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POINT-MAP-PROBABILITIES OF A POINT PROCESS AND MECKE'S INVARIANT MEASURE EQUATION[1]

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A compatible point-shift F maps, in a translation invariant way, each point of a stationary point process Φ to some point of Φ. It is fully determined by its associated point-map, f, which gives the image of the origin by F. It was proved by J. Mecke that if F is bijective, then the Palm probability of Φ is left invariant by the translation of $-f$. The initial question motivating this paper is the following generalization of this invariance result: in the nonbijective case, what probability measures on the set of counting measures are left invariant by the translation of $-f$? The point-map-probabilities of Φ are defined from the action of the semigroup of point-map translations on the space of Palm probabilities, and more precisely from the compactification of the orbits of this semigroup action. If the point-map-probability exists, is uniquely defined and if it satisfies certain continuity properties, it then provides a solution to this invariant measure problem. Point-map-probabilities are objects of independent interest. They are shown to be a strict generalization of Palm probabilities: when F is bijective, the point-map-probability of Φ boils down to the Palm probability of Φ. When it is not bijective, there exist cases where the point-map-probability of Φ is singular with respect to its Palm probability. A tightness based criterion for the existence of the point-map-probabilities of a stationary point process is given. An interpretation of the point-map-probability as the conditional law of the point process given that the origin has F-pre-images of all orders is also provided. The results are illustrated by a few examples.

Introduction. A *point-shift* is a mapping which is defined on all discrete subsets ϕ of $\mathbb{R}^d$ and maps each point $x \in \phi$ to some point $y \in \phi$, that is, if F is a point-shift, for all discrete $\phi \subset \mathbb{R}^d$ and all $x \in \phi$, $F(\phi, x) \in \phi$. Bijective point-shifts were studied in a seminal paper by J. Mecke in [14]. The concept of the point-map was introduced by H. Thorisson (see [17] and the references therein). Points-maps were further studied by M. Heveling and G. Last [9]. The latter reference also contains a short proof of Mecke's invariance theorem. Point-shifts are also known as allocation rules (see, e.g., [13]). A point-shift is *compatible with the translations of* $\mathbb{R}^d$ or simply *compatible* if

$$\forall t \in \mathbb{R}^d, \qquad F(\phi + t, x + t) = F(\phi, x) + t.$$

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As will be seen, a translation invariant point-shift F is fully determined by its point-map f which associates to all ϕ containing the origin the image of the latter by F, that is, $f(\phi) = F(\phi, 0)$. The point-shift F is called bijective on the point process Φ if, for almost all realizations ϕ of the point process, $F(\phi, \cdot)$ is bijective on the set ϕ.

The Palm probability of a translation invariant point process Φ is often intuitively described as the distribution of Φ conditionally on the presence of a point at the origin. This definition was formalized by C. Ryll-Nardzewski [16] based on the Matthes definition of Palm probabilities (see, e.g., [3]). This is the so-called *local interpretation* of the latter. The presence of a point at the origin makes the Palm distribution of Φ singular with respect to (w.r.t.) the translation invariant distribution of Φ.

The present paper is focused on the point-map-probabilities (or the f-probabilities) of Φ. Under conditions described in the paper, the f-probabilities can be described as the law of Φ conditionally on the event that the origin has F-pre-images of all orders (Theorem 2.12). This event is not of positive probability in general, and hence it is not possible to define this conditional probability in the usual way.

The first aim of this paper is to make this definition rigorous. The proposed construction is based on dynamical system theory. The action of the semigroup of translations by $-f$ on probability distributions on counting measures having a point at the origin is considered; the f-probabilities of Φ are then defined as the ω-limits of the orbit of this semigroup action on the Palm distribution of Φ (Definition 2.6). As the space of probability distributions on counting measures is not compact, the existence of f-probabilities of Φ is not granted. A necessary and sufficient condition for their existence is given in Lemma 2.9. Uniqueness is not granted either. An instance of construction of the f-probabilities of Poisson point processes where one has existence and uniqueness is given in Theorem 2.25.

It is shown in Section 2 that, when they exist, point-map-probabilities generalize Palm probabilities. A key notion to see this is that of *evaporation*. One says that there is evaporation when the image of Φ by the nth iterate of F tends to the empty counting measure for n tending to infinity.

When there is no evaporation, the f-probabilities of Φ are just the Palm distributions of Φ w.r.t. certain translation invariant thinnings of Φ and they are then absolutely continuous w.r.t. the Palm distribution $\mathcal{P}_0$ of Φ; in particular, if F is bijective, then the f-probability of Φ exists, is uniquely defined, and coincides with $\mathcal{P}_0$. However, in the evaporation case, the f-probabilities of Φ do not admit a representation of this type and they are actually singular w.r.t. $\mathcal{P}_0$ (Theorem 2.16).

It is also shown in Theorem 2.19 that, under appropriate continuity properties on f, a certain mixture of the f-probabilities of Φ is left invariant by the shift of $-f$. This generalizes Mecke's point stationarity theorem which states that if $F(\Phi, \cdot)$ is bijective and if Φ is distributed according to $\mathcal{P}_0$, then so is $\Phi - f$.

Section 1 contains the basic definitions and notation used in the paper, together with a small set of key examples. Section 2 gathers the main results and proofs. Several more examples are discussed in Section 3. The basic tools of point process theory and dynamical system theory used in the paper are summarized in the Appendix.

1. Preliminaries and notation.

1.1. *General notation.* Each measurable mapping $h : (X, \mathcal{X}) \to (X', \mathcal{X}')$ between two measurable spaces induces a measurable mapping $h_* : M(X) \to M(X')$, where $M(X)$ is the set of all measures on X: if μ is a measure on $(X, \mathcal{X})$, $h_*\mu$ is the measure on $(X', \mathcal{X}')$ defined by

$$h_*\mu(A) := (h_*\mu)(A) = \mu\big(h^{-1}A\big). \tag{1.1}$$

Note that if μ is a probability measure, $h_*\mu$ is also a probability measure.

1.2. *Point processes.* Let $\mathbf{N} = \mathbf{N}(\mathbb{R}^d)$ be the space of all locally finite counting measures (not necessarily simple) on $\mathbb{R}^d$. One can identify each element of $\mathbf{N}$ with the associated multi-subset of $\mathbb{R}^d$. The notation ϕ will be used to denote either the measure or the related multi-set. Let $\mathcal{N}$ be the Borel σ-field with respect to the vague topology on the space of counting measures (see Appendix A for more on this subject). The measurable space $(\mathbf{N}, \mathcal{N})$ is the *canonical space* of point processes.

The *support* of a counting measure ϕ is the same set as the multi-set related to ϕ, but without the multiplicities, and it is denoted by $\overline{\phi}$. The set of all counting measure supports is denoted by $\overline{\mathbf{N}}$, that is, $\overline{\mathbf{N}}$ is the set of all *simple* counting measures. $\mathcal{N}$ naturally induces a σ-field $\overline{\mathcal{N}}$ on $\overline{\mathbf{N}}$.

Let $\mathbf{N}^0$ (resp., $\overline{\mathbf{N}}^0$) denote the set of all elements of $\mathbf{N}$ (resp., $\overline{\mathbf{N}}$) which contain the origin, that is, for all $\phi \in \mathbf{N}^0$ (resp., $\phi \in \overline{\mathbf{N}}^0$), one has $0 \in \phi$.

A *point process* is a couple $(\Phi, \mathbb{P})$ where $\mathbb{P}$ is a probability measure on a measurable space $(\Omega, \mathcal{F})$ and Φ is a measurable mapping from $(\Omega, \mathcal{F})$ to $(\mathbf{N}, \mathcal{N})$. If $(\Omega, \mathcal{F}) = (\mathbf{N}, \mathcal{N})$ and Φ is the identity on $\mathbf{N}$, the point process is defined on the canonical space. Calligraphic letters $\mathcal{P}, \mathcal{Q}, \dots$ (resp., blackboard bold letters $\mathbb{P}, \mathbb{Q}, \dots$) will be used for probability measures defined on the canonical space [resp., on $(\Omega, \mathcal{F})$]. The *canonical version* of a point process $(\Phi, \mathbb{P})$ is the point process $(\mathrm{id}, \Phi_*\mathbb{P})$ which is defined on the canonical space. Here, id denotes the identity on $\mathbf{N}$.

1.3. *Stationary point processes.* Whenever $(\mathbb{R}^d, +)$ acts (in a measurable way) on a space, the action of $t \in \mathbb{R}^d$ on that space will be denoted by θ_t. It is assumed that $(\mathbb{R}^d, +)$ acts on the reference probability space $(\Omega, \mathcal{F})$, or equivalently that this space is equipped with a measurable flow $\theta_t : \Omega \to \Omega$, with t ranging over $\mathbb{R}^d$. This is a family of mappings such that $(\omega, t) \mapsto \theta_t \omega$ is measurable, θ_0 is the identity on Ω and

$$\theta_s \circ \theta_t = \theta_{s+t}.$$

A point process Φ is then said to be *compatible* if

$$\Phi(\theta_t \omega, B - t) = \Phi(\omega, B) \qquad \forall \omega \in \Omega, t \in \mathbb{R}^d, B \in \mathcal{B}, \tag{1.2}$$

where by convention, $\Phi(\omega, B) := (\Phi(\omega))(B)$. Here, $\mathcal{B}$ denotes the Borel σ-algebra on $\mathbb{R}^d$.

The action θ_t of $t \in \mathbb{R}^d$ can also be used on the space of counting measures to denote the translation by $-t$. For a counting measure $\phi \in \mathbf{N}(\mathbb{R}^d)$, $\theta_t \phi$ is then the counting measure defined by $\theta_t \phi(B) = \phi(B + t)$. Using this notation, the compatibility criterion (1.2) can be rewritten as

$$\Phi \circ \theta_t = \theta_t \circ \Phi.$$

Note that for consistency reasons, the action θ_t of $t \in \mathbb{R}^d$ on $\mathbb{R}^d$ itself is then $\theta_t x = x - t, \forall x \in \mathbb{R}^d$.

The probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is θ_t-invariant if $(\theta_t)_* \mathbb{P} = \mathbb{P}$. If, for all $t \in \mathbb{R}^d$, $\mathbb{P}$ is θ_t-invariant, it is called stationary. Below, a *stationary point process* is a point process $(\Phi, \mathbb{P})$ such that Φ is compatible and $\mathbb{P}$ is stationary.

When the point process is simple and stationary with a nondegenerate (positive and finite) intensity, its *Palm probability* is a classical object in the literature.

The Palm probability of a general (i.e., not necessarily simple) point process Φ is defined by

$$\mathbb{P}_\Phi[A] := \frac{1}{\lambda |B|} \int_\Omega \int_B \mathbf{1}\{\theta_t \omega \in A\} \Phi(\omega, \mathrm{d}t) \mathbb{P}[\mathrm{d}\omega], \tag{1.3}$$

for all $A \in \mathcal{F}$, and for all Borel sets $B \subset \mathbb{R}^d$ with a nondegenerate (positive and finite) Lebesgue measure. Note that the multiplicity of the atoms of Φ is taken into account in the last definition. If a point process $(\Phi, \mathbb{P})$ is stationary and has a nondegenerate intensity, the pair $(\Phi, \mathbb{P}_\Phi)$ is called the *Palm version* of $(\Phi, \mathbb{P})$. Expectation w.r.t. $\mathbb{P}_\Phi$ will be denoted by $\mathbb{E}_\Phi$.

