QUADRATIC BSDE WITH $\mathbb{L}^2$-TERMINAL DATA: KRYLOV'S ESTIMATE, ITÔ–KRYLOV'S FORMULA AND EXISTENCE RESULTS[1]

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We establish a Krylov-type estimate and an Itô–Krylov change of variable formula for the solutions of one-dimensional quadratic backward stochastic differential equations (QBSDEs) with a measurable generator and an arbitrary terminal datum. This allows us to prove various existence and uniqueness results for some classes of QBSDEs with a square integrable terminal condition and sometimes a merely measurable generator. It turns out that neither the existence of exponential moments of the terminal datum nor the continuity of the generator are necessary to the existence and/or uniqueness of solutions. We also establish a comparison theorem for solutions of a particular class of QBSDEs with measurable generator. As a byproduct, we obtain the existence of viscosity solutions for a particular class of quadratic partial differential equations (QPDEs) with a square integrable terminal datum.

1. Introduction. Let $(W_t)_{0 \le t \le T}$ be a d-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $(\mathcal{F}_t)_{0 \le t \le T})$ the natural filtration of W augmented with $\mathbb{P}$-negligible sets. Let $H(t, \omega, y, z)$ be a real valued $\mathcal{F}_t$-progressively measurable process defined on $[0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d$. Let ξ be an $\mathcal{F}_T$-measurable $\mathbb{R}$-valued random variable.

In this paper, the one-dimensional BSDE under consideration is

$$Y_t = \xi + \int_t^T H(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s, \qquad 0 \le t \le T.$$

This equation will be labeled eq(ξ, H). The data ξ and H are respectively called the terminal condition and the coefficient or the generator.

A BSDE is called quadratic if its generator has at most a quadratic growth in the z variable.

For given real numbers a and b, we set $a \wedge b := \min(a, b)$, $a \vee b := \max(a, b)$, $a^- := \max(0, -a)$ and $a^+ := \max(0, a)$. For $p > 0$, we denote: $\mathbb{L}^p_{\text{loc}}(\mathbb{R}) :=$ the space of (classes) of functions u defined on $\mathbb{R}$ which are p-integrable on bounded set of $\mathbb{R}$.

$\mathcal{W}^2_{p,\text{loc}} :=$ the Sobolev space of (classes) of functions u defined on $\mathbb{R}$ such that both u and its generalized derivatives u' and u'' belong to $\mathbb{L}^p_{\text{loc}}(\mathbb{R})$.

$\mathcal{C} :=$ the space of continuous and $\mathcal{F}_t$-adapted processes.

$\mathcal{S}^2 :=$ the space of continuous, $\mathcal{F}_t$-adapted processes φ such that

$$\mathbb{E} \sup_{0 \leq t \leq T} |\varphi_t|^2 < \infty.$$

$\mathcal{M}^2 :=$ the space of $\mathcal{F}_t$-adapted processes φ satisfying $\mathbb{E} \int_0^T |\varphi_s|^2 \, ds < +\infty$.

$\mathcal{L}^2 :=$ the space of $\mathcal{F}_t$-adapted processes φ satisfying $\int_0^T |\varphi_s|^2 \, ds < +\infty \mathbb{P}$-a.s.

DEFINITION 1.1. A solution of eq(ξ, H) is a process (Y, Z) which belongs to $\mathcal{C}(\mathbb{R}) \times \mathcal{L}^2(\mathbb{R}^d)$ such that $\int_0^T |H(s, Y_s, Z_s)| \, ds < \infty$ $\mathbb{P}$-a.s. and satisfies eq(ξ, H) for each $t \in [0, T]$.

The BSDEs with linear coefficient were studied in [10] and extended to the Lipschitz case in [25]. The first results on the existence of solutions to QBSDEs were obtained independently in [19] and in [12] by two different methods. The approach developed in [19] is based on the monotone stability and consists to find bounded solutions. Later, many authors have extended the result of [19] in different directions; see, for example, [8, 9, 11, 15, 16, 23, 26]. In [11], the existence of solutions was proved, in the case where the exponential moments of the terminal datum are finite, by using a localization procedure. In [26], a fixed-point method is developed to directly show the existence and uniqueness of a bounded solution when the terminal datum is bounded and the generator satisfies the (so-called) Lipschitz–quadratic condition. The BMO theory plays a key role in [26]. Recently, a monotone stability result for quadratic semimartingales was established in [9] then applied to derive the existence of solutions in the framework of exponential integrability of the terminal data. More precisely, the following stability result is established in [9]: If a sequence of quadratic semimartingales is a Cauchy sequence with respect to the a.s. uniform convergence, then its limit is again a quadratic semimartingale. Moreover, strong convergence results hold for the martingale parts and the finite variation parts of the semimartingale decomposition. As application, they prove existence of solutions to their quadratic BSDEs. Generalized stochastic QBSDEs were studied in [16] under more or less similar exponential integrability assumptions on the terminal datum.

It should be noted that all the previous papers were developed in the framework of continuous generators and bounded terminal data or at least having finite exponential moments. It is natural to ask the following questions:

1. Are there quadratic BSDEs that have solutions without assuming the existence of exponential moments of the terminal datum? In the affirmative, in which space?

2. Are there quadratic BSDEs with measurable but not necessarily continuous generator that have solutions without assuming the existence of exponential moments of the terminal datum? In the affirmative, in which space?

Let us start with a simple example which gives a positive answer to question 1. This example is covered by the present work and, to the best of our knowledge, is not covered by previous papers.

EXAMPLE 1.1.   Assume that:

(**H1**) ξ is square integrable.

Let $f : \mathbb{R} \longmapsto \mathbb{R}$ be a given continuous function with compact support, and set $M := \sup_{y \in \mathbb{R}} |f(y)|$. The BSDE eq$(\xi, f(y)|z|^2)$ is then of quadratic growth since $|f(y)||z|^2 \leq M|z|^2$. Let

$$u(x) := \int_0^x \exp\left(2\int_0^y f(t)\,dt\right) dy.$$

If (Y, Z) is a solution to eq$(\xi, f(y)|z|^2)$, then Itô's formula applied to $u(Y_t)$ shows that

$$u(Y_t) = u(\xi) - \int_t^T u'(Y_s)Z_s\,dW_s.$$

If we set $\bar{Y}_t := u(Y_t)$ and $\bar{Z}_t := u'(Y_t)Z_t$, then $(\bar{Y}, \bar{Z})$ solves the BSDE

$$\bar{Y}_t = u(\xi) - \int_t^T \bar{Z}_s\,dW_s.$$

Since both u and its inverse are $\mathcal{C}^2$ functions which are globally Lipschitz and one to one from $\mathbb{R}$ onto $\mathbb{R}$, we then deduce that eq$(\xi, f(y)|z|^2)$ admits a solution (resp., a unique solution) if and only if eq$(u(\xi), 0)$ admits a solution (resp., a unique solution). But, eq$(u(\xi), 0)$ has a unique solution in $\mathcal{S}^2 \times \mathcal{M}^2$ whenever $u(\xi)$ is square integrable. Since u and its inverse are globally Lipschitz, it follows that $u(\xi)$ is square integrable if and only if ξ is square integrable. Therefore, even when all the exponential moments are infinite eq$(\xi, f(y)|z|^2)$ has a unique solution in $\mathcal{S}^2 \times \mathcal{M}^2$ whenever ξ is square integrable. Note that, since the sign of f is not constant, our example also shows that the convexity of the generator is not necessary to uniqueness. Assume now that ξ is merely $\mathcal{F}_T$-measurable but not necessarily integrable. Then, thanks to Dudley's representation theorem [13], one can show by using the above transformation u that, eq$(\xi, f(y)|z|^2)$ has at least one solution (Y, Z) which belongs to $\mathcal{C} \times \mathcal{L}^2$.

