

THE SCALING LIMIT OF RANDOM SIMPLE TRIANGULATIONS AND RANDOM SIMPLE QUADRANGULATIONS

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Let M_n be a simple triangulation of the sphere \mathbb{S}^2 , drawn uniformly at random from all such triangulations with n vertices. Endow M_n with the uniform probability measure on its vertices. After rescaling graph distance by $(3/(4n))^{1/4}$, the resulting random measured metric space converges in distribution, in the Gromov–Hausdorff–Prokhorov sense, to the Brownian map. In proving the preceding fact, we introduce a labelling function for the vertices of M_n . Under this labelling, distances to a distinguished point are essentially given by vertex labels, with an error given by the winding number of an associated closed loop in the map. We establish similar results for simple quadrangulations.

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1. Introduction. We begin by heading straight for a statement of our main result.³ A graph is *simple* if it has no loops or multiple edges. For integer $n \geq 3$, let Δ_n° be the set of pairs (M, ξ) , where M is an n -vertex simple triangulation of the sphere \mathbb{S}^2 , and ξ is a corner of M . Also, for integer $n \geq 4$, let \square_n° be the set of pairs (M, ξ) with M an n -vertex simple quadrangulation of \mathbb{S}^2 and ξ a corner of M . Then let $\mathcal{M} = (\mathcal{M}_n, n \geq 4)$ be one of the sequences $(\Delta_n^\circ, n \geq 4)$ or $(\square_n^\circ, n \geq 4)$.

THEOREM 1.1. *For $n \geq 4$, let (M_n, ξ_n) be a uniformly random element of \mathcal{M}_n . Write $V(M_n)$ for the set of vertices of M_n , let $d_n : V(M_n) \rightarrow \mathbb{N}$ be graph distance in M_n and let μ_n be the uniform probability measure on $V(M_n)$. Finally, let $c = (3/4)^{1/4}$ if $\mathcal{M} = (\Delta_n^\circ, n \geq 4)$ and let $c = (3/8)^{1/4}$ if $\mathcal{M} = (\square_n^\circ, n \geq 4)$. Then, as $n \rightarrow \infty$,*

$$(V(M_n), cn^{-1/4}d_n, \mu_n) \xrightarrow{d} (S, d, \mu),$$

for the Gromov–Hausdorff–Prokhorov distance, where (S, d, μ) is the Brownian map.

We recall the definition of the Brownian map in Section 1.1, below. Our proof relies upon the remarkable work of Miermont [29] and, independently, Le Gall

³Precise definitions of almost all the terminology used in the Introduction appear in Sections 2 and 3. After stating our main result, the remainder of the Introduction provides motivation and an overview of its proof, particularly the novel aspects of said proof.

[22], which both established convergence for general (nonsimple) random quadrangulations. In particular, our results do not constitute an independent proof of uniqueness of the limit object. A discussion of the constants in the above theorem, and their relation with those from [22, 29], appears in Appendix A.

The part of Theorem 1.1 pertaining to simple triangulations (sometimes called *type-III* triangulations; see [3]) answers a question of Le Gall [22] and Le Gall and Beltran [5]. One general motivation for establishing convergence to the Brownian map is its conjectured role as a universal limit object for a wide range of random map ensembles. However, the case of simple triangulations holds additional interest due to the conjectured link between the Brownian map and the Liouville quantum gravity constructed by Duplantier and Sheffield [12]; see [16] for further discussion of this connection. Le Gall [20] proved that the Brownian map is almost surely homeomorphic to the 2-sphere (see also [23, 27]). However, homeomorphism equivalence is too weak, for example, to deduce conformal information or to prove dimensional scaling relations. For these, a canonical embedding of the Brownian map in S^2 is needed (or at least would be very useful).

For any simple triangulation M of S^2 , the Koebe–Andreev–Thurston theorem (see, e.g., [33], Chapter 7) provides a *canonical circle packing* in S^2 , unique up to conformal automorphism, whose tangency graph is M ; see Figure 1 for an illustration of a random circle packing. [This uniqueness holds only for simple triangulations; for a uniformly random (nonsimple) triangulation N with n vertices, for example, the number of degrees of freedom in a circle packing with tangency graph N is typically linear in n .] The uniqueness provides hope that the conformal properties of the Brownian map can be accessed by studying the circle packings associated to large random simple triangulations.

We deduce Theorem 1.1 from a result which provides more general sufficient conditions for a sequence $(M_n, n \in \mathbb{N})$ of random planar maps to converge in distribution to the Brownian map. More precisely, Theorem 4.1 states conditions under which, after suitably rescaling distances, and endowed with the uniform probability measure on its vertex set, M_n converges in distribution to the Brownian map for the Gromov–Hausdorff–Prokhorov distance.

The approach of Theorem 4.1 is based on bijective codings of maps by labelled plane trees. Its proof is a fairly routine generalization of existing arguments (mostly due to Jean-François Le Gall). We have formulated Theorem 4.1 in a general form as we expect it to be useful in proving convergence for other random map models, in particular, for models falling within the framework of the “master bijection” of Bernardi and Fusy [6] and of the general bijection for blossoming trees recently described by Albenque and Poulalhon [1]. We sketch the conditions under which Theorem 4.1 applies in Section 1.2.

While the conditions under which we establish convergence to the Brownian map are rather general, *verifying* that a discrete random map ensemble satisfies these conditions can be rather involved. In many map ensembles of interest, the primary missing link is a labelling rule for the vertices of a canonical spanning

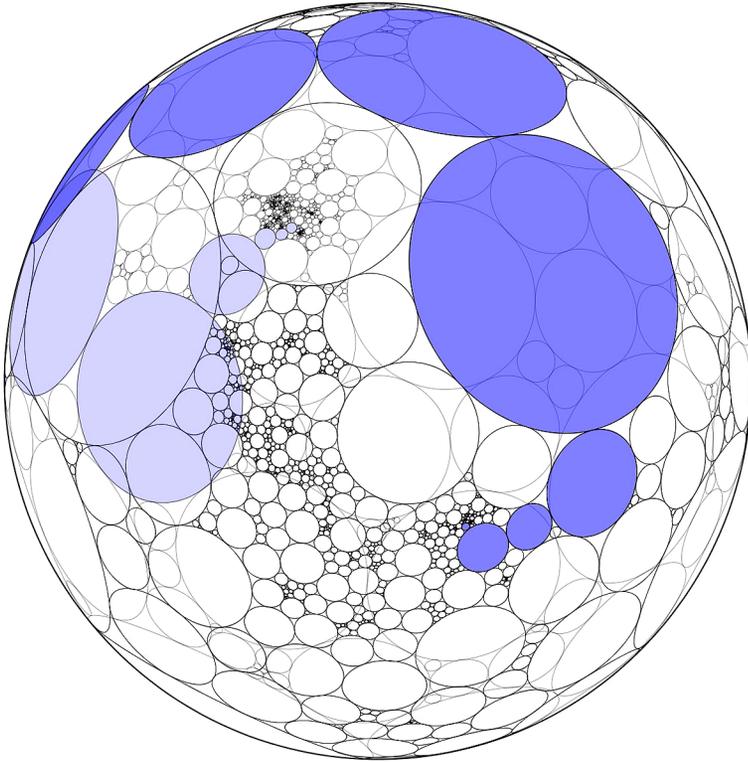


FIG. 1. The circle packing associated to a uniformly random simple triangulation of \mathbb{S}^2 with 10^5 vertices. Blue shaded circles form a shortest path between two uniformly random vertices (circles). Created using Ken Stephenson's CirclePack program.

tree of the map, such that vertex labels encode distances to a specified root vertex. For the case of random simple triangulations and quadrangulations, we provide a labelling that does not *precisely* encode distances, but we show that the error is insignificant in the limit. Intriguingly, for distances to a specified root vertex, the error in the label is bounded by the *winding number* of an associated closed loop in the map. In Section 1.3, we briefly describe the bijection between simple triangulations and certain labelled trees, on which our proof of Theorem 1.1 is based, and further discuss the role of winding numbers. The appearance of a winding number hints that a discrete complex-analytic perspective may shed further light on the shape of geodesics in random simple triangulations and eventually in the Brownian map.

One requirement of Theorem 4.1 is the convergence of a suitable spatial branching process, after renormalization, to the Brownian snake. Such convergence is known in many settings, but in others lack of symmetry (symmetry between the labels of children of a single node, in the coding of maps by labelled trees) has posed an obstacle. We introduce a technique we call *partial symmetrization*, in

which we choose a “representative subtree”, then randomly permute the children of as many nodes of the subtree as possible without affecting the subtree’s plane embedding. This introduces enough symmetry that we may appeal to known results to establish convergence to the Brownian snake. On the other hand, fixing a large subtree allows the partially symmetrized process to be related to the original labelled tree and so to the associated map. A detailed explanation of the partial symmetrization technique is easier to provide for a specific bijection, and we defer it to Section 6.

We believe partial symmetrization may be used to show that the multi-type spatial branching processes coding random p -angulations (for odd $p \geq 5$) converge to the Brownian snake. Given the work of Miermont [29] and of Le Gall [22], this is the only missing element in a proof that p -angulations (and perhaps more general random maps with degrees given by suitable Boltzmann weights) converge to the Brownian map. We expect to return to this in a subsequent work.

1.1. *The Brownian map.* Given an interval $I \subset \mathbb{R}$ or $I \subset \mathbb{N}$ and a function $f : I \rightarrow \mathbb{R}$, for $s, t \in I$ with $s < t$ we write $\check{f}(s, t) = \inf_{x \in I \cap [s, t]} f(x)$, $\check{f}(t, s) = \inf_{x \in I \setminus (s, t)} f(x)$. We additionally let $\check{f}(s, s) = f(s)$ for all $s \in I$.

Let $\mathbf{e} = (\mathbf{e}(t), 0 \leq t \leq 1)$ be a standard Brownian excursion and, conditionally given \mathbf{e} , let $Z = (Z(t), 0 \leq t \leq 1)$ be a centred Gaussian process such that $Z(0) = 0$ and for $0 \leq s \leq t \leq 1$,

$$\text{Cov}(Z(s), Z(t)) = \check{\mathbf{e}}(s, t).$$

We may and shall assume Z is a.s. continuous; see [18], Section IV, for a more detailed description of the construction of the pair (\mathbf{e}, Z) .

Next, define an equivalence relation $\sim_{\mathbf{e}}$ as follows. For $0 \leq x \leq y \leq 1$, let $x \sim_{\mathbf{e}} y$ if $\mathbf{e}(x) = \mathbf{e}(y) = \check{\mathbf{e}}(x, y)$. The Brownian continuum random tree $(\mathcal{T}_{\mathbf{e}}, d_{\mathcal{T}_{\mathbf{e}}})$ introduced in [2] is defined as $[0, 1] / \sim_{\mathbf{e}}$ equipped with distance $d_{\mathcal{T}_{\mathbf{e}}}(x, y) = \mathbf{e}(x) + \mathbf{e}(y) - \check{\mathbf{e}}(x, y)$ for $0 \leq x \leq y \leq 1$.

It can be verified that almost surely, for all $x, y \in [0, 1]$, if $x \sim_{\mathbf{e}} y$ then $Z(x) = Z(y)$, so we may view Z as having domain $\mathcal{T}_{\mathbf{e}}$. Furthermore, Z remains a.s. continuous on this domain. Next, for $x, y \in [0, 1]$ let

$$(1) \quad d_Z(x, y) = Z(x) + Z(y) - 2 \max(\check{Z}(x, y), \check{Z}(y, x)).$$

Then let d^* be the largest pseudo-metric on $[0, 1]$ satisfying that (a) for all $s, t \in [0, 1]$, if $s \sim_{\mathbf{e}} t$ then $d_Z(s, t) = 0$, and (b) $d^* \leq d_Z$. Let $S = [0, 1] / \{d^* = 0\}$, and let d be the push-forward of d^* to S . Finally, let μ be the push-forward of Lebesgue measure on $[0, 1]$ to S . The (measured) *Brownian map* is (a random variable with the law of) the triple (S, d, μ) . This name was first used by Marckert and Mokkadem [26], who considered a notion of convergence for random maps different from that of the present work.

For later use, let $\rho \in S$ be the equivalence class of the point 0, and, writing $s^* \in [0, 1]$ for the point where Z attains its minimum value (this point is almost

surely unique), let $u^* \in S$ be the equivalence class of s^* . Then Corollary 7.3 of [22] states that for U and V uniformly distributed on $[0, 1]$, independent of Z and of each other,

$$(2) \quad d^*(U, V) \stackrel{d}{=} d^*(U, s^*) \stackrel{d}{=} -\check{Z}(0, 1) \stackrel{d}{=} Z(V) - \check{Z}(0, 1).$$

1.2. *Sufficient conditions for convergence to the Brownian map.* Our argument leans heavily on the *rerooting invariance* of the Brownian map [(2), above]. Given the convergence of some discrete ensemble to the Brownian map, if the discrete ensemble possesses rerooting invariance then this can be transferred to the Brownian map. However, to date this is the only known technique for establishing rerooting invariance of the Brownian map (and the key reason why our results depend on those of [22, 29]).

Informally, to prove convergence we need that the random rooted map M_n can in some sense be described by a suitable pair of random functions $C_n : [0, 1] \rightarrow [0, \infty)$ and $Z_n : [0, 1] \rightarrow \mathbb{R}$. Often C_n will be the (spatially and temporally rescaled, clockwise) contour process of some canonical rooted spanning tree (T_n, ξ_n) of M_n , and for the sake of this informal description we assume this to be so. To establish convergence, we require (versions of) the following. In what follows, let $r_n \in [0, 1]$ be such that $Z_n(r_n) = \min(Z_n(x), 0 \leq x \leq 1)$, and write d_{M_n} for (suitably rescaled) graph distance on $V(M_n)$:

1. *Distances to the minimum given by Z_n .* There is a vertex $u_n \in V(M_n)$ such that for all vertices v , if a clockwise contour exploration of T_n visits v at time t then $Z_n(t) - Z_n(r_n)$ is $d_{M_n}(v, u_n) + o_n(1)$, where $o_n(1)$ represents an error that tends to zero in probability as $n \rightarrow \infty$.

2. *Distance bound via clockwise geodesics to the minimum.* For any pair of vertices v, v' of M_n , if a clockwise contour exploration of T_n visits v and v' at times t and t' , respectively, then $d_{M_n}(v, v')$ is bounded from above by

$$Z_n(t) + Z_n(t') - 2 \max(\check{Z}_n(t, t'), \check{Z}_n(t', t)) + o_n(1).$$

3. *Coding by the Brownian snake.* The pair (C_n, Z_n) converges in distribution to (\mathbf{e}, Z) , for the topology of uniform convergence on $C([0, 1], \mathbb{R})^2$.

4. *Invariance under rerooting.* If U_n, V_n are independent, uniformly random vertices of M_n , then $d_{M_n}(U_n, V_n)$ is asymptotically equal in distribution to $d_{M_n}(u_n, V_n)$.

Briefly, given these properties the proof then proceeds as follows. Our argument closely follows one used by Le Gall to prove convergence of rescaled random (nonsimple) triangulations to the Brownian map, once convergence for quadrangulations is known ([22], Section 8). It is useful to reparameterize so that all the metrics and pseudo-metrics under consideration are functions from $[0, 1]^2$ to $[0, \infty)$; this can be accomplished by identifying the vertices of each metric space M_n with a subset of $[0, 1]$ and using bilinear interpolation.

First, 1 and 2 together can be used to prove tightness of the sequence of laws of the functions $(d_{M_n}, n \in \mathbb{N})$, which implies convergence along subsequences. Thus, let $d : [0, 1]^2 \rightarrow [0, \infty)$ be a subsequential limit of d_{M_n} . Our aim is to show that almost surely d and d^* (defined in Section 1.1) are equal in law.

Next, 1 says that distances to the point of minimum label are given by Z_m , a limiting analogue of which is also true in the Brownian map. Invariance under rerooting 4 and (2) then yields that for U, V independent and uniform on $[0, 1]$, $d(U, V)$ is the limit in distribution of $-Z_n(r_n)$, so by 3 we obtain $d(U, V) \stackrel{d}{=} -\min(Z(x), 0 \leq x \leq 1) = d^*(U, V)$.

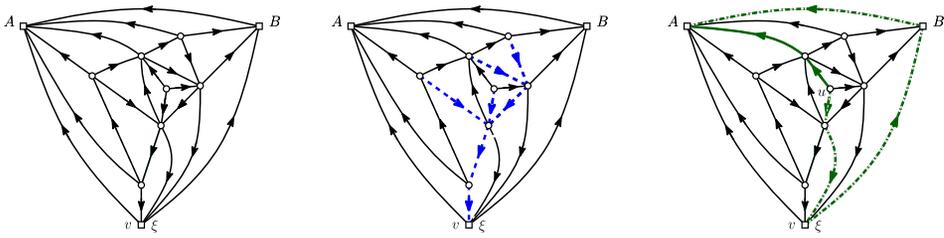
Finally, 2 gives a bound for d_{M_n} that is a finite- n analogue of the bound (1) for d_Z . Since d^* is maximal subject to $d^* \leq d_Z$, 3 then yields that d is stochastically dominated by d^* . In other words, by working in a suitable probability space, we may assume $d(x, y) \leq d^*(x, y)$ for almost every $(x, y) \in [0, 1]^2$. The fact that $d(U, V) \stackrel{d}{=} d^*(U, V)$ then implies d and d^* are almost everywhere equal, so have the same law.

1.3. *Labels and geodesics, and an overview of the proof.* In this section (and throughout much of the rest of the paper), we restrict our attention to simple triangulations, as the details for simple quadrangulations are nearly identical.

Fix a pair (G, ξ) with G a simple triangulation of \mathbb{S}^2 and ξ a corner of G . View G as embedded in \mathbb{R}^2 so the face containing ξ is the unique unbounded (outer) face. With this embedding, list the vertices of the face containing ξ in clockwise order as v, A, B , with v incident to ξ . A 3-orientation of (G, ξ) is an orientation \vec{E} of $E(G)$ such that in \vec{E} , A, B and v have outdegrees 0, 1 and 2, respectively, and all other vertices have outdegree three.⁴ Schnyder [32] showed (G, ξ) admits a 3-orientation if and only if G is simple, and in this case admits a *unique* 3-orientation containing no counterclockwise cycles (we say an oriented cycle is *clockwise* if ξ is on its left, and otherwise say it is counterclockwise); this 3-orientation is called *minimal*. Let \vec{E} be the minimal 3-orientation of (G, ξ) .

The definitions of the following paragraph are illustrated in Figure 2. A subtree of G containing the vertex v incident to ξ is *oriented* if all edges of the subtree are oriented towards v in \vec{E} . It turns out there is a unique oriented subtree T of G on vertices $V(G) \setminus \{A, B\}$ which is *minimal* in the sense that for all edges $uw \in \vec{E}$ with $\{u, w\} \notin E(T)$, if uw attaches to u and w in corners c and c' , respectively, then c precedes c' in a clockwise contour exploration of T starting from ξ . We endow this tree T with a labelling $Y : V(T) \rightarrow \mathbb{N}$ as follows. For $e = uw \in \vec{E}$ with $\{u, w\} \in E(G)$, the *leftmost oriented path* from e to A is the unique oriented path (u_0, u_1, \dots, u_k) with the following two properties: (i) $u_0 = u, u_1 = w$; (ii) for $1 \leq i < k$, if $\{u_i, y\} \in E(G)$ and this edge attaches to the path (u_0, \dots, u_k) on the

⁴This is equivalent to, but differs very slightly from, the standard definition.



(a) A simple triangulation endowed with its unique 3-orientation with no counter-clockwise cycles.

(b) The minimal oriented tree is drawn in dashed blue lines.

(c) The thick green paths are both leftmost oriented paths from u to A ; the solid path is $P(u)$, so $Y(u) = 3$.

FIG. 2. Orientations, spanning trees and leftmost paths in simple triangulations.

left, then $yu_i \in \vec{E}$. For each vertex $u \in V(T)$ distinct from v , there are three such paths starting at u (since u has outdegree three in \vec{E}); we let $P(u) = P_{G,\xi}(u)$ be one of the shortest such paths. Then let $Y(u) = |P(u)|$, the number of vertices in $P(u)$.

Surprisingly, (G, ξ) may be recovered from the pair (T, Y) . More strongly, the above transformation is a bijection mapping planted simple planar triangulations to a certain set of “validly labelled” planted plane trees. This bijection is essentially due to Poulalhon and Schaeffer [31], but the connection of vertex labels with the lengths of certain oriented paths is new.

Since $Y(u)$ is the number of vertices on a certain path from u to A , $Y(u) - 1$ is an upper bound on $d_G(u, A)$, the graph distance between u and A in G . It turns out that $Y(u) - d_G(u, A) - 1$ is bounded by twice the number of times a shortest path in G from u to A winds clockwise around the leftmost path $P_{G,\xi}(u)$. More strongly, if $P(u) = (u_0, u_1, \dots, u_k)$ and Q is a path from u_i to u_j disjoint from $P(u)$ except at its endpoints, then $|Q| \geq j - i - 1$, and $|Q| \leq j - i + 1$ (i.e., Q is a shortcut from u_i to u_j) only if Q leaves u_i on the right and rejoins u_j on the left. This fact allows $Y(u) - d_G(u, A) - 1$ to be controlled as follows.

Let $n = |V(G)|$. If Q is a shortcut from u_i to u_j then the union of Q and u_{i+1}, \dots, u_{j-1} forms a cycle C with $2(j - i) - 1$ or $2(j - i) - 2$ vertices. If there are $2k$ shortcuts between u and A and Q is the k th one, then all vertices of C have distance at least k both from A and from u . It will follow that typically (i.e., for random G), when k and $d_G(u_j, A)$ are both large (of order $n^{1/4}$) then $j - i$ should also be large (of order $n^{1/4}$), or else G would contain a cycle of length $o(n^{1/4})$ separating two macroscopic regions. On the other hand, a “shortcut” of length of order $n^{1/4}$ is rather long; we will straightforwardly show that typically the diameter of G will be $O(n^{1/4})$, in which case there can be at most a bounded number of such long shortcuts on any path. A rigorous version of this argument allows us to show that typically, for all $u \in V(T) \setminus \{v\} = V(G) \setminus \{v, A, B\}$, $Y(u) -$

$d_G(u, A) - 1$ is much smaller than $n^{1/4}$. In other words, after rescaling, the labels Y with high probability provide good approximations for distances to the root A . This essentially proves 1 from Section 1.2.

A modification of the above argument establishes without too much difficulty that for $u, w \in V(T)$ with u preceding w in lexicographic order, $d_G(u, w)$ is bounded by $Y(u) + Y(w) - 2\check{Y}(u, w) + 2$, where $\check{Y}(u, w)$ is the smallest value $Y(y)$ for any vertex y following u and preceding w in lexicographic order. This will establish 2. from Section 1.2.

To establish 3, we use “partial symmetrization” as previously discussed. Finally, rerooting invariance, 4 will be a straightforward consequence of choosing a random root corner. Having verified all the conditions of our general convergence result (the proof of which was already sketched), Theorem 1.1 for simple triangulations then follows immediately. An essentially identical development establishes Theorem 1.1 for simple quadrangulations.

1.4. *Outline.* We conclude the [Introduction](#) by fixing some basic notation, in Section 1.5. In Section 2, we provide definitions related to planar maps and plane trees, many of which are standard. In Section 3, we introduce the Gromov–Hausdorff distance and mention some of its basic properties. In Section 4, we formally state our “universality” result, providing general sufficient conditions for a random map ensemble to converge to the Brownian map; proofs are deferred to [Appendix B](#). In Section 5, we describe the bijections for simple triangulations and quadrangulations on which our proof of Theorem 1.1 is based. In Section 6, we prove convergence of the spatial branching process associated to a random simple triangulation to the Brownian snake; this is where partial symmetrization appears. In Section 7, we study the relation of distances with labels; this is where winding numbers appear. In Section 8, we use the bounds of Section 7 to show that our labelling provides a sufficiently close approximation of distances in random simple triangulations that the associated conditions of Theorem 4.1 are satisfied. In Section 9, we establish rerooting invariance and so complete the proof of Theorem 1.1. Finally, Section 10 proves Theorem 1.1 for quadrangulations, and [Appendix A](#) contains a derivation of the numerical constants from Theorem 1.1.

1.5. *Notation.* For the remainder of the paper, all graphs are connected, finite, simple (i.e., without loops nor multiple edges) and planar. Let $G = (V(G), E(G))$ be such a graph. Given a vertex $v \in V(G)$ we write $\deg_G(v) = |\{e \in E(G) : v \in e\}|$ for the degree of v in G . If $v \in e$, we say e is *incident* to v . We write $d_G : V(G) \times V(G) \rightarrow \mathbb{N}$ for graph distance on G . Given $W \subset V(G)$, we write $G[W]$ for the graph with vertices W and edges $\{\{u, v\} \in E(G) : u, v \in W\}$.

An *oriented edge* of G is an ordered pair uw , where $\{u, w\} \in E(G)$; we call uw an *orientation* of $\{u, w\}$. An orientation of G is a set $\vec{E} = \{\vec{e} : e \in E(G)\}$, where for each $e \in E(G)$, \vec{e} is an orientation of e . The *outdegree* of $v \in V(G)$ (with respect to \vec{E}) is $\deg^+(v) = \deg_{\vec{E}}^+(v) = |\{w \in V(G) : vw \in \vec{E}\}|$.