Whenever the context specifies a reference point process $(\Phi, \mathbb{P})$, the short notation $\mathcal{P}$ will be used to denote its distribution, that is, $\mathcal{P} = \Phi_* \mathbb{P}$. If in addition, Φ is stationary and with a nondegenerate intensity, the distribution of its Palm version will be denoted by $\mathcal{P}_0$, that is, $\mathcal{P}_0 = \Phi_* \mathbb{P}_\Phi$, and expectation w.r.t. $\mathcal{P}_0$ will be denoted by $\mathcal{E}_0$. In the canonical setup, the Palm version of $(\Phi, \mathbb{P}) = (\mathrm{id}, \mathcal{P})$ is $(\Phi, \mathbb{P}_\Phi) = (\mathrm{id}, \mathcal{P}_0)$.

1.4. *Compatible point-shifts.*

1.4.1. *Point-maps.* A point-shift on $\mathbf{N}$ is a measurable function $F : \mathbf{N} \times \mathbb{R}^d \to \mathbb{R}^d$, which is defined for all pairs (ϕ, x), where $\phi \in \mathbf{N}$ and $x \in \phi$, and satisfies the relation $F(\phi, x) \in \phi$ for all $x \in \phi$.

In order to define compatible point-shifts, it is convenient to use the notion of *point-map*. A measurable function $f : \mathbf{N}^0 \to \mathbb{R}^d$ is called a point-map if for all ϕ in $\mathbf{N}^0$, one has $f(\phi) = f(\overline{\phi})$, that is, it depends only on $\overline{\phi}$, and if $f(\phi) \in \overline{\phi}$.

If f is a point-map, the associated *compatible point-shift, F, is*

$$F(\phi, x) = f(\theta_x \phi) + x = \theta_{-x} f(\theta_x \phi).$$

The point-shift F is compatible in the sense that

$$\begin{aligned} F(\theta_t \phi, \theta_t x) &= F(\theta_t \phi, x - t) \\ &= f\bigl(\theta_{x-t}(\theta_t \phi)\bigr) + x - t \\ &= f(\theta_x \phi) + x - t = F(\phi, x) - t = \theta_t\bigl(F(\phi, x)\bigr). \end{aligned} \tag{1.4}$$

In the rest of this article, point-shift always means compatible point-shift. Small letters will be used for point-maps and capital letters for the associated point-shifts.

For the point-map f, the *action of the point-map* on $\mathbf{N}^0(\mathbb{R}^d)$ will be denoted by θ_f and defined by

$$\forall \phi \in \mathbf{N}^0(\mathbb{R}^d); \qquad \theta_f(\phi) = \theta_{f(\phi)}(\phi).$$

1.4.2. *Iterates of a point-shift.* For all $n \geq 0$, all $\phi \in \mathbf{N}$ and $x \in \phi$, the nth order iterate of the point-shift F is defined inductively by $F^0(\phi, x) = x$ and

$$F^{k+1}(\phi, x) = F\bigl(\phi, F^k(\phi, x)\bigr), \qquad k \geq 0.$$

For all n, F^n is a compatible point-shift and the associated point-map, which will be denoted by f^n, satisfies

$$f^n(\phi) = f^{n-1}(\phi) + f\bigl(\theta_{f^{n-1}}(\phi)\bigr), \qquad n \geq 1, \tag{1.5}$$

with $f^0(\phi) = 0$ and $\phi \in \mathbf{N}^0$. It is easy to verify that for all $n \in \mathbb{N}$, on $\mathbf{N}^0$,

$$\theta_{f^n} = \theta_f^n,$$

and hence

$$\theta_{f^{m+n}} = \theta_{f^m} \circ \theta_{f^n}. \tag{1.6}$$

In accordance with the definition of F^n, for all $n \geq 1$, let

$$F^{-n}(\phi, x) = \bigl\{y \in \phi; F^n(\phi, y) = x\bigr\}.$$

1.4.3. *Image point processes.* Let f be a point-map. For all $\phi \in \mathbf{N}$ and all nonnegative integers n, let

$$m_f^n(\phi, y) = \phi\big(F^{-n}(\phi, y)\big) = \sum_{x \in F^{-n}(\phi,y)} \phi\big(\{x\}\big) \qquad \forall y \in \phi, \tag{1.7}$$

where by convention, the summation over the empty set is zero. Note that if ϕ is simple, then $m_f^n(\phi, y) = \operatorname{card}(F^{-n}(\phi, y))$.

DEFINITION 1.1. Assume that $m_f^n(\phi, y) < \infty$ for all $y \in \phi$. The nth image counting measure (of ϕ by F) is then defined as the counting measure ϕ_f^n with support $\{y \in \phi; F^{-n}(\phi, y) \neq \varnothing\}$, and such that the multiplicity of y in the support of ϕ_f^n is $m_f^n(\phi, y)$.

It will be shown below that, for all stationary point processes $(\Phi, \mathbb{P})$, for all $n \geq 0$, $(\Phi_f^n, \mathbb{P})$ is a stationary point process (item 1 in Remark 2.4) with the same intensity as Φ (item 2 in Remark 2.4). The point process Φ_f^n will be referred to as the nth *image point process* (of Φ by the point-map).

1.5. *First point-shift examples.* This subsection presents a few basic examples of point-shifts. These examples will allow one to illustrate the main results in Section 2. More details on these examples and further examples can be found in Section 3.

1.5.1. *Strip point-shift.* The strip point-shift was introduced by Ferrari, Landim and Thorisson [8]. For all points $x = (x_1, x_2)$ in the plane, let $T(x)$ denote the half-strip $(x_1, \infty) \times [x_2 - 1, x_2 + 1]$. Then $S(\phi, x)$ is the left most point of ϕ in $T(x)$ (see Figure 1). It is easy to verify that S is compatible. It is not bijective. Its point-map will be denoted by s.

REMARK 1.2. The strip point-shift is not well-defined when there are more than one left most point in $T(x)$, or when there is no point of ϕ in $T(x)$. However, there is no problem if we consider the strip point-shift (and all other point-shifts) on point processes for which the point-shift is almost surely well-defined. Note that these two difficulties can always be taken care of by fixing, in some translation invariant manner, the choice of the image in the case of ambiguity, and by defining $F(\phi, x) = x$ in the case of non-existence. By doing so, one gets a point-shift defined for all (ϕ, x).

1.5.2. *Strip point-shift on the random geometric graph.* The *strip point-map on the random geometric graph* with the neighborhood radius r is

$$g(\phi) = \begin{cases} s(\phi), & \|s(\phi)\| < r, \\ 0, & \text{otherwise}, \end{cases}$$

where s is the strip point-map. The associated point-shift is depicted in Figure 1. It will be denoted by G. It is not bijective. Its point-map will be denoted by g.

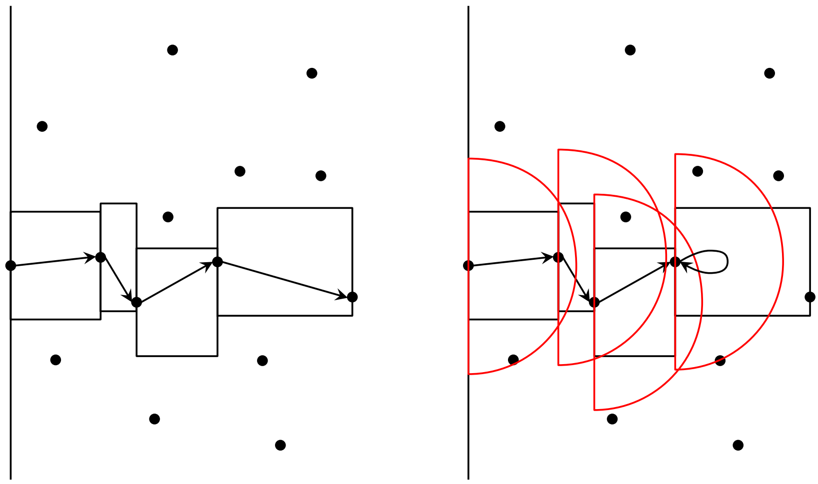

FIG. 1. *Left*: *Iterates of the strip point-shift S. Right*: *Iterates of G, the strip point-shift on the random geometric graph. In both cases, the point $G^4(\phi, 0)$ is that at the end of the directed path.*

1.5.3. *Closest point-shift*. The *closest point-shift*, C, maps each point of $x \in \phi$ to the point $y \neq x$ of ϕ which is the closest. This point-shift is not bijective either. The associated point-map will be denoted by c. It is depicted in Figure 2.

1.5.4. *Mutual-neighbor point-shift*. The *mutual-neighbor point-shift*, N, maps each point $x \in \phi$ to the point y of ϕ which is the closest to x, if x is also the point of ϕ which is the closest to y. Otherwise, it maps x to itself. It is easy to see that N is bijective and involutive: $N^2 \equiv \mathrm{id}$. The associated point-map will be denoted by n. It is depicted in Figure 2.

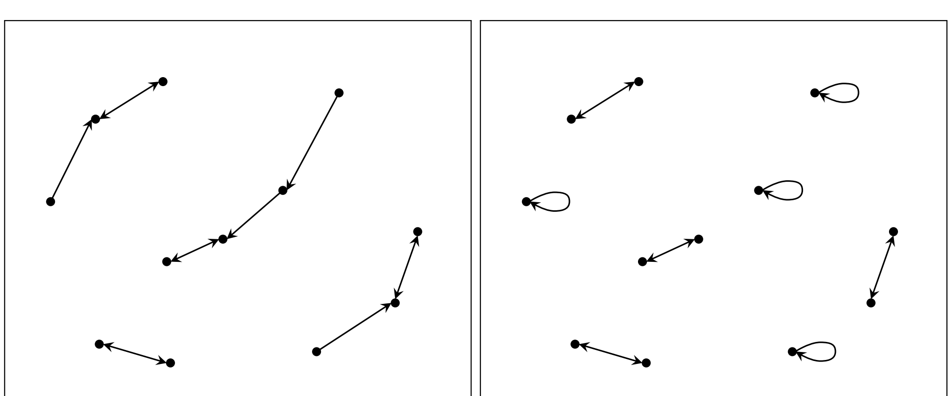

FIG. 2. *Left*: *the closest point-shift C. Right*: *the mutual-neighbor point-shift N. The directed edge emanating from a point indicates the image of the point.*

1.6. *Mecke's point stationarity theorem.* One of the motivations of this work is to extend the following proposition proved by J. Mecke in [14].

THEOREM 1.3 (Point stationarity). *Let $(\Phi, \mathbb{P})$ be a simple stationary point process and let F be a point-shift such that $F(\Phi, \cdot)$ is $\mathbb{P}$-a.s. bijective. Then the Palm probability of the point-process is invariant under the action of θ_f, that is,*

$$\mathbb{P}_\Phi = (\theta_{f(\Phi)})_* \mathbb{P}_\Phi, \tag{1.8}$$

with $\theta_{f(\Phi)}$ seen as a map from Ω to itself defined by

$$(\theta_{f(\Phi)})(\omega) := \theta_{f(\Phi(\omega))}\omega.$$

Since $\mathbb{P}_\Phi[\Phi(\{0\}) > 0] = 1$, $\theta_{f(\Phi)}$ is $\mathbb{P}_\Phi$-almost surely well-defined.

REMARK 1.4. The fact that θ_f is bijective $\Phi_* \mathbb{P}_\Phi$-a.s. is equivalent to the fact that F is bijective on $\Phi_* \mathbb{P}$-almost all realizations of the point process.

