The Annals of Probability
2017, Vol. 45, No. 3, 1842–1898
DOI: 10.1214/16-AOP1103
© Institute of Mathematical Statistics, 2017

RANDOM WALKS ON INFINITE PERCOLATION CLUSTERS IN MODELS WITH LONG-RANGE CORRELATIONS

BY ARTEM SAPOZHNIKOV

University of Leipzig

For a general class of percolation models with long-range correlations on $\mathbb{Z}^d$, $d \geq 2$, introduced in [*J. Math. Phys.* **55** (2014) 083307], we establish regularity conditions of Barlow [*Ann. Probab.* **32** (2004) 3024–3084] that mesoscopic subballs of all large enough balls in the unique infinite percolation cluster have regular volume growth and satisfy a weak Poincaré inequality. As immediate corollaries, we deduce quenched heat kernel bounds, parabolic Harnack inequality, and finiteness of the dimension of harmonic functions with at most polynomial growth. Heat kernel bounds and the quenched invariance principle of [*Probab. Theory Related Fields* **166** (2016) 619–657] allow to extend various other known results about Bernoulli percolation by mimicking their proofs, for instance, the local central limit theorem of [*Electron. J. Probab.* **14** (209) 1–27] or the result of [*Ann. Probab.* **43** (2015) 2332–2373] that the dimension of at most linear harmonic functions on the infinite cluster is $d + 1$.

In terms of specific models, all these results are new for random interlacements at every level in any dimension $d \geq 3$, as well as for the vacant set of random interlacements [*Ann. of Math. (2)* **171** (2010) 2039–2087; *Comm. Pure Appl. Math.* **62** (2009) 831–858] and the level sets of the Gaussian free field [*Comm. Math. Phys.* **320** (2013) 571–601] in the regime of the so-called local uniqueness (which is believed to coincide with the whole supercritical regime for these models).

CONTENTS

Received October 2014; revised February 2016.

MSC2010 subject classifications. Primary 60K37; secondary 58J35.

Key words and phrases. Percolation, random walk, heat kernel, Harnack inequality, harmonic function, local limit theorem, Poincaré inequality, isoperimetric inequality, long-range correlations, random interlacements, Gaussian free field.

1. Introduction. Delmotte [14] proved that the transition density of the simple random walk on a graph satisfies Gaussian bounds and the parabolic Harnack inequality holds if all the balls have regular volume growth and satisfy a Poincaré inequality. Barlow [4] relaxed these conditions by imposing them only on all *large enough* balls, and showed that they imply large time Gaussian bounds and the elliptic Harnack inequality for large enough balls. Later, Barlow and Hambly [7] proved that the parabolic Harnack inequality also follows from Barlow's conditions. Barlow [4] verified these conditions for the supercritical cluster of Bernoulli percolation on $\mathbb{Z}^d$, which lead to the almost sure Gaussian heat kernel bounds and parabolic Harnack inequality. By using stationarity and heat kernel bounds, the quenched invariance principle was proved in [9, 25, 38], which lead to many further results about supercritical Bernoulli percolation, including the local central limit theorem [7] and the fact that the dimension of harmonic functions of at most linear growth is $d+1$ [8].

The independence property of Bernoulli percolation was essential in verifying Barlow's conditions, and up to now it has been the only example of percolation model for which the conditions were verified. On the other hand, once the conditions are verified, the derivation of all the further results uses rather robust methods and allows for extension to other stationary percolation models.

The aim of this paper is to develop an approach to verifying Barlow's conditions for infinite clusters of percolation models, which on the one hand, applies to supercritical Bernoulli percolation, but on the other, does not rely on independence and extends beyond models which are in any stochastic relation with Bernoulli percolation. Motivating examples for us are random interlacements, vacant set of random interlacements, and the level sets of the Gaussian free field [35, 39, 40]. In all these models, the spatial correlations decay only polynomially with distance, and classical Peierls-type arguments do not apply. A unified framework to study percolation models with strong correlations was proposed in [19], within which the shape theorem for balls [19] and the quenched invariance principle [32] were proved. In this paper, we prove that Barlow's conditions are satisfied by infinite percolation clusters in the general setting of [19]. In particular, all the above mentioned properties of supercritical Bernoulli percolation extend to all the models satisfying assumptions from [19], which include supercritical Bernoulli percolation, random interlacements at every level in any dimension $d \geq 3$, the vacant set of random interlacements and the level sets of the Gaussian free field in the regime of local uniqueness.

1.1. *General graphs*. Let G be an infinite connected graph with the vertex set $V(G)$ and the edge set $E(G)$. For $x, y \in V(G)$, define the weights

$$\nu_{xy} = \begin{cases} 1, & \{x, y\} \in E(G), \\ 0, & \text{otherwise}, \end{cases} \qquad \mu_x = \sum_y \nu_{xy},$$

and extend ν to the measure on $E(G)$ and μ to the measure on $V(G)$.

For functions $f : V(G) \to \mathbb{R}$ and $g : E(G) \to \mathbb{R}$, let $\int f \, d\mu = \sum_{x \in V(G)} f(x)\mu_x$ and $\int g \, d\nu = \sum_{e \in E(G)} g(e)\nu_e$, and define $|\nabla f| : E(G) \to \mathbb{R}$ by $|\nabla f|(\{x, y\}) = |f(x) - f(y)|$ for $\{x, y\} \in E(G)$.

Let d_G be the graph distance on G, and define $\mathrm{B}_G(x, r) = \{y \in V(G) : \mathrm{d}_G(x, y) \leq r\}$. We assume that $\mu(\mathrm{B}_G(x, r)) \leq C_0 r^d$ for all $x \in V(G)$ and $r \geq 1$. In particular, this implies that the maximal degree in G is bounded by C_0.

We say that a graph G satisfies the *volume regularity* and the *Poincaré inequality* if for all $x \in V(G)$ and $r > 0$, $\mu(\mathrm{B}_G(x, 2r)) \leq C_1 \cdot \mu(\mathrm{B}_G(x, r))$ and, respectively, $\min_a \int_{\mathrm{B}_G(x,r)} (f - a)^2 \, d\mu \leq C_2 \cdot r^2 \cdot \int_{E(\mathrm{B}_G(x,r))} |\nabla f|^2 \, d\nu$, with some constants C_1 and C_2. Graphs satisfying these conditions are very well understood. Delmotte proved in [14] the equivalence of such conditions to Gaussian bounds on the transition density of the simple random walk and to the parabolic Harnack inequality for solution to the corresponding heat equation, extending results of Grigoryan [20] and Saloff-Coste [36] for manifolds. Under the same assumptions, he also obtained in [15] explicit bounds on the dimension of harmonic functions on G of at most polynomial growth. Results of this flavor are classical in geometric analysis, with seminal ideas going back to the work of De Giorgi [17], Nash [30] and Moser [28, 29] on the regularity of solutions of uniformly elliptic second-order equations in divergence form.

The main focus of this paper is on random graphs, and more specifically on random subgraphs of $\mathbb{Z}^d$, $d \geq 2$. Because of local defects in such graphs caused by randomness, it is too restrictive to expect that various properties (e.g., Poincaré inequality, Gaussian bounds, or Harnack inequality) should hold globally. An illustrative example is the infinite cluster $\mathcal{C}_\infty$ of supercritical Bernoulli percolation [21] defined as follows. For $p \in [0, 1]$, remove vertices of $\mathbb{Z}^d$ independently with probability $(1 - p)$. The graph induced by the retained vertices almost surely contains an infinite connected component (which is unique) if $p > p_c(d) \in (0, 1)$, and contains only finite components if $p < p_c(d)$. It is easy to see that for any $p > p_c(d)$

with probability 1, $\mathcal{C}_\infty$ contains copies of any finite connected subgraph of $\mathbb{Z}^d$ attached to $\mathcal{C}_\infty$ by one edge, and thus, none of the above global properties can hold.

Barlow [4] proposed the following relaxed assumption which takes into account possible exceptional behavior on microscopic scales.

DEFINITION 1.1 ([4], Definition 1.7). Let C_V, C_P, and $C_W \geq 1$ be fixed constants. For $r \geq 1$ integer and $x \in V(G)$, we say that $\mathrm{B}_G(x,r)$ is (C_V, C_P, C_W)-*good* if $\mu(\mathrm{B}_G(x,r)) \geq C_V r^d$ and the weak Poincaré inequality

$$\min_a \int_{\mathrm{B}_G(x,r)} (f-a)^2 \, d\mu \leq C_P \cdot r^2 \cdot \int_{E(\mathrm{B}_G(x,C_W r))} |\nabla f|^2 \, d\nu$$

holds for all $f : \mathrm{B}_G(x, C_W r) \to \mathbb{R}$.

We say the ball $\mathrm{B}_G(x,R)$ is (C_V, C_P, C_W)-*very good* if there exists $N_{\mathrm{B}_G(x,R)} \leq R^{\frac{1}{d+2}}$ such that $\mathrm{B}_G(y,r)$ is (C_V, C_P, C_W)-good whenever $\mathrm{B}_G(y,r) \subseteq \mathrm{B}_G(x,R)$, and $N_{\mathrm{B}_G(x,R)} \leq r \leq R$.

REMARK 1.2. For any finite $H \subset V(G)$ and $f : H \to \mathbb{R}$, the minimum $\min_a \int_H (f-a)^2 \, d\mu$ is attained by the value $a = \overline{f}_H = \frac{1}{\mu(H)} \int_H f \, d\mu$.

For a very good ball, the conditions of volume growth and Poincaré inequality are allowed to fail on microscopic scales. Thus, if all large enough balls are very good, the graph can still have rather irregular local behavior. Despite that, on large enough scales it looks as if it was regular on all scales, as the following results from [4, 7, 8] illustrate.

Let $X = (X_n)_{n \geq 0}$ and $Y = (Y_t)_{t \geq 0}$ be the discrete and continuous time simple random walks on G. X is a Markov chain with transition probabilities $\frac{\nu_{xy}}{\mu_x}$, and Y is the Markov process with generator $\mathcal{L}_G f(x) = \frac{1}{\mu_x} \sum_y \nu_{xy}(f(y) - f(x))$. In words, the walker X (resp., Y) waits a unit time (resp., an exponential time with mean 1) at each vertex x, and then jumps to a uniformly chosen neighbor of x in G. For $x \in V(G)$, we denote by $\mathrm{P}_x = \mathrm{P}_{G,x}$ (resp., $\mathrm{Q}_x = \mathrm{Q}_{G,x}$) the law of X (resp., Y) started from x. The transition density of X (resp., Y) with respect to μ is denoted by $p_n(x,y) = p_{G,n}(x,y) = \frac{\mathrm{P}_{G,x}[X_n = y]}{\mu_y}$ (resp., $q_t(x,y) = q_{G,t}(x,y) = \frac{\mathrm{Q}_{G,x}[Y_t = y]}{\mu_y}$).

The first implications of Definition 1.1 are large time Gaussian bounds for q_t and p_n.

THEOREM 1.3 ([4], Theorem 5.7(a) and [7], Theorem 2.2). *Let $x \in V(G)$. If there exists $R_0 = R_0(x,G)$ such that $\mathrm{B}_G(x,R)$ is (C_V, C_P, C_W)-very good with $N_{\mathrm{B}_G(x,R)}^{3(d+2)} \leq R$ for each $R \geq R_0$, then there exist constants $C_i = C_i(d, C_0, C_V, C_P, C_W)$ such that for all $t \geq R_0^{3/2}$ and $y \in V(G)$,*

$$F_t(x,y) \leq C_1 \cdot t^{-\frac{d}{2}} \cdot e^{-C_2 \cdot \frac{\mathrm{d}_G(x,y)^2}{t}}, \qquad \text{if } t \geq \mathrm{d}_G(x,y), \tag{1}$$

$$F_t(x,y) \geq C_3 \cdot t^{-\frac{d}{2}} \cdot e^{-C_4 \cdot \frac{\mathrm{d}_G(x,y)^2}{t}}, \qquad \text{if } t \geq \mathrm{d}_G(x,y)^{\frac{3}{2}}, \tag{2}$$

where F_t stands for either q_t or $p_{\lfloor t\rfloor} + p_{\lfloor t\rfloor+1}$.

The next result gives an elliptic Harnack inequality.

THEOREM 1.4 ([4], Theorem 5.11). *There exists a constant $C_{\mathrm{ehi}} = C_{\mathrm{ehi}}(d, C_0, C_V, C_P, C_W)$ such that for any $x \in V(G)$ and $R \ge 1$, if $\mathrm{B}_G(x, R\log R)$ is (C_V, C_P, C_W)-very good with $N^{4(d+2)}_{\mathrm{B}_G(x,R\log R)} \le R$, then for any $y \in \mathrm{B}_G(x, \frac{1}{3}R\log R)$, and $h : \mathrm{B}_G(y, R+1) \to \mathbb{R}$ nonnegative and harmonic in $\mathrm{B}_G(y, R)$,*

$$\sup_{\mathrm{B}_G(y,\frac{1}{2}R)} h \le C_{\mathrm{ehi}} \cdot \inf_{\mathrm{B}_G(y,\frac{1}{2}R)} h. \tag{3}$$

In fact, more general parabolic Harnack inequality also takes place. (For the definition of parabolic Harnack inequality, see, e.g., [7], Section 3.)

THEOREM 1.5 ([7], Theorem 3.1). *There exists a constant $C_{\mathrm{phi}} = C_{\mathrm{phi}}(d, C_0, C_V, C_P, C_W)$ such that for any $x \in V(G)$, $R \ge 1$, and $R_1 = R\log R \ge 16$, if $\mathrm{B}_G(x, R_1)$ is (C_V, C_P, C_W)-very good with $N^{2(d+2)}_{\mathrm{B}_G(x,R_1)} \le \frac{R_1}{2\log R_1}$, then for any $y \in \mathrm{B}_G(x, \frac{1}{3}R_1)$, the parabolic Harnack inequality (in both discrete and continuous time settings) holds with constant C_{phi} for $(0, R^2] \times \mathrm{B}_G(y, R)$. In particular, the elliptic Harnack inequality (3) also holds.*

The next result is about the dimension of the space of harmonic functions on G with at most polynomial growth.

THEOREM 1.6 ([8], Theorem 4). *Let $x \in V(G)$. If there exists $R_0 = R_0(x, G)$ such that $\mathrm{B}_G(x, R)$ is (C_V, C_P, C_W)-very good for each $R \ge R_0$, then for any positive k, the space of harmonic functions h with $\limsup_{\mathrm{d}_G(x,y)\to\infty} \frac{h(y)}{\mathrm{d}_G(x,y)^k} < \infty$ is finite dimensional, and the bound on the dimension only depends on k, d, C_0, C_V, C_P and C_W.*

The notion of very good balls is most useful in studying random subgraphs of $\mathbb{Z}^d$. Up to now, it was only applied to the unique infinite connected component of supercritical Bernoulli percolation; see [4, 7]. Barlow [4], Section 2, showed that on an event of probability 1, for every vertex of the infinite cluster, all large enough balls centered at it are very good. Thus, all the above results are immediately transferred into the almost sure statements for all vertices of the infinite cluster.

Despite the conditions of Definition 1.1 are rather general, their validity up to now has only been shown for the independent percolation. The reason is that most of the analysis developed for percolation is tied very sensitively with the independence property of Bernoulli percolation. One usually first reduces combinatorial complexity of patterns by a coarse graining, and then balances the complexity out by exponential bounds coming from the independence; see, for example, [4], Section 2.

The main purpose of this paper is to develop an approach to verifying properties of Definition 1.1 for random graphs which does not rely on independence or any comparison with Bernoulli percolation, and, as a result, extending the known results about Bernoulli percolation to models with strong correlations. Our primal motivation comes from percolation models with strong correlations, such as random interlacements, vacant set of random interlacements, or the level sets of the Gaussian free field; see, for example, [35, 39, 40].

REMARK 1.7. (1) The lower bound of Theorem 1.3 can be slightly generalized by following the proof of [4], Theorem 5.7(a). Let $\epsilon \in (0, \frac{1}{2}]$ and $K > \frac{1}{\epsilon}$. If there exists $R_0 = R_0(x, G)$ such that $\mathrm{B}_G(x, R)$ is (C_V, C_P, C_W)-very good with $N_{\mathrm{B}_G(x,R)}^{K(d+2)} \le R$ for each $R \ge R_0$, then for all $t \ge R_0^{1+\epsilon}$,

$$F_t(x, y) \ge C_3 \cdot t^{-\frac{d}{2}} \cdot e^{-C_4 \cdot \frac{\mathrm{d}_G(x,y)^2}{t}}, \qquad \text{if } t \ge \mathrm{d}_G(x, y)^{1+\epsilon}. \tag{4}$$

The constants C_3 and C_4 are the same as in (2), in particular, they do not depend on K and ϵ. For $\epsilon = \frac{1}{2}$ and $K = 3$, we recover (2). (There is a small typo in the statements of [4], Theorem 5.7(a) and [7], Theorem 2.2: $R_0^{2/3}$ should be replaced by $R_0^{3/2}$.)

Indeed, the proof of [4], Theorem 5.7(a), is reduced to verifying assumptions of [4], Theorem 5.3, for some choice of R. The original choice of Barlow is $R = t^{\frac{2}{3}}$, and it implies (2). By restricting the choice of $N_{\mathrm{B}_G(x,R)}$ as above, one notices that all the conditions of [4], Theorem 5.3, are satisfied by $R = t^{\frac{1}{1+\epsilon}}$, implying (4).

(2) In order to prove the lower bound of (2) for the same range of t's as in the upper bound (1), one needs to impose a stronger assumption on the regularity of the balls $\mathrm{B}_G(x, R)$ (see, for instance, [4], Definition 5.4, of the exceedingly good ball and [4], Theorem 5.7(b)). In fact, the recent result of [5], Theorem 1.10, states that the volume doubling property and the Poincaré inequality satisfied by large enough balls are equivalent to certain partial Gaussian bounds (and also to the parabolic Harnack inequality in large balls).

(3) Under the assumptions of Theorem 1.5, various estimates of the heat kernels for the processes X and Y killed on exiting from a box are given in [7], Theorem 2.1.

(4) Theorem 1.6 holds under much weaker assumptions, although reminiscent of the ones of Definition 1.1 (see [8], Theorem 4). Roughly speaking, one assumes that the conditions from Definition 1.1 hold with $N_{\mathrm{B}_G(x,R)}$ only sublinear in R, that is, a volume growth condition and the weak Poincaré inequality should hold only for macroscopic subballs of $\mathrm{B}_G(x, R)$.

1.2. *The model.* We consider the measurable space $\Omega = \{0,1\}^{\mathbb{Z}^d}$, $d \geq 2$, equipped with the sigma-algebra $\mathcal{F}$ generated by the coordinate maps $\{\omega \mapsto \omega(x)\}_{x\in\mathbb{Z}^d}$. For any $\omega \in \{0,1\}^{\mathbb{Z}^d}$, we denote the induced subset of $\mathbb{Z}^d$ by

$$\mathcal{S} = \mathcal{S}(\omega) = \{x \in \mathbb{Z}^d : \omega(x) = 1\} \subseteq \mathbb{Z}^d.$$

We view $\mathcal{S}$ as a subgraph of $\mathbb{Z}^d$ in which the edges are drawn between any two vertices of $\mathcal{S}$ within ℓ^1-distance 1 from each other, where the ℓ^1 and ℓ^∞ norms of $x = (x(1), \dots, , x(d)) \in \mathbb{R}^d$ are defined in the usual way by $|x|_1 = \sum_{i=1}^d |x(i)|$ and $|x|_\infty = \max\{|x(1)|, \dots |x(d)|\}$, respectively. For $x \in \mathbb{Z}^d$ and $r \in \mathbb{R}_+$, we denote by $\mathrm{B}(x,r) = \{y \in \mathbb{Z}^d : |x-y|_\infty \leq \lfloor r \rfloor\}$ the closed ℓ^∞-ball in $\mathbb{Z}^d$ with radius $\lfloor r \rfloor$ and center at x.

DEFINITION 1.8. For $r \in [0,\infty]$, we denote by $\mathcal{S}_r$, the set of vertices of $\mathcal{S}$ which are in connected components of $\mathcal{S}$ of ℓ^1-diameter $\geq r$. In particular, $\mathcal{S}_\infty$ is the subset of vertices of $\mathcal{S}$ which are in infinite connected components of $\mathcal{S}$.

1.2.1. *Assumptions.* On $(\Omega, \mathcal{F})$ we consider a family of probability measures $(\mathbb{P}^u)_{u\in(a,b)}$ with $0 < a < b < \infty$, satisfying the following assumptions **P1**–**P3** and **S1**–**S2** from [19]. Parameters d, a and b are considered fixed throughout the paper, and dependence of various constants on them is omitted.

An event $G \in \mathcal{F}$ is called *increasing* (resp., *decreasing*), if for all $\omega \in G$ and $\omega' \in \{0,1\}^{\mathbb{Z}^d}$ with $\omega(y) \leq \omega(y')$ (resp., $\omega(y) \geq \omega(y')$) for all $y \in \mathbb{Z}^d$, one has $\omega' \in G$.

P1 (*Ergodicity*). For each $u \in (a,b)$, every lattice shift is measure preserving and ergodic on $(\Omega, \mathcal{F}, \mathbb{P}^u)$.

P2 (*Monotonicity*). For any $u, u' \in (a,b)$ with $u < u'$, and any increasing event $G \in \mathcal{F}$, $\mathbb{P}^u[G] \leq \mathbb{P}^{u'}[G]$.

P3 (*Decoupling*). Let $L \geq 1$ be an integer and $x_1, x_2 \in \mathbb{Z}^d$. For $i \in \{1,2\}$, let $A_i \in \sigma(\{\omega \mapsto \omega(y)\}_{y\in\mathrm{B}(x_i,10L)})$ be decreasing events, and $B_i \in \sigma(\{\omega \mapsto \omega(y)\}_{y\in\mathrm{B}(x_i,10L)})$ increasing events. There exist $R_{\mathrm{P}}, L_{\mathrm{P}} < \infty$ and $\varepsilon_{\mathrm{P}}, \chi_{\mathrm{P}} > 0$ such that for any integer $R \geq R_{\mathrm{P}}$ and $a < \widehat{u} < u < b$ satisfying

$$u \geq (1 + R^{-\chi_{\mathrm{P}}}) \cdot \widehat{u},$$

if $|x_1 - x_2|_\infty \geq R \cdot L$, then

$$\mathbb{P}^u[A_1 \cap A_2] \leq \mathbb{P}^{\widehat{u}}[A_1] \cdot \mathbb{P}^{\widehat{u}}[A_2] + e^{-f_{\mathrm{P}}(L)},$$

and

$$\mathbb{P}^{\widehat{u}}[B_1 \cap B_2] \leq \mathbb{P}^u[B_1] \cdot \mathbb{P}^u[B_2] + e^{-f_{\mathrm{P}}(L)},$$

where f_{P} is a real valued function satisfying $f_{\mathrm{P}}(L) \geq e^{(\log L)^{\varepsilon_{\mathrm{P}}}}$ for all $L \geq L_{\mathrm{P}}$.

S1 (*Local uniqueness*). There exists a function $f_S : (a, b) \times \mathbb{Z}_+ \to \mathbb{R}$ such that for each $u \in (a, b)$,

$$\text{there exist} \quad \Delta_S = \Delta_S(u) > 0 \text{ and } R_S = R_S(u) < \infty \tag{5}$$
$$\text{such that} \quad f_S(u, R) \geq (\log R)^{1+\Delta_S} \text{ for all } R \geq R_S,$$

and for all $u \in (a, b)$ and $R \geq 1$, the following inequalities are satisfied:

$$\mathbb{P}^u[\mathcal{S}_R \cap \mathrm{B}(0, R) \neq \varnothing] \geq 1 - e^{-f_S(u,R)},$$

and

$$\mathbb{P}^u[\text{for all } x, y \in \mathcal{S}_{R/10} \cap \mathrm{B}(0, R), x \text{ is connected to } y \text{ in } \mathcal{S} \cap \mathrm{B}(0, 2R)]$$
$$\geq 1 - e^{-f_S(u,R)}.$$

S2 (*Continuity*). Let $\eta(u) = \mathbb{P}^u[0 \in \mathcal{S}_\infty]$. The function $\eta(\cdot)$ is positive and continuous on (a, b).

REMARK 1.9. (1) The use of assumptions **P2**, **P3**, and **S2** will not be explicit in this paper. They are only used to prove likeliness of certain patterns in $\mathcal{S}_\infty$ produced by a multi-scale renormalization; see (37). (Of course, they are also used in already known results of Theorems 1.10 and 1.11.) Roughly speaking, we use **P3** repeatedly on multiple scales for a convergent sequence of parameters u_k and use **P2** and **S2** to establish convergence of iterations.

(2) If the family $\mathbb{P}^u$, $u \in (a, b)$, satisfies **S1**, then a union bound argument gives that for any $u \in (a, b)$, $\mathbb{P}^u$-a.s., the set $\mathcal{S}_\infty$ is nonempty and connected, and there exist constants $C_i = C_i(u)$ such that for all $R \geq 1$,

$$\mathbb{P}^u[\mathcal{S}_\infty \cap \mathrm{B}(0, R) \neq \varnothing] \geq 1 - C_1 \cdot e^{-C_2 \cdot (\log R)^{1+\Delta_S}}. \tag{6}$$

1.2.2. *Examples*. Here, we briefly list some motivating examples (already announced earlier in the paper) of families of probability measures satisfying assumptions **P1**–**P3** and **S1**–**S2**. All these examples were considered in details in [19], and we refer the interested reader to [19], Section 2, for the proofs and further details.

(1) Bernoulli percolation with parameter $u \in [0, 1]$ corresponds to the product measure $\mathbb{P}^u$ with $\mathbb{P}^u[\omega(x) = 1] = 1 - \mathbb{P}^u[\omega(x) = 0] = u$. The family $\mathbb{P}^u$, $u \in (a, b)$, satisfies assumptions **P1**–**P3** and **S1**–**S2** for any $d \geq 2$ and $p_c(d) < a < b \leq 1$; see [21].
(2) Random interlacements at level $u > 0$ is the random subgraph of $\mathbb{Z}^d$, $d \geq 3$, corresponding to the measure $\mathbb{P}^u$ defined by the equations

$$\mathbb{P}^u[\mathcal{S} \cap K = \varnothing] = e^{-u \cdot \mathrm{cap}(K)}, \qquad \text{for all finite } K \subset \mathbb{Z}^d,$$

where $\mathrm{cap}(\cdot)$ is the discrete capacity. It follows from [33, 40, 41] that the family $\mathbb{P}^u$, $u \in (a, b)$, satisfies assumptions **P1**–**P3** and **S1**–**S2** for any $0 < a <$

$b < \infty$. Curiously, for any $u > 0$, $\mathcal{S}$ is $\mathbb{P}^u$-almost surely connected [40], that is, $\mathcal{S}_\infty = \mathcal{S}$.

(3) Vacant set of random interlacements at level $u > 0$ is the complement of the random interlacements at level u in $\mathbb{Z}^d$. It corresponds to the measure $\mathbb{P}^u$ defined by the equations

$$\mathbb{P}^u[K \subseteq \mathcal{S}] = e^{-u \cdot \mathrm{cap}(K)}, \qquad \text{for all finite } K \subset \mathbb{Z}^d.$$

Unlike random interlacements, the vacant set undergoes a percolation phase transition in u [39, 40]. If $u < u_*(d) \in (0, \infty)$ then $\mathbb{P}^u$-almost surely $\mathcal{S}_\infty$ is nonempty and connected, and if $u > u_*(d)$, $\mathcal{S}_\infty$ is $\mathbb{P}^u$-almost surely empty. It is known that the family $\mathbb{P}^{\frac{1}{u}}$, $u \in (a, b)$, satisfies assumptions **P1**–**P3** for any $0 < a < b < \infty$ [40, 41], **S2** for any $\frac{1}{u_*(d)} < a < b < \infty$ [42], and **S1** for some $\frac{1}{u_*(d)} < a < b < \infty$ [18].

(4) The Gaussian free field on $\mathbb{Z}^d$, $d \ge 3$, is a centered Gaussian field with covariances given by the Green function of the simple random walk on $\mathbb{Z}^d$. The excursion set above level $h \in \mathbb{R}$ is the random subset of $\mathbb{Z}^d$ where the fields exceeds h. Let $\mathbb{P}^h$ be the measure on Ω for which $\mathcal{S}$ has the law of the excursion set above level h. The model exhibits a non-trivial percolation phase transition [12, 35]. If $h < h_*(d) \in [0, \infty)$, then $\mathbb{P}^h$-almost surely $\mathcal{S}_\infty$ is nonempty and connected, and if $h > h_*(d)$, $\mathcal{S}_\infty$ is $\mathbb{P}^h$-almost surely empty. It was proved in [19, 35] that the family $\mathbb{P}^{h_*(d)-h}$, $h \in (a, b)$, satisfies assumptions **P1**–**P3** and **S2** for any $0 < a < b < \infty$, and **S1** for some $0 < a < b < \infty$.

The last three examples are particularly interesting, since they have *polynomial decay of spatial correlations* and cannot be studied by comparison with Bernoulli percolation on any scale. In particular, many of the methods developed for Bernoulli percolation do not apply. As we see from the examples, assumptions **P1**–**P3** and **S2** are satisfied by all the 4 models through their *whole* supercritical phases. However, assumption **S1** is currently verified for the whole range of interesting parameters only in the cases of Bernoulli percolation and random interlacements, and only for a nonempty subset of interesting parameters in the last two examples. We call all the parameters u for which $\mathbb{P}^u$ satisfies **S1** the regime of *local uniqueness* (since under **S1**, there is a unique giant cluster in each large box). It is a challenging open problem to verify if the regime of local uniqueness coincides with the supercritical phase for the vacant set of random interlacements and the level sets of the Gaussian free field. A positive answer to this question will imply that all the results of this paper hold unconditionally also for the last two considered examples through their whole supercritical phases.

