\delta$ with the C from Lemma 3.2, and apply the lemma with this α and $\beta = a^2\gamma^{k-1}/M^2$. If $X_{k-1,a}^* \leq \alpha\beta$ and $U_{k-1,a} \leq \beta$, then by the iterative inequality (2.1) in Proposition 2.1 we have (after canceling $a^{2\delta}$!)

$$U_{k,a} \leq \frac{C_1^k}{a^{2\delta}}(\beta + \alpha\beta)\beta^\delta = \frac{(C_1\gamma^\delta)^k(1 + (2C)^{k/2}M^\delta)}{\gamma^{1+\delta}M^{2\delta}} \cdot \gamma\beta \leq \gamma\beta.$$

The last inequality holds if we choose γ sufficiently small such that $(C_1\gamma^\delta)^k(1 + (2C)^{k/2}M^\delta) \leq M^\delta$ for all $k \geq 1$ and $M \geq 1$ and then M sufficiently large such that $\gamma^{1+\delta}M^\delta \geq 1$.

Now the above inequality implies that $\mathcal{E}_k^c \cap \mathcal{E}_{k-1} \subset \{X_{k-1,a}^* > \alpha\beta, U_{k-1,a} \leq \beta\}$. Its probability is estimated by Lemma 3.2 and we have

$$\mathbb{P}\{\mathcal{E}_k^c \cap \mathcal{E}_{k-1}\} \leq Ce^{-\alpha^2/C^k} = Ce^{-2^k M^{2\delta}}.$$

Using this in (3.2) we obtain, again for sufficiently large M,

$$\mathbb{P}\{G_a^c \cap \mathcal{E}_0^c\} \leq C\sum_{k=1}^\infty e^{-2^k M^{2\delta}} \leq e^{-M^\delta}.$$

This completes the proof of Proposition 3.3. $\square$

4. Moment estimate and Hölder continuity. In this section, we first prove our main result, namely the moment estimate of the solution of the SPDE (1.1) subject to the conditions stated in Section 1. Then we will prove the almost surely Hölder continuity of the solution. We restate the moment estimate here.

THEOREM 4.1. *Let u be a (stochastically strong) solution of the SPDE (1.1) with (nonrandom) initial data $u(0) = u_0$. Then for every $p > 0$ there is a constant $C = C(n, \lambda, \Lambda, T, p)$ such that*

$$\mathbb{E} \int_0^{2T} \|u(t)\|_2^p \, dt + \mathbb{E}\|u\|_{\infty,[T,2T]\times\mathbb{R}^n}^p \leq C\big(\|u_0\|_2 + \|K\|_2 + \|K\|_\infty\big)^p.$$

PROOF. By scaling it suffices to consider the case $T = 1$, $\|K\|_2 + \|K\|_\infty \leq 1$, and $\|u_0\|_2 \leq 1$. We need to show that there exists a constant C (depending on p of course) such that

(4.1) $$\mathbb{E} \int_0^2 \|u(t)\|_2^p \, dt \leq C \quad \text{and} \quad \mathbb{E}\|u\|_{\infty,[1,2]\times\mathbb{R}^n}^p \leq C.$$

As $\mathbb{P}$ is a probability measure, we may assume $p \geq 4$. We start with the first inequality. Let $\varphi(t) = \|u(t)\|_2^2 + 1$. By Itô's formula,

(4.2) $$d\varphi(t) = \varphi(t)\big(F(t)\, dt + dG_t\big),$$

where

$$F(t) = \frac{-\langle A\nabla u, \nabla u\rangle + \langle f(u), u\rangle + \|g(u)\|_2^2}{\|u\|_2^2 + 1} \quad \text{and} \quad G_t = \int_0^t \frac{\langle g_i(u), u\rangle}{\|u\|_2^2 + 1}\, dw_s^i.$$

The solution of SDE (4.2) is explicitly given by

$$\varphi(t) = \varphi(0) \exp\left[\int_0^t F(s)\, ds + G_t - \frac{1}{2}\langle G\rangle_t\right].$$

By the assumptions, we have $\langle G\rangle_t \leq 2(\Lambda + 1)^2$ for all $t \leq 2$, therefore, Novikov's condition ensures that

$$\exp\left[pG_t - \frac{p^2}{2}\langle G\rangle_t\right]$$

is a martingale for any $p > 0$ and $0 \leq t \leq 2$. This plus the fact $F(t) \leq 4(\Lambda + 1)^2$ give

$$\mathbb{E}\varphi^p(t) = \varphi(0)^p \mathbb{E}\left[\exp p\left(\int_0^t F(s)\, ds + G_t - \frac{1}{2}\langle G\rangle_t\right)\right] \leq C\varphi(0)^p.$$

This implies the first inequality in (4.1). Next, we show the second inequality in (4.1). Let

$$X = \|u\|_{\infty,[1,2]\times\mathbb{R}^n} \quad \text{and} \quad Y = \left(\int_0^2 \|u\|_2^4 \, dt\right)^{1/4}.$$