2. Results.

2.1. *Semigroup actions of a point-map.* Below, $\mathbf{N}^0 = \mathbf{N}^0(\mathbb{R}^d)$ and $\mathbf{M}^1(\mathbf{N}^0)$ denotes the set of probability measures on $\mathbf{N}^0$. For all point-maps f on $\mathbf{N}^0$, consider the following actions $\pi = \{\pi_n\}$ of $(\mathbb{N}, +)$:

1. $X = \mathbf{N}^0$, equipped with the vague topology, and for all $\phi \in \mathbf{N}^0$ and $n \in \mathbb{N}$,

$$\pi_n(\phi) = \theta_f^n(\phi) \in \mathbf{N}^0,$$

where θ_f^n is defined in Section 1.4.2.

2. $X = \mathbf{M}^1(\mathbf{N}^0)$, equipped with the weak convergence of probability measures on $\mathbf{N}^0$, and for all $\mathcal{Q} \in \mathbf{M}^1(\mathbf{N}^0)$ and $n \in \mathbb{N}$,

$$\pi_n(\mathcal{Q}) = (\theta_f)_*^n \mathcal{Q} = (\theta_f^n)_* \mathcal{Q} \in \mathbf{M}^1(\mathbf{N}^0).$$

2.2. *Periodicity and evaporation.* The point-map f will be said to be *periodic* on the stationary point process $(\Phi, \mathbb{P})$ if for $\Phi_* \mathbb{P}_\Phi$-almost all ϕ, the action of θ_f^n is periodic on ϕ, namely if there exists integers $p = p(\phi)$ and $K = K(\phi)$ such that for all $n \geq K$, $\theta_f^n(\phi) = \theta_f^{n+p}(\phi)$. The case where p is independent of ϕ is known as p-periodicity. The special case of 1-periodicity is that where, $\theta_f^n(\phi)$ is *stationary* (in the dynamical system sense) after some steps, that is, such that for all $n > K(\phi)$, $\theta_f^n(\phi) = \theta_f^K(\phi)$. Note that if for all $x \in \phi$, the trajectory $F^n(\phi, x)$ is stationary, that is, such that for all $n > K(\phi, x)$, $F^n(\phi, x) = F^K(\phi, x)$, then f is 1-periodic.

The mutual-neighbor point-map n on a homogeneous Poisson point process is 2-periodic.

Similarly, for the closest point-map c, the iterates of this point-shift form a *descending chain*, namely a sequence of point of the support of the point process such that the distance between the $k+1$-st and the kth is non-increasing in $k \geq 0$. The well-known fact that there are no infinite descending chains in the homogeneous Poisson point process (see [6]) implies that c is 2-periodic on such a point process, with the points of the period being mutual-neighbors.

If g is the strip point-map on the random geometric graph defined in Section 1.5.2, the strong Markov property of the stationary Poisson point process on $\mathbb{R}^d$ (see [18] for details on the strong Markov property of Poisson point process) gives that the point process on the right half-plane of $G(0)$ is distributed as the original Poisson point process. Hence, G is a.s. 1-periodic, even when the underlying random geometric graph is supercritical.

REMARK 2.1. Note that there are other ways of defining periodicity, possibly leading to other periods. For instance, for the mutual-neighbor point-map on a Poisson point process, the sequence of image point processes $\{\Phi_n^f\}_{n\geq 0}$ (defined in Section 1.4.3) is 1-periodic whereas f is 2-periodic according to the definition proposed above.

The point process $(\Phi, \mathbb{P})$ will be said to *evaporate* under the action of the point-map f if $\overline{\Phi_f^n}$ converges a.s. to the null measure as n tends to infinity, that is, for $\mathbb{P}$-almost surely, the set

$$\overline{\Phi_f^\infty} := \bigcap_{n=1}^{\infty} \overline{\Phi_f^n} \tag{2.1}$$

is equal to the empty set (note that $\overline{\Phi_f^n}$ is a nonincreasing sequence of sets). Consider the following set:

$$\begin{aligned} I &:= \{\phi \in \mathbf{N}^0; \forall n \in \mathbb{N}, F^{-n}(\phi, 0) \neq \varnothing\} \\ &= \{\phi \in \mathbf{N}^0; \forall n \in \mathbb{N}, m_f^n(\phi, 0) > 0\} \end{aligned} \tag{2.2}$$

[see Section 1.4.2 for the definition of $F^{-n}(\phi, y)$ and Section 1.4.3 for that of m_f^n].

LEMMA 2.2. *For all point-maps f, the stationary point process $(\Phi, \mathbb{P})$ evaporates under the action of f if and only if $\mathbb{P}_\Phi[\Phi \in I] = 0$.*

PROOF. Let $\mathcal{P} = \Phi_* \mathbb{P}$ and $\mathcal{P}_0 = \Phi_* \mathbb{P}_\Phi$. If $\chi(\phi, x)$ is the indicator of the fact that x has F-pre-images of all orders, then χ is a compatible marking of the point process [i.e., $\chi(\phi, x) = \chi(\theta_x \phi, 0)$ for all $x \in \phi$]. Therefore, if Ψ denotes the sub-point process of the points with mark χ equal to 1, then $(\Psi, \mathbb{P})$ is a stationary point process and by Campbell's theorem,

$$\lambda_\Psi = \lambda_\Phi \mathbb{E}_\Phi[\chi(\Phi, 0)] = \lambda_\Phi \mathbb{P}_\Phi[\Phi \in I]. \tag{2.3}$$

The evaporation of $(\Phi, \mathbb{P})$ by f means that Ψ has zero intensity. According to (2.3) this is equivalent to $\mathbb{P}_\Phi[\Phi \in I] = 0$. $\square$

The homogeneous Poisson point process on $\mathbb{R}^2$ evaporates under the action of the strip point-map s (see Section 3).

2.3. *Action of* $(\theta_f)_*$.

2.3.1. *Image palm probabilities.* Let Φ be a stationary point process on $\mathbb{R}^d$ and f be a point-map. Consider the action of $(\theta_f)_*$ [see equation (1.1)] when $\mathcal{Q} = \mathcal{P}_0$, the Palm distribution of Φ. It follows from the definition and from (1.3) that, for all $n \geq 1$, for all $G \in \mathcal{N}$ and for all Borel sets B with nondegenerate Lebesgue measure

$$(2.4) \qquad (\theta_f^n)_* \mathcal{P}_0[G] = \frac{1}{\lambda |B|} \int_{\mathbf{N}} \int_B \mathbf{1}\{\theta_f^n \circ \theta_t(\phi) \in G\} \phi(\mathrm{d}t) \mathcal{P}[\mathrm{d}\phi].$$

In what follows, $\mathcal{P}_0^{f,n}$ is a short notation for the probability on $\mathbf{N}^0$ defined in the last equation. This probability will be referred to as the *nth image Palm probability (w.r.t. f)* of the point process.

It follows from the semigroup property (1.6) that

$$(2.5) \qquad (\theta_f)_* \mathcal{P}_0^{f,n} = \mathcal{P}_0^{f,n+1} \qquad \forall n \in \mathbb{N},$$

when letting $\mathcal{P}_0^{f,0} := \mathcal{P}_0$. From the mass transport relation [13], and using the image counting measure ϕ_f^n defined in Section 1.4.3, one gets the following.

LEMMA 2.3. *For all* $n \geq 0$, *and all* $G \in \mathcal{N}$,

$$(2.6) \qquad \mathcal{P}_0^{f,n}[G] = \frac{1}{\lambda |B|} \int_{\mathbf{N}} \int_B \mathbf{1}\{\theta_t \phi \in G\} \phi_f^n(\mathrm{d}t) \mathcal{P}[\mathrm{d}\phi].$$

Note that, in general, the nth image Palm probability $\mathcal{P}_0^{f,n}$ is *not* the Palm probability of the nth image point process Φ_f^n (which is the distribution of Φ_f^n given that the origin belongs to Φ_f^n when using the local interpretation of the Palm probability). It is rather is the distribution of Φ given that the origin is in the nth image process. In both cases, point multiplicities should be taken into account.

REMARK 2.4. Equation (2.6) has several important implications:

1. If $\mathcal{P}$ is the distribution of a simple stationary point process, equation (2.6) gives

$$(2.7) \qquad \mathcal{P}_0^{f,n}[G] = \mathcal{E}_0[m_f^n 1_G] \qquad \forall G,$$

with m_f^n the random variable $m_f^n(\phi, 0)$ [see equation (1.7)] and 1_G the indicator function $1_G(\phi)$. So taking $G = \mathbf{N}_0$ gives

$$\mathcal{E}_0[m_f^n] = 1, \tag{2.8}$$

which shows that, $\mathcal{P}_0$ a.s., $m_f^n(\phi) < \infty$. This in turn implies that, $\mathcal{P}$ a.s., for all $y \in \phi$, $m_f^n(\phi, y) < \infty$.

2. Equation (2.8) together with the Campbell–Mecke formula imply that the intensity of Φ_f^n is equal to that of Φ, as already mentioned.

3. Equation (2.7) shows that $\mathcal{P}_0^{f,n}$ is absolutely continuous w.r.t. $\mathcal{P}_0$, with the Radon–Nikodym derivative

$$m_f^n := m_f^n(\phi, 0).$$

PROPOSITION 2.5. *For all simple point processes $\mathcal{P}$, for all n and G,*

$$\mathcal{P}_0^{f,n}[G] = \mathcal{E}_0\left[\frac{m_f^n}{\mathcal{E}_0[m_f^n \mid m_f^n > 0]} 1_G \,\middle|\, m_f^n > 0\right]. \tag{2.9}$$

PROOF. Equation (2.6) implies that

$$\mathcal{P}_0^{f,n}[G] = \mathcal{E}_0[m_f^n 1_G] = \mathcal{E}_0[m_f^n 1_G 1_{m_f^n > 0}].$$

Taking $G = \mathbf{N}_0$ gives

$$\mathcal{P}_0[m_f^n > 0] = \frac{1}{\mathcal{E}_0[m_f^n \mid m_f^n > 0]}.$$

Equation (2.9) then follows immediately. □

2.3.2. *Definition and existence of point-map-probabilities.*

DEFINITION 2.6. Let f be a point-map and let $\mathcal{P}$ be a stationary point process with Palm distribution $\mathcal{P}_0$. Every element of the ω-limit set [see (B.1)] of $\mathcal{P}_0$ (where limits are w.r.t. the topology of the convergence in distribution of probability measures on $\mathbf{N}^0$—cf. Appendix A) under the action of $\{(\theta_f^n)_*\}_{n\in\mathbb{N}}$ will be called a *f-probability of* $\mathcal{P}_0$. In particular, if the limit of the sequence $((\theta_f^n)_*\mathcal{P}_0)_{n=1}^{\infty} = (\mathcal{P}_0^{f,n})_{n=1}^{\infty}$ exists, it will be called *the* f-probability of $\mathcal{P}_0$ and denoted by $\mathcal{P}_0^f$.

Let $A_{\mathcal{P}_0}$ denote the orbit of $\mathcal{P}_0$. The set of f-probabilities of $\mathcal{P}_0$ is hence the set of all accumulation points of the closure $\mathrm{cl}(A_{\mathcal{P}_0})$ of $A_{\mathcal{P}_0}$, or equivalently the elements of $\mathbf{M}^1(\mathbf{N}^0)$ the neighborhoods of which contain infinitely many elements of $A_{\mathcal{P}_0}$; see the definitions in Section 1.

