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Perturbation by non-local operators

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Abstract. Suppose that $d \geq 1$ and $0 < \beta < \alpha < 2$. We establish the existence and uniqueness of the fundamental solution $q^b(t, x, y)$ to a class of (typically non-symmetric) non-local operators $\mathcal{L}^b = \Delta^{\alpha/2} + \mathcal{S}^b$, where

$$\mathcal{S}^b f(x) := \mathcal{A}(d, -\beta) \int_{\mathbb{R}^d} \big(f(x+z) - f(x) - \nabla f(x) \cdot z \mathbb{1}_{\{|z| \leq 1\}} \big) \frac{b(x, z)}{|z|^{d+\beta}}\, dz$$

and $b(x, z)$ is a bounded measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ with $b(x, z) = b(x, -z)$ for $x, z \in \mathbb{R}^d$. Here $\mathcal{A}(d, -\beta)$ is a normalizing constant so that $\mathcal{S}^b = \Delta^{\beta/2}$ when $b(x, z) \equiv 1$. We show that if $b(x, z) \geq -\frac{\mathcal{A}(d,-\alpha)}{\mathcal{A}(d,-\beta)} |z|^{\beta-\alpha}$, then $q^b(t, x, y)$ is a strictly positive continuous function and it uniquely determines a conservative Feller process X^b, which has strong Feller property. The Feller process X^b is the unique solution to the martingale problem of $(\mathcal{L}^b, \mathcal{S}(\mathbb{R}^d))$, where $\mathcal{S}(\mathbb{R}^d)$ denotes the space of tempered functions on $\mathbb{R}^d$. Furthermore, sharp two-sided estimates on $q^b(t, x, y)$ are derived. In stark contrast with the gradient perturbations, these estimates exhibit different behaviors for different types of $b(x, z)$. The model considered in this paper contains the following as a special case. Let Y and Z be (rotationally) symmetric α-stable process and symmetric β-stable processes on $\mathbb{R}^d$, respectively, that are independent to each other. Solution to stochastic differential equations $dX_t = dY_t + c(X_{t-})\, dZ_t$ has infinitesimal generator $\mathcal{L}^b$ with $b(x, z) = |c(x)|^\beta$.

Résumé. Supposons que $d \geq 1$ et $0 < \beta < \alpha < 2$. Nous établissons l'existence et l'unicité de la solution fondamentale $q^b(t, x, y)$ pour une classe d'opérateurs non locaux (typiquement non symétriques) $\mathcal{L}^b = \Delta^{\alpha/2} + \mathcal{S}^b$, où

$$\mathcal{S}^b f(x) := \mathcal{A}(d, -\beta) \int_{\mathbb{R}^d} \big(f(x+z) - f(x) - \nabla f(x) \cdot z \mathbb{1}_{\{|z| \leq 1\}} \big) \frac{b(x, z)}{|z|^{d+\beta}}\, dz$$

et $b(x, z)$ est une fonction mesurable bornée sur $\mathbb{R}^d \times \mathbb{R}^d$ telle que $b(x, z) = b(x, -z)$ pour $x, z \in \mathbb{R}^d$. Ici $\mathcal{A}(d, -\beta)$ est la constante de normalisation telle que $\mathcal{S}^b = \Delta^{\beta/2}$ quand $b(x, z) \equiv 1$. Nous montrons que si $b(x, z) \geq -\frac{\mathcal{A}(d,-\alpha)}{\mathcal{A}(d,-\beta)} |z|^{\beta-\alpha}$, alors $q^b(t, x, y)$ est une fonction continue strictement positive qui détermine uniquement un processus de Feller conservatif X^b, satisfaisant la propriété forte de Feller. Le processus de Feller X^b est l'unique solution du problème martingale $(\mathcal{L}^b, \mathcal{S}(\mathbb{R}^d))$, où $\mathcal{S}(\mathbb{R}^d)$ est l'espace des fonctions tempérées sur $\mathbb{R}^d$. De plus, des estimées précises supérieures et inférieures sur $q^b(t, x, y)$ sont obtenues. En opposition radicale avec le cas des perturbations gradients, ces estimées montrent des comportements différents pour différents types de $b(x, z)$. Le modèle considéré dans l'article contient le modèle suivant comme cas particulier. Soient Y et Z des processus indépendants α-stable, resp. β-stable, sur $\mathbb{R}^d$, symétriques par rotation. La solution de l'équation différentielle stochastique $dX_t = dY_t + c(X_{t-})\, dZ_t$ a $\mathcal{L}^b$ pour générateur infinitésimal avec $b(x, z) = |c(x)|^\beta$.

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1. Introduction

Let $d \geq 1$ be an integer and $0 < \beta < \alpha < 2$. For integer $k \geq 1$, denote by $C_b^k(\mathbb{R}^d)$ (resp. $C_c^k(\mathbb{R}^d)$) the space of continuous functions on $\mathbb{R}^d$ that have bounded continuous partial derivatives up to order k (resp. the space of continuous functions on $\mathbb{R}^d$ with compact support that have continuous partial derivatives up to order k). Recall that a stochastic process $Y = (Y_t, \mathbb{P}_x, x \in \mathbb{R}^d)$ on $\mathbb{R}^d$ is called a (rotationally) symmetric α-stable process on $\mathbb{R}^d$ if it is a Lévy process having

$$\mathbb{E}_x\big[e^{i\xi\cdot(Y_t-Y_0)}\big] = e^{-t|\xi|^\alpha} \quad \text{for every } x, \xi \in \mathbb{R}^d.$$

Let $\widehat{f}(\xi) := \int_{\mathbb{R}^d} e^{i\xi\cdot x} f(x)\,dx$ denote the Fourier transform of a function f on $\mathbb{R}^d$. The fractional Laplacian $\Delta^{\alpha/2}$ on $\mathbb{R}^d$ is defined as

$$\Delta^{\alpha/2} f(x) = \int_{\mathbb{R}^d} \big(f(x+z) - f(x) - \nabla f(x)\cdot z\mathbb{1}_{\{|z|\leq 1\}}\big)\frac{\mathcal{A}(d,-\alpha)}{|z|^{d+\alpha}}\,dz \tag{1.1}$$

for $f \in C_b^2(\mathbb{R}^d)$. Here $\mathcal{A}(d,-\alpha) = \Gamma((d+\alpha)/2)/(2^{-\alpha}\pi^{d/2}|\Gamma(-\alpha/2)|)$, which is the normalizing constant so that $\widehat{\Delta^{\alpha/2} f}(\xi) = -|\xi|^\alpha \widehat{f}(\xi)$. Hence $\Delta^{\alpha/2}$ is the infinitesimal generator for the symmetric α-stable process on $\mathbb{R}^d$.

Throughout this paper, $b(x,z)$ is a real-valued bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying

$$b(x,z) = b(x,-z) \quad \text{for every } x, z \in \mathbb{R}^d. \tag{1.2}$$

This paper is concerned with the existence, uniqueness and *sharp two-sided estimates* on the "fundamental solution" of the following non-local operator on $\mathbb{R}^d$,

$$\mathcal{L}^b f(x) = \Delta^{\alpha/2} f(x) + \mathcal{S}^b f(x), \quad f \in C_b^2\big(\mathbb{R}^d\big),$$

where

$$\mathcal{S}^b f(x) := \mathcal{A}(d,-\beta)\int_{\mathbb{R}^d} \big(f(x+z) - f(x) - \nabla f(x)\cdot z\mathbb{1}_{\{|z|\leq 1\}}\big)\frac{b(x,z)}{|z|^{d+\beta}}\,dz. \tag{1.3}$$

We point out that since $b(x,z)$ satisfies condition (1.2), the truncation $|z| \leq 1$ in (1.3) can be replaced by $|z| \leq \lambda$ for any $\lambda > 0$; that is, for every $\lambda > 0$,

$$\mathcal{S}^b f(x) = \mathcal{A}(d,-\beta)\int_{\mathbb{R}^d} \big(f(x+z) - f(x) - \langle\nabla f(x), z\rangle\mathbb{1}_{\{|z|\leq\lambda\}}\big)\frac{b(x,z)}{|z|^{d+\beta}}\,dz. \tag{1.4}$$

In fact, under condition (1.2),

$$\begin{aligned}\mathcal{S}^b f(x) &= \mathcal{A}(d,-\beta)\text{p.v.}\int_{\mathbb{R}^d}\big(f(x+z)-f(x)\big)\frac{b(x,z)}{|z|^{d+\beta}}\,dz \\ &:= \mathcal{A}(d,-\beta)\lim_{\varepsilon\to 0}\int_{\{z\in\mathbb{R}^d:|z|>\varepsilon\}}\big(f(x+z)-f(x)\big)\frac{b(x,z)}{|z|^{d+\beta}}\,dz.\end{aligned} \tag{1.5}$$

Condition (1.2) allows us to reduce general bounded measurable function b on $\mathbb{R}^d \times \mathbb{R}^d$ to the situation where $\|b\|_\infty$ is sufficiently small through a scaling argument (see (3.15) and Lemma 3.5). The operator $\mathcal{L}^b$ is in general non-symmetric. Clearly, $\mathcal{L}^b = \Delta^{\alpha/2}$ when $b \equiv 0$ and $\mathcal{L}^b = \Delta^{\alpha/2} + \Delta^{\beta/2}$ when $b \equiv 1$.

We are led to the study of this non-local operator $\mathcal{L}^b$ by the consideration of the following stochastic differential equation (SDE) on $\mathbb{R}^d$:

$$dX_t = dY_t + c(X_{t-})\,dZ_t, \tag{1.6}$$

where Y is a symmetric α-stable process on $\mathbb{R}^d$ and Z is an independent symmetric β-stable process with $0 < \beta < \alpha$. Such SDE arises naturally in applications when there are more than one sources of random noises. When c is a bounded Lipschitz function on $\mathbb{R}^d$, it is easy to show using Picard's iteration method that for every $x \in \mathbb{R}^d$, SDE (1.6) has a unique strong solution with $X_0 = x$. We denote the law of such a solution by $\mathbb{P}_x$. The collection of the solutions $(X_t, \mathbb{P}_x, x \in \mathbb{R}^d)$ forms a strong Markov process X on $\mathbb{R}^d$. Using Ito's formula, one concludes that the infinitesimal generator of X is $\mathcal{L}^b$ with $b(x,z) = |c(x)|^\beta$ and so in this case X solves the martingale problem for $(\mathcal{L}^b, C_b^2(\mathbb{R}^d))$. The following questions arise naturally: does the Markov process X have a transition density function? If so, what is its sharp two-sided estimates? Is there a solution to the martingale problem for $\Delta^{\alpha/2} + |c(x)|^\beta \Delta^{\beta/2}$ when c is not Lipschitz continuous? We will address these questions for the more general operator $\mathcal{L}^b$ in this paper.

Heat kernel analysis is an important subject in analysis and in probability theory, as heat kernel encodes all the information about the corresponding infinitesimal generator and the corresponding Markov processes. Since explicit formula can only be derived in some very special and limited cases, the main focus of the heat kernel analysis is on its sharp estimates. While it is relatively easy to get some crude bounds, obtaining sharp two-sided bounds on the heat kernel is typically quite challenging. It requires deep understanding of the corresponding generator. Heat kernel estimates for discontinuous Markov processes have been under intense study recently. Most results obtained so far are mainly for symmetric Markov processes. See [7] for a recent survey. It is well known that the study of non-symmetric operators requires different approaches and techniques than that for symmetric operators. Results of this paper can also be viewed as an attempt in establishing heat kernel estimates for non-symmetric discontinuous Markov processes. For example, Corollary 1.4 and Theorem 1.5 can be viewed as the non-symmetric analogy, though in a restricted setting, of the two-sided heat kernel estimates for symmetric stable-like processes and mixed stable-like processes established in [11] and [12], respectively. See Remark 1.7 below for more information on heat kernel analysis.