1.2.3. *Known results*. Below we recall some results from [19, 32] about the large scale behavior of graph distances in $\mathcal{S}_\infty$ and the quenched invariance principle for the simple random walk on $\mathcal{S}_\infty$. Both results are formulated in the form suitable for our applications.

THEOREM 1.10 ([19], Theorem 1.3). *Let $d \geq 2$ and $\theta_{\mathrm{chd}} \in (0,1)$. Assume that the family of measures $\mathbb{P}^u$, $u \in (a,b)$, satisfies assumptions* **P1**–**P3** *and* **S1**–**S2**. *Let $u \in (a,b)$. There exist $\Omega_{\mathrm{chd}} \in \mathcal{F}$ with $\mathbb{P}^u[\Omega_{\mathrm{chd}}] = 1$, constants C_{chd}, $c_{1.10}$ and $C_{1.10}$ all dependent on u and θ_{chd}, and random variables $R_{\mathrm{chd}}(x)$, $x \in \mathbb{Z}^d$, such that for all $\omega \in \Omega_{\mathrm{chd}} \cap \{0 \in \mathcal{S}_\infty\}$ and $x \in \mathcal{S}_\infty(\omega)$:*

(a) $R_{\mathrm{chd}}(x,\omega) < \infty$,

(b) *for all $R \geq R_{\mathrm{chd}}(x,\omega)$ and $y, z \in \mathrm{B}_{\mathbb{Z}^d}(x,R) \cap \mathcal{S}_\infty(\omega)$,*

$$\mathrm{d}_{\mathcal{S}_\infty(\omega)}(y,z) \leq C_{\mathrm{chd}} \cdot \max\{\mathrm{d}_{\mathbb{Z}^d}(y,z),\ R^{\theta_{\mathrm{chd}}}\},$$

(c) *for all $z \in \mathbb{Z}^d$ and $r \geq 1$,*

$$\mathbb{P}^u[R_{\mathrm{chd}}(z) \geq r] \leq C_{1.10} \cdot e^{-c_{1.10} \cdot (\log r)^{1+\Delta_{\mathrm{S}}}},$$

where Δ_{S} is defined in (5).

For $T > 0$, let $C[0,T]$ be the space of continuous functions from $[0,T]$ to $\mathbb{R}^d$, and $\mathcal{W}_T$ the Borel sigma-algebra on it. Let

$$\widetilde{B}_n(t) = \frac{1}{\sqrt{n}}\big(X_{\lfloor tn \rfloor} + (tn - \lfloor tn \rfloor) \cdot (X_{\lfloor tn \rfloor + 1} - X_{\lfloor tn \rfloor})\big). \tag{7}$$

THEOREM 1.11 ([32], Theorem 1.1, Lemma A.1, and Section 5). *Let $d \geq 2$. Assume that the family of measures $\mathbb{P}^u$, $u \in (a,b)$, satisfies assumptions* **P1**–**P3** *and* **S1**–**S2**. *Let $u \in (a,b)$ and $T > 0$. There exist $\Omega_{\mathrm{qip}} \in \mathcal{F}$ with $\mathbb{P}^u[\Omega_{\mathrm{qip}}] = 1$ and a nondegenerate matrix $\Sigma = \Sigma(u)$, such that for all $\omega \in \Omega_{\mathrm{qip}} \cap \{0 \in \mathcal{S}_\infty\}$:*

(a) *there exists $\chi : \mathcal{S}_\infty(\omega) \to \mathbb{R}^d$ such that $x \mapsto x + \chi(x)$ is harmonic on $\mathcal{S}_\infty(\omega)$, and $\lim_{n\to\infty} \frac{1}{n} \max_{x \in \mathcal{S}_\infty \cap \mathrm{B}(0,n)} |\chi(x)| = 0$,*

(b) *the law of $(\widetilde{B}_n(t))_{0 \leq t \leq T}$ on $(C[0,T], \mathcal{W}_T)$ converges weakly (as $n \to \infty$) to the law of Brownian motion with zero drift and covariance matrix Σ.*

In addition, if reflections and rotations of $\mathbb{Z}^d$ by $\frac{\pi}{2}$ preserve $\mathbb{P}^u$, then the limiting Brownian motion isotropic, that is, $\Sigma = \sigma^2 \cdot \mathrm{I}_d$ with $\sigma^2 > 0$.

REMARK 1.12. [32], Theorem 1.1, is stated for the ("blind") random walk which jumps to a neighbor with probability $\frac{1}{2d}$ and stays put with probability $1 - \frac{1}{2d} \cdot$ (number of neighbors). Since the blind walk and the simple random walk are time changes of each other, the invariance principle for one process implies the one for the other (see, for instance, [9], Lemma 6.4).

1.3. *Main results*. The main contribution of this paper is Theorem 1.13, where we prove that under the assumptions **P1**–**P3** and **S1**–**S2**, all large enough balls in $\mathcal{S}_\infty$ are very good in the sense of Definition 1.1. This result has many immediate applications, including Gaussian heat kernel bounds, Harnack inequalities, and finiteness of the dimension of harmonic functions on $\mathcal{S}_\infty$ with prescribed polynomial growth; see Theorems 1.3, 1.5, 1.4, 1.6. In fact, all the results from [7, 8] can be easily translated from Bernoulli percolation to our setting, since (as also pointed out by the authors) their proofs only rely on (some combinations of) stationarity, Gaussian heat kernel bounds, and the invariance principle. Among such results are estimates on the gradient of the heat kernel (Theorem 1.16) and on the Green function (Theorem 1.17), which will be deduced from the heat kernel bounds by replicating the proofs of [8],Theorem 6, and [7], Theorem 1.2(a), the fact that the dimension of at most linear harmonic functions on $\mathcal{S}_\infty$ is $d+1$ (Theorem 1.18), the local central limit theorem (Theorem 1.19), and the asymptotic for the Green function (Theorem 1.20), which we derive from the heat kernel bounds and the quenched invariance principle by mimicking the proofs of [8], Theorem 5, [7], Theorem 1.1, and [7], Theorem 1.2(b,c).

We begin by stating the main result of this paper.

THEOREM 1.13. *Let $d \geq 2$ and $\theta_{\mathrm{vgb}} \in (0, \frac{1}{d+2})$. Assume that the family of measures $\mathbb{P}^u$, $u \in (a,b)$, satisfies assumptions* **P1**–**P3** *and* **S1**–**S2**. *Let $u \in (a,b)$. There exist $\Omega_{\mathrm{vgb}} \in \mathcal{F}$ with $\mathbb{P}^u[\Omega_{\mathrm{vgb}}] = 1$, constants C_V, C_P, C_W, $c_{1.13}$ and $C_{1.13}$ all dependent on u and θ_{vgb}, and random variables $R_{\mathrm{vgb}}(x)$, $x \in \mathbb{Z}^d$, such that for all $\omega \in \Omega_{\mathrm{vgb}} \cap \{0 \in \mathcal{S}_\infty\}$ and $x \in \mathcal{S}_\infty(\omega)$:*

(a) $R_{\mathrm{vgb}}(x,\omega) < \infty$,

(b) *for all $R \geq R_{\mathrm{vgb}}(x,\omega)$, $\mathrm{B}_{\mathcal{S}_\infty(\omega)}(x,R)$ is (C_V, C_P, C_W)-very good with $N_{\mathrm{B}_{\mathcal{S}_\infty(\omega)}(x,R)} \leq R^{\theta_{\mathrm{vgb}}}$,*

(c) *for all $z \in \mathbb{Z}^d$ and $r \geq 1$,*

$$\mathbb{P}^u\big[R_{\mathrm{vgb}}(z) \geq r\big] \leq C_{1.13} \cdot e^{-c_{1.13}\cdot(\log r)^{1+\Delta_{\mathrm{S}}}}, \tag{8}$$

where Δ_{S} is defined in (5).

Theorem 1.13 will immediately follow from a certain isoperimetric inequality; see Definition 4.1, Claim 4.2 and Proposition 4.3. This isoperimetric inequality is more than enough to imply the weak Poincaré inequality that we need. In fact, as we learned from a discussion with Jean-Dominique Deuschel, it implies stronger Sobolev inequalities, and may be useful in situations beyond the goals of this paper (see, e.g., [31], Section 3).

COROLLARY 1.14. *Theorem 1.13 immediately implies that all the results of Theorems 1.3, 1.5, 1.4 and 1.6 hold almost surely for $G = \mathcal{S}_\infty$. Since the constants C_V, C_P and C_W in the statement of Theorem 1.13 are deterministic, all the constants in Theorems 1.3, 1.5, 1.4 and 1.6 are also deterministic.*

Combining Corollary 1.14 with Theorem 1.10 and Remark 1.7(1), we notice that the quenched heat kernel bounds of Theorem 1.3 hold almost surely for $G =$

$\mathcal{S}_\infty$ with d_G replaced by $\mathrm{d}_{\mathbb{Z}^d}$ in (1), (2) and (4). Since we will use the quenched heat kernel bounds often in the paper, we give a precise statement here.

THEOREM 1.15. *Let $d \geq 2$. Assume that the family of measures $\mathbb{P}^u$, $u \in (a,b)$, satisfies assumptions* **P1–P3** *and* **S1–S2**. *Let $u \in (a,b)$ and $\epsilon > 0$. There exist $\Omega_{\mathrm{hk}} \in \mathcal{F}$ with $\mathbb{P}^u[\Omega_{\mathrm{hk}}] = 1$, constants $C_i = C_i(u)$, $C_{1.15} = C_{1.15}(u,\epsilon)$, and $c_{1.15} = c_{1.15}(u,\epsilon)$, and random variables $T_{\mathrm{hk}}(x,\epsilon)$, $x \in \mathbb{Z}^d$, such that for all $\omega \in \Omega_{\mathrm{hk}} \cap \{0 \in \mathcal{S}_\infty\}$ and $x \in \mathcal{S}_\infty(\omega)$:*

(a) $T_{\mathrm{hk}}(x,\epsilon,\omega) < \infty$,
(b) *for all $t \geq T_{\mathrm{hk}}(x,\epsilon,\omega)$ and $y \in \mathcal{S}_\infty(\omega)$,*

$$F_t(x,y) \leq C_1 \cdot t^{-\frac{d}{2}} \cdot e^{-C_2 \cdot \frac{\mathrm{D}(x,y)^2}{t}}, \qquad \text{if } t \geq \mathrm{D}(x,y), \tag{9}$$

$$F_t(x,y) \geq C_3 \cdot t^{-\frac{d}{2}} \cdot e^{-C_4 \cdot \frac{\mathrm{D}(x,y)^2}{t}}, \qquad \text{if } t \geq \mathrm{D}(x,y)^{1+\epsilon}, \tag{10}$$

where F_t stands for either q_t or $p_{\lfloor t \rfloor} + p_{\lfloor t \rfloor + 1}$, and D for either $\mathrm{d}_{\mathcal{S}_\infty(\omega)}$ or $\mathrm{d}_{\mathbb{Z}^d}$,
(c) *for all $z \in \mathbb{Z}^d$ and $r \geq 1$,*

$$\mathbb{P}^u\big[T_{\mathrm{hk}}(z,\epsilon) \geq r\big] \leq C_{1.15} \cdot e^{-c_{1.15} \cdot (\log r)^{1+\Delta_{\mathrm{S}}}}, \tag{11}$$

where Δ_{S} is defined in (5).

In the applications of Theorem 1.15 in this paper, *we always take $\epsilon = \frac{1}{2}$* (the original choice of Barlow) and omit the dependence on ϵ from the notation. For instance, we will always write $T_{\mathrm{hk}}(x)$ meaning $T_{\mathrm{hk}}(x, \frac{1}{2})$. Any other choice of ϵ would also do.

It is well known that the parabolic Harnack inequality of Theorem 1.5 implies Hölder continuity of caloric functions (e.g., q_t and p_n); see [7], Proposition 3.2, in particular, by Corollary 1.14 this is true almost surely for $G = \mathcal{S}_\infty$. The next result is a sharp bound on the discrete gradient of the heat kernel, proved in [8], Theorem 6, for supercritical Bernoulli percolation using an elegant entropy argument.

THEOREM 1.16. *Let $d \geq 2$. Assume that the family of measures $\mathbb{P}^u$, $u \in (a,b)$, satisfies assumptions* **P1–P3** *and* **S1–S2**. *Let $u \in (a,b)$. There exist constants $C_i = C_i(u)$, such that for all $x, x', y \in \mathbb{Z}^d$ and $n > \max\{\mathrm{d}_{\mathbb{Z}^d}(x,y), \mathrm{d}_{\mathbb{Z}^d}(x',y)\}$,*

$$\begin{aligned} &\mathbb{E}^u\big[\big(p_n(x,y) - p_{n-1}(x',y)\big)^2 \cdot \mathbb{1}_{\{y \in \mathcal{S}_\infty\}} \cdot \mathbb{1}_{\{x \text{ and } x' \text{ are neighbors in } \mathcal{S}_\infty\}}\big] \\ &\quad \leq \frac{C_1}{n^{d+1}} \cdot e^{-C_2 \cdot \frac{\mathrm{d}_{\mathbb{Z}^d}(x,y)^2}{n}}. \end{aligned}$$

The heat kernel bounds of Theorem 1.15 imply also the following quenched estimates on the Green function $g_G(x,y) = \int_0^\infty q_{G,t}(x,y)\,dt = \sum_{n \geq 0} p_{G,n}(x,y)$ for almost all $G = \mathcal{S}_\infty$. It is proved in [7], Theorem 1.2 for supercritical Bernoulli percolation, but extension to our setting is rather straightforward.

THEOREM 1.17. *Let $d \geq 3$. Assume that the family of measures $\mathbb{P}^u$, $u \in (a,b)$, satisfies assumptions* **P1**–**P3** *and* **S1**–**S2**. *Let $u \in (a,b)$. There exist constants $C_i = C_i(u)$ such that for all $\omega \in \Omega_{\mathrm{hk}}$ and distinct $x, y \in \mathcal{S}_\infty(\omega)$, if $\mathrm{d}_{\mathbb{Z}^d}(x,y)^2 \geq \min\{T_{\mathrm{hk}}(x), T_{\mathrm{hk}}(y)\} \cdot (1 + C_3 \cdot \log \mathrm{d}_{\mathbb{Z}^d}(x,y))$, then*

$$C_1 \cdot \mathrm{d}_{\mathbb{Z}^d}(x,y)^{2-d} \leq g_{\mathcal{S}_\infty(\omega)}(x,y) \leq C_2 \cdot \mathrm{d}_{\mathbb{Z}^d}(x,y)^{2-d}.$$

The remaining results are derived from the Gaussian heat kernel bounds and the quenched invariance principle. In the setting of supercritical Bernoulli percolation, all of them were obtained in [7, 8], but all the proofs extend directly to our setting.

We begin with results about harmonic functions on $\mathcal{S}_\infty$. It is well known that Theorems 1.13 and 1.4 imply the almost sure Liouville property for positive harmonic functions on $\mathcal{S}_\infty$. The absence of nonconstant sublinear harmonic functions on $\mathcal{S}_\infty$ is even known assuming just stationary of $\mathcal{S}$ (see [8], Theorem 3 and discussion below). In particular, it implies the uniqueness of the function χ in Theorem 1.11(a). The following result about the dimension of at most linear harmonic functions is classical on $\mathbb{Z}^d$. It was extended to supercritical Bernoulli percolation on $\mathbb{Z}^d$ in [8], Theorem 5.

THEOREM 1.18. *Let $d \geq 2$. Assume that the family of measures $\mathbb{P}^u$, $u \in (a,b)$, satisfies assumptions* **P1**–**P3** *and* **S1**–**S2**. *Let $u \in (a,b)$. There exist $\Omega_{\mathrm{hf}} \in \mathcal{F}$ with $\mathbb{P}^u[\Omega_{\mathrm{hf}}] = 1$ such that for all $\omega \in \Omega_{\mathrm{hf}} \cap \{0 \in \mathcal{S}_\infty\}$, the dimension of the vector space of harmonic functions on $\mathcal{S}_\infty(\omega)$ with at most linear growth equals $d+1$.*

Since the parabolic Harnack inequality for solutions to the heat equation on $\mathcal{S}_\infty$ implies Hölder continuity of p_n and q_t, it is possible to replace the weak convergence of Theorem 1.11 by pointwise convergence. [7], Theorems 4.5 and 4.6, give general sufficient conditions for the local central limit theorem on general graphs. They were verified in [7], Theorem 1.1, for supercritical Bernoulli percolation. Theorems 1.11 and 1.15 allow to check these conditions in our setting leading to the following (same as for Bernoulli percolation) result. For $x \in \mathbb{R}^d$, $t > 0$, the Gaussian heat kernel with covariance matrix Σ is defined as

$$k_{\Sigma,t}(x) = \frac{1}{\sqrt{(2\pi t)^d \det(\Sigma)}} \cdot \exp\left(-\frac{x'\Sigma^{-1}x}{2t}\right),$$

where x' is the transpose of x.

THEOREM 1.19. *Let $d \geq 2$. Assume that the family of measures $\mathbb{P}^u$, $u \in (a,b)$, satisfies assumptions* **P1**–**P3** *and* **S1**–**S2**. *Let $u \in (a,b)$, $m = \mathbb{E}^u[\mu_0 \cdot \mathbb{1}_{0 \in \mathcal{S}_\infty}]$, and $T > 0$. There exist $\Omega_{\mathrm{lclt}} \in \mathcal{F}$ with $\mathbb{P}^u[\Omega_{\mathrm{lclt}}] = 1$, and a nondegenerate covariance matrix $\Sigma = \Sigma(u)$ such that for all $\omega \in \Omega_{\mathrm{lclt}} \cap \{0 \in \mathcal{S}_\infty\}$,*

$$\lim_{n\to\infty} \sup_{x \in \mathbb{R}^d} \sup_{t \geq T} \left| n^{\frac{d}{2}} \cdot F_{nt}(0, g_n(x)) - \frac{C(F)}{m} \cdot k_{\Sigma,t}(x) \right| = 0, \tag{12}$$

where F_s stands for q_s or $p_{\lfloor s\rfloor} + p_{\lfloor s\rfloor+1}$, $C(F)$ is 1 if $F = q$ and 2 otherwise, and $g_n(x)$ is the closest point in $\mathcal{S}_\infty$ to $\sqrt{n}x$.

Theorems 1.15 and 1.19 imply the following asymptotic for the Green function, extending results of [7], Theorem 1.2(b,c), to our setting. For a covariance matrix Σ, let $\mathrm{G}_\Sigma(x) = \int_0^\infty k_{\Sigma,t}(x)\,dt$ be the Green function of a Brownian motion with covariance matrix Σ. In particular, if $\Sigma = \sigma^2 \cdot \mathrm{I}_d$, then $\mathrm{G}_\Sigma(x) = (2\sigma^2\pi^{\frac{d}{2}})^{-1}\Gamma(\frac{d}{2} - 1)|x|^{2-d}$ for all $x \neq 0$, where $|\cdot|$ stands for the Euclidean norm on $\mathbb{R}^d$.

THEOREM 1.20. *Let $d \geq 3$. Assume that the family of measures $\mathbb{P}^u$, $u \in (a,b)$, satisfies assumptions* **P1**–**P3** *and* **S1**–**S2**. *Let $u \in (a,b)$, m and Σ as in Theorem 1.19, and $\varepsilon > 0$. There exist $\Omega_{\mathrm{gf}} \in \mathcal{F}$ with $\mathbb{P}^u[\Omega_{\mathrm{gf}}] = 1$ and a proper random variable $M = M(\varepsilon)$, such that for all $\omega \in \Omega_{\mathrm{gf}} \cap \{0 \in \mathcal{S}_\infty\}$:*

(a) *for all $x \in \mathcal{S}_\infty(\omega)$ with $|x| \geq M$,*

$$\frac{(1-\varepsilon)\mathrm{G}_\Sigma(x)}{m} \leq g_{\mathcal{S}_\infty(\omega)}(0,x) \leq \frac{(1+\varepsilon)\mathrm{G}_\Sigma(x)}{m},$$

(b) *for all $y \in \mathbb{R}^d$, $\lim_{k\to\infty} k^{2-d} \cdot \mathbb{E}^u[g_{\mathcal{S}_\infty(\omega)}(0,\lfloor ky\rfloor)|0 \in \mathcal{S}_\infty] = \frac{\mathrm{G}_\Sigma(y)}{m}$.*

REMARK 1.21. (1) Let us emphasize that our method does not allow to replace $(\log r)^{1+\Delta_{\mathrm{S}}}$ in (8) by $f_{\mathrm{S}}(u,R)$ from **S1**. In particular, even if $f_{\mathrm{S}}(u,R)$ growth polynomially with R, we are not able to improve the bound in (8) to stretched exponential. In the case of independent Bernoulli percolation, it is known from [4], Section 2, that the result of Theorem 1.13 holds with a stretched exponential bound in (8).

(2) The fact that the right-hand side of (11) decays faster than any polynomial will be crucially used in the proofs of Theorems 1.16, 1.18 and 1.20. Quenched bounds on the diagonal $p_n(x,x)$ under the assumptions **P1**–**P3** and **S1**–**S2** were obtained in [32] [see Remark 1.3(4) and (5) there] for all $n \geq n_0(\omega)$, although without any control on the tail of $n_0(\omega)$.

(3) In the case of supercritical Bernoulli percolation, Barlow showed in [4], Theorem 1, that the bound (10) holds for all $t \geq \max\{T_{\mathrm{hk}}(x), \mathrm{D}(x,y)\}$. The step "from $\epsilon > 0$ to $\epsilon = 0$" is highly nontrivial and follows from the fact that very good boxes on *microscopic* scales are dense; see [4], Definition 5.4 and Theorem 5.7(b). We do not know if such property can be deduced from the assumptions **P1**–**P3** and **S1**–**S2** or proved for any of the specific models considered in Section 1.2.2 (except for Bernoulli percolation). Our renormalization does not exclude the possibility of dense mesoscopic traps in $\mathcal{S}_\infty$, but we do not have a counterexample either. For comparison, let us mention that the heat kernel bounds (9) and (10) were obtained in [1, 6] for the random conductance model with i.i.d. weights, where it is also stated in [6], Remark 3.4, and [1], Remark 4.12, that the lower bound for times

comparable with $\mathrm{D}(x, y)$ can likely be obtained by adapting Barlow's proof, but omitted there because of a considerable amount of extra work and few applications.

(4) The first proofs of the quenched invariance principle for random walk on the infinite cluster of Bernoulli percolation [9, 25, 38] relied significantly on the quenched upper bound on the heat kernel. It was then observed in [11] that it is sufficient to control only the diagonal of the heat kernel (proved for Bernoulli percolation in [26]). This observation was essential in proving the quenched invariance principle for percolation models satisfying **P1**–**P3** and **S1**–**S2** in [32], where the desired upper bound on the diagonal of the heat kernel was obtained by means of an isoperimetric inequality (see [32], Theorem 1.2). Theorem 1.15 allows now to prove the quenched invariance principle of [32] by following the original path, for instance, by a direct adaptation of the proof of [9], Theorem 1.1.

(5) Our proof of Theorem 1.19 follows the approach of [7] in the setting of supercritical Bernoulli percolation, namely, it is deduced from the quenched invariance principle, parabolic Harnack inequality, and the upper bound on the heat kernel. If we replace in (12) $\sup_x$ by $\sup_{|x|<K}$ for any fixed $K > 0$, then it is not necessary to assume the upper bound on the heat kernel; see [13], Theorem 1.

(6) A new approach to limit theorems and Harnack inequalities for the elliptic random conductance model under assumptions on moments of the weights and their reciprocals has been recently developed in [2, 3]. It relies on Moser's iteration and new weighted Sobolev and Poincaré inequalities, and is applicable on general graphs satisfying globally conditions of regular volume growth and an isoperimetric inequality (see [3], Assumption 1.1). The method of [2] was recently used in [31] to prove the quenched invariance principle for the random conductance model on the infinite cluster of supercritical Bernoulli percolation under the same assumptions on moments of the weights as in [2]. It would be interesting to extend this result to the random conductance model on percolation clusters satisfying conditions **P1**–**P3** and **S1**–**S2**; see Section 6.

1.4. *Some words about the proof of Theorem* 1.13. Theorem 1.10 is enough to control the volume growth, thus we only discuss here the weak Poincaré inequality. A finite subset H of $V(G)$ satisfies the (strong) Poincaré inequality $P(C, r)$, if for any function $f : H \to \mathbb{R}$, $\min_a \int_H (f - a)^2 \, d\mu \le C \cdot r^2 \cdot \int_{E(H)} |\nabla f|^2 \, d\nu$. The well-known sufficient condition for $P(C, r)$ is the following isoperimetric inequality for subsets of H (see, e.g., [24], Proposition 3.3.10, or [37], Lemma 3.3.7):

$$\text{there exists } c > 0 \text{ such that for all } A \subset H \text{ with } |A| \le \tfrac{1}{2}|H|, \\ \text{the number of edges between } A \text{ and } H \setminus A \text{ is at least } \tfrac{c}{r}|A|. \tag{13}$$

It is too difficult (if not impossible) to show that subsets of a large ball $\mathrm{B}_G(y, r)$ satisfy such condition, since the boundary of $\mathrm{B}_G(y, r)$ may be quite rough. However, if there exists a subset $\mathcal{C}(y, r)$ of $V(G)$ such that $\mathrm{B}_G(y, r) \subseteq \mathcal{C}(y, r) \subseteq \mathrm{B}_G(y, C'r)$ and (13) holds for subsets of $\mathcal{C}(y, r)$, then it is easy to see that the weak Poincaré inequality with constants C and C' holds for $\mathrm{B}_G(y, r)$ (see Claim 4.2).

In the case $G = \mathcal{S}_\infty \subset \mathbb{Z}^d$, a possible choice of $\mathcal{C}(y, r)$ is the cluster of y in $\mathcal{S}_\infty \cap \mathrm{B}(y, r)$, which turns out to be also the largest cluster $\mathcal{C}_{\max}(y, r)$ in $\mathcal{S} \cap \mathrm{B}(y, r)$ (here and below, we implicitly assume that r is large enough). In the setting of Bernoulli percolation, it is known that subsets of $\mathcal{C}_{\max}(y, r)$ satisfy (13) (see [4], Proposition 2.11). In our setting, Theorem 1.10 implies that $\mathrm{B}_G(y, r) \subseteq \mathcal{C}_{\max}(y, r) \subseteq \mathrm{B}_G(y, C'r)$, thus we only need to prove that subsets of $\mathcal{C}_{\max}(y, r)$ satisfy (13). The first isoperimetric inequality for subsets of $\mathcal{C}_{\max}(y, r)$ was proved in [32], Theorem 1.2. It states that for any $A \subset \mathcal{C}_{\max}(y, r)$ with $|A| \geq r^\delta$, the number of edges between A and $\mathcal{S}_\infty \setminus A$ is at least $c|A|^{\frac{d-1}{d}}$ (thus, also at least $\frac{c'}{r}|A|$). Note the key difference, the edges are taken between A and $\mathcal{S}_\infty \setminus A$, not just between A and $\mathcal{C}_{\max}(y, r) \setminus A$. The above isoperimetric inequality implies certain Nash-type inequalities sufficient to prove a diffusive upper bound on the heat kernel (see [27], Theorem 2, [11], Proposition 6.1, [10], Lemma 3.2, [32], (A.4)), but it is too weak to imply the Poincaré inequality (see, e.g., [24], Sections 3.2 and 3.3, for an overview of the two isoperimetric inequalities and their relation to various functional inequalities). Let us also mention that in the setting of Bernoulli percolation, the "weak" isoperimetric inequality admits a simple proof ([10], Theorem A.1), but the proof of the "strong" one is significantly more involved ([4], Proposition 2.11).

We have not succeeded in proving (13) for subsets of $\mathcal{C}_{\max}(y, r)$ under our general conditions and do not know if it can be done. Instead, we bypass the issue of rough boundary of $\mathcal{C}_{\max}(y, r)$ by considering a certain *enlargement*, $\widetilde{\mathcal{C}}_{\max}(y, r)$ of $\mathcal{C}_{\max}(y, r)$ with a sufficiently regular boundary. We obtain $\widetilde{\mathcal{C}}_{\max}(y, r)$ by adding to $\mathcal{C}_{\max}(y, r)$ all vertices from $\mathcal{S}_\infty$ which are *locally* connected to $\mathcal{C}_{\max}(y, r)$. In particular, the inclusion $\mathrm{B}_G(y, r) \subseteq \widetilde{\mathcal{C}}_{\max}(y, r) \subseteq \mathrm{B}_G(y, C'r)$ is preserved. A large part of the work is then to prove that subsets of $\widetilde{\mathcal{C}}_{\max}(y, r)$ satisfy (13) (see Proposition 4.3, Theorem 3.8 and Corollaries 3.9 and 3.17). The general outline of this proof is similar to the one of the proof of the weak isoperimetric inequality for $\mathcal{C}_{\max}(y, r)$ in [32], but we have to modify renormalization and coarse graining of subsets of $\widetilde{\mathcal{C}}_{\max}(y, r)$ and rework some arguments to get good control of the boundary and the volume of subsets of $\widetilde{\mathcal{C}}_{\max}(y, r)$ in terms of the boundary and the volume of the corresponding coarse grainings. For instance, it is crucial for us (but not for [32]) that the coarse graining of a big set (say, of size $\frac{1}{2}|\widetilde{\mathcal{C}}_{\max}(y, r)|$) should not be too big (see, e.g., the proof of Claim 3.13).