REMARK 2.7. In view of (2.7), for all $\mathcal{P}$ simple, the existence of a unique f-probability $\mathcal{P}_0^f$ is equivalent to

$$\lim_{n\to\infty}\int_{\mathbf{N}_0} h(\phi)m_f^n(\phi)\mathcal{P}_0(\mathrm{d}\phi)=\int_{\mathbf{N}_0} h(\phi)\mathcal{P}_0^f(\mathrm{d}\phi), \tag{2.10}$$

for all bounded and continuous functions $h:\mathbf{N}_0\to\mathbb{R}$.

COROLLARY 2.8. *Let $\mathcal{Q}$ be a f-probability. Let I be the set defined in* (2.1). *If for all positive integers n, $\phi\to\mathbf{1}_{m_f^n(\phi)>0}$ is $\mathcal{Q}$-a.s. continuous, then $\mathcal{Q}[I]=1$.*

PROOF. The statement is an immediate consequence of Lemma A.4. □

The relative compactness of $A_{\mathcal{P}_0}$ (and the existence of f-probabilities) is not granted in general. The next lemmas give conditions for this relatively compactness to hold. From Lemma 4.5. in [10], one gets the following.

LEMMA 2.9. *A necessary and sufficient condition for the set $A_{\mathcal{P}_0}$ to be relatively compact in $\mathbf{M}^1(\mathbf{N}^0(\mathbb{R}^d))$ is that for all bounded Borel subsets B of $\mathbb{R}^d$,*

$$\lim_{r\to\infty}\limsup_{n\to\infty}\mathcal{P}_0^{f,n}\left[\phi\in\mathbf{N}^0 \text{ s.t. } \phi(B)>r\right]=0. \tag{2.11}$$

Examples of point-map and point process pairs where the last relative compactness property does not hold are provided in Section 3.4. On stationary point processes, all the point-maps discussed in Section 1.5.1 satisfy this relative compactness property. For the periodic cases (e.g., c, n and g on Poisson point processes), the result follows from Proposition 2.10 below, whereas for the strip point-map s, the proof is given in Section 3.3.

The point-map f will be said to have *finite orbits* on the stationary point process $(\Phi,\mathbb{P})$ if for $\Phi_*\mathbb{P}_\Phi$-almost all ϕ, $\{\theta_f^n(\phi)\}_{n\in\mathbb{N}}$ is finite.

PROPOSITION 2.10. *If f has finite orbits on the stationary point process $(\Phi,\mathbb{P})$, then the set $A_{\mathcal{P}_0}$ is relatively compact.*

PROOF. For all bounded Borel subsets B of $\mathbb{R}^d$ and $\phi\in\mathbf{N}^0$, let

$$R_B(\phi):=\max_{n=0}^{\infty}\left\{\left(\theta_f^n\phi\right)(B)\right\}.$$

Since f has finite orbits, the RHS is the maximum over finite number of terms, and hence R_B is well-defined and finite. Clearly, $R_B(\phi)\geq R_B(\theta_f\phi)$ and, therefore, the distribution of the random variable R_B under $\mathcal{P}_0^{f,n}$ is stochastically decreasing w.r.t. n. Hence,

$$\begin{aligned}\lim_{r\to\infty}\limsup_{n\to\infty}\mathcal{P}_0^{f,n}\big[\phi\in\mathbf{N}^0 \text{ s.t. } \phi(B)>r\big]\\ \le \lim_{r\to\infty}\limsup_{n\to\infty}\mathcal{P}_0^{f,n}\big[\phi\in\mathbf{N}^0 \text{ s.t. } R_B(\phi)>r\big]\\ \le \lim_{r\to\infty}\mathcal{P}_0\big[\phi\in\mathbf{N}^0 \text{ s.t. } R_B(\phi)>r\big]=0.\end{aligned}$$ □

REMARK 2.11. It is easy to check that the following statements are equivalent. (1) f has finite orbits, (2) f is periodic and (3) for $\Phi_*\mathbb{P}$-almost all ϕ, for all $x\in\phi$, $\{F^n(\phi,x)\}_{n\in\mathbb{N}}$ has finitely many different points.

So, for instance, for the directional point-map on the random geometric graph g, $A_{\mathcal{P}_0}$ is relatively compact as this point-map is 1-periodic.

2.4. *On palm and point-map-probabilities*. This subsection is focused on the relation between Palm probabilities and point-map-probabilities. Throughout the subsection, f is a point-map, and $(\Phi,\mathbb{P})$ is a simple and stationary point process with nondegenerate intensity. The distribution of Φ is denoted by $\mathcal{P}$ and its Palm probability by $\mathcal{P}_0$.

2.4.1. *Conditional interpretation of the point-map-probability*. The next theorem, which immediately follows from equation (2.9), gives a conditional definition of the f-probability from $\mathcal{P}_0$.

THEOREM 2.12. *Let $\mathcal{P}$ be a simple stationary point process on $\mathbb{R}^d$. For all $n\in\mathbb{N}$ and $\phi\in\mathbf{N}$, let $m_f^n(\phi):=m_f^n(\phi,0)$. For all n, $m_f^n(\phi)$ is $\mathcal{P}_0$ a.s. finite. If there exists a unique f-probability $\mathcal{P}_0^f$ for $\mathcal{P}$, then for all G such that $\mathcal{P}_0^{f,n}[G]$ tends to $\mathcal{P}_0^f[G]$ as n tends to infinity, one has*

$$\mathcal{P}_0^f[G]=\lim_{n\to\infty}\mathcal{E}_0\bigg[\frac{m_f^n}{\mathcal{E}_0[m_f^n\mid m_f^n>0]}1_G\,\bigg|\,m_f^n>0\bigg]. \tag{2.12}$$

Notice that, in addition to the conditioning, there is a Radon–Nikodym derivative (w.r.t. $\mathcal{P}_0[\cdot\mid m_f^n>0]$) equal to $m_f^n(\phi)/\mathcal{E}_0[m_f^n\mid m_f^n>0]$.

2.4.2. *The periodic case*. Below, for all stationary point processes $(\Psi,\mathbb{P})$ defined on $(\Omega,\mathcal{F})$ with a positive intensity, $\mathbb{P}_\Psi$ denotes the Palm probability w.r.t. Ψ on $(\Omega,\mathcal{F})$.

LEMMA 2.13. *If f is 1-periodic on the (simple) stationary point process $(\Phi,\mathbb{P})$, then a.s., for all $x\in\Phi$, $\lim_n\Phi_f^n(\{x\})$ exists and is finite.*

PROOF. If x is a trap of Φ, that is, $F(\Phi, x) = x$, then $(\Phi_n^f(x))_{n=1}^{\infty}$ is nondecreasing in n. Let Ψ be the thinning of Φ to traps of Φ for which the above limit is not finite. The compatibility of F implies that $(\Psi, \mathbb{P})$ is a stationary point process. If B is the unit box in $\mathbb{R}^d$ and K is a positive integer, for n large enough, one has

$$\lambda_\Phi = \int_\Omega \Phi_f^n(B)\mathbb{P}(\mathrm{d}\omega) \geq \int_\Omega \Phi_f^n(\Psi \cap B)\mathbb{P}(\mathrm{d}\omega) \geq \int_\Omega K\Psi(B)\mathbb{P}(\mathrm{d}\omega) = K\lambda_\Psi,$$

where λ_Φ and λ_Ψ denote the intensities of the point processes. Therefore, $\lambda_\Psi \leq \lambda_\Phi/K$, which proves that $\lambda_\Psi = 0$. Hence, a.s., at the traps of Φ, the limit exists and is finite. Given this, it is easy to verify that if $y \in \Phi$ is not a trap, for n large enough, $\Phi_f^n(y) = 0$ and hence the limit exists for all points of Φ. $\square$

When $\lim_n \phi_f^n$ exists and is a counting measure, it is denoted it by Φ_f^∞. Hence, in the 1-periodic case, $(\Phi_f^\infty, \mathbb{P})$ is well-defined and a nondegenerate stationary point process.

THEOREM 2.14. *If f is* 1*-periodic on* $(\Phi, \mathbb{P})$, *then the f-probability $\mathcal{P}_0^f$ of $\mathcal{P}_0 = \Phi_* \mathbb{P}_\Phi$ exists and is given by*

$$\mathcal{P}_0^f = \Phi_* \mathbb{P}_{\Phi_f^\infty}. \tag{2.13}$$

Let $m_f^\infty(\Phi)$ denote the multiplicity of the origin under $\mathbb{P}_{\Phi_f^\infty}$. Then $\mathcal{P}_0^f$ is absolutely continuous with respect to $\mathcal{P}_0$, with

$$\frac{\mathrm{d}\mathcal{P}_0^f}{\mathrm{d}\mathcal{P}_0}(\phi) = m_f^\infty(\phi). \tag{2.14}$$

In addition, $\mathcal{P}_0^f = (\theta_f)_* \mathcal{P}_0^f$.

PROOF. In the 1-periodic case, for all bounded Borel sets B, $\Phi_f^n(B)$ a.s. coincides with $\Phi_f^\infty(B)$ for n large enough, so that by letting n to infinity in (2.6), one gets that for all $G \in \mathcal{N}$, the limit

$$\lim_n \mathcal{P}_0^{f,n}[G] = \frac{1}{\lambda|B|} \int_{\mathbf{N}} \int_B \mathbf{1}\{\theta_t \phi \in G\} \phi_f^\infty(\mathrm{d}t) \mathcal{P}[\mathrm{d}\phi] \tag{2.15}$$

exists. Since ϕ_f^∞ is a stationary point process with the same intensity as the original point process (because of the conservation of intensity), $\mathbb{P}_\Phi^f$ is the distribution of Φ with respect to the Palm distribution of Φ_f^∞ indeed. In addition, for all $H \in \mathcal{F}$

$$\begin{aligned}
\mathbb{P}_{\Phi_f^\infty}[H] &= \frac{1}{\lambda|B|} \int_\Omega \int_B \mathbf{1}\{\theta_t \omega \in H\} \Phi_f^\infty(\omega, \mathrm{d}t) \mathbb{P}[\mathrm{d}\omega] \\
&= \frac{1}{\lambda|B|} \int_\Omega \int_B \Phi_f^\infty(\omega, \{t\}) \mathbf{1}\{\theta_t \omega \in H\} \Phi(\omega, \mathrm{d}t) \mathbb{P}[\mathrm{d}\omega]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\lambda|B|}\int_\Omega\int_B \Phi_f^\infty(\theta_t\omega,\{0\})\mathbf{1}\{\theta_t\omega\in H\}\Phi(\omega,\mathrm{d}t)\mathbb{P}[\mathrm{d}\omega] \\
&= \mathbb{E}_\Phi\big[\Phi_f^\infty(\{0\})\mathbf{1}_H(\Phi)\big] = \mathbb{E}_\Phi\big[m_f^\infty(\Phi)\mathbf{1}_H(\Phi)\big],
\end{aligned}
$$

where the second equality stems from the facts that $\overline{\Phi_f^\infty}\subset\Phi$ and that Φ is simple. This proves (2.14) when $H=\Phi^{-1}G$. Finally, since f is 1-periodic, $\mathcal{P}_0^f$-almost surely, $f\equiv 0$ which proves that $\mathcal{P}_0^f$ is invariant under the action of $(\theta_f)_*$. $\square$

The point-map g provides an examples where Theorem 2.14 holds. See Section 3.5. Note that similar statements hold in the p-periodic case. In this case, f^p is 1-periodic on the point processes $\{(\Phi,\mathcal{P}_0^{f,n})\}_{n=0}^{p-1}$, and hence there exists at most p point-map-probabilities. Details on this fact are omitted.