For $a \ge 0$, denote by $p_a(t,x,y)$ the fundamental function of $\Delta^{\alpha/2} + a\Delta^{\beta/2}$ (or equivalently, the transition density function of the Lévy process $Y_t^a := Y_t + a^{1/\beta} Z_t$). Clearly, $p_a(t,x,y)$ is a function of t and $x-y$, so sometimes we also write it as $p_a(t, x-y)$. Note that for every $\lambda > 0$, $\{\lambda^{-1} Y^a_{\lambda^\alpha t}, t \ge 0\}$ has the same distribution as $\{Y_t^{a\lambda^{(\alpha-\beta)}}, t \ge 0\}$. Consequently, for any $\lambda > 0$, we have

$$p_{a\lambda^{(\alpha-\beta)}}(t,x,y) = \lambda^d p_a\big(\lambda^\alpha t, \lambda x, \lambda y\big) \quad \text{for } t > 0 \text{ and } x, y \in \mathbb{R}^d. \tag{1.7}$$

It is recently proven in [12] that on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$,

$$p_0(t,x,y) \asymp t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}, \tag{1.8}$$

$$p_a(t,x,y) \asymp \big(t^{-d/\alpha} \wedge (at)^{-d/\beta}\big) \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{at}{|x-y|^{d+\beta}}\right). \tag{1.9}$$

Here for two non-negative functions f and g, the notation $f \asymp g$ means that there is a constant $c \ge 1$ so that $c^{-1} f \le g \le cf$ on their common domain of definitions. For real numbers $a, c \in \mathbb{R}$, we use $a \vee c$ and $a \wedge c$ to denote $\max\{a,c\}$ and $\min\{a,c\}$, respectively. We point out that the comparison constants in (1.9) is independent of $a > 0$ by the scaling property (1.7). Note that $(at)^{-d/\beta} \ge t^{-d/\alpha}$ whenever $0 < t \le a^{-\alpha/(\alpha-\beta)}$. Thus for every $k > 0$,

$$p_a(t,x,y) \asymp t^{-d/\alpha} \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{at}{|x-y|^{d+\beta}}\right) \quad \text{on } \big(0, ka^{-\alpha/(\alpha-\beta)}\big] \times \mathbb{R}^d \times \mathbb{R}^d, \tag{1.10}$$

with the comparison constants depending only on d, α, β and k.

Since $\mathcal{L}^b = \Delta^{\alpha/2} + \mathcal{S}^b$ is a lower order perturbation of $\Delta^{\alpha/2}$ by $\mathcal{S}^b$, heuristically the fundamental solution (or kernel) $q^b(t,x,y)$ of $\mathcal{L}^b$ should satisfy the following Duhamel's formula:

$$q^b(t,x,y) = p_0(t,x,y) + \int_0^t \int_{\mathbb{R}^d} q^b(t-s,x,z) \mathcal{S}_z^b p_0(s,z,y)\,dz\,ds \tag{1.11}$$

for $t>0$ and $x,y\in\mathbb{R}^d$. Here the notation $S^b_z p_0(s,z,y)$ means the non-local operator $\mathcal{S}^b$ is applied to the function $z\mapsto p_0(s,z,y)$. Similar notation will also be used for other operators, for example, $\Delta_z^{\alpha/2}$. Applying (1.11) recursively, it is reasonable to conjecture that $\sum_{n=0}^\infty q_n^b(t,x,y)$, if convergent, is a solution to (1.11), where $q_0^b(t,x,y):=p_0(t,x,y)$ and

$$q_n^b(t,x,y):=\int_0^t\int_{\mathbb{R}^d} q_{n-1}^b(t-s,x,z)\mathcal{S}_z^b p_0(s,z,y)\,dz\,ds \quad \text{for } n\ge 1. \tag{1.12}$$

For each bounded function $b(x,z)$ on $\mathbb{R}^d\times\mathbb{R}^d$ and $\lambda>0$, define

$$m_{b,\lambda}=\operatorname*{essinf}_{x,z\in\mathbb{R}^d,|z|>\lambda} b(x,z) \quad\text{and}\quad M_{b,\lambda}=\operatorname*{esssup}_{x,z\in\mathbb{R}^d,|z|>\lambda}\big|b(x,z)\big|. \tag{1.13}$$

The followings are the main results of this paper.

Theorem 1.1. *For every bounded function b on $\mathbb{R}^d\times\mathbb{R}^d$ satisfying condition (1.2), there is a unique continuous function $q^b(t,x,y)$ on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ that satisfies (1.11) on $(0,\varepsilon]\times\mathbb{R}^d\times\mathbb{R}^d$ with $|q^b(t,x,y)|\le cp_1(t,x,y)$ on $(0,\varepsilon]\times\mathbb{R}^d\times\mathbb{R}^d$ for some $\varepsilon,c>0$, and that*

$$\int_{\mathbb{R}^d} q^b(t,x,y)q^b(s,y,z)\,dy=q^b(t+s,x,z)\quad \textit{for every } t,s>0 \textit{ and } x,z\in\mathbb{R}^d. \tag{1.14}$$

Moreover, the following holds.

(i) *There is a constant $A_0=A_0(d,\alpha,\beta)>0$ so that $q^b(t,x,y)=\sum_{n=0}^\infty q_n^b(t,x,y)$ on $(0,(A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}]\times\mathbb{R}^d\times\mathbb{R}^d$, where $q_n^b(t,x,y)$ is defined by (1.12).*

(ii) *$q^b(t,x,y)$ satisfies the Duhamel's formula (1.11) for all $t>0$ and $x,y\in\mathbb{R}^d$. Moreover, $\mathcal{S}^b_x q^b(t,x,y)$ exists pointwise in the sense of (1.5) and*

$$q^b(t,x,y)=p_0(t,x,y)+\int_0^t\int_{\mathbb{R}^d} p_0(t-s,x,z)\mathcal{S}_z^b q^b(s,z,y)\,dz\,ds \tag{1.15}$$

for $t>0$ and $x,y\in\mathbb{R}^d$.

(iii) *For each $t>0$ and $x\in\mathbb{R}^d$, $\int_{\mathbb{R}^d} q^b(t,x,y)\,dy=1$.*

(iv) *For every $f\in C_b^2(\mathbb{R}^d)$,*

$$T_t^b f(x)-f(x)=\int_0^t T_s^b\mathcal{L}^b f(x)\,ds,$$

where $T_t^b f(x)=\int_{\mathbb{R}^d} q^b(t,x,y)f(y)\,dy$.

(v) *Let $A>0$ and $\lambda>0$. There is a positive constant $C=C(d,\alpha,\beta,A,\lambda)\ge 1$ so that for any b satisfying (1.2) with $\|b\|_\infty\le A$,*

$$\big|q^b(t,x,y)\big|\le Ce^{Ct}p_{M_{b,\lambda}}(t,x,y)\quad \textit{on } (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d. \tag{1.16}$$

We remark that estimate (1.16) allows one to get sharper bound on $|q^b(t,x,y)|$ by selecting optimal $\lambda>0$. When Z_t is the deterministic process t and c is an $\mathbb{R}^d$-valued bounded Lipschitz function on $\mathbb{R}^d$, the solution of (1.6) is a symmetric α-stable process with drift. Its infinitesimal generator is $\Delta^{\alpha/2}+c(x)\nabla$. Existence of integral kernel to $\Delta^{\alpha/2}+c(x)\nabla$ and its estimates have been studied recently in [6] (in fact, c there can be an $\mathbb{R}^d$-valued function in certain Kato class).

Unlike the gradient perturbation for $\Delta^{\alpha/2}$, in general the kernel $q^b(t,x,y)$ in Theorem 1.1 can take negative values. For example, this is the case when $b\equiv -1$, that is, when $\mathcal{L}^b=\Delta^{\alpha/2}-\Delta^{\beta/2}$, according to the next theorem. Observe that

$$\mathcal{L}^b f(x)=\int_{\mathbb{R}^d}\big(f(x+z)-f(x)-\langle\nabla f(x),z\rangle\mathbb{1}_{\{|z|\le 1\}}\big)j^b(x,z)\,dz,$$

where

$$j^b(x,z)=\frac{\mathcal{A}(d,-\alpha)}{|z|^{d+\alpha}}\left(1+\frac{\mathcal{A}(d,-\beta)}{\mathcal{A}(d,-\alpha)}b(x,z)|z|^{\alpha-\beta}\right). \tag{1.17}$$

The next result gives a necessary and sufficient condition for the kernel $q^b(t,x,y)$ in Theorem 1.1 to be non-negative when $b(x,z)$ is continuous in x for a.e. z.

Theorem 1.2. *Let b be a bounded function on $\mathbb{R}^d\times\mathbb{R}^d$ that satisfies (1.2) and that*

$$x\mapsto b(x,z) \text{ is continuous for a.e. } z\in\mathbb{R}^d. \tag{1.18}$$

Then $q^b(t,x,y)\ge 0$ on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ if and only if for each $x\in\mathbb{R}^d$, $j^b(x,z)\ge 0$ for a.e. $z\in\mathbb{R}^d$; that is, if and only if

$$b(x,z)\ge -\frac{\mathcal{A}(d,-\alpha)}{\mathcal{A}(d,-\beta)}|z|^{\beta-\alpha}\quad \text{for a.e. } z\in\mathbb{R}^d. \tag{1.19}$$

In particular, if $b(x,z)=b(x)$ is a function of x only, then $q^b(t,x,y)\ge 0$ on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ if and only if $b(x)\ge 0$ on $\mathbb{R}^d$.

Next theorem drops the assumption (1.18), gives lower bound estimates and refines upper bound estimates on $q^b(t,x,y)$ for $b(x,z)$ satisfying condition (1.19) and makes connections to the martingale problem for $\mathcal{L}^b$. To state it, we need first to recall some definitions.

Let $\mathbb{D}([0,\infty),\mathbb{R}^d)$ be the space of right continuous $\mathbb{R}^d$-valued functions having left limits on $[0,\infty)$, equipped with Skorokhod topology. Denote by X_t the projection coordinate map on $\mathbb{D}([0,\infty),\mathbb{R}^d)$. Let $\mathcal{C}$ be a subspace of $C_b^2(\mathbb{R}^d)$. A probability measure Q on the Skorokhod space $\mathbb{D}([0,\infty),\mathbb{R}^d)$ is said to to be a solution to the martingale problem for $(\mathcal{L}^b,\mathcal{C})$ with initial value $x\in\mathbb{R}^d$ if $Q(X_0=x)=1$ and for every $f\in\mathcal{C}$,

$$M_t^f:=f(X_t)-f(X_0)-\int_0^t\mathcal{L}^bf(X_s)\,ds$$

is a Q-martingale. The martingale problem $(\mathcal{L}^b,\mathcal{C})$ with initial value $x\in\mathbb{R}^d$ is said to be well-posed if it has a unique solution.

Let $C_\infty(\mathbb{R}^d)$ be the space of continuous functions on $\mathbb{R}^d$ that vanish at infinity, equipped with supremum norm. Set

$$C_\infty^2(\mathbb{R}^d)=\{f\in C_\infty(\mathbb{R}^d): \text{the first and second derivatives of } f \text{ are all in } C_\infty(\mathbb{R}^d)\}.$$

A Markov process on $\mathbb{R}^d$ is called a Feller process if its transition semigroup is a strongly continuous semigroup in $C_\infty(\mathbb{R}^d)$. Feller processes is a class of nice strong Markov processes, called Hunt processes (see [16]). Let $\overline{p}_0(t,x,y)$ be the fundamental solution of the truncated operator

$$\overline{\Delta}^{\alpha/2}f(x)=\int_{|z|\le 1}\left(f(x+z)-f(x)-\nabla f(x)\cdot z\mathbb{1}_{\{|z|\le 1\}}\right)\frac{\mathcal{A}(d,-\alpha)}{|z|^{d+\alpha}}\,dz;$$

or, equivalently, $\overline{p}_0(t,x,y)$ is the transition density function for the finite range α-stable (Lévy) process with Lévy measure $\mathcal{A}(d,-\alpha)|z|^{-(d+\alpha)}\mathbb{1}_{\{|z|\le 1\}}$. It is established in [8] that $\overline{p}_0(t,x,y)$ is jointly continuous and enjoys the following two sided estimates:

$$\overline{p}_0(t,x,y)\asymp t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}} \tag{1.20}$$

for $t\in(0,1]$ and $|x-y|\le 1$, and there are constants $c_k=c_k(d,\alpha)>0$, $k=1,2,3,4$ so that

$$c_1\left(\frac{t}{|x-y|}\right)^{c_2|x-y|}\le\overline{p}_0(t,x,y)\le c_3\left(\frac{t}{|x-y|}\right)^{c_4|x-y|} \tag{1.21}$$

for $t\in(0,1]$ and $|x-y|>1$.