We partition the lattice $\mathbb{Z}^d$ into large boxes of equal size. For each configuration $\omega \in \Omega$, we subdivide all the boxes into *good* and *bad*. Restriction of $\mathcal{S}$ to a good box contains a unique largest *in volume* cluster, and the largest clusters in two adjacent good boxes are connected in $\mathcal{S}$ in the union of the two boxes. Traditionally, in the study of Bernoulli percolation, the good boxes are defined to contain a unique cluster of large diameter. In our case, the existence of several clusters of large diameter in good boxes is not excluded. The reason to work with volumes is that the existence of a unique giant cluster in a box can be expressed as an intersection of two events, an increasing (existence of cluster with big volume) and decreasing (smallness of the total volume of large clusters). Assumption **P3** gives us control of correlations between monotone events, which is sufficient to set up two multi-scale renormalization schemes with scales L_n (one for increasing and one for decreasing events) and conclude that bad boxes tend to organize in blobs on multiple scales, so that the majority of boxes of size L_n contain at most 2 blobs of diameter bigger than L_{n-1} each, but even their diameters are much smaller than the actual scale L_n. By removing two boxes of size $r_{n-1}L_{n-1} \ll L_n$ containing the biggest blobs of an L_n-box, then by removing from each of the remaining L_{n-1}-boxes two boxes of size $r_{n-2}L_{n-2} \ll L_{n-1}$ containing its biggest blobs, and so on, we end up with a subset of good boxes, which is a dense in $\mathbb{Z}^d$, locally well connected, and well structured coarse graining of $\mathcal{S}_\infty$; see Figure 2 for an illustration. Similar renormalization has been used in [19, 32, 34]. By reworking some arguments from [32], we prove that large subsets of the restriction of the coarse graining to any large box satisfy a d-dimensional isoperimetric inequality, if the scales L_n grow sufficiently fast (Theorem 2.5).

We deduce from it the desired isoperimetric inequality for large subsets A of $\widetilde{\mathcal{C}}_{\max}(y,r)$ (Theorem 3.8) as follows. If A is spread out in $\widetilde{\mathcal{C}}_{\max}(y,r)$, then it has large boundary, otherwise, we associate with it a set of those good boxes from the coarse graining, the unique largest cluster of which is entirely contained in A. It turns out that the boundary and the volume of the resulting set are comparable with those of A. Moreover, if $|A| \le \frac{1}{2}|\widetilde{\mathcal{C}}_{\max}(y,r)|$, then the volume of its coarse graining is also only a fraction of the total volume of the coarse graining of $\widetilde{\mathcal{C}}_{\max}(y,r)$. The isoperimetric inequality then follows from the one for subsets of the coarse graining.

1.5. *Structure of the paper.* In Section 2, we define perforated sublattices of $\mathbb{Z}^d$ and state an isoperimetric inequality for subsets of perforations. The main definition there is (19), and the main result is Theorem 2.5. The proof of Theorem 2.5 is given in Section 5. In Section 3, we define a coarse graining of $\mathcal{S}_\infty$ and study certain extensions of largest clusters of $\mathcal{S}_\infty$ in boxes (Definition 3.5). Particularly, we prove that they satisfy the desired isoperimetric inequality (Theorem 3.8) and the volume growth (Corollary 3.16). In Section 4, we introduce the notions of *regular* and *very regular* balls, so that a (very) regular ball is always (very) good, and use it to prove the main result of the paper. In fact, in Proposition 4.3 we prove that large balls are very likely to be very regular, which is stronger than Theorem 1.13. Some open problems are discussed in Section 6. In Section 6, we sketch the proofs of Theorems 1.16–1.20. At the end of the paper, we provide an index of commonly used notation.

Finally, let us make a convention about constants. As already said, we omit from the notation dependence of constants on a, b, and d. We usually also omit the dependence on ε_{P}, χ_{P} and Δ_{S}. Dependence on other parameters is reflected in the notation, for example, as $c(u, \theta_{\mathrm{vgb}})$. Sometimes we use C, C', c, etc., to denote "intermediate" constants, their values may change from line to line, and even within a line.

2. Perforated lattices. In this section, we define lattices perforated on multiple scales and study their isoperimetric properties. Informally, for a sequence of scales $L_n = l_{n-1} \cdot L_{n-1}$, we define a perforation of the box $[0, L_n)^d$ by removing small rectangular regions of L_{n-1}-boxes from it, then removing small rectangular regions of L_{n-2}-boxes from each of the remaining L_{n-1}-boxes, and so on down to scale L_0. The precise definition is given in (19). Such perforated lattices will be used in Section 3 as coarse approximations of largest connected components of $\mathcal{S}$ in boxes. The main result of this section is an isoperimetric inequality for subsets of perforations; see Theorem 2.5.

The rules for perforation (the shape and location of removed regions) are determined by certain cascading events, which we define first; see (14) and Definition 2.1. The recursive construction of the perforated lattice is given in Section 2.2, where the main definition is (19).

Let l_n, r_n, L_n, $n \geq 0$ be sequences of positive integers such that $l_n > r_n$ and $L_n = l_{n-1} \cdot L_{n-1}$, for $n \geq 1$. To each L_n we associate the rescaled lattice

$$\mathbb{G}_n = L_n \cdot \mathbb{Z}^d = \{L_n \cdot x : x \in \mathbb{Z}^d\},$$

with edges between any pair of (ℓ^1-)nearest neighbor vertices of $\mathbb{G}_n$.

2.1. *Cascading events.* Let $\overline{\mathrm{E}} = (\overline{\mathrm{E}}_{x,L_0} : L_0 \geq 1, x \in \mathbb{G}_0)$ be a family of events from some sigma-algebra. For each $L_0 \geq 1, n \geq 0$, $x \in \mathbb{G}_n$, define recursively the events $\overline{\mathrm{G}}_{x,n,L_0}(\overline{\mathrm{E}})$ by $\overline{\mathrm{G}}_{x,0,L_0}(\overline{\mathrm{E}}) = \overline{\mathrm{E}}_{x,L_0}$ and

$$\overline{\mathrm{G}}_{x,n,L_0}(\overline{\mathrm{E}}) = \bigcup_{\substack{x_1,x_2 \in \mathbb{G}_{n-1} \cap (x+[0,L_n)^d) \\ |x_1-x_2|_\infty \geq r_{n-1} \cdot L_{n-1}}} \overline{\mathrm{G}}_{x_1,n-1,L_0}(\overline{\mathrm{E}}) \cap \overline{\mathrm{G}}_{x_2,n-1,L_0}(\overline{\mathrm{E}}). \tag{14}$$

The events in (14) also depend on the scales l_n and r_n, but we omit this dependence from the notation, since these sequences will be properly chosen and fixed later.

DEFINITION 2.1. Given sequences l_n, r_n, L_n, $n \geq 0$, as above, and two families of events $\overline{\mathrm{D}}$ and $\overline{\mathrm{I}}$, we say that for $n \geq 0$, $x \in \mathbb{G}_n$ is $(\overline{\mathrm{D}}, \overline{\mathrm{I}}, n)$-*bad* [resp., $(\overline{\mathrm{D}}, \overline{\mathrm{I}}, n)$-*good*], if the event $\overline{\mathrm{G}}_{x,n,L_0}(\overline{\mathrm{D}}) \cup \overline{\mathrm{G}}_{x,n,L_0}(\overline{\mathrm{I}})$ occurs (resp., does not occur).

Good vertices give rise to certain geometrical structures on $\mathbb{Z}^d$ (perforated lattices), which we define in the next subsection.

The choice of the families $\overline{\mathrm{D}}$ and $\overline{\mathrm{I}}$ throughout the paper is either irrelevant for the result (as in Sections 2 and 5) or fixed (as in Section 3.1). Thus, from now on we write n-bad (resp., n-good) instead of $(\overline{\mathrm{D}}, \overline{\mathrm{I}}, n)$-*bad* [resp., $(\overline{\mathrm{D}}, \overline{\mathrm{I}}, n)$-*good*], hopefully without causing any confusions.

REMARK 2.2. Definition 2.1 can be naturally generalized to k families of events $\overline{\mathrm{E}}_1, \ldots, \overline{\mathrm{E}}_k$, for any fixed k, and all the results of Sections 2 and 5 still hold (with suitable changes of constants). For our applications, it suffices to consider only two families of events (see Section 3.1). Thus, for simplicity of notation, we restrict to this special case.

2.2. *Recursive construction*. Throughout this subsection, we fix sequences l_n, r_n, L_n, $n \geq 0$, such that $l_n > 8r_n$ and l_n is divisible by r_n for all n. We also fix two local families of events $\overline{\mathrm{D}}$ and $\overline{\mathrm{I}}$, and integers $s \geq 0$ and $K \geq 1$. Recall Definition 2.1 of n-good vertices in $\mathbb{G}_n$. For $x \in \mathbb{Z}^d$, define

$$Q_{K,s}(x) = x + \mathbb{Z}^d \cap [0, KL_s)^d, \tag{15}$$

and write $Q_{K,s}$ for $Q_{K,s}(0)$. We also fix $x_s \in \mathbb{G}_s$ and assume that

$$\text{all the vertices in } \mathbb{G}_s \cap Q_{K,s}(x_s) \text{ are } s\text{-good.} \tag{16}$$

Our aim is to construct a subset of 0-good vertices in the lattice box $\mathbb{G}_0 \cap Q_{K,s}(x_s)$ by recursively perforating it on scales $L_s, L_{s-1}, \ldots, L_1$. We use Definition 2.1 to determine the rules of perforation on each scale.

We first recursively define certain subsets of i-good vertices in $\mathbb{G}_i \cap Q_{K,s}(x_s)$ for $i \leq s$; see (17) and (18). Let

$$\mathcal{G}_{K,s,s}(x_s) = \mathbb{G}_s \cap Q_{K,s}(x_s). \tag{17}$$

By (16), all $z_s \in \mathcal{G}_{K,s,s}(x_s)$ are s-good.

Assume that $\mathcal{G}_{K,s,i}(x_s) \subset \mathbb{G}_i$ is defined for some $i \leq s$ so that all $z_i \in \mathcal{G}_{K,s,i}(x_s)$ are i-good. By Definition 2.1, for each $z_i \in \mathcal{G}_{K,s,i}(x_s)$, there exist

$$a_{z_i}, b_{z_i} \in (r_{i-1}L_{i-1}) \cdot \mathbb{Z}^d \cap \big(z_i + [0, L_i)^d\big)$$

such that the boxes $(a_{z_i} + [0, 2r_{i-1}L_{i-1})^d)$ and $(b_{z_i} + [0, 2r_{i-1}L_{i-1})^d)$ are contained in $(z_i + [0, L_i)^d)$, and all the vertices in

$$\big(\mathbb{G}_{i-1} \cap \big(z_i + [0, L_i)^d\big)\big) \setminus \big(\big(a_{z_i} + [0, 2r_{i-1}L_{i-1})^d\big) \cup \big(b_{z_i} + [0, 2r_{i-1}L_{i-1})^d\big)\big)$$

are $(i-1)$-good. If the choice is not unique, we choose the pair arbitrarily. All the results below hold for any allowed choice of a_{z_i} and b_{z_i}. To save notation, we will not mention it in the statements.

Define $\mathcal{R}_{z_i} \subseteq \mathbb{G}_{i-1}$ to be:

(a) the empty set, if all the vertices in $\mathbb{G}_{i-1} \cap (z_i + [0, L_i)^d)$ are $(i-1)$-good, or
(b) $\mathbb{G}_{i-1} \cap ((a_{z_i} + [0, 2r_{i-1}L_{i-1})^d) \cup (b_{z_i} + [0, 2r_{i-1}L_{i-1})^d))$ if $|a_{z_i} - b_{z_i}|_\infty > 2r_{i-1}L_{i-1}$, or
(c) a box $\mathbb{G}_{i-1} \cap (c_{z_i} + [4r_{i-1}L_{i-1})^d)$ in $\mathbb{G}_{i-1} \cap (z_i + [0, L_i)^d)$, with $c_{z_i} \in (r_{i-1}L_{i-1}) \cdot \mathbb{Z}^d$, which contains $\mathbb{G}_{i-1} \cap ((a_{z_i} + [0, 2r_{i-1}L_{i-1})^d) \cup (b_{z_i} + [0, 2r_{i-1}L_{i-1})^d))$.

Possible outcomes (b) and (c) of $\mathcal{R}_{z_i}$ are illustrated on Figure 1.

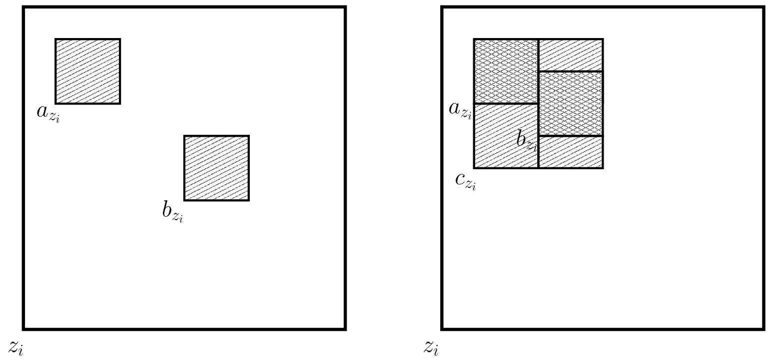

FIG. 1. *Two possible outcomes of $\mathcal{R}_{z_i}$. On the left, the points a_{z_i} and b_{z_i} are far from each other, on the right, they are close.*

REMARK 2.3. By construction, the set $\mathcal{R}_{z_i}$ is the disjoint union of 0, 2, or 2^d boxes $\mathbb{G}_{i-1} \cap (x + [0, 2r_{i-1}L_{i-1})^d)$ with $x \in (r_{i-1}L_{i-1}) \cdot \mathbb{Z}^d$.

To complete the construction, let

$$\mathcal{G}_{K,s,i-1}(x_s) = \mathbb{G}_{i-1} \cap \bigcup_{z_i \in \mathcal{G}_{K,s,i}(x_s)} \big((z_i + [0, L_i)^d) \setminus \mathcal{R}_{z_i}\big). \tag{18}$$

Note that all $z_{i-1} \in \mathcal{G}_{K,s,i-1}(x_s)$ are $(i-1)$-good.

Now that the sets $(\mathcal{G}_{K,s,j}(x_s))_{j \le s}$, are constructed by (17) and (18), we define the multiscale perforations of $\mathbb{G}_0 \cap Q_{K,s}(x_s)$ by

$$\mathcal{Q}_{K,s,j}(x_s) = \mathbb{G}_0 \cap \bigcup_{z_j \in \mathcal{G}_{K,s,j}(x_s)} \big(z_j + [0, L_j)^d\big), \qquad j \le s. \tag{19}$$

See Figure 2 for an illustration. By construction:

(a) for all j, $\mathcal{Q}_{K,s,j-1}(x_s) \subseteq \mathcal{Q}_{K,s,j}(x_s)$,
(b) all the vertices of $\mathcal{Q}_{K,s,0}(x_s)$ are 0-good.

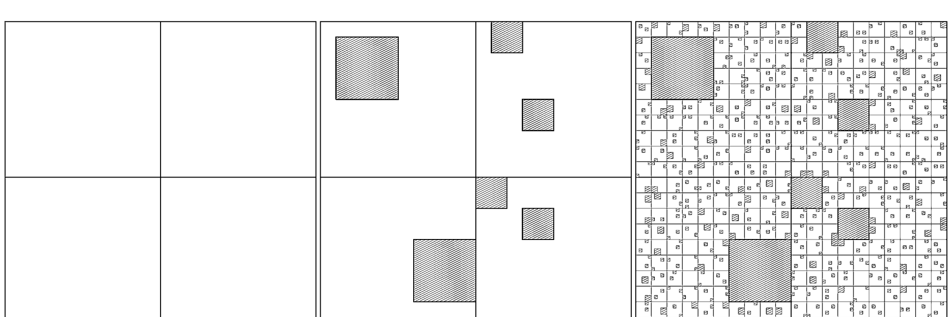

FIG. 2. *Perforations $\mathcal{Q}_{2,s,s}$, $\mathcal{Q}_{2,s,s-1}$, and $\mathcal{Q}_{2,s,s-2}$ of $Q_{2,s}$.*

We will view the sets $\mathcal{Q}_{K,s,j}(x_s)$ as subgraphs of $\mathbb{G}_0$ with edges drawn between any two vertices of the set which are at ℓ^1 distance L_0 from each other. The next lemma summarizes some basic properties of $\mathcal{Q}_{K,s,0}(x_s)$'s, which are immediate from the construction.

LEMMA 2.4. *Let $d \geq 2$, $K \geq 1$, and $s \geq 0$. For any choice of scales l_n, r_n, L_n, $n \geq 0$, such that $l_n > 8r_n$ and l_n is divisible by r_n for all n, and for any admissible choice of a_{z_i}, b_{z_i} or c_{z_i} in the construction of $\mathcal{Q}_{K,s,0}(x_s)$:*

(a) *$\mathcal{Q}_{K,s,0}(x_s)$ is connected in $\mathbb{G}_0$,*
(b) $|\mathcal{Q}_{K,s,0}(x_s)| \geq \prod_{j=0}^{\infty}(1 - (\frac{4r_j}{l_j})^d) \cdot |Q_{K,s}|$.

2.3. *Isoperimetric inequality.* For a graph G and a subset A of G, the boundary of A in G is the subset of edges of G, $E(G)$, defined as

$$\partial_G A = \{\{x, y\} \in E(G) : x \in A,\ y \in G \setminus A\}.$$

The next theorem states that under assumption (16) and some assumptions on l_n and r_n (basically that $\sum_{n\geq 0} \frac{r_n}{l_n}$ is sufficiently small), there exist $\gamma > 0$ such that for all large enough $A \subset \mathcal{Q}_{K,s,0}(x_s)$ with $|A| \leq \frac{1}{2} \cdot |\mathcal{Q}_{K,s,0}(x_s)|$, $|\partial_{\mathcal{Q}_{K,s,0}(x_s)} A| \geq \gamma \cdot |A|^{\frac{d-1}{d}}$.

THEOREM 2.5. *Let $d \geq 2$. Let l_n and r_n, $n \geq 0$, be integer sequences such that for all n, $l_n > 8r_n$, l_n is divisible by r_n, and*

$$\prod_{j=0}^{\infty}\left(1 - \left(\frac{4r_j}{l_j}\right)^2\right) \geq \max\left\{\frac{15}{16}, e^{-\frac{1}{16(d-1)}}, \frac{1 - \frac{1}{2^{d+2}}}{1 - \frac{1}{2^{d+3}}}\right\} \quad \text{and}$$

$$3456 \cdot \sum_{j=0}^{\infty} \frac{r_j}{l_j} \leq \frac{1}{10^6}. \tag{20}$$

Then for any integers $s \geq 0$, $L_0 \geq 1$, and $K \geq 1$, $x_s \in \mathbb{G}_s$, and two families of events $\overline{\mathrm{D}}$ and $\overline{\mathrm{I}}$, if all the vertices in $\mathbb{G}_s \cap Q_{K,s}(x_s)$ are s-good, then any $\mathcal{A} \subseteq \mathcal{Q}_{K,s,0}(x_s)$ with

$$\left(\frac{L_s}{L_0}\right)^{d^2} \leq |\mathcal{A}| \leq \frac{1}{2} \cdot |Q_{K,s} \cap \mathbb{G}_0|$$

satisfies

$$|\partial_{\mathcal{Q}_{K,s,0}(x_s)} \mathcal{A}| \geq \frac{1}{2d \cdot 32^d \cdot 27^d \cdot 10^6} \cdot \left(1 - \left(\frac{2}{3}\right)^{\frac{1}{d}}\right) \cdot \left(1 - e^{-\frac{1}{16(d-1)}}\right) \cdot |\mathcal{A}|^{\frac{d-1}{d}}.$$

REMARK 2.6. In the setting of Theorem 2.5, if $\mathcal{A} \subseteq \mathcal{Q}_{K,s,0}(x_s)$ satisfies

$$\left(\frac{L_s}{L_0}\right)^{d^2} \le |\mathcal{A}| \le C \cdot |\mathcal{Q}_{K,s,0}(x_s)|,$$

for some $\frac{1}{2} < C < 1$, then

$$|\partial_{\mathcal{Q}_{K,s,0}(x_s)}\mathcal{A}| \ge \frac{1-C}{2d \cdot 32^d \cdot 27^d \cdot 10^6} \cdot \left(1 - \left(\frac{2}{3}\right)^{\frac{1}{d}}\right) \cdot \left(1 - e^{-\frac{1}{16(d-1)}}\right) \cdot |\mathcal{A}|^{\frac{d-1}{d}}.$$

This easily follows from Theorem 2.5 by passing, if necessary, to the complement of $\mathcal{A}$ in $\mathcal{Q}_{K,s,0}(x_s)$; see, for instance, Remark 5.2.

We postpone the proof of Theorem 2.5 to Section 5. In fact, in two dimensions, we are able to prove the analogue of Theorem 2.5 for *all* subsets $\mathcal{A} \subseteq \mathcal{Q}_{K,s,0}(x_s)$ with $1 \le |\mathcal{A}| \le \frac{1}{2} \cdot |Q_{K,s} \cap \mathbb{G}_0|$; see Lemma 5.6. We believe that also in any dimension $d \ge 3$, the isoperimetric inequality of Theorem 2.5 holds for all subsets $\mathcal{A} \subseteq \mathcal{Q}_{K,s,0}(x_s)$ with $1 \le |\mathcal{A}| \le \frac{1}{2} \cdot |Q_{K,s} \cap \mathbb{G}_0|$, but cannot prove it. Theorem 2.5 follows immediately from a more general isoperimetric inequality in Theorem 5.10.

3. Properties of the largest clusters. In this section, we study properties of the largest subset of $\mathcal{S} \cap Q_{K,s}$ [where $Q_{K,s}$ is defined in (15)]. Our ultimate goal is to prove that under general conditions, all large enough balls $\mathrm{B}_{\mathcal{S}}(y,r)$ contained in the largest cluster have regular volume growth and allow for local extensions $\mathcal{C}(y,r)$ satisfying the inclusion $\mathrm{B}_{\mathcal{S}}(y,r) \subseteq \mathcal{C}(y,r) \subseteq \mathrm{B}_{\mathcal{S}}(y,r)$ and the isoperimetric inequality (13). As discussed in Section 1.4, this is enough to conclude that a large ball contained in the largest cluster is very good in the sense of Definition 1.1.

We first define two families of events such that the corresponding perforated lattices defined in (19) serve as a "skeleton" of the largest subset of $\mathcal{S} \cap Q_{K,s}$. Then we provide sufficient conditions for the uniqueness of the largest subset of $\mathcal{S} \cap Q_{K,s}$ (Lemma 3.3). To avoid problems, which may be caused by roughness of the boundary of the largest subset of $\mathcal{S} \cap Q_{K,s}$, we *enlarge* it by adding to it all the points of $\mathcal{S}$ which are *locally* connected to it (Definition 3.5). For the enlarged set, we prove under some general conditions (Definition 3.7) that its subsets satisfy an isoperimetric inequality (Theorem 3.8 and Corollary 3.9). Such enlargements of largest clusters will be precisely the enlarged sets $\mathcal{C}(y,r)$ discussed in Section 1.4.

Indeed, under the same condition we prove that the distances in the largest cluster are comparable to those on $\mathbb{Z}^d$ (Lemma 3.15), all large enough balls in the largest cluster have regular volume growth (Corollary 3.16) and have local extensions (obtained as local extensions of the largest cluster in some box) satisfying an isoperimetric inequality (Corollary 3.17).

3.1. *Special sequences of events.* Recall Definition 1.8 of $\mathcal{S}_r$. Consider an ordered pair of real numbers

$$\eta = (\eta_1, \eta_2), \qquad \text{with } \eta_1 \in (0, 1) \text{ and } \eta_1 \le \eta_2 < 2\eta_1. \tag{21}$$

Two families of events $\overline{\mathrm{D}}^\eta = (\overline{\mathrm{D}}^\eta_{x,L_0}, L_0 \ge 1, x \in \mathbb{G}_0)$ and $\bar{\mathrm{I}}^\eta = (\bar{\mathrm{I}}^\eta_{x,L_0}, L_0 \ge 1, x \in \mathbb{G}_0)$ are defined as follows:

- The complement of $\overline{\mathrm{D}}^\eta_{x,L_0}$ is the event that for each $y \in \mathbb{G}_0$ with $|y - x|_1 \le L_0$, the set $\mathcal{S}_{L_0} \cap (y + [0, L_0)^d)$ contains a connected component $\mathcal{C}_y$ with at least $\eta_1 L_0^d$ vertices such that for all $y \in \mathbb{G}_0$ with $|y - x|_1 \le L_0$, $\mathcal{C}_y$ and $\mathcal{C}_x$ are connected in $\mathcal{S} \cap ((x + [0, L_0)^d) \cup (y + [0, L_0)^d))$.
- The event $\bar{\mathrm{I}}^\eta_{x,L_0}$ occurs if $|\mathcal{S}_{L_0} \cap (x + [0, L_0)^d)| > \eta_2 L_0^d$.

Note that $\overline{\mathrm{D}}^\eta_{x,L_0}$ are decreasing and $\bar{\mathrm{I}}^\eta_{x,L_0}$ increasing events. From now on, we fix these two local families, and say that $x \in \mathbb{G}_n$ is n-bad/n-good, if it is n-bad/n-good for the two local families $\overline{\mathrm{D}}^\eta$ and $\bar{\mathrm{I}}^\eta$ in the sense of Definition 2.1. In particular, $x \in \mathbb{G}_0$ is 0-good if both $\overline{\mathrm{D}}^\eta_{x,L_0}$ and $\bar{\mathrm{I}}^\eta_{x,L_0}$ do not occur; see Figure 3.

The following lemma is immediate from the definition of 0-good vertex and the conditions (21) on η. (See, e.g., [19], Lemma 6.2, for a similar result.)

LEMMA 3.1. *Let $L_0 \ge 1$ and η as in (21).*

(a) *For any 0-good vertex $x \in \mathbb{G}_0$, connected component $\mathcal{C}_x$ in $\mathcal{S}_{L_0} \cap (x + [0, L_0)^d)$ with at least $\eta_1 L_0^d$ vertices is defined uniquely.*

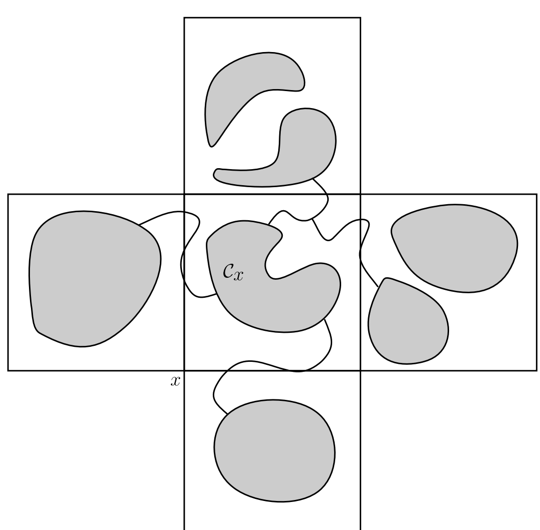

FIG. 3. *A 0-good vertex x. A unique connected component $\mathcal{C}_x$ of size $\ge \eta_1 L_0^d$ in $(x + [0, L_0)^d)$ is connected to a connected component of size $\ge \eta_1 L_0^d$ in each of the adjacent boxes.*

(b) *For any* 0*-good* $x, y \in \mathbb{G}_0$ *with* $|x-y|_1 = L_0$, (*uniquely chosen*) $\mathcal{C}_x$ *and* $\mathcal{C}_y$ *are connected in the graph* $\mathcal{S} \cap ((x+[0, L_0)^d) \cup (y+[0, L_0)^d))$.

3.2. *Uniqueness of the largest cluster.*

DEFINITION 3.2. Let $(L_n)_{n\ge 0}$ be an increasing sequence of scales. For $x \in \mathbb{Z}^d$ and $r \ge 1$, let $\mathcal{C}_{K,s,r}(x)$ be the largest connected component in $\mathcal{S}_r \cap Q_{K,s}(x)$ (with ties broken arbitrarily), and write $\mathcal{C}_{K,s,r} = \mathcal{C}_{K,s,r}(0)$.

Fix η as in (21) and two families of events $\overline{\mathrm{D}}^\eta$ and $\overline{\mathrm{I}}^\eta$ as in Section 3.1.

LEMMA 3.3. *Let* l_n *and* r_n *be integer sequences such that for all* n, l_n *is divisible by* r_n, $l_n > 8r_n$, *and*

$$\prod_{i=0}^{\infty}\left(1-\left(\frac{4r_i}{l_i}\right)^d\right) > \frac{1+\eta_2}{1+2\eta_1}. \tag{22}$$

Let $L_0 \ge 1$, $K \ge 1$, *and* $s \ge 0$ *integers*, $x_s \in \mathbb{G}_s$. *If all the vertices in* $\mathbb{G}_s \cap Q_{K,s}(x_s)$ *are* s*-good, then* $\mathcal{C}_{K,s,L_0}(x_s)$ *is uniquely defined and*

$$\left|\mathcal{C}_{K,s,L_0}(x_s)\right| \ge \frac{1}{2}\eta_2 \cdot |Q_{K,s}|. \tag{23}$$

PROOF. Without loss of generality, we assume that $x_s = 0$. Since all vertices in $\mathbb{G}_s \cap Q_{K,s}$ are s-good, we can define the perforation $\mathcal{Q}_{K,s,0}$ by (19). By definition, all the vertices of $\mathcal{Q}_{K,s,0}$ are 0-good, and by Lemma 2.4, $\mathcal{Q}_{K,s,0}$ is connected in $\mathbb{G}_0$.