2.4.3. *The evaporation case.* The next theorem shows that in contrast to Theorem 2.14 where the f-probability is absolutely continuous with respect to the Palm probability, there are cases where the f-probability and the Palm probability are singular. This theorem is based on the following lemma.

LEMMA 2.15. *Let I be the set defined in* (2.2). *If $\mathcal{Q}$ is a probability distribution on $\mathbf{N}^0$ which satisfies $(\theta_f)_*\mathcal{Q}=\mathcal{Q}$, then $\mathcal{Q}[I]=1$. In this case, $\mathcal{Q}$-almost surely, there exists a bi-infinite path* (*which can be a periodic orbit*) *which passes through the origin, that is, $\{y_i=y_i(\phi)\}_{i\in\mathbb{Z}}$ is such that $y_0=0$ and $F(\phi,y_i)=y_{i+1}$.*

PROOF. Let $M_n:=\{\phi\in\mathbf{N}^0;F^{-n}(\phi,0)=\varnothing\}$, where $F^n(\phi,\cdot)$ is defined in Section 1.4.2. It is sufficient to show that, for all $n>0$, $\mathcal{Q}[M_n]=0$. But the invariance of $\mathcal{Q}$ under the action of $(\theta_f)_*$ gives

$$
\begin{aligned}
\mathcal{Q}[M_n] &= (\theta_f)_*^n\mathcal{Q}[M_n] \\
&= \mathcal{Q}\big[(\theta_f)^{-n}M_n\big] \\
&= \mathcal{Q}\big[\{\phi\in\mathbf{N}^0;F^{-n}(\phi,F^n(\phi,0))=\varnothing\}\big]=0.
\end{aligned}
$$

The proof of the second statement is clear if the orbit of ϕ is periodic under the action of θ_f and if not, it is an immediate consequence of König's infinity lemma [12]. $\square$

THEOREM 2.16. *If the stationary point process $(\Phi,\mathbb{P})$ evaporates under the action of f, and if the f-probability $\mathcal{P}_0^f$ of $\mathcal{P}_0=\Phi_*\mathbb{P}_\Phi^0$ exists and satisfies $\mathcal{P}_0^f=(\theta_f)_*\mathcal{P}_0^f$, then $\mathcal{P}_0^f$ is singular with respect to $\mathcal{P}_0$.*

PROOF. The result is obtained when combining Lemmas 2.15 and 2.2. $\square$

It is shown in Section 3.3 that the assumptions of Theorem 2.16 are satisfied by the strip point-map s on Poisson point processes in $\mathbb{R}^2$.

REMARK 2.17. The case with evaporation is that where the conditioning representation given in equation (2.12) is w.r.t. an event whose probability w.r.t. $\mathcal{P}_0$ tends to 0 as n tends to infinity.

REMARK 2.18. The singularity property established in Theorem 2.16 can be completed by the following observation: under the assumptions of this theorem, there is no finite and measurable $U = U(\phi) \in \phi$ [resp., $V = V(\phi) \in \phi$] such that $\mathcal{P}_0^f = (\theta_U)_*\mathcal{P}$ [resp., $\mathcal{P}_0^f = (\theta_V)_*\mathcal{P}_0$], that is, there is no *shift-coupling* giving $\mathcal{P}_0^f$ as function of $\mathcal{P}$ (resp., $\mathcal{P}_0$). The proof is by contradiction: evaporation implies that $\mathcal{P}$ (resp., $\mathcal{P}_0$) a.s., $\theta_x\phi \notin I$ for all $x \in \phi$. But this together with $\mathcal{P}_0^f = (\theta_U)_*\mathcal{P}$ [resp., $\mathcal{P}_0^f = (\theta_V)_*\mathcal{P}_0$] imply that $\mathcal{P}_0^f[I] = 0$, which contradicts the fact that, under the assumptions of Theorem 2.16, $\mathcal{P}_0^f[I] = 1$.

2.5. *Mecke's point-stationarity revisited.*

2.5.1. *Mecke's invariant measure equation.*

Consider the following point-map invariant measure equation:

$$(\theta_f)_*\mathcal{Q} = \mathcal{Q}, \tag{2.16}$$

where the unknown is $\mathcal{Q} \in M^1(\mathbf{N}^0)$. From Mecke's point stationarity Theorem 1.3, if θ_f (or equivalently F) is bijective, then the Palm probability $\mathcal{P}_0$ of any simple stationary point process solves (2.16). From Theorem 2.14, if f is 1-periodic on $(\Phi, \mathbb{P})$, then the f-probability of Φ exists and from the last statement of this theorem, it satisfies (2.16). More precisely, a solution to (2.16) was built from the Palm probability $\mathcal{P}_0$ of Φ by equation (2.14).

Equation (2.16) will be referred to as *Mecke's invariant measure equation*. The bijective case shows that the solution of (2.16) is not unique in general (all Palm probabilities are solution).

A natural question is whether one can construct a solution of (2.16) from the Palm probability of a stationary point process beyond the bijective and the 1-periodic cases, for instance when Φ evaporates under the action of f.

Consider the Cesàro sums

$$\widetilde{\mathcal{P}}_0^{f,n} := \frac{1}{n}\sum_{i=0}^{n-1}\mathcal{P}_0^{f,i}, \qquad n \in \mathbb{N}. \tag{2.17}$$

When the limit of $\widetilde{\mathcal{P}}_0^{f,n}$ as n tends to infinity exists [w.r.t. the topology of $\mathbf{M}^1(\mathbf{N}^0)$], let

$$\widetilde{\mathcal{P}}_0^{f} := \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathcal{P}_0^{f,i}. \tag{2.18}$$

In general, $\widetilde{\mathcal{P}}_0^{f}$ is not a f-probability.

THEOREM 2.19. *Assume there exists a subsequence* $(\widetilde{\mathcal{P}}_0^{f,n_i})_{i=1}^{\infty}$ *which converges to a probability measure* $\widetilde{\mathcal{P}}_0^f$. *If* $(\theta_f)_*$ *is continuous at* $\widetilde{\mathcal{P}}_0^f$, *then* $\widetilde{\mathcal{P}}_0^f$ *solves Mecke's invariant measure equation* (2.16).

PROOF. From (2.5),

$$(2.19)\quad \begin{aligned}(\theta_f)_*\widetilde{\mathcal{P}}_0^{f,n} - \widetilde{\mathcal{P}}_0^{f,n} &= \frac{1}{n}\left(\sum_{i=0}^{n-1}(\theta_f)_*\mathcal{P}_0^{f,i} - \sum_{i=0}^{n-1}\mathcal{P}_0^{f,i}\right)\\ &= \frac{1}{n}\left(\sum_{i=1}^{n}\mathcal{P}_0^{f,i} - \sum_{i=0}^{n-1}\mathcal{P}_0^{f,i}\right) = \frac{1}{n}(\mathcal{P}_0^{f,n} - \mathcal{P}_0).\end{aligned}$$

Therefore, if the subsequence $(\widetilde{\mathcal{P}}_0^{f,n_i})_{i=1}^{\infty}$ converges in distribution w.r.t. the vague topology to a probability measure $\widetilde{\mathcal{P}}_0^f$, then (2.19) implies that the sequence $((\theta_f)_*\widetilde{\mathcal{P}}_0^{f,n_i})_{i=1}^{\infty}$ converges to $\widetilde{\mathcal{P}}_0^f$ as well. Now the continuity of $(\theta_f)_*$ at $\widetilde{\mathcal{P}}_0^f$ implies that $((\theta_f)_*\widetilde{\mathcal{P}}_0^{f,n_i})_{i=1}^{\infty}$ converges to $(\theta_f)_*\widetilde{\mathcal{P}}_0^f$ and, therefore, $(\theta_f)_*\widetilde{\mathcal{P}}_0^f = \widetilde{\mathcal{P}}_0^f$. □

REMARK 2.20. Here are some comments on the last theorem:

1. A sufficient condition for the existence of a converging subsequence in Theorem 2.19 is the relative compactness condition of Lemma 2.9.
2. When the sequence $(\mathcal{P}_0^{f,n})_{n=1}^{\infty}$ converges to $\mathcal{P}_0^f$, then $(\widetilde{\mathcal{P}}_0^{f,n})_{n=1}^{\infty}$ converges to $\mathcal{P}_0^f$, too, and hence Theorem 2.19 implies the invariance of the f-probability $\mathcal{P}_0^f$ under the action of $(\theta_f)_*$, whenever $(\theta_f)_*$ has the required continuity.
3. If instead of $(\widetilde{\mathcal{P}}_0^{f,n})_{n=1}^{\infty}$, $(\mathcal{P}_0^{f,n})_{n=1}^{\infty}$ has convergent subsequences with different limits, that is, if the set of f-probabilities is not a singleton, then none of the f-probabilities satisfies (2.16). However, it follows from Lemma B.1 in Appendix B that if $(\theta_f)_*$ is continuous, and if $(\mathcal{P}_0^{f,n})_{n=1}^{\infty}$ is relatively compact, then the set of f-probabilities of $\mathcal{P}_0$ is compact, nonempty and $(\theta_f)_*$-invariant.
4. The conditions listed in Theorem 2.19 are all required. There exist point-maps f such that $(\widetilde{\mathcal{P}}_0^{f,n})_{n=1}^{\infty}$ has no convergent subsequence (see Section 3.4); there also exist point-maps f such that $(\mathcal{P}_0^{f,n})_{n=1}^{\infty}$ is convergent, but $(\theta_f)_*$ is not continuous at the limit and $\mathcal{P}_0^f$ is not invariant under the action of $(\theta_f)_*$ (see Section 3.1). The use of Cesàro limits is required too as there exist point-maps f such that $(\mathcal{P}_0^{f,n})_{n=1}^{\infty}$ is not convergent, whereas $(\widetilde{\mathcal{P}}_0^{f,n})_{n=1}^{\infty}$ converges to a limit which satisfies (2.16) (see Section 3.6).

2.5.2. *Continuity condition.* In case of existence of $\widetilde{\mathcal{P}}_0^f$, Theorem 2.19 gives a sufficient condition for $\widetilde{\mathcal{P}}_0^f$ to solve (2.16); however since $\widetilde{\mathcal{P}}_0^f$ lives in the space of probability measures on counting measures, the verification of the continuity of $(\theta_f)_*$ at $\widetilde{\mathcal{P}}_0^f$ can be difficult. The following propositions give more handy tools to verify the continuity criterion.