Define $b^+(x,z)=\max\{b(x,z),0\}$.

Theorem 1.3. *For every $A>0$ and $\lambda>0$, there are positive constants $C_k=C_k(d,\alpha,\beta,A)$, $k=1,2$, and $C_3=C_3(d,\alpha,\beta,A,\lambda)$ such that for any bounded b satisfying (1.2) and (1.19) with $\|b\|_\infty\le A$,*

$$C_1\overline{p}_0(t,C_2x,C_2y)\le q^b(t,x,y)\le C_3 p_{M_{b^+,\lambda}}(t,x,y)\quad \text{for } t\in(0,1] \text{ and } x,y\in\mathbb{R}^d. \tag{1.22}$$

Moreover, for every $\varepsilon>0$, there is a positive constant $C_4=C_4(d,\alpha,\beta,A,\lambda,\varepsilon)$ such that for any b on $\mathbb{R}^d\times\mathbb{R}^d$ satisfying (1.2) with $\|b\|_\infty\le A$ so that

$$j^b(x,z)\ge\varepsilon|z|^{-(d+\alpha)}\quad \text{for a.e. } x,z\in\mathbb{R}^d \tag{1.23}$$

we have

$$C_4 p_{m_{b^+,\lambda}}(t,x,y)\le q^b(t,x,y)\le C_3 p_{M_{b^+,\lambda}}(t,x,y)\quad \text{for } t\in(0,1] \text{ and } x,y\in\mathbb{R}^d. \tag{1.24}$$

The kernel $q^b(t,x,y)$ uniquely determines a Feller process $X^b=(X_t^b,t\ge0,\mathbb{P}_x,x\in\mathbb{R}^d)$ on the canonical Skorokhod space $\mathbb{D}([0,\infty),\mathbb{R}^d)$ such that

$$\mathbb{E}_x\big[f\big(X_t^b\big)\big]=\int_{\mathbb{R}^d}q^b(t,x,y)f(y)\,dy$$

for every bounded continuous function f on $\mathbb{R}^d$. The Feller process X^b is conservative and has a Lévy system $(J^b(x,y)\,dy,t)$, where $J^b(x,y)=j^b(x,y-x)$,

$$J^b(x,y)=j^b(x,y-x)=\frac{\mathcal{A}(d,-\alpha)}{|x-y|^{d+\alpha}}+\frac{\mathcal{A}(d,-\beta)b(x,y-x)}{|x-y|^{d+\beta}}. \tag{1.25}$$

Moreover, for each $x\in\mathbb{R}^d$, $(X^b,\mathbb{P}_x)$ is the unique solution to the martingale problem $(\mathcal{L}^b,\mathcal{S}(\mathbb{R}^d))$ with initial value x. Here $\mathcal{S}(\mathbb{R}^d)$ denotes the space of tempered functions on $\mathbb{R}^d$.

Here we say $(J^b(x,y)\,dy,t)$ is a Lévy system for X^b if for any non-negative measurable function f on $\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}^d$ with $f(s,y,y)=0$ for all $y\in\mathbb{R}^d$, any stopping time T (with respect to the filtration of X^b) and any $x\in\mathbb{R}^d$,

$$\mathbb{E}_x\bigg[\sum_{s\le T}f\big(s,X_{s-}^b,X_s^b\big)\bigg]=\mathbb{E}_x\bigg[\int_0^T\bigg(\int_{\mathbb{R}^d}f\big(s,X_s^b,y\big)J^b\big(X_s^b,y\big)\,dy\bigg)\,ds\bigg]. \tag{1.26}$$

A Lévy system for X^b describes the jumps of the process X^b. A Markov process on $\mathbb{R}^d$ is said to have strong Feller property if its transition semigroup maps bounded measurable functions on $\mathbb{R}^d$ into bounded continuous functions on $\mathbb{R}^d$. Since $q^b(t,x,y)$ is a continuous function, one has by Theorem 1.1 and the dominated convergence theorem that the Feller process X^b of Theorem 1.3 has strong Feller property.

Condition (1.23) is always satisfied if $b(x,z)$ is non-negative. We emphasize the $m_{b^+,\lambda}$ and $M_{b^+,\lambda}$ terms appeared in the estimates in Theorem 1.3. Under condition (1.23) and the assumption that $\|b\|_\infty\le A$, the value of $b(x,z)$ on $\mathbb{R}^d\times\{z\in\mathbb{R}^d:|z|\le\lambda\}$ is irrelevant in the estimates of $q^b(t,x,y)$ in (1.24). By selecting suitable $\lambda>0$ in (1.24), one can get optimal two-sided estimates on $q^b(t,x,y)$. The following follows immediately from Theorem 1.3 by taking a suitable $\lambda>0$.

Corollary 1.4. *Let $A\ge0$ and $\varepsilon>0$. There is a positive constant $C=C(d,\alpha,\beta,A,\varepsilon)\ge1$ so that for any bounded b satisfying (1.2) with $\|b\|_\infty\le A$ and*

$$j^b(x,z)\ge\varepsilon\bigg(\frac{1}{|z|^{d+\alpha}}+\frac{1}{|z|^{d+\beta}}\bigg)\quad \text{for a.e. } x,z\in\mathbb{R}^d,$$

we have

$$C^{-1}p_1(t,x,y)\le q^b(t,x,y)\le Cp_1(t,x,y)\quad \text{for } t\in(0,1] \text{ and } x,y\in\mathbb{R}^d.$$

Theorem 1.3 in particular implies that if $b(x,\cdot)$ is a bounded function satisfying (1.2) and (1.19) so that $b(x,z)=0$ for every $x\in\mathbb{R}^d$ and $|z|\geq R$ for some $R>0$; or, equivalently if $\mathcal{L}^b=\Delta^{\alpha/2}+\mathcal{S}^b$ is a lower order perturbation of $\Delta^{\alpha/2}$ by finite range non-local operator $\mathcal{S}^b$, then the upper bound of the kernel $q^b(t,x,y)$ is dominated by $p_0(t,x,y)$ for each $(t,x,y)\in(0,1]\times\mathbb{R}^d\times\mathbb{R}^d$. In fact, we have the following more general result.

Theorem 1.5. *For every $A>0$ and $M\geq 1$, there is a constant $C_5=C_5(d,\alpha,\beta,A,M)\geq 1$ such that for any bounded b satisfying (1.2) with $\|b\|_\infty\leq A$ and*

$$M^{-1}|z|^{-(d+\alpha)}\leq j^b(x,z)\leq M|z|^{-(d+\alpha)}\quad\text{for a.e. } x,z\in\mathbb{R}^d,\tag{1.27}$$

or equivalently,

$$-\big(1-M^{-1}\big)\frac{\mathcal{A}(d,-\alpha)}{\mathcal{A}(d,-\beta)}|z|^{\beta-\alpha}\leq b(x,z)\leq (M-1)\frac{\mathcal{A}(d,-\alpha)}{\mathcal{A}(d,-\beta)}|z|^{\beta-\alpha}\quad\text{for a.e. } x,z\in\mathbb{R}^d,\tag{1.28}$$

we have

$$C_5^{-1}p_0(t,x,y)\leq q^b(t,x,y)\leq C_5 p_0(t,x,y)\quad\text{for } t\in(0,1] \text{ and } x,y\in\mathbb{R}^d.\tag{1.29}$$

We can restate some of results from Theorems 1.1, 1.2, 1.3 and 1.5 as follows.

Theorem 1.6. *Let $b(x,z)$ be a bounded function on $\mathbb{R}^d\times\mathbb{R}^d$ satisfying (1.2) and (1.19). For each $x\in\mathbb{R}^d$, the martingale problem for $(\mathcal{L}^b,\mathcal{S}(\mathbb{R}^d))$ with initial value x is well-posed. These martingale problem solutions $\{\mathbb{P}_x,x\in\mathbb{R}^d\}$ form a strong Markov process X^b, which has infinite lifetime and possesses a jointly continuous transition density function $q^b(t,x,y)$ with respect to the Lebesgue measure on $\mathbb{R}^d$. Moreover, the transition density function $q^b(t,x,y)$ is the same as the fundamental solution given in Theorem 1.1 and so all the conclusions there as well as that of Theorems 1.3 and 1.5 hold for $q^b(t,x,y)$.*

Remark 1.7.

(i) In general, we can not expect q^b to have comparable lower and upper bound estimates. The estimates in (1.22) and (1.24) are sharp in the sense that $q^b(t,x,y)=p_0(t,x,y)$ when $b\equiv 0$, $q^b(t,x,y)=p_1(t,x,y)$ when $b\equiv 1$, and $q^b(t,x,y)=\overline{p}_0(t,x,y)$ when $b(x,z)=0$ for $|z|\leq 1$ and $b(x,z)=-\frac{\mathcal{A}(d,-\alpha)}{\mathcal{A}(d,-\beta)}|z|^{\beta-\alpha}$ for $|z|\geq 1$. Clearly, by (1.8)–(1.9), $p_0(t,x,y)$ and $p_1(t,x,y)$ are not comparable on $(0,1]\times\mathbb{R}^d\times\mathbb{R}^d$. We point out that it follows from (1.9) and (1.24) that every $A\geq 1$, there is a constant $\widetilde{C}=\widetilde{C}(d,\alpha,\beta,A)\geq 1$ so that for any non-negative b on $\mathbb{R}^d\times\mathbb{R}^d$ satisfying (1.2) with $1/A\leq b(x,z)\leq A$ a.e.

$$(1/\widetilde{C})p_1(t,x,y)\leq q^b(t,x,y)\leq \widetilde{C}p_1(t,x,y)\quad\text{for } t\in(0,1] \text{ and } x,y\in\mathbb{R}^d.\tag{1.30}$$

(ii) Heat kernel estimates for fractional Laplacian $\Delta^{\alpha/2}$ under gradient perturbation and (possibly non-local) Feynman–Kac perturbation have recently been studied in [6,9,10,32]. In both of these cases, under a Kato class condition on the coefficients, the fundamental solution of the perturbed operator is always strictly positive and is comparable to the fundamental solution $p_0(t,x,y)$ of the fractional Laplacian $\Delta^{\alpha/2}$ on $(0,1]\times\mathbb{R}^d\times\mathbb{R}^d$.

The novelty of this paper is on non-local perturbations. The analysis of non-local perturbations with infinite jumping intensity measure is much harder and is in fact very challenging. While the idea of using Duhamel's method (1.11) in the study of operator perturbation is not new, the key is how to rigorously establish it and implement it to obtain two-sided sharp heat kernel estimates where the lower bound is comparable to the upper bound, and to establish the uniqueness of the fundamental solution. It requires precise estimates on the non-local derivatives of the heat kernel for fractional Laplacian, which turns out to be quite delicate and challenging. To the best of authors' knowledge, this is the first paper on the study of heat kernels under non-local perturbation with infinite jump intensity measure in a systematic way. We emphasize that the function $b(x,z)$ in 1.3 is only measurable. Our Theorems 1.2 and 1.3 reveal some new phenomenon that heat kernels under non-local perturbation $\mathcal{S}^b$ are typically unstable. This is is in stark contrast with $\Delta^{\alpha/2}$ under either gradient (local) perturbations or (possibly non-local) Feynman–Kac perturbations. However, Theorem 1.5 of this paper in particular indicates that the heat kernel estimate for $\Delta^{\alpha/2}$ is stable under finite range lower order perturbation.

(iii) Kolokoltsov [21] studied heat kernel estimates for symmetric pseudo-differential operators (or stable-like jump diffusions) with smooth symbols. However neither the results nor the approach in [21] applies to our case even when $b(x,z)$ is assumed to be smooth. In addition to the smooth symbol requirement, the operators (1.8)–(1.9) considered in [21] would require $\alpha = \beta$, excluding the case where there are two different stable scales as are considered in this paper. In particular, it does not apply to SDE (1.6). For information on the connection between pseudo-differential operators and discontinuous Markov processes, we refer the reader to [18–20,27] and the references therein.