By Lemma 3.1, for any $x \in \mathcal{Q}_{K,s,0}$, there is a uniquely defined connected subset $\mathcal{C}_x$ of $\mathcal{S}_{L_0} \cap (x+[0, L_0)^d)$ with at least $\eta_1 L_0^d$ vertices. Since $\mathcal{Q}_{K,s,0}$ is connected in $\mathbb{G}_0$, by Lemma 3.1, the set $\bigcup_{x\in\mathcal{Q}_{K,s,0}} \mathcal{C}_x$ is contained in a connected component of $\mathcal{S}_{L_0} \cap Q_{K,s}$ and

$$\left|\bigcup_{x\in\mathcal{Q}_{K,s,0}} \mathcal{C}_x\right| \ge \eta_1 \cdot |\mathcal{Q}_{K,s,0}| \ge \eta_1 \cdot \prod_{i=0}^{\infty}\left(1-\left(\frac{4r_i}{l_i}\right)^d\right) \cdot |Q_{K,s}|, \tag{24}$$

where the second inequality follows from Lemma 2.4.

On the other hand, since for any 0-good vertex x, the set $x+[0, L_0)^d$ contains at most $\eta_2 L_0^d$ vertices from $\mathcal{S}_{L_0}$,

$$\begin{aligned} |\mathcal{S}_{L_0} \cap Q_{K,s}| &\le \eta_2 L_0^d \cdot |\mathcal{Q}_{K,s,0}| + L_0^d \cdot \left(|Q_{K,s} \cap \mathbb{G}_0| - |\mathcal{Q}_{K,s,0}|\right) \\ &\le \left(\eta_2 + 1 - \prod_{i=0}^{\infty}\left(1-\left(\frac{4r_i}{l_i}\right)^d\right)\right) \cdot |Q_{K,s}| \\ &< 2\eta_1 \cdot \prod_{i=0}^{\infty}\left(1-\left(\frac{4r_i}{l_i}\right)^d\right) \cdot |Q_{K,s}|, \end{aligned} \tag{25}$$

where the second inequality follows from the inequality $|\mathcal{Q}_{K,s,0}| \le |\mathcal{Q}_{K,s,s}| = \frac{|Q_{K,s}|}{L_0^d}$ and Lemma 2.4, and the third inequality follows from the assumption (22).

We have shown that the connected component of $\mathcal{S}_{L_0} \cap Q_{K,s}$ which contains $\bigcup_{x\in\mathcal{Q}_{K,s,0}} \mathcal{C}_x$ has volume $> \frac{1}{2} \cdot |\mathcal{S}_{L_0} \cap Q_{K,s}|$. In particular, it is the unique largest in volume connected component of $\mathcal{S}_{L_0} \cap Q_{K,s}$. Moreover, by (24), its volume is

$$\ge \eta_1 \cdot \prod_{i=0}^{\infty}\left(1 - \left(\frac{4r_i}{l_i}\right)^d\right) \cdot |Q_{K,s}| \overset{(22)}{\ge} \eta_1 \cdot \frac{1+\eta_2}{1+2\eta_1} \cdot |Q_{K,s}| \overset{(21)}{\ge} \frac{1}{2}\eta_2 \cdot |Q_{K,s}|,$$

which proves (23). $\square$

COROLLARY 3.4. *From the proof of Lemma 3.3, if the conditions of Lemma 3.3 are satisfied, then*

$$\bigcup_{x\in\mathcal{Q}_{K,s,0}} \mathcal{C}_x \subseteq \mathcal{C}_{K,s,L_0}. \tag{26}$$

In particular, for any $1 \le K' \le K'' \le K$ and $x', x'' \in \mathbb{G}_s \cap Q_{K,s}$ such that $Q_{K',s}(x') \subseteq Q_{K'',s}(x'') \subseteq Q_{K,s}$, $\mathcal{C}_{K',s,L_0}(x') \subseteq \mathcal{C}_{K'',s,L_0}(x'') \subseteq \mathcal{C}_{K,s,L_0}$.

3.3. *Isoperimetric inequality.* Under the conditions of Definition 3.7, the largest cluster $\mathcal{C}_{K,s,L_0}(x)$ is well connected locally in the bulk, but may still be quite "hairy" near its boundary, which may have negative effect on its isoperimetric properties. To bypass this issue, we consider a local extension $\widetilde{\mathcal{C}}_{K,s,L_0}(x)$ of $\mathcal{C}_{K,s,L_0}(x)$ obtained by adding to $\mathcal{C}_{K,s,L_0}(x)$ all the vertices which are connected to it locally. Unlike $\mathcal{C}_{K,s,L_0}(x)$, its local extension $\widetilde{\mathcal{C}}_{K,s,L_0}(x)$ is everywhere locally well connected. In this section, we prove a desired isoperimetric inequality for subsets of $\widetilde{\mathcal{C}}_{K,s,L_0}(x)$ (see Theorem 3.8 and Corollary 3.9).

DEFINITION 3.5. Let $\mathcal{E}_{K,s,r}(x)$ be the set of vertices $y' \in \mathcal{S}$ such that for some $y \in \mathcal{C}_{K,s,r}(x)$, y' is connected to y in $\mathcal{S} \cap \mathrm{B}(y, 2L_s)$, and define

$$\widetilde{\mathcal{C}}_{K,s,r}(x) = \mathcal{C}_{K,s,r}(x) \cup \mathcal{E}_{K,s,r}(x).$$

REMARK 3.6. Mind that $\widetilde{\mathcal{C}}_{K,s,r}(x)$ is contained in $x + [-2L_s, (K+2)L_s)^d$, but it is different from the largest cluster of $\mathcal{S}_r \cap (x + [-2L_s, (K+2)L_s)^d)$.

We study isoperimetric properties of $\widetilde{\mathcal{C}}_{K,s,L_0}(x)$ for configurations from the following event.

DEFINITION 3.7. Let η be as in (21), $K \ge 1$ and $s \ge 0$ integers, $x_s \in \mathbb{G}_s$. The event $\mathcal{H}^{\eta}_{K,s}(x_s) \in \mathcal{F}$ occurs if:

(a) all the vertices in $\mathbb{G}_s \cap (x_s + [-2L_s, (K+2)L_s)^d)$ are s-good,
(b) any $x, y \in \mathcal{S}_{L_s} \cap Q_{K,s}(x_s)$ with $|x-y|_\infty \le L_s$ are connected in $\mathcal{S} \cap \mathrm{B}(x, 2L_s)$.

We write $\mathcal{H}^{\eta}_{K,s}$ for $\mathcal{H}^{\eta}_{K,s}(0)$.

Here is the main result of this section.

THEOREM 3.8. *Let η be as in (21). Assume that the sequences l_n and r_n satisfy the conditions of Theorem 2.5 and*

$$\prod_{i=0}^{\infty}\left(1-\left(\frac{4r_i}{l_i}\right)^d\right) \geq \frac{1+\eta_2}{1+\frac{\eta_2+2\eta_1}{2}}. \tag{27}$$

Let $L_0 \geq 1$, $K \geq 1$, and $s \geq 0$ integers, $x_s \in \mathbb{G}_s$. If $\mathcal{H}^{\eta}_{K,s}(x_s)$ occurs, then $\mathcal{C}_{K,s,L_0}(x_s)$ is uniquely defined and there exists $\gamma_{3.8} = \gamma_{3.8}(\eta, L_0) \in (0,1)$ such that for any $A \subseteq \widetilde{\mathcal{C}}_{K,s,L_0}(x_s)$ with $L_s^{d(d+1)} \leq |A| \leq \frac{1}{2}\cdot|\widetilde{\mathcal{C}}_{K,s,L_0}(x_s)|$,

$$|\partial_{\widetilde{\mathcal{C}}_{K,s,L_0}(x_s)} A| \geq \gamma_{3.8}\cdot|A|^{\frac{d-1}{d}}.$$

In the applications, we will not use directly the result of Theorem 3.8, but only the following corollary, which estimates from below the size of the boundary of *any* subset of $\widetilde{\mathcal{C}}_{K,s,L_0}(x_s)$ with volume $\leq \frac{1}{2}\cdot|\widetilde{\mathcal{C}}_{K,s,L_0}(x_s)|$ precisely as in (13). In the future (see the proof of Corollary 3.17), we will use $\widetilde{\mathcal{C}}_{K,s,L_0}(x_s)$ as a local extension of a large ball in $\mathcal{C}_{K,s,L_0}(x_s)$. As discussed in Section 1.4, this will be sufficient to conclude that this ball satisfies the weak Poincaré inequality.

COROLLARY 3.9. *Let η be as in (21) and $\epsilon \in (0, \frac{1}{d}]$. Assume that the sequences l_n and r_n satisfy the conditions of Theorem 3.8. Assume that*

$$K \geq L_s^{d+\frac{d^2-1}{\epsilon d}}.$$

If $\mathcal{H}^{\eta}_{K,s}(x_s)$ occurs, then for any $A \subseteq \widetilde{\mathcal{C}}_{K,s,L_0}(x_s)$ with $|A| \leq \frac{1}{2}\cdot|\widetilde{\mathcal{C}}_{K,s,L_0}(x_s)|$,

$$|\partial_{\widetilde{\mathcal{C}}_{K,s,L_0}(x_s)} A| \geq \gamma_{3.8}\cdot|A|^{\frac{d-1}{d}+\epsilon}\cdot\big((K+4)L_s\big)^{-\epsilon d}.$$

In particular, if $\epsilon = \frac{1}{d}$, then $|\partial_{\widetilde{\mathcal{C}}_{K,s,L_0}(x_s)} A| \geq \gamma_{3.8}\cdot\frac{|A|}{(K+4)L_s}$.

PROOF. If $|A| \geq L_s^{d(d+1)}$, then we apply Theorem 3.8,

$$|\partial_{\widetilde{\mathcal{C}}_{K,s,L_0}(x_s)} A| \geq \gamma_{3.8}\cdot|A|^{\frac{d-1}{d}} \geq \gamma_{3.8}\cdot|A|^{\frac{d-1}{d}+\epsilon}\cdot\big((K+4)L_s\big)^{-\epsilon d}.$$

If $|A| \leq L_s^{d(d+1)}$, then we use the trivial bound $|\partial_{\widetilde{\mathcal{C}}_{K,s,L_0}(x_s)} A| \geq 1$. By the assumption on K,

$$\big((K+4)L_s\big)^{\epsilon d} \geq \big(L_s^{d(d+1)}\big)^{\frac{d-1}{d}+\epsilon},$$

which implies, using the assumption on $|A|$, that $|A|^{\frac{d-1}{d}+\epsilon} \leq ((K+4)L_s)^{\epsilon d}$. Thus, in this case,

$$|\partial_{\widetilde{\mathcal{C}}_{K,s,L_0}(x_s)} A| \geq 1 \geq \gamma_{3.8}\cdot|A|^{\frac{d-1}{d}+\epsilon}\cdot\big((K+4)L_s\big)^{-\epsilon d}.$$

The proof of corollary is complete. $\square$

The proof of Theorem 3.8 is subdivided into several claims. In Claim 3.10, we prove that $\widetilde{\mathcal{C}}_{K,s,L_0}$ is locally connected and in Claims 3.12 and 3.13 we reduce the isoperimetric problem for subsets of $\widetilde{\mathcal{C}}_{K,s,L_0}$ to the one for subsets of a perforated lattice.

CLAIM 3.10. *Any $x, y \in \widetilde{\mathcal{C}}_{K,s,L_0}$ with $|x-y|_\infty \le L_s$ are connected in $\widetilde{\mathcal{C}}_{K,s,L_0} \cap \mathrm{B}(x, 15L_s)$.*

PROOF. Fix $x, y \in \widetilde{\mathcal{C}}_{K,s,L_0}$ with $|x-y|_\infty \le L_s$, and take $x', y' \in \mathcal{C}_{K,s,L_0}$ such that x and x' are connected in $\widetilde{\mathcal{C}}_{K,s,L_0} \cap \mathrm{B}(x', 2L_s)$, y and y' are connected in $\widetilde{\mathcal{C}}_{K,s,L_0} \cap \mathrm{B}(y', 2L_s)$. By the triangle inequality, $|x'-y'|_\infty \le 5L_s$.

Since all the vertices in $\mathbb{G}_s \cap Q_{K,s}$ are s-good, there exist $x'', y'' \in \bigcup_{z \in \mathcal{Q}_{K,s,0}} \mathcal{C}_z$ such that $|x'-x''|_\infty \le L_s$ and $|y'-y''|_\infty \le L_s$. By the definitions of $\mathcal{H}^{\eta}_{K,s}$ and $\widetilde{\mathcal{C}}_{K,s,L_0}$, x'' is connected to x' in $\widetilde{\mathcal{C}}_{K,s,L_0} \cap \mathrm{B}(x', 2L_s)$ and y'' is connected to y' in $\widetilde{\mathcal{C}}_{K,s,L_0} \cap \mathrm{B}(y', 2L_s)$.

By the triangle inequality, $|x''-y''|_\infty \le 7L_s$. Let $(z + [0, 8L_s)^d)$ be a box in $Q_{K,s}$ which contains both x'' and y'', where $z \in \mathbb{G}_s$. Since all the vertices in $\mathbb{G}_s \cap (z + [0, 8L_s)^d)$ are s-good, the perforation $\mathcal{Q}_{8,s,0}(z) = \mathcal{Q}_{K,s,0} \cap (z + [0, 8L_s)^d)$ of $(z + [0, 8L_s)^d)$ is connected in $\mathbb{G}_0$ by Lemma 2.4. Thus, by Lemma 3.1, the set $\bigcup_{w \in \mathcal{Q}_{8,s,0}(z)} \mathcal{C}_w$ is contained in a connected component of $\mathcal{S} \cap (z + [0, 8L_s)^d)$. In particular, x'' and y'' are connected in $\mathcal{S} \cap (z + [0, 8L_s)^d)$. By (26) and the fact that (27) implies (22), the set $\bigcup_{w \in \mathcal{Q}_{8,s,0}(z)} \mathcal{C}_w$ is contained in $\mathcal{C}_{K,s,L_0}$. Therefore, x'' is connected to y'' in $\mathcal{C}_{K,s,L_0} \cap (z + [0, 8L_s)^d) \subset \mathcal{C}_{K,s,L_0} \cap \mathrm{B}(x'', 8L_s)$.

We conclude that x is connected to y in $\widetilde{\mathcal{C}}_{K,s,L_0} \cap \mathrm{B}(x, 15L_s)$. □

Let

$$x'_s = (-2L_s, \dots, -2L_s) \in \mathbb{G}_s \quad \text{and} \quad K' = K + 4.$$

Since all the vertices in $\mathbb{G}_s \cap Q_{K',s}(x'_s)$ are s-good, we can define its perforation $\mathcal{Q}_{K',s,0}(x'_s)$ as in (19). By definition, $\mathcal{Q}_{K',s,0}(x'_s)$ is a subset of 0-good vertices in $\mathbb{G}_0 \cap Q_{K',s}(x'_s)$, and by Lemma 2.4, $\mathcal{Q}_{K',s,0}(x'_s)$ is connected in $\mathbb{G}_0$.

By the fact that (27) implies (22), Lemma 3.1, (26), and the definition of $\widetilde{\mathcal{C}}_{K,s,L_0}$,

$$\bigcup_{x \in \mathcal{Q}_{K',s,0}(x'_s)} \mathcal{C}_x \subseteq \widetilde{\mathcal{C}}_{K,s,L_0}. \tag{28}$$

The next two claims allow to reduce the isoperimetric problem for subsets of $\widetilde{\mathcal{C}}_{K,s,L_0}$ to the isoperimetric problem for subsets of $\mathcal{Q}_{K',s,0}(x'_s)$. The crucial step for the proof is the following definition of $\mathcal{A}$ and A'.

DEFINITION 3.11. For $A \subseteq \widetilde{\mathcal{C}}_{K,s,L_0}$, let $\mathcal{A}$ be the set of all $x \in \mathcal{Q}_{K',s,0}(x'_s)$ such that $\mathcal{C}_x \subseteq A$, and A' the set of $x \in A$ such that there exists $y \in \widetilde{\mathcal{C}}_{K,s,L_0} \setminus A$ with $|x-y|_\infty \le L_s$.

CLAIM 3.12.

$$|\partial_{\widetilde{\mathcal{C}}_{K,s,L_0}} A| \geq \max\left\{\frac{1}{2d} \cdot |\partial_{\mathcal{Q}_{K',s,0}(x'_s)}\mathcal{A}|, \frac{|A'|}{(31 \cdot L_s)^d}\right\} \tag{29}$$

and

$$|A| \leq 2 \cdot 3^d \cdot L_0^d \cdot |\mathcal{A}| + |A'|. \tag{30}$$

PROOF. We begin with the proof of (29). For any $x \in \mathcal{A}$ and $y \in \mathcal{Q}_{K',s,0}(x'_s) \setminus \mathcal{A}$ such that $|x-y|_1 = L_0$, $\mathcal{C}_x \subseteq A$ and $\mathcal{C}_y \nsubseteq A$. By Lemma 3.1 and (28), $\mathcal{C}_x$ and $\mathcal{C}_y$ are connected in $\widetilde{\mathcal{C}}_{K,s,L_0} \cap ((x + [0, L_0)^d) \cup (y + [0, L_0)^d))$. Each path in $\widetilde{\mathcal{C}}_{K,s,L_0}$ connecting $\mathcal{C}_x$ and $\mathcal{C}_y \setminus A$ contains an edge from $\partial_{\widetilde{\mathcal{C}}_{K,s,L_0}} A$. This implies that

$$|\partial_{\widetilde{\mathcal{C}}_{K,s,L_0}} A| \geq \frac{1}{2d} \cdot |\partial_{\mathcal{Q}_{K',s,0}(x'_s)}\mathcal{A}|. \tag{31}$$

Next, by the definition of A', for any $x \in A'$, there exists $y \in \widetilde{\mathcal{C}}_{K,s,L_0} \setminus A$ such that $|x-y|_\infty \leq L_s$. By Claim 3.10, x and y are connected in $\widetilde{\mathcal{C}}_{K,s,L_0} \cap \mathrm{B}(x, 15L_s)$. In particular, the ball $\mathrm{B}(x, 15L_s)$ contains an edge from $\partial_{\widetilde{\mathcal{C}}_{K,s,L_0}} A$. Since every edge from $\partial_{\widetilde{\mathcal{C}}_{K,s,L_0}} A$ is within ℓ^∞ distance $15L_s$ from at most $(31L_s)^d$ vertices of A',

$$|\partial_{\widetilde{\mathcal{C}}_{K,s,L_0}} A| \geq \frac{|A'|}{(31 \cdot L_s)^d}. \tag{32}$$

Inequalities (31) and (32) imply (29).

We proceed with the proof of (30). We need to show that

$$|A \setminus A'| \leq 2 \cdot 3^d \cdot L_0^d \cdot |\mathcal{A}|. \tag{33}$$

Let $z \in A \setminus A'$. By the definition of $\widetilde{\mathcal{C}}_{K,s,L_0}$, there exists $z_s \in \mathbb{G}_s \cap Q_{K',s}(x'_s)$ such that

$$z_s + [0, L_s)^d \subset \mathrm{B}(z, L_s).$$

By the definition of A' and (28), for any $x \in \mathcal{Q}_{K',s,0}(x'_s) \cap (z_s + [0, L_s)^d)$, $\mathcal{C}_x \subset A$. Thus, $\mathcal{Q}_{K',s,0}(x'_s) \cap (z_s + [0, L_s)^d) \subseteq \mathcal{A}$. By Lemma 2.4 and (27),

$$\begin{aligned} |\mathcal{Q}_{K',s,0}(x'_s) \cap (z_s + [0, L_s)^d)| = |\mathcal{Q}_{1,s,0}(z_s)| &\geq \frac{1+\eta_2}{1+\frac{\eta_2+2\eta_1}{2}} \cdot \left(\frac{L_s}{L_0}\right)^d \\ &\geq \frac{1}{2} \cdot \left(\frac{L_s}{L_0}\right)^d. \end{aligned}$$

Thus,

$$|\mathcal{A} \cap \mathrm{B}(z, L_s)| \geq \frac{1}{2} \cdot \left(\frac{L_s}{L_0}\right)^d,$$

and we conclude that

$$\frac{1}{2}\left(\frac{L_s}{L_0}\right)^d |A \setminus A'| \le \big|\{z \in A \setminus A', x \in \mathcal{A} : x \in \mathrm{B}(z, L_s)\}\big| \le \big|\mathrm{B}(0, L_s)\big| \cdot |\mathcal{A}|,$$

which implies (33). □

Let $\gamma_{2.5}$ be the isoperimetric constant from Theorem 2.5:

$$\gamma_{2.5} = \frac{1}{2d \cdot 32^d \cdot 27^d \cdot 10^6} \cdot \left(1 - \left(\frac{2}{3}\right)^{\frac{1}{d}}\right) \cdot \left(1 - e^{-\frac{1}{16(d-1)}}\right).$$

CLAIM 3.13. Let $c_\eta = \frac{2\eta_1 - \eta_2}{4\eta_1}$. Then

$$\max\left\{|\partial_{\mathcal{Q}_{K',s,0}(x'_s)}\mathcal{A}|, \frac{|A'|}{L_s^d}\right\} \ge c_\eta \cdot \gamma_{2.5} \cdot \max\left\{|\mathcal{A}|^{\frac{d-1}{d}}, \frac{|A'|}{L_s^d}\right\}. \tag{34}$$

PROOF. If $|\mathcal{A}|^{\frac{d-1}{d}} < \frac{|A'|}{L_s^d}$, then (34) trivially holds. Thus, we assume that $|\mathcal{A}|^{\frac{d-1}{d}} \ge \frac{|A'|}{L_s^d}$. We will deduce (34) from Theorem 2.5. By (32),

$$|A| \le 2 \cdot 3^d \cdot L_0^d \cdot |\mathcal{A}| + L_s^d \cdot |\mathcal{A}|^{\frac{d-1}{d}} \le 3^{d+1} \cdot L_s^d \cdot |\mathcal{A}|.$$

Since $|A| \ge L_s^{d(d+1)}$, we obtain that $|\mathcal{A}| \ge (\frac{L_s}{L_0})^{d^2}$.

Since $\mathcal{A} \subseteq \mathcal{Q}_{K',s,0}(x'_s)$, for all $x \in \mathcal{A}$, $|\mathcal{C}_x| \ge \eta_1 L_0^d$. Thus, $|A| \ge \eta_1 L_0^d \cdot |\mathcal{A}|$. Since also all the vertices in $\mathbb{G}_s \cap Q_{K',s}(x'_s)$ are s-good, we obtain as in (25) that

$$\begin{aligned}
|A| &\le \frac{1}{2} \cdot |\widetilde{\mathcal{C}}_{K,s,L_0}| \le \frac{1}{2} \cdot \left(\eta_2 + 1 - \prod_{i=0}^{\infty}\left(1 - \left(\frac{4r_i}{l_i}\right)^d\right)\right) \cdot |Q_{K',s}(x'_s)| \\
&\overset{(27)}{\le} \frac{\eta_2 + 2\eta_1}{4} \cdot \prod_{i=0}^{\infty}\left(1 - \left(\frac{4r_i}{l_i}\right)^d\right) \cdot |Q_{K',s}(x'_s)| \\
&\le \frac{\eta_2 + 2\eta_1}{4} L_0^d \cdot |\mathcal{Q}_{K',s,0}(x'_s)|,
\end{aligned}$$

where the last inequality follows from Lemma 2.4. Thus, $|\mathcal{A}| \le (1 - c_\eta) \cdot |\mathcal{Q}_{K',s,0}(x'_s)|$. By Theorem 2.5 and Remark 2.6,

$$|\partial_{\mathcal{Q}_{K',s,0}(x'_s)}\mathcal{A}| \ge c_\eta \cdot \gamma_{2.5} \cdot |\mathcal{A}|^{\frac{d-1}{d}},$$

completing the proof of (34). □

We are now ready to prove Theorem 3.8. It easily follows from Claims 3.12 and 3.13.

PROOF OF THEOREM 3.8. By (29), (30) and (34),

$$\frac{|\partial_{\widetilde{\mathcal{C}}_{K,s,L_0}} A|}{|A|^{\frac{d-1}{d}}} \geq \frac{\frac{1}{31^d} \cdot c_\eta \cdot \gamma_{2.5} \cdot \max\{|\mathcal{A}|^{\frac{d-1}{d}}, \frac{|A'|}{L_s^d}\}}{(2 \cdot 3^d \cdot L_0^d \cdot |\mathcal{A}| + |A'|)^{\frac{d-1}{d}}}$$

$$\geq \frac{\frac{1}{31^d} \cdot c_\eta \cdot \gamma_{2.5} \cdot \max\{|\mathcal{A}|^{\frac{d-1}{d}}, \frac{|A'|}{L_s^d}\}}{2 \cdot 3^{d-1} \cdot L_0^{d-1} \cdot |\mathcal{A}|^{\frac{d-1}{d}} + |A'|^{\frac{d-1}{d}}}.$$

On the one hand, if $L_0^d \cdot |\mathcal{A}| \geq |A'|$, then

$$\frac{|\partial_{\widetilde{\mathcal{C}}_{K,s,L_0}} A|}{|A|^{\frac{d-1}{d}}} \geq \frac{\frac{1}{31^d} \cdot c_\eta \cdot \gamma_{2.5} \cdot |\mathcal{A}|^{\frac{d-1}{d}}}{2 \cdot 3^{d-1} \cdot L_0^{d-1} \cdot |\mathcal{A}|^{\frac{d-1}{d}} + |A'|^{\frac{d-1}{d}}} \geq \frac{\frac{1}{31^d} \cdot c_\eta \cdot \gamma_{2.5}}{3 \cdot (3 \cdot L_0)^{d-1}}.$$

On the other hand, if $L_0^d \cdot |\mathcal{A}| \leq |A'|$, then by (30), $|A'| \geq \frac{1}{3^{d+1}} \cdot |A| \geq \frac{1}{3^{d+1}} \cdot L_s^{d(d+1)} \geq L_s^{d^2}$, and

$$\frac{|\partial_{\widetilde{\mathcal{C}}_{K,s,L_0}} A|}{|A|^{\frac{d-1}{d}}} \geq \frac{\frac{1}{31^d} \cdot c_\eta \cdot \gamma_{2.5} \cdot |A'|^{\frac{1}{d}}}{3^d \cdot L_s^d} \geq \frac{1}{93^d} \cdot c_\eta \cdot \gamma_{2.5}.$$

The proof of Theorem 3.8 is complete with $\gamma_{3.8} = \frac{1}{93^d \cdot L_0^{d-1}} \cdot c_\eta \cdot \gamma_{2.5}$. $\square$

REMARK 3.14. With a more careful analysis and assuming that Theorem 2.5 holds for all subsets of size at least $(\frac{L_s}{L_0})^{2d}$ (see Remark 5.11), condition on A in Theorem 3.8 can be relaxed to $|A| \geq L_s^{2d}$. Assuming that Theorem 2.5 holds for all subsets (see Remark 5.11), condition on A in Theorem 3.8 can be relaxed to $|A| \geq L_s^d$. Since for our purposes the current statement of Theorem 3.8 suffices, we do not prove the stronger statement here.

3.4. *Graph distance*. In this section, we study the graph distances $\mathrm{d}_{\mathcal{S}}$ in $\mathcal{S}$ between vertices of $\mathcal{C}_{K,s,L_0}(x_s)$ for configurations in $\mathcal{H}^\eta_{K,s}(x_s)$. As a consequence, we prove that large enough balls centered at vertices of $\mathcal{C}_{K,s,L_0}(x_s)$ have regular volume growth (Corollary 3.16) and allow for local extensions which satisfy an isoperimetric inequality (Corollary 3.17). These results will be used in Section 4 to prove our main result.

LEMMA 3.15. *Let $d \geq 2$ and η as in (21). Let l_n and r_n, $n \geq 0$, be integer sequences such that for all n, $l_n > 16r_n$ and $\prod_{n\geq 0}(1 + \frac{32r_n}{l_n}) \leq 2$. Let $L_0 \geq 1$, $K \geq 1$, and $s \geq 0$ integers, $x_s \in \mathbb{G}_s$. There exists $C_{3.15} = C_{3.15}(L_0)$ such that if $\mathcal{H}^\eta_{K,s}(x_s)$ occurs, then for all $y, y' \in \mathcal{C}_{K,s,L_0}(x_s)$,*

$$\mathrm{d}_{\mathcal{S}}(y, y') \leq C_{3.15} \cdot \max\{|y - y'|_\infty, L_s^d\}.$$

PROOF. Let $y_s, y'_s \in Q_{K,s}(x_s) \cap \mathbb{G}_s$ be such that $(y_s + [0, L_s)^d) \subset \mathrm{B}(y, L_s)$ and $(y'_s + [0, L_s)^d) \subset \mathrm{B}(y', L_s)$. By [19], Lemma 6.3 (applied to sequences l_n and $4r_n$), there exist $y_0 \in \mathcal{Q}_{K,s,0}(x_s) \cap (y_s + [0, L_s)^d)$ and $y'_0 \in \mathcal{Q}_{K,s,0}(x_s) \cap (y'_s + [0, L_s)^d)$ which are connected by a nearest neighbor path of 0-good vertices $z_1 = y_0, z_2, \dots, z_{k-1}, z_k = y'_0$ in $\mathcal{Q}_{K,s,0}(x_s)$, where $k \le \prod_{n\ge 0}(1 + \frac{32r_n}{l_n}) \cdot \frac{|y_s - y'_s|_1 + L_s}{L_0}$.