PROPOSITION 2.21. *If θ_f is $\widetilde{\mathcal{P}}_0^f$-a.s. continuous, then $(\theta_f)_*$ is continuous at $\widetilde{\mathcal{P}}_0^f$.*

PROOF. The proof is an immediate consequence of Proposition A.6 in Appendix A, as the space $\mathbf{N}(\mathbb{R}^d)$ is a Polish space. □

PROPOSITION 2.22. *If f is $\widetilde{\mathcal{P}}_0^f$-almost surely continuous, then $(\theta_f)_*$ is $\widetilde{\mathcal{P}}_0^f$-continuous.*

PROOF. One can verify that $\theta : \mathbb{R}^d \times \mathbf{N} \to \mathbf{N}$ defined by $\theta(t, \phi) = \theta_t \phi$ is continuous. Also $h : \mathbf{N}^0 \to \mathbb{R}^d \times \mathbf{N}$ defined by $h(\phi) = (f(\phi), \phi)$ is continuous at continuity points of f in $\mathbf{N}^0$. Hence, $\theta_f = \theta \circ h$ is continuous at continuity points of f. □

The converse of the statement of Proposition 2.22 does not hold in general (see Section 3.4). Combining the last propositions and Theorem 2.19 gives:

COROLLARY 2.23. *If the limit $\widetilde{\mathcal{P}}_0^f$ defined in* (2.18) *exists and if in addition f is $\widetilde{\mathcal{P}}_0^f$-almost surely continuous, then $(\theta_f)_*$ is continuous at $\widetilde{\mathcal{P}}_0^f$, and $\widetilde{\mathcal{P}}_0^f$ then solves Mecke's invariant measure equation* (2.16).

In Theorem 2.19 and the last propositions, the continuity of the mapping $(\theta_f)_*$ is required at some specific point only. The continuity of f is a stronger requirement which does not hold for most interesting cases as shown by the following proposition (see Appendix C for a proof).

PROPOSITION 2.24. *For $d \geq 2$, there is no continuous point-map on the whole space $\mathbf{N}^0$ other than the point-map of the identity point-shift, that is, the point-map which maps all $\phi \in \mathbf{N}^0$ to the origin.*

2.5.3. *Regeneration.* In certain cases, the existence of $\mathcal{P}_0^f$ can be established using the theory of regenerative processes [1]. This method can be used when the point process satisfies the *strong Markov property* such as Poisson point processes [15].

Assume f is a fixed point-map and $(\Phi, \mathbb{P}_\Phi)$ is the Palm version of a stationary point process. For $n \geq 0$, let

$$X_n = X_n(f, \Phi) = f^n(\Phi) \in \mathbb{R}^d, \tag{2.20}$$

where f^n is defined in (1.5). Note that $\mathbb{P}_\Phi$-almost surely, $\Phi \in \mathbf{N}^0$, and hence X_n is well-defined. Finally, denote $\theta_{X_n}\Phi$ by Φ_n (this point process should not be confused with Φ_f^n defined in Section 1.4.3) and by Φ_n^r the restriction of Φ_n to the sphere of radius r centered at the origin. Using this notation, Lemma 2.3 gives $(\theta_{X_n})_*\mathcal{P}_0 = \mathcal{P}_0^{f,n}$ or equivalently $(\Phi_n)_*\mathbb{P}_\Phi = \mathcal{P}_0^{f,n}$.

The following theorem leverages classical results in the theory of regenerative processes.

THEOREM 2.25. *If, for all $r > 0$, there exists a strictly increasing sequence of nonlattice integer-valued random variables $(\eta_i)_{i=1}^\infty$, which may depend on r, such that*

1. *$(\eta_{i+1} - \eta_i)_{i=1}^\infty$ is a sequence of i.i.d. random variables with finite mean,*
2. *the sequence $Y_i := (\Phi_{\eta_i}^r, \Phi_{\eta_i+1}^r, \dots, \Phi_{\eta_{i+1}-1}^r)$ is an i.i.d. sequence and Y_{i+1} is independent of $\eta_1, \dots, \eta_i$,*

then the f-probability $\mathcal{P}_0^f$ exists and, for all bounded and measurable functions h and for $\mathcal{P}_0$-almost all ϕ,

$$\lim_{k\to\infty} \frac{1}{k} \sum_{n=0}^{k-1} h(\theta_f^n \phi) = \int_{\mathbf{N}^0} h(\psi)\mathcal{P}_0^f(\mathrm{d}\psi). \tag{2.21}$$

If in addition, for all n, f is $\mathcal{P}_0^{f,n}$-almost surely continuous, then $\mathcal{P}_0^f$ is invariant under the action of $(\theta_f)_*$ and θ_f is ergodic on $(\mathbf{N}^0, \mathcal{N}^0, \mathcal{P}_0^f)$.

PROOF. In order to prove the weak convergence of $\mathcal{P}_0^{f,n}$ to $\mathcal{P}_0^f$, it is sufficient to show the convergence in all balls of integer radius r around the origin. Note that $\mathcal{P}_0^{f,n}$ is the distribution of Φ_n and hence, to prove the existence of $\mathcal{P}_0^f$, it is sufficient to prove the convergence of the distribution of Φ_n^r for all $r \in \mathbb{N}$.

Note that the sequence $(\eta_i)_{i=1}^\infty$ forms a sequence of regenerative times for the configurations in $B_r(0)$. Since $\mathbf{N}^0$ is metrizable (cf. [1], Theorem B.1.2), the distribution of Φ_n^r converges to a distribution $\mathcal{P}_{0,r}^f$ on configurations of points in $B_r(0)$ satisfying

$$\frac{1}{\mathcal{E}_0[\eta_2 - \eta_1]}\mathcal{E}_0\left[\sum_{n=\eta_1}^{\eta_2-1} h(\Phi_n^r)\right] = \int_{\mathbf{N}^0} h(\psi \cap B_r(0))\mathcal{P}_{0,r}^f \,\mathrm{d}(\psi \cap B_r(0)), \tag{2.22}$$

for all $h : \mathbf{N}_0 \to \mathbb{R}^+$. Since the distributions $(\mathcal{P}_{0,r}^f)_{r=1}^\infty$ are the limits of $(\Phi_n^r)_{r=1}^\infty$, they satisfy the consistency condition of Kolmogorov's extension theorem and therefore there exists a probability distribution $\mathcal{P}_0^f$ on $\mathbf{N}^0$ having $\mathcal{P}_{0,r}^f$ as the distribution of its restriction to $B_r(0)$. This proves the existence of the f-probability.

The left-hand side of (2.22) can be replaced by an ergodic average (cf. [1] Theorem B.3.1), that is, for all $r \in \mathbb{N}$, for $\mathcal{P}_0$-almost all $\phi \in \mathbf{N}^0$,

$$\begin{aligned}\lim_{k\to\infty} \frac{1}{k}\sum_{n=0}^{k-1} h\big(\theta_f^n \phi \cap B_r(0)\big) &= \int_{\mathbf{N}^0} h\big(\psi \cap B_r(0)\big)\mathcal{P}_0^{f,r}\,\mathrm{d}\big(\psi \cap B_r(0)\big)\\ &= \int_{\mathbf{N}^0} h\big(\psi \cap B_r(0)\big)\mathcal{P}_0^f(\mathrm{d}\psi).\end{aligned}$$

Finally, r varies in the integers, and hence the last equation gives (2.21), for $\mathcal{P}_0$-almost all ϕ.

By defining h as the continuity indicator of f, the $\mathcal{P}_0^{f,n}$-almost sure continuity of f and (2.21) give its $\mathcal{P}_0^f$-almost sure continuity, and hence that of $(\theta_f)_*$ at $\mathcal{P}_0^f$. Therefore, $\mathcal{P}_0^f$ is invariant under the action of $(\theta_f)_*$. Also ergodicity is clear from regeneration. □

The main technical difficulty for using Theorem 2.25 consists in finding an appropriate sequence $(\eta_i)_{i=1}^\infty$. Proposition 3.1 below leverages the strong Markov property of Poisson point processes to find appropriate sequences and prove the existence of the point-map probability for the point-map s for homogeneous Poisson point processes. Proposition 3.2 uses the same approach to show that the same holds true for the directional point-map d_α. Other examples can be found in Section 3.

3. More on examples.

3.1. *Strip point-shift.* Let $\mathcal{P}_0$ denote the Palm distribution of the homogeneous Poisson point process on $\mathbb{R}^2$. It follows from results in [8] (in Theorem 3.1. of this reference, the authors proved that the graph of this point-shift has finite branches, which is equivalent to evaporation) that $\mathcal{P}_0$ evaporates under the action of the strip point-map s. It is also shown in Proposition 3.1 below that $\mathcal{P}_0$, admits a unique s-probability which satisfies the continuity requirements of Theorem 2.19.

This point-shift also allows one to illustrate the need of the continuity property in Theorem 2.19. Consider the setup of Proposition 3.1. For all $\phi \in \mathbf{N}^0$ such that the origin has infinitely many pre-images, change the definition of the point-map s as follows: it is now the closest point on the right half-plane which has no other point of ϕ in the ball of radius 1 around it. Due to evaporation, this changes the definition of s on a set of measure zero under $\mathcal{P}_0^{s,n}$, for all $n \in \mathbb{N}$, and hence, the sequence $(\mathcal{P}_0^{s,n})_{n=1}^\infty$ is again converging to the same limit as that defined in the proof of Proposition 3.1. But under the action of the new s, $(\theta_s)_*\mathcal{P}_0^s$ is not equal to $\mathcal{P}_0^s$ due to the facts that (i) 0 has infinitely many pre-images $\mathcal{P}_0^s$-a.s. and (ii) there is no point of the point process in the ball of radius 1. This does not agree with the fact that, in the right half-plane, the distribution of $\mathcal{P}_0^s$ is a Poisson point process (see the proof of Proposition 3.1). Hence, $\mathcal{P}_0^s$ is not invariant under the action of $(\theta_s)_*$.

3.2. *Directional point-shift*. The directional point-shift was introduced in [2]. Let e_1 be the first coordinate unit vector The *directional point-map* d maps the origin to the nearest point in the right half-space, defined by e_1, that is, for all $\phi \in \mathbf{N}^0$,

$$d(\phi) := \operatorname{argmin}\{\|y\|; y \in \phi, y \cdot e_1 > 0\}. \tag{3.1}$$

The associated point-shift will be denoted by D.

The directional point-map on $\mathbb{R}^2$ with deviation limit α, d_α, is similar to d, except that the point y is chosen in the cone with angle 2α and central direction e_1 rather than in a half-plane, that is,

$$d_\alpha(\phi, x) := \operatorname{argmin}\left\{\|y\|; y \in \phi, \frac{y}{\|y\|} \cdot u > \cos\alpha\right\}. \tag{3.2}$$

When $\alpha = \frac{\pi}{2}$ one has $d_\alpha = d$. Its point-shift is denoted by D_α.

When $\alpha < \pi/2$, it can be shown that the homogeneous Poisson point process on $\mathbb{R}^2$ evaporates under the action of d_α, and from Proposition 3.2 below, it admits a unique d_α-probability which satisfies the continuity requirements of Theorem 2.19.

3.3. *Regeneration*. This subsection is focused on the existence of point-map-probabilities for point-maps defined on Poisson point processes. It is based on Theorem 2.25 and is illustrated by two examples.