(iv) Martingale problem for non-local operators (with or without elliptic differential operator component) has been studied by many authors. See, e.g., [3–5,22,23,25,26,29–31] and the references therein. In particular, Komatsu [23] and Mikulevicius–Pragarauskas [25] considered martingale problem for a class of non-local operators that is directly related to $\mathcal{L}^b$. In fact, the uniqueness of the martingale problem for $(\mathcal{L}^b, \mathcal{S}(\mathbb{R}^d))$ stated in Theorem 1.3 above is a direct consequence of [23, Theorem 3], while it follows from [25, Theorem 5] that for any bounded b satisfying (1.2) and (1.19), there is a unique solution to the martingale problem $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$. The main contribution of Theorem 1.3 is on the two-sided transition density function estimates for the martingale problem solution X_t^b. We also mention that the well-posedness of martingale problem for $(\Delta^{\alpha/2} + b(x)\cdot\nabla, C_c^\infty(\mathbb{R}^d))$ with $b(x)$ an $\mathbb{R}^d$-valued Kato class function has recently been established in [14].

(v) There are several directions to extend our results. For example, one can replace $\frac{\mathcal{A}(d,-\alpha)}{|z|^{d+\alpha}}$ in (1.1) and $1/|z|^{d+\beta}$ in (1.3) by the Lévy kernels of pure jump subordinate Brownian motions. This should be doable by following the ideas and approach of this paper. Another direction is to consider Laplacian under non-local perturbation; that is, to replace $\Delta^{\alpha/2}$ in $\mathcal{L}^b$ by Laplacian operator Δ. This has recently been carried out in Wang [33].

The rest of the paper is organized as follows. In Section 2, we derive some estimates on $\overline{\Delta}_x^{\beta/2} p_0(t,x,y)$ and $\Delta_x^{\beta/2} p_0(t,x,y)$ that will be used in later. The existence and uniqueness of the fundamental solution $q^b(t,x,y)$ of $\mathcal{L}^b$ are given in Section 3. This is done through a series of lemmas and theorems, which provide more detailed information on $q^b(t,x,y)$ and $q_n^b(t,x,y)$. Theorem 1.1 then follows from these results. We show in Section 4 that the semigroup $\{T_t^b; t>0\}$ associated with $q^b(t,x,y)$ is a strongly continuous semigroup in $C_\infty(\mathbb{R}^d)$. We then apply Hille–Yosida–Ray theorem and Courrége's first theorem to establish Theorem 1.2. When b satisfies (1.2), (1.18) and (1.19), $q^b(t,x,y)$ determines a conservative Feller process X^b. We first derive a Lévy system of X^b and also prove $(X^b, \mathbb{P}_x)$ is the unique solution to the martingale problem for $(\mathcal{L}^b, \mathcal{S}(\mathbb{R}^d))$ in Section 5. We next establish, for any given $A>0$, the equi-continuity of $q^b(t,x,y)$ on each $[1/M, M]\times\mathbb{R}^d\times\mathbb{R}^d$ for any b that satisfies (1.2) with $\|b\|_\infty \le A$. Using this, we can drop the condition (1.18) and establish the Feller process X^b with transition density $q^b(t,x,y)$ for general bounded b that satisfies (1.2) and (1.19) by approximating it with a sequence of $\{k_n(x,z), n\ge 1\}$ that satisfy (1.2), (1.18) and (1.19). The upper bound estimate for $q^b(t,x,y)$ in (1.22) and (1.24) can be obtained from that of $q^{\widehat{b}_\lambda}(t,x,y)$ due to the Meyer's construction of $X^{\widehat{b}_\lambda}$ from X^b, where $\widehat{b}_\lambda(x,z) = b(x,z)1_{\{|z|\le\lambda\}}(z) + b^+(x,z)1_{\{|z|>\lambda\}}(z)$. The lower bound estimates in (1.22) and (1.24) are established by the Lévy system of X^b and some probability estimates. Finally, we use the estimates in (1.24) for b with support in $\{(x,z)\in\mathbb{R}^d\times\mathbb{R}^d : |z|\le 1\}$ and the non-local Feynman–Kac perturbation results from [10] to obtain Theorem 1.5.

Throughout this paper, we use the capital letters $C_1, C_2, \ldots$ to denote constants in the statement of the results, and their labeling will be fixed. The lowercase constants $c_1, c_2, \ldots$ will denote generic constants used in the proofs, whose exact values are not important and can change from one appearance to another. We will use ":=" to denote a definition. For a differentiable function f on $\mathbb{R}^d$, we use $\partial_i f$ and $\partial_{ij}^2 f$ to denote $\frac{\partial f}{\partial x_i}$ and $\frac{\partial^2 f}{\partial x_i \partial x_j}$.

In this paper, details of some proofs are omitted after giving sufficient information on how to carry them out. We believe the reader should have no problem in filling in these details. Nevertheless, a longer version [13] of this paper is available on arXiv where the reader can find these details.

2. Preliminaries

Recall that $p_0(t,x,y) = p_0(t,x-y)$ is the transition density function of the symmetric α-stable process Y^0.

Lemma 2.1. *There exists a constant $C_6 = C_6(d, \alpha) > 0$ such that for every $t > 0$, $x \in \mathbb{R}^d$ and $i, j = 1, \dots, d$,*

$$\left|\frac{\partial}{\partial x_i} p_0(t, x)\right| \le C_6 t^{-(d+1)/\alpha}\left(1 \wedge \frac{t^{1/\alpha}}{|x|}\right)^{d+1+\alpha},$$

$$\left|\frac{\partial^2}{\partial x_i\, \partial x_j} p_0(t, x)\right| \le C_6 t^{-(d+2)/\alpha}\left(1 \wedge \frac{t^{1/\alpha}}{|x|}\right)^{d+2+\alpha}.$$

Proof. By [6, Lemma 5], there is a positive constant c_1 so that for all $t > 0$ and $x, y \in \mathbb{R}^d$

$$\left|\nabla_x p_0(t, x)\right| \le c_1 |x| \left(t^{-(d+2)/\alpha} \wedge \frac{t}{|x|^{d+2+\alpha}}\right) \le c_1 \left(t^{-(d+1)/\alpha} \wedge \frac{t}{|x|^{d+1+\alpha}}\right).$$

That is, the first inequality holds. Let $\eta_t(r)$ be the density function of the $\alpha/2$-stable subordinator at time t and $g(t, x) = (4\pi t)^{-d/2} e^{-|x|^2/4t}$ be the Gaussian kernel on $\mathbb{R}^d$. There is a constant c so that $\eta_t(r) \le c t r^{-1-\alpha/2}$ for all $r, t > 0$, see [6, Lemma 5]. Note that

$$\left|\frac{\partial^2}{\partial x_i\, \partial x_j} g(s, x)\right| \le \left(\frac{|x|^2}{s^2} + \frac{2}{s}\right) g(s, x) = (4\pi)^2 |x|^2 g^{(d+4)}(s, x_1) + 8\pi g^{(d+2)}(s, x_2),$$

where $x_1 \in \mathbb{R}^{d+4}$ and $x_2 \in \mathbb{R}^{d+2}$ with $|x_1| = |x_2| = |x|$, $g^{(d+2)}(s, x_2)$ and $g^{(d+4)}(s, x_1)$ are the Gaussian kernels on $\mathbb{R}^{d+2}$ and $\mathbb{R}^{d+4}$, respectively. Since $p_0(t, x) = \int_0^\infty g(s, x)\eta_t(s)\, ds$, we have by the dominated convergence theorem that there is a positive constant c_2 so that for all $t > 0$ and $x \in \mathbb{R}^d$

$$\begin{aligned}\left|\frac{\partial^2}{\partial x_i\, \partial x_j} p_0(t, x)\right| &\le \int_0^\infty \left|\frac{\partial^2}{\partial x_i\, \partial x_j} g(s, x)\right| \eta_t(s)\, ds \\ &\le (4\pi)^2 |x|^2 p_0^{(d+4)}(t, x_1) + 8\pi p_0^{(d+2)}(t, x_2) \\ &\le c_2 \left(t^{-(d+2)/\alpha} \wedge \frac{t}{|x|^{d+2+\alpha}}\right),\end{aligned}$$

where $p_0^{(d+2)}(t, x_2)$ and $p_0^{(d+4)}(t, x_1)$ are the transition density functions of the symmetric α-stable processes in $\mathbb{R}^{d+2}$ and $\mathbb{R}^{d+4}$, respectively. This establishes the second inequality in Lemma 2.1. □

Define for $t > 0$ and $x, y \in \mathbb{R}^d$, the function

$$\left|\Delta_x^{\beta/2}\right| p_0(t, x, y) = \begin{cases} \mathcal{A}(d, -\beta)\Big(\int_{|z| \le t^{1/\alpha}} \big|p_0(t, x+z, y) - p_0(t, x, y) - \frac{\partial}{\partial x} p_0(t, x, y) \cdot z\big| \frac{1}{|z|^{d+\beta}}\, dz \\ \quad + \int_{|z| > t^{1/\alpha}} |p_0(t, x+z, y) - p_0(t, x, y)| \frac{dz}{|z|^{d+\beta}}\Big) \quad \text{for } |x-y|^\alpha \le t, \\ \mathcal{A}(d, -\beta)\Big(\int_{|z| \le |x-y|/2} \big|p_0(t, x+z, y) - p_0(t, x, y) - \frac{\partial}{\partial x} p_0(t, x, y) \cdot z\big| \frac{1}{|z|^{d+\beta}}\, dz \\ \quad + \int_{|z| > |x-y|/2} |p_0(t, x+z, y) - p_0(t, x, y)| \frac{dz}{|z|^{d+\beta}}\Big) \quad \text{for } |x-y|^\alpha > t. \end{cases}$$

Let

$$f_0(t, x, y) := \left(t^{1/\alpha} \vee |x-y|\right)^{-(d+\beta)} = t^{-(d+\beta)/\alpha}\left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\beta}. \tag{2.1}$$

Lemma 2.2. *There exists a constant $C_7 = C_7(d, \alpha, \beta) > 0$ such that*

$$\left|\Delta_x^{\beta/2}\right| p_0(t, x, y) \le C_7 f_0(t, x, y) \quad \text{on } (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d. \tag{2.2}$$

Proof. We only need to prove $|\Delta_x^{\beta/2}| p_0(t, x) \le C_7 f_0(t, x, 0)$ for all $t > 0$ and $x \in \mathbb{R}^d$.