If $S = (s_1, \dots, s_r)$ is any sequence of objects, we say that S has length r and write $|S| = r$. A *path* in G is a sequence $P = (u_0, u_1, \dots, u_k)$ of vertices of G with $\{u_i, u_{i+1}\} \in E(G)$ for $0 \leq i < k$; we say P is a path from u_0 to u_k , and note that $|P| = k + 1$. A path is *simple* if all its vertices are distinct. A *cycle* in G is a path $(u_0, u_1, \dots, u_k, u_{k+1})$ such that $u_{k+1} = u_0$; it is *simple* if (u_0, \dots, u_k) is a simple path. If G is a *tree* (connected and acyclic), then for $u, w \in G$ we write $\llbracket u, v \rrbracket$ for the unique (shortest) path in G from u to v . Finally, for a nonnegative integer k , write $[k] = \{0, 1, \dots, k\}$.

2. Planar maps and plane trees.

2.1. *Planar maps.* A *planar embedding* of G is a function $\phi : V(G) \cup E(G) \rightarrow \mathbb{S}^2$ satisfying the following properties:

- (1) The restriction $\phi|_{V(G)}$ is injective.
- (2) For each $e = uv \in E(G)$, $\phi(e)$ is a simple curve with endpoints $\phi(u)$ and $\phi(v)$.
- (3) For any two distinct edges $e, f \in E(G)$, the curves $\phi(e)$ and $\phi(f)$ are disjoint except possibly at their endpoints.

The pair (G, ϕ) is called a *planar map*. The faces of (G, ϕ) are the connected components of $\mathbb{S}^2 \setminus \bigcup_{x \in V(G) \cup E(G)} \phi(x)$. Given a face f the vertices and edges incident to f are given by the set $\phi^{-1}(\partial f)$, where ∂f is the boundary of f .

Two planar maps are *isomorphic* if there exists an orientation-preserving homeomorphism of \mathbb{S}^2 that sends one to the other. It is easily verified that planar map isomorphism is an equivalence relation.

For any planar map (G, ϕ) , for each vertex $v \in V(G)$ there is a unique cyclic (clockwise) ordering \mathcal{O}_v of the edges incident to v . Furthermore, up to isomorphism, the set of orderings $\{\mathcal{O}_v : v \in V(G)\}$ uniquely determines (G, ϕ) . We may therefore specify the isomorphism equivalence class of (G, ϕ) by providing G and the set of cyclic orderings associated to (G, ϕ) . We will henceforth denote (a representative from the isomorphism equivalence class of) a planar map simply by G , leaving implicit both ϕ and its associated cyclic orderings.

For the remainder of Section 2.1, consider a fixed planar map G . A *corner* of G is an ordered pair $\xi = (e, e')$ where e and e' are incident to a common vertex v , and e' immediately follows e in the clockwise order around v .⁵ We write $v(\xi) = v_G(\xi) = v$ and say that ξ is incident to v (and also to e and e'). We write $\mathcal{C}(G)$ for the set of corners of G . For $\xi, \xi' \in \mathcal{C}(G)$ we let $d_G(\xi, \xi') = d_G(v(\xi), v(\xi'))$ be the graph distance between the vertices incident to ξ and ξ' , and likewise let $d_G(\xi, w) = d_G(v(\xi), w)$ for $w \in V(G)$.

If $e = \{u, v\}$ and $e' = \{v, w\}$, and f is the face on the left when following e and e' from u through v to w , then we say $\xi = (e, e')$ is incident to f and vice

⁵We allow that $e = e'$, which can happen if $d_G(v) = 1$.

versa. The *degree* of f is the number of corners incident to f . The planar map G is a *triangulation* or a *quadrangulation* if all its faces have respectively degree 3 or degree 4.

Given $e = \{u, v\} \in E(G)$, write $\kappa^\ell(u, v) = \kappa_G^\ell(u, v)$ [resp., $\kappa^r(u, v) = \kappa_G^r(u, v)$] for the corner incident to u and to $\{u, v\}$ that is on the left (resp., on the right) when following e from u to v .

A *planted planar map* is a pair (G, ξ) , where G is a planar map and $\xi \in \mathcal{C}(G)$. We call ξ the *root corner* of (G, ξ) , call $v(\xi)$ its *root vertex*, and call the face of G incident to ξ its *root face*. If G' is a connected subgraph of G containing ξ , then (G', ξ) is again a planar map, and we call it a *planted submap* of (G, ξ) .

2.2. *Plane trees.* A plane tree (resp., planted plane tree) is a planar map G [resp., planted planar map (G, ξ)] such that G is a tree.⁶ If $T = (T, \xi)$ is a planted plane tree, then recalling that $v(\xi)$ is the root vertex of T , we may speak of parents, children, ancestors, descendants in the usual way. For each $w \in V(T)$, we write $|w| = d_T(\xi, w)$, and call $|w|$ the generation of w . We also write $k(w) = k_T(w)$ for the number of children of w , and if $w \neq v(\xi)$ then we write $p(w) = p_T(w)$ for the parent of w .

The *Ulam–Harris encoding* is the injective function $U = U_T : V(T) \rightarrow \bigcup_{i \geq 0} \mathbb{N}^i$ defined as follows (let $\mathbb{N}^0 = \{\emptyset\}$ by convention). First, set $U(v(\xi)) = \emptyset$. For every other vertex $w \in V(T)$, consider the unique path $v(\xi) = v_0, v_1, \dots, v_k = w$ from $v(\xi)$ to w . For $1 \leq i \leq k$ let n_i be such that v_i is the n_i th child of v_{i-1} , in cyclic order around v_{i-1} starting from $\kappa^r(v_{i-1}, v_{i-2})$ if $i \geq 2$ or from ξ if $i = 1$. Then set $U(w) = n_1 n_2 \dots n_k \in \mathbb{N}^k$. In other words, the root receives label \emptyset and for each $i \geq 1$ the label of any i th child is obtained recursively by concatenating the integer i to the label of its parent. It is easily verified that (the isomorphism class of) T can be recovered from the set of labels $\{U(v) : v \in V(T)\}$.

When there is no ambiguity, we identify planted plane trees with their Ulam–Harris encodings. In particular, in this case the root vertex is denoted \emptyset and if v is a vertex of T , then its children are denoted $v1, \dots, vk$, where $k = k_T(v)$.

The *lexicographic ordering* $\preceq_{\text{lex}} = \preceq_{\text{lex}, T}$ of $V(T)$ is the total order of $V(T)$ induced by the lexicographic order on $\{U(v) : v \in V(T)\}$. This ordering induces a lexicographic ordering of $E(T)$ (also denoted $\preceq_{\text{lex}} = \preceq_{\text{lex}, T}$ by a slight abuse of notation) by defining $\{u, v\} \preceq_{\text{lex}, T} \{u', v'\}$ if and only if $u, v \preceq_{\text{lex}, T} u'$ or $u, v \preceq_{\text{lex}, T} v'$. These are the orders in which a clockwise contour exploration of the plane tree T starting from ξ first visits the vertices and edges of T , respectively.

The *contour exploration* $r = r_T : [2|V(T)| - 2] \rightarrow V(T)$ is inductively defined as follows. Let $r(0) = v(\xi)$. Then, for $1 \leq i \leq 2|V(T)| - 2$, let $r(i)$ be the lexicographically first child of $r(i - 1)$ that is not an element of $\{r(0), \dots, r(i - 1)\}$,

⁶It is relatively common to define a planted plane tree as a pair (T, v) where T is a plane tree and v is a degree-one vertex of T . Our definition, which is equivalent, can be recovered by deleting the plant vertex and its incident edge, and rooting at the corner thereby created.

or let $r(i)$ be the parent of $r(i - 1)$ if no such node exists. Note that each vertex $v \in V(T) \setminus \{v(\xi)\}$ appears $\deg_T(v)$ times in the contour exploration, and $v(\xi)$ appears $\deg_T(v(\xi)) + 1$ times.

The contour exploration induces an ordering of $\mathcal{C}(T)$, as follows. For $0 \leq i < 2|V(T)| - 2$, let $e(i) = e_T(i) = \{r(i), r(i + 1)\}$. Then let $\xi(0) = \xi_T(0) = \xi$, and for $1 \leq i < 2|V(T)| - 2$ let $\xi(i) = \xi_T(i) = (e(i - 1), e(i))$. The *contour ordering*, denoted $\preceq_{\text{ctr}} = \preceq_{\text{ctr}, T}$, is the total order of $\mathcal{C}(T)$ induced by $(\xi(i), 0 \leq i < 2|V(T)| - 2)$. For convenience, also let $\xi(2|V(T)| - 2) = \xi_T(2|V(T)| - 2) = \xi$. Finally, write $\preceq_{\text{cyc}} = \preceq_{\text{cyc}, T}$ for the cyclic order on $\mathcal{C}(T)$ induced by $\preceq_{\text{ctr}, T}$. It can be verified that \preceq_{cyc} does not depend on the choice of root corner ξ . We define cyclic intervals accordingly: for $c, c' \in \mathcal{C}(T)$, let

$$[c, c']_{\text{cyc}} = \begin{cases} \{c'' : c \preceq_{\text{ctr}} c'' \preceq_{\text{ctr}} c'\}, & \text{if } c \preceq_{\text{ctr}} c', \\ \{c'' : c'' \preceq_{\text{ctr}} c \text{ or } c' \preceq_{\text{ctr}} c''\}, & \text{if } c' \preceq_{\text{ctr}} c. \end{cases}$$

Given $u, v \in V(T)$, we say that v is the *successor* of u if $u \preceq_{\text{lex}} v$ and for all $w \in V(T)$, if $u \preceq_{\text{lex}} w \preceq_{\text{lex}} v$ then $w = u$ or $w = v$. We define successorship for corners similarly.

Given a plane tree $T = (T, \xi)$ and a set $R \subset V(T)$ with $v(\xi) \in R$. Also, the subtree of T *spanned* by R , denoted $T\langle R \rangle$, is the subtree of T induced by the union of the shortest paths between all pairs of vertices in R . Note that $T\langle R \rangle$ naturally inherits a planted plane tree structure from T .

2.3. *The contour process and spatial plane trees.* A *spatial plane tree* is a triple $T = (T, \xi, D)$, where (T, ξ) is a planted plane tree and $D : E(T) \rightarrow \mathbb{R}$ is an arbitrary function. Given a labelled plane tree, define a function $X = X_T : V(T) \rightarrow \mathbb{R}$ as follows. First, let $X(v(\xi)) = 0$. Next, given $u \in V(T)$ with $X(u)$ already defined, for $1 \leq i \leq k_T(u)$ let $X(ui) = X(u) + D(u, ui)$. We call X_T the *labelling function* of T .

Now define $C([0, 1], \mathbb{R})$ functions C_T and Z_T by setting

$$C_T(i/(2|V(T)| - 2)) = d_T(\xi, r_{(T, \xi)}(i)) \quad \text{and} \\ Z_T(i/(2|V(T)| - 2)) = X_T(r_{(T, \xi)}(i)),$$

for $i \in \{0, 1, \dots, 2|V(T)| - 2\}$, and extending each function to $[0, 1]$ by linear interpolation. We refer to C_T and Z_T as the *contour* and *labelling* processes of T , respectively. Note that the definition of C_T does not depend on the function D .

2.4. *Spanning trees in planar maps.* Given a planar map G , a *spanning tree* of G is a subgraph T of G such that T is a tree with $V(T) = V(G)$. If (G, ξ) is a planted planar map and T is a spanning tree of G , then we call (T, ξ) a *planted spanning tree* of (G, ξ) .

Finally, given a planted planar map $G = (G, \xi)$ and an orientation \vec{E} of $E(G)$, we say that a planted spanning tree (T, ξ) of G is *oriented* with respect to \vec{E} if

in the orientation of $E(T)$ obtained from \vec{E} by restriction, all edges are oriented towards $v(\xi)$.

3. Distances between metric spaces: Gromov, Hausdorff and Prokhorov.

3.1. *The Gromov–Hausdorff distance.* For proofs of the assertions in this section, and for further details, we refer the reader to [11, 28]. Let $X = (X, d)$ and $X' = (X', d')$ be compact metric spaces. Given $C \subset X \times X'$, the *distortion* of C , denoted $\text{dis}(C)$, is the quantity

$$\text{dis}(C) = \sup\{|d(x, y) - d'(x', y')| : (x, x') \in C, (y, y') \in C\}.$$

A *correspondence* between X and X' is a set $C \subset X \times X'$ such that for every $x \in X$ there is $x' \in X'$ such that $(x, x') \in C$ and vice versa. We write $C(X, X')$ for the set of correspondences between X and X' . The Gromov–Hausdorff distance $d_{\text{GH}}(X, X')$ between metric spaces $X = (X, d)$ and $X' = (X', d')$ is

$$d_{\text{GH}}(X, X') = \frac{1}{2} \inf\{\text{dis}(C) : C \in C(X, X')\}.$$

We list without proof some basic properties of d_{GH} . Let \mathcal{M} be the set of isometry classes of compact metric spaces:

- (1) Given metric spaces $X = (X, d)$ and $X' = (X', d')$, there exists $C \in C(X, X')$ such that $d_{\text{GH}}(X, X') = \text{dis}(C)/2$.
- (2) If X_1 and X_2 are isometric, and X'_1 and X'_2 are isometric, then $d_{\text{GH}}(X_1, X'_1) = d_{\text{GH}}(X_2, X'_2)$. In other words, d_{GH} is a *class function* for \mathcal{M} .
- (3) The push-forward of d_{GH} to \mathcal{M} (which we continue to denote d_{GH}) is a distance on \mathcal{M} , and $(\mathcal{M}, d_{\text{GH}})$ is a complete separable metric space.

A *k-pointed metric space* is a triple $(X, d, (x_1, \dots, x_k))$ where (X, d) is a metric space and $x_i \in X$ for $1 \leq i \leq k$. We say *k-pointed metric spaces* $X = (X, d, (x_1, \dots, x_k))$ and $X' = (X', d', (x'_1, \dots, x'_k))$ are *isometry-equivalent* if there exists a bijective isometry $f : X \rightarrow X'$ such that $f(x_i) = x'_i$ for $1 \leq i \leq k$. The *k-pointed Gromov–Hausdorff distance* d_{GH}^k between X, X' is given by

$$d_{\text{GH}}^k(X, X') = \frac{1}{2} \inf\{\text{dis}(C) : C \in C(X, X') \text{ and } (x_i, x'_i) \in C, 1 \leq i \leq k\}.$$

Much as before, if $\mathcal{M}^{(k)}$ is the set of isometry-equivalence classes of *k-pointed compact metric spaces*, then d_{GH}^k is a class function for $\mathcal{M}^{(k)}$ so may be viewed as having domain $\mathcal{M}^{(k)}$, and $(\mathcal{M}^{(k)}, d_{\text{GH}}^k)$ then forms a complete separable metric space.

3.2. *The Gromov–Hausdorff–Prokhorov distance.* Following [28], a *weighted metric space* is a triple (X, d, μ) such that (X, d) is a metric space and μ is a Borel probability measure on (X, d) . Weighted metric spaces (X, d, μ) and (X', d', μ') are *isometry-equivalent* if there exists a measurable bijective isometry $\phi : X \rightarrow X'$ such that $\phi_*\mu = \mu'$, where $\phi_*\mu$ denotes the push-forward of μ under ϕ . Write \mathcal{M}_w for the set of isometry-equivalence classes of weighted compact metric spaces.

Given weighted metric spaces $X = (X, d, \mu)$ and $X' = (X', d', \mu')$, a *coupling* between μ and μ' is a Borel measure ν on $X \times X'$ (for the product metric) with $\pi_*\nu = \mu$ and $\pi'_*\nu = \mu'$, where $\pi : X \times X' \rightarrow X$ and $\pi' : X \times X' \rightarrow X'$ are the projection maps. Let $M(\mu, \mu')$ be the set of couplings between μ and μ' . The *Gromov–Hausdorff–Prokhorov distance* is defined by

$$d_{\text{GHP}}(X, X') = \inf\{\varepsilon > 0 : \exists C \in \mathcal{C}(X, X'), \exists \nu \in M(\mu, \mu'), \nu(C) \geq 1 - \varepsilon, \text{dis}(C) \leq 2\varepsilon\}.$$

The push-forward of d_{GHP} to \mathcal{M}_w , which we again denote d_{GHP} , is a distance on \mathcal{M}_w , and $(\mathcal{M}_w, d_{\text{GHP}})$ is a complete separable metric space (see [28], Section 6 and [13], Section 2).

4. Map encodings. The purpose of this section is to state sufficient conditions for a family of random maps to converge to the Brownian map after rescaling. The framework we describe enables us to use the convergence argument the same line of argument as in Le Gall [22] with only minor modifications (which are essentially to ensure that the convergence holds in the Gromov–Hausdorff–Prokhorov sense and not only in the Gromov–Hausdorff sense). Our choice to work in a slightly more abstract setting was motivated by potential applications to several models of maps for which convergence to the Brownian map is yet to be established. We return to this point at the end of the section.

A *map encoding* is a pair $P = (M, T)$ where $M = (M, \xi)$ is a planted planar map and $T = (T, \xi, D)$ is a spatial plane tree with $V(T) \subset V(M)$. Note that although T shares its vertices with M , it need not be a subgraph of M .

Fix a sequence $P = (P_n, n \geq 1)$ of random map encodings. Write $P_n = (M_n, T_n)$, write C_n and Z_n for the contour and label processes of T_n , respectively, and write X_n and r_n for the labelling function of T_n and for the contour exploration of T_n , respectively. The sequence P is *good* if there exist sequences $(a_n, n \in \mathbb{N})$ and $(b_n, n \in \mathbb{N})$ such that the following three properties hold.

1. As $n \rightarrow \infty$, $(a_n C_n, b_n Z_n) \xrightarrow{d} (\mathbf{e}, Z)$ in the topology of uniform convergence on $\mathcal{C}([0, 1], \mathbb{R})^2$, where (\mathbf{e}, Z) is as described in Section 1.1.
2. (i) For all $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbf{P}\left\{b_n \cdot \max_{v \in V(M_n)} d_{M_n}(v, V(T_n)) > \varepsilon\right\} = 0.$$

(ii) Write d_{Prok} for the Prokhorov distance between Borel measures on \mathbb{R} . For each n , conditionally given \mathbb{P}_n , let U_n, V_n be independent uniformly random elements of $V(T_n)$. Then

$$\lim_{n \rightarrow \infty} b_n \cdot d_{\text{Prok}}(d_{M_n}(\zeta_n, \xi_n), d_{M_n}(U_n, V_n)) = 0.$$

3. (i) Let $m = m(n) = 2|V(T_n)| - 2$. Then for all $\varepsilon > 0$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbf{P}\{ \exists i, j \in [m] : d_{M_n}(r_n(i), r_n(j)) \geq Z_n(i/m) + Z_n(j/m) \\ - 2 \max(\check{Z}_n(i/m, j/m), \check{Z}_n(j/m, i/m)) + \varepsilon b_n^{-1} \} \\ = 0. \end{aligned}$$

(ii) For all $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbf{P}\{ \exists j \in [m] : d_{M_n}(r_n(j), \zeta_n) \leq Z_n(j/m) - \check{Z}_n(0, 1) - \varepsilon b_n^{-1} \} = 0.$$

For later use, we note one consequence of 3. Let I_n be minimal such that $Z_n(I_n/m) = \check{Z}_n(0, 1)$, 3(ii) implies that

$$\lim_{n \rightarrow \infty} \mathbf{P}\{ d_{M_n}(r_n(I_n), \zeta_n) > \varepsilon b_n^{-1} \} = 0.$$

Together with 3(i) and 3(ii) this yields that, for all $\varepsilon > 0$,

$$(3) \quad \lim_{n \rightarrow \infty} \mathbf{P}\left\{ \exists j \in [m] : |d_{M_n}(r_n(j), \zeta_n) - (X_n(r_n(j)) - X_n(r_n(I_n)))| > \frac{\varepsilon}{b_n} \right\} = 0.$$

In other words, for $u \in V(T_n)$, the distance $d_{M_n}(u, \zeta_n)$ is essentially given by the difference between the label of u and the infimum of labels in T_n .

THEOREM 4.1. *If \mathbb{P} is a good sequence of random map encodings then, writing μ_n for the uniform probability measure on $V(T_n) \subset V(M_n)$, we have*

$$(V(M_n), b_n d_{M_n}, \mu_n) \xrightarrow{d} (S, d, \mu)$$

for d_{GHP} , where (S, d, μ) is the Brownian map, as defined in Section 1.1.

The proof of Theorem 4.1, which closely follows an argument of Le Gall [22] (as mentioned above), appears in Appendix B. We conclude the section by mentioning one corollary of the theorem; we are slightly informal to avoid notational excess and as the argument is straightforward. For $n, k \geq 1$, conditionally given \mathbb{P}_n , let $U_{n,1}, \dots, U_{n,k}$ be independent with law μ_n . Proposition 10 of [28] implies that if the convergence in Theorem 4.1 holds then also

$$(V(M_n), b_n d_{M_n}, (U_{n,1}, \dots, U_{n,k})) \xrightarrow{d} (S, d, (U_1, \dots, U_k)),$$

for d_{GH}^k , where conditionally given (S, d, μ) , U_1, \dots, U_k are independent with law μ . By Proposition 8.2 of [21], conditionally given (S, d, μ) , the points $\rho, u^* \in S$ are independent with law μ ; by 2(ii) it follows that $(V(M_n), b_n d_{M_n}, (\xi_n, \zeta_n)) \xrightarrow{d} (S, d, (\rho, u^*))$ for d_{GH}^2 .

REMARK 4.2. The motivation underlying the introduction of good random map encodings is to define a general framework which can be used in future work as a “black box” to establish the convergence of various families of maps towards the Brownian map. In order to justify this, we provide some specific examples (though not an exhaustive list) of settings where we believe our generalization will be of use.

Condition 2(i) states that after rescaling distances by b_n , all vertices of the map M_n are with high probability close to some tree vertex. In the present work, it turns out that only two vertices of the M_n do not belong to T_n . In some models of maps, it only holds that at least one vertex per face of the map belongs to the associated tree. The maximum face degree in a random simple map is typically logarithmic in the size of the map, so in that setting the strength of Condition 2(i) is useful.

Condition 2(ii) requires the distance between the root of the map and the root of the tree to be asymptotically equal in distribution to the distance between two uniform vertices of the map. In the case of simple triangulations, the distance between the two roots is actually exactly distributed as the distance between two uniformly random points. However, it happens frequently that in bijections between maps and trees, the root of the map plays a special role and is not precisely uniformly distributed. For example, in studying 3-connected maps, a family of maps naturally arises for which all nonroot faces are quadrangles, but the root face is a hexagon [15].

Finally, for the classical case of uniform quadrangulations, conditions 3(i) and 3(ii) hold true without the term εb_n^{-1} . However, in the present work, we can only prove that labels of the tree control distances in the maps up to an error term which is $o(b_n)$ in probability, so we require the full strength of 3(i) and 3(ii).

5. Bijections for simple triangulations. We start with a summary of the results of the section; to do so some definitions are needed. For integer $k \geq 1$, a plane tree T is a k -blossoming tree if each vertex of degree greater than one is incident to exactly k vertices of degree one. If T is a k -blossoming tree (for some k), we write $\mathcal{B} = \mathcal{B}(T)$ for the set of degree-one vertices of T . When it causes no ambiguity, we identify vertices of \mathcal{B} with their incident corners. Note that both k and \mathcal{B} are uniquely determined by T . We call \mathcal{B} the blossoms of T , and $V(T) \setminus \mathcal{B}$ the inner vertices of T . Also, an edge between two inner vertices is called an inner edge, and an edge between an inner vertex and a blossom is a stem. A corner c is an inner corner if $c \notin \mathcal{B}$. A planted k -blossoming tree is a planted plane tree (T, ξ) such that T is a k -blossoming tree and ξ is an inner corner of T . The bijections of

Section 5 concern 2-blossoming trees, which we simply call blossoming trees for the remainder of the section.

Write \mathcal{T}_n for the set of planted blossoming trees (T, ξ) with n inner vertices. Fix $(T, \xi) \in \mathcal{T}_n$, and note that $|E(T)| = |V(T)| - 1 = 3n - 1$ so $|\mathcal{C}(T)| = 6n - 2 = 3|\mathcal{B}(T)| - 2$. We say (T, ξ) is *balanced* if $\xi = (e, e')$ for distinct stems e, e' , and for all $c \in \mathcal{C}(T)$,

$$(4) \quad 3|[\xi, c]_{\text{cyc}} \cap \mathcal{B}| + 1 \geq |[\xi, c]_{\text{cyc}}|$$

(recall the definition of $[\xi, c]_{\text{cyc}}$ from Section 2.2). For $n \geq 1$, let $\mathcal{T}_n^\circ \subset \mathcal{T}_n$ be the set of balanced blossoming trees with n inner vertices.

A *valid labelling* of a planted plane tree $T = (T, \xi)$ is a labelling $d = (d_e, e \in E(T))$ of the edges of T by elements of $\{-1, 0, 1\}$ such that for all $v \in V(T)$, writing $k = k_T(v)$, the sequence $d_{\{v, v1\}}, \dots, d_{\{v, vk\}}$ is nondecreasing. Let $\mathcal{T}_n^{\text{vl}}$ be the set of validly labelled plane trees with n vertices. We emphasize that a validly labelled plane tree is a “normal” tree, not a blossoming tree.