The present paper is a development and a continuation of our announced results [4, 5]. We do not aim to generalize the previous papers on QBSDEs. Our goal is to give another approach which allows us to establish the existence of solutions, in the space $\mathcal{S}^2 \times \mathcal{M}^2$, for a large class of QBSDEs with a square integrable terminal datum and sometimes a measurable generator. It turns out that neither the existence of exponential moments of the terminal datum nor the continuity of the generator are necessary to the existence and/or uniqueness of solutions. Krylov's inequality and Itô–Krylov's formula, which we establish here for solutions of QBSDEs, play an important role in the proofs. To be more precise, let us consider the following assumptions:

(**H2**) There exist an $\mathcal{F}_t$-adapted positive stochastic process η satisfying $\mathbb{E}\int_0^T \eta_s\, ds < \infty$ and a locally integrable positive function f such that for a.e. (t, ω) and every (y, z),

$$\big|H(t, y, z)\big| \le \eta_t + f(y)|z|^2.$$

(**H3**) The function f, defined in assumption (**H2**), belongs to $\mathbb{L}^1(\mathbb{R})$.

In the first part, we use the occupation time formula to show that if assumption (**H2**) holds, then for any solution (Y, Z) of eq(ξ, H), the time spent by Y in a Lebesgue negligible set is negligible with respect to the measure $|Z_t|^2\, dt$; that is, the following Krylov estimate holds for any positive measurable function ψ:

$$(1.1) \qquad \mathbb{E}\int_0^{T \wedge \tau_R} \psi(Y_s)|Z_s|^2\, ds \le C\|\psi\|_{\mathbb{L}^1([-R, R])},$$

where τ_R is the first exit time of Y from the interval $[-R, R]$ and C is a constant depending on T, $\|\xi\|_{\mathbb{L}^1(\Omega)}$ and $\|f\|_{\mathbb{L}^1([-R, R])}$.

This inequality allows us to show that for any solution (Y, Z) of eq(ξ, H) and any function φ in $\mathcal{W}^2_{1,\mathrm{loc}}(\mathbb{R})$ the following change of variable formula holds:

$$(1.2) \qquad \varphi(Y_t) = \varphi(Y_0) + \int_0^t \varphi'(Y_s)\, dY_s + \frac{1}{2}\int_0^t \varphi''(Y_s)|Z_s|^2\, ds.$$

This formula is usually called the Itô–Krylov formula.

Inequality (1.1) as well as formula (1.2) are interesting in their own and can have potential applications in BSDEs, since they are established here with minimal conditions on the data ξ and H. Note that, although inequality (1.1) may be established by adapting the method developed by Krylov, which is based on partial differential equations [20] (see also [1, 7, 21, 24]), the proof we give here is purely probabilistic and very simple.

As first applications of formulas (1.1) and (1.2), we establish the existence of solutions in $\mathcal{S}^2 \times \mathcal{M}^2$ for the two classes of QBSDEs, eq$(\xi, f(y)|z|^2)$ and eq$(\xi, a + by + cz + f(y)|z|^2)$, by merely assuming that f is globally integrable, ξ is square integrable and a, b, c are arbitrary real numbers. It should be noted that, when f is not continuous, the function $u(x) := \int_0^x \exp(2\int_0^y f(t)\, dt)\, dy$ is not

of class $\mathcal{C}^2$, and hence the classical Itô formula cannot be applied. Nevertheless, when f belongs to $\mathbb{L}^1(\mathbb{R})$, then the function u belongs to the space $\mathcal{W}^2_{1,\text{loc}}(\mathbb{R})$, and hence formula (1.2) can be applied to u. This enables us to derive the existence of a minimal and a maximal solution for eq(ξ, Φ_f), with

$$\Phi_f(y, z) := a + b|y| + c|z| + f(y)|z|^2, \tag{1.3}$$

where a, b and c are some nonnegative constants. We also prove a comparison theorem for BSDEs of type eq(ξ, $f(y)|z|^2$) whenever we can compare their terminal data and a.e. their generators. We then deduce the uniqueness of solutions for eq(ξ, $f(y)|z|^2$) when ξ is square integrable and f belongs to $\mathbb{L}^1(\mathbb{R})$, that is, even when f is merely defined a.e. The following example illustrates this fact.

EXAMPLE 1.2. The BSDE with data (ξ, H) admits a unique solution in $\mathcal{S}^2 \times \mathcal{M}^2$ when ξ is square integrable and H is one of the following generators:

$H_1(y, z) := \sin(y)|z|^2$ if $y \in [-\pi, \frac{\pi}{2}]$ and $H_1(y, z) := 0$ otherwise,

$H_2(y, z) := (\mathbf{1}_{[a,b]}(y) - \mathbf{1}_{[c,d]}(y))|z|^2$ for a given $a < b$ and $c < d$,

$H_3(y, z) := \frac{1}{(1+y^2)\sqrt{|y|}}|z|^2$ if $y \neq 0$ and $H_3(y, z) := 1$ otherwise.

It is worth noting that the generator $H_3(y, z)$ is neither continuous nor locally bounded and eq(ξ, H_3) has a unique solution in $\mathcal{S}^2 \times \mathcal{M}^2$ whenever ξ is square integrable. Note also that the generators H_1 and H_2 are not convex and the uniqueness of solution holds in $\mathcal{S}^2 \times \mathcal{M}^2$ for eq(ξ, H_1) and eq(ξ, H_2). This gives a positive answer to question 2.

In the second part, we deal with eq(ξ, H) when ξ is square integrable, H is continuous and satisfies

$$|H(s, y, z)| \leq \Phi_f(y, z), \tag{1.4}$$

where f is a positive and globally integrable function on $\mathbb{R}$. Hence, f cannot be a constant.

Our approach consists then to use the extremal solutions of eq($-\xi^-$, $-\Phi_f$) and eq(ξ^+, Φ_f) as barriers between which we establish the existence of a solution to eq(ξ, H). This approach allows us to control more precisely the integrability we impose to the terminal datum. It turns out that there is a balance to be maintained between the integrability of ξ and the integrability of f. That is, when f is globally integrable then, less integrability is required for ξ. And when f is locally integrable, then more integrability is required to ξ. We mention that, in contrast to the most previous papers, our result also covers the BSDEs with linear growth (by putting $f = 0$). It therefore provides a unified treatment for quadratic BSDEs and those of linear growth, keeping ξ square integrable in both cases.

The idea behind the proof is: when $|H(s, y, z)| \leq F(s, y, z)$, then the solvability of eq(ξ, H) is reduced to the solvability of eq(ξ, F). The method consists to deduce the solvability of a BSDE (without barriers) from that of a suitable QBSDE with two reflecting barriers, see for instance the references [6, 17, 18] for QBSDEs

with two reflecting barriers. The barriers we consider are defined by one solution of eq($-\xi^-, -\Phi_f$) and one solution of eq(ξ^+, Φ_f). For this approach, neither a priori estimates nor approximation are needed.

In the third part, we give an application to quadratic partial differential equations (QPDEs). We first assume that f is continuous. We then prove the existence of a viscosity solution for a class of nondivergence form semilinear PDEs with quadratic nonlinearity of type $f(u)|\nabla u|^2$. This is done with an unbounded terminal datum. It surprisingly turns out that there is a gap between the BSDEs driven by a generator of type $f(y)|z|^2$ and their associated semilinear PDEs (see Remark 5.2, Section 5). Note that the class of quadratic PDEs under study in this paper can be used as a simplified model in some incomplete financial markets; see, for example, [14].