Let $\widetilde{z}_i$ be an arbitrary vertex in $\mathcal{C}_{z_i}$. (Recall the definition of $\mathcal{C}_z$ from Lemma 3.1.) By Lemma 3.1, for all $1 \le i < k$, $\widetilde{z}_i$ is connected to $\widetilde{z}_{i+1}$ in $\mathcal{S} \cap ((z_i + [0, L_0)^d) \cup (z_{i+1} + [0, L_0)^d)$. Therefore, any vertices $\widetilde{y} \in \mathcal{C}_{y_0}$ and $\widetilde{y}' \in \mathcal{C}_{y'_0}$ are connected by a nearest neighbor path in $\mathcal{S} \cap \bigcup_{i=1}^k (z_i + [0, L_0)^d)$. Any such path consists of at most $L_0^d \cdot \prod_{n\ge 0}(1 + \frac{32r_n}{l_n}) \cdot \frac{|y_s - y'_s|_1 + L_s}{L_0}$ vertices.

By Corollary 3.4, $\widetilde{y} \in \mathcal{C}_{K,s,L_0}(x_s) \cap \mathrm{B}(y, L_s)$ and $\widetilde{y}' \in \mathcal{C}_{K,s,L_0}(x_s) \cap \mathrm{B}(y', L_s)$. Thus, by the definition of $\mathcal{H}^\eta_{K,s}(x_s)$, y is connected to $\widetilde{y}$ in $\mathcal{S} \cap \mathrm{B}(y, 2L_s)$ and y' is connected to $\widetilde{y}'$ in $\mathcal{S} \cap \mathrm{B}(y', 2L_s)$.

By putting all the arguments together, we obtain that y is connected to y' by a nearest neighbor path in $\mathcal{S}$ of at most $2 \cdot |\mathrm{B}(0, 2L_s)| + L_0^d \cdot \prod_{n\ge 0}(1 + \frac{32r_n}{l_n}) \cdot \frac{|y_s - y'_s|_1 + L_s}{L_0}$ vertices. Since $|y_s - y'_s|_1 \le d \cdot |y - y'|_\infty + 2dL_s$, the result follows. $\square$

COROLLARY 3.16. *In the setup of Lemmas 3.3 and 3.15, there exists $c_{3.16} = c_{3.16}(\eta, L_0) > 0$ such that for any $C_{3.15} L_s^d \le r \le K L_s$ and $y \in \mathcal{C}_{K,s,L_0}(x_s)$,*

$$\mu\big(\mathrm{B}_{\mathcal{S}}(y, r)\big) \ge c_{3.16} \cdot r^d.$$

PROOF. Let $K' = \max\{k : kL_s \le \frac{r}{C_{3.15}}\}$. There exists $y_s \in Q_{K,s}(x_s) \cap \mathbb{G}_s$ such that $Q_{K',s}(y_s) \subset \mathrm{B}(y, \frac{r}{C_{3.15}}) \cap Q_{K,s}(x_s)$. Since $\mathcal{H}^\eta_{K,s}(x_s)$ occurs, we can define the perforation $\mathcal{Q}_{K,s,0}(x_s)$ of $Q_{K,s}(x_s)$ as in (19). Consider also the perforation $\mathcal{Q}_{K',s,0}(y_s) = \mathcal{Q}_{K,s,0}(x_s) \cap Q_{K',s}(y_s)$ of $Q_{K',s}(y_s)$. By (26),

$$\bigcup_{x \in \mathcal{Q}_{K',s,0}(y_s)} \mathcal{C}_x \subset \mathcal{C}_{K,s,L_0}(x_s).$$

Since also $\bigcup_{x \in \mathcal{Q}_{K',s,0}(y_s)} \mathcal{C}_x \subset \mathrm{B}(y, \frac{r}{C_{3.15}})$, Lemma 3.15 implies that

$$\bigcup_{x \in \mathcal{Q}_{K',s,0}(y_s)} \mathcal{C}_x \subset \mathrm{B}_{\mathcal{S}}(y, r).$$

By applying Lemma 2.4 to $\mathcal{Q}_{K',s,0}(y_s)$ and using the fact that $|\mathcal{C}_x| \ge \eta_1 L_0^d$, we conclude from the above inclusion that

$$\begin{aligned} \big|\mathrm{B}_{\mathcal{S}}(y, r)\big| &\ge \eta_1 \cdot (K'L_s)^d \cdot \prod_{i\ge 0}\left(1 - \left(\frac{4r_i}{l_i}\right)^d\right) \\ &\overset{(27)}{\ge} \eta_1 \cdot (K'L_s)^d \cdot \frac{1+\eta_2}{1 + \frac{\eta_2 + 2\eta_1}{2}} \overset{(21)}{\ge} \frac{1}{2}\eta_2 \cdot \left(\frac{r}{2C_{3.15}}\right)^d. \end{aligned}$$

Since $\mu(\mathrm{B}_{\mathcal{S}}(y,r)) \ge |\mathrm{B}_{\mathcal{S}}(y,r)|$, the result follows with $c_{3.16} = \frac{1}{2}\eta_2 \cdot \frac{1}{(2C_{3.15})^d}$. $\square$

COROLLARY 3.17. *Let $\epsilon \in (0, \frac{1}{d}]$. In the setup of Theorem 3.8 and Lemma 3.15, if $\mathcal{H}^{\eta}_{5K,s}(x'_s)$ occurs with $x'_s = x_s + (-2KL_s, \dots, -2KL_s)$, then for all $L_s^{d+1+\frac{d^2-1}{\epsilon d}} \le r \le KL_s$ and $y \in \mathcal{C}_{K,s,L_0}(x_s)$, there exists $\mathcal{C}_{\mathrm{B}_{\mathcal{S}}(y,r)}$ such that $\mathrm{B}_{\mathcal{S}}(y,r) \subseteq \mathcal{C}_{\mathrm{B}_{\mathcal{S}}(y,r)} \subseteq \mathrm{B}_{\mathcal{S}}(y, 8C_{3.15}r)$ and for all $A \subset \mathcal{C}_{\mathrm{B}_{\mathcal{S}}(y,r)}$ with $|A| \le \frac{1}{2} \cdot |\mathcal{C}_{\mathrm{B}_{\mathcal{S}}(y,r)}|$,*

$$|\partial_{\mathcal{C}_{\mathrm{B}_{\mathcal{S}}(y,r)}} A| \ge \gamma_{3.8} \cdot |A|^{\frac{d-1}{d}+\epsilon} \cdot (8r)^{-\epsilon d}.$$

In particular, if $\epsilon = \frac{1}{d}$, then $|\partial_{\mathcal{C}_{\mathrm{B}_{\mathcal{S}}(y,r)}} A| \ge \gamma_{3.8} \cdot \frac{|A|}{8r}$.

PROOF. Let $K' = \min\{k : kL_s \ge 2r+1\} + 1$. (Note that $K'L_s \le 4r$.) For $y \in \mathcal{C}_{K,s,L_0}(x_s)$, let $y_s \in \mathbb{G}_s \cap Q_{5K,s}(x'_s)$ be such that $\mathrm{B}(y,r) \subseteq Q_{K',s}(y_s) \subseteq Q_{5K,s}(x'_s)$.

We will prove that $\mathcal{C}_{\mathrm{B}_{\mathcal{S}}(y,r)} = \widetilde{\mathcal{C}}_{K',s,L_0}(y_s)$ satisfies all the requirements. Since $\mathcal{H}^{\eta}_{K',s}(y_s)$ occurs, by Corollary 3.4, $\mathrm{B}_{\mathcal{S}}(y,r) \subseteq \mathcal{C}_{K',s,L_0}(y_s) \subseteq \widetilde{\mathcal{C}}_{K',s,L_0}(y_s)$. By Lemma 3.15, for $r \ge L_s^d$,

$$\widetilde{\mathcal{C}}_{K',s,L_0}(y_s) \subseteq \mathrm{B}_{\mathcal{S}}\big(y, C_{3.15}(K'+4)L_s\big) \subseteq \mathrm{B}_{\mathcal{S}}(y, 8C_{3.15}r).$$

By Corollary 3.9, since $K' \ge L_s^{d+\frac{d^2-1}{\epsilon d}}$, for any $A \subset \widetilde{\mathcal{C}}_{K',s,L_0}(y_s)$ with $|A| \le \frac{1}{2}|\widetilde{\mathcal{C}}_{K',s,L_0}(y_s)|$,

$$|\partial_{\widetilde{\mathcal{C}}_{K',s,L_0}(y_s)} A| \ge \gamma_{3.8} \cdot |A|^{\frac{d-1}{d}+\epsilon} \cdot \big((K'+4)L_s\big)^{-\epsilon d} \ge \gamma_{3.8} \cdot |A|^{\frac{d-1}{d}+\epsilon} \cdot (8r)^{-\epsilon d}.$$

Since $\mathcal{C}_{\mathrm{B}_{\mathcal{S}}(y,r)} = \widetilde{\mathcal{C}}_{K',s,L_0}(y_s)$ satisfies all the necessary conditions, the proof of Corollary 3.17 is complete. $\square$

4. Proof of Theorem 1.13. In this section, we collect together the deterministic results that large enough balls have regular volume growth (Corollary 3.16) and allow for local extensions satisfying an isoperimetric inequality (Corollary 3.17) to deduce Theorem 1.13. In fact, the result that we prove here is stronger. In Definition 4.1, we introduce the notions of regular and very regular balls, so that (very) regular ball is always (very) good (see Claim 4.2), and then prove in Proposition 4.3 that large balls are likely to be very regular. The main result is an immediate consequence of Proposition 4.3.

The following definition will only be used for the special choice of $\epsilon = \frac{1}{d}$; see Claim 4.2. Nevertheless, we choose to work with the more general definition involving arbitrary $\epsilon \in (0, \frac{1}{d}]$, since smaller ϵ's give better isoperimetric inequalities, and could be used to prove stronger functional inequalities than the Poincaré inequality, as we learned from Jean-Dominique Deuschel (see, e.g., [31], Section 3.2).

DEFINITION 4.1. Let C_V, C_P, and $C_W \geq 1$ be fixed constants. Let $\epsilon \in (0, \frac{1}{d}]$. For $r \geq 1$ integer and $x \in V(G)$, we say that $\mathrm{B}_G(x,r)$ is $(C_V, C_P, C_W, \epsilon)$-*regular* if

$$\mu\big(\mathrm{B}_G(x,r)\big) \geq C_V r^d$$

and there exists a set $\mathcal{C}_{\mathrm{B}_G(x,r)}$ such that $\mathrm{B}_G(x,r) \subseteq \mathcal{C}_{\mathrm{B}_G(x,r)} \subseteq \mathrm{B}_G(x, C_W r)$ and for any $A \subset \mathcal{C}_{\mathrm{B}_G(x,r)}$ with $|A| \leq \frac{1}{2} \cdot |\mathcal{C}_{\mathrm{B}_G(x,r)}|$,

$$|\partial_{\mathcal{C}_{\mathrm{B}_G(x,r)}} A| \geq \frac{1}{\sqrt{C_P}} \cdot |A|^{\frac{d-1}{d}+\epsilon} \cdot r^{-\epsilon d}.$$

We say $\mathrm{B}_G(x,R)$ is $(C_V, C_P, C_W, \epsilon)$-*very regular* if there exists $N_{\mathrm{B}_G(x,R)} \leq R^{\frac{1}{d+2}}$ such that $\mathrm{B}_G(y,r)$ is (C_V, C_P, C_W)-regular whenever $\mathrm{B}_G(y,r) \subseteq \mathrm{B}_G(x,R)$, and $N_{\mathrm{B}_G(x,R)} \leq r \leq R$.

In the special case $\epsilon = \frac{1}{d}$, we omit ϵ from the notation and call $(C_V, C_P, C_W, \frac{1}{d})$-(very) regular ball simply (C_V, C_P, C_W)-(very) regular.

CLAIM 4.2. *If $\mathrm{B}_G(x,r)$ is (C_V, C_P, C_W)-regular, then it is (C_V, C_P, C_W)-good.*

PROOF. By [24], Proposition 3.3.10, and Remark 1.2,

$$\begin{aligned}\min_a \int_{\mathcal{C}_{\mathrm{B}_G(x,r)}} (f-a)^2\, d\mu &= \int_{\mathcal{C}_{\mathrm{B}_G(x,r)}} (f - \overline{f}_{\mathcal{C}_{\mathrm{B}_G(x,r)}})^2\, d\mu \\ &\leq C_P \cdot r^2 \cdot \int_{E(\mathcal{C}_{\mathrm{B}_G(x,r)})} |\nabla f|^2\, d\nu.\end{aligned}$$

Thus, again by Remark 1.2,

$$\begin{aligned}\min_a \int_{\mathrm{B}_G(x,r)} (f-a)^2\, d\mu &\leq \int_{\mathrm{B}_G(x,r)} (f - \overline{f}_{\mathcal{C}_{\mathrm{B}_G(x,r)}})^2\, d\mu \\ &\leq \int_{\mathcal{C}_{\mathrm{B}_G(x,r)}} (f - \overline{f}_{\mathcal{C}_{\mathrm{B}_G(x,r)}})^2\, d\mu \\ &\leq C_P \cdot r^2 \cdot \int_{E(\mathcal{C}_{\mathrm{B}_G(x,r)})} |\nabla f|^2\, d\nu \\ &\leq C_P \cdot r^2 \cdot \int_{E(\mathrm{B}_G(x,C_W r))} |\nabla f|^2\, d\nu. \qquad \square\end{aligned}$$

Theorem 1.13 is immediate from Claim 4.2 and the following proposition, in which one needs to take $\epsilon = \frac{1}{d}$.

PROPOSITION 4.3. *Let $d \geq 2$, $u \in (a,b)$, and $\theta_{\mathrm{vgb}} \in (0, \frac{1}{d+2})$. Let $\epsilon \in (0, \frac{1}{d}]$. Assume that the family of measures $\mathbb{P}^u$, $u \in (a,b)$, satisfies assumptions* **P1–P3** *and* **S1**–**S2**. *There exist constants* C_V, C_P, *and* C_W, $c_{4.3}$ *and* $C_{4.3}$ *depending on* u, θ_{vgb}, *and* ϵ, *such that for all* $R \geq 1$,

$$\mathbb{P}^u\big[\mathrm{B}_{\mathcal{S}}(0,R) \text{ is } (C_V, C_P, C_W, \epsilon)\text{-very regular with } N_{\mathrm{B}_{\mathcal{S}}(0,R)} \leq R^{\theta_{\mathrm{vgb}}} | 0 \in \mathcal{S}_\infty\big]$$
$$\geq 1 - C_{4.3} \cdot e^{-c_{4.3}(\log R)^{1+\Delta_{\mathrm{S}}}}.$$

PROOF. We first make a specific choice of various parameters. Fix $u \in (a,b)$. We take

$$\eta_1 = \frac{3}{4}\eta(u) \quad \text{and} \quad \eta_2 = \frac{5}{4}\eta(u), \tag{35}$$

where $\eta(u)$ is defined in **S2**. It is easy to see that η_1 and η_2 satisfy assumptions (21). We fix this choice of $\eta = (\eta_1, \eta_2)$ throughout the proof.

Next, we choose the scales for renormalization. For positive integers l_0, r_0, and L_0, we take

$$\begin{aligned} &\theta_{\mathrm{sc}} = \lceil 1/\varepsilon_{\mathrm{P}} \rceil, \qquad l_n = l_0 \cdot 4^{n^{\theta_{\mathrm{sc}}}}, \qquad r_n = r_0 \cdot 2^{n^{\theta_{\mathrm{sc}}}}, \\ &L_n = l_{n-1} \cdot L_{n-1}, \qquad n \geq 1, \end{aligned} \tag{36}$$

where ε_{P} is defined in **P3**. By [19], Lemmas 5.2 and 5.4, under the assumptions **P1**–**P3** and **S1**–**S2**, there exist $C_1 = C_1(u) < \infty$ and $C_2 = C_2(u, l_0) < \infty$ such that for all $l_0, r_0 \geq C_1$, $L_0 \geq C_2$, and $n \geq 0$,

$$\mathbb{P}^u[0 \text{ is } n\text{-bad}] \leq 2 \cdot 2^{-2^n}. \tag{37}$$

We choose $l_0, r_0 \geq C_1$ so that the scales l_n and r_n defined in (36) satisfy the conditions of Lemma 3.3, Theorem 3.8 and Lemma 3.15, and choose $L_0 \geq C_2$. Thus, (37) is also satisfied.

Next, we choose s and K. Fix $R \geq 1$. Without loss of generality, we can assume that

$$R^{\theta_{\mathrm{vgb}}} \geq \max\big(C_{3.15} L_0^d, L_0^{d+1+\frac{d^2-1}{\epsilon d}}\big).$$

Let

$$s = \max\big\{s' : \max\big\{C_{3.15} L_{s'}^d, L_{s'}^{d+1+\frac{d^2-1}{\epsilon d}}\big\} \leq R^{\theta_{\mathrm{vgb}}}\big\}.$$

With this choice of s, let $K = \min\{k : kL_s \geq 2R+1\} + 1$, $x_s \in \mathbb{G}_s$ such that $\mathrm{B}(0,R) \subseteq Q_{K,s}(x_s)$, and $x_s' = x_s + (-2KL_s, \ldots, -2KL_s)$.

We begin with the proof. If the event $\mathcal{H}^\eta_{5K,s}(x_s') \cap \{0 \in \mathcal{S}_\infty\}$ occurs, then $\mathrm{B}_{\mathcal{S}}(0,R) \subseteq \mathcal{C}_{K,s,L_0}(x_s)$. Therefore, for all $y \in \mathrm{B}_{\mathcal{S}}(0,R)$ and $R^{\theta_{\mathrm{vgb}}} \leq r \leq R$, by Corollaries 3.16 and 3.17, the ball $\mathrm{B}_{\mathcal{S}}(y,r)$ is $(c_{3.16}, \frac{64^{\epsilon d}}{\gamma_{3.8}^2}, 8C_{3.15}, \epsilon)$-regular. Thus,

$$\begin{aligned} &\text{if the event } \mathcal{H}^\eta_{5K,s}(x_s') \cap \{0 \in \mathcal{S}_\infty\} \text{ occurs, then the ball } \mathrm{B}_{\mathcal{S}}(0,R) \\ &\text{is } \Big(c_{3.16}, \frac{64^{\epsilon d}}{\gamma_{3.8}^2}, 8C_{3.15}, \epsilon\Big)\text{-very regular with } N_{\mathrm{B}_{\mathcal{S}}(0,R)} \leq R^{\theta_{\mathrm{vgb}}}. \end{aligned} \tag{38}$$

Let

$$C_V = c_{3.16}, \qquad C_P = \frac{64^{\epsilon d}}{\gamma_{3.8}^2}, \qquad C_W = 8C_{3.15}.$$

By (38), it suffices to prove that there exist constants $c = c(u, \theta_{\mathrm{vgb}}, \epsilon, \varepsilon_{\mathrm{P}}) > 0$ and $C = C(u, \theta_{\mathrm{vgb}}, \epsilon, \varepsilon_{\mathrm{P}}) < \infty$ such that for all $R \ge 1$,

$$\mathbb{P}^u[\mathcal{H}^\eta_{5K,s}(x'_s) | 0 \in \mathcal{S}_\infty] \ge 1 - Ce^{-c(\log R)^{1+\Delta_{\mathrm{S}}}}. \tag{39}$$

By Definition 3.7, (37) and **S1**, there exists $C = C(u) < \infty$ such that

$$\mathbb{P}^u[\mathcal{H}^\eta_{5K,s}(x'_s)^c] \le (5K+4)^d \cdot 2 \cdot 2^{-2^s} + (5KL_s)^d \cdot C \cdot e^{-f_{\mathrm{S}}(u, 2L_s)}.$$

Thus, it remains to show that for our choice of all the parameters, the right-hand side of the above display is at most $Ce^{-c(\log R)^{1+\Delta_{\mathrm{S}}}}$.

Let $D = d + 1 + \frac{d^2-1}{\epsilon d}$. By (36) and the choice of s, for all $R \ge C_{3.15} \cdot L_0^{D/\theta_{\mathrm{vgb}}}$,

$$\left(\frac{R}{C_{3.15}}\right)^{\frac{\theta_{\mathrm{vgb}}}{D}} \le L_{s+1} = l_s \cdot L_s \le l_0 \cdot 4 \cdot (L_s)^{1+2^{\theta_{\mathrm{sc}}}},$$

which implies that

$$L_s \ge \frac{1}{4l_0}\left(\frac{R}{C_{3.15}}\right)^{\frac{\theta_{\mathrm{vgb}}}{D(1+2^{\theta_{\mathrm{sc}}})}}. \tag{40}$$

By (36) and (40), there exists a constant $c = c(\theta_{\mathrm{vgb}}, \theta_{\mathrm{sc}}, l_0, L_0, \epsilon) > 0$ such that for all $R \ge C_{3.15} \cdot L_0^{D/\theta_{\mathrm{vgb}}}$,

$$s \ge c \cdot (\log R)^{\frac{1}{1+\theta_{\mathrm{sc}}}} - 1. \tag{41}$$

Using (5), (40) and (41), we deduce that there exist $c' = c'(u, \theta_{\mathrm{vgb}}, \theta_{\mathrm{sc}}, \epsilon) > 0$ and $C' = C'(u, \theta_{\mathrm{vgb}}, \theta_{\mathrm{sc}}, l_0, L_0, \epsilon) < \infty$ such that for all $R \ge C'$,

$$2^s \ge (\log R)^{1+\Delta_{\mathrm{S}}} \quad \text{and} \quad f_{\mathrm{S}}(u, 2L_s) \ge c'(\log R)^{1+\Delta_{\mathrm{S}}}.$$

By the choice of K, $KL_s \le 4R$. Therefore, there exist $c'' = c''(u, \theta_{\mathrm{vgb}}, \theta_{\mathrm{sc}}, \epsilon) > 0$ and $C'' = C''(u, \theta_{\mathrm{vgb}}, \theta_{\mathrm{sc}}, l_0, L_0, \epsilon) < \infty$ such that for all $R \ge C''$,

$$\mathbb{P}^u[\mathcal{H}^\eta_{5K,s}(x'_s)^c] \le C'' e^{-c''(\log R)^{1+\Delta_{\mathrm{S}}}}. \tag{42}$$

Since $\mathbb{P}^u[0 \in \mathcal{S}_\infty] = \eta(u) > 0$, (42) implies (39). The proof is complete. □

REMARK 4.4. The events $\overline{\mathrm{D}}^\eta_{x,L_0}$ and $\overline{\mathrm{I}}^\eta_{x,L_0}$ slightly differ from the corresponding events $\overline{A}^u_x$ and $\overline{B}^u_x$ in [19], but only minor modifications are needed to adapt [19], Lemmas 4.2 and 4.4, to our setting.

There is room for flexibility in the choice of η. For instance, if $\epsilon = \epsilon(u) \ge 0$ is chosen so that $\eta(u(1-\epsilon)) > \frac{5}{6} \cdot \eta(u(1+\epsilon))$, Then $\eta_1 = \frac{3}{4}\eta(u(1-\epsilon))$ and $\eta_2 = \frac{5}{4}\eta(u(1+\epsilon))$ satisfy (21), and (37) remains true for this choice of η by monotonicity.

5. Proof of Theorem 2.5. The rough outline of the proof is the following. We first prove the isoperimetric inequality for all subsets of perforated lattices in two dimensions; see Lemma 5.6. In dimensions $d \geq 3$, we proceed in two steps. We first consider only macroscopic subsets $\mathcal{A}$ of the perforated lattice, that is, those with the volume comparable with the volume of the perforated lattice. By applying a selection lemma (see Lemma 5.3), we identify a large number of disjoint two-dimensional slices in the ambient box which on the one hand have a small nonempty intersection with $\mathcal{A}$, and on the other, all together contain a positive fraction of the volume of $\mathcal{A}$. We estimate the boundary of $\mathcal{A}$ in each of the slices using the two-dimensional result, and conclude by estimating the boundary of $\mathcal{A}$ in the perforation by the sum of the boundaries of $\mathcal{A}$ in each of the slices. Finally, we treat the general case by constructing a suitable coarse graining of $\mathcal{A}$ from mesoscopic boxes in which $\mathcal{A}$ has positive density. The restriction of the boundary of $\mathcal{A}$ to such boxes is estimated by using the result from the first case. Both isoperimetric inequalities in $d \geq 3$ are stated in Theorem 5.10.

We begin with a number of auxiliary ingredients for the proof: (a) some general facts about isoperimetric inequalities (Section 5.1.1) and (b) a combinatorial selection lemma (Section 5.1.2).

5.1. *Auxiliary results.*

5.1.1. *General facts about isoperimetric inequalities.* Here, we collect some isoperimetric inequalities that we will frequently use.

LEMMA 5.1. *Let $d \geq 2$, $n_1, \dots, n_d \geq 1$ integers with $\max_i n_i \leq N \cdot \min_i n_i$, and C a positive real such that $N \cdot C^{\frac{1}{d}} < 1$. Then, for any subset A of $G = \mathbb{Z}^d \cap [0, n_1) \times \cdots \times [0, n_d)$ with $|A| \leq C \cdot |G|$,*

$$|\partial_G A| \geq \max\{(1 + 2d \cdot (1 - NC^{\frac{1}{d}})^{-1})^{-1} \cdot |\partial_{\mathbb{Z}^d} A|, (1 - NC^{\frac{1}{d}}) \cdot |A|^{\frac{d-1}{d}}\}.$$

PROOF. The proof is similar to that of [16], Proposition 2.2. Let π_i be the projection of $\mathbb{Z}^d$ onto the $(d-1)$ dimensional sublattice of vertices with ith coordinate equal to 0. Let $P_i = \pi_i(A)$, i' be a coordinate corresponding to P_i with the maximal size, and $P' = P_{i'}$. Let $P'' = P' \cap \pi_{i'}(G \setminus A)$, that is, the projection of those i'-columns that contain vertices from both A and $G \setminus A$. Note that $|\partial_G A| \geq |P''|$ and $|\partial_{\mathbb{Z}^d} A| \leq |\partial_G A| + 2d \cdot |P'|$. Also note that $|P' \setminus P''| \leq \frac{|A|}{n_{i'}} \leq N \cdot C^{\frac{1}{d}} \cdot |A|^{\frac{d-1}{d}}$. By the Loomis–Whitney inequality, $|A|^{\frac{d-1}{d}} \leq |P'|$. Thus, $|\partial_G A| \geq |P''| \geq (1 - N \cdot C^{\frac{1}{d}}) \cdot |P'| \geq (1 - N \cdot C^{\frac{1}{d}}) \cdot |A|^{\frac{d-1}{d}}$ and $|\partial_{\mathbb{Z}^d} A| \leq |\partial_G A| \cdot (1 + 2d \cdot (1 - N \cdot C^{\frac{1}{d}})^{-1})$. $\square$

REMARK 5.2. Let G be a finite graph, and assume that for all $A \subseteq G$ with $c_1 \cdot |G| \leq |A| \leq \frac{1}{2} \cdot |G|$, $|\partial_G A| \geq c_2 \cdot |A|^{\frac{d-1}{d}}$. Then for any $A' \subset G$ with $\frac{1}{2} \cdot |G| \leq$

$|A'| \le (1 - c_1) \cdot |G|$, $|\partial_G A'| = |\partial_G (G \setminus A')| \ge c_2 \cdot |G \setminus A'|^{\frac{d-1}{d}} \ge (c_1 c_2) \cdot |A'|^{\frac{d-1}{d}}$. Thus, any such A' also satisfies an isoperimetric inequality, but possibly with a smaller constant.

5.1.2. *Selection lemma.* The aim of this section is to prove the following combinatorial lemma. Its Corollaries 5.4 and 5.5 together with the two-dimensional isoperimetric inequality of Lemma 5.6 will be crucially used in the proof of the isoperimetric inequality for macroscopic subsets of perforated lattices in any dimension $d \ge 3$ in Theorem 5.10.