Finally, recall that for $n \geq 3$, Δ_n° is the set of planted triangulations with n inner vertices. The following diagram summarizes the bijective relations between $\mathcal{T}_n, \mathcal{T}_n^\circ$, and Δ_{n+2}° established in [31] and recalled in the current section:

$$(5) \quad \mathcal{T}_n^{\text{vl}} \xleftarrow[\text{bij}]{\phi_n; \text{Proposition 5.4}} \mathcal{T}_n \xrightarrow[(4n-2) \text{ to } -2]{\text{projection}} \mathcal{T}_n^\circ \xrightarrow[\chi_n; \text{Proposition 5.1}]{\text{bij}} \Delta_{n+2}^\circ.$$

After concluding with bijective arguments, in Section 5.4 we explain how to sample uniformly random triangulations using conditioned Galton–Watson trees. We end the section by describing the inverse of the bijection $\chi_n : \mathcal{T}_n^\circ \rightarrow \Delta_{n+2}^\circ$, which we use later.

5.1. *A bijection between triangulations and blossoming trees.* We first describe a bijection of Poulalhon and Schaeffer [31] between balanced blossoming trees and simple, planted triangulations of the sphere (see Figure 3; the orientations of the arrows in the figure are explained in Section 5.5). Fix a blossoming tree T . Given a stem $\{b, u\}$ with $b \in \mathcal{B}(T)$, if bu is followed by two inner edges in a clockwise contour exploration of T — uv and vw , say—then the *local closure* of $\{b, u\}$ consists in removing the blossom b and its stem, and adding a new edge $\{u, w\}$ [such that $\kappa^r(u, w) = (\{u, w\}, \{u, v\})$ and $\kappa^\ell(w, u) = (\{w, v\}, \{w, u\})$]. After performing the local closure, uw always has a triangle on its right. The edge $\{u, w\}$ is considered to be an inner edge in subsequent local closures.

The *partial closure* of a blossoming tree is the planar map obtained by performing all possible local closures. Equivalently, for each corner $c \in \mathcal{B}$, let $s(c)$ be the inner corner c' minimizing $|[c, c']_{\text{cyc}}|$ subject to the condition that

$$(6) \quad 3|[c, c']_{\text{cyc}} \cap \mathcal{B}| < |[c, c']_{\text{cyc}}|,$$

if such a corner exists (recall the definition of \preceq_{cyc} from Section 2.2). The partial closure operation identifies $v(c)$ with $v(s(c))$ whenever $c \in \mathcal{B}$ and $s(c)$ is defined;

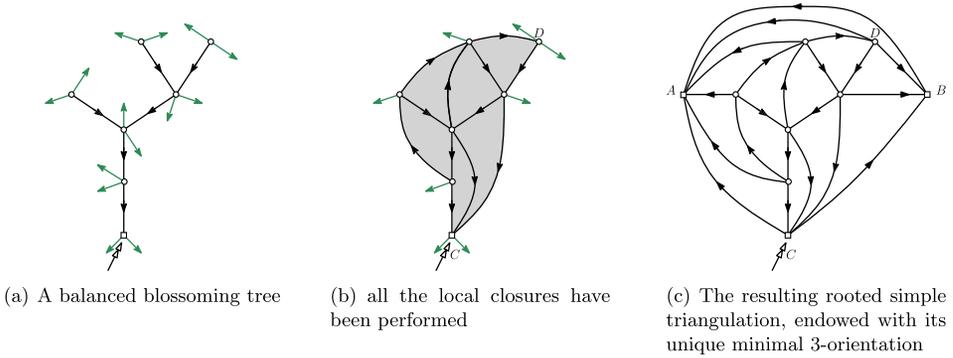


FIG. 3. The closure of a balanced tree into a simple triangulation.

it follows from the latter description that the partial closure does not depend on the order in which local closures take place. Say $v(c)$ is *closed* if $s(c)$ is defined, and otherwise say $v(c)$ is *unclosed*.

It can be checked that the partial closure is a simple map and contains precisely one face f of degree greater than three, and all unclosed blossoms are incident to f . Furthermore, simple counting arguments show that each inner corner incident to f is adjacent to at least one unclosed blossom, and that there are precisely two corners, say ξ^C and ξ^D , that are incident to two unclosed blossoms. Note that ξ^C and ξ^D are both corners of T (i.e., they are not created while performing the partial closure). Let $C = v(\xi^C)$ and $D = v(\xi^D)$.

Let (T, ξ) be a balanced blossoming tree such that $\xi = (e, e')$, $e = (v(\xi), v)$ and $e' = (v(\xi), v')$. It follows straightforwardly from (6) that v and v' are unclosed, or equivalently ξ is equal to ξ_C or ξ_D . We now suppose $\xi \in \{\xi^C, \xi^D\}$. Let S_{CD} (resp. S_{DC}) be the set of nonblossom vertices v of the distinguished face f of the partial closure such that in the planted tree (T, C) [resp. (T, D)] we have $v \preceq_{\text{ctr}} D$ (resp., $v \preceq_{\text{ctr}} C$). In other words, vertices of S_{CD} lie after C and before D in a clockwise tour of f , and likewise for S_{DC} .

To finish the construction, remove the remaining blossoms and their stems. Add two additional vertices A and B within f , then add an edge between A (resp., B) and each of the vertices of S_{CD} (resp. of S_{DC}). In the resulting map, define a corner c by $c = (\{C, B\}, \{C, A\})$ if $v(\xi) = C$ or $c = (\{D, A\}, \{D, B\})$ if $v(\xi) = D$. Finally, add an edge between A and B in such a way that, after its addition, A, B , and $v(\xi)$ lie on the same face f . The result is a planar map, rooted at ξ , called the *closure* of T . For later use, define a function $s' : V(T) \rightarrow V(T)$ as follows. First, set $s'(v) = v$ for $v \in V(T) \setminus \mathcal{B}$. For $v \in \mathcal{B}$, let u be the unique neighbour of v and let k be the unique corner incident to v . If $s(k)$ is defined then let $s'(v) = v(s(k))$; otherwise, if $u \in S_{CD}$ let $s'(v) = A$ and if $u \in S_{DC}$ let $s'(v) = B$.

Write $\chi : \bigcup_{n \geq 1} \mathcal{T}_n^\circ \rightarrow \bigcup_{n \geq 1} \Delta_{n+2}^\circ$ for the function sending a balanced blossoming tree to its closure, and for $n \geq 1$ let $\chi_n : \mathcal{T}_n^\circ \rightarrow \Delta_{n+2}^\circ$ be the restriction of χ to \mathcal{T}_n° .

PROPOSITION 5.1 ([31]). *For all $n \geq 1$, χ_n is a bijection between \mathcal{T}_n° and Δ_{n+2}° .*

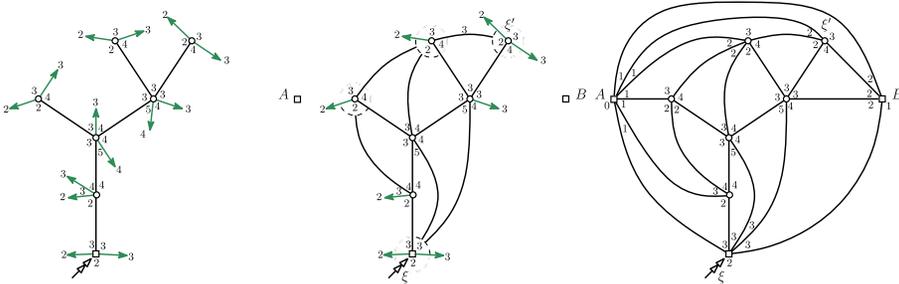
Note that if (T, ξ) is a blossoming tree and $\chi(T, \xi) = (G, c)$ then it is natural to identify the inner vertices and inner edges of T with subsets of $V(G)$ and $E(G)$, respectively. More formally, we may choose representatives from the isomorphism equivalence classes of the tree and its closure so that $V(T) \setminus \mathcal{B}(T) = V(G) \setminus \{A, B\}$ and $\{\{u, v\} \in E(T) : u, v \notin \mathcal{B}\} \subset E(G)$. We will adopt this perspective in the remainder of the paper.

5.2. *Bijection with labels.* We now present an alternative description of the bijection from Proposition 5.1, based on (6). Given a blossoming tree (T, ξ) , write $\mathsf{T} = (T, \xi)$ and define $\lambda = \lambda_{\mathsf{T}} : \mathcal{C}(T) \rightarrow \mathbb{Z}$ as follows. Recall the definition of the contour ordering $(\xi(i), 0 \leq i \leq 2|V(T)| - 2)$ from Section 2.2, and in particular that $\xi(0) = \xi$. Let $\lambda(\xi(0)) = 2$ and, for $0 \leq i < 2|V(T)| - 3$, set

$$\lambda(\xi(i + 1)) = \begin{cases} \lambda(\xi(i)) - 1, & \text{if } \xi(i) \notin \mathcal{B}(T), \xi(i + 1) \notin \mathcal{B}(T), \\ \lambda(\xi(i)), & \text{if } \xi(i) \notin \mathcal{B}(T), \xi(i + 1) \in \mathcal{B}(T), \\ \lambda(\xi(i)) + 1, & \text{if } \xi(i) \in \mathcal{B}(T), \xi(i + 1) \notin \mathcal{B}(T). \end{cases}$$

This labelling is depicted in Figure 4(a). Informally, we perform a clockwise contour exploration of the tree and label the corners as we go. When leaving an inner vertex and arriving at an inner vertex, decrease the label by one; when leaving an inner vertex and arriving at a blossom, leave the label unchanged; when leaving a blossom and arriving at an inner vertex, increase the label by one.

It is not hard to see that $\mathsf{T} = (T, \xi)$ is balanced if and only if ξ is incident to two stems and $\lambda(c) \geq 2$ for all $c \in \mathcal{C}(T)$ [see Figure 4(a)]. Assume (T, ξ) is balanced and write ξ' for the unique corner in $\mathcal{C}(T) \setminus \{\xi\}$ for which (T, ξ') is also balanced. Given a corner $c \in \mathcal{C}(T)$ with $v(c) \in \mathcal{B}(T)$, recall the definition of $s(c)$ from (6). A counting argument shows that when $s(c)$ is defined, it is equal to the



(a) The corner labelling of a balanced blossoming tree (b) The labelled partial closure (c) The resulting corner-labelled simple triangulation

FIG. 4. Closing a balanced tree via the corner labelling.

first corner c' following c in clockwise order for which $\lambda(c') < \lambda(c)$ [and in fact $\lambda(s(c)) = \lambda(c) - 1$]. Furthermore, $s(c)$ is defined if and only if either $\lambda(c) > 2$ and $c \preceq_{\text{ctr}} \xi'$, or $\lambda(c) > 3$ and $\xi' \preceq_{\text{ctr}} c$.

Next, add two vertices, say A and B , within the unique face of the partial closure with degree greater than three. For each $c \in \mathcal{C}(T)$ with $v(c) \in \mathcal{B}(T)$ and $s(c)$ undefined, identify $v(c)$ with A if $\lambda(c) = 2$, and with B if $\lambda(c) = 3$. At this point, the unique face of degree greater than three is incident to ξ , A , ξ' and B in cyclic order. Finally, add a single edge between A and B . The following fact, the straightforward proof of which is omitted, states that the resulting planar map is $\chi(T)$.

FACT 5.2. The triangulation obtained from a balanced blossoming tree by iterating local closures and the one obtained by the label procedure coincide.

The closure contains corners not present in the blossoming tree, and the new corners are labelled as follows. For any bud corner c for which $s(c)$ is defined, closing $v(c)$ may be viewed as splitting a single corner in two, and the two new corners inherit the label of the corner that was split. An example is shown in Figure 4(b); the dashed arcs denote corners that are “split” by the partial closure operation. Let f be the face of $\chi(T, \xi)$ incident to ξ . Give the corner of A (resp., B) incident to f label 0 (resp., 1), and give all other corners incident to A (resp., B) label 1 (resp. 2). We write $\lambda^* = \lambda^*_{(T, \xi)}$ for this corner labelling of $\chi(T, \xi)$, and note that $\lambda^* : \mathcal{C}(\chi(T, \xi)) \rightarrow \mathbb{Z}^{\geq 0}$ since we have assumed (T, ξ) is balanced. An example of the resulting corner-labelled triangulation is depicted in Figure 4(c).

5.3. From labels to displacement vectors. We next explain the connection between blossoming trees and validly labelled plane trees. Fix $n \geq 1$, let $(T, \xi) \in \mathcal{T}_n$ and let λ be the labelling of corners as defined in Section 5.2. We define a function $Y = Y_{(T, \xi)} : V(T) \rightarrow \mathbb{Z}$ by setting $Y(v) = \min\{\lambda(c) : c \in \mathcal{C}(T), v(c) = v\}$ for all $v \in V(T)$. Next, for each inner edge $e \in E(T)$, writing $e = \{v, p(v)\}$, with $v \in V(T) \setminus \{v(\xi)\}$, set $D_e = D_e(T, \xi) = Y(v) - Y(p(v))$. The following easy fact, the proof of which is omitted, allows us to recover the locations of stems from the edge labels.

FACT 5.3. For all $e = \{v, p(v)\} \in E(T)$, $D_e + 1 = |\{e' \preceq_{\text{lex}} e : e' \text{ a stem incident to } p(v)\}|$.

Now fix $v \in V(T)$, let $k = k_{(T, \xi)}(v)$, and for $1 \leq i \leq k$ let $e_i = \{v, vi\}$. It follows from the above fact that for $1 \leq i \leq k$ the number of stems e incident to v with $e \preceq_{\text{lex}} e_i$ is $D_{e_i} + 1$. In particular, $(D_{e_i}, 1 \leq i \leq k)$ is a nondecreasing sequence of elements of $\{-1, 0, 1\}$; this is what allows us to connect blossoming trees with validly labelled trees.

For $n \geq 1$, define a map $\phi_n : \mathcal{T}_n \rightarrow \mathcal{T}_n^{\text{vl}}$ as follows. Given $(T, \xi) \in \mathcal{T}_n$, write $\xi = (e_-, e_+)$. Let e be the last inner edge incident to $v(\xi)$ preceding e_- in clockwise

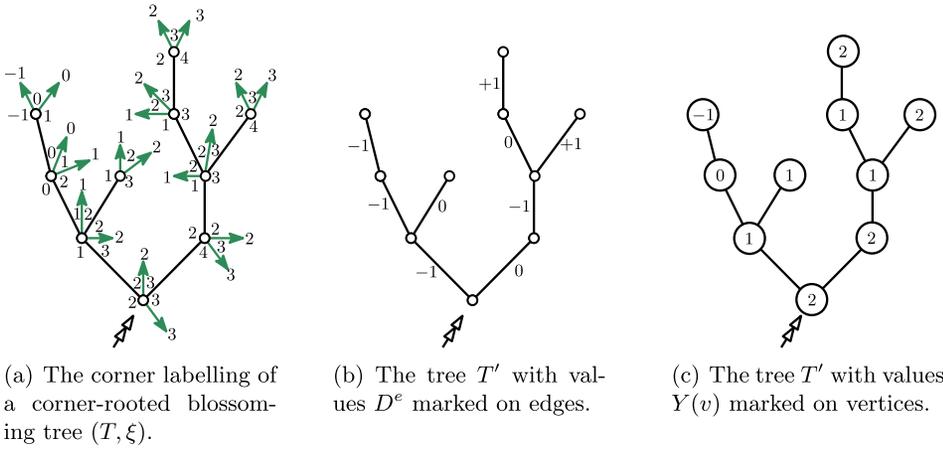


FIG. 5. The equivalence between blossoming trees and validly vector-labelled plane trees. The root corner is indicated via a double arrow.

order (with $e = e_-$ if e_- is an inner edge), and let e' be the first inner edge incident to $v(\xi)$ following e_+ in clockwise order (with $e = e_+$ if e_+ is an inner edge). Write $\xi' = (e, e')$, let T' be the subtree of T induced by the inner vertices, let $D = (D_e(T, \xi), e \in E(T))$, and let $\phi_n(T, \xi) = (T', \xi', D)$. The following proposition is an immediate consequence of Fact 5.3.

PROPOSITION 5.4. The map $\phi_n : \mathcal{T}_n \rightarrow \mathcal{T}_n^{v1}$ is a bijection.

The above bijection and definitions are illustrated in Figure 5. In the next section, we explain how the above functions can be used to sample random simple triangulations with the aid of conditioned Galton–Watson trees.

5.4. *Corner-rooted triangulations via conditioned Galton–Watson trees.* Let (T_n, ξ_n) be uniformly distributed on \mathcal{T}_n . We are now able to describe the law of (T_n, ξ_n) as a modification of the law of a critical Galton–Watson tree conditioned to have a given size. (Galton–Watson trees are naturally viewed as planted plane trees; see, e.g., [19].) Let $G \stackrel{d}{=} \text{Geometric}(3/4)$, and let B have law given by

$$(7) \quad \mathbf{P}\{B = c\} = \frac{\binom{c+2}{2} \mathbf{P}\{G = c\}}{\mathbf{E}\binom{G+2}{2}}, \quad \text{for } c \in \mathbb{N}.$$

FACT 5.5. The distribution B is critical, that is, $\mathbf{E}B = 1$.

This fact follows from simple computations involving the 3 first moments of a geometric law; its proof is omitted.

PROPOSITION 5.6. *Let (T', ξ') be a Galton–Watson tree with offspring distribution B conditioned to have n vertices. For each vertex v of T' , add two stems incident to v , uniformly at random from among the $\binom{k_{T'}(v)+2}{2}$ possibilities. The resulting planted plane tree (T, ξ) is uniformly distributed over \mathcal{T}_n .*

PROOF. Fix $t \in \mathcal{T}_n$ and let $t' = t \setminus \mathcal{B}(T)$ be the tree t with its blossoms removed. List the vertices of t' in lexicographic order as v_1, \dots, v_n and recall that $k_{t'}(v_i)$ is the number of children of v_i in t' .

Then (T, ξ) is equal to t if and only if $(T', \xi') = t'$ and for each $v \in V(T')$, the stems are inserted at the right place. Hence,

$$\begin{aligned} \mathbf{P}\{(T, \xi) = t\} &\propto \prod_{i=1}^n \frac{1}{\binom{k_{t'}(v_i)+2}{2}} \mathbf{P}\{B = k_{t'}(v_i)\} \\ &= \prod_{i=1}^n \frac{1}{\binom{k_{t'}(v_i)+2}{2}} \binom{k_{t'}(v_i)+2}{2} \mathbf{P}\{G = k_{t'}(v_i)\} \\ &= \frac{3^{n-1}}{4^{2n-1}}. \end{aligned}$$

The last equality holds since G is geometric and $\sum_{i=1}^n k_{t'}(v_i) = n - 1$. Since the last term does not depend on the shape of t , all elements of \mathcal{T}_n appear with the same probability. \square

COROLLARY 5.7. *Let $(T, \hat{\xi})$ be uniformly random in \mathcal{T}_n and let $\xi^1, \xi^2 \in \mathcal{C}(T)$ be such that (T, ξ^i) is balanced for $i \in \{1, 2\}$. Conditionally given $(T, \hat{\xi})$ choose $\xi \in \{\xi^1, \xi^2\}$ uniformly at random. Then $(G, c) = \chi(T, \xi)$ is uniformly distributed in Δ_{n+2}° .*

PROOF. Let $t = (t, c)$ be a balanced blossoming tree of size n . Consider the set of triples $t^\bullet = \{(t, c, \hat{c}) : \hat{c} \text{ an inner corner of } t\}$. Then we have

$$\mathbf{P}\{(T, \xi) = t\} = \mathbf{P}\{(T, \xi, \hat{\xi}) \in t^\bullet\} = \frac{1}{2} \sum_{(t, c, \hat{c}) \in t^\bullet} \mathbf{P}\{(T, \hat{\xi}) = (t, \hat{c})\} = \frac{2n - 1}{|\mathcal{T}_n|},$$

where the last equality comes from the fact that t has $4n - 2$ inner corners (hence, $|t^\bullet| = 4n - 2$) and that $(T, \hat{\xi})$ is uniformly random in \mathcal{T}_n . It follows that (T, ξ) is uniformly random in \mathcal{T}_n° , which concludes the proof since χ is a bijection between \mathcal{T}_n° and Δ_{n+2}° . \square

Proposition 5.4 now allows us to describe the distribution of a uniformly random element (T', ξ', D) of \mathcal{T}_n^{vl} . For each $k \geq 1$, let ν_k be the uniform law over nondecreasing vectors $(d_1, \dots, d_k) \in \{-1, 0, 1\}^k$.

COROLLARY 5.8. *Let (T', ξ') be a Galton–Watson tree with offspring distribution B conditioned to have n vertices. Conditionally given (T', ξ') , independently for each $v \in V(T')$ let $(D_{\{v, v_j\}}, 1 \leq j \leq k(v))$ be a random vector with law $\nu_{k(v)}$. Finally, let $D = (D_e, e \in E(T'))$. Then (T', ξ', D) is uniformly distributed in $\mathcal{T}_n^{\text{vl}}$.*

PROOF. By Proposition 5.6, the tree (T', ξ') has the same law as the subtree obtained from a uniformly random element of \mathcal{T}_n by removing all stems. The corollary then follows by Proposition 5.4. \square

For later use, we note the following fact. Recall the definition of the labelling function X_T for T a spatial plane tree, from Section 2.3.

FACT 5.9. Fix two inner corners ξ_1, ξ_2 of $\mathcal{C}(T)$, and let $T_1 = (T', \xi'_1, D_1) = \phi_n(T, \xi_1)$ and $T_2 = (T', \xi'_2, D_2) = \phi_n(T, \xi_2)$. Then for all $v \in V(T')$, $X_{T_1}(v) = Y_{(T, \xi_1)}(v) - 2$, and $|(Y_{(T, \xi_1)}(v) - Y_{(T, \xi_2)}(v)) - X_{(T, \xi_1)}(v(\xi_2))| \leq 3$.

In other words, the labellings X_T and Y_T are related by an additive constant of 2, and rerooting shifts all labels according to the label of the new root under the old labelling, up to an additive error of 3. This is a direct consequence of Fact 5.3 and the definitions of X_T and Y_T ; its proof is omitted.

We conclude Section 5 by explaining the inverse of the bijection χ_n . The description of the inverse relies the properties of so called 3-orientations for simple triangulations. We make use of such orientations in Section 7 when studying the relation between vertex labels and geodesics.

5.5. Orientations and the opening operation. In a planted map endowed with an orientation, a directed cycle is said to be clockwise if the root corner is situated on its left and counterclockwise otherwise. An orientation is called *minimal* if it has no counterclockwise cycles. Let (G, ξ) be a planted planar triangulation, and recall from Section 1.3 that (G, ξ) admits a unique minimal 3-orientation. We next describe how to obtain this 3-orientation via the bijection described in Proposition 5.1.

Given a balanced 2-blossoming tree $T = (T, \xi)$, orient all stems towards their incident blossom, and orient all other edges towards $v(\xi)$. In the triangulation $\chi(T)$, all edges except $\{A, B\}$ inherit an orientation from T ; orient $\{A, B\}$ from B to A . Then all inner vertices of T not incident to ξ have outdegree 3 in T and the closure operation does not change this outdegree. It follows easily that the resulting orientation of $\chi(T)$ is a 3-orientation. Furthermore, the “clockwise direction” of the local closures implies that closure never creates counterclockwise cycles, so the 3-orientation is minimal.

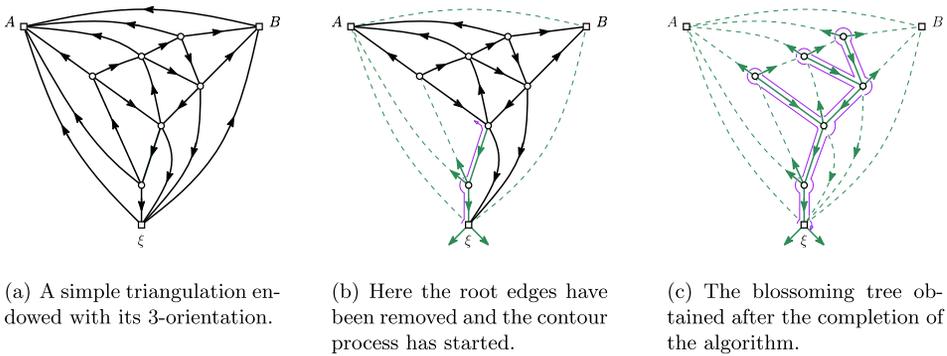


FIG. 6. *The opening of a simple triangulation into a 2-blossoming tree.*

Given a planted planar triangulation $G = (G, \xi)$, the balanced blossoming tree $\chi^{-1}(G, \xi)$ can be recovered as follows. Let \vec{E} be the unique minimal 3-orientation of $E(G)$. List the vertices of the face incident to ξ in clockwise order as $(v(\xi), A, B)$. Remove the edge $\{A, B\}$, and perform a clockwise contour exploration of G starting from ξ . Each time we see an edge uv for the first time, if it is oriented in the opposite direction from the contour process then keep it; otherwise replace it by a stem $\{u, b_{uv}\}$. This procedure is depicted in Figure 6.

6. Convergence to the Brownian snake. Fix a probability distribution μ on \mathbb{N} , and a sequence $v = (v_k, k \geq 1)$, where for each $k \geq 1$, v_k is a probability distribution on \mathbb{R}^k . We denote v_k^i , the i th marginal of v_k .