(i) We first consider the case $|x|^\alpha \le t$. In this case,

$$\begin{aligned}\big|\Delta_x^{\beta/2}\big|p_0(t,x) &= \mathcal{A}(d,-\beta)\int_{|z|\le t^{1/\alpha}}\Big|p_0(t,x+z)-p_0(t,x)-\frac{\partial}{\partial x}p_0(t,x)\cdot z\Big|\frac{dz}{|z|^{d+\beta}}\\ &\quad+\mathcal{A}(d,-\beta)\int_{|z|\ge t^{1/\alpha}}\big|p_0(t,x+z)-p_0(t,x)\big|\frac{dz}{|z|^{d+\beta}}\\ &= I+II.\end{aligned}$$

Note that by Lemma 2.1, $\sup_{u\in\mathbb{R}^d}|\frac{\partial^2}{\partial u_i\,\partial u_j}p_0(t,u)|\le C_6 t^{-(d+2)/\alpha}$, and so by Taylor's formula,

$$I\le \mathcal{A}(d,-\beta)\sup_{u\in\mathbb{R}^d}\Big|\frac{\partial^2}{\partial u_i\,\partial u_j}p_0(t,u)\Big|\int_{|z|\le t^{1/\alpha}}\frac{|z|^2}{|z|^{d+\beta}}\,dz\le c_1 t^{-(d+2)/\alpha}t^{(2-\beta)/\alpha}\le c_1 t^{-(d+\beta)/\alpha}.$$

On the other hand, by (1.8)

$$II\le \mathcal{A}(d,-\beta)\int_{|z|\ge t^{1/\alpha}}\big(p_0(t,x+z)+p_0(t,x)\big)\frac{dz}{|z|^{d+\beta}}\le c_2 t^{-d/\alpha}\int_{|z|\ge t^{1/\alpha}}\frac{1}{|z|^{d+\beta}}\,dz\le c_3 t^{-(d+\beta)/\alpha}.$$

(ii) Next, we consider the case $|x|^\alpha\ge t$. In this case,

$$\begin{aligned}\big|\Delta_x^{\beta/2}\big|p_0(t,x) &= \mathcal{A}(d,-\beta)\int_{|z|\le |x|/2}\Big|p_0(t,x+z)-p_0(t,x)-\frac{\partial}{\partial x}p_0(t,x)\cdot z\Big|\frac{dz}{|z|^{d+\beta}}\\ &\quad+\mathcal{A}(d,-\beta)\int_{|z|\ge |x|/2}\big|p_0(t,x+z)-p_0(t,x)\big|\frac{dz}{|z|^{d+\beta}}\\ &=: I+II.\end{aligned}$$

Note that $|x+z|\ge|x|/2$ for $|z|\le|x|/2$. So by Lemma 2.1,

$$\sup_{|z|\le|x|/2}\Big|\frac{\partial^2}{\partial x_i\,\partial x_j}p_0(t,x+z)\Big|\le C_6\sup_{|z|\le|x|/2}t|x+z|^{-(d+2+\alpha)}\le 2^{(d+2+\alpha)}C_6 t|x|^{-(d+2+\alpha)}.$$

Hence, by Taylor's formula

$$\begin{aligned}I&\le \mathcal{A}(d,-\beta)\sup_{|z|\le|x|/2}\Big|\frac{\partial^2}{\partial x_i\,\partial x_j}p_0(t,x+z)\Big|\int_{|z|\le|x|/2}\frac{|z|^2}{|z|^{d+\beta}}\,dz\\ &\le c_4 t|x|^{-(d+2+\alpha)}|x|^{2-\beta}=c_4 t|x|^{-(d+\alpha+\beta)}.\end{aligned}\tag{2.3}$$

Noting that $|x|^\alpha\ge t$, thus $I\le c_4|x|^{-(d+\beta)}$. On the other hand, note that symmetric α-stable process is a subordinate Brownian motion, so $p_0(t,x+z)\le p_0(t,x)$ if $|x+z|\ge|x|$ and $p_0(t,x)\le p_0(t,x+z)$ if $|x+z|\le|x|$. Hence, by (1.8) and the condition that $|x|^\alpha\ge t$, we obtain

$$\begin{aligned}II&\le \mathcal{A}(d,-\beta)\int_{|z|\ge|x|/2,|x+z|\ge|x|}2p_0(t,x)\frac{dz}{|z|^{d+\beta}}+\mathcal{A}(d,-\beta)\int_{|z|\ge|x|/2,|x+z|\le|x|}2p_0(t,x+z)\frac{dz}{|z|^{d+\beta}}\\ &\le 2\mathcal{A}(d,-\beta)p_0(t,x)\int_{|z|\ge|x|/2}\frac{dz}{|z|^{d+\beta}}+2^{d+1+\beta}\mathcal{A}(d,-\beta)|x|^{-(d+\beta)}\int_{z\in\mathbb{R}^d}p_0(t,x+z)\,dz\\ &\le c_5 t|x|^{-(d+\alpha)}|x|^{-\beta}+2^{d+1+\beta}\mathcal{A}(d,-\beta)|x|^{-(d+\beta)}\le c_6|x|^{-(d+\beta)}.\end{aligned}\tag{2.4}$$

This establishes the lemma. □

In order to get the upper bound estimates in (1.16) in terms of weight $M_{b,\lambda}$ rather than $\|b\|_\infty$, we define, for $t>0$, $\lambda>0$ and $x,y\in\mathbb{R}^d$, the function

$$\big|\Delta^{\beta/2}_{\lambda,x}\big|p_0(t,x,y)=\begin{cases}\mathcal{A}(d,-\beta)\big(\int_{|z|\le\lambda\wedge t^{1/\alpha}}|p_0(t,x+z,y)-p_0(t,x,y)-\frac{\partial}{\partial x}p_0(t,x,y)\cdot z|\frac{1}{|z|^{d+\beta}}\,dz\\ \quad+\int_{\lambda>|z|>(\lambda\wedge t^{1/\alpha})}|p_0(t,x+z,y)-p_0(t,x,y)|\frac{dz}{|z|^{d+\beta}}\big)\quad\text{for } |x-y|^\alpha\le t,\\ \mathcal{A}(d,-\beta)\big(\int_{|z|\le\lambda\wedge|x-y|/2}|p_0(t,x+z,y)-p_0(t,x,y)-\frac{\partial}{\partial x}p_0(t,x,y)\cdot z|\frac{1}{|z|^{d+\beta}}\,dz\\ \quad+\int_{\lambda>|z|>(\lambda\wedge|x-y|/2)}|p_0(t,x+z,y)-p_0(t,x,y)|\frac{dz}{|z|^{d+\beta}}\big)\quad\text{for } |x-y|^\alpha> t.\end{cases}$$

Observe that

$$\big|\Delta^{\beta/2}_{\lambda,x}\big|p_0(t,x,y)\le\big|\Delta^{\beta/2}_{x}\big|p_0(t,x,y).$$

Set

$$f_{0,\lambda}(t,x,y)=\begin{cases}t^{-(d+\beta)/\alpha} & \text{when } |x-y|\le t^{1/\alpha},\\ |x-y|^{-(d+\beta)}\mathbb{1}_{\{|x-y|\le\lambda\}}+|x-y|^{-(d+\alpha)}\mathbb{1}_{\{|x-y|>\lambda\}} & \text{when } |x-y|> t^{1/\alpha}.\end{cases}$$

Observe that when $\lambda=\infty$, $f_{0,\infty}$ is just the function f_0 defined in (2.1).

Lemma 2.3. *For each $\lambda>0$ and $T>0$, there exists a constant $C_8=C_8(d,\alpha,\beta,\lambda,T)>0$ such that*

$$\big|\Delta^{\beta/2}_{\lambda,x}\big|p_0(t,x,y)\le C_8 f_{0,\lambda}(t,x,y)\quad\text{on } (0,T]\times\mathbb{R}^d\times\mathbb{R}^d. \tag{2.5}$$

Proof. (i) We first consider the case $|x-y|^\alpha\le t$. Note that $|\Delta^{\beta/2}_{\lambda,x}|p_0(t,x,y)\le|\Delta^{\beta/2}_{x}|p_0(t,x,y)$. Hence, by the first part (i) in the proof of Lemma 2.2, there exists a positive constant c_1 so that

$$\big|\Delta^{\beta/2}_{\lambda,x}\big|p_0(t,x,y)\le c_1t^{-(d+\beta)/\alpha}.$$

(ii) Next, we consider the case $|x-y|^\alpha>t$. In this case

$$\begin{aligned}\big|\Delta^{\beta/2}_{\lambda,x}\big|p_0(t,x,y)\ &\le\ \mathcal{A}(d,-\beta)\int_{|z|\le|x-y|/2}\bigg|p_0(t,x+z,y)-p_0(t,x,y)-\frac{\partial}{\partial x}p_0(t,x,y)\cdot z\bigg|\frac{dz}{|z|^{d+\beta}}\\ &\qquad+\mathcal{A}(d,-\beta)\int_{\lambda\ge|z|\ge(\lambda\wedge|x-y|/2)}\big|p_0(t,x+z,y)-p_0(t,x,y)\big|\frac{dz}{|z|^{d+\beta}}\\ &=:I+II.\end{aligned}$$

By (2.3), there is a positive constant c_2 so that

$$I\le c_2t|x-y|^{-(d+\alpha+\beta)}\le c_3\big(|x-y|^{-(d+\beta)}\mathbb{1}_{\{|x-y|\le2\lambda\}}+|x-y|^{-(d+\alpha)}\mathbb{1}_{\{|x-y|>2\lambda\}}\big).$$

Here the last inequality holds since $t|x-y|^{-(d+\alpha+\beta)}\le T(2\lambda)^{-\beta}|x-y|^{-(d+\alpha)}$ when $|x-y|>2\lambda$ and $t|x-y|^{-(d+\alpha+\beta)}\le|x-y|^{-(d+\beta)}$ due to $|x-y|^\alpha\ge t$.

It is clear that $II=0$ if $|x-y|>2\lambda$. On the other hand, if $|x-y|\le2\lambda$, then there exists a positive constant c_4 so that $II\le c_4|x-y|^{-(d+\beta)}$ by (2.4). Finally, we note that $|x-y|^{-(d+\beta)}\asymp|x-y|^{-(d+\alpha)}$ for $\lambda<|x-y|\le2\lambda$. This establishes the lemma. □

For each $\lambda>0$ and $a\ge0$, we extend the definition of $f_{0,\lambda}(t,x,y)$ to define

$$f_{a,\lambda}(t,x,y):=\begin{cases}t^{-(d+\beta)/\alpha} & \text{when } |x-y|\le t^{1/\alpha},\\ |x-y|^{-(d+\beta)}\mathbb{1}_{\{|x-y|\le\lambda\}}+(|x-y|^{-(d+\alpha)}+a|x-y|^{-(d+\beta)})\mathbb{1}_{\{|x-y|>\lambda\}} & \text{when } |x-y|> t^{1/\alpha}.\end{cases} \tag{2.6}$$

Note that $f_{a,\infty}(t,x,y)=f_0(t,x,y)$.

Lemma 2.4. *For each $\lambda > 0$, there is a constant $C_9 = C_9(d, \alpha, \beta, \lambda) > 0$ such that for every $a \in [0, 1]$,*

$$\int_0^t \int_{\mathbb{R}^d} f_{a,\lambda}(s, z, y)\, dz\, ds \le C_9\bigl(t^{1-\beta/\alpha} + t\bigr), \quad t \in (0, \infty), y \in \mathbb{R}^d. \tag{2.7}$$

Proof. By the definition of $f_{a,\lambda}$,

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} f_{a,\lambda}(s, z, y)\, dz\, ds \\
&\quad\le \int_0^t \int_{|y-z|\le s^{1/\alpha}} s^{-(d+\beta)/\alpha}\, dz\, ds + \int_0^t \int_{\lambda \ge |y-z|>s^{1/\alpha}} \frac{1}{|y-z|^{d+\beta}}\, dz\, ds \\
&\qquad+ \int_0^t \int_{|y-z|\ge\lambda} \bigl(|y-z|^{-(d+\alpha)} + |y-z|^{-(d+\beta)}\bigr)\, dz\, ds \\
&\quad\le c_1 \int_0^t \bigl(s^{-\beta/\alpha} + 1\bigr)\, ds \le c_2\bigl(t^{1-\beta/\alpha} + t\bigr).
\end{aligned}$$

□

For every $a \ge 0$, define

$$g_a(t, x, y) = \begin{cases} t^{-d/\alpha} & \text{when } |x-y| \le t^{1/\alpha}, \\ \frac{t}{|x-y|^{d+\alpha}} + \frac{at}{|x-y|^{d+\beta}} & \text{when } |x-y| > t^{1/\alpha}. \end{cases} \tag{2.8}$$

Observe that

$$\int_{\mathbb{R}^d} g_a(t, x, y)\, dy \asymp 1 + a t^{1-\beta/\alpha} \quad \text{on } (0, \infty) \times \mathbb{R}^d. \tag{2.9}$$

Recall that $p_a(t, x, y)$ is the heat kernel of the operator $\Delta^{\alpha/2} + a\Delta^{\beta/2}$. Moreover, in view of (1.10),

$$g_a(t, x, y) \asymp p_a(t, x, y) \quad \text{on } (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d. \tag{2.10}$$

Lemma 2.5. *For each $\lambda > 0$ and $T > 0$, there exists $C_{10} = C_{10}(d, \alpha, \beta, \lambda, T) > 0$ such that for every $a \in [0, 1]$ and all $t \in (0, T]$, $x, y \in \mathbb{R}^d$,*

$$\int_0^t \int_{\mathbb{R}^d} g_a(t-s, x, z) f_{a,\lambda}(s, z, y)\, dz\, ds \le C_{10} g_a(t, x, y).$$

Proof. Denote by $I = \int_0^t \int_{\mathbb{R}^d} g_a(t-s, x, z) f_{a,\lambda}(s, z, y)\, dz\, ds$.