The paper is organized as follows. In Section 2, Krylov's estimate and Itô–Krylov's formula are proved for solutions of QBSDEs. In Section 3, the solvability of eq($\xi, f(y)|z|^2$) and eq($\xi, a + by + cz + f(y)|z|^2$) are established in $\mathcal{S}^2 \times \mathcal{M}^2$ in the case where f is globally integrable but merely defined a.e. In the same section, a comparison theorem and the uniqueness of solution are established for eq($\xi, f(y)|z|^2$). In Section 4, the solvability of eq(ξ, H) is established in $\mathcal{S}^2 \times \mathcal{M}^2$ when the generator H is continuous and dominated by Φ_f, with f globally integrable but not necessarily continuous. In Section 5, the existence of a viscosity solution is proved for quadratic PDEs (QPDEs) associated to the Markovian version of eq($\xi, f(y)|z|^2$). In the Appendix, we give some auxiliary results which are used throughout the paper.

2. Krylov's estimates and Itô–Krylov's formula in BSDEs.
Krylov's estimate as well as Itô–Krylov's formula are established here for one-dimensional BSDEs with a merely measurable generator and an $\mathcal{F}_T$-measurable terminal datum. Moreover, they are valid although the martingale part of Y can degenerate. Actually, the martingale part of Y can degenerate with respect to the Lebesgue measure but remains nondegenerate with respect to the measure $|Z_t|^2 \, dt$. Note that the Krylov estimate we state in the next proposition can be established by using Krylov's method [20] (see also [1, 7, 21, 24]), which is based on partial differential equations. The proof we give here is probabilistic and more simple. It is based on the occupation time formula.

2.1. *Krylov's estimates in BSDEs.*

PROPOSITION 2.1 (Local estimate). *Assume* (**H2**) *holds. Let* (Y, Z) *be a solution to* eq(ξ, H). *Let* $R > |Y_0|$. *Then there exists a positive constant* C *depending on* T, R, $\|\eta\|_{\mathbb{L}^1([0,T] \times \Omega)}$ *and* $\|f\|_{\mathbb{L}^1([-R,R])}$ *such that for any nonnegative measurable function* ψ,

$$\mathbb{E} \int_0^{T \wedge \tau_R} \psi(Y_s)|Z_s|^2 \, ds \leq C \|\psi\|_{\mathbb{L}^1([-R,R])},$$

where $\tau_R := \inf\{t > 0 : |Y_t| \geq R\}$.

PROOF. We put $\tau := \tau_R \wedge \tau_N' \wedge \tau_M''$, where $\tau_N' := \inf\{t > 0, \int_0^t |Z_s|^2 \, ds \geq N\}$ and $\tau_M'' := \inf\{t > 0, \int_0^t |H(s, Y_s, Z_s)| \, ds \geq M\}$. Let a be a real number such that $a \leq R$ and $L_\cdot^a(Y)$ be the local time of Y at the level a. By Tanaka's formula, we have

$$(Y_{t \wedge \tau} - a)^- = (Y_0 - a)^- + \int_0^{t \wedge \tau} \mathbf{1}_{\{Y_s < a\}} \, dY_s + \frac{1}{2} L_{t \wedge \tau}^a(Y)$$

$$= (Y_0 - a)^- - \int_0^{t \wedge \tau} \mathbf{1}_{\{Y_s < a\}} H(s, Y_s, Z_s) \, ds$$

$$+ \int_0^{t \wedge \tau} \mathbf{1}_{\{Y_s < a\}} Z_s \, dW_s + \frac{1}{2} L_{t \wedge \tau}^a(Y).$$

Since the map $y \mapsto (y - a)^-$ is Lipschitz, it follows that

$$\frac{1}{2} L_{t \wedge \tau}^a(Y) \leq |Y_{t \wedge \tau} - Y_0| + \int_0^{t \wedge \tau} \mathbf{1}_{\{Y_s \leq a\}} H(s, Y_s, Z_s) \, ds$$

(2.1)

$$- \int_0^{t \wedge \tau} \mathbf{1}_{\{Y_s \leq a\}} Z_s \, dW_s.$$

Passing to expectation, we get

(2.2) $$\sup_a \mathbb{E}\big[L_{t \wedge \tau}^a(Y)\big] \leq 4R + 2M.$$

Since $-R \leq Y_{t \wedge \tau} \leq R$ for each t, then $\mathrm{Support}(L_\cdot^a(Y_{\cdot \wedge \tau})) \subset [-R, R]$. Therefore, using inequality (2.1), assumption (**H2**) and the occupation time formula, we obtain

$$\frac{1}{2} L_{t \wedge \tau}^a(Y) \leq |Y_{t \wedge \tau} - Y_0| + \int_0^T \eta_s \, ds + \int_0^{t \wedge \tau} \mathbf{1}_{\{Y_s \leq a\}} f(Y_s) \, d\langle Y \rangle_s$$

$$- \int_0^{t \wedge \tau} \mathbf{1}_{\{Y_s \leq a\}} Z_s \, dW_s$$

$$= |Y_{t \wedge \tau} - Y_0| + \int_0^T \eta_s \, ds + \int_{-R}^a f(x) L_{t \wedge \tau}^x(Y) \, dx$$

$$- \int_0^{t \wedge \tau} \mathbf{1}_{\{Y_s \leq a\}} Z_s \, dW_s.$$

Taking expectation, it holds that

$$\frac{1}{2} \mathbb{E}\big[L_{t \wedge \tau}^a(Y)\big] \leq \mathbb{E}|Y_{t \wedge \tau} - Y_0| + \mathbb{E} \int_0^T \eta_s \, ds + \int_{-R}^a |f(x)| \mathbb{E}\big[L_{t \wedge \tau}^x(Y)\big] \, dx < \infty.$$

Thanks to inequality (2.2) and Gronwall's lemma, we deduce that

$$\mathbb{E}\big[L_{t \wedge \tau}^a(Y)\big] \leq 2\bigg[\mathbb{E}\big(|Y_{t \wedge \tau} - Y_0|\big) + \mathbb{E} \int_0^T \eta_s \, ds\bigg] \exp\bigg(2 \int_{-R}^a |f(x)| \, dx\bigg)$$

$$\leq \bigg[4R + 2\mathbb{E} \int_0^T \eta_s \, ds\bigg] \exp\big(2\|f\|_{\mathbb{L}^1([-R, R])}\big).$$

Passing successively to the limit on N and M (keeping in mind that $\tau := \tau_R \wedge \tau'_N \wedge \tau''_M$) then using Beppo–Levi's theorem, we get

$$(2.3) \qquad \mathbb{E}[L^a_{t \wedge \tau_R}(Y)] \leq \left[4R + 2\mathbb{E} \int_0^T \eta_s \, ds \right] \exp(2\|f\|_{\mathbb{L}^1([-R,R])}).$$

Now, let ψ be an arbitrary positive function. We use the previous inequality and the time occupation formula to show that

$$\mathbb{E} \int_0^{T \wedge \tau_R} \psi(Y_s)|Z_s|^2 \, ds$$

$$= \mathbb{E} \int_0^{T \wedge \tau_R} \psi(Y_s) \, d\langle Y \rangle_s$$

$$\leq \mathbb{E} \int_{-R}^R \psi(a) L^a_{T \wedge \tau_R}(Y) \, da$$

$$\leq \left[4R + 2\mathbb{E} \int_0^T \eta_s \, ds \right] \exp(2\|f\|_{\mathbb{L}^1([-R,R])}) \|\psi\|_{\mathbb{L}^1([-R,R])}.$$

Proposition 2.1 is proved. □

Arguing as previously, one can establish the following *global estimate*.