LEMMA 5.3. *Let $\frac{6}{7} \le C_2 < 1$, and for $d \ge 2$, let*

$$C_d = \frac{C_2^{d-1}}{\prod_{j=1}^{d-2}(1 + \frac{3}{9^j})}, \qquad \delta_d = \frac{1}{9^{d-2}}.$$

Let $R_1, \dots, R_d$ be positive integers. Then, for any subset A of $Q = [0, R_1) \times \cdots \times [0, R_d) \cap \mathbb{Z}^d$ satisfying

$$1 \le |A| \le C_d \cdot |Q|,$$

there exist $S_1, \dots, S_k$, disjoint two-dimensional subrectangles of Q such that

$$\left| A \cap \bigcup_i S_i \right| \ge \delta_d \cdot |A|,$$

and for all $1 \le i \le k$,

$$1 \le |A \cap S_i| \le C_2 \cdot |S_i|.$$

COROLLARY 5.4. *Note that $\prod_{j=1}^{d-2}(1 + \frac{3}{9^j}) \le e^{\sum_{j \ge 1} \frac{3}{9^j}} = e^{\frac{3}{8}}$. Thus, if we take $C_2 = e^{-\frac{1}{8(d-1)}} > \frac{6}{7}$, then $C_d > e^{-\frac{1}{2}} > \frac{1}{2}$, and Lemma 5.3 implies that for any $A \subset Q$ with $|A| \le \frac{1}{2} \cdot |Q|$, there exist disjoint two-dimensional rectangles $S_1, \dots, S_k$ such that $|A \cap \bigcup_i S_i| \ge \frac{1}{9^{d-2}} \cdot |A|$ and $1 \le |A \cap S_i| \le e^{-\frac{1}{8(d-1)}} \cdot |S_i|$.*

COROLLARY 5.5. *If $R_1 = \cdots = R_d = R$, and $|A| \ge c_d \cdot R^d$ for some $c_d > 0$, then at least $\frac{\delta_d c_d}{2} R^{d-2}$ of the S_i's contain at least $\frac{\delta_d c_d}{2} R^2$ vertices from A. Indeed, if such a choice did not exist, then we would have*

$$\delta_d c_d R^d \le \delta_d \cdot |A| \le \left| A \cap \bigcup_i S_i \right| < R^2 \cdot \frac{\delta_d c_d}{2} R^{d-2} + \frac{\delta_d c_d}{2} R^2 \cdot \left(k - \frac{\delta_d c_d}{2} R^{d-2} \right)$$
$$\le \delta_d c_d R^d.$$

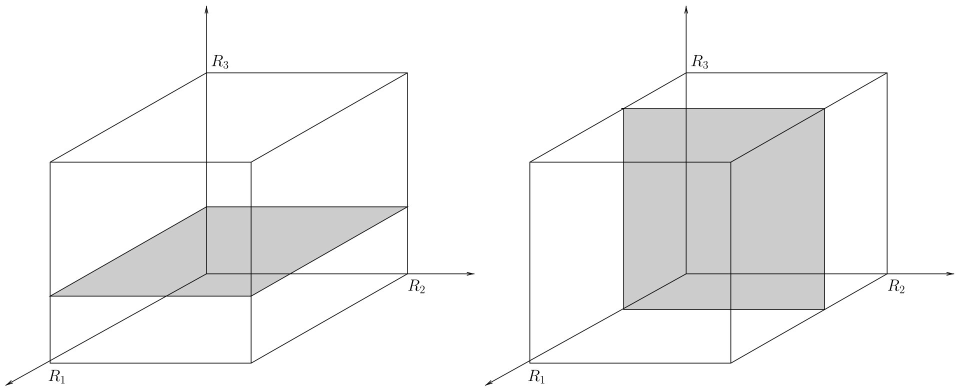

FIG. 4. *An illustration of a slice* $[0, R_1) \times [0, R_2) \times z$, $z \in [0, R_3)$ (*left*), *and a rectangle* $x \times [0, R_2) \times [0, R_3)$, $x \in [0, R_1)$ *from* $\mathcal{M}$ (*right*) *in* 3 *dimensions*.

PROOF OF LEMMA 5.3. The proof is by induction on d. For $d = 2$, the statement is obvious. We assume that $d \geq 3$.

Consider all two-dimensional slices of the form $[0, R_1) \times [0, R_2) \times x$, $x \in [0, R_3) \times \cdots \times [0, R_d)$, see Figure 4(left). If among them there exist slices $S_1, \dots, S_k$ such that $|A \cap \bigcup_i S_i| \geq \delta_d \cdot |A|$ and for all i, $1 \leq |A \cap S_i| \leq C_2 \cdot R_1 R_2$, then we are done.

Thus, assume the contrary. Let $\mathcal{S}_1$ be the subset of those slices that contain $> C_2 \cdot R_1 R_2$ vertices from A, and $\mathcal{S}_2$ the rest. By definition, $|\mathcal{S}_1| \leq \frac{|A|}{C_2 \cdot R_1 R_2}$, and by assumption, $|A \cap \bigcup_{S \in \mathcal{S}_2} S| < \delta_d \cdot |A|$.

Consider $(d-1)$ dimensional rectangles

$$\mathcal{M} = \{x \times [0, R_2) \times \cdots \times [0, R_d), x \in [0, R_1)\}$$

[see Figure 4(right)], and consider separately their intersections with $\mathcal{S}_1$ and $\mathcal{S}_2$.

First, consider intersections with $\mathcal{S}_1$. Each of the rectangles from $\mathcal{M}$ intersects $\bigcup_{S \in \mathcal{S}_1} S$ in at most $R_2 \cdot \frac{|A|}{C_2 \cdot R_1 R_2} = \frac{|A|}{C_2 \cdot R_1}$ vertices. Since $|A \cap \bigcup_{S \in \mathcal{S}_1} S| \geq (1 - \delta_d) \cdot |A|$, the number of rectangles $M \in \mathcal{M}$ with $|M \cap A| \geq \frac{|A|}{3 \cdot R_1}$ is at least $\frac{2}{3} R_1$. Indeed, if not, then at least $\frac{1}{3} R_1$ of rectangles from $\mathcal{M}$ contain $< \frac{|A|}{3 \cdot R_1}$ vertices from A, and

$$\begin{aligned} \left| A \cap \bigcup_{S \in \mathcal{S}_1} S \right| &< \frac{1}{3} R_1 \cdot \frac{|A|}{3 \cdot R_1} + \frac{2}{3} R_1 \cdot \frac{|A|}{C_2 \cdot R_1} = \left(\frac{1}{9} + \frac{2}{3 \cdot C_2} \right) \cdot |A| \leq \frac{8}{9} \cdot |A| \\ &\leq (1 - \delta_d) \cdot |A|, \end{aligned}$$

which is a contradiction.

Next, consider intersections with $\mathcal{S}_2$. Since $|A \cap \bigcup_{S \in \mathcal{S}_2} S| \leq \delta_d \cdot |A|$, the number of rectangles $M \in \mathcal{M}$ with $|A \cap M \cap \bigcup_{S \in \mathcal{S}_2} S| \leq 3\delta_d \cdot \frac{|A|}{R_1}$ is at least $\frac{2}{3} R_1$. Indeed, if not, then for at least $\frac{1}{3}R_1$ of them, $|A \cap M \cap \bigcup_{S\in\mathcal{S}_2} S| > 3\delta_d \cdot \frac{|A|}{R_1}$, and

$$\left|A \cap \bigcup_{S\in\mathcal{S}_2} S\right| > \frac{1}{3}R_1 \cdot 3\delta_d \cdot \frac{|A|}{R_1} = \delta_d \cdot |A|,$$

which is a contradiction.

Therefore, we can choose $M_1, \dots, M_{\frac{1}{3}R_1} \in \mathcal{M}$ so that for each $1 \le i \le \frac{1}{3}R_1$,

$$|A \cap M_i| \ge \frac{|A|}{3R_1}, \qquad \left|A \cap M_i \cap \bigcup_{S\in\mathcal{S}_1} S\right| \le \frac{|A|}{C_2 R_1},$$

$$\left|A \cap M_i \cap \bigcup_{S\in\mathcal{S}_2} S\right| \le 3\delta_d \cdot \frac{|A|}{R_1}.$$

In particular, for each $1 \le i \le \frac{1}{3}R_1$,

$$\begin{aligned} |A \cap M_i| &= \left|A \cap M_i \cap \bigcup_{S\in\mathcal{S}_1} S\right| + \left|A \cap M_i \cap \bigcup_{S\in\mathcal{S}_2} S\right| \\ &\le \frac{|A|}{C_2 \cdot R_1} + 3\delta_d \cdot \frac{|A|}{R_1} \le \frac{C_d}{C_2} \cdot \prod_{j=2}^{d} R_j \cdot \left(1 + \frac{3}{9^{d-2}}\right) = C_{d-1} \cdot \prod_{j=2}^{d} R_j \end{aligned}$$

and

$$\left|A \cap \bigcup_i M_i\right| = \sum_i |A \cap M_i| \ge \frac{1}{3}R_1 \cdot \frac{|A|}{3 \cdot R_1} = \frac{|A|}{9}.$$

If $d = 3$, then M_i are disjoint two-dimensional rectangles satisfying all the requirements of the lemma. If $d > 3$, consider the sets $A_i = A \cap M_i$, $1 \le i \le \frac{1}{3}R_1$. They satisfy assumption of the lemma with d replaced by $d - 1$. Therefore, there exist disjoint two-dimensional rectangles $(S_{ij})_{1\le j\le k_j}$ in M_i such that for all $1 \le j \le k_i$,

$$|A_i \cap S_{ij}| \le C_2 \cdot |S_{ij}|,$$

and

$$\left|A_i \cap \bigcup_j S_{ij}\right| \ge \delta_{d-1} \cdot |A_i|.$$

It is easy to conclude that the two-dimensional rectangles $(S_{ij})_{1\le j\le k_i, 1\le i\le \frac{1}{3}R_1}$ satisfy all the requirements of the lemma. Indeed, they are disjoint,

$$\left|A \cap \bigcup_{ij} S_{ij}\right| = \sum_i \left|A_i \bigcup_j S_{ij}\right| \ge \frac{1}{3}R_1 \cdot \delta_{d-1} \cdot |A_i| \ge \frac{1}{3}R_1 \cdot \delta_{d-1} \cdot \frac{|A|}{3 \cdot R_1} = \delta_d \cdot |A|,$$

and for each i and j,

$$|A \cap S_{ij}| = |A_i \cap S_{ij}| \le C_2 \cdot |S_{ij}|.$$

The proof is complete. $\square$

5.2. *Isoperimetric inequality in two dimensions.* The main goal of this section is to prove the following lemma. It immediately implies Theorem 2.5 in the case $d = 2$, but actually gives an isoperimetric inequality which holds for *all* $\mathcal{A} \in \mathcal{Q}_{K,s,0}(x_s)$ with $1 \le |\mathcal{A}| \le \frac{1}{2} \cdot |Q_{K,s} \cap \mathbb{G}_0|$.

LEMMA 5.6. *Let $d = 2$. Let l_n and r_n, $n \ge 0$, be integer sequences such that for all n, $l_n > 8r_n$, l_n is divisible by r_n and*

$$\prod_{j=0}^{\infty}\left(1-\left(\frac{4r_j}{l_j}\right)^2\right) \ge \frac{15}{16} \quad \text{and} \quad 3456 \cdot \sum_{j=0}^{\infty} \frac{r_j}{l_j} \le \frac{1}{10^6}. \tag{43}$$

Then for any integers $s \ge 0$, $L_0 \ge 1$, and $K \ge 1$, $x_s \in \mathbb{G}_s$, and two families of events $\overline{\mathrm{D}}$ and $\bar{\mathrm{I}}$, if all the vertices in $\mathbb{G}_s \cap Q_{K,s}(x_s)$ are s-good, then for any $\mathcal{A} \subseteq \mathcal{Q}_{K,s,0}(x_s)$ such that $1 \le |\mathcal{A}| \le \frac{1}{2} \cdot |Q_{K,s}(x_s) \cap \mathbb{G}_0|$,

$$|\partial_{\mathcal{Q}_{K,s,0}(x_s)} \mathcal{A}| \ge \frac{1}{10^6} \cdot |\mathcal{A}|^{\frac{1}{2}}.$$

REMARK 5.7. (1) Assumptions (43) and the constant $\frac{1}{10^6}$ in the result of Lemma 5.6 are not optimal for our proof, but rather chosen to simplify calculations.

(2) We believe that an analogue of Lemma 5.6 holds for all $d \ge 2$, but cannot prove it. There is only one place in the proof where the assumption $d = 2$ is used; see Remark 5.9.

PROOF. Fix $s \ge 0$ and $K \ge 1$ integers, $x_s \in \mathbb{G}_s$. Recall the definition of $\mathcal{Q}_{K,s,i}(x_s)$ from (19), and write $\mathcal{Q}_i$ for $\mathcal{Q}_{K,s,i}(x_s)$ throughout the proof. Note that $\mathcal{Q}_s = Q_{K,s}(x_s) \cap \mathbb{G}_0$ and for all i, $\mathcal{Q}_{i-1} \subseteq \mathcal{Q}_i$.

Let $\mathcal{A}$ be a subset of $\mathcal{Q}_0$ such that $1 \le |\mathcal{A}| \le \frac{1}{2} \cdot |\mathcal{Q}_s|$. We need to prove that $|\partial_{\mathcal{Q}_0} \mathcal{A}| \ge \frac{1}{10^6} \cdot |\mathcal{A}|^{\frac{1}{2}}$. First of all, without loss of generality we can assume that both $\mathcal{A}$ and $\mathcal{Q}_0 \setminus \mathcal{A}$ are connected in $\mathbb{G}_0$. (For the proof of this claim, see page 112 in [26], Section 3.1.)

Let $B, B_1, \dots, B_m$ be all the connected components (in $\mathbb{G}_0$) of $\mathcal{Q}_s \setminus \mathcal{A}$, of which B is the unique component intersecting $\mathcal{Q}_0$, and B_i's are the "holes" in $\mathcal{Q}_s$ completely surrounded by $\mathcal{A}$. (See Figure 5.) The boundary of $\mathcal{A}$ in $\mathcal{Q}_0$ does not contain any edges adjacent to B_i's. It is convenient to absorb all the holes B_i's into $\mathcal{A}$ to get the set $\mathcal{A}'$ with the same boundary in $\mathcal{Q}_0$, but with an important feature that its exterior vertex boundary in $\mathcal{Q}_s$ is $*$-connected. More precisely, let

$$\mathcal{A}' = \mathcal{A} \cup \bigcup_{i=1}^{m} B_i \quad \text{and} \quad \mathcal{Q}'_i = \mathcal{Q}_i \cup \bigcup_{i=1}^{m} B_i.$$

Then (a) $\partial_{\mathcal{Q}'_0} \mathcal{A}' = \partial_{\mathcal{Q}_0} \mathcal{A}$, (b) $|\mathcal{A}'| \ge |\mathcal{A}|$, (c) $\mathcal{A}'$ is connected in $\mathbb{G}_0$, (d) $\mathcal{Q}'_0 \setminus \mathcal{A}' = \mathcal{Q}_0 \setminus \mathcal{A}$ (in particular, connected in $\mathbb{G}_0$), and (e) for any $x, x' \in \mathcal{E} = \{y \in \mathcal{Q}_s :$

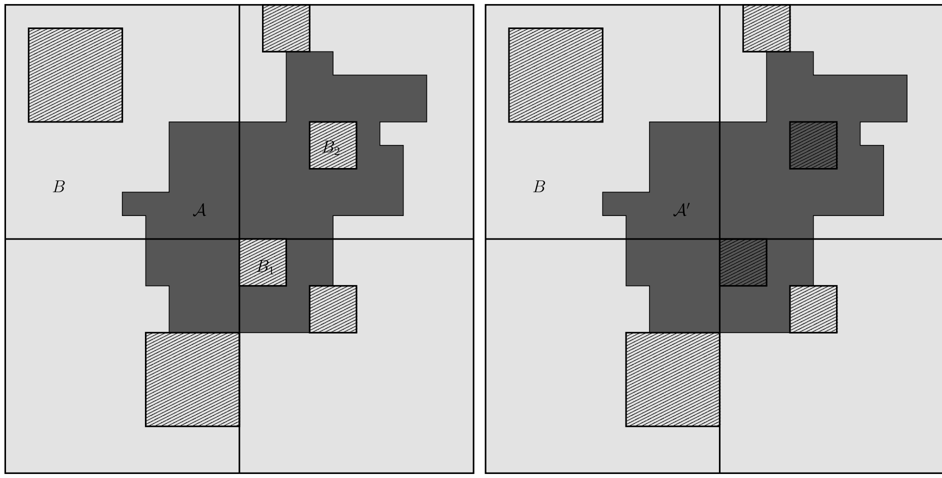

FIG. 5. *The set $\mathcal{A}'$ is obtained from $\mathcal{A}$ by adding to it all the holes B_i completely surrounded by $\mathcal{A}$. This operation does not change the boundary of $\mathcal{A}$ in $\mathcal{Q}_0$.*

$\{x, y\} \in \partial_{\mathcal{Q}_s}\mathcal{A}'$ for some $x \in \mathcal{A}'\}$ (the exterior vertex boundary of $\mathcal{A}'$ in $\mathcal{Q}_s$) there exist $z_0 = x, z_1, \dots, z_m = x' \in \mathcal{E}$ such that $|z_k - z_{k+1}|_\infty = L_0$ for all k (i.e., $\mathcal{E}$ is $*$-connected). Properties (a)–(d) are immediate from the definition of $\mathcal{A}'$, and property (e) follows from [16], Lemma 2.1(ii), and the facts that $\mathcal{A}'$ and $\mathcal{Q}_s \setminus \mathcal{A}'$ are connected in $\mathbb{G}_0$.

By properties (a)–(b) of $\mathcal{A}'$, it suffices to prove that

$$|\partial_{\mathcal{Q}'_0}\mathcal{A}'| \geq \frac{1}{10^6} \cdot |\mathcal{A}'|^{\frac{1}{2}}.$$

By Lemma 2.4 and the first part of (43), $|\mathcal{A}'| \leq |\mathcal{A}| + |\mathcal{Q}_s \setminus \mathcal{Q}_0| \leq \frac{9}{16} \cdot |\mathcal{Q}_s|$. Thus, by Lemma 5.1,

$$|\partial_{\mathcal{Q}_s}\mathcal{A}'| \geq \frac{1}{4} \cdot |\mathcal{A}'|^{\frac{1}{2}}. \tag{44}$$

Therefore, it suffices to prove that

$$|\partial_{\mathcal{Q}'_0}\mathcal{A}'| \geq \frac{2}{5 \cdot 10^5} \cdot |\partial_{\mathcal{Q}_s}\mathcal{A}'|. \tag{45}$$

The proof of (45) is done by partitioning $\partial_{\mathcal{Q}_s}\mathcal{A}' \setminus \partial_{\mathcal{Q}'_0}\mathcal{A}'$ into the sets δ_i of edges with one end vertex in $\mathcal{A}'$ and the other in $\mathcal{Q}_i \setminus \mathcal{Q}_{i-1}$ and comparing the cardinality of δ_i's with that of $\partial_{\mathcal{Q}_s}\mathcal{A}'$. If $\partial_{\mathcal{Q}_s}\mathcal{A}'$ is very large (macroscopic), then all δ_i are negligibly small in comparison to $\partial_{\mathcal{Q}_s}\mathcal{A}'$. It is more delicate to estimate the size of δ_i's if $\partial_{\mathcal{Q}_s}\mathcal{A}'$ is small, as the contribution of some δ_i's to the boundary $\partial_{\mathcal{Q}_s}\mathcal{A}'$ may be quite significant. In this case, we will introduce a suitable scale on which $\partial_{\mathcal{Q}_s}\mathcal{A}'$ is large, and view $\mathcal{A}'$ as a disjoint union of subsets of boxes on the new scale. Let

$$\delta_i = \partial_{\mathcal{Q}'_i}\mathcal{A}' \setminus \partial_{\mathcal{Q}'_{i-1}}\mathcal{A}'.$$

Then, δ_j's are disjoint and for any $1 \le i \le s$,

$$|\partial_{\mathcal{Q}'_0}\mathcal{A}'| = |\partial_{\mathcal{Q}'_i}\mathcal{A}'| - \sum_{j=1}^{i} |\delta_j|. \tag{46}$$

Let

$$t = \max\left\{0 \le i \le s : |\partial_{\mathcal{Q}_s}\mathcal{A}'| \ge \frac{1}{12} \cdot \frac{L_i}{L_0}\right\}.$$

The scale L_t is the correct scale to study $\partial_{\mathcal{Q}_s}\mathcal{A}'$. As we will see below in (47), the intersection of $\partial_{\mathcal{Q}_s}\mathcal{A}'$ with $\mathcal{Q}_i \setminus \mathcal{Q}_{i-1}$, $i \le t$ (holes of size significantly smaller than L_t), is negligible in comparison to $\partial_{\mathcal{Q}_s}\mathcal{A}'$. In particular, it will be enough to conclude (45) in the case $t = s$, see (50). If $t < s$, then $\partial_{\mathcal{Q}_s}\mathcal{A}'$ is small and may have a significant intersection with $\mathcal{Q}_{t+1} \setminus \mathcal{Q}_t$. An additional argument will be used to deal with this case; see below (51).

We begin with an estimation of the part of $\partial_{\mathcal{Q}_s}\mathcal{A}'$ adjacent to "small holes".

CLAIM 5.8. For all $1 \le i \le t$,

$$|\delta_i| \le 3456 \cdot \frac{r_{i-1}}{l_{i-1}} \cdot |\partial_{\mathcal{Q}_s}\mathcal{A}'|. \tag{47}$$

PROOF. By the definition of $\mathcal{Q}_i$'s (see also Figure 2), the set $\mathcal{Q}_i \setminus \mathcal{Q}_{i-1}$ can be expressed as the disjoint union of boxes $S_j = \mathbb{G}_0 \cap (y_j + [0, 2r_{i-1}L_{i-1})^2)$, for some $y_1, \dots, y_k \in (r_{i-1}L_{i-1}) \cdot \mathbb{Z}^2$, such that every box S_j is within ℓ^∞ distance L_i from at most 36 $S_{j'}$'s. (By Remark 2.3, each L_i-box contains at most 4 S_j's, and it is adjacent to at most 8 other L_i-boxes, hence $4 \cdot 9 = 36$.)

To estimate the size of δ_i, we consider two cases: (a) $\partial_{\mathcal{Q}_s}\mathcal{A}'$ is adjacent to few S_j's, in which case δ_i is very small, (b) $\partial_{\mathcal{Q}_s}\mathcal{A}'$ is adjacent to many S_j's, in which case many of the S_j's will be well-separated and $\mathcal{A}'$ will be spread out. To handle this case, we will use the fact that the exterior vertex boundary of $\mathcal{A}'$ is $*$-connected, thus the majority of edges in $\partial_{\mathcal{Q}_s}\mathcal{A}'$ will be "in between" S_j's. (See Figure 6.)

Let N_i be the total number of those S_j's which are adjacent (in $\mathbb{G}_0$) to $\mathcal{A}'$. Since for each j, $|\partial_{\mathcal{Q}_s}S_j| \le 8\frac{r_{i-1}L_{i-1}}{L_0}$, it follows that $|\delta_i| \le N_i \cdot 8\frac{r_{i-1}L_{i-1}}{L_0}$. We consider separately the cases $N_i \le 36$ and $N_i > 36$.

If $N_i \le 36$, then

$$|\delta_i| \le N_i \cdot 8\frac{r_{i-1}L_{i-1}}{L_0} \le 36 \cdot 8 \cdot \frac{r_{i-1}}{l_{i-1}} \cdot \frac{L_i}{L_0} \le 36 \cdot 8 \cdot 12 \cdot \frac{r_{i-1}}{l_{i-1}} \cdot |\partial_{\mathcal{Q}_s}\mathcal{A}'|, \tag{48}$$

where the last inequality follows from the definition of t and the fact that $i \le t$.

If $N_i > 36$, then $\mathcal{A}'$ is adjacent to at least $\lceil \frac{N_i}{36} \rceil (\ge 2)$ of S_j's which are pairwise at ℓ^∞ distance at least L_i from each other. Recall from property (e) of $\mathcal{A}'$ that $\mathcal{E}$ is the exterior vertex boundary of $\mathcal{A}'$, which is $*$-connected. Since $\mathcal{E}$ intersects each

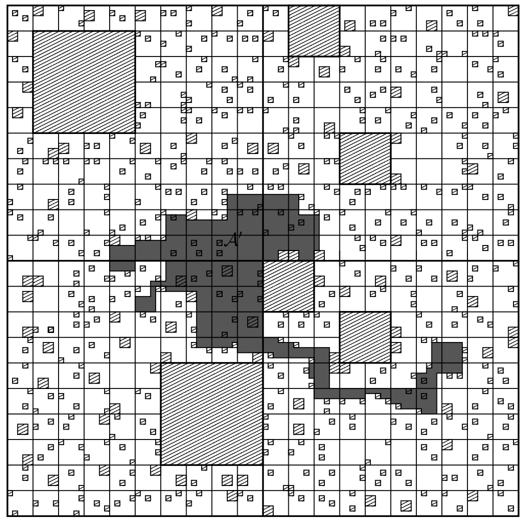

FIG. 6. *Since every S_j is within L_i distance from at most 35 other S_j's, if the set $\mathcal{A}'$ is adjacent to many S_j's then it must be adjacent to some sufficiently separated S_j (drawn in light grey), and its boundary is thus stretched between these S_j's. In two (and only two) dimensions, this is sufficient to conclude that the boundary of $\mathcal{A}'$ is much larger than its part adjacent to all the S_j's, which we call δ_i.*

of the $\lceil \frac{N_i}{36} \rceil$ well separated S_j's, the intersections of $\mathcal{E}$ with $\frac{1}{3}L_i$-neighborhoods of the S_j's are disjoint sets of vertices of cardinality $\geq \frac{1}{3}\frac{L_i}{L_0}$ each. Therefore, $|\mathcal{E}| \geq \frac{1}{3}\frac{L_i}{L_0} \cdot \frac{N_i}{36}$, and we obtain that

$$|\delta_i| \leq N_i \cdot 8\frac{r_{i-1}L_{i-1}}{L_0} \leq 36 \cdot 3 \cdot 8 \cdot \frac{r_{i-1}}{l_{i-1}} \cdot |\mathcal{E}| \leq 36 \cdot 3 \cdot 8 \cdot 4 \cdot \frac{r_{i-1}}{l_{i-1}} \cdot |\partial_{\mathcal{Q}_s}\mathcal{A}'|, \tag{49}$$

where the last inequality follows from the fact that each vertex of $\mathcal{E}$ is adjacent to at most 4 edges from $\partial_{\mathcal{Q}_s}\mathcal{A}'$.

Combining (48) and (49), we get (47). $\square$

If the boundary $\partial_{\mathcal{Q}_s}\mathcal{A}'$ is macroscopic, namely, if $t = s$, then the intersection of $\partial_{\mathcal{Q}_s}\mathcal{A}'$ with any hole is negligible, and Claim 5.8 immediately implies (45). Indeed, by (46) and (47),

$$|\partial_{\mathcal{Q}_0'}\mathcal{A}'| = |\partial_{\mathcal{Q}_t'}\mathcal{A}'| - \sum_{j=1}^{t}|\delta_j| \geq \left(1 - 3456 \cdot \sum_{j=0}^{\infty}\frac{r_j}{l_j}\right) \cdot |\partial_{\mathcal{Q}_s}\mathcal{A}'|, \tag{50}$$

and (45) follows from (50) and the second part of (43).

In the rest of the proof, we consider the case of small $\partial_{\mathcal{Q}_s}\mathcal{A}'$, namely $t < s$. In this case,

$$\frac{1}{12} \cdot \frac{L_t}{L_0} \leq |\partial_{\mathcal{Q}_s}\mathcal{A}'| < \frac{1}{12} \cdot \frac{L_{t+1}}{L_0} \leq \frac{1}{12} \cdot \frac{L_s}{L_0}. \tag{51}$$

As already mentioned, this case is more delicate, since $\partial_{\mathcal{Q}_s}\mathcal{A}'$ may have large intersection with big holes, for instance, δ_{t+1} is generally not negligible in comparison to $\partial_{\mathcal{Q}_s}\mathcal{A}'$.

We first consider the case when $\partial_{\mathcal{Q}_s}\mathcal{A}'$ is still relatively big in comparison to the boundary of holes in $\mathcal{Q}_{t+1}\setminus\mathcal{Q}_t$. Assume that $|\partial_{\mathcal{Q}_s}\mathcal{A}'| > 14\cdot 36\cdot 8\cdot\frac{r_t L_t}{L_0}$. In this case, we will show that

$$(52)\qquad |\delta_{t+1}| \le \frac{1}{14}\cdot|\partial_{\mathcal{Q}_s}\mathcal{A}'| \quad\text{and}\quad |\partial_{\mathcal{Q}'_{t+1}}\mathcal{A}'| \ge \frac{1}{7}\cdot|\partial_{\mathcal{Q}_s}\mathcal{A}'|.$$

Together with Claim 5.8, (52) is sufficient for (45). Indeed, by (46),

$$(53)\qquad |\partial_{\mathcal{Q}'_0}\mathcal{A}'| = |\partial_{\mathcal{Q}'_{t+1}}\mathcal{A}'| - \sum_{j=1}^{t+1}|\delta_j| \ge \left(\frac{1}{14} - 3456\cdot\sum_{j=0}^{\infty}\frac{r_j}{l_j}\right)\cdot|\partial_{\mathcal{Q}_s}\mathcal{A}'|,$$

and (45) follows from (53) and the second part of (43).

PROOF OF (52). To estimate the size of δ_{t+1}, we proceed as in the proof of (47). The set $\mathcal{Q}_{t+1}\setminus\mathcal{Q}_t$ can be expressed as a disjoint union of boxes $S_j = \mathbb{G}_0 \cap (y_j + [0, 2r_t L_t)^2)$, for some $y_1,\dots,y_k \in (r_t L_t)\cdot\mathbb{Z}^2$, such that every box is within ℓ^∞ distance L_{t+1} from at most 36 of the boxes. Since $|\partial_{\mathcal{Q}_s}\mathcal{A}'| < \frac{1}{12}\cdot\frac{L_{t+1}}{L_0}$ and the exterior vertex boundary of $\mathcal{A}'$ is $*$-connected, the set $\mathcal{A}'$ can be adjacent (in $\mathbb{G}_0$) to at most 36 such boxes (in fact, to at most $4\cdot 4 = 16$), which implies that

$$(54)\qquad |\delta_{t+1}| \le 36\cdot 8\frac{r_t L_t}{L_0} \le \frac{1}{14}\cdot|\partial_{\mathcal{Q}_s}\mathcal{A}'|,$$

where the last inequality follows from the assumption on $|\partial_{\mathcal{Q}_s}\mathcal{A}'|$.