(i) Suppose that $|x-y| \le t^{1/\alpha}$. Then

$$\begin{aligned}
I &= \int_0^t \int_{|x-z|\le 2t^{1/\alpha}} g_a(t-s, x, z) f_{a,\lambda}(s, z, y)\, dz\, ds \\
&\quad+ \int_0^t \int_{|x-z|>2t^{1/\alpha}} g_a(t-s, x, z) f_{a,\lambda}(s, z, y)\, dz\, ds \\
&=: I_1 + I_2.
\end{aligned}$$

We write I_1 as

$$\begin{aligned}
I_1 &= \int_0^{t/2} \int_{|x-z|\le 2t^{1/\alpha}} g_a(t-s, x, z) f_{a,\lambda}(s, z, y)\, dz\, ds \\
&\quad+ \int_{t/2}^t \int_{|x-z|\le 2t^{1/\alpha}} g_a(t-s, x, z) f_{a,\lambda}(s, z, y)\, dz\, ds \\
&= I_{11} + I_{12}.
\end{aligned}$$

If $s \in (0, t/2)$, then $t - s \in (t/2, t)$. In this case, $g_a(t-s, x, z) \le c_1 t^{-d/\alpha}$ when $|x - z| \le 2t^{1/\alpha}$ by (2.8). Hence, by Lemma 2.4,

$$I_{11} \le c_1 t^{-d/\alpha} \int_0^t \int_{\mathbb{R}^d} f_{a,\lambda}(s, z, y)\, dz\, ds \le c_2 \big(T^{1-\beta/\alpha} + T\big) t^{-d/\alpha}.$$

When $s \in [t/2, t]$, since $|x - y| \le t^{1/\alpha}$ and $|x - z| \le 2t^{1/\alpha}$, $|y - z| \le 3t^{1/\alpha} \le 3(2s)^{1/\alpha}$. Thus $f_{a,\lambda}(s, z, y) \le c_3 s^{-(d+\beta)/\alpha} \le 2^{(d+\beta)/\alpha} c_3 t^{-(d+\beta)/\alpha}$. Hence,

$$I_{12} \le 2^{(d+\beta)/\alpha} c_3 t^{-(d+\beta)/\alpha} \int_0^t \int_{\mathbb{R}^d} g_a(t-s, x, z)\, dz\, ds \le c_4 T^{1-\beta/\alpha} \big(1 + T^{1-\beta/\alpha}\big) t^{-d/\alpha}.$$

Next we consider I_2. Noting that $|x - z| > 2t^{1/\alpha}$, so we have by (2.8) and Lemma 2.4,

$$\begin{aligned}
I_2 &\le c_5 \int_0^t \int_{|x-z|>2t^{1/\alpha}} \left(\frac{t-s}{|x-z|^{d+\alpha}} + \frac{t-s}{|x-z|^{d+\beta}} \right) f_{a,\lambda}(s, z, y)\, dz\, ds \\
&\le c_6 t^{-d/\alpha} \big(1 + t^{1-\beta/\alpha}\big) \int_0^t \int_{\mathbb{R}^d} f_{a,\lambda}(s, z, y)\, dz\, ds \\
&\le c_7 \big(1 + T^{1-\beta/\alpha}\big)\big(T^{1-\beta/\alpha} + T\big) t^{-d/\alpha}.
\end{aligned}$$

We thus conclude from the above that there is a $c_8 = c_8(d, \alpha, \beta, \lambda, T) > 0$ such that $I \le c_8 t^{-d/\alpha}$ for every $t \in (0, T]$ whenever $|x - y| \le t^{1/\alpha}$.

(ii) Next assume that $|x - y| > t^{1/\alpha}$. Then

$$\begin{aligned}
I &= \int_0^t \int_{|x-z|\le|x-y|/2} g_a(t-s, x, z) f_{a,\lambda}(s, z, y)\, dz\, ds \\
&\quad + \int_0^t \int_{|x-z|>|x-y|/2} g_a(t-s, x, z) f_{a,\lambda}(s, z, y)\, dz\, ds \\
&=: I_1 + I_2.
\end{aligned}$$

If $|x - z| \le |x - y|/2$, then $|y - z| \ge |x - y|/2 > t^{1/\alpha}/2$. Hence, there is a constant c_9 so that

$$f_{a,\lambda}(s, z, y) \le c_9 \big(|x-y|^{-(d+\alpha)} + a|x-y|^{-(d+\beta)}\big)$$

for $s \in (0, t)$. Therefore,

$$\begin{aligned}
I_1 &\le c_9 \big(|x-y|^{-(d+\alpha)} + a|x-y|^{-(d+\beta)}\big) \cdot \int_0^t \int_{\mathbb{R}^d} g_a(t-s, x, z)\, dz\, ds \\
&\le c_{10} \big(1 + T^{1-\beta/\alpha}\big) \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{at}{|x-y|^{d+\beta}} \right).
\end{aligned}$$

If $|x - z| > |x - y|/2$, then $|x - z| > t^{1/\alpha}/2$. Hence $g_a(t-s, x, z) \le c_{11}\big(\frac{t}{|x-y|^{d+\alpha}} + \frac{at}{|x-y|^{d+\beta}}\big)$ by (2.8). Thus by Lemma 2.4, we obtain

$$\begin{aligned}
I_2 &\le c_{11} \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{at}{|x-y|^{d+\beta}} \right) \int_0^t \int_{\mathbb{R}^d} f_{a,\lambda}(s, z, y)\, dz\, ds \\
&\le c_{12} \big(T^{1-\beta/\alpha} + T\big) \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{at}{|x-y|^{d+\beta}} \right).
\end{aligned}$$

This completes the proof of the lemma. □

3. Fundamental solution

Throughout the rest of this paper, $b(x,z)$ is a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying condition (1.2). Recall the definition of the non-local operator $\mathcal{S}^b$ from (1.3). Let $|q^b|_0(t,x,y) = p_0(t,x,y)$, and define for each $n \ge 1$,

$$|q^b|_n(t,x,y) = \int_0^t \int_{\mathbb{R}^d} |q^b|_{n-1}(t-s,x,z) \big|\mathcal{S}_z^b p_0(s,z,y)\big|\, dz\, ds.$$

For each $\lambda > 0$, define $b_\lambda(x,z) = b(x,z) 1_{\{|z|>\lambda\}}(z)$.

In view of (1.8), there exists a constant $C_{11} = C_{11}(d,\alpha,\beta) > 0$ such that $p_0(t,x,y) \le C_{11} g_a(t,x,y)$ for all $t > 0$, $a \in [0,1]$ and $x, y \in \mathbb{R}^d$, where g_a is the function defined by (2.8). On the other hand, note that

$$\begin{aligned}
\big|\mathcal{S}^b f(x)\big| &= \left|\mathcal{A}(d,-\beta) \int_{\mathbb{R}^d} \big(f(x+z) - f(x) - \langle \nabla f(x), z\rangle \mathbb{1}_{\{|z|\le\lambda\}}\big) \frac{b(x,z)}{|z|^{d+\beta}}\, dz\right| \\
&\le \left|\mathcal{A}(d,-\beta) \int_{|z|\le\lambda} \big(f(x+z) - f(x) - \langle \nabla f(x), z\rangle\big) \frac{b(x,z)}{|z|^{d+\beta}}\, dz\right| \\
&\quad + \left|\mathcal{A}(d,-\beta) \int_{\mathbb{R}^d} \big(f(x+z) - f(x)\big) \frac{b_\lambda(x,z)}{|z|^{d+\beta}}\, dz\right| \\
&\le \|b\|_\infty \cdot \big|\Delta_{\lambda,x}^{\beta/2}\big| f(x) + \|b_\lambda\|_\infty \cdot \big|\Delta_x^{\beta/2}\big| f(x),
\end{aligned}$$

where $|\Delta_{\lambda,x}^{\beta/2}| f(x)$ is defined in the similar way as $|\Delta_{\lambda,x}^{\beta/2}| p_0(t,x,y)$. Then by Lemma 2.2 and Lemma 2.3, for every $A > 0$, $\lambda > 0$ and $T > 0$ and every bounded function b with $\|b\|_\infty \le A$,

$$\begin{aligned}
\big|\mathcal{S}_z^b p_0(t,z,y)\big| &\le \|b\|_\infty \cdot \big|\Delta_{\lambda,z}^{\beta/2}\big| p_0(t,z,y) + \|b_\lambda\|_\infty \cdot \big|\Delta_z^{\beta/2}\big| p_0(t,z,y) \\
&\le C_8 A f_{0,\lambda}(t,z,y) + C_7 M_{b,\lambda} f_0(t,z,y) \\
&\le (C_7 + C_8) A f_{M_{b,\lambda}/A,\lambda}(t,z,y), \quad t \in (0,T].
\end{aligned} \tag{3.1}$$

Here recall that $M_{b,\lambda} = \operatorname{esssup}_{x,z\in\mathbb{R}^d, |z|>\lambda} |b(x,z)|$, $f_{a,\lambda}$ is the function defined in (2.6). The above estimate is a refinement of Lemma 2.2. The latter corresponds to the case where $\lambda = \infty$.

Lemma 3.1. *For each $\lambda > 0$, $A > 0$ and $T > 0$ and every bounded function b on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying condition (1.2) with $\|b\|_\infty \le A$,*

$$|q^b|_n(t,x,y) \le C_{11} \big(A(C_7 + C_8) C_{10}\big)^n g_{M_{b,\lambda}/A}(t,x,y) < \infty, \quad t \in (0,T], x, y \in \mathbb{R}^d. \tag{3.2}$$

Proof. We prove this lemma by induction. Since $p_0(t,x,y) \le C_{11} g_{M_{b,\lambda}/A}(t,x,y)$ and $M_{b,\lambda}/A \le 1$, in view of Lemma 2.5 and (3.1), (3.2) clearly holds for $n = 1$. Suppose that (3.2) holds for $n = j \ge 1$. Then by Lemma 2.5 and (3.1),

$$\begin{aligned}
&|q^b|_{j+1}(t,x,y) \\
&\quad \le C_{11} \big(A(C_7 + C_8) C_{10}\big)^j \int_0^t \int_{\mathbb{R}^d} g_{M_{b,\lambda}/A}(t-s,x,z) \big|\mathcal{S}_z^b p_0(s,z,y)\big|\, dz\, ds \\
&\quad \le C_{11} \big(A(C_7 + C_8) C_{10}\big)^j (C_7 + C_8) A \int_0^t \int_{\mathbb{R}^d} g_{M_{b,\lambda}/A}(t-s,x,z) f_{M_{b,\lambda}/A,\lambda}(s,z,y)\, dz\, ds \\
&\quad \le C_{11} \big(A(C_7 + C_8) C_{10}\big)^{j+1} g_{M_{b,\lambda}/A}(t,x,y)
\end{aligned}$$

for $t \in (0,T]$ and $x, y \in \mathbb{R}^d$. This proves that (3.2) holds for $n = j+1$ and thus for every $n \ge 1$. □

Now we define $q_n^b : (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ as follows. For $t>0$ and $x,y\in\mathbb{R}^d$, let $q_0^b(t,x,y)=p_0(t,x,y)$, and for each $n\ge 1$, define

$$q_n^b(t,x,y)=\int_0^t\int_{\mathbb{R}^d} q_{n-1}^b(t-s,x,z)\mathcal{S}_z^b p_0(s,z,y)\,dz\,ds. \tag{3.3}$$

Clearly by Lemma 3.1, each $q_n^b(t,x,y)$ is well defined.