For $n \in \mathbb{N}$, we write $\text{LGW}(\mu, v, n)$ for the law on spatial plane trees $T = (T, \xi, D)$ such that:

- The pair (T, ξ) has the law of the genealogical tree of a Galton–Watson process with offspring distribution μ , conditioned to have total progeny n .⁷
- Conditionally on (T, ξ) , $D : E(T) \rightarrow \mathbb{R}$ has the following law. Independently for each $u \in V(T)$, if u has k children then $(D(\{u, u1\}), \dots, D(\{u, uk\}))$ is distributed according to v_k .

Here is the connection with random simple triangulations. If (T_n, ξ_n, D_n) is uniformly distributed in \mathcal{T}_n^{v1} , then Corollary 5.8 states that the law of (T_n, ξ_n, D_n) is $\text{LGW}(\mu, v, n)$, where μ is the law defined in (7) and for $k \geq 1$, v_k is the uniform law on nondecreasing vectors in $\{-1, 0, 1\}^k$.

Recall the definition of the pair (\mathbf{e}, Z) from Section 1.1, and the definitions of the functions C_T , X_T and Z_T from Section 2.3. We establish the following convergence.

⁷To avoid trivial technicalities, we assume μ is such that the support of μ has greatest common divisor 1, so that such conditioning is well defined for all n sufficiently large.

PROPOSITION 6.1. For $n \geq 1$, let $T_n = (T_n, \xi_n, D_n)$ be uniformly random in \mathcal{T}_n^{vl} . Then as $n \rightarrow \infty$,

$$(8) \quad ((3n)^{-1/2} C_{T_n}(t), (4n/3)^{-1/4} Z_{T_n}(t))_{0 \leq t \leq 1} \xrightarrow{d} (\mathbf{e}(t), Z(t))_{0 \leq t \leq 1},$$

for the topology of uniform convergence on $C([0, 1], \mathbb{R}^2)$.

Before proving this proposition, we place it in the context of the existing literature on invariance principles for spatial branching processes. Fix μ and ν and let $(T_n, n \in \mathbb{N})$ be such that $T_n = (T_n, \xi_n, D_n)$ has law $\text{LGW}(\mu, \nu, n)$ for $n \in \mathbb{N}$. In what follows, given a measure η on \mathbb{R} and $p > 0$ write $|\eta|_p = (\int_{\mathbb{R}} |x|^p d\eta)^{1/p}$. Aldous ([2], Theorem 2) showed that if $|\mu|_1 = 1$ and $\sigma_\mu^2 := |\mu|_2^2 - |\mu|_1^2 \in (0, \infty)$, then

$$(9) \quad \left(\frac{\sigma_\mu C_{T_n}(t)}{2 n^{1/2}} \right)_{0 \leq t \leq 1} \xrightarrow{d} \mathbf{e}$$

as $n \rightarrow \infty$, for the topology of uniform convergence on $C([0, 1], \mathbb{R})$. Now additionally suppose that for each k , the marginals $\{v_k^i : 1 \leq i \leq k\}$ of ν_k are identically distributed, that $|\nu_k^1|_1 < \infty$, that ν_k is *centred* [i.e., $\int_{\mathbb{R}} x d\nu_k^1(x) = 0$ for every $1 \leq i \leq k$], and that

$$\sup_k v_k^1([y, \infty]) = o(y^{-4}) \quad \text{for every } k \geq 1.$$

Under these conditions, writing $\sigma_\nu = \sigma_{\nu^1}$, Janson and Marckert ([17], Theorem 2) prove that

$$(10) \quad \left(\frac{\sigma_\mu C_{T_n}(t)}{2 n^{1/2}}, \frac{(\sigma_\mu/2)^{1/2} Z_{T_n}(t)}{\sigma_\nu n^{1/4}} \right)_{0 \leq t \leq 1} \xrightarrow[n \rightarrow \infty]{(d)} (\mathbf{e}(t), Z(t))_{0 \leq t \leq 1}$$

in the same topology as in Proposition 6.1. (In fact, Theorem 2 of [17] is stated with the additional assumption that ν_k is a product measure for all k . However, it is not difficult to see, and was explicitly noted in [17], that straightforward modifications of the proof allow this additional assumption to be removed.) Under the same assumptions, the convergence in (10) can also be obtained as a special case of [25], Theorem 8. In the latter article, the marginals of ν_k are not required to be identically distributed but they are assumed to be *locally centred* meaning that for all $1 \leq i \leq k$, $\int_{\mathbb{R}} x d\nu_k^i(x) = 0$. In our setting, the law of the spatial plane tree is given by Corollary 5.8. In this case, the entries are clearly not identically distributed, and neither are they locally centred: observe for instance that $\int_{\mathbb{R}} x d\nu_2^1(x) = -1/3$.

In [24], the “locally centred” assumption is replaced by a *global centering* assumption, namely that

$$\sum_{k \geq 0} \mu(\{k\}) \sum_{i=1}^k \int_{\mathbb{R}} x d\nu_k^i(x) = 0,$$

which is satisfied by our model. However, [24] requires that μ has bounded support, which is not the case in Corollary 5.8.

We expect that the technique we use to prove Proposition 6.1 can be used to extend the results of [24] to a broad range of laws $\text{LGW}(\mu, \nu, n)$ for which μ does not have compact support, under the slightly stronger centering assumption that $\sum_{i=1}^k \int_{\mathbb{R}} x \, d\nu_k^i(x) = 0$ for every k . However, for the sake of concision we have chosen to focus on the random spatial plane trees that arise from random simple triangulations (the treatment for random simple quadrangulations differs only microscopically and is omitted).

For the remainder of Section 6, let μ be as defined in (7), and for $k \geq 1$ let ν_k be the uniform law on nondecreasing vectors in $\{-1, 0, 1\}^k$. For $n \geq 1$ let $T_n = (T_n, \xi_n, D_n)$ be uniformly random in $\mathcal{T}_n^{\nu_1}$ as in Proposition 6.1; by the comments preceding that proposition, T_n has law $\text{LGW}(\mu, \nu, n)$. To prove Proposition 6.1, we establish the following facts.

LEMMA 6.2 (Random finite-dimensional distributions). *Let $(U_i, i \geq 1)$ be independent Uniform $[0, 1]$ random variables, independent of the trees $(T_n, n \geq 1)$, and for $j \geq 1$ let $(U_i^\uparrow, 1 \leq i \leq j)$ be the increasing ordering of U_1, \dots, U_j . Then for all $j \geq 1$,*

$$(11) \quad ((3n)^{-1/2} C_{T_n}(U_i^\uparrow), (4n/3)^{-1/4} Z_{T_n}(U_i^\uparrow))_{1 \leq i \leq j} \xrightarrow{d} (\mathbf{e}(U_i^\uparrow), Z(U_i^\uparrow))_{1 \leq i \leq j},$$

as $n \rightarrow \infty$.

LEMMA 6.3 (Tightness). *The family of laws of the processes $((4n/3)^{-1/4} Z_{T_n}, n \geq 1)$ is tight for the space of probability measures on $C([0, 1], \mathbb{R})$.*

PROOF OF PROPOSITION 6.1. It is immediate from (9) and Lemma 6.3 that the collection of laws of the processes $((3n)^{-1/2} C_{T_n}, (4n/3)^{-1/4} Z_{T_n}, n \geq 1)$ forms a tight family in the space of probability measures on $C([0, 1], \mathbb{R}^2)$. It therefore remains to establish convergence of (nonrandom) finite-dimensional distributions. For this, it suffices to show that for all $m \geq 1, 0 \leq t_1 < t_2 < \dots < t_m \leq 1$ and for all bounded Lipschitz functions $F : (\mathbb{R}^2)^m \rightarrow \mathbb{R}$,

$$(12) \quad \mathbf{E}[F(((3n)^{-1/2} C_{T_n}(t_i), (4n/3)^{-1/4} Z_{T_n}(t_i))_{1 \leq i \leq m})] \\ \rightarrow \mathbf{E}[F((\mathbf{e}(t_i), Z(t_i))_{1 \leq i \leq m})].$$

For the remainder of the proof, we fix m, F and $(t_i, 1 \leq i \leq m)$ as above. Write $\|F\|_\infty$ for the uniform norm of F , and let $\|F\|_{\text{lip}}$ be the Lipschitz constant of F with respect to the uniform norm on \mathbb{R}^{2m} .

Tightness implies (see [7], Theorem 8.2) that for all $\delta > 0$ there exists $\alpha = \alpha(\delta)$ such that

$$(13) \quad \limsup_{n \rightarrow \infty} \mathbf{P} \left\{ \sup_{x, y \in [0, 1], |x - y| \leq \alpha} \left(\frac{|C_{T_n}(x) - C_{T_n}(y)|}{(3n)^{1/2}} \right. \right. \\ \left. \left. + \frac{|Z_{T_n}(x) - Z_{T_n}(y)|}{(4n/3)^{1/4}} \right) > \delta \right\} < \delta.$$

Since \mathbf{e} and Z are almost surely uniformly continuous, by decreasing $\alpha(\delta)$ if necessary we may additionally ensure that

$$(14) \quad \mathbf{P}\left\{ \sup_{x, y \in [0, 1], |x-y| \leq \alpha} (|\mathbf{e}(x) - \mathbf{e}(y)| + |Z(x) - Z(y)|) > \delta \right\} < \delta.$$

Given $\delta > 0$, let $\alpha = \alpha(\delta)$ be as above, and let $j = j(\delta) > 2/\alpha$ be large enough that

$$\mathbf{P}\left\{ \max_{1 \leq i \leq j} \left| U_i^\uparrow - \frac{i}{j} \right| \geq \frac{\alpha}{2} \right\} < \delta.$$

Since $j > 2/\alpha$, we may choose integers k_1, \dots, k_m so that for $1 \leq i \leq m$, $|k_i/j - t_i| < \alpha/2$. It follows that

$$(15) \quad \mathbf{P}\left\{ \max_{1 \leq i \leq m} |U_{k_i}^\uparrow - t_i| \geq \alpha \right\} < \delta.$$

Write A, B and C for the events whose probabilities are bounded in (13), (14) and (15), respectively, and let $E = (A \cup B \cup C)^c$. Note that $\mathbf{P}\{E\} > 1 - 3\delta$. When E occurs, we have

$$\begin{aligned} & |F(((3n)^{-1/2}C_{T_n}(t_i), (4n/3)^{-1/4}Z_{T_n}(t_i))_{1 \leq i \leq m}) \\ & \quad - F(((3n)^{-1/2}C_{T_n}(U_{k_i}^\uparrow), (4n/3)^{-1/4}Z_{T_n}(U_{k_i}^\uparrow))_{1 \leq i \leq m})| \leq \delta \|F\|_{\text{lip}}. \end{aligned}$$

We thus have

$$(16) \quad \begin{aligned} & \mathbf{E}[F(((3n)^{-1/2}C_{T_n}(t_i), (4n/3)^{-1/4}Z_{T_n}(t_i))_{1 \leq i \leq m})] \\ & \quad - \mathbf{E}[F(((3n)^{-1/2}C_{T_n}(U_{k_i}^\uparrow), (4n/3)^{-1/4}Z_{T_n}(U_{k_i}^\uparrow))_{1 \leq i \leq m})] \\ & \quad \leq \delta \|F\|_{\text{lip}} + 6\delta \|F\|_\infty. \end{aligned}$$

Furthermore, when E occurs, $|\mathbf{e}(t_i) - \mathbf{e}(U_{k_i}^\uparrow)| + |Z(t_i) - Z(U_{k_i}^\uparrow)| < \delta$, thus we get similarly:

$$(17) \quad \begin{aligned} & \mathbf{E}[F((\mathbf{e}(t_i), Z(t_i))_{1 \leq i \leq m}) - F((\mathbf{e}(U_{k_i}^\uparrow), Z(U_{k_i}^\uparrow))_{1 \leq i \leq m})] \\ & \quad \leq \delta \|F\|_{\text{lip}} + 6\delta \|F\|_\infty. \end{aligned}$$

Finally, by Lemma 6.2, as $n \rightarrow \infty$,

$$\begin{aligned} & \mathbf{E}[F(((3n)^{-1/2}C_{T_n}(U_{k_i}^\uparrow), (4n/3)^{-1/4}Z_{T_n}(U_{k_i}^\uparrow))_{1 \leq i \leq m})] \\ & \quad \rightarrow \mathbf{E}[F((\mathbf{e}(U_{k_i}^\uparrow), Z(U_{k_i}^\uparrow))_{1 \leq i \leq m})] \end{aligned}$$

which, together with (16) and (17), implies (12). \square

The remainder of the section is devoted to proving Lemmas 6.2 and 6.3. Before proceeding to this, we state a definition which plays a key role. Given a probability

measure π on \mathbb{R}_k , its *symmetrization* $\hat{\pi}$ is obtained by permuting the marginals uniformly at random. More precisely, if (X_1, \dots, X_k) has law π and, independently, σ is a uniformly random permutation of $\{1, \dots, k\}$, then $(X_{\sigma(1)}, \dots, X_{\sigma(k)})$ has law $\hat{\pi}$.

For the remainder of Section 6, let μ be the law of the random variable B defined in (7). Also, for $k \geq 1$ let ν_k be as in Corollary 5.8, and let $\hat{\nu}_k$ be the symmetrization of ν_k . Note that since $\sum_i \int_{\mathbb{R}} x d\nu_k^i(x) = 0$, we have $\int_{\mathbb{R}} x d\hat{\nu}_k^i(x) = 0$ for each $1 \leq i \leq k$; in other words, $\hat{\nu} = (\hat{\nu}_k, k \geq 1)$ is locally centred. The proofs of Lemmas 6.2 and 6.3 both rely on couplings between $\text{LGW}(\mu, \nu, n)$ and $\text{LGW}(\mu, \hat{\nu}, n)$.

6.1. *Symmetrization of plane trees.* Fix a spatial plane tree $t = (t, \xi, d)$. For the remainder of the section it is convenient to conflate t and its Ulam–Harris encoding. This allows us to identify t with its vertex set; also, since with this coding the root vertex is always $\emptyset = v(\xi)$, we write $t = (t, d)$ instead of $t = (t, \xi, d)$.

Denote by $\mathfrak{S}(t)$ the set of vectors $\sigma = (\sigma^v : v \in t, k_t(v) > 0)$ indexed by the nonleaf vertices of t , with σ^v a permutation of $\{1, \dots, k_t(v)\}$. For $\sigma \in \mathfrak{S}(t)$, the *symmetrization of t with respect to σ* is the tree t^σ obtained from t by permuting the order of the subtrees rooted at the children of v according to σ^v , for each $v \in t$. More formally,

$$t^\sigma = \{\sigma(v), v \in t\},$$

where if $v = n_1 n_2 \dots n_k \in t$ then

$$\sigma(v) = \sigma^{\emptyset}(n_1) \sigma^{n_1}(n_2) \dots \sigma^{n_1 \dots n_{k-1}}(n_k).$$

We then let $t^\sigma = (t^\sigma, d^\sigma)$, where $d^\sigma(\sigma(u), \sigma(ui)) = d(u, ui)$ for all edges $\{u, ui\}$ of t . Visually, displacements are attached to edges, and follow their edges when the tree is permuted. Observe that t and t^σ are isomorphic as rooted edge-labelled trees (but need not be isomorphic as spatial plane trees). The local effect of symmetrization is depicted in Figure 7(a).

CLAIM 6.4. *Let $T = (T, D)$ have law $\text{LGW}(\mu, \nu, n)$, and let Σ be a uniformly random element of $\mathfrak{S}(T)$. Then T^Σ has law $\text{LGW}(\mu, \hat{\nu}, n)$.*

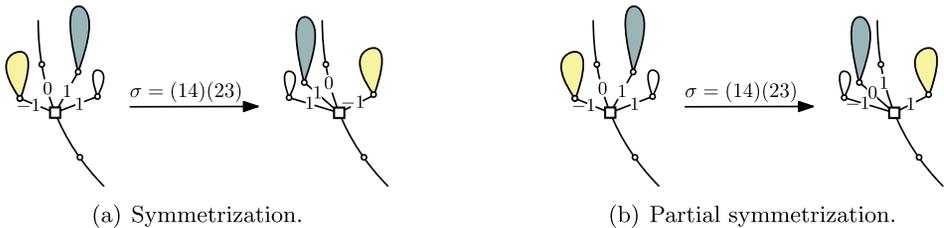


FIG. 7. Examples of the local rules for symmetrization and partial symmetrization at the square vertex [which in 7(b) is presumed to lie in $\text{path}(t, r)$].

PROOF. Since T and T^Σ are isomorphic as rooted trees, it follows from the branching property of Galton–Watson processes that they have the same law. The definition of \hat{v} , and the fact that Σ uniformly permutes labels at every vertex, then imply that T^Σ has law $\text{LGW}(\mu, \hat{v}, n)$. \square

COROLLARY 6.5. For $n \geq 1$ let $T_n = (T_n, D_n)$ be uniformly random in $\mathcal{T}_n^{v_1}$, and let Σ_n be a uniformly random element of $\mathfrak{S}(T_n)$. Then as $n \rightarrow \infty$,

$$(18) \quad ((3n)^{-1/2} C_{T_n^{\Sigma_n}}(t), (4n/3)^{-1/4} Z_{T_n^{\Sigma_n}}(t))_{0 \leq t \leq 1} \xrightarrow{d} (\mathbf{e}(t), Z(t))_{0 \leq t \leq 1},$$

for the topology of uniform convergence on $C([0, 1], \mathbb{R}^2)$.

PROOF. By Corollary 5.8, T_n has law $\text{LGW}(\mu, v, n)$, so $T_n^{\Sigma_n}$ has law $\text{LGW}(\mu, \hat{v}, n)$ by Claim 6.4. Since \hat{v} is bounded and locally centred and its marginals are uniformly distributed on $\{-1, 0, 1\}$, the result follows by (10), and the computation (see Appendix A) that $|\mu|_1 = 1$, $\sigma_\mu/2 = 3^{-1/2}$, and $\sigma_v = (2/3)^{1/2}$. \square

Now fix a vector r of vertices of t , and let $\text{path}(t, r)$ be the set of “path-points” of $t \langle r \rangle$: the vertices of $t \langle r \rangle$ that have at exactly one child in $t \langle r \rangle$. Write $\mathfrak{S}(t, r)$ for the set of vectors $\sigma = (\sigma^v : v \in \text{path}(t, r))$ with each σ^v a permutation of $\{1, \dots, k(v)\}$. Given $\sigma \in \mathfrak{S}(t, r)$, extend σ to a vector $\tau \in \mathfrak{S}(t)$ by setting

$$\tau^v = \begin{cases} \sigma^v, & \text{if } v \in \text{path}(t, r), \\ \text{Id}^{k(v)}, & \text{otherwise.} \end{cases}$$

Then the partial symmetrization of t with respect to r and σ is the labelled tree $\bar{t} = (\bar{t}, \bar{d})$ with vertices $\bar{t} = \{\tau(v), v \in t\} = t^\tau$ and displacements \bar{d} given by $\bar{d}(\tau(u), \tau(u)i) = d(u, ui)$ for all edges $\{u, ui\}$ of t . Visually, the vector $(d(v, vi), 1 \leq i \leq k(v))$ is now attached to the vertex v ; this vector follows the vertex when the tree is permuted, but does not change the order of its entries. The partial symmetrization depends on σ and on r , but we omit this from the notation. The local rule for partial symmetrization is illustrated in Figure 7(b), and Figure 8 contains an example of partial symmetrization of an entire tree.

In what follows, for $v \in t$ we write $\bar{v} = \tau(v)$ for the image of v under the partial symmetrization. If $r = (r_1, \dots, r_j)$, then we write $\bar{r} = (\bar{r}_1, \dots, \bar{r}_j)$. We also let $\bar{\sigma}$ be the pushforward of σ to $\text{path}(\bar{t}, \bar{r})$, so $\bar{\sigma}(\bar{v}) = \sigma(v)$ for $v \in \text{path}(\bar{t}, \bar{r})$. Note that we then have $(\bar{\sigma}(v) : v \in \text{path}(\bar{t}, \bar{r})) \in \mathfrak{S}(\bar{t}, \bar{r})$.

We remark that t and \bar{t} are isomorphic as rooted trees, but need not be isomorphic as plane trees or as labelled trees. Here are some comments regarding partial symmetrization:

- In forming \bar{t} , the order of the children at branchpoints of $t \langle r \rangle$ is not changed. This implies that $t \langle r \rangle$ and $\bar{t} \langle \bar{r} \rangle$ are isomorphic as plane trees.

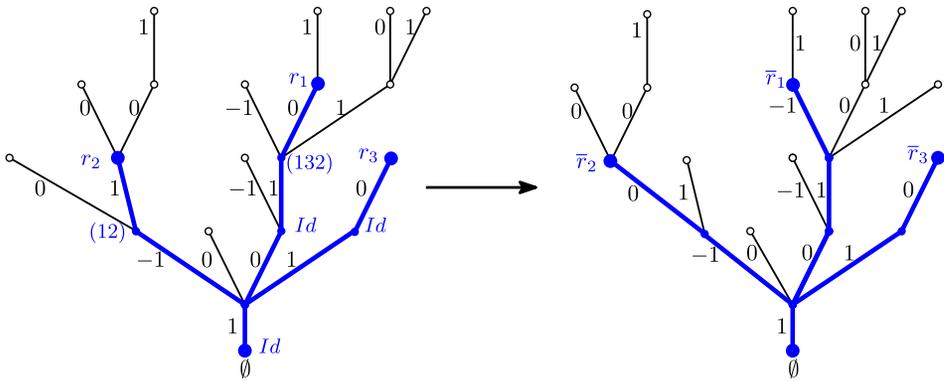


FIG. 8. Illustration of partial symmetrization of a tree $t = (t, d)$. The vertices in r are represented by bigger, solid blue disks. The edges of $t(r)$ are thicker and are blue. On the left, $\sigma \in \mathfrak{S}(t, r)$ is indicated by listing the permutation at each vertex of $\text{path}(t, r)$. On the right, the partial symmetrization \bar{t} of t with respect to r and σ is shown, as are the images of the elements of r .

- In particular, if $r = (r_1, \dots, r_k)$ is increasing with respect to lexicographic order in t then $\bar{r} = (\bar{r}_1, \dots, \bar{r}_k)$ is likewise ordered lexicographically in \bar{t} .
- Partial symmetrization is invertible: t, r and σ may be recovered from \bar{t}, \bar{r} and $\bar{\sigma}$. When we wish to make the dependencies of the symmetrization more explicit, we write $\text{sym}(t, r, \sigma)$ instead of $(\bar{t}, \bar{r}, \bar{\sigma})$.

PROPOSITION 6.6. Let $T = (T, D)$ have law $\text{LGW}(\mu, \nu, n)$, let $R = (R_1, \dots, R_j)$ be a vector of j independent and uniformly random vertices of T , and let $\Sigma = (\Sigma^v, v \in \text{path}(T, R))$ be a uniformly random element of $\mathfrak{S}(T, R)$. Write $\bar{T} = (\bar{T}, \bar{D})$ for the partial symmetrization of T with respect to Σ and R , and write $\bar{R} = (\bar{R}_1, \dots, \bar{R}_j)$ for the images of (R_1, \dots, R_j) in \bar{T} . Then (T, R) and (\bar{T}, \bar{R}) are identically distributed.

PROOF. Fix any pair $t' = (t', d') \in \mathcal{T}_n^{v_1}$, and any vector $r' = (r'_1, \dots, r'_j)$ of nodes of t' . Next, fix $\sigma' \in \mathfrak{S}(t', r')$, and let (t, r, σ) be the unique triple for which $\text{sym}(t, r, \sigma) = (t', r', \sigma')$. Then

$$\begin{aligned} \mathbf{P}\{(T, R, \Sigma) = (t, r, \sigma)\} &= \mathbf{P}\{T = t\} \cdot \frac{1}{n^j} \cdot \prod_{v \in \text{path}(t, r)} \frac{1}{k_t(v)!} \\ &= \mathbf{P}\{T = t'\} \cdot \frac{1}{n^j} \cdot \prod_{v' \in \text{path}(t', r')} \frac{1}{k_{t'}(v')!}. \end{aligned}$$

The second equality holds since t and t' are isomorphic as rooted trees and the vector of labels at each vertex of t is the same as at the one at its image in t' . Since

$\sigma' \in \mathfrak{S}(t', r')$ is arbitrary and $|\mathfrak{S}(t', r')| = \prod_{v' \in \text{path}(t', r')} k_{r'}(v')!$, it follows that

$$\mathbf{P}\{(\bar{T}, \bar{R}) = (t', r')\} = \frac{\mathbf{P}\{T = t'\}}{n^j},$$

as required. \square

COROLLARY 6.7. *Let $T = (T, D)$ and $R = (R_1, \dots, R_j)$ be as in Proposition 6.6. Let $\hat{T} = (\hat{T}, \hat{D})$ have law $\text{LGW}(\mu, \hat{v}, n)$ and let $\hat{R} = (\hat{R}_1, \dots, \hat{R}_j)$ be a vector of j uniformly random vertices of \hat{T} .*

Write (V_1, \dots, V_j) and $(\hat{V}_1, \dots, \hat{V}_j)$ for the lexicographic orderings of R and of \hat{R} . For $1 \leq i \leq j$, let

$$A_i = \sum_{v \in [\emptyset, V_i]: p(v) \in \text{path}(T, R)} D(p(v), v) \quad \text{and}$$

$$\hat{A}_i = \sum_{v \in [\emptyset, \hat{V}_i]: p(v) \in \text{path}(\hat{T}, \hat{R})} \hat{D}(p(v), v).$$

Then

$$(|V_1|, \dots, |V_j|, A_1, \dots, A_j) \stackrel{d}{=} (|\hat{V}_1|, \dots, |\hat{V}_j|, \hat{A}_1, \dots, \hat{A}_j).$$

PROOF. Let (\bar{T}, \bar{R}) be as in Proposition 6.6. If $\{p(u), u\}$ is an edge of T and $p(u) \in \text{path}(T, R)$ then the partial symmetrization uniformly permutes the children of $p(u)$. Since displacements are not permuted, and are independent on child edges of distinct vertices, it follows that the random variables

$$\{\bar{D}(p(\bar{u}), \bar{u}) : p(u) \in \text{path}(T, R)\} = \{\bar{D}(p(\bar{u}), \bar{u}) : p(\bar{u}) \in \text{path}(\bar{T}, \bar{R})\}$$

are independent and uniformly distributed on $\{-1, 0, 1\}$. The conclusion of Proposition 6.6 then implies the same holds for the random variables $\{D(p(u), u) : p(u) \in \text{path}(T, R)\}$.