To estimate $|\partial_{\mathcal{Q}'_{t+1}}\mathcal{A}'|$ from below, we view $\mathcal{A}'$ as a disjoint union of subsets $\mathcal{A}'_j$ of L_{t+1}-boxes, and estimate from below the relative boundary of each $\mathcal{A}'_j$ in the corresponding box. By definition, $\mathcal{Q}_{t+1}$ is the disjoint union of boxes $\mathbb{G}_0\cap(z_j + [0, L_{t+1})^2)$, $z_j \in \mathcal{G}_{K,s,t+1}(x_s)$. Let $\mathcal{A}'_j$ be the restriction of $\mathcal{A}'$ to the box $(z_j + [0, L_{t+1})^2)$. By (44) and (51), for every j,

$$|\mathcal{A}'_j| \le |\mathcal{A}'| \le 16\cdot|\partial_{\mathcal{Q}_s}\mathcal{A}'|^2 \le \frac{1}{9}\cdot|\mathbb{G}_0\cap[0, L_{t+1})^2|.$$

By applying Lemma 5.1 in each of $\mathbb{G}_0\cap(z_j + [0, L_{t+1})^2)$,

$$(55)\qquad |\partial_{\mathcal{Q}'_{t+1}}\mathcal{A}'| \ge \sum_j |\partial_{\mathbb{G}_0\cap(z_j+[0,L_{t+1})^d)}\mathcal{A}'_j| \ge \frac{1}{7}\cdot\sum_j|\partial_{\mathbb{G}_0}\mathcal{A}'_j| \ge \frac{1}{7}\cdot|\partial_{\mathcal{Q}_s}\mathcal{A}'|.$$

The combination of (54) and (55) gives (52). $\square$

It remains to consider the case $|\partial_{\mathcal{Q}_s}\mathcal{A}'| \le 14\cdot 36\cdot 8\cdot\frac{r_t L_t}{L_0}$. In this case, $\partial_{\mathcal{Q}_s}\mathcal{A}'$ is comparable to the boundary of holes in $\mathcal{Q}_{t+1}\setminus\mathcal{Q}_t$. We will show that

$$(56)\qquad |\partial_{\mathcal{Q}'_t}\mathcal{A}'| \ge \frac{1}{2\cdot 10^5}\cdot|\partial_{\mathcal{Q}_s}\mathcal{A}'|.$$

Together with Claim 5.8, (56) is sufficient for (45). Indeed, by (46),

$$|\partial_{\mathcal{Q}'_0}\mathcal{A}'| = |\partial_{\mathcal{Q}'_t}\mathcal{A}'| - \sum_{j=1}^{t}|\delta_j| \geq \left(\frac{1}{2\cdot 10^5} - 3456\cdot\sum_{j=0}^{\infty}\frac{r_j}{l_j}\right)\cdot|\partial_{\mathcal{Q}_s}\mathcal{A}'|, \tag{57}$$

and (45) follows from (57) and the second part of (43).

PROOF OF (56). Since $\partial_{\mathcal{Q}_s}\mathcal{A}'$ is comparable to the boundary of holes in $\mathcal{Q}_{t+1}\setminus\mathcal{Q}_t$, this time we will look at $\mathcal{A}'$ on the scale r_tL_t. By Lemma 2.4 and the assumption that l_t is divisible by r_t, $\mathcal{Q}_t$ can be expressed as a disjoint union of boxes $(z_j+[0,r_tL_t)^2)$, $z_j\in(r_tL_t)\cdot\mathbb{Z}^2$. Let $\mathcal{A}'_j$ be the restriction of $\mathcal{A}'$ to the box $(z_j+[0,r_tL_t)^2)$. We will compare the boundary $\partial_{\mathcal{Q}_s}\mathcal{A}'$ to the relative boundary of $\mathcal{A}'_j$'s in the respective boxes.

If for all j, $|\mathcal{A}'_j|\leq\frac{1}{4}\cdot|\mathbb{G}_0\cap[0,r_tL_t)^2|$, see Figure 7(left), then by Lemma 5.1 applied in each of $\mathbb{G}_0\cap(z_j+[0,r_tL_t)^2)$,

$$|\partial_{\mathbb{G}_0\cap(z_j+[0,r_tL_t)^2)}\mathcal{A}'_j| \geq \frac{1}{9}\cdot|\partial_{\mathbb{G}_0}\mathcal{A}'_j|.$$

Since the sets $\partial_{\mathbb{G}_0\cap(z_j+[0,r_tL_t)^2)}\mathcal{A}'_j$ are disjoint subsets of $\partial_{\mathcal{Q}'_t}\mathcal{A}'$,

$$|\partial_{\mathcal{Q}'_t}\mathcal{A}'| \geq \sum_j|\partial_{\mathbb{G}_0\cap(z_j+[0,r_tL_t)^2)}\mathcal{A}'_j| \geq \frac{1}{9}\cdot\sum_j|\partial_{\mathbb{G}_0}\mathcal{A}'_j| \geq \frac{1}{9}\cdot|\partial_{\mathcal{Q}_s}\mathcal{A}'|,$$

which implies (56).

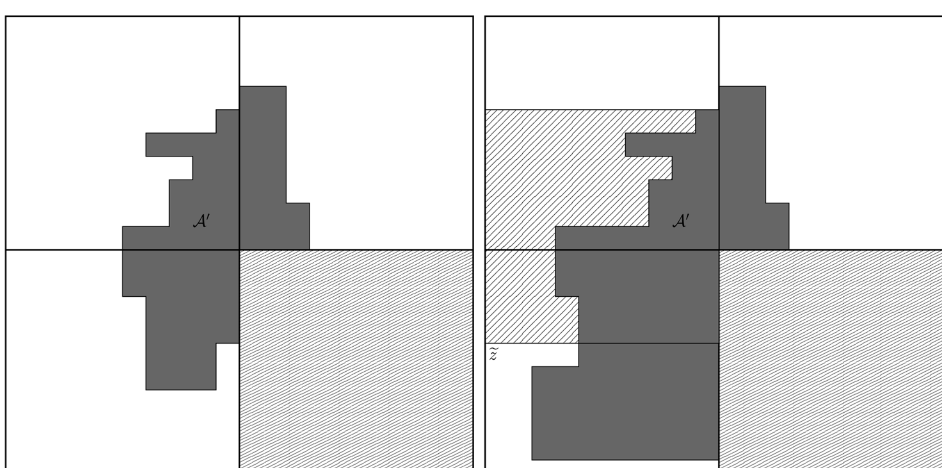

FIG. 7. *The case when the boundary of $\mathcal{A}'$ is comparable to the boundary of holes on the scale of $\mathcal{A}'$. Two subcases: $\mathcal{A}'$ has small intersection with every r_tL_t-box (left) or large intersection with some r_tL_t-box (right). In the second subcase, we can identify a box $(\tilde{z}+[0,r_tL_t)^d)$ in which $\mathcal{A}'$ has non-trivial density.*

On the other hand, if $|\mathcal{A}'_j| > \frac{1}{4} \cdot |\mathbb{G}_0 \cap [0, r_t L_t)^2|$ for at least one j, see Figure 7(right), then there exists $\widetilde{z} \in \mathbb{G}_t$ such that:

- $\mathbb{G}_0 \cap (\widetilde{z} + [0, r_t L_t)^2) \subset \mathcal{Q}_t$ and
- $\frac{1}{4} \cdot |\mathbb{G}_0 \cap [0, r_t L_t)^2| \leq |\mathcal{A}' \cap (\widetilde{z} + [0, r_t L_t)^2)| \leq \frac{3}{4} \cdot |\mathbb{G}_0 \cap [0, r_t L_t)^2|$.

Indeed, if none of z_j's satisfies the two requirements, then there exist j_1 and j_2 such that $|z_{j_1} - z_{j_2}|_\infty = r_t L_t$, $|\mathcal{A}'_{j_1}| > \frac{3}{4} \cdot |\mathbb{G}_0 \cap [0, r_t L_t)^2|$ and $|\mathcal{A}'_{j_2}| \leq \frac{1}{4} \cdot |\mathbb{G}_0 \cap [0, r_t L_t)^2|$. Then $\widetilde{z} = \lambda \cdot z_{j_1} + (1 - \lambda) \cdot z_{j_2}$ satisfies the two requirements for some $\lambda \in (0, 1)$. (If r_t is divisible by 2, then one can take $\lambda = \frac{1}{2}$.)

By applying Lemma 5.1 to $\mathbb{G}_0 \cap (\widetilde{z} + [0, r_t L_t)^2)$,

$$\begin{aligned}
|\partial_{\mathcal{Q}'_t} \mathcal{A}'| &\geq |\partial_{\mathbb{G}_0 \cap (\widetilde{z} + [0, r_t L_t)^2)} (\mathcal{A}' \cap (\widetilde{z} + [0, r_t L_t)^2))| \\
&\geq \left(1 - \frac{\sqrt{3}}{2}\right) \cdot |\mathcal{A}' \cap (\widetilde{z} + [0, r_t L_t)^2)|^{\frac{1}{2}} \geq \left(1 - \frac{\sqrt{3}}{2}\right) \cdot \frac{1}{2} \cdot \frac{r_t L_t}{L_0} \\
&\geq \frac{1}{16} \cdot \frac{r_t L_t}{L_0} \geq \frac{1}{16 \cdot 14 \cdot 36 \cdot 8} \cdot |\partial_{\mathcal{Q}_s} \mathcal{A}'|,
\end{aligned}$$

where the last inequality follows from the assumption on $|\partial_{\mathcal{Q}_s} \mathcal{A}'|$. This inequality completes the proof of (56). □

To summarize, the desired relation (45) between $\partial_{\mathcal{Q}'_0} \mathcal{A}'$ and $\partial_{\mathcal{Q}_s} \mathcal{A}'$ follows from the three inequalities (50) (the boundary $\partial_{\mathcal{Q}_s} \mathcal{A}'$ is macroscopic), (53) (the boundary $\partial_{\mathcal{Q}_s} \mathcal{A}'$ is small, but much bigger than the boundaries of holes on the given scale) and (57) (the boundary $\partial_{\mathcal{Q}_s} \mathcal{A}'$ is small and comparable to the boundaries of holes on the given scale). The proof of Lemma 5.6 is complete. □

REMARK 5.9. The only step in the proof of Lemma 5.6 that uses (crucially!) the assumption $d = 2$ is the derivation of (49). More precisely, the fact that the boundary of a set is well approximated by simple paths. In higher dimensions, this is clearly not the case (the dimension of the boundary is generally bigger than the dimension of a simple path), and the above argument breaks down. See Figure 6.

5.3. *Isoperimetric inequality in any dimension for large enough subsets.* In this section, we prove the following theorem, which includes Theorem 2.5 as a special case.

THEOREM 5.10. *Let $d \geq 2$, $c > 0$. Let l_n and r_n, $n \geq 0$, be integer sequences satisfying assumptions of Lemma 5.6 and such that*

$$\prod_{i=0}^{\infty} \left(1 - \left(\frac{4r_i}{l_i}\right)^2\right) \geq e^{-\frac{1}{16(d-1)}} \quad \textit{and} \quad \prod_{i=0}^{\infty} \left(1 - \left(\frac{4r_i}{l_i}\right)^d\right) \geq \frac{1 - \frac{1}{2^{d+2}}}{1 - \frac{1}{2^{d+3}}}. \tag{58}$$

Then for any integers $s \geq 0$, $L_0 \geq 1$, *and* $K \geq 1$, $x_s \in \mathbb{G}_s$, *and two families of events* $\overline{\mathrm{D}}$ *and* $\bar{\mathrm{I}}$, *if all the vertices in* $\mathbb{G}_s \cap Q_{K,s}(x_s)$ *are* s*-good*, *then any* $\mathcal{A} \subseteq \mathcal{Q}_{K,s,0}(x_s)$ *with*

$$\min\left\{c \cdot |Q_{K,s} \cap \mathbb{G}_0|, \left(\frac{L_s}{L_0}\right)^{d^2}\right\} \leq |\mathcal{A}| \leq \frac{1}{2} \cdot |Q_{K,s} \cap \mathbb{G}_0|$$

satisfies

$$|\partial_{\mathcal{Q}_{K,s,0}(x_s)} \mathcal{A}| \geq \frac{c^2}{2d \cdot 32^d \cdot 27^d \cdot 10^6} \cdot \left(1 - \left(\frac{2}{3}\right)^{\frac{1}{d}}\right) \cdot \left(1 - e^{-\frac{1}{16(d-1)}}\right) \cdot |\mathcal{A}|^{\frac{d-1}{d}}.$$

PROOF. Fix $s \geq 0$ and $K \geq 1$ integers, $x_s \in \mathbb{G}_s$ and assume that all the vertices in $\mathbb{G}_s \cap Q_{K,s}(x_s)$ are s-good. Take $\mathcal{A} \subseteq \mathcal{Q}_{K,s,0}(x_s)$ such that $|\mathcal{A}| \leq \frac{1}{2} \cdot |Q_{K,s} \cap \mathbb{G}_0|$.

We consider separately the cases $|\mathcal{A}| \geq c \cdot |Q_{K,s} \cap \mathbb{G}_0|$ and $|\mathcal{A}| \geq (\frac{L_s}{L_0})^{d^2}$. In fact, we will use the result for the first case to prove the result for the second.

In the first case, we use Corollaries 5.4 and 5.5 to the selection lemma from Section 5.1.2 to identify a large number of disjoint two-dimensional slices in the ambient box $Q_{K,s} \cap \mathbb{G}_0$ which on the one hand have a small nonempty intersection with $\mathcal{A}$, and on the other, all together contain a positive fraction of the volume of $\mathcal{A}$. We estimate the boundary of $\mathcal{A}$ in each of the slices using the two-dimensional isoperimetric inequality of Lemma 5.6. Since the slices are pairwise disjoint, we can estimate the boundary of $\mathcal{A}$ by the sum of the boundaries of $\mathcal{A}$ in each of the slices.

In the second case, we consider a coarse graining of $\mathcal{A}$ by densely occupied L_s-boxes. If the number of densely occupied L_s-boxes is small, then $\mathcal{A}$ is scattered in $Q_{K,s} \cap \mathbb{G}_0$ and has big boundary. If, on the other hand, the number of densely occupied L_s-boxes is big, then the set of such boxes has large boundary (the poorly occupied boxes adjacent to some densely occupied ones). Each pair of adjacent densely and poorly occupied L_s-boxes are contained in a $2L_s$-box. Vertices from $\mathcal{A}$ occupy a nontrivial fraction of vertices in this $2L_s$-box. Thus, we can estimate the boundary of $\mathcal{A}$ restricted to this box using the first part of the theorem. By summing over all pairs of adjacent densely and poorly occupied L_s-boxes, we obtain a desired lower bound on the size of the boundary of $\mathcal{A}$.

We first consider the case $|\mathcal{A}| \geq c \cdot |Q_{K,s} \cap \mathbb{G}_0|$. By Corollaries 5.4 and 5.5, there exist

$$\geq \frac{c}{2 \cdot 9^{d-2}} \cdot \left(\frac{KL_s}{L_0}\right)^{d-2}$$

two-dimensional subrectangles S_i in $Q_{K,s} \cap \mathbb{G}_0$ (see Figure 8) such that for all i,

$$|\mathcal{A} \cap S_i| \geq \frac{c}{2 \cdot 9^{d-2}} \cdot \left(\frac{KL_s}{L_0}\right)^2 \quad \text{and} \quad |\mathcal{A} \cap S_i| \leq e^{-\frac{1}{8(d-1)}} \cdot \left(\frac{KL_s}{L_0}\right)^2.$$

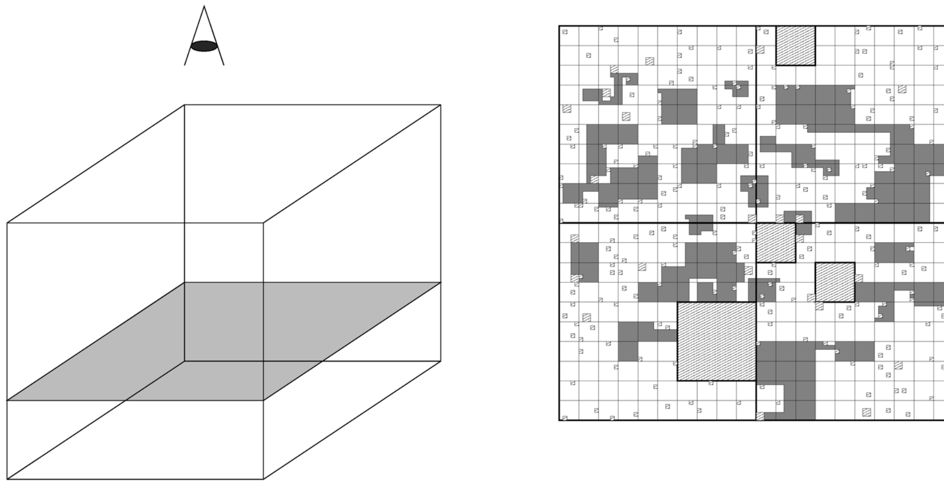

FIG. 8. *Left*: *a two-dimensional slice* S_i. *Right*: *perforation* $\mathcal{Q}_{K,s,0}(x_s) \cap S_i$ *of* S_i *and the intersection of* $\mathcal{A}$ *with* S_i (*drawn in grey*).

By Lemma 2.4 [applied to the perforation $\mathcal{Q}_{K,s,0}(x_s) \cap S_i$ of S_i] and the first part of (58), $|\mathcal{Q}_{K,s,0}(x_s) \cap S_i| \ge e^{-\frac{1}{16(d-1)}} \cdot (\frac{KL_s}{L_0})^2$, which implies that for all i,

$$|\mathcal{A} \cap S_i| \le e^{-\frac{1}{16(d-1)}} \cdot \big|\mathcal{Q}_{K,s,0}(x_s) \cap S_i\big|.$$

We apply the two-dimensional isoperimetric inequality of Lemma 5.6 and Remark 5.2 to each of the sets $\mathcal{A} \cap S_i$ in $\mathcal{Q}_{K,s,0}(x_s) \cap S_i$, and obtain that for all i,

$$\begin{aligned}\big|\partial_{\mathcal{Q}_{K,s,0}(x_s)\cap S_i}(\mathcal{A} \cap S_i)\big| &\ge \frac{1}{10^6} \cdot \big(1 - e^{-\frac{1}{16(d-1)}}\big) \cdot |\mathcal{A} \cap S_i|^{\frac{1}{2}} \\ &\ge \frac{1}{10^6} \cdot \big(1 - e^{-\frac{1}{16(d-1)}}\big) \cdot \frac{c}{2 \cdot 3^{d-2}} \cdot \frac{KL_s}{L_0}.\end{aligned}$$

Since all $\partial_{\mathcal{Q}_{K,s,0}(x_s)\cap S_i}(\mathcal{A} \cap S_i)$ are disjoint subsets of $\partial_{\mathcal{Q}_{K,s,0}(x_s)}\mathcal{A}$,

$$\begin{aligned}|\partial_{\mathcal{Q}_{K,s,0}(x_s)}\mathcal{A}| &\ge \sum_i \big|\partial_{\mathcal{Q}_{K,s,0}(x_s)\cap S_i}(\mathcal{A} \cap S_i)\big| \\ &\ge \frac{c}{2 \cdot 9^{d-2}} \cdot \left(\frac{KL_s}{L_0}\right)^{d-2} \cdot \frac{1}{10^6} \cdot \big(1 - e^{-\frac{1}{16(d-1)}}\big) \cdot \frac{c}{2 \cdot 3^{d-2}} \cdot \frac{KL_s}{L_0} \qquad (59) \\ &\ge \frac{c^2}{4 \cdot 27^{d-2} \cdot 10^6} \cdot \big(1 - e^{-\frac{1}{16(d-1)}}\big) \cdot |\mathcal{A}|^{\frac{d-1}{d}}.\end{aligned}$$

This completes the proof of Theorem 5.10 for sets with $|\mathcal{A}| \ge c \cdot |Q_{K,s} \cap \mathbb{G}_0|$.

Next, we consider the case $|\mathcal{A}| \ge (\frac{L_s}{L_0})^{d^2}$. Let

$$\mathbb{A}_s = \{x \in \mathbb{G}_s : \mathcal{A} \cap (x + [0, L_s)^d) \ne \varnothing\}$$

be the set of bottom-left corners of L_s-boxes which contain a vertex from $\mathcal{A}$. Note that $|\mathbb{A}_s| \geq |\mathcal{A}| \cdot (\frac{L_0}{L_s})^d$. We also define the subset $\widetilde{\mathbb{A}}_s$ of $\mathbb{A}_s$ corresponding to the densely occupied boxes,

$$\widetilde{\mathbb{A}}_s = \left\{ x \in \mathbb{G}_s : \left|\mathcal{A} \cap \left(x + [0, L_s)^d\right)\right| \geq \frac{3}{4} \cdot \left(\frac{L_s}{L_0}\right)^d \right\}.$$

We consider separately the cases when $|\widetilde{\mathbb{A}}_s| \geq \frac{1}{2} \cdot |\mathbb{A}_s|$ and $|\widetilde{\mathbb{A}}_s| \leq \frac{1}{2} \cdot |\mathbb{A}_s|$.

We first consider the case $|\widetilde{\mathbb{A}}_s| \geq \frac{1}{2} \cdot |\mathbb{A}_s|$, that is, the number of densely occupied boxes is large.

Since $\frac{3}{4} \cdot (\frac{L_s}{L_0})^d \cdot |\widetilde{\mathbb{A}}_s| \leq |\mathcal{A}| \leq \frac{1}{2} \cdot |Q_{K,s} \cap \mathbb{G}_0|$,

$$|\widetilde{\mathbb{A}}_s| \leq \frac{2}{3} \cdot |Q_{K,s} \cap \mathbb{G}_0| \cdot \left(\frac{L_0}{L_s}\right)^d = \frac{2}{3} \cdot \left|Q_{K,s}(x_s) \cap \mathbb{G}_s\right|.$$

By applying Lemma 5.1 to $\widetilde{\mathbb{A}}_s \subset Q_{K,s}(x_s) \cap \mathbb{G}_s$, we get

$$|\partial_{Q_{K,s}(x_s)\cap\mathbb{G}_s} \widetilde{\mathbb{A}}_s| \geq \left(1 - \left(\frac{2}{3}\right)^{\frac{1}{d}}\right) \cdot |\widetilde{\mathbb{A}}_s|^{\frac{d-1}{d}}. \tag{60}$$

Next, we zoom in onto the boundary $\partial_{Q_{K,s}(x_s)\cap\mathbb{G}_s} \widetilde{\mathbb{A}}_s$. Take any pair $x \in \widetilde{\mathbb{A}}_s$ and $y \in (Q_{K,s}(x_s) \cap \mathbb{G}_s) \setminus \widetilde{\mathbb{A}}_s$ from $\partial_{Q_{K,s}(x_s)\cap\mathbb{G}_s} \widetilde{\mathbb{A}}_s$. Note that

$$\left|\mathcal{A} \cap \left(x + [0, L_s)^d\right)\right| \geq \frac{3}{4} \cdot \left(\frac{L_s}{L_0}\right)^d$$

and

$$\left|\mathcal{A} \cap \left(y + [0, L_s)^d\right)\right| < \frac{3}{4} \cdot \left(\frac{L_s}{L_0}\right)^d.$$

Take a box $(z + [0, 2L_s)^d)$ in $Q_{K,s}(x_s) \cap \mathbb{G}_0$ containing both $(x + [0, L_s)^d)$ and $(y + [0, L_s)^d)$, where $z \in (Q_{K,s}(x_s) \cap \mathbb{G}_s)$. Note that $\mathcal{A}$ occupies a nontrivial fraction of vertices in $(z + [0, 2L_s)^d)$. More precisely,

$$\begin{aligned}\frac{3}{2^{d+2}} \cdot \left|\left(z + [0, 2L_s)^d\right) \cap \mathbb{G}_0\right| &\leq \left|\mathcal{A} \cap \left(z + [0, 2L_s)^d\right)\right| \\ &\leq \left(1 - \frac{1}{2^{d+2}}\right) \cdot \left|\left(z + [0, 2L_s)^d\right) \cap \mathbb{G}_0\right|.\end{aligned}$$

Moreover, all the vertices in $(z + [0, 2L_s)^d \cap \mathbb{G}_s$ are s-good. We are in a position to apply the first part of the theorem to $\mathcal{A} \cap (z + [0, 2L_s)^d)$ in $(z + [0, 2L_s)^d)$. Combining the upper bound on $|\mathcal{A} \cap (z + [0, 2L_s)^d)|$ with the lower bound on the volume of the perforation $\mathcal{Q}_{2,s,0}(z) = \mathcal{Q}_{K,s,0}(x_s) \cap (z + [0, 2L_s)^d)$ given by Lemma 2.4 and the second part of the assumption (58), we obtain that

$$\left|\mathcal{A} \cap \left(z + [0, 2L_s)^d\right)\right| \leq \left(1 - \frac{1}{2^{d+3}}\right) \cdot \left|\mathcal{Q}_{K,s,0}(x_s) \cap \left(z + [0, 2L_s)^d\right)\right|.$$

Therefore, by the first part of the theorem (with $c = \frac{3}{2^{d+2}}$) applied to the subset $\mathcal{A} \cap (z + [0, 2L_s)^d)$ of $\mathcal{Q}_{2,s,0}(z)$ and Remark 5.2,

$$
\begin{aligned}
&\bigl|\partial_{\mathcal{Q}_{K,s,0}(x_s)\cap(z+[0,2L_s)^d)}\bigl(\mathcal{A} \cap \bigl(z + [0, 2L_s)^d\bigr)\bigr)\bigr| \\
&\quad \ge \frac{1}{2^{d+3}} \cdot \frac{9}{4 \cdot 4^{d+2} \cdot 27^{d-2} \cdot 10^6} \cdot \bigl(1 - e^{-\frac{1}{16(d-1)}}\bigr) \cdot \bigl|\mathcal{A} \cap \bigl(z + [0, 2L_s)^d\bigr)\bigr|^{\frac{d-1}{d}} \\
&\quad \ge \frac{3}{4} \cdot \frac{9}{8^{d+3} \cdot 27^{d-2} \cdot 10^6} \cdot \bigl(1 - e^{-\frac{1}{16(d-1)}}\bigr) \cdot \left(\frac{L_s}{L_0}\right)^{d-1}.
\end{aligned}
$$

This inequality gives us an estimate on the part of the boundary $\partial_{\mathcal{Q}_{K,s,0}(x_s)}\mathcal{A}$ contained in $(z + [0, 2L_s)^d)$ for each $z \in (Q_{K,s}(x_s) \cap \mathbb{G}_s)$ such that the cube $(z + [0, 2L_s)^d)$ contains an overcrowded and undercrowded adjacent L_s-boxes $(x + [0, L_s)^d)$ and $(y + [0, L_s)^d)$ with $x \in \widetilde{\mathbb{A}}_s$ and $y \in (Q_{K,s}(x_s) \cap \mathbb{G}_s) \setminus \widetilde{\mathbb{A}}_s$. By (60), the total number of such z's is

$$
\ge \frac{1}{d2^{d-1}} \cdot |\partial_{Q_{K,s}(x_s)\cap\mathbb{G}_s}\widetilde{\mathbb{A}}_s| \ge \frac{1}{d2^{d-1}} \cdot \left(1 - \left(\frac{2}{3}\right)^{\frac{1}{d}}\right) \cdot |\widetilde{\mathbb{A}}_s|^{\frac{d-1}{d}},
$$

where the factor $\frac{1}{d2^{d-1}}$ counts for possible overcounting, since every cube $(z + [0, 2L_s)^d)$, $z \in \mathbb{G}_s$, contains at most $d2^{d-1}$ pairs x, y with $\{x, y\} \in \partial_{Q_{K,s}(x_s)\cap\mathbb{G}_s}\widetilde{\mathbb{A}}_s$.

Moreover, every edge from $\partial_{\mathcal{Q}_{K,s,0}(x_s)}\mathcal{A}$ belongs to at most 2^d cubes $(z + [0, 2L_s)^d)$, $z \in \mathbb{G}_s$. Thus,

$$
|\partial_{\mathcal{Q}_{K,s,0}(x_s)}\mathcal{A}| \ge \frac{1}{2^d} \cdot \sum_{z\in\mathbb{G}_s} \bigl|\partial_{\mathcal{Q}_{K,s,0}(x_s)\cap(z+[0,2L_s)^d)}\bigl(\mathcal{A} \cap \bigl(z + [0, 2L_s)^d\bigr)\bigr)\bigr|.
$$

By putting all the estimates together, we obtain that

$$
\begin{aligned}
|\partial_{\mathcal{Q}_{K,s,0}(x_s)}\mathcal{A}| &\ge \frac{1}{2^d} \cdot \sum_{z\in\mathbb{G}_s} \bigl|\partial_{\mathcal{Q}_{K,s,0}(x_s)\cap(z+[0,2L_s)^d)}\bigl(\mathcal{A} \cap \bigl(z + [0, 2L_s)^d\bigr)\bigr)\bigr| \\
&\ge \frac{1}{2^d} \cdot \frac{1}{d2^{d-1}} \cdot \left(1 - \left(\frac{2}{3}\right)^{\frac{1}{d}}\right) \cdot |\widetilde{\mathbb{A}}_s|^{\frac{d-1}{d}} \\
&\qquad \times \frac{3}{4} \cdot \frac{9}{8^{d+3} \cdot 27^{d-2} \cdot 10^6} \cdot \bigl(1 - e^{-\frac{1}{16(d-1)}}\bigr) \cdot \left(\frac{L_s}{L_0}\right)^{d-1} \\
&\ge \frac{1}{2d \cdot 32^d \cdot 27^d \cdot 10^6} \cdot \left(1 - \left(\frac{2}{3}\right)^{\frac{1}{d}}\right) \cdot \bigl(1 - e^{-\frac{1}{16(d-1)}}\bigr) \cdot |\mathcal{A}|^{\frac{d-1}{d}},
\end{aligned} \tag{61}
$$

where the last inequality follows from the case assumption $|\widetilde{\mathbb{A}}_s| \ge \frac{1}{2} \cdot |\mathbb{A}_s| \ge \frac{1}{2} \cdot |\mathcal{A}| \cdot (\frac{L_0}{L_s})^d$.