Lemma 3.2. *For every $n\ge 0$, $q_n^b(t,x,y)$ is jointly continuous on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$.*

Proof. We prove it by induction. Clearly $q_0^b(t,x,y)$ is continuous on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$. Suppose that $q_n^b(t,x,y)$ is continuous on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$. For every $M\ge 2$, it follows from (3.1), Lemma 3.1 and the dominated convergence theorem that for $\varepsilon<1/(2M)$, $(t,x,y)\mapsto\int_\varepsilon^{t-\varepsilon}\int_{\mathbb{R}^d} q_n^b(t-s,x,z)\mathcal{S}_z^b p_0(s,z,y)\,dz\,ds$ is jointly continuous on $[1/M,M]\times\mathbb{R}^d\times\mathbb{R}^d$.

On the other hand, it follows from (3.1) and (2.9) that

$$\begin{aligned}
&\sup_{t\in[1/M,M]}\sup_{x,y}\int_{t-\varepsilon}^t\int_{\mathbb{R}^d} g_{M_b,\lambda}(t-s,x,z)\big|\mathcal{S}_z^b p_0(s,z,y)\big|\,dz\,ds\\
&\quad\le c_1A\Big(\sup_{t\in[1/M,M]}\big[(t-\varepsilon)^{-(d+\beta)/\alpha}+(t-\varepsilon)^{-(d+\alpha)/\alpha}\big]\Big)\sup_{x\in\mathbb{R}^d}\sup_{t\in[1/M,M]}\int_{t-\varepsilon}^t\int_{\mathbb{R}^d} g_{M_b,\lambda}(t-s,x,z)\,dz\,ds\\
&\quad\le c_2A(2M)^{(d+\alpha)/\alpha}\int_0^\varepsilon\big(1+r^{1-\beta/\alpha}\big)\,dr\le c_3A(2M)^{(d+\alpha)/\alpha}\varepsilon,
\end{aligned}$$

which goes to zero as $\varepsilon\to 0$; while by (3.1) and (2.7), there exist c_4 and c_5 such that

$$\begin{aligned}
&\sup_{t\in[1/M,M]}\sup_{x,y}\int_0^\varepsilon\int_{\mathbb{R}^d} g_{M_b,\lambda}(t-s,x,z)\big|\mathcal{S}_z^b p_0(s,z,y)\big|\,dz\,ds\\
&\quad\le c_4\Big(\sup_{t\in[1/M,M]}(t-\varepsilon)^{-d/\alpha}\Big)\sup_{y\in\mathbb{R}^d}\int_0^\varepsilon\int_{\mathbb{R}^d}\big|\mathcal{S}_z^b p_0(s,z,y)\big|\,dz\,ds\\
&\quad\le c_5(2M)^{d/\alpha}\|b\|_\infty\varepsilon^{1-\beta/\alpha}\to 0
\end{aligned} \tag{3.4}$$

as $\varepsilon\to 0$. We conclude from Lemma 3.1, (3.3) and the above argument that $q_{n+1}^b(t,x,y)$ is jointly continuous in $(t,x,y)\in[1/M,M]\times\mathbb{R}^d\times\mathbb{R}^d$ and so in $(t,x,y)\in(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$. This completes the proof of the lemma. $\square$

Recall $f_0(t,x,y)$ is the function defined in (2.1) and $|\Delta_x^{\beta/2}|p_0(t,x,y)\le C_7 f_0(t,x,y)$ on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$.

Lemma 3.3. *There is a constant $C_{12}=C_{12}(d,\alpha,\beta)>0$ so that for every $A>0$ and every bounded function b on $\mathbb{R}^d\times\mathbb{R}^d$ with $\|b\|_\infty\le A$ and for every integer $n\ge 0$ and $\varepsilon>0$,*

$$\left|\int_{\{z\in\mathbb{R}^d:|z|>\varepsilon\}}\big(q_n^b(t,x+z,y)-q_n^b(t,x,y)\big)\frac{\mathcal{A}(d,-\beta)b(x,z)}{|z|^{d+\beta}}\,dz\right|\le (C_{12}A)^{n+1}f_0(t,x,y) \tag{3.5}$$

for $(t,x,z)\in(0,1]\times\mathbb{R}^d\times\mathbb{R}^d$, and $\mathcal{S}_x^b q_n^b(t,x,y)$ exists pointwise for $(t,x,z)\in(0,1]\times\mathbb{R}^d\times\mathbb{R}^d$ in the sense of (1.5) with

$$\mathcal{S}_x^b q_{n+1}^b(t,x,y)=\int_0^t\int_{\mathbb{R}^d}\mathcal{S}_x^b q_n^b(t-s,x,z)\mathcal{S}_z^b p_0(s,z,y)\,dz\,ds \tag{3.6}$$

and

$$\big|\mathcal{S}_x^b q_n^b(t,x,y)\big|\le (C_{12}A)^{n+1}f_0(t,x,y)\quad\text{on }(0,1]\times\mathbb{R}^d\times\mathbb{R}^d. \tag{3.7}$$

Moreover,

$$q^b_{n+1}(t,x,y)=\int_0^t\int_{\mathbb{R}^d} p_0(t-s,x,z)\mathcal{S}^b_z q^b_n(s,z,y)\,dz\,ds \quad \textit{for } (t,x,y)\in(0,1]\times\mathbb{R}^d\times\mathbb{R}^d. \tag{3.8}$$

Proof. Let $q(t,x,y)$ denote the transition density function of the symmetric β-stable process on $\mathbb{R}^d$. Then by (1.8) but with β in place of α, we have $q(t,x,y)\asymp t^{-d/\beta}(1\wedge\frac{t^{1/\beta}}{|x-y|})^{d+\beta}$ for $(t,x,y)\in(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$. Observe that (2.1) yields $f_0(t,x,y)\asymp t^{-\beta/\alpha}q(t^{\beta/\alpha},x,y)$ for $(t,x,y)\in(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$. Hence on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$,

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^d} f_0(t-s,x,z)f_0(s,z,y)\,ds\,dz\\
&\quad\asymp\int_0^t (t-s)^{-\beta/\alpha}s^{-\beta/\alpha}\left(\int_{\mathbb{R}^d} q\big((t-s)^{\beta/\alpha},x,z\big)q\big(s^{\beta/\alpha},z,y\big)\,dz\right)ds\\
&\quad\asymp q\big(t^{\beta/\alpha},x,y\big)\int_0^t (t-s)^{-\beta/\alpha}s^{-\beta/\alpha}\,ds\\
&\quad= q\big(t^{\beta/\alpha},x,y\big)t^{1-(2\beta/\alpha)}\int_0^1 (1-u)^{-\beta/\alpha}u^{-\beta/\alpha}\,du\\
&\quad\asymp t^{1-\beta/\alpha}f_0(t,x,y).
\end{aligned}$$

In the second $\asymp$ above, we used the fact that $(t/2)^{\beta/\alpha}\le(t-s)^{\beta/\alpha}+s^{\beta/\alpha}\le 2t^{\beta/\alpha}$ for every $s\in(0,t)$, while in the last equality, we used a change of variable $s=tu$. So there is a constant $c_1=c_1(d,\alpha,\beta)>0$ so that

$$\int_0^t\int_{\mathbb{R}^d} f_0(t-s,x,z)f_0(s,z,y)\,ds\,dz\le c_1 f_0(t,x,y)\quad\text{for every } t\in(0,1] \text{ and } x,y\in\mathbb{R}^d. \tag{3.9}$$

By increasing the value of c_1 if necessary, we may and do assume that c_1 is larger than 1.

We now proceed by induction. Let $C_{12}:=c_1C_7$. Note that

$$\big|\mathcal{S}^b_x p_0(t,x,y)\big|\le A\big|\Delta_x^{\beta/2}\big|p_0(t,x,y)\le C_7Af_0(t,x,y). \tag{3.10}$$

When $n=0$, (3.8) holds by definition. By Lemma 2.2, (3.5) and (3.7) hold for $n=0$. Suppose that (3.5) and (3.7) hold for $n=j$. Then for every $\varepsilon>0$, by the definition of q^b_{j+1}, Lemma 3.1, (3.9) and Fubini's theorem,

$$\begin{aligned}
&\int_{\{w\in\mathbb{R}^d:|\omega|>\varepsilon\}}\big(q^b_{j+1}(t,x+w,y)-q^b_{j+1}(t,x,y)\big)\frac{\mathcal{A}(d,-\beta)b(x,w)}{|w|^{d+\beta}}\,dw\\
&\quad=\int_0^t\int_{\mathbb{R}^d}\left(\int_{\{w\in\mathbb{R}^d:|w|>\varepsilon\}}\big(q^b_j(t-s,x+w,z)-q^b_j(t-s,x,z)\big)\frac{\mathcal{A}(d,-\beta)b(x,w)}{|w|^{d+\beta}}\,dw\right)\\
&\qquad\times\mathcal{S}^b_z p_0(s,z,y)\,dz\,ds\\
&\quad\le\int_0^t\int_{\mathbb{R}^d}(C_{12}A)^{j+1}f_0(t-s,x,z)\big|\mathcal{S}^b_z p_0(s,z,y)\big|\,dz\,ds\\
&\quad\le\int_0^t\int_{\mathbb{R}^d}(C_{12}A)^{j+1}f_0(t-s,x,z)C_7Af_0(s,z,y)\,dz\,ds\le(C_{12}A)^{j+2}f_0(t,x,y).
\end{aligned}$$

We conclude that

$$\begin{aligned}
&S^b_x q^b_{j+1}(t,x,y)\\
&\quad:=\lim_{\varepsilon\to0}\int_{\{w\in\mathbb{R}^d:|w|>\varepsilon\}}\big(q^b_{j+1}(t,x+w,y)-q^b_{j+1}(t,x,y)\big)\frac{\mathcal{A}(d,-\beta)b(x,w)}{|w|^{d+\beta}}\,dw
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \int_{\mathbb{R}^d} \left(\lim_{\varepsilon \to 0} \int_{\{w\in\mathbb{R}^d: |w|>\varepsilon\}} \big(q_j^b(t-s,x+w,z) - q_j^b(t-s,x,z)\big) \frac{\mathcal{A}(d,-\beta)b(x,w)}{|w|^{d+\beta}}\,dw \right) \\
&\quad \times \mathcal{S}_z^b p_0(s,z,y)\,dz\,ds \\
&= \int_0^t \int_{\mathbb{R}^d} \mathcal{S}_x^b q_j^b(t-s,x,z) \mathcal{S}_z^b p_0(s,z,y)\,dz\,ds
\end{aligned}
$$

exists and (3.6) as well as (3.7) holds for $n = j + 1$. (The same proof verifies (3.6) when $n = 0$.) On the other hand, using (3.3) for $n = j + 1$ and (3.6)–(3.8) for $n = j$, we have by Fubini's theorem that (3.8) also holds for $n = j + 1$. (See arXiv version [13] of this paper for details.) The lemma is now established by induction. □

Recall that $M_{b,\lambda} = \operatorname{esssup}_{x,z\in\mathbb{R}^d,|z|>\lambda} |b(x,z)| = \|b_\lambda(x,z)\|_\infty$.

Lemma 3.4. *For each $\lambda > 0$, there are positive constants $A_0 = A_0(d,\alpha,\beta,\lambda)$ and $C_{13} = C_{13}(d,\alpha,\beta,\lambda)$ so that if $\|b\|_\infty \le A_0$, then for every integer $n \ge 0$,*

$$
\big|q_{n+1}^b(t,x,y)\big| \le C_{13} 2^{-n} p_{M_{b,\lambda}}(t,x,y) \quad \textit{for } t\in(0,1] \textit{ and } x,y\in\mathbb{R}^d, \tag{3.11}
$$

(3.5) holds and so $\mathcal{S}_x^b q_n^b(t,x,y)$ exists pointwise in the sense of (1.5) with

$$
\big|\mathcal{S}_x^b q_n^b(t,x,y)\big| \le 2^{-n} f_0(t,x,y) \quad \textit{for } t\in(0,1] \textit{ and } x,y\in\mathbb{R}^d, \tag{3.12}
$$

and

$$
\sum_{n=0}^\infty q_n^b(t,x,y) \ge \frac{1}{2} p_0(t,x,y) \quad \textit{for } t\in(0,1] \textit{ and } |x-y| \le 3t^{1/\alpha}. \tag{3.13}
$$

Proof. We take a positive constant A_0 so that $A_0 \le 1 \wedge [2(C_7 + C_8)C_{10} + 2C_{12}]^{-1}$. We have by Lemma 3.1 and Lemma 3.3 that for every b with $\|b\|_\infty \le A_0$,

$$
\big|q_{n+1}^b(t,x,y)\big| \le C_{11} 2^{-n} g_{M_{b,\lambda}/A_0}(t,x,y) \le C_{11} A_0^{-1} 2^{-n} g_{M_{b,\lambda}}(t,x,y) \quad \text{and} \quad \big|\mathcal{S}_x^b q_n^b(t,x,y)\big| \le 2^{-n} f_0(t,x,y)
$$

for every $t\in(0,1]$ and $x,y\in\mathbb{R}^d$. This together with (2.10) establishes (3.11) and (3.12).