PROPOSITION 3.1. *If s is the strip point-map, and $(\Phi, \mathbb{P})$ is a homogeneous Poisson point process in the plane with distribution $\mathcal{P}$, then the s-probability exists and is given by* (2.21). *In addition, for all n, s is $\mathcal{P}_0^{s,n}$-almost surely continuous. Therefore, the action of $(\theta_s)_*$ preserves $\mathcal{P}_0^s$ and is ergodic.*

PROOF. The random vector, $X_1 = X_1(\Phi)$ defined in equation (2.20), depends only on the points of Φ which belong to the rectangle $R_0(\Phi) = [0, x_1] \times [-1, 1]$, where x_1 is the first coordinate of the left most point of $\Phi \cap T(0)$. It is easy to verify that $R_0(\Phi)$ is a stopping set (cf. [15] and [18]). Let $R_n(\Phi)$ be the rectangle which is needed to determine the image of the origin in $\theta_{X_n}\Phi$ under the action of S. Let $R_n + X_n$ be the translation of the set R_n by the vector X_n Then it is clear that

$$U_k = \bigcup_{n=0}^{k} (R_n + X_n) \tag{3.3}$$

is also a stopping set. As a consequence, the strong Markov property of Poisson point process (cf. [18]) implies, given $X_0, \dots, X_n$, the point process on the right half-plane of X_n is distributed as the original Poisson point process. Let

$$p_n = \pi_1(X_{n+1} - X_n),$$

where π_1 is the projection on the first coordinate. Since $\mathcal{P}_0^{s,n}$, restricted to the right half-plane, is the distribution of a Poisson point process and since the sequence $(p_n)_{n=1}^{\infty}$ depends only on the configuration of points in the right half-plane, $(p_n)_{n=1}^{\infty}$ is a sequence of i.i.d. exponential random variables with parameter 2λ, where λ is the intensity of the point process. Also if η_i is the integer n such that, for the ith time, p_n is larger than $2r$, then the sequence $(\eta_i)_{i=1}^{\infty}$ forms a sequence of regenerative times for configuration of points in $B_r(0)$. Combining this with the distribution of p_n gives that $(\eta_i)_{i=1}^{\infty}$ satisfies the required conditions in Theorem 2.25.

Finally, consider the discontinuity points of s. Let $\phi \in \mathbf{N}^1$ with $s(\phi) = x = (x_1, x_2)$. It is shown below that if ϕ is a discontinuity point of s, then either x lies on the boundary of $T(0)$ or there is a point of ϕ other than the origin and x which lies on the perimeter of the rectangle $[0, x_1] \times [-1, 1]$. This proves that, for all n, the discontinuity points of s are of $\mathcal{P}_0^{f,n}$-zero measure. To prove the continuity claim, assume that ϕ satisfies none of the above condition. Hence, there exists $\varepsilon > 0$ such that, $x_1 > \varepsilon$, $x_2 \in [-(1-\varepsilon), 1-\varepsilon]$ and there is no other point of ϕ in $[-\varepsilon, x_1 + 2\varepsilon], [-1-\varepsilon, 1+\varepsilon]$. Therefore, for $\psi \in \mathbf{N}^0$ close enough to ϕ in the vague topology, there is a point $y = (y_1, y_2) \in \psi$ in an ε-neighborhood of x, which gives $y \in (0, x_1 + \varepsilon) \times (-1, 1)$ and since there is no point of ψ other than 0 and y in $[0, x_1 + \varepsilon] \times [-1, 1]$, $s(\psi) = y$, which proves the claim.

Therefore, all conditions of Theorem 2.25 are satisfied, which proves the proposition. □

Note that the proof shows that the distribution $\mathcal{P}_0^s$ on the right half-plane is homogeneous Poisson with the original intensity.

PROPOSITION 3.2. *Let d_α be the directional point-map defined in Section 3.2 with $\alpha < \pi/2$. Under the assumptions of Proposition 3.1, the d_α-probability exists and is given by (2.21). In addition, for all n, d_α is $\mathcal{P}_0^{d_\alpha,n}$-almost surely continuous, and hence the action of $(\theta_{d_\alpha})_*$ preserves $\mathcal{P}_0^{d_\alpha}$ and is ergodic.*

PROOF. The proof is similar to that of Proposition 3.1, but more subtle. It uses the same notation as that of Theorem 2.25.

Let C^α denote the cone with angle 2α, central direction e_1, and apex at the origin. Let $X_1(\phi)$ be the point of $C^\alpha \cap \phi$ which is the closest to the origin (other than the origin itself). Let $C_0^\alpha(\phi)$ be the closed subset of C^α consisting of all points of C^α which are not farther to the origin than $X_1(\phi)$. This set will be referred to as a bounded cone below. One may verify that $C_0^\alpha(\phi)$ is a stopping set and that X_1 is determined by C_0^α. Let $C_n^\alpha(\phi)$ be the closed bounded cone which is needed to determine the image of the origin in $\theta_{X_n}\phi$ under the action of d_α. It is easy to verify that

$$U_k = \bigcup_{n=0}^{k} (C_n^\alpha + X_n), \tag{3.4}$$

is also a stopping set. It is a simple geometric fact that

$$U_{n-1} \cap C^{\pi/2-\alpha}(X_n) = \{X_n\}, \tag{3.5}$$

and as a consequence, given U_{n-1}, the point process in $C^{\pi/2-\alpha} + X_n$ is distributed as the original point process. This fact together with the facts that U_n is a stopping set and C_n^α has no point of the point process other than X_n and X_{n+1}, give that, in the nth step, with probability at least $\min\{1, (\pi/2 - \alpha)/(\alpha)\}$, X_{n+1} is in $C^{\pi/2-\alpha}(X_n)$. Let η_i be the ith time for which $X_{n+1} \in C^{\pi/2-\alpha}(X_n)$ and has a distance more than $2r$ from the edges of $C^{\pi/2-\alpha}(X_n)$. The Poisson distribution of points in $C^{\pi/2-\alpha}(X_n)$ gives that the random variables $\eta_{i+1} - \eta_i$ are stochastically bounded by an exponential random variable, and hence they satisfy all requirements of Theorem 2.25.

As in the case of the strip point-shift, it can be shown that if d_α is not continuous at $\phi \in \mathbf{N}^0$ then either there is no point in the interior of C^α or there is a point on the perimeter of $C_0^\alpha(\phi)$.

Note that since U_{n-1} is a stopping set and $(C^\alpha + X_n) \cap U_{n-1}$ has no point of the point process other than X_n, X_{n+1} is distributed as in a Poisson point process in $C_n^\alpha + X_n$ given the fact that some parts contain no point. Therefore, since the discontinuities of d_α are of probability zero under the Poisson distribution, they are of probability zero under all $\mathcal{P}_0^{d_\alpha,n}$, and hence Theorem 2.25 proves the statements of the proposition. □

The statement of Proposition 3.2 is also true in the case $\alpha = \pi/2$ and can be proved using ideas similar to those in the proof for $\alpha < \pi/2$. However, the technical details of the proof in this case may hide the main idea and this case is hence ignored in the proposition.

3.4. *Condenser and expander point-shift.* Assume each point $x \in \phi$ is marked with

$$\nu_p(x) = \#\big(\phi \cap B_1(x)\big) \qquad \big(\text{resp., } \nu_m(x) = \sup\big\{r > 0 : \phi \cap B_r(x) = \{x\}\big\}\big),$$

where $B_r(x) = \{y \in \mathbb{R}^2 : \|x - y\| < r\}$. Note that $\nu_p(x)$ and $\nu_m(x)$ are always positive. The *condenser point-shift P* (resp., *expander point-shift M*) acts on counting measures as follows: it goes from each point $x \in \phi$ to the closest point y such that $\nu_p(y) \geq 2\nu_p(x)$ [resp., $\nu_m(y) \geq 2\nu_m(x)$]. It is easy to verify that both point-shifts are compatible and almost surely well-defined on the homogeneous Poisson point process.

Poisson point processes evaporate under the action of both point-shifts P and M.

The condenser point-map provides an example where no f-probability exists. Let $(\mathrm{id}, \mathcal{P})$ be the Poisson point process with intensity one on $\mathbb{R}^2$ and let p be the condenser point-map. Clearly,

$$\mathcal{P}_0^{p,n}\big[\phi\big(B_1(0)\big) > 2^n\big] = 1.$$

Therefore, the tightness criterion is not satisfied, and thus there is no convergent subsequence of $(\mathcal{P}_0^{p,n})_{n=1}^{\infty}$.

Similarly, the expander point-map allows one to show that there is no converse to Proposition 2.22. More precisely, θ_m is continuous $\mathcal{P}_0^m$-almost surely but the point-map is $\mathcal{P}_0^m$-almost surely discontinuous. Hence, the converse of the statement of Proposition 2.22 does not hold in general. Consider m on the homogeneous Poisson point process. One can verify that $(\mathcal{P}_0^{m,n})_{n=1}^{\infty}$ converges to the probability measure concentrated on the counting measure δ_0 with a single point at the origin. In this example, θ_m is $\mathcal{P}_0^m$-a.s. continuous. This follows from the fact that when looking at the point process in any bounded subset of $\mathbb{R}^d$, it will be included in some ball of radius r around the origin and, therefore, the configuration of points in it will be constant (only one point at the origin) after finitely many application of θ_m. But the point-map m makes larger and larger steps, and hence the sequence of laws of m under $\mathcal{P}_0^{m,n}$ diverges. Hence, m is almost surely not continuous at the realization δ_0 on which $\mathcal{P}_0^m$ is concentrated.

3.5. *Closest hard core point-shift.* By definition, the image of $x \in \phi$ by the *closest hard core point-shift* H is the closest point y of ϕ (including x itself) such that $\phi(B_1(y)) = 1$. Its point-map will be denoted by h.

The point-map h is 1-periodic. It provides an illustration of Theorem 2.14. Consider h acting on a stationary Poisson point process of intensity one in the plane. For the simple counting measure ϕ, let $\Psi(\phi)$ denote sub-point process of ϕ made of points y of ϕ such that $\phi(B_1(y)) = 1$. If ϕ is chosen w.r.t. $\mathcal{P}$, then $\Psi(\phi)$ is also a stationary point process. Let $\mathcal{Q}_0$ denote the Palm probability of $\Psi(\phi)$. Then $\mathcal{P}_0^f$ is absolutely continuous w.r.t. $\mathcal{Q}_0$ and its Radon–Nikodym derivative at each $\Psi(\phi) \in \mathbf{N}^0$ is proportional to the number of points of ϕ in the Voronoi cell of the origin in $\Psi(\phi)$.

3.6. *Quadri-void grid point-shift.* Let $\psi = \mathbb{Z} \setminus 4\mathbb{Z}$, that is, those integers which are not multiple of 4. If U is a uniform random variable in $[0, 4)$, then $\psi + U$ is a stationary point process on the real line which will be called the quadri-void grid below. The Palm distribution of this point process has mass of $\frac{1}{3}$ on $\theta_1\psi$, $\theta_2\psi$ and $\theta_3\psi$.

Let q be the point-map defined by

$$q(\theta_1\psi) = 2, \qquad q(\theta_2\psi) = 1 \quad \text{and} \quad q(\theta_3\psi) = -2.$$

For odd values of $n > 0$, one has

$$\mathcal{P}_0^{q,n}[\phi = \theta_3\psi] = \frac{2}{3}, \qquad \mathcal{P}_0^{q,n}[\phi = \theta_1\psi] = \frac{1}{3},$$

whereas for even values of $n > 0$,

$$\mathcal{P}_0^{q,n}[\phi = \theta_3\psi] = \frac{1}{3}, \qquad \mathcal{P}_0^{q,n}[\phi = \theta_1\psi] = \frac{2}{3}.$$

Therefore, $(\mathcal{P}_0^{q,n})_{n=1}^{\infty}$ has two convergent subsequences with different limits, one for even and one for odd values of n, and none of these limits is invariant under the action of $(\theta_q)_*$. However, the sequence $(\widetilde{\mathcal{P}}_0^{q,n})_{n=1}^{\infty}$ converges to a limit $\widetilde{\mathcal{P}}_0^{q}$ which is the mean of the odd and even g-probabilities, that is,

$$\widetilde{\mathcal{P}}_0^{q}[\phi = \theta_3\psi] = \frac{1}{2}, \qquad \widetilde{\mathcal{P}}_0^{q}[\phi = \theta_1\psi] = \frac{1}{2},$$

and it is invariant under the action of $(\theta_q)_*$.