By considering u and $-u$, we have from Proposition 3.3 with δ defined there,

$$(4.3) \qquad \mathbb{P}\left\{ X > a, Y \leq \frac{a}{M} \right\} \leq 2e^{-M^{\delta}}$$

for all $a \geq 1$ and $M \geq M_0$, hence

$$\mathbb{P}\{X > a, Y \leq \sqrt{a}\} \leq 2e^{-a^{\delta/2}}$$

for $a \geq M_0^2$, assuming that $M_0 \geq 1$. By the first inequality in (4.1), we have

$$\mathbb{E} Y^{2p} \leq 2^{(p-2)/2} \mathbb{E} \int_0^2 \|u(t)\|_2^{2p}\, dt \leq C.$$

Hence,

$$\mathbb{E} \|u\|_{\infty,[1,2]\times\mathbb{R}^n}^p = p \int_0^\infty \mathbb{P}(X > a) a^{p-1}\, da$$

$$\leq M_0^{2p} + p \int_{M_0^2}^\infty \mathbb{P}(Y > \sqrt{a}) a^{p-1}\, da$$

$$+ p \int_{M_0^2}^\infty \mathbb{P}\{X > a, Y \leq \sqrt{a}\} a^{p-1}\, da.$$

The second term is bounded by $\mathbb{E} Y^{2p}$, and the third term is finite by (4.3). This proves the second inequality in (4.1). $\square$

We can now prove the almost sure Hölder continuity result, which we state again for easy reference.

THEOREM 4.2. *Let u be a solution of the SPDE*

$$\partial_t u = \operatorname{div}(A\nabla u) + f(t, x, u) + g_i(t, x, u)\dot{w}_t^i$$

whose coefficients satisfy the conditions stated in Section 1. *Then there exists a positive exponent $\alpha = \alpha(n, \lambda, \Lambda)$ such that almost surely $u \in C^\alpha([T, 2T] \times \mathbb{R}^n)$ for all $T > 0$. Furthermore, for every $p > 0$, there is a constant $C = C(n, \lambda, \Lambda, T, p)$ such that*

$$\mathbb{E} \|u\|_{C^\alpha([T,2T]\times\mathbb{R}^n)}^p \leq C \big(\|u_0\|_{L^2(\mathbb{R}^n)} + \|K\|_{L^2(\mathbb{R}^n)} + \|K\|_{L^\infty(\mathbb{R}^n)} \big)^p.$$

PROOF. By scaling it suffices to assume $T = 1$, $\|u_0\|_2 \leq 1$ and $\|K\|_\infty + \|K\|_2 \leq 1$. Following a suggestion of Professor Nicolai Krylov and the approaches used in Debussche, De Moor and Hofmanova [6], we consider the solution v of an SPDE with the same stochastic perturbation but simpler diffusion coefficients:

$$dv = \Delta v\, dt + g_i(u)\, dw_t^i, \qquad v(2^{-1}) = 0.$$

The function $\phi = u - v$ satisfies

$$(4.4) \quad \partial_t \phi = \operatorname{div}(A\nabla\phi) + f(\phi + v) + \operatorname{div}(A\nabla v) - \Delta v \qquad \text{on } [2^{-1}, 2] \times \mathbb{R}^n.$$

From the linear growth assumption (1) for g and Proposition 4.1, we have

$$\mathbb{E} \int_{2^{-1}}^{2} \|g(u)\|_p^p \, dt \leq C.$$

According to Krylov's $W^{1,p}$-theory (see Krylov [8]) $v \in C^{\alpha_1}([2^{-1}, 2] \times \mathbb{R}^n)$ for some exponent α_1. Furthermore, we have the estimates

$$(4.5) \qquad \mathbb{E}\|v\|_{C^{\alpha_1}([2^{-1},2]\times\mathbb{R}^n)}^p \leq \mathbb{E}\|g(u)\|_{L^p([2^{-1},2]\times\mathbb{R}^n)} \leq C$$

and

$$(4.6) \qquad \mathbb{E} \int_{1/2}^{2} \|D^2 v\|_{W^{-1,p}}^p \, dt \leq C_p.$$

Since (4.4) does not have a stochastic perturbation, the usual regularity theory (see Lieberman [11], Section VI.13, pages 143–149) applies and we have $\phi \in C^{\alpha_2}([1, 2] \times \mathbb{R}^n)$ for some small exponent $\alpha_2 \in (0, 1)$ and

$$\|\phi\|_{C^{\alpha_2}([1,2]\times\mathbb{R}^n)}$$
$$\leq C\big(\|\phi\|_{\infty,[2^{-1},2]\times\mathbb{R}^n} + \|D^2 v\|_{L^p([1,2],W^{-1,p})}\big)$$
$$\leq C\big(\|u\|_{\infty,[2^{-1},2]\times\mathbb{R}^n} + \|v\|_{\infty,[2^{-1},2]\times\mathbb{R}^n} + \|D^2 v\|_{L^p([2^{-1},2],W^{-1,p})}\big).$$

Using the estimates (4.5) and (4.6), we conclude that $\mathbb{E}\|\phi\|_{C^{\alpha_2}([1,2]\times\mathbb{R}^n)}^p \leq C$. From this inequality, (4.5) and $u = \phi + v$, we obtain the desired inequality $\mathbb{E}\|u\|_{C^{\alpha}[1,2]\times\mathbb{R}^n}^p \leq C$ with $\alpha = \min\{\alpha_1, \alpha_2\}$. $\quad\square$

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