PROPOSITION 2.2 (Global estimate). *Assume that* (**H2**) *and* (**H3**) *are satisfied. Let* (Y, Z) *be a solution of* $\mathrm{eq}(\xi, H)$ *such that* $\mathbb{E}(\sup_{t \leq T} |Y_t|)$ *is finite. Then there exists a positive constant C depending on T, $\|\eta\|_{\mathbb{L}^1([0,T] \times \Omega)}$, $\|f\|_{\mathbb{L}^1(\mathbb{R})}$ and $\mathbb{E}(\sup_{t \leq T} |Y_t|)$ such that, for any nonnegative measurable function ψ,*

$$\mathbb{E} \int_0^T \psi(Y_s)|Z_s|^2 \, ds \leq C \|\psi\|_{\mathbb{L}^1(\mathbb{R})}.$$

2.2. *An Itô–Krylov change of variable formula in BSDEs.* In this subsection, we shall establish an Itô–Krylov change of variable formula for the solutions of one dimensional BSDEs. This will allow us to treat some QBSDEs with measurable generator. Let us give a summarized explanation on Itô–Krylov's formula. The Itô change of variable formula expresses that the image of a semimartingale, by a $\mathcal{C}^2$-class function, is a semimartingale. In the case where

$$X_t := X_0 + \int_0^t \sigma(s, \omega) \, dW_s + \int_0^t b(s, \omega) \, ds$$

is an Itô semimartingale, the so-called Itô–Krylov formula (established by N. V. Krylov) expresses that if $\sigma \sigma^*$ is uniformly elliptic, then Itô's formula also remains valid when u merely belongs to $\mathcal{W}^2_{p,\mathrm{loc}}$ with p strictly larger than the dimension of the process X. The Itô–Krylov formula was extended in [1] and [7] to continuous semimartingales $X_t := X_0 + M_t + V_t$ with a nondegenerate martingale part and some additional conditions. The nondegeneracy means that the matrix of the increasing processes $\langle M^i, M^j \rangle$ is uniformly elliptic.

THEOREM 2.1. *Assume that* (**H2**) *is satisfied. Let* (Y, Z) *be a solution of* eq(ξ, H). *Then, for any function u belonging to the space* $\mathcal{W}^2_{1,\mathrm{loc}}(\mathbb{R})$, *we have*

$$(2.4) \qquad u(Y_t) = u(Y_0) + \int_0^t u'(Y_s)\,dY_s + \frac{1}{2}\int_0^t u''(Y_s)|Z_s|^2\,ds.$$

Let us note that the functions of $\mathcal{W}^2_{1,\mathrm{loc}}(\mathbb{R})$ have a representant which belongs to $\mathcal{C}^1(\mathbb{R})$. This representant will be considered from now on.

PROOF OF THEOREM 2.1. For $R > |Y_0|$, let $\tau_R := \inf\{t > 0 : |Y_t| \geq R\}$. Since τ_R tends to infinity as R tends to infinity, it is then enough to establish formula (2.4) for $u(Y_{t\wedge\tau_R})$. Using Proposition 2.1, the term $\int_0^{t\wedge\tau_R} u''(Y_s)|Z_s|^2\,ds$ is well defined. Let u_n be a sequence of $\mathcal{C}^2$-class functions satisfying:

  (i) u_n converges uniformly to u in the interval $[-R, R]$,
  (ii) u'_n converges uniformly to u' in the interval $[-R, R]$,
  (iii) u''_n converges in $\mathbb{L}^1([-R, R])$ to u''.

Note that the sequence (u_n) can be obtained via a classical regularization by convolution. We use Itô's formula to show that

$$u_n(Y_{t\wedge\tau_R}) = u_n(Y_0) + \int_0^{t\wedge\tau_R} u'_n(Y_s)\,dY_s + \frac{1}{2}\int_0^{t\wedge\tau_R} u''_n(Y_s)|Z_s|^2\,ds.$$

Passing to the limit (on n) in the previous identity then using the above properties (i), (ii), (iii) and Proposition 2.1, we get

$$u(Y_{t\wedge\tau_R}) = u(Y_0) + \int_0^{t\wedge\tau_R} u'(Y_s)\,dY_s + \frac{1}{2}\int_0^{t\wedge\tau_R} u''(Y_s)|Z_s|^2\,ds.$$

Indeed, the limit for the left-hand side term, as well as those of the first and the second right-hand side terms can be obtained by using properties (i) and (ii). The limit for the third right-hand side term follows from property (iii) and Proposition 2.1. $\qquad\square$

3. QBSDEs with $\mathbb{L}^2$ terminal data and measurable generators.

In this section, we deal with eq$(\xi, f(y)|z|^2)$ and eq(ξ, Φ_f) where the function f is assumed merely integrable but not necessarily continuous. We then prove the existence, uniqueness and a comparison of solutions for eq$(\xi, f(y)|z|^2)$ as well as, the existence of solutions for eq(ξ, Φ_f). We also prove the existence of a minimal and a maximal for eq(ξ, Φ_f).

3.1. eq$(\xi, f(y)|z|^2)$ *with f integrable on* $\mathbb{R}$. The following proposition allows us to see that neither the existence of exponential moments of ξ nor the continuity of the generator are needed to the unique solvability of eq$(\xi, f(y)|z|^2)$.

PROPOSITION 3.1. *Assume that ξ is square integrable and f is globally integrable on $\mathbb{R}$. Then* $\mathrm{eq}(\xi, f(y)|z|^2)$ *has a unique solution* (Y, Z) *which belongs to* $\mathcal{S}^2 \times \mathcal{M}^2$.

PROOF. Let u be the function defined in Lemma A.1 (in the Appendix). Since ξ is square integrable then $u(\xi)$ is square integrable also. Therefore, $\mathrm{eq}(u(\xi), 0)$ has a unique solution in $\mathcal{S}^2 \times \mathcal{M}^2$. Since u^{-1} belongs to $\mathcal{W}^2_{1,\mathrm{loc}}(\mathbb{R})$, we then apply Theorem 2.1 to u^{-1} and the solution of $\mathrm{eq}(u(\xi), 0)$ to get the desired result. □

The following proposition allows us to compare the solutions for QBSDEs of type $\mathrm{eq}(\xi, f(y)|z|^2)$. The novelty is that the comparison of solutions holds whenever we can compare the generators a.e. in the y-variable. Both the generators can be non-Lipschitz.

PROPOSITION 3.2 (Comparison). *Let ξ_1, ξ_2 be $\mathcal{F}_T$-measurable and satisfy assumption* (**H1**). *Let f_1, f_2 be two elements of $\mathbb{L}^1(\mathbb{R})$. Let (Y^{f_1}, Z^{f_1}), (Y^{f_2}, Z^{f_2}) be respectively the solution of* $\mathrm{eq}(\xi_1, f_1(y)|z|^2)$ *and* $\mathrm{eq}(\xi_2, f_2(y)|z|^2)$. *Assume that $\xi_1 \leq \xi_2$ a.s. and $f_1 \leq f_2$ a.e. Then $Y_t^{f_1} \leq Y_t^{f_2}$ for all t $\mathbb{P}$-a.s.*

PROOF. According to Proposition 3.1, the solutions (Y^{f_1}, Z^{f_1}) and (Y^{f_2}, Z^{f_2}) belong to $\mathcal{S}^2 \times \mathcal{M}^2$. For a given function h, we put

$$u_h(x) := \int_0^x \exp\left(2 \int_0^y h(t)\,dt\right) dy.$$

The idea consists to suitably use Theorem 2.1 to the function $u_{f_1}(Y_T^{f_2})$. This gives

$$u_{f_1}(Y_T^{f_2}) = u_{f_1}(Y_t^{f_2}) + \int_t^T u'_{f_1}(Y_s^{f_2})\,dY_s^{f_2} + \frac{1}{2}\int_t^T u''_{f_1}(Y_s^{f_2})\,d\langle Y^{f_2}\rangle_s$$

$$= u_{f_1}(Y_t^{f_2}) + M_T - M_t - \int_t^T u'_{f_1}(Y_s^{f_2})f_2(Y_s^{f_2})|Z_s^{f_2}|^2\,ds$$

$$+ \frac{1}{2}\int_t^T u''_{f_1}(Y_s^{f_2})|Z_s^{f_2}|^2\,ds,$$

where $(M_t)_{0\leq t\leq T}$ is a martingale.