APPENDIX A: RANDOM MEASURES

This subsection summarizes the results about random measures which are used in this paper in order to have a self-contained paper. The interested reader should refer to [10, 11]. No proofs are given.

Let S be a locally compact (all points have a compact neighborhood) second countable (has a countable base) Hausdorff space. In this case, S is known to be Polish, that is, there exists some separable and complete metrization ρ of S.

Let $\mathcal{B}(S)$ be the Borel algebra of S and $\mathcal{B}_b(S)$ be all bounded elements of $\mathcal{B}(S)$, that is, all $B \in \mathcal{B}(S)$ such that the closure of B is compact. Let $\mathbf{M}(S)$ be the class of all Radon measures on $(S, \mathcal{B}(S))$, that is, all measures μ such that for all $B \in \mathcal{B}_b(S)$, $\mu B < \infty$ and let $\mathbf{N}(S)$ be the subspace of all $\mathbb{N}$-valued measures in $\mathbf{M}(S)$. The elements of $\mathbf{N}(S)$ are *counting measures*. For all μ in $\mathbf{M}(S)$, define

$$\mathcal{B}_b(S)^{\mu} := \{B \in \mathcal{B}_b(S); \mu(\partial B) = 0\}.$$

Let $C_b(S)$ [resp., $C_c(S)$] be the class of all continuous and bounded (resp., continuous and compact support) $h : S \to \mathbb{R}^+$. Let

$$\mu h := \int_S h(x)\mu(\mathrm{d}x),$$

where the latter is equal to $\sum_{x\in\mu} h(x)$ when μ is a counting measure. Note that in the summation one takes the multiplicity of points into account. The class of all finite intersections of $\mathbf{M}(S)$-sets [or $\mathbf{N}(S)$-sets] of the form $\{\mu : s < \mu h < t\}$ with real r and s and arbitrary $h \in C_c(S)$ forms a base of a topology on $\mathbf{N}(S)$ which is known as the *vague topology*. In the vague topology, $\mathbf{N}(S)$ is closed in $\mathbf{M}(S)$ ([10], page 94, A 7.4.). A necessary and sufficient condition for the convergence in this topology ([10], page 93) is

$$\mu_n \xrightarrow{v} \mu \quad \Leftrightarrow \quad \forall h \in C_c(S), \qquad \mu_n h \to \mu h.$$

If one considers the subspace of all bounded measures in $\mathbf{N}(S)$, one may replace $C_c(S)$ by $C_b(s)$. This leads to the *weak topology* for which

$$\mu_n \xrightarrow{w} \mu \quad \Leftrightarrow \quad \forall h \in C_b(S), \qquad \mu_n h \to \mu h.$$

The convergence in distribution of the random variables $\xi_1, \xi_2, \ldots,$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and taking their values in $(S, \mathcal{B}(S))$, to the random element ξ is defined as follows:

$$\xi_n \xrightarrow{d} \xi \quad \Leftrightarrow \quad (\xi_n)_* \mathbb{P} \xrightarrow{w} (\xi)_* \mathbb{P}.$$

The next lemma describes the relation between the convergences in the vague topology and the weak one.

LEMMA A.1 ([10], page 95, A 7.6). *For all bounded $\mu, \mu_1, \mu_2, \ldots \in \mathbf{M}(S)$, one has*

$$\mu_n \xrightarrow{w} \mu \quad \Leftrightarrow \quad \mu_n \xrightarrow{v} \mu \quad \text{and} \quad \mu_n S \to \mu S.$$

According to Lemma A.1, when discussing the convergence of probability measures, there is no difference between the vague and the weak convergence.

The following proposition is a key point in the development of the theory of random measures and random point processes ([10], page 95, A 7.7.).

PROPOSITION A.2. *Both $\mathbf{M}(S)$ and $\mathbf{N}(S)$ are Polish in the vague topology. Also the subspaces of bounded measures in $\mathbf{M}(S)$ and $\mathbf{N}(S)$ are Polish in the weak topology.*

Proposition A.2 allows one to define measures on $\mathbf{M}(S)$ or $\mathbf{N}(S)$ which are Polish spaces and use for them the theory available for S. If $\mathcal{M}$ (resp., $\mathcal{N}$) is the σ-algebra generated by the vague topology on $\mathbf{M}(S)$ [resp., $\mathbf{N}(S)$], a *random measure* (resp., *random point process*) on S is simply a random element of $(\mathbf{M}(S), \mathcal{M})$ [resp., $(\mathbf{N}(S), \mathcal{N})$]. Note that a random point process is a special case of a random measure.

The next theorem and lemmas give handy tools to deal with convergence in distribution of random measures on S.

THEOREM A.3 ([10], page 22, Theorem 4.2). *If $\mu, \mu_1, \mu_2, \ldots$ are random measures on S [i.e., random elements of $(\mathbf{M}(S), \mathcal{M})$], then*

$$\mu_n \xrightarrow{d} \mu \quad \Leftrightarrow \quad \mu_n h \xrightarrow{d} \mu h \qquad \forall h \in C_c(S).$$

LEMMA A.4 ([10], page 22, Lemma 4.4). *If $\mu, \mu_1, \mu_2, \dots$ are random measures on S satisfying $\mu_n \xrightarrow{d} \mu$, then $\mu_n h \xrightarrow{d} \mu h$ for every bounded measurable function $h : S \to \mathbb{R}^+$ with bounded support satisfying $\mu(D_h) = 0$ almost surely, where D_h is the set of all discontinuity points of h. Furthermore*,

$$(\mu_n B_1, \dots, \mu_n B_k) \xrightarrow{d} (\mu B_1, \dots \mu B_k), \qquad k \in \mathbb{N}, B_1, \dots B_k \in \mathcal{B}_b(S)^{\mu}.$$

LEMMA A.5 ([10], page 23, Lemma 4.5). *A sequence $(\mu_n)_{n=1}^{\infty}$ of random measures on S is relatively compact w.r.t. the convergence in distribution in the vague topology if and only if*

$$\lim_{t\to\infty} \limsup_{n\to\infty} \mathbb{P}[\mu_n B > t] = 0 \qquad \forall B \in \mathcal{B}_b(S).$$

Denote by $P(S)$ the set of all probability measures on S. Clearly, $P(S) \subset M(S)$ and according to Lemma A.1, the weak and the vague topologies on $P(S)$ coincide.

PROPOSITION A.6 ([4], page 30, Theorem 5.1). *If S and T are Polish spaces and $h : (S, \mathcal{B}(S)) \to (T, \mathcal{B}(T))$ is a measurable mapping, then h_* is continuous w.r.t. the weak topology at point $\mathbb{P} \in P(S)$ if h is $\mathbb{P}$-almost surely continuous.*

Note that the version of Proposition A.6 which is in [4], is expressed for metric spaces. But, as noted in the beginning of the Appendix, Polish spaces are metrizable, and hence one can apply the same statement for such spaces.

APPENDIX B: SEMIGROUP ACTIONS

Let X be a Hausdorff space. An action of $(\mathbb{N}, +)$ on X is a collection π of mappings $\pi_n : X \to X$, $n \in \mathbb{N}$, such that for all $x \in X$, and $m, n \in \mathbb{N}$, $\pi_m \circ \pi_n(x) = \pi_{m+n}(x)$. When each of the mappings π_n is continuous, π is also often referred to as a discrete time dynamical system.

On a Hausdorff space X, one can endow the set X^X with a topology, for example, that of pointwise convergence. The closure of the action of $\mathbb{N}$ is then the closure $\overline{\Pi}$ of the set $\Pi = \{\pi_n, n \in \mathbb{N}\} \subset X^X$ w.r.t. this topology. A classical instance (see, e.g., [7]) is that where the space X is compact, the mappings π_n are all continuous, and the topology on X^X is that of pointwise convergence. Then $\overline{\Pi}$ is compact.

Denote the orbit $\{x, \pi(x), \pi_2(x), \dots\}$ of $x \in X$ by A_x. For all $x \in X$, the closure $\mathrm{cl}A_x$ of A_x is a closed π-invariant subset of X. If, for all n, π_n is continuous, then the restriction of π to $\mathrm{cl}A_x$ defines a semigroup action of $\mathbb{N}$. The compactness of $\mathrm{cl}A_x$ is not granted when X is noncompact. When it holds, several important structural properties follow as illustrated by the next lemmas where X is a metric space with distance d. Let

$$\omega_x = \{y \in X \text{ s.t. } \exists n_1 < n_2 < \cdots \in \mathbb{N} \text{ with } \pi_{n_i}(x) \to y\} \tag{B.1}$$

denote the *ω-limit set* of x.

LEMMA B.1 (Lemma 4.2, page 134, and page 166 in [5]). *Assume that π_n is continuous for all n and that* $\mathrm{cl}A_x$ *is compact. Then, for all neighborhoods U of ω_x, there exists an $N = N(U, x)$ such that $\pi_n(x) \in U$ for all $n \geq N$. Moreover ω_x is nonempty, compact and π-invariant.*

In words, under the compactness and continuity conditions, the orbit is attracted to the ω-limit set.

LEMMA B.2 (Lemma 2.9, page 95 in [5]). *If* $\mathrm{cl}A_x$ *is compact, then the following property holds: for all $\varepsilon > 0$, there exists $N = N(\varepsilon, x) \in \mathbb{N}$ such that for all $y \in \mathrm{cl}A_x$, the set $\{\pi_n(x), 0 \leq n \leq N\}$ contains a point z such that $d(y, z) \leq \varepsilon$. If in addition π_n is continuous for all n, then the last property is equivalent to the compactness of* $\mathrm{cl}A_x$.

In words, under the compactness condition, in a long enough interval, the trajectory $\pi_n(x)$ visits a neighborhood of every point of $\mathrm{cl}A_x$.

APPENDIX C: PROOF OF PROPOSITION 2.24

Let g be a point-map the image of which at $\phi \in \mathbf{N}^0$ is $x \in \phi$, with $x \neq 0$. Assume there is a point $y \in \phi$ with $y \notin \{0, x\}$. Since ϕ is a discrete subset of $\mathbb{R}^d$ and $d \geq 2$ there exist curves $\gamma_1, \gamma_2 : [0, 1] \to \mathbb{R}^d$ such that:

1. $\gamma_1(0) = \gamma_2(1) = x$ and $\gamma_2(0) = \gamma_1(1) = y$;
2. γ_1 and γ_2 only intersect at their end-points;
3. γ_1 and γ_2 contain no point of ϕ other than x and y.

Now let Γ be a closed curve in $\mathbf{N}^0$ defined as

$$\Gamma : [0, 1] \to \mathbf{N}^0; \qquad \Gamma(t) = \bigl(\phi \setminus \{x, y\}\bigr) \cup \bigl\{\gamma_1(t), \gamma_2(t)\bigr\}, \qquad t \in [0, 1].$$

The continuity of g, items 2 and 3 imply that for all $t \in [0, 1]$, $g(\Gamma(t)) = \gamma_1(t)$. Hence, $g(\Gamma(0)) = x$ and $g(\Gamma(1)) = y$. But it follows from item 1 that $\Gamma(0) = \Gamma(1) = \phi$, which contradicts the fact that x and y are different points of ϕ. When $\phi = \{0, x\}$, one obtains the contradiction by letting x go to infinity whereas in this situation, $\{0, x\}$ converges to $\{0\}$ in the vague topology.

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