Finally, the trees T and \hat{T} have the same law, so $(|V_1|, \dots, |V_j|) \stackrel{d}{=} (|\hat{V}_1|, \dots, |\hat{V}_j|)$. More strongly, the subtrees $T \langle R \rangle = T \langle V \rangle$ and $\hat{T} \langle \hat{R} \rangle = \hat{T} \langle \hat{V} \rangle$ are identically distributed. By the definition of \hat{v} , the displacements $\{\hat{D}(p(\hat{u}), \hat{u}) : p(\hat{u}) \in \text{path}(\hat{T}, \hat{R})\}$ are independent and uniform on $\{-1, 0, 1\}$, and the result follows. \square

6.2. Proof of Lemma 6.2. For $n \geq 1$ let $T_n = (T_n, D_n)$ have law $\text{LGW}(\mu, v, n)$. Fix $j \geq 1$, let U_1, \dots, U_j be independent Uniform $[0, 1]$ random variables independent of the trees T_n , and let $U_1^\uparrow, \dots, U_j^\uparrow$ be the increasing ordering of U_1, \dots, U_j .

For $1 \leq i \leq j$, let u_i be such that

$$\{r_{T_n}(\lfloor (2n - 2) \cdot U_i \rfloor), r_{T_n}(\lceil (2n - 2) \cdot U_i \rceil)\} = \{p(u_i), u_i\},$$

so that $\{p(u_i), u_i\} \in E(T_n)$ is the edge of T_n being traversed at time U_i by C_{T_n} .

Next, write $u_1^\uparrow, \dots, u_j^\uparrow$ for the lexicographic ordering of u_1, \dots, u_j . It is straightforward that if none of u_1, \dots, u_j is an ancestor of another, then the order statistics of u_1, \dots, u_j and of U_1, \dots, U_j coincide. In this case, for each $1 \leq i \leq j$, at time U_i^\uparrow the edge $\{p(u_i^\uparrow), u_i^\uparrow\}$ is being traversed by C_{T_n} . Furthermore, the probability that one of u_1, \dots, u_j is an ancestor of another is easily seen to tend to zero as $n \rightarrow \infty$.

Recalling the notation $|u_i| = d_{T_n}(\emptyset, u_i)$, now observe that $|C_{T_n}(U_i) - |u_i|| \leq 1$ and $|Z_{T_n}(U_i) - X_{T_n}(u_i)| \leq 1$ for all $1 \leq i \leq j$. If none of u_1, \dots, u_j is an ancestor of another, then it follows from the preceding paragraph that $|C_{T_n}(U_i^\uparrow) - |u_i^\uparrow|| \leq 1$ and $|Z_{T_n}(U_i^\uparrow) - X_{T_n}(u_i^\uparrow)| \leq 1$ for all $1 \leq i \leq j$. As this occurs with probability tending to one, to prove the lemma it suffices to show that

$$(19) \quad ((3n)^{-1/2}|u_i^\uparrow|, (4n/3)^{-1/4}X_{T_n}(u_i^\uparrow))_{1 \leq i \leq j} \xrightarrow{d} (\mathbf{e}(U_i^\uparrow), Z(U_i^\uparrow))_{1 \leq i \leq j}.$$

The elements of (u_1, \dots, u_j) are independent and uniformly distributed over $T_n \setminus \{\emptyset\}$. We may thus couple (u_1, \dots, u_j) with a sequence $\mathbf{W} = (w_1, \dots, w_j)$ of independent uniformly random elements of T_n so that $\mathbf{P}\{(u_1, \dots, u_j) \neq (w_1, \dots, w_j)\} \rightarrow 0$ as $n \rightarrow \infty$. [Here and below, we suppress the dependence of (w_1, \dots, w_j) on n for readability.] But if $(u_1, \dots, u_j) = (w_1, \dots, w_j)$ then the lexicographic reorderings of these vectors are also equal. Writing $(w_1^\uparrow, \dots, w_j^\uparrow)$ for the lexicographic ordering of (w_1, \dots, w_j) , it follows that replacing u_i^\uparrow by w_i^\uparrow for $1 \leq i \leq j$ does not affect the convergence (or lack thereof) in (19).

Next, for each $1 \leq i \leq j$ write $A_i = \sum_{v \in \llbracket \emptyset, w_i^\uparrow \rrbracket : p(v) \in \text{path}(T_n, \mathbf{W})} D_n(p(v), v)$. The tree $T_n \langle R \rangle$ has at most j leaves, so $|A_i - X_{T_n}(w_i^\uparrow)| \leq j - 1$. It follows that replacing $X_{T_n}(w_i^\uparrow)$ by A_i for each $1 \leq i \leq j$ likewise does not affect whether or not (19) converges in distribution. It thus suffices to establish the convergence

$$((3n)^{-1/2}|w_i^\uparrow|, (4n/3)^{-1/4}A_i)_{1 \leq i \leq j} \xrightarrow{d} (\mathbf{e}(U_i^\uparrow), Z(U_i^\uparrow))_{1 \leq i \leq j}.$$

Now let $\hat{T}_n = (\hat{T}_n, \hat{D}_n)$ have law $\text{LGW}(\mu, \hat{v}, n)$, let $\hat{\mathbf{W}} = (\hat{w}_1, \dots, \hat{w}_j)$ be uniformly random vertices of T_n and let $(\hat{w}_1^\uparrow, \dots, \hat{w}_j^\uparrow)$ be their lexicographic reordering. With $\hat{A}_i = \sum_{v \in \llbracket \emptyset, \hat{w}_i^\uparrow \rrbracket : p(v) \in \text{path}(\hat{T}_n, \hat{\mathbf{R}})} \hat{D}_n(p(v), v)$, Corollary 6.7 implies that we may replace w_i^\uparrow by \hat{w}_i^\uparrow and A_i by \hat{A}_i , without affecting distributional convergence. We may even replace A_i by $X_{\hat{T}_n}(\hat{w}_i^\uparrow)$, since $|\hat{A}_i - X_{\hat{T}_n}(\hat{w}_i^\uparrow)| \leq j - 1$.

In sum, by the above reductions, it suffices to prove that

$$(20) \quad ((3n)^{-1/2}|\hat{w}_i^\uparrow|, (4n/3)^{-1/4}X_{\hat{T}_n}(\hat{w}_i^\uparrow))_{1 \leq i \leq j} \xrightarrow{d} (\mathbf{e}(U_i^\uparrow), Z(U_i^\uparrow))_{1 \leq i \leq j}.$$

To accomplish this, we essentially reverse the above chain of reductions, and conclude by applying a known convergence result for globally centered snakes.

Let V_1, \dots, V_j be independent Uniform[0, 1] random variables independent of everything else and let $V_1^\uparrow, \dots, V_j^\uparrow$ be the increasing ordering of V_1, \dots, V_j . Then let v_i be such that $\{p(v_i), v_i\}$ is being traversed at time V_i by $C_{\hat{T}_n}$, and let $(v_1^\uparrow, \dots, v_j^\uparrow)$ be the lexicographic ordering of (v_1, \dots, v_j) .

Reprising the argument from the start of the proof, we see that with probability tending to one, for all $1 \leq i \leq j$ the edge $\{p(v_i^\uparrow), v_i^\uparrow\}$ is being traversed at time V_i^\uparrow . When this occurs, we have $|C_{\hat{T}_n}(V_i^\uparrow) - d_{\hat{T}_n}(\emptyset, v_i^\uparrow)| \leq 1$ and $|Z_{\hat{T}_n}(V_i^\uparrow) - X_{\hat{T}_n}(v_i^\uparrow)| \leq 1$. It then follows from Corollary 6.5 that

$$(21) \quad ((3n)^{-1/2}d_{\hat{T}_n}(\emptyset, v_i^\uparrow), (4n/3)^{-1/4}X_{\hat{T}_n}(v_i^\uparrow))_{1 \leq i \leq j} \xrightarrow{d} (\mathbf{e}(V_i^\uparrow), Z(V_i^\uparrow))_{1 \leq i \leq j}.$$

Finally, v_1, \dots, v_j are independent uniformly random nonroot vertices of \hat{T} , so the total variation distance between the laws of (v_1, \dots, v_j) and of $(\hat{w}_1, \dots, \hat{w}_j)$ tends to zero. It follows that the total variation distance between the laws of $(v_1^\uparrow, \dots, v_j^\uparrow)$ and $(\hat{w}_1^\uparrow, \dots, \hat{w}_j^\uparrow)$ also tends to zero, so we may replace v_i^\uparrow by \hat{w}_i^\uparrow in (21) without changing the limit. The right-hand sides of (20) and (21) are identically distributed, so this completes the proof.

6.3. *Proof of Lemma 6.3.* For $n \geq 1$, let $T_n = (T_n, D_n)$ have law LGW(μ, ν, n). Recall that Z_{T_n} is obtained from X_{T_n} by the identity $Z_{T_n}(i/(2n - 2)) = X_{T_n}(r(i))$ and by linear interpolation. We shall prove that for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(22) \quad \limsup_{n \rightarrow \infty} \mathbf{P} \left\{ \sup_{|i-j| \leq \delta n} |X_{T_n}(r(i)) - X_{T_n}(r(j))| > \varepsilon n^{1/4} \right\} < \varepsilon.$$

In the above supremum, it should be understood that we restrict to $i, j \in [2n - 2]$, but we omit this from the notation. Due to the relation between Z_{T_n} and X_{T_n} , this immediately implies tightness of the family laws of $(Z_{T_n}, n \geq 1)$, and so proves the lemma.

For each $n \geq 1$, let Σ_n be a uniformly random element of $\mathfrak{S}(T_n)$, and let \hat{T}_n be the symmetrization of T_n with respect to Σ_n . By Corollary 6.5, as $n \rightarrow \infty$,

$$(23) \quad ((3n)^{-1/2}C_{\hat{T}_n}(t), (4n/3)^{-1/4}Z_{\hat{T}_n}(t))_{0 \leq t \leq 1} \xrightarrow{d} (\mathbf{e}(t), Z(t))_{0 \leq t \leq 1},$$

for the topology of uniform convergence on $C([0, 1], \mathbb{R})^2$. It follows in particular that the family of laws of the processes $(Z_{\hat{T}_n}, n \geq 1)$ is tight. Since $Z_{\hat{T}_n}$ and $X_{\hat{T}_n}$ are related in the same way as Z_{T_n} and X_{T_n} , this implies that for all $\varepsilon > 0$ there exists $\alpha = \alpha(\varepsilon) > 0$ such that

$$(24) \quad \sup_{n \geq 1} \mathbf{P} \left\{ \sup_{|i-j| \leq \alpha n} |X_{\hat{T}_n}(\hat{r}(i)) - X_{\hat{T}_n}(\hat{r}(j))| > \varepsilon n^{1/4} \right\} < \varepsilon,$$

where we write \hat{r} for the contour exploration of \hat{T}_n . We also fix $\varepsilon > 0$ and let $\alpha = \alpha(\varepsilon)$ be small enough that (24) holds.

Recall from Section 1.1 the definition of the Brownian CRT $(\mathcal{T}_e, d_{\mathcal{T}_e})$. As noted in that section, the process Z can be seen as having domain $(\mathcal{T}_e, d_{\mathcal{T}_e})$ and remains a.s. uniformly continuous on this domain. It follows that for all $\varepsilon > 0$, there exists $\beta > 0$ such that

$$\mathbf{P}\{\sup\{|Z(x) - Z(y)|, \text{ for } x, y \in [0, 1] \text{ such that } d_{\mathcal{T}_e}(x, y) \leq \beta\} > \varepsilon\} < \varepsilon.$$

Together with the convergence in (23) and the relation between $Z_{\hat{T}_n}$ and $X_{\hat{T}_n}$, this implies that for all $\varepsilon > 0$ there exists $\beta = \beta(\varepsilon) > 0$ such that

$$(25) \quad \sup_{n \geq 1} \mathbf{P}\left\{ \sup_{u, v \in \hat{T}_n: d_{\hat{T}_n}(u, v) \leq \beta n^{1/2}} |X_{\hat{T}_n}(u) - X_{\hat{T}_n}(v)| > \varepsilon n^{1/4} \right\} < \varepsilon.$$

For $v \in T_n$, we write \hat{v} for the image of v in \hat{T}_n . The only subtlety of the proof is that v and \hat{v} may be visited at very different times in the contour explorations of T_n and \hat{T}_n . In other words, we typically do not have $\hat{r}(i) = r(\widehat{i})$. Observe, however, that for any i and j , the paths $\llbracket r(i), r(j) \rrbracket$ in T_n and $\llbracket r(\widehat{i}), r(\widehat{j}) \rrbracket$ in \hat{T}_n are identical: they have the same length and visit edges with the same labels, in the same order. In particular, we have

$$\begin{aligned} d_{T_n}(r(i), r(j)) &= d_{\hat{T}_n}(r(\widehat{i}), r(\widehat{j})), \\ X_{T_n}(r(i)) - X_{T_n}(r(j)) &= X_{\hat{T}_n}(r(\widehat{i})) - X_{\hat{T}_n}(r(\widehat{j})). \end{aligned}$$

Taking $\beta = \beta(\varepsilon)$ as above, (25), then yields that for all $n \geq 1$,

$$\begin{aligned} &\mathbf{P}\left\{ \sup_{|i-j| \leq \delta n} |X_{T_n}(r(i)) - X_{T_n}(r(j))| > \varepsilon n^{1/4} \right\} \\ &= \mathbf{P}\left\{ \sup_{|i-j| \leq \delta n} |X_{\hat{T}_n}(r(\widehat{i})) - X_{\hat{T}_n}(r(\widehat{j}))| > \varepsilon n^{1/4} \right\} \\ &\leq \mathbf{P}\{\exists u, v \in \hat{T}_n : d_{\hat{T}_n}(u, v) \leq \beta n^{1/2}, |X_{\hat{T}_n}(u) - X_{\hat{T}_n}(v)| \geq \varepsilon n^{1/4}\} \\ &\quad + \mathbf{P}\{\exists i, j : |i - j| \leq \delta n, d_{\hat{T}_n}(r(\widehat{i}), r(\widehat{j})) > \beta n^{1/2}\} \\ &\leq \varepsilon + \mathbf{P}\{\exists i, j : |i - j| \leq \delta n, d_{\hat{T}_n}(r(\widehat{i}), r(\widehat{j})) > \beta n^{1/2}\} \\ &= \varepsilon + \mathbf{P}\{\exists i, j : |i - j| \leq \delta n, d_{T_n}(r(i), r(j)) > \beta n^{1/2}\}. \end{aligned}$$

Now note that

$$\sup\{d_{T_n}(r(i), r(j)) : |i - j| \leq \delta n\} \leq 2 \sup\{|C_{T_n}(x) - C_{T_n}(y)| : |x - y| \leq \delta\}.$$

By the distributional convergence in (9) and the a.s. continuity of Brownian excursion, it follows that if $\delta > 0$ is sufficiently small then

$$\sup_n \mathbf{P}\{\exists i, j : |i - j| \leq \delta n, d_{T_n}(r(i), r(j)) > \beta n^{1/2}\} < \varepsilon.$$

For such δ , we then have

$$\sup_n \mathbf{P} \left\{ \sup_{|i-j| \leq \delta n} |X_{T_n}(r(i)) - X_{T_n}(r(j))| > \varepsilon n^{1/4} \right\} < 2\varepsilon,$$

which establishes (22) and completes the proof.

7. Blossoming trees, labelling, and distances. The goal of this section is to *deterministically* relate labels in a validly-labelled plane tree with the distances in the corresponding triangulation. For the remainder of Section 7, we fix $n \in \mathbb{N}$ and $(T, \hat{\xi}) \in \mathcal{T}_n$, let $\xi \in \mathcal{C}(T)$ be such that $T = (T, \xi)$ is balanced. Finally, let $G = (G, c) = \chi(T, \xi)$ and let $(T', \xi', D) = \phi_n(T, \hat{\xi})$.

Writing \mathcal{B} for the buds of T , we suppose throughout that $V(T') = V(T) \setminus \mathcal{B} = V(G) \setminus \{A, B\}$. Finally, define $Y = Y_T$ as in Section 5.3, and note that since T is balanced, $Y(v) \geq 2$ for all $v \in V(T)$. It will be useful to extend the domain of Y by setting $Y(A) = 1$ and $Y(B) = 2$, and we adopt this convention.

7.1. *Bounding distances using leftmost paths.* To warm up, we prove a basic lemma bounding the difference between labels of adjacent vertices.

LEMMA 7.1. *For all $\{u, w\} \in E(G)$, $|Y(u) - Y(w)| \leq 3$.*

PROOF. First, recall from page 2786 that if $u \in V(T)$ and $\{u, A\} \in E(G)$ or $\{u, B\} \in E(G)$ then there is a corner ζ incident to u with $\lambda(\zeta) \leq 3$, so $Y(u) \leq 3$. From this, if $\{u, w\} \cap \{A, B\} \neq \emptyset$ then the result is immediate. Next, if $\{u, v\} \in E(T)$ then it is an inner edge of T , in which case $Y(u) - Y(v) = D_{\{u,v\}}(T, \xi) \in \{-1, 0, 1\}$. Finally, if $\{u, w\} \notin E(T)$ but $u, w \in V(T)$ then there are corners c^1, c^2 of T such that $v(c^1) = u$, $v(c^2) = w$, and either $c^2 = s(c^1)$ or $c^1 = s(c^2)$. Assuming by symmetry that $c^2 = s(c^1)$, we have $\lambda(c^2) = \lambda_T(c^1) - 1$. Since the labels on corners incident to a single vertex differ by at most two, the result follows in this case. \square

The above lemma, though simple, already allows us to prove the labels provide a lower bound for the graph distance to A in G , up to a constant factor.

COROLLARY 7.2. *For all $u \in V(G)$, $d_G(u, A) \geq Y(u)/3$.*

PROOF. Let (u_0, u_1, \dots, u_l) be a shortest path from $u = u_0$ to $A = u_l$ in G . By Lemma 7.1, since $Y(A) = 1$ we have $Y(u) = |Y(u_0) - Y(u_l) - 1| < 3l$, so $d_G(u, A) = l \geq Y(u)/3$. \square

We next aim to prove a corresponding upper bound. For this, we use the leftmost paths briefly introduced in Section 1.3. Let $(G, c) = \chi(T, \xi)$ as above, and let \vec{E} be its unique minimal 3-orientation (defined in Section 5.5). Given an oriented

edge $e = uw$ with $\{u, w\} \in E$ and $x \in V(G)$, a *path* from e to x is a path $Q = (v_0, v_1, \dots, v_m)$ in G with $v_0v_1 = uw$ and $v_m = x$. (In the preceding, we do not require that $uw \in \vec{E}$.) Given $e = \{u_0, u_1\} \in E(G)$ with $u_0u_1 \in \vec{E}$, the *leftmost path* from e to A is the unique directed path $P(e) = P_{(G,c)}(e) = (u_0, u_1, \dots, u_\ell)$ with $u_\ell = A$ such that for each $1 \leq i \leq \ell - 1$, u_iu_{i+1} is the first outgoing edge incident to u_i when considering the edges incident to u_i in clockwise order starting from $\{u_{i-1}, u_i\}$. The following fact establishes two basic properties of leftmost paths.

FACT 7.3. For all $e \in E(G)$, $P(e)$ is a simple path. Furthermore, if $P(e) = (u_0, u_1, \dots, u_\ell)$ and $P(e') = (v_0, v_1, \dots, v_m)$ are distinct leftmost paths to A with $u_0 = v_0 = u$, and $u_i = v_j$ for some $i, j > 0$, then $u_{i+k} = v_{j+k}$ for all $0 \leq k \leq \ell - i = m - j$.

PROOF. Let $P(e) = (u_0, u_1, \dots, u_\ell)$ be the leftmost path from u_0u_1 to A . Suppose there are $0 \leq i < j \leq \ell$ such that $u_i = u_j$, and choose such i, j for which $|j - i|$ is minimum. Then $C = (u_i, u_{i+1}, \dots, u_j)$ is an oriented cycle with $j - i$ vertices; let $V' \subset V(G)$ be the vertices lying on or to the right of this cycle. Since \vec{E} is minimal, C is necessarily a clockwise cycle, so $v(c) \notin V'$. Also, neither A nor B are in any directed cycles, and it follows that $\{A, B, v(c)\} \cap V' = \emptyset$. Since \vec{E} is a 3-orientation it follows that for all $x \in V'$, $\deg_{\vec{E}}^+(x) = 3$. Furthermore, for all $x \in V' \setminus \{u_i\}$, since $P(e)$ is a leftmost path, all out-neighbours of x are elements of V' . Writing G' for the sub-map of G induced by V' , it follows that $|E(G')| \geq 3|V' \setminus \{u_i\}|$. But G' is a simple planar map, and C is a face of G' of degree $j - i \geq 3$. It follows by Euler's formula that $|E(G')| \leq 3|V'| - 3 - (j - i) \leq 3|V' \setminus \{u_i\}| - 3$, a contradiction.

The proof that two leftmost paths merge if they meet after their starting point follows the same lines and is left to the reader. \square

The next proposition provides a key connection between the corner labelling $\lambda = \lambda_T$ and the lengths of leftmost paths.

PROPOSITION 7.4. For any edge $e = \{u, w\} \in E(G)$ with $uw \in \vec{E}$ and $u \neq B$, $\lambda(\kappa^\ell(u, w)) = |P(e)|$.

PROOF. First, a simple counting argument shows that if $\{x, y_1\}$ is an inner edge of T , and $\{x, y_2\}$ is the first stem following $\{x, y_1\}$ in clockwise order around x , then writing ζ for the corner of T incident to y_2 we have $\lambda(\kappa^r(x, y_1)) = \lambda(\zeta)$. Recall the definition of the *successor* function s from (6) and the equivalent definition from Section 5.2. Since $\{x, y_2\}$ is a stem, in G , y_2 is identified with $s(\zeta)$, and by definition $\lambda(s(\zeta)) = \lambda(c) - 1$.

Next, recall the definition of the labelling $\lambda^* = \lambda_T^* : \mathcal{C}(G) \rightarrow \mathbb{Z}^{\geq 0}$ from the end of Section 5.2. It follows from that definition that for any oriented edge $xy \in \vec{E}$, $\lambda^*(\kappa^r(y, x)) = \lambda^*(\kappa^\ell(x, y)) - 1$. In other words, the label on the left decreases by exactly one when following any oriented edge.

Now write $P(e) = (u_0, u_1, \dots, u_\ell)$. Since there are no edges oriented away from $P(e)$ leaving $P(e)$ to the left, it follows from the two preceding paragraphs that for $0 < i < \ell$ we have $\lambda^*(\kappa^r(u_{i+1}, u_i)) = \lambda^*(\kappa^\ell(u_i, u_{i+1})) - 1 = \lambda^*(\kappa^r(u_i, u_{i-1})) - 1$, so

$$\lambda^*(\kappa^r(u_\ell, u_{\ell-1})) = \lambda^*(\kappa^\ell(u_0, u_1)) - \ell = \lambda^*(\kappa^\ell(u, w)) - \ell.$$

Finally, $\lambda^*(\kappa^r(u_\ell, u_{\ell-1})) = 1$ by definition since $u_\ell = A$ and $u_{\ell-1} \neq v(\xi)$. We thus obtain $\lambda^*(\kappa^\ell(u, w)) = \ell + 1 = |P(e)|$. \square

COROLLARY 7.5. *For all $u \in V(G)$, $d_G(u, A) \leq Y(u) - 1$.*

PROOF. Recall the convention that $Y(B) = 2$ and $Y(A) = 1$; since also $Y(v(\xi)) = 2$, it suffices to prove the result for $u \in V(G) \setminus \{A, B, v(\xi)\}$. For such u , if $\{u, w\}$ is the first stem incident to u in clockwise order around u starting from $\{u, p(u)\}$, then $Y(u) = \lambda(\kappa^\ell(u, w))$. The claim then follows from Proposition 7.4. \square

7.2. Bounding distances between two points using modified leftmost paths. In this section, we use arguments similar to those of the preceding section; this time to prove deterministic upper bounds on pairwise distances in G . Fix two inner corners ζ_1, ζ_2 of T , and define

$$\check{Y}_T(\zeta_1, \zeta_2) = \min\{Y_T(w) : \exists \zeta \in [\zeta_1, \zeta_2]_{\text{cyc}}, w = v(\zeta)\}.$$

PROPOSITION 7.6. *For all $u, v \in V(G) \setminus \{A, B\}$, and for any corners ζ_u, ζ_v of T respectively incident to u and v , we have*

$$d_G(u, v) \leq Y_T(u) + Y_T(v) - 2 \max\{\check{Y}(\zeta_u, \zeta_v), \check{Y}(\zeta_v, \zeta_u)\} + 6.$$

Before proving the proposition, we establish some preliminary results. Given an edge $e = u_0u_1$ with $\{u_0, u_1\} \in E(T)$ for which $u_0 \notin B(T)$, we define the *modified leftmost path* from e to A to be the unique (not necessarily oriented) path $Q(e) = (u_0, u_1, \dots, u_\ell)$ in G with $u_\ell = A$ and such that for each $1 \leq i \leq \ell - 1$, u_iu_{i+1} is the first edge (considering the edges incident to u_i in clockwise order starting from $\{u_{i-1}, u_i\}$) which is either an outgoing edge (with respect to the orientation \vec{E}) incident to u_i or an *inner edge* of T . Equivalently, it is the leftmost oriented path, with the modified orientation obtained by viewing edges of $E(T)$ as unoriented (or as oriented in both directions).

