The Annals of Probability
2017, Vol. 45, No. 3, 2056–2058
DOI: 10.1214/16-AOP1096

ERRATUM TO "SCALING FOR A ONE-DIMENSIONAL DIRECTED POLYMER WITH BOUNDARY CONDITIONS"

BY TIMO SEPPÄLÄINEN

University of Wisconsin-Madison

Annals of Probability (2012) **40** 19–73

The published version contains two errors which are corrected in a new version [1] posted on the `arXiv` and also on the author's homepage.

The published version has a mistake on lines 3–5 of page 52. Namely, the reversal mapping has to be applied in a fixed rectangle, but here the rectangle varies with k.

This issue can be circumvented with a coupling of polymer paths that gives the following inequality (this is Lemma 5.4 in the corrected version).

LEMMA 1. *For each fixed* ω, $Q^{\omega}_{m_1,n}(\xi_x > 0) \le Q^{\omega}_{m_2,n}(\xi_x > 0)$ *for all* $0 < m_1 < m_2$ *and* $n \ge 0$.

PROOF. Fix ω. We construct a coupling of polymer paths. On the full lattice $\mathbb{Z}_+^2$, define a backward Markov kernel:

$$\overleftarrow{\pi}_{x,x-e} = \frac{Y_x Z_{x-e}}{Z_x} = \frac{Z_{x-e}}{Z_{x-e_1} + Z_{x-e_2}}, \qquad x \in \mathbb{N}^2, e \in \{e_1, e_2\},$$

with the obvious degenerate transitions $\overleftarrow{\pi}_{(i,0),(i-1,0)} = \overleftarrow{\pi}_{(0,j),(0,j-1)} = 1$ on the axes and absorption $\overleftarrow{\pi}_{(0,0),(0,0)} = 1$ at the origin. For each $x \in \mathbb{Z}_+^2 \setminus \{(0,0)\}$, pick a jump to $v(x) \in \{x - e_1, x - e_2\}$ according to these transition probabilities. Fix an endpoint (m, n). Construct a path $x_{0,m+n}$ from the origin to (m, n) backwards, beginning with $x_{m+n} = (m, n)$ and then iterating $x_k = v(x_{k+1})$ for $k = m + n - 1$, $m + n - 2, \dots, 0$. The process ends at $x_0 = 0$. The probability of the path is

$$\prod_{k=1}^{m+n} \overleftarrow{\pi}_{x_k, x_{k-1}} = \frac{1}{Z_{m,n}} \prod_{k=1}^{m+n} Y_{x_k} = Q^{\omega}_{m,n}(x_{0,m+n}).$$

In other words, specifying the jumps $\{v(x)\}$ constructs a simultaneous realization of the polymer paths under all quenched measures $Q^{\omega}_{m,n}$ for a fixed ω.

Suppose $m_1 < m_2$ and the path between the origin and (m_1, n) goes through the point $(1, 0)$. Then the same is true for the path between the origin and (m_2, n).

Received September 2015.

This is because the path from (m_2, n) cannot reach $(0, 1)$ without intersecting the path from (m_1, n), and once they intersect they merge by the construction. $\square$

Utilizing this lemma, the argument proceeds from the top display of page 52 of the published version as follows, by restricting the sum in the probability to $k \le \bar{u}$ as done originally:

$$\begin{aligned}
(5.10) &\le \mathbb{P}\biggl[U_{1,0}\sum_{k=u+1}^{m}\biggl(\prod_{i=2}^{k}\frac{U_{i,0}}{U^{\widetilde{\omega}}_{m-i+2,n}}\biggr)Q^{\widetilde{\omega}}_{m-k+1,n}(\xi_x > 0) < e^{\eta N^{1/3}}\biggr] \\
&\le \mathbb{P}\biggl[Q^{\widetilde{\omega}}_{m-\bar{u}+1,n}(\xi_x > 0)U_{1,0}\sum_{k=u+1}^{\bar{u}}\biggl(\prod_{i=2}^{k}\frac{U_{i,0}}{U^{\widetilde{\omega}}_{m-i+2,n}}\biggr) < e^{\eta N^{1/3}}\biggr] \\
&\le \mathbb{P}\biggl[Q^{\widetilde{\omega}}_{m-\bar{u}+1,n}(\xi_x > 0) \le \frac{1}{2}\biggr] \\
&\qquad + \mathbb{P}\biggl[U_{1,0}\sum_{k=u+1}^{\bar{u}}\biggl(\prod_{i=2}^{k}\frac{U_{i,0}}{U^{\widetilde{\omega}}_{m-i+2,n}}\biggr) \le 2e^{\eta N^{1/3}}\biggr].
\end{aligned}$$

The last line above is the same as (5.13) in the original paper and it is estimated as before. By the distribution preserving $*$-mapping the next to last line above equals $\mathbb{P}[Q^{*,\widetilde{\omega}}_{m,n}(\xi^*_x \ge \bar{u}) \le \frac{1}{2}]$. This is the same as (5.12) in the original paper and it is estimated as before.

The second error is that the published version is missing part of the proof of Lemma 5.4(ii). The proof is complete for b-values in the range $1 \le b \le CN^{2/3}$. To get the estimate for $b \ge CN^{2/3}$, proceed as follows. Transpose the first inequality of (5.2) and let $U_{i,0} \to 0$ for $i > u$ to obtain the inequality

$$\frac{Z_{m,n}(0 < \xi_x \le u)}{Z_{m,n-1}(0 < \xi_x \le u)} \le \frac{Z^{\square}_{(1,1),(m,n)}}{Z^{\square}_{(1,1),(m,n-1)}}.$$

Rearrange this inequality and iterate it to drive the n-coordinate to 1, to arrive at

$$\begin{aligned}
\frac{Z_{m,n}(0 < \xi_x \le u)}{Z^{\square}_{(1,1),(m,n)}} &\le \frac{Z_{m,n-1}(0 < \xi_x \le u)}{Z^{\square}_{(1,1),(m,n-1)}} \le \cdots \le \frac{Z_{m,1}(0 < \xi_x \le u)}{Z^{\square}_{(1,1),(m,1)}} \\
&= \sum_{k=1}^{u} U_{k,0}\prod_{i=1}^{k-1}\frac{U_{i,0}}{Y_{i,1}} \le u\exp\Bigl[\max_{1\le k\le u} S_k\Bigr],
\end{aligned}$$

where

$$S_k = \log U_{k,0} + \sum_{i=1}^{k-1}(\log U_{i,0} - \log Y_{i,1}).$$

The argument is completed by applying Kolmogorov's inequality to the random walk S_k, as was done in the first part of the proof of the lemma.

Lemma 5.4 of the published paper has become Lemma 5.5 in the corrected version [1], where the reader can find more details.

REFERENCES

[1] SEPPÄLÄINEN, T. (2012). Scaling for a one-dimensional directed polymer with boundary conditions. *Ann. Probab.* **40** 19–73. Corrected version available at http://arxiv.org/abs/0911.2446.

MATHEMATICS DEPARTMENT
UNIVERSITY OF WISCONSIN-MADISON
VAN VLECK HALL
480 LINCOLN DR.
MADISON, WISCONSIN 53706-1388
USA
E-MAIL: seppalai@math.wisc.edu