Since $u''_{f_2}(x) - 2f_2(x)u'_{f_2}(x) = 0$, $u''_{f_1}(x) - 2f_1(x)u'_{f_1}(x) = 0$ and $u'_{f_1}(x) \geq 0$, then

$$u_{f_1}(Y_T^{f_2}) = u_{f_1}(Y_t^{f_2}) + M_T - M_t - \int_t^T u'_{f_1}(Y_s^{f_2})[f_2(Y_s^{f_2}) - f_1(Y_s^{f_2})]|Z_s^{f_2}|^2\,ds.$$

Since the term $\int_t^T u'_{f_1}(Y_s^{f_2})[f_2(Y_s^{f_2}) - f_1(Y_s^{f_2})]|Z_s^{f_2}|^2\,ds$ is positive, then

$$u_{f_1}(Y_t^{f_2}) \geq u_{f_1}(Y_T^{f_2}) - (M_T - M_t).$$

Passing to conditional expectation then using the fact that u_{f_1} is an increasing function and $\xi_2 \geq \xi_1$, we get

$$
\begin{aligned}
u_{f_1}(Y_t^{f_2}) &\geq \mathbb{E}[u_{f_1}(Y_T^{f_2})/\mathcal{F}_t] \\
&= \mathbb{E}[u_{f_1}(\xi_2)/\mathcal{F}_t] \\
&\geq \mathbb{E}[u_{f_1}(\xi_1)/\mathcal{F}_t] \\
&= u_{f_1}(Y_t^{f_1}).
\end{aligned}
$$

Taking $u_{f_1}^{-1}$ in both sides, we get $Y_t^{f_2} \geq Y_t^{f_1}$. Proposition 3.2 is proved. $\quad\square$

COROLLARY 3.1 (Uniqueness). *Let ξ satisfies* (**H1**). *Let f_1, f_2 be two integrable functions. Let (Y^{f_1}, Z^{f_1}), (Y^{f_2}, Z^{f_2}) respectively denote arbitrary solutions of* eq$(\xi, f_1(y)|z|^2)$ *and* eq$(\xi, f_2(y)|z|^2)$. *If $f_1 = f_2$-a.e., then $(Y^{f_1}, Z^{f_1}) = (Y^{f_2}, Z^{f_2})$ in the space $\mathcal{S}^2 \times \mathcal{M}^2$.*

REMARK 3.1. Proposition 3.2 and Corollary 3.1 will be used (in PDEs part) in order to show the existence of a gap in the classical relation between the BSDEs and their corresponding PDEs.

3.2. eq(ξ, Φ_f), *with f integrable on $\mathbb{R}$.*

PROPOSITION 3.3. *Assume that* (**H1**) *is satisfied. Assume moreover that f is globally integrable on $\mathbb{R}$, but not necessarily continuous. Then,* eq(ξ, Φ_f) *has a minimal and a maximal solution which belong to $\mathcal{S}^2 \times \mathcal{M}^2$. In particular, all solutions are in $\mathcal{S}^2 \times \mathcal{M}^2$.*

PROOF. Let u be the function defined in Lemma A.1(I) in the Appendix. Let $\bar{\xi} := u(\xi)$ and $G(\bar{y}, \bar{z}) := (a + b|u^{-1}(\bar{y})|)u'[u^{-1}(\bar{y})] + c|\bar{z}|$. Consider the BSDE:

$$
(3.1) \qquad \bar{Y}_t = \bar{\xi} + \int_t^T G(\bar{Y}_s, \bar{Z}_s)\,ds - \int_t^T \bar{Z}_s\,dW_s.
$$

Using Assumption (**H1**) and Lemma A.1(I), we show that $\bar{\xi}$ is square integrable and G is continuous and of linear growth. Hence, according to [22], the BSDE (3.1) has a minimal and a maximal solutions in $\mathcal{S}^2 \times \mathcal{M}^2$. Since u^{-1} belongs to $\mathcal{W}_{1,\text{loc}}^2(\mathbb{R})$, then applying Theorem 2.1 to $u^{-1}(\bar{Y}_t)$, we deduce that eq(ξ, Φ_f) has a solution. Since u is a strictly increasing function, the minimality and the maximality of solutions are preserved between eq(ξ, Φ_f) and the BSDE (3.1). We shall show that all solutions of eq(ξ, Φ_f) lie in $\mathcal{S}^2 \times \mathcal{M}^2$. Let (Y, Z) be an arbitrary solution to eq(ξ, H). Let (Y^m, Z^m) and (Y^M, Z^M) be respectively the minimal and the maximal solution to this BSDE. Since Y^m and Y^M belong to $\mathcal{S}^2$, so does for Y. Thanks to Lemma A.1(I), it follows that $\bar{Y} := u(Y)$ also belongs to $\mathcal{S}^2$. Since $(\bar{Y}, \bar{Z}) := (u(Y), Zu'(Y))$ is a solution of eq$(\bar{\xi}, G)$ and G is at most

of linear growth, then standard arguments of BSDEs allow to show that $\bar{Z}$ belongs to $\mathcal{M}^2$. Using inequalities (A.3) in the Appendix, we show that Z belongs to $\mathcal{M}^2$. $\square$

4. eq(ξ, H) with $|H| \leq \Phi_f$ and f integrable and locally bounded.
Throughout this section, the following assumptions will be in force:

(**H4**) For a.e. (s, ω), H is continuous in (y, z).

(**H5**) There exists a locally bounded function f which is moreover positive and globally integrable on $\mathbb{R}$ such that for every s, y, z, $|H(s, y, z)| \leq \Phi_f(y, z)$.

THEOREM 4.1. *Assume that* (**H1**), (**H4**) *and* (**H5**) *are satisfied. Then* eq(ξ, H) *has at least one solution* (Y, Z) *in* $\mathcal{S}^2 \times \mathcal{M}^2$ *such that* $Y^L \leq Y \leq Y^U$.

The approach we give here shows that the solvability of eq(ξ, H) is reduced to the solvability of eq(ξ, Φ_f). It then allows us to control more precisely the integrability of the terminal datum. Note that neither a priori estimates nor the approximation of the solution will be used. Moreover, the comparison theorem will not be used. Technically, the idea of the proof consists to deduce the solvability of eq(ξ, H) from the solvability of a suitable reflected BSDE with two reflecting barriers.