We view $Q(e)$ as an oriented path from e to A (though the edge orientations given by the path need not agree with \vec{E}); we may thus speak of the left-hand and right-hand side of $Q(e)$.

FACT 7.7. For $1 \leq i \leq \ell - 1$, $\lambda^*(\kappa^\ell(u_i, u_{i+1})) = \lambda^*(\kappa^\ell(u_{i-1}, u_i)) - 1$. In other words, the labels along the left of a modified leftmost path decrease by one along each edge.

PROOF. First, by the definitions of λ and λ^* , for any edge $\{u_{i-1}, u_i\}$ of a modified leftmost path, $\lambda^*(\kappa^r(u_i, u_{i-1})) = \lambda^*(\kappa^\ell(u_{i-1}, u_i)) - 1$. Moreover, from the definition of a modified leftmost path, there is no stem incident to u_i in T that lies strictly between $\{u_{i-1}, u_i\}$ and $\{u_i, u_{i+1}\}$ (in clockwise order around u_i starting from $\{u_{i-1}, u_i\}$). Hence, $\kappa^\ell(u_i, u_{i+1}) = \kappa^r(u_i, u_{i-1})$ in T (see the proof of Proposition 7.4 for more details). The result follows. \square

Given $\{u, v\} \in E(G)$, if $\{u, v\} \notin E(T)$ and $\{u, v\} \neq \{A, B\}$ then by symmetry we may assume there is an edge $\{u, b\} \in E(T)$ such that $v = v(s(b))$. In this case, by a slight abuse of notation we write $\kappa^\ell(u, v) = \kappa^\ell(u, b)$.

In the statement and proof of the next fact, write $L = \lambda^*(\kappa^\ell(u_0, u_1))$ and let $M = \min\{\lambda(\xi) : \xi \in \mathcal{C}(T), \kappa^\ell(u_0, u_1) \preceq_{\text{ctr}} \xi\}$, where we view $\{u_0, u_1\}$ as an edge of $E(T)$. By the discussion on page 2786, $M \in \{2, 3\}$ and $M = 3$ precisely if $\bar{\xi} \preceq_{\text{ctr}} \kappa^\ell(u_0, u_1)$, where $\bar{\xi}$ is the unique element of $\mathcal{C}(T) \setminus \{\xi\}$ for which $(T, \bar{\xi})$ is balanced.

Let $c_e^*(0) = \kappa^\ell(u_0, u_1)$, and for $1 \leq j \leq L - M$ let $c_e^*(j)$ be the first corner ζ following $c_e^*(0)$ in T for which $\lambda(\zeta) = L - j$. For $1 \leq j \leq L - M$, $c_e^*(j)$ is necessarily an inner corner of T .

FACT 7.8. For all $0 \leq j \leq L - M$, $c_e^*(j) = \kappa^\ell(u_j, u_{j+1})$, so $v_T(c_e^*(j)) = u_j \in Q(e)$.

Before giving the proof, observe that this property need not hold for a regular leftmost path; this is the reason we require modified leftmost paths.

PROOF OF FACT 7.8. For $j = 0$, this holds by definition; we now fix $j \geq 1$ and argue by induction. The definition of λ yields that $c_e^*(j) \preceq_{\text{ctr}, T} c_e^*(j + 1)$, for any $0 \leq j < L - 2$. We consider two cases. First, suppose $\{u_{j-1}, u_j\}$ is an inner edge of T . Let $w \in V(T)$ be such that $\kappa^r(u_j, u_{j-1}) = (\{u_{j-1}, u_j\}, \{u_j, w\})$. If w is an inner vertex, then $w = u_{j+1}$. Likewise, if w is a blossom then $v(s(w)) = u_{j+1}$. In either case, $\kappa^\ell(u_j, u_{j+1}) = \kappa^r(u_j, u_{j-1})$ in T . Hence, $\kappa^\ell(u_j, u_{j+1})$ is the corner immediately following $\kappa^\ell(u_{j-1}, u_j)$ in the contour exploration of T . By Fact 7.7, $\lambda^*(\kappa^\ell(u_j, u_{j+1})) = \lambda^*(\kappa^\ell(u_{j-1}, u_j)) - 1$, and $c_e^*(j - 1) = \kappa^\ell(u_{j-1}, u_j)$ by the inductive hypothesis. It follows that $c_e^*(j) = \kappa^\ell(u_j, u_{j+1})$.

Second, suppose $\{u_{j-1}, u_j\}$ is not an inner edge. By definition, there is no edge in T incident to u_j and lying strictly between $\{u_{j-1}, u_j\}$ and $\{u_j, u_{j+1}\}$ in clockwise order around u_j . Hence, in T , $s(u_{j-1}) = \kappa^\ell(u_j, u_{j+1})$. In this case, the result follows by the definition of $s(u_{j-1})$ and by induction. \square

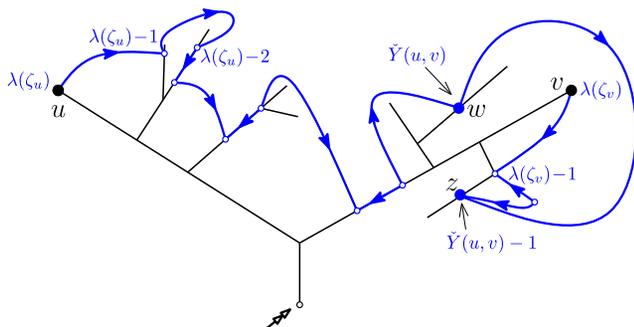


FIG. 9. Path between u and v formed by concatenating sections of two modified leftmost paths. Arrows indicate orientation in \vec{E} . Straight arrows along the path are edges of $E(T)$; curved arrows are edges of $E(G) \setminus E(T)$.

PROOF OF PROPOSITION 7.6. By symmetry, we may assume that $\zeta_u \leq_{\text{ctr},T} \zeta_v$. Let s_u and s_v be the edges lying to the right of these corners in T , and let e_u and e_v be the corresponding edges in G . Write $Q(e_u) = (u_0, u_1, \dots, u_{\lambda(\zeta_u)-1})$ with $u_0 = u$ and $e_u = \{u_0, u_1\}$, and likewise write $Q(e_v) = (v_0, v_1, \dots, v_{\lambda(\zeta_v)-1})$. Observe that $\lambda^*(\kappa^\ell(u_0, u_1)) = \lambda(\zeta_u)$ and likewise $\lambda^*(\kappa^\ell(u_0, u_1)) = \lambda(\zeta_u)$.

We assume for simplicity that $\check{Y}(u, v) > 3$ [when $\check{Y}(u, v) \leq 3$ there is a minor case analysis involving the presence of vertices A and B in $Q(e_u)$ and $Q(e_v)$; the details are straightforward and we omit them]. By Fact 7.8 and the definition of $\check{Y}(u, v)$, necessarily $c_{e_u}(\lambda(\zeta_u) - \check{Y}(u, v) + 1)$ and $c_{e_v}^*(\lambda(\zeta_v) - \check{Y}(u, v) + 1)$ are incident to the same vertex $z = u_{\lambda(\zeta_u) - \check{Y}(u, v) + 1} = v_{\lambda(\zeta_v) - \check{Y}(u, v) + 1}$. Let P be the concatenation of the subpath of $Q(e_u)$ from $u = u_0$ to z with the subpath of $Q(e_v)$ from z to $v_0 = v$ (see Figure 9 for an illustration). Then P connects u and v in G , so

$$\begin{aligned} d_G(u, v) &\leq |P| - 1 \\ &= \lambda(\zeta_u) + \lambda(\zeta_v) - 2\check{Y}(\zeta_u, \zeta_v) + 2 \\ &\leq Y(u) + Y(v) - 2\check{Y}(\zeta_u, \zeta_v) + 6, \end{aligned}$$

where the last inequality comes from the fact that $|\lambda(\zeta_u) - Y(u)| \leq 2$ and $|\lambda(\zeta_v) - Y(v)| \leq 2$. A symmetric argument proves the existence of a path P' in G between v and u of length $\lambda(\zeta_u) + \lambda(\zeta_v) - 2\check{Y}(v, u) + 6$; this gives the desired bound. \square

7.3. Winding numbers and distance lower bounds. It turns out that the lower bound on $d_G(u, A)$ given by Corollary 7.2 can be improved by considering winding numbers around u ; we now remind the reader of their definition.

Consider a closed curve $\gamma : [0, 1] \rightarrow \mathbb{R}^2 \setminus \{0\}$, and parametrize γ in polar coordinates as $((r(t), \theta(t)), 0 \leq t \leq 1)$ so that θ is a continuous function. We define the *winding number* of γ around zero to be $(\theta(1) - \theta(0))/(2\pi)$. Next, fix a reference point $r \in \mathbb{S}^2$. For $x \in \mathbb{S}^2 \setminus \{r\}$ and a closed curve $\gamma : [0, 1] \rightarrow \mathbb{S}^2 \setminus \{r, x\}$,

let $\varphi : \mathbb{S}^2 \setminus \{r\}$ to \mathbb{R}^2 be a homeomorphism with $\varphi(x) = 0$, and define the winding number $\text{wind}_r(x, \gamma)$ of γ around x to be the winding number of $\varphi \circ \gamma : [0, 1] \rightarrow \mathbb{R}^2$ around zero. It is straightforward that this definition does not depend on the choice of $\varphi(x)$.

In what follows, it is useful to imagine having chosen a particular representative from the equivalence class of G , or in other words a particular planar embedding (it is straightforward to verify that the coming arguments do not depend on which embedding is chosen). Let r be any point in the interior of the face of G incident to c .

DEFINITION 7.9. Fix an oriented edge $e = uw \in \vec{E}$ and a simple path $Q = (v_0, v_1, \dots, v_m)$ from u to A . Define the *winding number* $w(Q, e) = w_G(Q, e)$ of Q around e as follows. Write $P(e) = (u_0, u_1, \dots, u_\ell)$. Note that $u_0 = v_0 = u$, $u_1 = w$ and $u_\ell = v_m = A$. Form a cycle $C = (v_0, v_1, \dots, v_m = u_\ell, u_{\ell-1}, \dots, u_0)$. Then fix a point x in the interior of the face incident to $\kappa^r(u, w)$, and let $w(Q, e) = \text{wind}_r(x, C)$.

In the preceding definition, we conflate C with its image in \mathbb{S}^2 under the embedding of G (and likewise with x); it is straightforward to verify that $w(Q, e)$ does not depend on the choice of such embeddings.

PROPOSITION 7.10. For all $e = uw \in \vec{E}$, if Q is a simple path from u to A then $|Q| \geq |P(e)| + 2(w(Q, e) - 2)$.

PROOF. Write $P(e) = (u_0, u_1, \dots, u_\ell)$. Let $R = (w_0, w_1, \dots, w_k)$ be a simple path meeting $P(e)$ only at w_0 and w_k , with $w_0 = u_i$, $w_k = u_j$ for some $0 \leq i < j \leq \ell$. If $j < \ell$ then let $\hat{c} = \kappa^r(u_j, u_{j+1})$ and if $j = \ell$ (so $u_j = A$) then let \hat{c} be the corner of the root face of (G, c) incident to A .

We say R leaves $P(e)$ from the right if $i > 0$ and the corner $\kappa_G^r(u_i, u_{i+1})$ precedes $\kappa_G^r(u_i, w_1)$ in clockwise order around u_i starting from $\kappa_G^r(u_i, u_{i-1})$. Otherwise say that R leaves $P(e)$ from the left; in particular, if $i = 0$ then R leaves from the left by convention. Likewise, R returns to $P(e)$ from the right if \hat{c} precedes $\kappa^r(u_j, w_{k-1})$ in clockwise order around u_j starting from $\kappa^r(u_j, u_{j-1})$; otherwise say that R returns to $P(e)$ from the left.

The key to the proof is the following set of inequalities. Note that $k = |R| - 1$:

- (1) If R leaves $P(e)$ from the right and returns from the left, then $k \geq j - i - 2$.
- (2) If R leaves $P(e)$ from the left and returns from the left, then $k \geq j - i$.
- (3) If R leaves $P(e)$ from the left and returns from the right, then $k \geq j - i - 1 + 2(\mathbf{1}_{[i>0]} + \mathbf{1}_{[j<\ell]})$.
- (4) If R leaves $P(e)$ from the right and returns from the right, then $k \geq j - i - 1 + 2\mathbf{1}_{[j<\ell]}$.

For later use, we say R has *type 1* if R leaves $P(e)$ from the right and returns from the left, and define types 2, 3 and 4, accordingly. Let $C = (w_0, \dots, w_k =$

u_j, \dots, u_i) be the cycle contained in the union of R and $P(e)$. We provide the details of the bounds from (1) and (3), as (2) and (4) are respectively similar.

Note that although C does not respect the orientation of edges given by \vec{E} , it is nonetheless an oriented cycle, so it makes sense to speak of the right- and left-hand sides of C . For (1), let V' be the set of vertices on or to the right of C , and let G' be the submap of G induced by V' . All faces of G' have degree three except C , which has degree $k + j - i$. By Euler's formula, it follows that $|E(G')| = 3|V'| - 3 - (k + j - i)$.

For $i < m < j$, since $P(e)$ is a leftmost path, $|\{x \in V' : u_m x \in \vec{E}\}| = 1$. Also, since R returns from the left, we must have $w_{k-1} u_j \in \vec{E}$ [or else $w_{k-1} = u_{j+1}$, which contradicts that $P(e)$ meets R only at its endpoints], so $|\{x \in V' : u_j x \in \vec{E}\}| = 0$. Since \vec{E} is a 3-orientation, it follows that $E(G') \leq 3|V'| - 2(j - i) - 1$, which combined with the equality of the preceding paragraph yields that $k \geq j - i - 2$.

For (3), let V' be the set of vertices on or to the left of C . Euler's formula again yields $|E(G')| = 3|V'| - 3 - (k + j - i)$. For $x \in V'$ not lying on C , we have $x \notin \{A, B, v(c)\}$, so since \vec{E} is a 3-orientation, $|\{y \in V' : xy \in \vec{E}\}| = 3$. For $i < m < j - 1$, we have $m < \ell - 1$, so u_m is not on the root face; since R returns from the right, it follows that $|\{y \in V' : u_m y \in \vec{E}\}| = 3$. Lastly, $|\{x \in V' : u_{j-1} x \in \vec{E}\}| \geq 1$ since u_{j-1}, u_j lies on C , and likewise $|\{x \in V' : u_i x \in \vec{E}\}| \geq 1$. The edges of R are disjoint from the sets of edges counted above, so

$$|E(G')| \geq 3|V' \setminus \{w_1, \dots, w_k, u_i, u_{j-1}\}| + 2 + k = 3|V'| - 2k - 4.$$

Combined with the equality given by Euler's formula, this yields $k \geq (j - i) - 1$. Next, since $P(e)$ is leftmost, $u_m \notin \{A, B, v(c)\}$ for $m < \ell - 1$, which is straightforwardly seen. Thus, if $j < \ell$ then since \vec{E} is a 3-orientation, we in fact have $|\{x \in V' : u_{j-1} x \in \vec{E}\}| = 3$, and the same counting argument yields that $k \geq (j - i) + 1$. Similarly, if $i > 0$ then $|\{x \in V' : u_i x \in \vec{E}\}| = 3$ and again $k \geq (j - i) + 1$. Finally, if $0 < i < j < \ell$ then the same argument yields $k \geq (j - i) + 3$.

To conclude, subdivide the path Q into edge-disjoint sub-paths R_1, \dots, R_t , each of which is either a sub-path of $P(e)$ or else meets $P(e)$ only at its endpoints. We assume R_1, \dots, R_t are ordered so that Q is the concatenation of R_1, \dots, R_t , so in particular, $u = u_0$ is the first vertex of R_1 , $A = u_\ell$ is the last vertex of R_t , and for $1 \leq s < t$ the last vertex of R_s is the first vertex of R_{s+1} .

For $1 \leq i \leq 4$, let n_i be the number of sub-paths of type i among $\{R_1, \dots, R_t\}$. Since R_t is the only sub-path that intersects the root face, and R_1 is the only sub-path which may contain $u = u_0$, the above inequalities and a telescoping sum give

$$\begin{aligned} |Q| &= 1 + \sum_{s=1}^t (|R_s| - 1) \\ &\geq |P(e)| - 2n_1 + 3n_3 + n_4 - 2(\mathbf{1}_{[R_1 \text{ has type } 3]} + \mathbf{1}_{[R_t \text{ has type } 3 \text{ or } 4]}). \end{aligned}$$

In particular, we obtain the bound $|Q| \geq |P(e)| + 2(n_3 - n_1 - 2)$. Finally, sub-paths that leave from the right and return from the left correspond to clockwise windings of C around u , and subpaths that leave from the left and return from the right correspond to counterclockwise windings of C around u . It follows that $n_3 - n_1$ is precisely the winding number $w(Q, e)$; this completes the proof. \square

In what follows, if C is an oriented cycle in G then we write $V^l(C)$ [resp., $V^r(C)$] for the sets of vertices lying on or to the left (resp., on or to the right) of C , and note that $V^l(C) \cap V^r(C) = V(C)$.

PROPOSITION 7.11. *For all $e = uw \in \vec{E}$, if Q is a shortest path from u to A and $w(Q, e) < 0$ then there is a cycle C in G such that $G[V^l(C)]$ and $G[V^r(C)]$ each have diameter at least $\lfloor -w(Q, e)/2 \rfloor - 2$, and such that $\max_{y \in V^h(C)} Y_T(y) - \min_{y \in V^h(C)} Y_T(y) \geq \lfloor -w(Q, e)/2 \rfloor$ for $h \in \{l, r\}$.*

PROOF. We write $Q = (u_0, u_1, \dots, u_\ell)$, and partition Q into edge-disjoint sub-paths R_1, \dots, R_t as at the end of the proof of Proposition 7.10. For $1 \leq s \leq t$ and $1 \leq i \leq 4$, let $n_i(s)$ be the number of sub-paths of type i among $\{R_1, \dots, R_s\}$. If $u_0 u_1 \neq e$, then by definition R_1 leaves $P(e)$ from the left, so $n_1(1) = 0$.

Let $m = \lfloor -w(Q, e)/2 \rfloor$, and let s be minimal so that $n_1(s) - n_3(s) = m$; necessarily, R_s has type 1. Also, $s \geq m + 1$, and since $n_1(t) - n_3(t) = -w(Q, e) \geq 2m$ we also have $m \leq t - m$. Write $R_s = (w_0, w_1, \dots, w_k)$, with $w_0 = u_i$, $w_1 = u_j$ for distinct $i, j \in \{1, \dots, \ell\}$. By reversing R_s if necessary, we may assume $i < j$,⁸ and write $C = (w_0, w_1, \dots, w_k, u_{j-1}, \dots, u_i = w_0)$. Since $m + 1 \leq s \leq t - m$, the concatenation of R_1, \dots, R_{s-1} has length at least m and so does the concatenation of R_{s+1}, \dots, R_t . Since Q is a shortest path from u to A , it follows that for $0 \leq i \leq k$, $d_G(u, w_i) \geq m + i + 1$ and $d_G(A, w_i) \geq m + (k - i)$.

Fix $0 < a < j - i$ and let S be a shortest path from u to u_{i+a} . The concatenation of $S, (u_{i+a}, \dots, u_j)$, and R_{s+1}, \dots, R_t has $d_G(u, u_{i+a}) + (j - i - a) + d_G(u_j, A)$ edges. On the other hand, by the inequality in (1) from the proof of Proposition 7.10, we have $k \geq j - i - 2$, so Q has at least $d_G(u, u_i) + j - i - 2 + d_G(u_j, A)$ edges. Since Q is a shortest path, it follows that

$$d_G(u, u_{i+a}) \geq d_G(u, u_i) + a - 2 \geq m + a - 2 \geq m - 1.$$

A similar argument shows that for all $0 < a < j - i$, $d_G(A, u_{i+a}) \geq m - 1$. Finally, one of $G[V^l(C)]$ or $G[V^r(C)]$ contains R_1, \dots, R_s , and the other contains R_s, \dots, R_t . Therefore, each of $G[V^l(C)]$ and $G[V^r(C)]$ contains at least m vertices of $P(e)$; since vertex labels strictly decrease along $P(e)$, the final claim of the proposition follows. \square

⁸It is not hard to prove that there is always some shortest path Q for which the ordered sequence of intersections with $P(e)$ respect the orientation of $P(e)$, so that there is no need to reverse R_s to ensure $i < j$. However, we do not require such a property for the current proof.

PROPOSITION 7.12. For all $e = uw \in \vec{E}$, if Q is a shortest path from u to A and $w(Q, e) < -2$ then there is an oriented cycle C in G of length at most $6(|Q| - 1)/(-w(Q, e) - 2)$ such that $G[V^l(C)]$ and $G[V^r(C)]$ each have diameter at least $\lfloor -w(Q, e)/3 \rfloor - 2$ and such that $\max_{y \in V^h(C)} Y_T(y) - \min_{y \in V^h(C)} Y_T(y) \geq \lfloor -w(Q, e)/3 \rfloor$ for $h \in \{l, r\}$.

PROOF. The proof is very similar to that of Proposition 7.11, so we omit most details. Partition Q into R_1, \dots, R_t and define $n_i(s)$, $1 \leq s \leq t$, $1 \leq i \leq 4$ as before. Let $m = \lfloor -w(Q, e)/3 \rfloor$. There are at least m values of s such that R_s has type 1 and $m + 1 \leq n_1(s) - n_3(s) \leq 2m$; among these, let s^* minimize $|R_{s^*}|$. Then

$$d_G(u, A) \geq m \cdot (|R_{s^*}| - 1) \geq \frac{-(w(Q, e) + 2)}{3} \cdot (|R_{s^*}| - 1).$$

The sub-path of $P(e)$ joining the endpoints of R_{s^*} has at most two more edges than R_{s^*} , so the cycle formed by this sub-path of $P(e)$ and R_{s^*} has at most $2|R_{s^*}| \leq 6(d_G(w, A) + 1)/(-w(Q, e) - 2) = 6(|Q| - 1)/(-w(Q, e) - 2)$ vertices. The remainder of the proof closely follows that of Proposition 7.11. \square

8. Labels approximate distances for random triangulations. Fix $n \in \mathbb{N}$, and let $(T, \hat{\xi})$ be uniformly distributed in \mathcal{T}_n . [We will later take $n \rightarrow \infty$, but suppress the dependence of $(T, \hat{\xi})$ on n for readability.] Define $(T', \xi', D) = \phi_n(T, \hat{\xi})$. Next, as in Corollary 5.7, let $\xi^1, \xi^2 \in \mathcal{C}(T)$ be such that (T, ξ^i) is balanced for $i \in \{1, 2\}$. Conditionally given $(T, \hat{\xi})$, choose $\xi \in \{\xi_1, \xi_2\}$ uniformly at random. Write $T = (T, \xi)$ and define $G = (G, c) = \chi(T)$. By Corollary 5.7, (G, c) is uniformly distributed in Δ_n° and $T' = (T', \xi', D')$ is uniformly distributed in \mathcal{T}_n^{vl} .

Again define $Y = Y_T$ as in Section 5.3 and again extend Y to $V(G)$ by taking $Y(A) = 1$ and $Y(B) = 2$. We note that, since the function ϕ_n identifies T' as a subtree of T , $Y(v)$ is defined for $v \in V(T')$.

Using Corollary 7.5 and Proposition 7.10, we now show that with high probability, the labelling $Y : V(G) \rightarrow \mathbb{Z}^{\geq 0}$ gives distances to A in G up to a uniform $o(n^{1/4})$ correction.

THEOREM 8.1. For all $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbf{P}\{\exists u \in V(G) : d_G(u, A) \notin [Y(u) - \varepsilon n^{1/4}, Y(u) - 1]\} = 0.$$

The upper bound $d_G(u, A) \leq Y(u) - 1$ holds deterministically by Corollary 7.5. To prove the lower bound (in probability), we begin by stating a lemma whose proof, postponed to the end of the section, is based on soft convergence arguments and the continuity of the Brownian snake. Recall the definition of the contour exploration $(r_T(j), 0 \leq j \leq 2n - 2)$. Given $0 \leq i \leq 2n - 2 = 2|V(T)| - 2$ and $\Delta > 0$, let

$$g_T(i, \Delta) = \sup\{j < i : |Y(r_T(j)) - Y(r_T(i))| \geq \Delta \text{ or } j = 0\},$$

$$d_T(i, \Delta) = \inf\{j > i : |Y(r_T(j)) - Y(r_T(i))| \geq \Delta \text{ or } j = 2n - 2\}.$$

Then let $N(i, \Delta) = \{v \in V(T) : \exists g_T(i, \delta) \leq j \leq d_T(i, \Delta), r_T(j) = v\}$ be the set of vertices of T visited by the contour exploration between times $g_T(i, \Delta)$ and $d_T(i, \Delta)$.