It remains to consider the case $|\widetilde{\mathbb{A}}_s| \le \frac{1}{2} \cdot |\mathbb{A}_s|$. In this case, $\mathcal{A}$ is scattered in $Q_{K,s}(x_s) \cap \mathbb{G}_0$, and should have big boundary. Indeed, for each $x \in \mathbb{A}_s \setminus \widetilde{\mathbb{A}}_s$,

$$1 \le \big|\mathcal{A} \cap \big(x + [0, L_s)^d\big)\big| < \frac{3}{4} \cdot \left(\frac{L_s}{L_0}\right)^d.$$

By the lower bound on the volume of the perforation $\mathcal{Q}_{1,s,0}(x) = \mathcal{Q}_{K,s,0}(x_s) \cap (x + [0, L_s)^d)$ given in Lemma 2.4 and the second part of (58),

$$\big|\mathcal{Q}_{K,s,0}(x_s) \cap \big(x + [0, L_s)^d\big)\big| \ge \frac{1 - \frac{1}{2^{d+2}}}{1 - \frac{1}{2^{d+3}}} \cdot \left(\frac{L_s}{L_0}\right)^d \ge \frac{3}{4} \cdot \left(\frac{L_s}{L_0}\right)^d.$$

Thus, $(x + [0, L_s)^d)$ contains vertices from both $\mathcal{A}$ and $\mathcal{Q}_{K,s,0}(x_s) \setminus \mathcal{A}$. By Lemma 2.4, $\mathcal{Q}_{1,s,0}(x) = \mathcal{Q}_{K,s,0}(x_s) \cap (x + [0, L_s)^d)$ is connected in $\mathbb{G}_0$, thus it contains an edge from $\partial_{\mathcal{Q}_{K,s,0}(x_s)}\mathcal{A}$. Since all $(x + [0, L_s)^d)$, $x \in \mathbb{A}_s \setminus \widetilde{\mathbb{A}}_s$ are disjoint, we conclude that

$$|\partial_{\mathcal{Q}_{K,s,0}(x_s)}\mathcal{A}| \ge |\mathbb{A}_s \setminus \widetilde{\mathbb{A}}_s| \ge \frac{1}{2} \cdot |\mathbb{A}_s| \ge \frac{1}{2} \cdot |\mathcal{A}| \cdot \left(\frac{L_0}{L_s}\right)^d \ge \frac{1}{2} \cdot |\mathcal{A}|^{\frac{d-1}{d}}, \tag{62}$$

where the last inequality follows from the case assumption.

The proof of Theorem 5.10 in the case $|\mathcal{A}| \ge (\frac{L_s}{L_0})^{d^2}$ is complete by (61) and (62). □

REMARK 5.11. We believe that Theorem 5.10 holds for all $\mathcal{A}$ with $|\mathcal{A}| \le \frac{1}{2} \cdot |Q_{K,s} \cap \mathbb{G}_0|$. With a more involved proof, we can relax the assumption $|\mathcal{A}| \ge (\frac{L_s}{L_0})^{d^2}$ of Theorem 5.10 to $|\mathcal{A}| \ge (\frac{L_s}{L_0})^{2d}$. Since this does not give us the result for all $\mathcal{A}$, and the current statement of Theorem 5.10 suffices for the applications in this paper, we do not include this proof here.

6. Open problems.

1. Consider the random conductance model on the edges of the infinite cluster $\mathcal{S}_\infty$ with ergodic conductances $\{c_e\}_{e\in\mathcal{S}_\infty}$ satisfying the moment conditions from [2], Theorem 1.3: $\mathbf{E}[c_e^p] < \infty$ and $\mathbf{E}[c_e^{-q}] < \infty$ with $p, q \in (1, \infty]$ and $\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$. Prove the quenched invariance principle, Harnack inequalities, and the local central limit theorem. We remark that the quenched invariance principle for the random conductance model on the infinite cluster of Bernoulli percolation under the above moment assumptions has been recently proved in [31].
2. The approach of [2] has been extended in [3] to a class of graphs satisfying [3], Assumption 1.1, which is reminiscent of Definition 4.1, but stronger. The main difference is that we do not require that an isoperimetric inequality is satisfied by subsets of a ball, but by those of a local extension of the ball. It would be interesting to see if the machinery developed in [2, 3] can be applied to the random conductance model on graphs with all large balls being very regular in the sense of Definition 4.1.

3. A classical example of percolation model with correlations is the random cluster model; see, for example, [22]. It is quite a challenging open problem to show that the supercritical random cluster model satisfies the assumptions **S1**–**S2**.
4. As remarked in Section 1.2.2, the vacant set of random interlacements and the level sets of the Gaussian free field satisfy the assumptions **P1**–**P3** and **S2** for all supercritical parameters and **S1** for a non-empty subset of parameters. It is a difficult open problem to prove that in both models, the assumption **S1** is satisfied for all supercritical parameters.
5. (This question was asked by Jürgen Jost.) The multiscale renormalization ideas developed in this paper seem to rely strongly on the geometry of the lattice $\mathbb{Z}^d$, for instance, the connectedness of the set $\mathcal{Q}_{K,s,0}(x_s)$ in Lemma 2.4. Can the techniques of this paper be extended to (a class of) Cayley graphs of Abelian groups?

APPENDIX: PROOFS OF THEOREMS 1.16–1.20

In this section, we give proof sketches of Theorems 1.16, 1.17, 1.18, 1.19 and 1.20. Their proofs are straightforward adaptations of main results in [7, 8] from Bernoulli percolation to our setup.

PROOF OF THEOREM 1.16. The proof is essentially the same as that of [8], Theorem 6. The only minor care that is required comes from the fact that the bound (11) is not stretched exponential. Since this fact is used several times, we provide a general outline of the proof. As in the proof of [8], Theorem 6, by stationarity **P1** and the ergodicity of $\mathcal{S}_\infty$ with respect to the shift by X_1 (see, e.g., [9], Theorem 3.1), it suffices to prove that

$$\mathbb{E}^u\big[\big(p_{2n}(0,x) - p_{2n-1}(X_1,x)\big)^2 \cdot \mathbb{1}_{x\in\mathcal{S}_\infty}\big] \le \frac{C}{n^{d+1}} \cdot e^{-c\frac{\mathrm{d}_{\mathbb{Z}^d}(0,x)^2}{n}},$$

where C and c only depend on d and u. If $\mathrm{d}_{\mathbb{Z}^d}(0,x) \ge n^{\frac{1}{2}}(\log n)^{\frac{1+\Delta_{\mathrm{S}}}{2}}$, where Δ_{S} is defined in (5), then by the general upper bound on the heat kernel (see, e.g., [4], (1.5)),

$$\begin{aligned}\mathbb{E}^u\big[\big(p_{2n}(0,x) - p_{2n-1}(X_1,x)\big)^2 \cdot \mathbb{1}_{x\in\mathcal{S}_\infty}\big] &\le C \cdot e^{-c\frac{\mathrm{d}_{\mathbb{Z}^d}(0,x)^2}{n}}\\ &\le \frac{C'}{n^{d+1}} \cdot e^{-c'\frac{\mathrm{d}_{\mathbb{Z}^d}(0,x)^2}{n}}.\end{aligned}$$

Thus, we can assume that $\mathrm{d}_{\mathbb{Z}^d}(0,x) \le n^{\frac{1}{2}}(\log n)^{\frac{1+\Delta_{\mathrm{S}}}{2}}$.

Let $N = N(\omega) = \max\{T_{\mathrm{hk}}(y) : y \in \mathrm{B}_{\mathbb{Z}^d}(0,n)\}$. By (11),

$$\begin{aligned}&\mathbb{E}^u\big[\big(p_{2n}(0,x) - p_{2n-1}(X_1,x)\big)^2 \cdot \mathbb{1}_{x\in\mathcal{S}_\infty} \cdot \mathbb{1}_{N(\omega)\ge n}\big]\\ &\quad\le \mathbb{P}^u\big[N(\omega) \ge n\big] \le Cn^d \cdot e^{-c\cdot(\log n)^{1+\Delta_{\mathrm{S}}}}\\ &\quad\le \frac{C'}{n^{d+1}} \cdot e^{-c'\cdot(\log n)^{1+\Delta_{\mathrm{S}}}} \le \frac{C'}{n^{d+1}} \cdot e^{-c'\frac{\mathrm{d}_{\mathbb{Z}^d}(0,x)^2}{n}}.\end{aligned}$$

It remains to bound $\mathbb{E}^u[(p_{2n}(0,x) - p_{2n-1}(X_1,x))^2 \cdot \mathbb{1}_{x\in\mathcal{S}_\infty} \cdot \mathbb{1}_{N(\omega)\le n}]$. As in [8], Section 2, define the quenched entropy of the simple random walk on $\mathcal{S}_\infty$ by $\mathbf{H}_n = \sum_x \phi(p_{\mathcal{S}_\infty,n}(0,x))$, where $\phi(0) = 0$ and $\phi(t) = -t\log t$ for $t > 0$, and the mean entropy by $H_n = \mathbb{E}^u[\mathbf{H}_n]$. By a general argument in the proof of [8], Theorem 6, the heat kernel upper bound (9) implies that

$$\mathbb{E}^u\big[\big(p_{2n}(0,x) - p_{2n-1}(X_1,x)\big)^2 \cdot \mathbb{1}_{x\in\mathcal{S}_\infty} \cdot \mathbb{1}_{N(\omega)\le n}\big]$$
$$\le (H_n - H_{n-1}) \cdot \frac{C}{n^d} \cdot e^{-c\frac{\mathrm{d}_{\mathbb{Z}^d}(0,x)^2}{n}}.$$

The proof of [8], Theorem 6, is completed by showing in [8], Lemma 20, that $H_n - H_{n-1} \le \frac{C}{n}$. Thus, in order to finish the proof of Theorem 1.16, it suffices to prove that $H_n - H_{n-1} \le \frac{C}{n}$ in our setting, too. This is a simple consequence of Theorem 1.15. Indeed, by writing $\mathbf{H}_n$ as the sums over x with $\mathrm{d}_{\mathbb{Z}^d}(0,x)^{\frac{3}{2}} \le n$ and $\mathrm{d}_{\mathbb{Z}^d}(0,x)^{\frac{3}{2}} \ge n$, applying (9) and (10) to the summands in the first sum, and showing smallness of the second sum by using, for instance, the general upper bound on the heat kernel (see, e.g., [4], (1.5)), we prove that for all $n \ge T_{\mathrm{hk}}(0)$, $\mathbf{H}_n = \frac{d}{2}\log n + O(1)$. For $n \le T_{\mathrm{hk}}(0)$, we use the crude bound $\mathbf{H}_n \le d\log(2n)$ (see the proof below [8], (25)). By integrating $\mathbf{H}_n$ and using (11), we get that $H_n = \frac{d}{2}\log n + O(1)$, which implies that $H_n - H_{\lfloor n/2\rfloor} \le C$ for some C. Since $H_n - H_{n-1}$ is decreasing by [8], Corollary 10, we conclude that $H_n - H_{n-1} \le \frac{2C}{n}$, completing the proof of Theorem 1.16. □

PROOF OF THEOREM 1.17. The proof of Theorem 1.17 is literally the same as the proof of [7], Theorem 1.2(a). For the upper bound, one splits the Green function into the integrals over $[0, \min\{T_{\mathrm{hk}}(x), T_{\mathrm{hk}}(y)\}]$ and $[\min\{T_{\mathrm{hk}}(x), T_{\mathrm{hk}}(y)\}, \infty)$. Using general bounds on the heat kernel [see [7], (6.4) and (6.5)], one shows that the first integral is $o(\mathrm{d}_{\mathbb{Z}^d}(x,y)^{2-d})$, and by (9), the second integral is bounded by $C\mathrm{d}_{\mathbb{Z}^d}(x,y)^{2-d}$. For the lower bound, one estimates the Green function from below by the integral of heat kernel over $[\mathrm{d}_{\mathbb{Z}^d}(x,y)^2, \infty)$, applies (10), and arrives at the desired bound. □

PROOF OF THEOREM 1.18. The proof of Theorem 1.18 is identical to the one of [8], Theorem 5. The constant functions and the projections of $x + \chi(x)$ (see Theorem 1.11(a)) on coordinates of $\mathbb{Z}^d$ are independent harmonic functions with at most linear growth. Thus, the dimension of such functions is at least $(d+1)$. It remains to show that the above functions form a basis. Let h be a harmonic function h on $\mathcal{S}_\infty$ with at most linear growth and $h(0) = 0$, and assume that it is extended on $\mathbb{R}^d$ (see above [8], Proposition 19). By Theorem 1.13 and the upper bound on the heat kernel (9), the proof of [8], Proposition 19, goes through without any changes in our setting, implying that the sequence $h_n(\cdot) = \frac{1}{n}h(n\cdot)$ is uniformly bounded and equicontinuous on compacts. Thus, there exists a sequence n_k such that h_{n_k} converges uniformly on compact sets to a continuous function $\widetilde{h}$. By using the quenched invariance principle of Theorem 1.11, one obtains by repeating the proof of [8], Theorem 5, that $\widetilde{h}$ is harmonic in $\mathbb{R}^d$. Since $\widetilde{h}$ has at most linear growth and $\widetilde{h}(0) = 0$, it is linear. Therefore, the function $f(x) = h(x) - \widetilde{h}(x + \chi(x))$ is harmonic on $\mathcal{S}_\infty$ and for every $\varepsilon > 0$ and all large enough k, $|f(x)| \le \varepsilon n_k$ for all $x \in \mathrm{B}_{\mathcal{S}_\infty}(0, n_k/\varepsilon)$. By (9), $\mathrm{E}_{\mathcal{S}_\infty,0}[f(X_{n_k^2})^2] \le \varepsilon n_k^2$ for all large k. The proof of [8], Theorem 5, is finished by applying [8], Corollary 21, which states that f must be constant. The proof of [8], Corollary 21, is rather general and only uses the fact that the mean entropy H_n (see the proof of Theorem 1.16) satisfies $H_n - H_{n-1} \le \frac{C}{n}$. We already proved this bound in the proof of Theorem 1.16. Thus, [8], Corollary 21, holds in our setting, and we conclude that f must be constant. The proof is complete. $\square$

PROOF OF THEOREM 1.19. Theorem 1.19 was proved in the case of supercritical Bernoulli percolation in [7], Theorem 1.1, by first providing general assumptions [7], Assumption 4.4, for the local limit theorem on infinite subgraphs of $\mathbb{Z}^d$ (see [7], Theorems 4.5 and 4.6), and then verifying these assumptions for the infinite cluster of Bernoulli percolation. [7], Assumption 4.4, is tailored for random subgraphs of $\mathbb{Z}^d$ with laws invariant under reflections with respect to coordinate axes and rotations by $\frac{\pi}{2}$. These assumptions only simplify the expression for the heat kernel of the limiting Brownian motion, and can be naturally extended to the case without such symmetries.

We only consider the case of discrete time random walk (the continuous time case is the same). As in [7], Theorem 4.5, to prove Theorem 1.19 it suffices to show that there exist an event $\Omega' \in \mathcal{F}$ with $\mathbb{P}^u[\Omega'] = 1$, positive constants δ, C_i, and C_H, and a covariance matrix Σ, such that for all $\omega \in \Omega' \cap \{0 \in \mathcal{S}_\infty\}$:

(a) for any $y \in \mathbb{R}^d$ and $r > 0$, as $n \to \infty$, $\mathrm{P}_{\mathcal{S}_\infty,0}[\widetilde{B}_n(t) \in (y + [-r, r]^d)]$ converges to $\int_{y+[-r,r]^d} k_{\Sigma,t}(y')\,dy'$ uniformly over compact subsets of $(0, \infty)$ ($\widetilde{B}_n(t)$ is as in (7)),

(b) there exists $T_1 = T_1(\omega) < \infty$ such that for all $n \ge T_1$ and $x \in \mathcal{S}_\infty$, $p_n(0, x) \le C_1 \cdot n^{-\frac{d}{2}} \cdot e^{-C_2 \cdot \frac{\mathrm{d}_{\mathcal{S}_\infty}(0,x)^2}{n}}$,

(c) for each $y \in \mathcal{S}_\infty$, there exists $R_H(y) = R_H(y, \omega) < \infty$ such that the parabolic Harnack inequality holds with constant C_H in $(0, R^2] \times \mathrm{B}_{\mathcal{S}_\infty}(y, R)$ for all $R \ge R_H(y)$,

(d) for $h(r) = \max\{r' : \exists y \in [-r, r]^d$ such that $\mathcal{S}_\infty \cap (y + [-r', r']^d) = \varnothing\}$, the ratio $\frac{h(r)}{r}$ tends to 0 as $r \to \infty$,

(e) for any $x \in \mathbb{Z}^d$ and $r > 0$, $\lim_{n\to\infty} \frac{\mu(\mathcal{S}_\infty \cap (\sqrt{n}x + [-\sqrt{n}r, \sqrt{n}r]^d))}{(2\sqrt{n}r)^d} = \mathbb{E}^u[\mu_0 \cdot \mathbb{1}_{0\in\mathcal{S}_\infty}]$,

(f) for each $x \in \mathbb{Z}^d$ and $r > 0$, there exists $T_2(x) = T_2(x, \omega) < \infty$ such that for all $n \ge T_2$, and $x', y' \in \mathcal{S}_\infty \cap (\sqrt{n}x + [-\sqrt{n}r, \sqrt{n}r]^d)$, $\mathrm{d}_{\mathcal{S}_\infty}(x', y') \le C_3 \cdot \max\{\mathrm{d}_{\mathbb{Z}^d}(x', y'), n^{\frac{1}{2}-\delta}\}$,

(g) for $x \in \mathbb{Z}^d$ and R_H as in (c), $\lim_{n\to\infty} n^{-\frac{1}{2}} R_H(g_n(x)) = 0$.

It is easy to see that the above assumptions are satisfied in our setting:

(a) follows from Theorem 1.11,
(b) follows from (9),
(c) follows from Theorems 1.5 and 1.13,
(d) follows from stationarity, (6), and the Borel–Cantelli lemma,
(e) follows from a spatial ergodic theorem [23], Theorem 2.8 in Chapter 6, since the sequence of boxes $(\sqrt{n}x + [-\sqrt{n}r, \sqrt{n}r]^d)_{n\ge 1}$ is regular in the sense of [23], Definition 2.4 in Chapter 6 (see [3], Lemma 5.1),
(f) follows from Theorem 1.10,
(g) follows from (8), Theorem 1.5, and the Borel–Cantelli lemma.

The proof of Theorem 1.19 is complete. □

PROOF OF THEOREM 1.20. Statement (a) follows from Theorem 1.19 and (9) by repeating the proof of [7], Theorem 1.2(b), without any changes. For the statement (c) we use bounds [7], (6.30) and (6.31), and (11), to get

$$\begin{aligned}
&\frac{(1-\varepsilon)\mathrm{G}_\Sigma(x)}{m}\mathbb{P}^u\big[M \le |x|\,|0 \in \mathcal{S}_\infty\big] \\
&\quad \le \mathbb{E}^u\big[g_{\mathcal{S}_\infty}(0,x)|0 \in \mathcal{S}_\infty\big] \\
&\quad \le \frac{(1+\varepsilon)\mathrm{G}_\Sigma(x)}{m} + \frac{C'\mathbb{P}^u[M > |x|\,|0 \in \mathcal{S}_\infty]}{|x|^{d-2}} \\
&\qquad + C'\big(\mathbb{E}^u\big[g_{\mathcal{S}_\infty}(0,x)^2|0 \in \mathcal{S}_\infty\big]\big)^{\frac{1}{2}} \cdot e^{-c'(\log|x|)^{1+\Delta_{\mathrm{S}}}},
\end{aligned}$$

where M is defined in the statement of Theorem 1.20. As in [7], (6.17), by (9),

$$g_{\mathcal{S}_\infty}(0,x) \le g_{\mathcal{S}_\infty}(0,0) \le T_0(0) + \int_{T_0(0)}^{\infty} C' t^{-\frac{d}{2}}\,dt \le \big(1+2C'\big)T_0(0).$$

Combining this bound with (11), we obtain that $\mathbb{E}^u[g_{\mathcal{S}_\infty}(0,x)^2|0 \in \mathcal{S}_\infty] < C''$. Let $x = ky$. Since $\mathrm{G}_\Sigma(ky) = k^{2-d}\mathrm{G}_\Sigma(y)$, by taking limits $k \to \infty$ and then $\varepsilon \to 0$, we compete the proof of statement (c). □

Acknowledgements. The author thanks Takashi Kumagai and Alain-Sol Sznitman for encouragements to write this paper and for valuable comments, Jean-Dominique Deuschel for sending him the master thesis of Tuan Anh Nguyen [31] and for a discussion about isoperimetric and Sobolev inequalities, and Jiří˜ Cerný for a suggestion to use general densities η_1 and η_2 in Section 3. Special thanks go to the anonymous referee for helpful suggestions and comments.

REFERENCES

[1] ANDRES, S., BARLOW, M. T., DEUSCHEL, J.-D. and HAMBLY, B. M. (2013). Invariance principle for the random conductance model. *Probab. Theory Related Fields* **156** 535–580. MR3078279

[2] ANDRES, S., DEUSCHEL, J.-D. and SLOWIK, M. (2015). Invariance principle for the random conductance model in a degenerate ergodic environment. *Ann. Probab.* **43** 1866–1891. MR3353817

[3] ANDRES, S., DEUSCHEL, J.-D. and SLOWIK, M. (2016). Harnack inequalities on weighted graphs and some applications to the random conductance model. *Probab. Theory Related Fields* **164** 931–977. MR3477784

[4] BARLOW, M. T. (2004). Random walks on supercritical percolation clusters. *Ann. Probab.* **32** 3024–3084. MR2094438

[5] BARLOW, M. T. and CHEN, X. (2016). Gaussian bounds and parabolic Harnack inequality on locally irregular graphs. *Math. Ann.* To appear.

[6] BARLOW, M. T. and DEUSCHEL, J.-D. (2010). Invariance principle for the random conductance model with unbounded conductances. *Ann. Probab.* **38** 234–276. MR2599199

[7] BARLOW, M. T. and HAMBLY, B. M. (2009). Parabolic Harnack inequality and local limit theorem for percolation clusters. *Electron. J. Probab.* **14** 1–27. MR2471657

[8] BENJAMINI, I., DUMINIL-COPIN, H., KOZMA, G. and YADIN, A. (2015). Disorder, entropy and harmonic functions. *Ann. Probab.* **43** 2332–2373. MR3395463

[9] BERGER, N. and BISKUP, M. (2007). Quenched invariance principle for simple random walk on percolation clusters. *Probab. Theory Related Fields* **137** 83–120. MR2278453

[10] BERGER, N., BISKUP, M., HOFFMAN, C. E. and KOZMA, G. (2008). Anomalous heat-kernel decay for random walk among bounded random conductances. *Ann. Inst. Henri Poincaré Probab. Stat.* **44** 374–392. MR2446329

[11] BISKUP, M. and PRESCOTT, T. M. (2007). Functional CLT for random walk among bounded random conductances. *Electron. J. Probab.* **12** 1323–1348. MR2354160

[12] BRICMONT, J., LEBOWITZ, J. L. and MAES, C. (1987). Percolation in strongly correlated systems: The massless Gaussian field. *J. Stat. Phys.* **48** 1249–1268. MR0914444

[13] CROYDON, D. A. and HAMBLY, B. M. (2008). Local limit theorems for sequences of simple random walks on graphs. *Potential Anal.* **29** 351–389. MR2453564

[14] DELMOTTE, T. (1999). Parabolic Harnack inequality and estimates of Markov chains on graphs. *Rev. Mat. Iberoam.* **15** 181–232. MR1681641

[15] DELMOTTE, T. (1998). Harnack inequalities on graphs. In *Séminaire de Théorie Spectrale et Géométrie, Vol.* 16, *Année* 1997–1998. *Sémin. Théor. Spectr. Géom.* **16** 217–228. Univ. Grenoble I, Saint-Martin-d'Hères, Grenoble. MR1666463

[16] DEUSCHEL, J.-D. and PISZTORA, A. (1996). Surface order large deviations for high-density percolation. *Probab. Theory Related Fields* **104** 467–482. MR1384041

[17] DE GIORGI, E. (1957). Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat.* (3) **3** 25–43. MR0093649

[18] DREWITZ, A., RÁTH, B. and SAPOZHNIKOV, A. (2014). Local percolative properties of the vacant set of random interlacements with small intensity. *Ann. Inst. Henri Poincaré Probab. Stat.* **50** 1165–1197. MR3269990

[19] DREWITZ, A., RÁTH, B. and SAPOZHNIKOV, A. (2014). On chemical distances and shape theorems in percolation models with long-range correlations. *J. Math. Phys.* **55** 083307, 30. MR3390739

[20] GRIGORYAN, A. A. (1992). Heat equation on a noncompact Riemannian manifold. *Math. USSR-Sb.* **72** 47–77.

[21] GRIMMETT, G. (1999). *Percolation*, 2nd ed. Springer, Berlin. MR1707339

[22] GRIMMETT, G. (2006). *The Random-Cluster Model*. Springer, Berlin. MR2243761

[23] KRENGEL, U. (1985). *Ergodic Theorems*. Walter de Gruyter, Berlin. MR0797411

[24] KUMAGAI, T. (2014). *Random Walks on Disordered Media and Their Scaling Limits*. *Lecture Notes in Math*. **2101**. Springer, Cham. MR3156983

[25] MATHIEU, P. and PIATNITSKI, A. (2007). Quenched invariance principles for random walks on percolation clusters. *Proc*. *R*. *Soc*. *Lond*. *Ser*. *A Math*. *Phys*. *Eng*. *Sci*. **463** 2287–2307. MR2345229

[26] MATHIEU, P. and REMY, E. (2004). Isoperimetry and heat kernel decay on percolation clusters. *Ann*. *Probab*. **32** 100–128. MR2040777

[27] MORRIS, B. and PERES, Y. (2005). Evolving sets, mixing and heat kernel bounds. *Probab*. *Theory Related Fields* **133** 245–266. MR2198701

[28] MOSER, J. (1961). On Harnack's theorem for elliptic differential equations. *Comm*. *Pure Appl*. *Math*. **14** 577–591. MR0159138

[29] MOSER, J. (1964). A Harnack inequality for parabolic differential equations. *Comm*. *Pure Appl*. *Math*. **17** 101–134. MR0159139

[30] NASH, J. (1958). Continuity of solutions of parabolic and elliptic equations. *Amer*. *J*. *Math*. **80** 931–954. MR0100158

[31] NGUYEN, T. A. (2014). Quenched invariance principle for random conductance model Master thesis Berlin.

[32] PROCACCIA, E., ROSENTHAL, R. and SAPOZHNIKOV, A. (2016). Quenched invariance principle for simple random walk on clusters in correlated percolation models. *Probab*. *Theory Related Fields* **166** 619–657. MR3568036

[33] RÁTH, B. and SAPOZHNIKOV, A. (2011). On the transience of random interlacements. *Electron*. *Commun*. *Probab*. **16** 379–391. MR2819660

[34] RÁTH, B. and SAPOZHNIKOV, A. (2013). The effect of small quenched noise on connectivity properties of random interlacements. *Electron*. *J*. *Probab*. **18** no. 4, 20. MR3024098

[35] RODRIGUEZ, P.-F. and SZNITMAN, A.-S. (2013). Phase transition and level-set percolation for the Gaussian free field. *Comm*. *Math*. *Phys*. **320** 571–601. MR3053773

[36] SALOFF-COSTE, L. (1992). A note on Poincaré, Sobolev, and Harnack inequalities. *Internat*. *Math*. *Res*. *Notices* **2** 27–38. MR1150597

[37] SALOFF-COSTE, L. (1997). Lectures on finite Markov chains. In *Lectures on Probability Theory and Statistics* (*Saint-Flour*, 1996). *Lecture Notes in Math*. **1665** 301–413. Springer, Berlin. MR1490046

[38] SIDORAVICIUS, V. and SZNITMAN, A.-S. (2004). Quenched invariance principles for walks on clusters of percolation or among random conductances. *Probab*. *Theory Related Fields* **129** 219–244. MR2063376

[39] SIDORAVICIUS, V. and SZNITMAN, A.-S. (2009). Percolation for the vacant set of random interlacements. *Comm*. *Pure Appl*. *Math*. **62** 831–858. MR2512613

[40] SZNITMAN, A.-S. (2010). Vacant set of random interlacements and percolation. *Ann*. *of Math*. (2) **171** 2039–2087. MR2680403

[41] SZNITMAN, A.-S. (2012). Decoupling inequalities and interlacement percolation on $G \times \mathbb{Z}$. *Invent*. *Math*. **187** 645–706. MR2891880

[42] TEIXEIRA, A. (2009). On the uniqueness of the infinite cluster of the vacant set of random interlacements. *Ann*. *Appl*. *Probab*. **19** 454–466. MR2498684

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LEIPZIG
AUGUSTUSPLATZ 10
04109 LEIPZIG
GERMANY
E-MAIL: artem.sapozhnikov@math.uni-leipzig.de