PROOF. Let (Y^L, Z^L) be the minimal solution of eq$(-\xi^-, -\Phi_f)$ and (Y^U, Z^U) be the maximal solution of eq(ξ^+, Φ_f). We know by Proposition 3.3 that Y^L and Y^U exist and belong to $\mathcal{S}^2$. Since $Y^L \leq 0 \leq Y^U$, we consider the reflected BSDE with Y^L and Y^U as barriers:

$$(4.1) \quad \begin{cases} \text{(i)} \quad Y_t = \xi + \int_t^T H(s, Y_s, Z_s) \, ds - \int_t^T Z_s \, dW_s \\ \qquad\qquad + \int_t^T dK_s^+ - \int_t^T dK_s^- \quad \text{for each } t \leq T, \\ \text{(ii)} \quad \text{for every } t \leq T, Y_t^L \leq Y_t \leq Y_t^U, \\ \text{(iii)} \quad \int_0^T (Y_t - Y_t^L) \, dK_t^+ = \int_0^T (Y_t^U - Y_t) \, dK_t^- = 0, \qquad \text{a.s.,} \\ \text{(iv)} \quad K_0^+ = K_0^- = 0, \\ \qquad\quad K^+, K^- \text{ are continuous and nondecreasing,} \\ \text{(v)} \quad dK^+ \perp dK^-. \end{cases}$$

Since f is locally bounded, then it can be majorized by a piecewise constant function which we denote h. Therefore, using a suitable linear interpolation, one can construct a continuous function g such that $g \geq h \geq f$. Therefore, according to Theorem 3.2 of [17] (see Theorem A.1 in the Appendix), with

$$\eta_t = a + b(|Y_t^L| + |Y_t^U|) + c^2$$

and

$$C_t = 1 + \sup_{s \le t} \sup_{\alpha \in [0,1]} g(\alpha Y_s^L + (1-\alpha) Y_s^U),$$

the previous reflected BSDE has a solution (Y, Z, K^+, K^-) such that (Y, Z) belongs to $\mathcal{C} \times \mathcal{L}^2$. In order to show that (Y, Z) is a solution to our nonreflected eq(ξ, H), it is enough to prove that $dK^+ = dK^- = 0$. Since (Y^U, Z^U) is a solution to eq(ξ, Φ_f), then Tanaka's formula shows that

$$
\begin{aligned}
(Y_t^U - Y_t)^+ = {}& (Y_0^U - Y_0)^+ + \int_0^t \mathbf{1}_{\{Y_s^U > Y_s\}} [H(s, Y_s, Z_s) - \Phi_f(Y_s^U, Z_s^U)] \, ds \\
& + \int_0^t \mathbf{1}_{\{Y_s^U > Y_s\}} (dK_s^+ - dK_s^-) + \int_0^t \mathbf{1}_{\{Y_s^U > Y_s\}} (Z_s^U - Z_s) \, dW_s \\
& + L_t^0(Y^U - Y),
\end{aligned}
$$

where $L_t^0(Y^U - Y)$ denotes the right local time at time t and level 0 of the semimartingale $(Y^U - Y)$.

Identifying the terms of $(Y_t^U - Y_t)^+$ with those of $(Y_t^U - Y_t)$, we show that $(Z_s - Z_s^U)\mathbf{1}_{\{Y_s^U = Y_s\}} = 0$ for a.e. (s, ω). Since $\int_0^t \mathbf{1}_{\{Y_s^U = Y_s\}} \, dK_s^+ = 0$ and $H(s, y, z) \le \Phi_f(y, z)$, we deduce that

$$
\begin{aligned}
0 \le {}& L_t^0(Y^U - Y) + \int_0^t \mathbf{1}_{\{Y_s^U = Y_s\}} [\Phi_f(Y_s^U, Z_s^U) - H(s, Y_s, Z_s)] \, ds \\
= {}& -\int_0^t \mathbf{1}_{\{Y_s^U = Y_s\}} \, dK_s^- \le 0.
\end{aligned}
$$

It follows that $\int_0^t \mathbf{1}_{\{Y_s^U = Y_s\}} \, dK_s^- = 0$, which implies that $dK^- = 0$. Arguing symmetrically, one can show that $dK^+ = 0$.

We now prove that (Y, Z) belongs to $\mathcal{S}^2 \times \mathcal{M}^2$. Since both Y^U and Y^L belong to $\mathcal{S}^2$, so does for Y. It remains to prove that Z belongs to $\mathcal{M}^2$. Let v be the function defined in Lemma A.1(II). For $N > 0$, let $\tau_N := \inf\{t > 0 : |Y_t| + \int_0^t |v'(Y_s)|^2 |Z_s|^2 \, ds \ge N\} \wedge T$. Set $\mathrm{sgn}(x) = 1$ if $x \ge 0$ and $\mathrm{sgn}(x) = -1$ if $x < 0$. Since the map $x \mapsto v(|x|)$ belongs to $\mathcal{W}_{1,\mathrm{loc}}^2(\mathbb{R})$, then thanks to Theorem 2.1, we have for any $t \in [0, T]$,

$$
\begin{aligned}
v(|Y_0|) = {}& v(|Y_{t \wedge \tau_N}|) + \int_0^{t \wedge \tau_N} \mathrm{sgn}(Y_s) v'(|Y_s|) H(s, Y_s, Z_s) \, ds \\
& - \int_0^{t \wedge \tau_N} \frac{1}{2} v''(|Y_s|) |Z_s|^2 \, ds - \int_0^{t \wedge \tau_N} \mathrm{sgn}(Y_s) v'(|Y_s|) Z_s \, dW_s.
\end{aligned}
$$

Assumption (**H5**) and Lemma A.1(II) allow to show that for any $N > 0$,

$$
\frac{1}{4} \mathbb{E} \int_0^{t \wedge \tau_N} |Z_s|^2 \, ds \le v(|Y_0|) + \mathbb{E} \int_0^T [(a + b|Y_s|) v'(|Y_s|) + 4c^2 (v'(|Y_s|))^2] \, ds.
$$

The proof is completed by using Lemma A.1(II) and Fatou's lemma. $\quad\square$

REMARK 4.1. In [11], the authors considered the case $|H(t, y, z)| \leq a + b|y| + \frac{\gamma}{2}|z|^2$, where γ is some strictly positive constant and e^ξ has some p-moment, $p > 0$. This situation is not covered by the present paper since the constants are not globally integrable. However, the results of [11] can be obtained by the method we developed in Theorem 4.1. For instance, when $\gamma = 1$, the solvability of eq(ξ, H) is reduced to that of eq$(\xi, a + b|y| + \frac{1}{2}|z|^2)$. Using an exponential transformation, one can see that the solvability of eq$(\xi, a + b|y| + \frac{1}{2}|z|^2)$ is equivalent to that of eq$(e^\xi, a|y| + b|y|| \ln |y||)$. According to [2], this last logarithmic BSDE admits a solution whenever e^ξ has finite p-moment for some $p > 0$. Details are given in [2] and [3].