LEMMA 8.2. *For all $\varepsilon > 0$ and $\beta > 0$, there exist $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that for $n \geq n_0$,*

$$\mathbf{P}\{\inf\{|N(i, \beta n^{1/4})| : 0 \leq i \leq 2n - 2\} \geq \alpha n\} \geq 1 - \varepsilon.$$

PROOF OF THEOREM 8.1. As mentioned, we need only prove the lower bound. It suffices to show that for all $\varepsilon > 0$,

$$\limsup_{n \rightarrow \infty} \mathbf{P}\{\exists e = uv \in \vec{E} : d_G(u, A) < Y_T(u) - 6(\varepsilon n^{1/4} + 2)\} \leq 4\varepsilon$$

(we have done a little anticipatory selection of constants in the preceding formula). Write $\text{diam}(G)$ for greatest distance between any two vertices of G . By Corollary 7.5, $\text{diam}(G) \leq 2 \max_{u \in V(G)} (Y_T(u) - 1) = 2(\max_{u \in V(T)} Y_T(u) - \min_{u \in V(T)} Y_T(u)) + 2$, so by Fact 5.9, $\text{diam}(G) \leq \max_{u \in V(T')} X_{(T', \xi', D)}(u) - \min_{u \in V(T')} X_{(T', \xi', D)}(u) + 8$. Finally,

$$\max_{u \in V(T')} X_{T'}(u) - \min_{u \in V(T')} X_{T'}(u) = \max_{x \in [0, 1]} Z_{T'}(x) - \min_{x \in [0, 1]} Z_{T'}(x),$$

and Proposition 6.1 implies that $(\max_{x \in [0, 1]} Z_{T'}(x) - \min_{x \in [0, 1]} Z_{T'}(x))n^{-1/4}$ converges in distribution as $n = |V(T')| \rightarrow \infty$, to an almost surely finite random variable. It follows that there is $y = y(\varepsilon) > 0$ such that $\mathbf{P}\{\text{diam}(G) \geq yn^{1/4}\} < \varepsilon$. Choose such y , and let B be the event that G contains a cycle C of length at most $2y/\varepsilon$ such that with $V^l(C)$ and $V^r(C)$ as defined earlier, for $h \in \{l, r\}$ we have

$$\max_{u \in V^h(C)} Y_T(u) - \min_{u \in V^h(C)} Y_T(u) \geq \varepsilon n^{1/4}.$$

Next, suppose there exists $e = uv \in \vec{E}$ for which $d_G(u, A) < Y_T(u) - 6(\varepsilon n^{1/4} + 2)$. Fix such an edge e , and any shortest path Q from u to A ; by Proposition 7.10 we have $w(Q, e) \leq -3\varepsilon n^{1/4} - 2$. It follows from Proposition 7.12 that either $\text{diam}(G) \geq yn^{1/4}$ or else B occurs. It thus suffices to show that

$$(26) \quad \mathbf{P}\{B, \text{diam}(G) \leq yn^{1/4}\} \leq 3\varepsilon.$$

Suppose B occurs, let C be as in the definition of B , and let F be the subgraph of T induced by $V(T) \setminus V(C)$. Then F is a forest, and each component of F is contained within $G[V^l(C)]$ or $G[V^r(C)]$ since T is a subgraph of G . Also, for $\{u, w\} \in E(G)$ we have $|Y_T(u) - Y_T(w)| \leq 1$. It follows that, for $h \in \{l, r\}$, if $G[V^h(C)]$ contains k components of F then one such component T_h must have

$$\max_{u \in V(T_h)} Y_T(u) - \min_{u \in V(T_h)} Y_T(u) > \varepsilon n^{1/4} / k - 1.$$

But F has at most $|E(C)| \leq 2(|Q| - 1)/(\varepsilon n^{1/4})$ connected components. When $\text{diam}(G) \leq yn^{1/4}$ we have $2(|Q| - 1)/(\varepsilon n^{1/4}) \leq 2y/\varepsilon$, so for each $h \in \{l, r\}$, some component T_h of F contained in $G[V^h(C)]$ must have

$$\max_{u \in V(T_h)} Y_T(u) - \min_{u \in V(T_h)} Y_T(u) \geq \frac{\varepsilon^2 n^{1/4}}{2y} - 1.$$

Using again that labels of adjacent vertices differ by at most one, if $\text{diam}(G) \leq yn^{1/4}$ then for $h \in \{l, r\}$ there is $v_h \in V^h(C)$ such that

$$\min_{v \in V(C)} |Y(v_h) - Y_T(v)| \geq \frac{\varepsilon^2 n^{1/4}}{4y} - \frac{1}{2} - \frac{2y}{\varepsilon}.$$

Now for $h \in \{l, r\}$ let $j_h = j_h(T) = \inf\{0 \leq i \leq 2n - 2 : r_T(i) = v_h\}$. Also, fix any $\beta \in (0, \varepsilon^2/2y)$. By Lemma 8.2 there is $\alpha > 0$ such that for n sufficiently large,

$$\mathbf{P}\{\min(N(j_l, \beta n^{1/4}), N(j_r, \beta n^{1/4})) \leq \alpha n\} \leq \varepsilon.$$

For n large enough that $\varepsilon^2 n^{1/4}/(4y) - 1/2 - 2y/\varepsilon > \beta n^{1/4}$, for $h \in \{l, r\}$ we also have $N(j_h, \beta n^{1/4}) \subset V^h(C)$, and it follows that for n sufficiently large

$$(27) \quad \mathbf{P}\{B, \text{diam}(G) \leq yn^{1/4}\} \leq 2\varepsilon + \mathbf{P}\left\{\exists C \text{ a cycle in } G, |C| \leq \frac{2y}{\varepsilon}, \min(|V^l(C)|, |V^r(C)|) \geq \alpha n\right\}.$$

The event in the last probability is that G contains a separating cycle of length at most $2y/\varepsilon$ that separates G into two subtriangulations, each of size at least αn . The number $t_{n,m}$ of simple triangulations of an $(m + 2)$ -gon with n inner vertices has been computed in [9], and has the asymptotic form $t_{n,m} \sim A_m \alpha^n n^{-5/2}$, where A_m and α are explicit constants. (Observe that, in this notation, the number of rooted simple triangulations with n vertices is equal to $t_{n-3,1}$.) For $K \in \mathbb{N}$ and $\alpha > 0$, denote by $\Gamma_K(\alpha)$ the event that a random simple triangulation with n vertices admits a separating cycle γ_n of length at most K that separates G_n into two components each of size at least αn . Then

$$(28) \quad \mathbf{P}\{\Gamma_K(\alpha)\} \sim (t_{n-3,1})^{-1} \sum_{k=1}^{K-2} \int_{\alpha}^{1-\alpha} t_{\lfloor un \rfloor, k} t_{\lfloor (1-u)n \rfloor, k} du \sim A_{K,\alpha} n^{-5/2},$$

where $A_{K,\alpha}$ depends only on α and K . The event within the last probability in (27) is contained within the event $\Gamma_{\lceil 2y/\varepsilon \rceil}(\alpha)$, so for n sufficiently large its probability is at most ε . In view of (26), this completes the proof. \square

PROOF OF LEMMA 8.2. Fix $\varepsilon > 0$ and $\beta > 0$ as in the statement of the lemma. List the elements of $V(T'_n)$ according to lexicographic order in T_n as $v_n(1), \dots, v_n(n)$, and for $1 \leq m \leq n$ let $i_n(m) = \inf\{i \geq 0 : r_{T_n}(i) = v_n(m)\}$.

By considering the height process, a straightforward argument (almost identical that given for equations (12) and (13) of [19]) shows that

$$\sup_{0 \leq t \leq 1} \left| \frac{i_n(\lfloor tn \rfloor)}{2n-2} - t \right| \xrightarrow{d} 0$$

from which it follows that for any $\delta > 0$,

$$\mathbf{P} \left\{ \frac{\inf\{|N(i, \beta n^{1/4})| : 0 \leq i \leq 2n-2\} + \delta n}{\inf\{d_{T_n}(i, \beta n^{1/4}) - g_{T_n}(i, \beta n^{1/4}) : 0 \leq i \leq 2n-2\}} < \frac{1}{2} \right\} \rightarrow 0.$$

In particular, given $\alpha > 0$, for n large, if $d_{T_n}(i, \beta n^{1/4}) - g_{T_n}(i, \beta n^{1/4}) > \alpha n$ for all $0 \leq i \leq 2n-2$ then with high probability $\inf\{|N(i, \beta n^{1/4})| : 0 \leq i \leq 2n-2\} > \alpha n/3$. It therefore suffices to prove there exists $\alpha > 0$ such that for all n sufficiently large,

$$(29) \quad \mathbf{P}\{\inf\{d_{T_n}(i, \beta n^{1/4}) - g_{T_n}(i, \beta n^{1/4}) : 0 \leq i \leq 2n-2\} \geq \alpha n\} > 1 - \varepsilon.$$

By Proposition 6.1 and Skorohod’s embedding theorem, we now work in a space in which

$$(30) \quad ((3n)^{-1/2}C_{T_n}(t), (4n/3)^{-1/4}Z_{T_n}(t))_{0 \leq t \leq 1} \xrightarrow{\text{a.s.}} (\mathbf{e}(t), Z(t))_{0 \leq t \leq 1}.$$

Let $A = A(Z) = \inf\{|x - y| : x, y \in [0, 1], |Z(x) - Z(y)| > \beta/(2 \cdot (4/3)^{1/4})\}$, or let $A(Z) = 1$ if the set in the preceding infimum is empty. When $(d_{T_n}(i, \beta n^{1/4}) - g_{T_n}(i, \beta n^{1/4}))/ (2n-2) < 1$ either $d_{T_n}(i, \beta n^{1/4}) \neq 0$ or $g_{T_n}(i, \beta n^{1/4}) \neq 2j-2$, so either $Z_{T_n}(d_{T_n}(i, \beta n^{1/4}))/ (2n-2) - Z_{T_n}(i/(2n-2)) > \beta n^{1/4}$ or $Z_{T_n}(i/(2n-2)) - Z_{T_n}(g_{T_n}(i, \beta n^{1/4}))/ (2n-2) > \beta n^{1/4}$. By (30), it follows that a.s.

$$(2n-2)^{-1} \cdot \inf\{d_{T_n}(i, \beta n^{1/4}) - g_{T_n}(i, \beta n^{1/4}) : 0 \leq i \leq 2n-2\} > A$$

for all n sufficiently large. Finally, since Z is a.s. uniformly continuous on $[0, 1]$, almost surely $A > 0$, and (29) follows immediately. \square

9. The proof of Theorem 1.1 for triangulations. For each $n \in \mathbb{N}$, construct a map encoding $P_n = (M_n, T_n)$ as follows. Let $(T, \hat{\xi})$ be uniformly random in \mathcal{T}_n and let $T_n = (T_n, \xi_n, D_n) := \phi_n(T, \hat{\xi})$. Next let $\xi^1, \xi^2 \in \mathcal{C}(T)$ be such that (T, ξ^i) is balanced for $i \in \{1, 2\}$. Conditionally given $(T, \hat{\xi})$ choose $\xi \in \{\xi^1, \xi^2\}$ uniformly at random. Then let $M_n = (M_n, \zeta_n) := \chi_n(T, \xi)$.

To prove Theorem 1.1 for triangulations, we verify that $\mathbf{P} = (P_n, n \in \mathbb{N})$ is a good sequence of map encodings, with sequences $a_n = (3n)^{-1/2}$ and $b_n = (4n/3)^{-1/4}$. Assuming this, to conclude note that, by Corollary 5.7, (M_n, ζ_n) is a uniformly random element of Δ_{n+2}° . Since $b_{n+2}/b_n \rightarrow 1$ as $n \rightarrow \infty$, the result then follows from Theorem 4.1.

In what follows, our usage of the notation C_n, Z_n, X_n and r_n agrees with that of Section 4. By Proposition 5.4 and Corollaries 5.7 and 5.8, T_n has law $\text{LGW}(\mu, \nu, n)$, where the law of μ is given by (7) and $\nu = (\nu_k, k \geq 1)$ is as in

Corollary 5.8. Condition 1 then holds by Proposition 6.1. Condition 2(i) is immediate from the construction as M_n contains only two vertices (A and B) that are not elements of $V(T_n)$.

Next, recall that $m = m(n) = 2|V(T_n)| - 2$. For $u, v \in V(M_n) \setminus \{A, B\}$, let i and j be such that $u = r_n(i)$ and $v = r_n(j)$. Then, combining Fact 5.9 and Proposition 7.6 [with $\zeta_u = \xi_{T_n}(i)$ and $\zeta_v = \xi_{T_n}(j)$] yields:

$d_{M_n}(u, v) \leq Z_n(i/m) + Z_n(j/m) - 2 \max(\check{Z}_n(i/m, j/m), \check{Z}_n(j/m, i/m)) + 18$, where the additive constant 18 arises from the 6 in Proposition 7.6, plus four times the additive error of 3 from Fact 5.9. It follows that for all $0 \leq i, j \leq m$,

$$d_{M_n}(r_n(i), r_n(j)) \leq Z_n(i/m) + Z_n(j/m) - 2 \max(\check{Z}_n(i/m, j/m), \check{Z}_n(j/m, i/m)) + 18,$$

which verifies 3(i). Condition 3(ii) follows directly from Theorem 8.1. It remains to establish 2(ii). Since $X_n(\xi_n) = 0$, it follows from (3) that

$$b_n \cdot |d_{M_n}(\zeta_n, \xi_n) + \check{Z}_n(0, 1)| \xrightarrow{d} 0,$$

so by 1,

$$(31) \quad b_n d_{M_n}(\zeta_n, \xi_n) \xrightarrow{d} -\check{Z}(0, 1) \stackrel{d}{=} Z(V) - \check{Z}(0, 1),$$

where $V \stackrel{d}{=} \text{Uniform}[0, 1]$ is independent of Z ; the last equality in distribution is from (2). Now let V_n be a uniformly random element of $V(T_n)$. Arguing from (3) and 1 as above, we obtain

$$(32) \quad b_n d_{M_n}(V_n, \zeta_n) \xrightarrow{d} Z(V) - \check{Z}(0, 1).$$

Next, recall that (M_n, ζ_n) is uniformly random in Δ_{n+2}° . It follows that, conditionally given M_n , ζ_n is a uniformly random element of $\mathcal{C}(M_n)$; let c_n be another uniformly random element of $\mathcal{C}(M_n)$, independent of ζ_n and of V_n . It follows that

$$(33) \quad d_{M_n}(V_n, \zeta_n) \stackrel{d}{=} d_{M_n}(V_n, c_n).$$

Let \vec{E} be the minimal 3-orientation associated to M_n . Writing $c_n = (\{x_n, y_n\}, \{y_n, z_n\})$, let $\tilde{v}(c_n) = y_n$ if $y_n z_n \in \vec{E}$, and $\tilde{v}(c_n) = z_n$ otherwise. Note that $\tilde{v}(c_n)$ is either equal to or incident to $v(c_n)$, so $|d_{M_n}(V_n, v(c_n)) - d_{M_n}(V_n, \tilde{v}(c_n))| \leq 1$. Further, since c_n is a uniformly random corner of M_n , $\{y_n, z_n\}$ is a uniformly random edge of M_n , so for all $v \in V(M_n)$, $\mathbf{P}\{\tilde{v}(c_n) = v\}$ is proportional to the outdegree of v in \vec{E} . Since all inner vertices of M_n have outdegree 3 in \vec{E} , and c_n is independent of V_n , we may couple $\tilde{v}(c_n)$ with a uniformly random element U_n of $V(T_n)$, independent of V_n , such that $\mathbf{P}\{U_n \neq \tilde{v}(c_n)\} \rightarrow 0$ as $n \rightarrow \infty$. Furthermore, $d_{M_n}(v(c_n), U_n) \leq 1$ on $\{U_n = \tilde{v}(c_n)\}$, so $b_n d_{M_n}(c_n, U_n) \rightarrow 0$ in probability, as $n \rightarrow \infty$. It then follows from (32) and (33) that

$$b_n d_{M_n}(V_n, U_n) \xrightarrow{d} Z(V) - \check{Z}(0, 1).$$

With (31), this establishes 2(ii) and completes the proof.

10. The proof of Theorem 1.1 for quadrangulations. The results on which the proof for simple triangulations rely all have nearly exact analogues for simple quadrangulations, which makes the proof for quadrangulations quite straightforward. In this section, we state the required results, with an emphasis on the details that differ between the two cases.

10.1. *Simple quadrangulations and blossoming trees.* The counterpart of the bijection between simple triangulations and 2-blossoming trees is a bijection between simple quadrangulations and 1-blossoming trees, due to Fusy [14]. In this section, by “blossoming trees” we mean 1-blossoming trees, and write $\mathcal{T}_{\square,n}$ for the set of blossoming trees with n inner vertices. Fix a blossoming tree T . Given a stem $\{b, u\}$ with $b \in \mathcal{B}(T)$, if bu is followed by *three* inner edges in a clockwise contour exploration of T — uv , vw and wz , say—then the *local closure* of $\{b, u\}$ consists in removing the blossom b [from both $V(T)$ and \mathcal{B}] and its stem, and adding a new edge $\{u, z\}$.

After all local closures have been performed, all unclosed blossoms are incident to a single face f . A simple counting argument shows that there exist exactly two edges $\{u, v\}$ and $\{x, y\}$ of f such that u, v, x and y are each incident to one unclosed stem; between any two other consecutive unclosed stems, there are two edges of f . Assume by symmetry that f lies to the left of both uv and xy , and write $\xi_C = \kappa^r(v, u)$, $\xi_D = \kappa^r(y, x)$, $C = v(\xi_C)$ and $D = v(\xi_D)$ [see Figure 10(b)].

Given $\xi \in \mathcal{C}(T)$, the planted blossoming tree (T, ξ) is *balanced* if $\xi = \xi_C$ or $\xi = \xi_D$. Suppose $\xi \in \{\xi_C, \xi_D\}$ and write $v = v(\xi)$. Let S_{CD} (resp. S_{DC}) be the set of nonblossom vertices u incident to an unclosed blossom in the partial closure, such that in the planted tree (T, C) [resp., (T, D)] we have $C \leq_{\text{ctr}} v <_{\text{ctr}} D$ (resp., $D \leq_{\text{ctr}} v <_{\text{ctr}} C$).

To finish the construction, remove the remaining blossoms and their stems. Add two additional vertices A and B within the outer face, and an edge between A

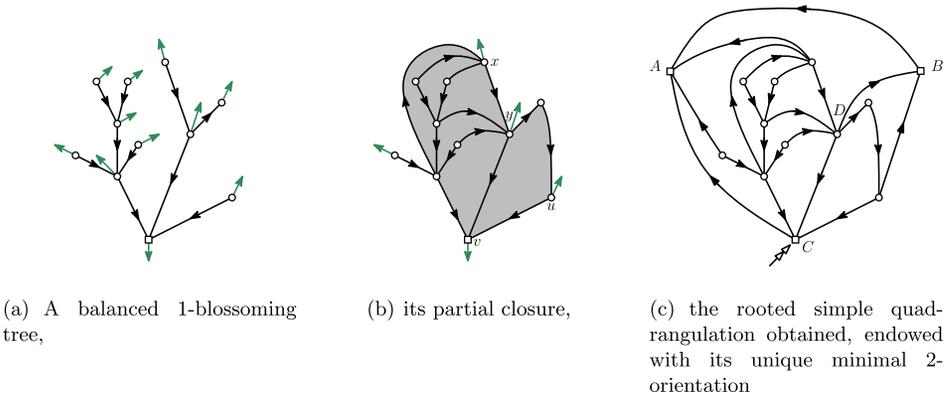


FIG. 10. The closure of a balanced 1-blossoming tree into a simple quadrangulation.

(resp., B) and each of the vertices of S_{CD} (resp., of S_{DC}). In the resulting map, define a corner c by $c = (\{C, B\}, \{C, A\})$ if $v = C$ or $c = (\{D, A\}, \{D, B\})$ if $v = D$. Finally, add an edge between A and B in such a way that, after its addition, c lies on the same face as A, B , and v [see Figure 10(c)]. Write $\chi_\square(T)$ for the resulting map.

Fix a planted planar quadrangulation (Q, ξ) , and view (Q, ξ) as embedded in \mathbb{R}^2 so that the face containing ξ is the unique unbounded face. A 2-orientation of a (Q, ξ) is an orientation for which $\alpha(v) = 2$ for each vertex v not incident to the root face and, listing the vertices of the unbounded face in clockwise order as v, A, B, w with $v = v(\xi)$, we have $\alpha(A) = 0, \alpha(B) = \alpha(v) = 1$ and $\alpha(w) = 2$. Write \vec{E}_\square for the resulting quadrangulation. Ossona de Mendez [30] showed that a quadrangulation admits a 2-orientation if and only if it is simple, and in this case admits a unique minimal 2-orientation.

PROPOSITION 10.1 ([14]). *The closure operation $\chi_{\square,n}$ is a bijection between the set $\mathcal{T}_{\square,n}^\circ$ of balanced 1-blossoming trees with n inner vertices and the set \square_{n+2}° of planted quadrangulations with $n + 2$ vertices. Furthermore, for $T \in \mathcal{T}_{\square,n}$, $\chi_{\square,n}(T)$ is naturally endowed with its minimal 2-orientation by viewing stems of T as oriented towards blossoms, and all other edges as oriented towards the root.*

10.2. *Sampling simple quadrangulations.* Given a blossoming tree $T_\square = (T, \xi)$, define $\lambda_\square := \lambda_{\square, T_\square} : \mathcal{C}(T) \rightarrow \mathbb{Z}$ as follows. Let $(\xi_\square(i), 0 \leq i \leq 2|V(T)| - 2)$ be the contour ordering from Section 2.2, with $\xi_0 = \xi$. Let $\lambda_\square(\xi_0) = 2$ and, for $0 \leq i < 2|V(T)| - 3$, set

$$\lambda_\square(\xi_\square(i + 1)) = \begin{cases} \lambda_\square(\xi_\square(i)) - 1, & \text{if } \xi_\square(i) \notin \mathcal{B}, \xi_\square(i + 1) \notin \mathcal{B}, \\ \lambda_\square(\xi_\square(i)), & \text{if } \xi_\square(i) \notin \mathcal{B}, \xi_\square(i + 1) \in \mathcal{B}, \\ \lambda_\square(\xi_\square(i)) + 2, & \text{if } \xi_\square(i) \in \mathcal{B}, \xi_\square(i + 1) \notin \mathcal{B}. \end{cases}$$

As opposed to Section 5.2, here the label increases by 2 after each stem.

It is not hard to see that $T_\square = (T, \xi)$ is balanced if and only if ξ is incident to one stem and $\lambda_\square(c) \geq 2$ for all $c \in \mathcal{C}(T)$. With the same definition of successors for corners, and the same construction as in Section 5.2, this labelling yields another description of the bijection from Section 10.1. Let $\mathcal{T}_{\square,n}^{v1}$ be the set of triples (T, ξ', d) where (T, ξ') is a planted plane tree and $d = (d_e, e \in E(T))$ is a ± 1 labelling of $E(T)$ such that for all $v \in V(T)$, listing the edges from v to its children in lexicographic order as e_1, \dots, e_k , the sequence d_{e_1}, \dots, d_{e_k} is nondecreasing.

Let $X_\square \stackrel{d}{=} \text{Geometric}(2/3)$, and let B_\square have law given by

$$(34) \quad \mathbf{P}\{B_\square = k\} = \frac{(k + 1)\mathbf{P}\{X_\square = k\}}{\mathbf{E}(X_\square + 1)}, \quad \text{for } k \in \mathbb{N}.$$

The following is the analogue of Corollary 5.8 for quadrangulations.

PROPOSITION 10.2. *Let (T', ξ') be a Galton–Watson tree with offspring distribution B_{\square} conditioned to have n vertices. Conditionally given (T', ξ') , independently for each $v \in V(T')$, list the children of v in clockwise order as v_1, \dots, v_k and let $(D_{\{v, v_j\}}, 1 \leq j \leq k)$ be a random vector with law $\nu_{\square, k}$, where $\nu_{\square, k}$ is the uniform law over nondecreasing vectors $(d_1, \dots, d_k) \in \{-1, 1\}^k$. Finally, let $D = (D_e, e \in E(T'))$. Then (T', ξ', D) is uniformly distributed in $\mathcal{T}_{\square, n}^{\text{vl}}$ and the closure $\chi_{\square, n}(T', \xi', D)$ is uniformly distributed in \square_{n+2}° .*

The proof of Proposition 6.1 extends immediately to this setting and we obtain the following convergence (see Appendix A for the computation of the constants).

PROPOSITION 10.3. *For $n \in \mathbb{N}$, let $T_n = (T_n, \xi_n, D_n)$ be a uniformly random element of $\mathcal{T}_{\square, n}^{\text{vl}}$. Then as $n \rightarrow \infty$,*

$$(35) \quad \left(\frac{3}{4n^{1/2}} C_{T_n}(t), \left(\frac{3}{8n} \right)^{1/4} Z_{T_n}(t) \right)_{0 \leq t \leq 1} \xrightarrow{d} (\mathbf{e}(t), Z(t))_{0 \leq t \leq 1},$$

for the topology of uniform convergence on $C([0, 1], \mathbb{R})^2$.

10.3. *Labels and distance in simple quadrangulations.* We next state analogues of the results of Sections 7 and 8 for quadrangulations. Fix $n \in \mathbb{N}$ and $(T, \xi) \in \mathcal{T}_{\square, n}^{\circ}$, let $(Q, c) = \chi_{\square, n}(T, \xi)$ be endowed with its minimal 2-orientation \vec{E}_{\square} and let $(T', \xi', D) \in \mathcal{T}_{\square, n}^{\text{vl}}$ be the validly-labelled tree associated to (T, ξ) . Finally, write $Q = (Q, c)$ and $T_{\square} = (T, \xi)$.

The definition of leftmost paths for simple quadrangulations is an obvious modification of that for triangulations. Together with the fact that (with Y_T defined as before) for $\{u, w\} \in E(Q)$, $|Y_T(u) - Y_T(w)| \leq 3$, we obtain the following facts. The lemma is a counterpart of Lemma 7.1 and Corollary 7.5; the proposition is a counterpart of Proposition 7.6, and uses an identical definition for $\check{Y}_{T_{\square}}(u, v)$.

LEMMA 10.4. *For all $u \in V(Q)$, $Y_{T_{\square}}(u)/3 \leq d_Q(u, A) \leq Y_{T_{\square}}(u) - 1$.*

PROPOSITION 10.5. *For all $u, v \in V(Q)$,*

$$d_Q(u, v) \leq Y_{T_{\square}}(u) + Y_{T_{\square}}(v) - 2 \max\{\check{Y}_{T_{\square}}(u, v), \check{Y}_{T_{\square}}(v, u)\} + 2.$$

The winding number introduced in Definition 7.9 is used in the following analogue of Proposition 7.10.

PROPOSITION 10.6. *For all $e = uw \in \vec{E}_{\square}$, if Q is a simple path from e to A then $|Q| \geq |P(e)| + 2(w(Q, e) - 1)$.*

PROOF. The proof of Proposition 7.10 extends readily to the case of quadrangulations. Keeping the same notation, the following inequalities (proofs of which are left to the reader) allow one to conclude along the same lines:

- (1) If R leaves $P(e)$ from the right and returns from the left, then $k \geq j - i - 2$.
- (2) If R leaves $P(e)$ from the left and returns from the left, then $k \geq j - i$.
- (3) If R leaves $P(e)$ from the left and returns from the right, then $k \geq j - i + 2(\mathbf{1}_{[i>0]} + \mathbf{1}_{[j<\ell]})$.
- (4) If R leaves $P(e)$ from the right and returns from the right, then $k \geq j - i + 2\mathbf{1}_{[j<\ell]}$. \square

Combining Lemma 10.4 and Proposition 10.6, we obtain that with probability tending to one, distances to A in Q are given by labels in T_\square up to a $o(n^{1/4})$ perturbation.

THEOREM 10.7. For all $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbf{P}\{\exists u \in V(Q) : d_Q(u, A) \notin [Y_{T_\square}(u) - \varepsilon n^{1/4}, Y_{T_\square}(u) - 1]\} = 0.$$

PROOF. The only element of the proof of Theorem 8.1 that cannot be directly applied here is the approximation of $\mathbf{P}\{\Gamma_K(\alpha)\}$ given in (36) that relies on the number $t_{n,m}$ of simple triangulations of an $(m + 2)$ -gon. This has an easy fix: for $\alpha > 0$, write $\Gamma_{\square,K}(\alpha)$ for the event that a uniformly random simple quadrangulation Q_n with n faces admits a separating cycle of length at most K , separating Q_n into two components each of size at least αn . An explicit expression for the number $q_{n,m}$ of simple quadrangulations of a $2m$ -gon with n inner vertices is derived in [10], and has the asymptotic form $q_{n,m} \sim A_m \alpha^n n^{-5/2}$, where A_m and α are explicit constants. (Observe that, in this notation, the number of rooted simple quadrangulations with n vertices is equal to $q_{n-4,2}$.) Then

$$(36) \quad \mathbf{P}\{\Gamma_K(\alpha)\} \sim (q_{n-4,2})^{-1} \sum_{k=0}^K \int_{\alpha}^{1-\alpha} t_{\lfloor un \rfloor, k} t_{\lfloor (1-u)n \rfloor, k} du \sim A_{\square, K, \alpha} n^{-5/2},$$

where $A_{\square, K, \alpha}$ depends only on α and K . \square

APPENDIX A: NOTES ABOUT CONSTANTS

In this section, we briefly derive the forms of the constant coefficients arising in Theorem 1.1 and Proposition 6.1.

For simple triangulations, we work with a critical Galton–Watson tree with a offspring distribution B uniquely specified by the following facts:

- (1) Criticality: $\mathbf{E}B = 1$.
- (2) There exists $p \in (0, 1)$ such that if G is Geometric(p) then the law of B is given by setting, for each $k \in \mathbb{N}$,

$$\mathbf{P}\{B = k\} = \frac{\binom{k+2}{2} \mathbf{P}\{G = k\}}{\mathbf{E}\binom{G+2}{2}}.$$

From these conditions, a straightforward calculation shows that $p = 3/4$, and another easy computation yields that $\mathbf{E}[B^2] = 7/3$ so $\mathbf{Var}\{B\} = 4/3$. In the notation of Section 6, this yields $\sigma_\mu = 2/3^{1/2}$.

Next, the displacement D between a node in our tree and a uniformly selected child is equal to one of $\{-1, 0, 1\}$, each with equal probability; it follows that $\mathbf{E}[D^2] = 2/3$. Let ν_k be the law of the displacement vector for a vertex with k children, then let $\hat{\nu}_k$ be its symmetrization (as defined in Section 6), and write $\hat{\nu}_k^i$ for the i th marginal of $\hat{\nu}_k$. It follows that $\sigma_{\hat{\nu}_k^i}^2 = 2/3$ for all $1 \leq i \leq k$, so $\sigma_v^2 = \sigma_v^2 = 2/3$ and $(\sigma_\mu/2)^{1/2}/\sigma_v = (3/4)^{1/4}$. Together with Theorem 4.1, this explains the values of constants relating to triangulations.

We remark that the scaling required for convergence of triangulations in Theorem 1.1 agrees with the intuition described in [8], Section 4.1. It differs by a factor $8^{1/4}$ from the scaling for general triangulations that arises in Theorem 1.1 of [22], which can be understood as follows. First, in [22], the index n denotes the number of faces rather than the number of vertices, which accounts for a factor $2^{1/4}$. The size of the simple core of a loopless triangulation with m vertices is typically $\sim m/2$ (see Table 4 of [4]); this explains another factor $2^{1/4}$. Finally, the loopless core of a simple triangulation with m vertices is again typically of order $\sim m/2$ (this is not proved in [4] but may be handled using the same technology); this explains the final factor $2^{1/4}$. The latter factor does not arise in considering quadrangulations, which cannot contain loops; this may be viewed as explaining the different form of the constant for triangulations versus those of bipartite maps in Theorem 1.1 of [22].

For simple quadrangulations, we work with a critical Galton–Watson tree with offspring distribution B uniquely specified by the following facts:

- (1) Criticality: $\mathbf{E}B = 1$.
- (2) There exists $p \in (0, 1)$ such that if G is Geometric(p) then the law of B is given by setting, for each $c \in \mathbb{N}$,

$$\mathbf{P}\{B = c\} = \frac{(c + 1)\mathbf{P}\{G = c\}}{\mathbf{E}[G + 1]}.$$

From these calculations, a straightforward calculation shows that $p = 2/3$, and another easy computation then yields that $\mathbf{E}[B^2] = 5/2$, so $\mathbf{Var}\{B\} = 3/2$. Next, the displacement D between a node v and a uniformly selected child is equal to -1 or to 1 , each with equal probability, so has $\mathbf{E}[D] = 0$ and $\mathbf{Var}\{D\} = 1$. Using Theorem 4.1 as above then yields the scaling for quadrangulations in Theorem 1.1, and agrees with the two-point calculation for simple quadrangulations by Bouttier and Guitter [8].

APPENDIX B: CONVERGENCE FOR GOOD SEQUENCES OF MAP ENCODINGS

Throughout the section, we let $\mathbf{P} = (\mathbf{P}_n, n \geq 0)$ be a good sequence of random map encodings. We write $\mathbf{P}_n = (\mathbf{M}_n, \mathbf{T}_n)$, $\mathbf{M}_n = (M_n, \zeta_n)$ and $\mathbf{T}_n = (T_n, \xi_n)$, and let C_n, Z_n, r_n and X_n be as in Section 4.

Next, for $n \geq 1$, list the vertices of T_n according to their lexicographic order as $v_n(1), \dots, v_n(|T_n|)$. Given $1 \leq j \leq |V(T_n)|$, let $i_n(j) = \inf\{i : r_n(i) = v_n(j)\}$ be the index at which $v_n(j)$ first appears in the contour exploration. Let $m = m_n = 2|V(T_n)| - 2$.

LEMMA B.1. *As $n \rightarrow \infty$, we have*

$$\sup_{0 \leq t \leq 1} \left| \frac{i_n(\lfloor t \cdot |V(T_n)| \rfloor)}{m_n} - t \right| \xrightarrow{d} 0.$$

PROOF. As in Lemma 8.2, a straightforward argument using the height process (following (12) and (13) of [19]) shows that when $m_n \geq 2$, deterministically

$$\sup_{0 \leq t \leq 1} \left| \frac{i_n(\lfloor t \cdot |V(T_n)| \rfloor)}{m_n} - t \right| \leq \frac{\max\{|v|, v \in V(T_n)\} + 4}{m_n}.$$

Since $m_n \rightarrow \infty$, it thus suffices to show that $(\max\{|v|, v \in V(T_n)\} + 4)/m_n \xrightarrow{d} 0$. To see this, let U and V be independent Uniform[0, 1] random variables. If the latter convergence fails to hold then for infinitely many n , with uniformly positive probability, a single path from the root in T_n contains a macroscopic proportion of the elements of the vertices of T_n . It follows easily that

$$\limsup_{n \rightarrow \infty} \mathbf{P}\left\{U < V, C_n(U) = \min_{U \leq x \leq V} C_n(x)\right\} > 0.$$

On the other hand, $\mathbf{P}\{U < V, \mathbf{e}(U) = \min_{U \leq x \leq V} \mathbf{e}(x)\} = 0$, so the preceding equation implies that \mathbf{e} is not the distributional limit of any rescaling of C_n . Thus $\mathbf{1}$ does not hold, a contradiction. \square

PROOF OF THEOREM 4.1. We claim that it suffices to establish

$$(37) \quad (V(T_n), b_n d_{M_n}, \mu_n) \xrightarrow{d} (S, d, \mu)$$

for d_{GHP} . [In the above, by d_{M_n} we really mean the distance on $V(T_n)$ induced by d_{M_n} . This slight notational abuse should cause no confusion.] Indeed, suppose the latter convergence holds. By Skorohod’s representation theorem, we may work in a space in which the convergence (37) is almost sure. Fix $\varepsilon > 0$, and let E_n be the event that $\max_{v \in V(M_n)} b_n \cdot d_{M_n}(v, V(T_n)) \leq \varepsilon/2$ and $d_{GH}^2((V(T_n), b_n d_{M_n}, \nu(\xi_n), \nu(\zeta_n)), (S, d, \rho, u^*)) \leq \varepsilon/2$. Now let $\mathcal{R}_n^0 = \{(x, y) \in$

$V(T_n) \times V(M_n)$; $b_n d_{M_n}(x, y) \leq \varepsilon/2$; then \mathcal{R}_n^0 has distortion at most ε . Furthermore, $(v(\zeta_n), v(\xi_n)) \in \mathcal{R}_n^0$ and $(v(\xi_n), v(\xi_n)) \in \mathcal{R}_n^0$. Let ν_n be the probability measure on $V(T_n) \times V(M_n)$ whose restriction to $\{(v, v) : v \in V(T_n)\}$ is the uniform probability measure. Then ν_n is a coupling of μ_n [as a measure on $V(T_n)$] and μ_n [as a measure on $V(M_n)$], and $\nu_n(\mathcal{R}_n^0) = 1$. On E_n we have that \mathcal{R}_n^0 is a correspondence, so on E_n ,

$$d_{\text{GHP}}((V(M_n), b_n d_{M_n}, \mu_n), (V(T_n), b_n d_{M_n}, \mu_n)) \leq \varepsilon/2,$$

and by the triangle inequality it follows that on E_n ,

$$d_{\text{GHP}}((V(M_n), b_n d_{M_n}, \mu_n), (S, d, \mu)) \leq \varepsilon.$$

Finally, in this space, since \mathbf{P} is good sequence of map encodings, $\mathbf{P}\{E_n\} \rightarrow 1$ as $n \rightarrow \infty$, and it follows that $(V(M_n), b_n d_{M_n}, \mu_n) \xrightarrow{d} (S, d, \mu)$ for d_{GHP} . We thus turn our attention to proving (37).

The first part of our argument is based on that of [20], Proposition 3.2; the second part follows closely the argument of Section 8.3 of [22]. Define a function $d_n : [0, 1]^2 \rightarrow [0, \infty)$ as follows. Define as above $m = m_n = 2|V(T_n)| - 2$, and for $i, j \in [m]$, let $d_n(i/m, j/m) = d_{M_n}(r_n(i), r_n(j))$. Then extend d_n to $[0, 1]^2$ by “bilinear interpolation”: if $(x, y) = ((i + \alpha)/m, (j + \beta)/m)$ for $0 \leq \alpha < 1, 0 \leq \beta < 1$ then let

$$\begin{aligned} d_n(x, y) &= \alpha\beta d_n((i + 1)/m, (j + 1)/m) + \alpha(1 - \beta) d_n((i + 1)/m, j/m) \\ &\quad + (1 - \alpha)\beta d_n(i/m, (j + 1)/m) + (1 - \alpha)(1 - \beta) d_n(i/m, j/m). \end{aligned}$$

Using **1**, we now work in a space in which

$$(38) \quad (a_n C_n, b_n Z_n) \xrightarrow{\text{a.s.}} (\mathbf{e}, Z).$$

We will show that in such a space, additionally

$$(39) \quad b_n d_n \xrightarrow{\text{a.s.}} d^*,$$

for the topology of uniform convergence on $C([0, 1]^2)$. Assume (39) holds, and for $n \in \mathbb{N}$, consider the correspondence \mathcal{R}_n between (S, d) and $(V(T_n), b_n d_{M_n})$ given by letting $[s] \in [0, 1]/\{d^* = 0\} = S$ correspond to $r_n(i)$ if and only if $\lfloor s \cdot m \rfloor = i$, for $0 \leq i \leq m$.⁹ By (39), the distortion of \mathcal{R}_n tends to zero.

Let μ_n^- be the uniform probability measure on $V(T_n) \setminus \{v(\xi_n)\}$. Define a coupling between μ_n^- and μ as follows. Fix $s \in [0, 1]$. Let $f_1(s) = [s] \in S$. If $s = i/m$ then let $f_2(s) = r_n(i)$. If $s \in (i/m, (i + 1)/m)$ and $\{r_n(i), r_n(i + 1)\} = \{u, p(u)\} \in E(T_n)$ then let $f_2(s) = u$. Finally, let $f = (f_1, f_2) : [0, 1] \rightarrow S \times V(T_n)$, let λ denote one-dimensional Lebesgue measure on $[0, 1]$, and let $\nu = f_* \lambda$. Write π and π' for the projection maps from $S \times V(T_n)$ to S and to $V(T_n)$, respectively. We

⁹A similar technique is used at the end of Section 8 of [22].

clearly have $\pi_*\nu = \mu$. Also, for each edge $e \in E(T)$, there are precisely two indices $i, j \in \{0, 1, \dots, m_n\}$ for which $\{r_n(i), r_n(i + 1)\} = \{u, p(u)\}$; it follows that $\pi'_*\nu = \mu_n^-$.

For any pair $([s], r_n(i)) \in \mathcal{R}_n$, either $f_2(s) = r_n(i)$ or $f_2(s) = p(r_n(i))$, the two possibilities due to the two directions in which the edge $\{p(r_n(i)), r_n(i)\}$ is traversed during the contour exploration. We thus let

$$\mathcal{R}_n^+ = \{([s], w) : ([s], w) \in \mathcal{R}_n \text{ or } ([s], p(w)) \in \mathcal{R}_n\}.$$

Since \mathcal{R}_n was a correspondence, \mathcal{R}_n^+ is again a correspondence, and $\nu(\mathcal{R}_n^+) = 1$. Finally, $\text{dis}(\mathcal{R}_n^+) \leq \text{dis}(\mathcal{R}_n) + 2b_n$ so $\text{dis}(\mathcal{R}_n^+) \rightarrow 0$ as $n \rightarrow \infty$. It follows by definition that

$$(V(M_n), b_n d_{M_n}, \mu_n^-) \xrightarrow{d} (S, d, \mu)$$

for d_{GHP} . Finally, the Prokhorov distance between μ_n^- and μ_n is $1/|V(T_n)|$, which tends to zero as $n \rightarrow \infty$. We may therefore replace μ_n^- by μ_n and the preceding convergence still holds, which establishes (37) and so proves the theorem. It thus remains to prove (39).

Define a function $d_n^\circ : [0, 1]^2 \rightarrow [0, \infty)$ as follows: for $x, y \in \{i/m, 0 \leq i \leq m\}$, let

$$d_n^\circ(x, y) = Z_n(x) + Z_n(y) - 2 \max(\check{Z}_n(x, y), \check{Z}_n(y, x)).$$

Then extend d_n° to $[0, 1]^2$ by bilinear interpolation as with d_n . Recalling that for integer $0 \leq i \leq m$, $Z_n(i/m) = X_n(r_n(i))$, it follows straightforwardly from **1** that for all $\varepsilon, \delta > 0$,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbf{P} \left\{ \sup_{|x-y| \leq \delta} b_n d_n^\circ(x, y) \geq \varepsilon \right\} \\ \leq \mathbf{P} \left\{ \sup_{|x-y| \leq \delta} (Z(x) + Z(y) - 2 \max(\check{Z}(x, y), \check{Z}(y, x))) \geq \varepsilon \right\} \end{aligned}$$

(the derivation of this inequality is spelled out in a little more detail in [20], Section 3). Since Z is almost surely continuous, it follows that for any $\eta > 0$ and $k \in \mathbb{N}$, there exist $\delta_k > 0$ and $n_k \in \mathbb{N}$ such that for all $n \geq n_k$,

$$\mathbf{P} \left\{ \sup_{|x-y| \leq \delta_k} b_n d_n^\circ(x, y) \geq 2^{-(k+1)} \right\} \leq \frac{\eta}{2^{k+1}}.$$

Next, by **3(i)**, after increasing n_k if necessary, for $n \geq n_k$,

$$(40) \quad \mathbf{P} \left\{ \sup_{x, y \in [0, 1]} b_n (d_n(x, y) - d_n^\circ(x, y)) \geq 2^{-(k+1)} \right\} \leq \frac{\eta}{2^{k+1}}.$$

By decreasing δ_k if necessary, we may also ensure that for $n < n_k$,

$$\mathbf{P} \left\{ \sup_{|x-y| \leq \delta_k} b_n \max(d_n(x, y), d_n^\circ(x, y)) \leq 2^{-(k+1)} \right\} = 1.$$

Combining the three preceding estimates yields that for all $n \geq 1$,

$$\mathbf{P}\left\{ \sup_{|x-y| \leq \delta_k} b_n d_n(x, y) \geq 2^{-k} \right\} \leq \frac{\eta}{2^k},$$

so for all n ,

$$\mathbf{P}\left\{ \forall k, \sup_{|x-y| \leq \delta_k} b_n d_n(x, y) < 2^{-k} \right\} \geq 1 - \eta.$$

In other words, with $(\delta_k)_{k \geq 0}$ as above, for all n , with probability at least $1 - \eta$ the function $b_n d_n$ belongs to the compact

$$K = \left\{ f \in C([0, 1]^2, \mathbb{R}) : f(0, 0) = 0, \forall k, \sup_{|x-y| \leq \delta_k} f(x, y) \leq 2^{-k} \right\},$$

which implies that $\{b_n d_n, n \in \mathbb{N}\}$ is tight in $C([0, 1]^2, \mathbb{R})$. For the remainder of the proof, we let $\tilde{d} \in C([0, 1]^2, \mathbb{R})$ be any almost sure subsequential limit of $b_n d_n$; we suppress the subsequence from the notation for readability.

Recall that we work in a space where (38) holds. In such a space, it follows from the continuity of Z that $b_n d_n \xrightarrow{\text{a.s.}} d_Z$, where $d_Z : [0, 1]^2 \rightarrow \mathbb{R}$ is as defined in Section 1.1. By (40), it follows that for any $\eta > 0$ and $p \geq 1$,

$$\limsup_{n \rightarrow \infty} \mathbf{P}\left\{ \sup_{x, y \in [0, 1]} (b_n d_n(x, y) - d_Z(x, y)) \geq 2^{-p} \right\} \leq \eta 2^{-p},$$

so a.s. $\tilde{d} \leq d_Z$.

We next claim that a.s. $\tilde{d}(x, y) = 0$ for all $x \neq y \in [0, 1]$ for which $x \sim_e y$. To see this, suppose that $x \sim_e y$ for some $x, y \in [0, 1]$, and assume by symmetry that $x < y$. Continuing to write $m = m_n = 2|V(T_n)| - 2$, (38) implies there exist random integer sequences $(x_n, n \in \mathbb{N})$ and $(y_n, n \in \mathbb{N})$ such that $x_n/m_n \xrightarrow{\text{a.s.}} x$, $y_n/m_n \xrightarrow{\text{a.s.}} y$, and

$$C_n(x_n/m_n) = C_n(y_n/m_n) = \min\{C_n(z) : x_n \leq m_n \cdot z \leq y_n\}.$$

It follows that $r_n(x_n) = r_n(y_n)$, or equivalently that $i_n(x_n) = i_n(y_n)$, so

$$d_n(x_n/m_n, y_n/m_n) = d_{M_n}(r_n(i_n(x_n)), r_n(i_n(y_n))) = 0.$$

Since $b_n d_n \xrightarrow{\text{a.s.}} \tilde{d}$ (along a subsequence) and $x_n/m_n \xrightarrow{\text{a.s.}} x$, $y_n/m_n \xrightarrow{\text{a.s.}} y$, this implies that

$$0 = d_n(x_n/m_n, y_n/m_n) \xrightarrow{\text{a.s.}} \tilde{d}(x, y),$$

so $\tilde{d}(x, y) \stackrel{\text{a.s.}}{=} 0$ as claimed.

Since, almost surely, $\tilde{d} = 0$ on $\{x, y\} : x \sim_e y\}$, and $\tilde{d} \leq d_Z$, we must have $\tilde{d} \leq d^*$ since d^* is the largest pseudo-metric on $[0, 1]$ satisfying these constraints. We now show that in fact, almost surely $\tilde{d} = d^*$.

Let U, V be independent and uniform on $[0, 1]$. Letting I_n be minimal such that $Z_n(I_n/m) = \check{Z}_n(0, 1)$, then by **1** we have

$$b_n X_n(r_n(I_n)) = b_n \check{Z}_n(0, 1) \xrightarrow{d} \check{Z}(0, 1) \stackrel{d}{=} -d^*(U, V),$$

the last identity by (2) (which is Corollary 7.3 of [22]). Since $v(\xi_n) = r_n(0)$ and $X_n(r_n(0)) = 0$, by (3) we also have

$$\lim_{n \rightarrow \infty} \mathbf{P}\{b_n \cdot |d_{M_n}(\zeta_n, \xi_n) + X_n(r_n(I_n))| > \varepsilon\} = 0,$$

so since $X_n(r_n(I_n)) = \check{Z}_n(0, 1)$, we obtain

$$b_n d_{M_n}(\zeta_n, \xi_n) \xrightarrow{d} d^*(U, V).$$

Since the Prokhorov metric topologizes weak convergence, by **2(ii)** it follows that for U_n and V_n two independent random elements of R_n , then

$$b_n d_{M_n}(U_n, V_n) \xrightarrow{d} d^*(U, V).$$

Now let $1 \leq J_n, K_n \leq |v(T_n)|$ be such that $v_n(J_n) = U_n$ and $v_n(K_n) = V_n$. The preceding convergence implies $b_n d_n(J_n, K_n) \xrightarrow{d} d^*(U, V)$. Lemma B.1 implies that $(J_n, K_n) \xrightarrow{d} (U, V)$, so the tightness of the collection $(b_n d_n, n \geq 1)$ then yields

$$b_n d_n(U, V) \xrightarrow{d} d^*(U, V).$$

Finally, along the subsequence where $b_n d_n \xrightarrow{\text{a.s.}} \tilde{d}$, we also have $b_n d_n(U, V) \xrightarrow{d} \tilde{d}(U, V)$, so it must be that $\tilde{d}(U, V) \stackrel{d}{=} d^*(U, V)$. Since a.s. $\tilde{d} \leq d^*$, it must therefore hold that $\tilde{d} \stackrel{\text{a.s.}}{=} d^*$.

We have now shown that in the space where (38) holds, any subsequential limit \tilde{d} of $b_n d_n$ must satisfy $\tilde{d} \stackrel{\text{a.s.}}{=} d^*$; this implies that in fact, in this space we have $b_n d_n \xrightarrow{\text{a.s.}} \tilde{d}$, which establishes (39) and so completes the proof. \square

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