COROLLARY 4.1 (BMO property). *Assume that the terminal condition ξ is bounded and the generator H satisfies* (**H1**), (**H4**) *and* (**H5**). *Then* eq(ξ, H) *has a solution* (Y, Z) *such that Y is bounded and $\int_0^\cdot Z_s \, dW_s$ is a BMO martingale*, *that is, there exists a positive constant C such that for any $\mathcal{F}_t$-stopping time $\tau \leq T$ we have*

$$\mathbb{E}\left(\int_\tau^T |Z_s|^2 \, ds \Big/ \mathcal{F}_\tau \right) \leq C.$$

PROOF. Let Y^L be the minimal solution of eq$(-\xi^-, -\Phi_f)$ and Y^U be the maximal solution of eq(ξ^+, Φ_f). According to Theorem 4.1, eq(ξ, H) has a solution (Y, Z) in $\mathcal{S}^2 \times \mathcal{M}^2$ which satisfies for each $t \leq T$, $Y_t^L \leq Y_t \leq Y_t^U$. Therefore, for every $t \leq T$,

(4.2) $$|Y_t| \leq |Y_t^L| + |Y_t^U|.$$

Let u be the function defined in Lemma A.1. According to the proof of Proposition 3.3, the processes $Y^1 := u(Y^L)$ and $Y^2 := u(Y^U)$ satisfy BSDEs whose generators are continuous and of linear growth. Since ξ is bounded, then $u(\xi)$ is bounded. Therefore, standard arguments of BSDEs allow to show that Y^1 and Y^2 are bounded. It follows that $u(Y)$ is bounded, and hence Y is bounded. Let v be the function defined in Lemma A.1(II). Since the map $x \longmapsto v(|x|)$ belongs to $\mathcal{W}_{1,\mathrm{loc}}^2(\mathbb{R})$, then Theorem 2.1 shows that for any $\mathcal{F}_t$-stopping time $\tau \leq T$,

$$v(|Y_T|) = v(|Y_\tau|) + \int_\tau^T \left[\frac{1}{2} v''(|Y_s|)|Z_s|^2 - \mathrm{sgn}(Y_s)v'(|Y_s|)H(s, Y_s, Z_s) \right] ds$$

$$+ \int_\tau^T \mathrm{sgn}(Y_s)v'(|Y_s|)Z_s \, dW_s.$$

Since Y is bounded and Z belongs to $\mathcal{M}^2$, it follows that the stochastic integral in the right-hand side term of previous equality is a square integrable $\mathcal{F}_t$-martingale. Passing to conditional expectation then arguing as in the proof of Theorem 4.1 and

using Lemma A.1-(II), one can show that there exists a positive constant K_1 such that

$$\mathbb{E}\left(\int_\tau^T |Z_s|^2\,ds\Big/\mathcal{F}_\tau\right)$$

$$\leq K_1 + \mathbb{E}\left(\int_\tau^T \left[(a+b|Y_s|)u'(|Y_s|) + 4c^2(u'(|Y_s|))^2\right]ds\Big/\mathcal{F}_\tau\right)$$

$$\leq K_1 + \mathbb{E}\left(\int_0^T \left[(a+bK_1)K_1 + 4c^2 K_1\right]ds\Big/\mathcal{F}_\tau\right)$$

$$\leq K_1 + K_2 T.$$

Corollary 4.1 is proved. $\quad\square$

5. Application to quadratic partial differential equations. Let σ, b be measurable functions defined on $\mathbb{R}^d$ with values in $\mathbb{R}^{d\times d}$ and $\mathbb{R}^d$ respectively. Let $a := \sigma\sigma^*$ and define the operator L by

$$L := \sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x)\frac{\partial}{\partial x_i}.$$

Let g be a measurable function from $\mathbb{R}^d$ to $\mathbb{R}$. Consider the following semilinear PDE:

$$(5.1)\qquad \begin{cases} \dfrac{\partial v}{\partial s}(s,x) = Lv(s,x) + f(v(s,x))|\nabla_x v(s,x)|^2, & \text{on } [0,T)\times\mathbb{R}^d, \\ v(T,x) = g(x). \end{cases}$$

Assumptions:

(**H6**) σ, b are uniformly Lipschitz.
(**H7**) σ, b are of linear growth and f is continuous and integrable.
(**H8**) The terminal condition g is continuous and with polynomial growth.

THEOREM 5.1. *Assume* (**H6**), (**H7**) *and* (**H8**) *hold. Then* $v(t,x) := Y_t^{t,x}$ *is a viscosity solution of PDE* (5.1).

REMARK 5.1. The conclusion of Theorem 5.1 remains valid when assumption (**H6**) is replaced by: the martingale problem is well posed for $a := \frac{1}{2}\sigma\sigma^*$ and b.

To prove the existence of viscosity solution, we will follow the idea of [19]. To this end, we need the following touching property. This allows us to avoid the comparison theorem. The proof of the touching property can be found for instance in [19].

LEMMA 5.1. *Let $(\xi_t)_{0 \leq t \leq T}$ be a continuous adapted process such that*

$$d\xi_t = \beta(t)\,dt + \alpha(t)\,dW_t,$$

where β and α are continuous adapted processes such that β, $|\alpha|^2$ are integrable. If $\xi_t \geq 0$ a.s. for all t, then for all t,

$$\mathbf{1}_{\{\xi_t=0\}}\alpha(t) = 0 \qquad a.s.,$$

$$\mathbf{1}_{\{\xi_t=0\}}\beta(t) \geq 0 \qquad a.s.$$

PROOF OF THEOREM 5.1. We first prove the continuity of $v(t,x) := Y_t^{t,x}$. Let u be the transformation defined in Lemma A.1 (Appendix). Let $(\bar{Y}_s^{t,x}, \bar{Z}_s^{t,x})$ be the unique solution of eq$(u(g(X_T^{t,x}), 0)$ in $\mathcal{S}^2 \times \mathcal{M}^2$. Thanks to assumption (**H6**), it follows that the map $(t,x) \mapsto \bar{Y}_t^{t,x}$ is continuous. Using Lemma A.1, we deduce that $v(t,x) := Y_t^{t,x}$ is continuous in (t,x). We now show that v is a viscosity subsolution of PDE (5.1). We denote $(X_s, Y_s, Z_s) := (X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$. Since $v(t,x) = Y_t^{t,x}$, then the Markov property of X and the uniqueness of Y show that for every $s \in [t,T]$,

$$(5.2) \qquad v(s, X_s) = Y_s.$$

Let $\phi \in \mathcal{C}^{1,2}$ and (t,x) be a local maximum of $v - \phi$ which we suppose global and equal to 0, that is,

$$\phi(t,x) = v(t,x) \quad \text{and} \quad \phi(\bar{t}, \bar{x}) \geq v(\bar{t}, \bar{x}) \qquad \text{for each } (\bar{t}, \bar{x}).$$

This and equality (5.2) imply that

$$(5.3) \qquad \phi(s, X_s) \geq Y_s.$$

By Itô's formula, we have

$$\phi(s, X_s) = \phi(t, X_t) + \int_t^s \left(\frac{\partial \phi}{\partial r} + L\phi\right)(r, X_r)\,dr + \int_t^s \sigma \nabla_x \phi(r, X_r)\,dW_r.$$

Since Y satisfies the BSDE

$$Y_t = Y_s + \int_t^s f(Y_r)|Z_r|^2\,dr - \int_t^s Z_r\,dW_r,$$

then using inequality (5.3) and the touching property, we show that for each s,

$$\mathbf{1}_{\{\phi(s,X_s)=Y_s\}}\left(\frac{\partial \phi}{\partial s} + L\phi\right)(s, X_s) + f(Y_s)|Z_s|^2 \geq 0 \qquad a.s.,$$

and

$$\mathbf{1}_{\{\phi(s,X_s)=Y_s\}}\big|\sigma^T \nabla_x \phi(s, X_s) - Z_s\big| = 0 \qquad a.s.$$

For $s = t$, the second equation gives $Z_t = \sigma \nabla_x \phi(t, X_t) := \sigma \nabla_x \phi(t,x)$. And the first inequality gives the desired result. $\quad\square$

REMARK 5.2. Corollary 3.1 shows that eq(ξ, $f_1(y)|z|^2$) and eq(ξ, $f_2(y)|z|^2$) generate the same solution when $f_1 = f_2$ a.e. Thereby, for a square integrable ξ and $f \in \mathbb{L}^1(\mathbb{R})$, eq($\xi$, $f(y)|z|^2$) has a unique solution in $\mathcal{S}^2 \times \mathcal{M}^2$; however, the meaning of the following PDE:

$$\begin{cases} \dfrac{\partial v}{\partial s}(s, x) = Lv(s, x) + f(v(s, x))|\nabla_x v(s, x)|^2, & \text{on } [0, T) \times \mathbb{R}^d, \\ v(T, x) = g(x) \end{cases}$$

is not clear since f is defined merely a.e.

Indeed, if we denote by $\mathcal{N}_f$ the negligible set of all real numbers y for which f is not defined, then the quantity $f(v(s, x))$ in the previous PDE could not have sense when $v(s, x)$ belongs to $\mathcal{N}_f$. The gap stays in the fact that the BSDE $(g(X_T^{t,x}), f(y)|z|^2)$ has a unique solution while its associated PDE is not well defined. We think that, in this case, the associated PDE to eq(ξ, $f(y)|z|^2$) would have the form

$$\begin{cases} \dfrac{\partial v}{\partial s}(s, x) = Lv(s, x) + f(v(s, x))|\nabla_x v(s, x)|^2, & \text{on } [0, T) \times \mathbb{R}^d, \\ v(T, x) = g(x), \\ \nabla v(t, x) = 0 & \text{if } v(t, x) \in \mathcal{N}_f. \end{cases}$$

APPENDIX: SOME AUXILIARY RESULTS

The following lemma is used in order to eliminate the quadratic term from eq(ξ, $f(y)|z|^2$) and eq(ξ, $a + b|y| + c|z| + f(y)|z|^2$).

LEMMA A.1. (I) *Let* $f \in \mathbb{L}^1(\mathbb{R})$ *but not necessarily continuous. Then the function*

(A.1) $$u(x) := \int_0^x \exp\left(2 \int_0^y f(t) \, dt\right) dy$$

satisfies the differential equation $\frac{1}{2}u''(x) - f(x)u'(x) = 0$ *a.e. on* $\mathbb{R}$, *and has the following properties*:

(j) *u is a quasi-isometry, that is there exist two positive constants m and M such that, for any $x, y \in \mathbb{R}$,*

$$m|x - y| \leq |u(x) - u(y)| \leq M|x - y|.$$

(jj) *u is a one to one function from $\mathbb{R}$ onto $\mathbb{R}$. Both u and its inverse function u^{-1} belong to $\mathcal{W}_{1,\text{loc}}^2(\mathbb{R})$. If moreover f is continuous, then both u and u^{-1} belongs to $\mathcal{C}^2(\mathbb{R})$.*

(II) *Set*

$$K(y) := \int_0^y \exp\left(-2\int_0^x f(r)\,dr\right) dx.$$

Then the function

(A.2) $$v(x) := \int_0^x K(y)\exp\left(2\int_0^y f(t)\,dt\right) dy$$

satisfies the differential equation $\frac{1}{2}v''(x) - f(x)v'(x) = \frac{1}{2}$ *a.e. on* $\mathbb{R}$ *and has the following properties*:

(jjj) v *and* v' *are positive on* $\mathbb{R}_+$, *and* v *belongs to* $\mathcal{W}^2_{1,\mathrm{loc}}(\mathbb{R})$.
Moreover:
(jv) *The map* $x \longmapsto v(|x|)$ *belongs to* $\mathcal{W}^2_{1,\mathrm{loc}}(\mathbb{R})$.
(v) *There exist two strictly positive constants* c_1 *and* c_2 *such that for every* $x \in \mathbb{R}$, $v(|x|) \leq c_1|x|^2$ *and* $v'(|x|) \leq c_2|x|$.

PROOF. (I) Clearly, u and its inverse u^{-1} are continuous, one to one, strictly increasing functions and we have $\frac{1}{2}u''(x) - f(x)u'(x) = 0$ a.e. on $\mathbb{R}$. Since $u'(x) := \exp(2\int_0^x f(t)\,dt)$, then

(A.3) for every $x \in \mathbb{R}$, $\exp(-2\|f\|_{\mathbb{L}^1(\mathbb{R})}) \leq |u'(x)| \leq \exp(2\|f\|_{\mathbb{L}^1(\mathbb{R})})$.

This shows that u is a quasi-isometry.

We shall prove (jj). Using the quasi-isometry property of u, one can show that both u and u^{-1} belong to $\mathcal{C}^1$. Since the second generalized derivative u'' satisfies $u''(x) = 2f(x)u'(x)$ for a.e. x, we get that u'' belongs to $\mathbb{L}^1_{\mathrm{loc}}(\mathbb{R})$. Therefore, u belongs to $\mathcal{W}^2_{1,\mathrm{loc}}(\mathbb{R})$. Using again assertion (j), we prove that u^{-1} belongs to $\mathcal{W}^2_{1,\mathrm{loc}}(\mathbb{R})$.

(II) Obviously v and v' are positive on $\mathbb{R}_+$ and v satisfies the differential equation $\frac{1}{2}v''(x) - f(x)v'(x) = \frac{1}{2}$ a.e. on $\mathbb{R}$. Since f is globally integrable on $\mathbb{R}$, one can easily check that v belongs to $\mathcal{W}^2_{1,\mathrm{loc}}(\mathbb{R})$. This proves assertions (jjj), from which we deduce assertion (jv).

We shall prove assertion (v). Since $K(y) \leq y\exp(2\|f\|_{\mathbb{L}^1(\mathbb{R})})$ for each $y \geq 0$ and $\exp(2\int_0^y f(t)\,dt) \leq \exp(2\|f\|_{\mathbb{L}^1(\mathbb{R})})$, it follows that for every $x \in \mathbb{R}$, $v(|x|) \leq c_1|x|^2$. We show that $v'(|x|) \leq c_2|x|$ by similar computations. Lemma A.1 is proved. $\square$

The following result on two barriers reflected QBSDEs was obtained by Essaky and Hassani in [17]. It establishes the existence of solutions for reflected QBSDEs without assuming any integrability condition on the terminal datum. Of course, these solutions are not square integrable but merely belong to $\mathcal{C} \times \mathcal{L}^2$.

THEOREM A.1 ([17], Theorem 3.2). *Let L and U be continuous processes and ξ be a $\mathcal{F}_T$-measurable random variable. Assume that*:

(1) *For every $t \in [0, T]$, $L_t \leq U_t$.*

(2) *$L_T \leq \xi \leq U_T$.*

(3) *There exists a continuous semimartingale which passes between the barriers L and U.*

(4) *The generator h is continuous in (y, z) and satisfies for every (s, ω), every $y \in [L_s(\omega), U_s(\omega)]$ and every $z \in \mathbb{R}^d$*

$$\left| h(s, \omega, y, z) \right| \leq \eta_s(\omega) + C_s(\omega)|z|^2,$$

where η and C are two $\mathcal{F}_t$-adapted processes such that $\mathbb{E} \int_0^T \eta_s \, ds < \infty$ and C is continuous.

Then the following RBSDE has a minimal and a maximal solution:

$$(A.4) \quad \begin{cases} \text{(i)} & Y_t = \xi + \int_t^T h(s, Y_s, Z_s) \, ds - \int_t^T Z_s \, dW_s \\ & \quad + \int_t^T dK_s^+ - \int_t^T dK_s^- \quad \text{for all } t \leq T, \\ \text{(ii)} & \forall t \leq T, \quad L_t \leq Y_t \leq U_t, \\ \text{(iii)} & \int_0^T (Y_s - L_s) \, dK_s^+ = \int_0^T (U_s - Y_s) \, dK_s^- = 0 \quad a.s., \\ \text{(iv)} & K_0^+ = K_0^- = 0, \quad K^+, K^- \text{ are continuous nondecreasing}, \\ \text{(v)} & dK^+ \perp dK^-. \end{cases}$$

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