On the other hand, by (2.8), there exists $c = c(d,\alpha,\beta) \ge 1$ so that $g_a(t,x,y) \le c p_0(t,x,y)$ for $a\in[0,1]$ and $|x-y| \le 3t^{1/\alpha}$ and $t\in(0,1]$. Take A_0 small enough so that $A_0 \le 1\wedge[2(C_7+C_8)C_{10}+2C_{12}]^{-1}$ and $\sum_{n=1}^\infty (A_0(C_7+C_8)C_{10})^n \le \frac{1}{2cC_{11}}$. Then for every b with $\|b\|_\infty \le A_0$, we have by Lemma 3.1 for $|x-y| \le 3t^{1/\alpha}$ and $t\in(0,1]$ that

$$
\sum_{n=1}^\infty \big|q^b\big|_n(t,x,y) \le cC_{11} \sum_{n=1}^\infty \big(A_0(C_7+C_8)C_{10}\big)^n p_0(t,x,y) \le \frac{1}{2} p_0(t,x,y).
$$

Consequently, for $|x-y| \le 3t^{1/\alpha}$ and $t\in(0,1]$,

$$
\sum_{n=0}^\infty q_n^b(t,x,y) \ge p_0(t,x,y) - \sum_{n=1}^\infty \big|q_n^b(t,x,y)\big| \ge \frac{1}{2} p_0(t,x,y).
$$

□

We now extend the results in Lemma 3.4 to any bounded b that satisfies condition (1.2). For $\lambda > 0$, define

$$
b^{(\lambda)}(x,z) = \lambda^{\beta/\alpha-1} b\big(\lambda^{-1/\alpha}x, \lambda^{-1/\alpha}z\big). \tag{3.14}
$$

For a function f on $\mathbb{R}^d$, set $f^{(\lambda)}(x) := f(\lambda^{-1/\alpha}x)$. By a change of variable, one has from (1.1) and (1.3) that

$$
\Delta^{\alpha/2} f^{(\lambda)}(x) = \lambda^{-1}\big(\Delta^{\alpha/2} f\big)\big(\lambda^{-1/\alpha}x\big) \quad \text{and} \quad \mathcal{S}^{b^{(\lambda)}} f^{(\lambda)}(x) = \lambda^{-1}\big(\mathcal{S}^b f\big)\big(\lambda^{-1/\alpha}x\big). \tag{3.15}
$$

We remark here that condition (1.2) used in establishing (3.15). Note that the transition density function $p_0(t,x,y)$ of the symmetric α-stable process has the following scaling property:

$$p_0(t,x,y) = \lambda^{-d/\alpha} p_0\big(\lambda^{-1}t, \lambda^{-1/\alpha}x, \lambda^{-1/\alpha}y\big). \tag{3.16}$$

Recall $q_n^b(t,x,y)$ is the function defined inductively by (3.3) with $q_0^b(t,x,y) := p_0(t,x,y)$.

Lemma 3.5. *Suppose that b is a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1.2). For every $\lambda > 0$ and for every integer $n \ge 0$ and $t > 0$,*

$$q_n^{b^{(\lambda)}}(t,x,y) = \lambda^{-d/\alpha} q_n^b\big(\lambda^{-1}t, \lambda^{-1/\alpha}x, \lambda^{-1/\alpha}y\big), \quad x, y \in \mathbb{R}^d; \tag{3.17}$$

or, equivalently,

$$q_n^b(t,x,y) = \lambda^{d/\alpha} q_n^{b^{(\lambda)}}\big(\lambda t, \lambda^{1/\alpha}x, \lambda^{1/\alpha}y\big), \quad x, y \in \mathbb{R}^d. \tag{3.18}$$

Proof. We prove it by induction. Clearly in view of (3.16), (3.17) holds when $n = 0$. Suppose that (3.17) holds for $n = j \ge 0$. Then by the definition (3.3), (3.15) and (3.16),

$$\begin{aligned}
q_{j+1}^{b^{(\lambda)}}(t,x,y) &= \int_0^t \int_{\mathbb{R}^d} q_j^{b^{(\lambda)}}(t-s,x,z) \mathcal{S}_z^{b^{(\lambda)}} p_0(s,z,y)\, dz\, ds \\
&= \int_0^t \int_{\mathbb{R}^d} \lambda^{-d/\alpha} q_j^b\big(\lambda^{-1}(t-s), \lambda^{-1/\alpha}x, \lambda^{-1/\alpha}z\big) \lambda^{-d/\alpha-1} \big(\mathcal{S}_z^b p_0\big(\lambda^{-1}s, \cdot, \lambda^{-1/\alpha}y\big)\big)\big(\lambda^{-1/\alpha}z\big)\, dz\, ds \\
&= \lambda^{-d/\alpha} \int_0^{\lambda^{-1}t} \int_{\mathbb{R}^d} q_j^b\big(\lambda^{-1}t - r, \lambda^{-1/\alpha}x, w\big) \big(\mathcal{S}_w^b p_0\big(r, \cdot, \lambda^{-1/\alpha}y\big)\big)(w)\, dw\, dr \\
&= \lambda^{-d/\alpha} q_{j+1}^b\big(\lambda^{-1}t, \lambda^{-1/\alpha}x, \lambda^{-1/\alpha}y\big).
\end{aligned}$$

This proves that (3.17) holds for $n = j+1$ and so, by induction, it holds for every $n \ge 0$. $\square$

Recall that A_0 is the positive constant in Lemma 3.4.

Theorem 3.6. *For every $\lambda > 0$ and $A > 0$, there is a positive constant $C_{14} = C_{14}(d,\alpha,\beta,A,\lambda) > 0$ so that for every bounded function b with $\|b\|_\infty \le A$, that satisfies condition (1.2) and $n \ge 0$,*

$$\big|q_n^b(t,x,y)\big| \le C_{14} 2^{-n} \left(t^{-d/\alpha} \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{M_{b,\lambda} t}{|x-y|^{d+\beta}} \right) \right) \tag{3.19}$$

for every $0 < t \le 1 \wedge (A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}$ and $x, y \in \mathbb{R}^d$, and

$$\sum_{n=0}^\infty q_n^b(t,x,y) \ge \frac{1}{2} p_0(t,x,y) \quad \text{for } 0 < t \le 1 \wedge \big(A_0/\|b\|_\infty\big)^{\alpha/(\alpha-\beta)} \text{ and } |x-y| \le 3t^{1/\alpha}. \tag{3.20}$$

Moreover, for every $n \ge 0$, (3.5) holds and so $\mathcal{S}_x^b q_n^b(t,x,y)$ exists pointwise in the sense of (1.5) with

$$\big|\mathcal{S}_x^b q_n^b(t,x,y)\big| \le 2^{-n}\big(\|b\|_\infty/A_0\big) f_0(t,x,y) \tag{3.21}$$

for every $0 < t \le 1 \wedge (A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}$ and $x, y \in \mathbb{R}^d$. Moreover, (3.6) and (3.8) hold.

Proof. In view of Lemma 3.4, it suffices to prove the theorem for $A_0 < \|b\|_\infty \le A$. Set $r = (\|b\|_\infty/A_0)^{\alpha/(\alpha-\beta)}$. The function $b^{(r)}$ defined by (3.14) has the property $\|b^{(r)}\|_\infty = A_0$. Thus by Lemma 3.4, there is a constant $C_{14} = C_{14}(d,\alpha,\beta,A,\lambda) := C_{13}(d,\alpha,\beta,r^{1/\alpha}\lambda) > 0$ so that for every integer $n \ge 0$,

$$\big|q_n^{b^{(r)}}(t,x,y)\big| \le C_{14} 2^{-n} p_{M_{b^{(r)},r^{1/\alpha}\lambda}}(t,x,y) \quad \text{for } t \in (0,1] \text{ and } x, y \in \mathbb{R}^d. \tag{3.22}$$

Noting $r^{1-\beta/\alpha} M_{b^{(r)}, r^{1/\alpha}\lambda} = M_{b,\lambda}$, we have by (3.18), (3.22) and (1.10) that for every $0 < t \le 1/r = (A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}$ and $x, y \in \mathbb{R}^d$,

$$
\begin{aligned}
\left|q_n^b(t,x,y)\right| &= r^{d/\alpha}\left|q_n^{b^{(r)}}\left(rt, r^{1/\alpha}x, r^{1/\alpha}y\right)\right| \\
&\le C_{14} 2^{-n} r^{d/\alpha} p_{M_{b^{(r)}, r^{1/\alpha}\lambda}}\left(rt, r^{1/\alpha}x, r^{1/\alpha}y\right) \\
&\le 2CC_{14}2^{-n}\left(t^{-d/\alpha} \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{r^{1-\beta/\alpha} M_{b^{(r)}, r^{1/\alpha}\lambda} t}{|x-y|^{d+\beta}}\right)\right) \\
&\le 2CC_{14}2^{-n}\left(t^{-d/\alpha} \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{M_{b,\lambda} t}{|x-y|^{d+\beta}}\right)\right),
\end{aligned}
$$

which establishes (3.19). Similarly, (3.20) follows from (3.16) and(3.13) with b replaced by $b^{(r)}$, while the conclusion of (3.21) is a direct consequence of (3.15), (3.18) and (3.12) with b replaced by $b^{(r)}$. That (3.6) and (3.8) hold follows directly from Lemma 3.3 and Lemma 3.5. □

Recall that $q^b(t,x,y) := \sum_{n=0}^{\infty} q_n^b(t,x,y)$, whenever it is convergent. The following theorem follows immediately from Lemmas 3.2, 3.4 and Theorem 3.6.

Theorem 3.7. *For every $\lambda > 0$ and $A > 0$, let $C_{14} = C_{14}(d, \alpha, \beta, A, \lambda)$ be the constant in Theorem 3.6. Then for every bounded function b with $\|b\|_\infty \le A$ that satisfies condition (1.2), $q^b(t,x,y)$ is well defined and is jointly continuous in $(0, 1 \wedge (A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}] \times \mathbb{R}^d \times \mathbb{R}^d$. Moreover,*

$$
\left|q^b(t,x,y)\right| \le 2C_{14}\left(t^{-d/\alpha} \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{M_{b,\lambda} t}{|x-y|^{d+\beta}}\right)\right) \tag{3.23}
$$

and $\mathcal{S}_x^b q^b(t,x,y)$ exists pointwise in the sense of (1.5) with

$$
\left|\mathcal{S}_x^b q^b(t,x,y)\right| \le 2\left(\|b\|_\infty/A_0\right) f_0(t,x,y)
$$

for every $0 < t \le 1 \wedge (A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}$ and $x, y \in \mathbb{R}^d$, and

$$
q^b(t,x,y) \ge \frac{1}{2} p_0(t,x,y) \quad \text{for } 0 < t \le 1 \wedge \left(A_0/\|b\|_\infty\right)^{\alpha/(\alpha-\beta)} \text{ and } |x-y| \le 3t^{1/\alpha}. \tag{3.24}
$$

Moreover, for every $0 < t \le 1 \wedge (A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}$ and $x, y \in \mathbb{R}^d$,

$$
\begin{aligned}
q^b(t,x,y) &= p_0(t,x,y) + \int_0^t \int_{\mathbb{R}^d} q^b(t-s,x,z) \mathcal{S}_z^b p_0(s,z,y)\,dz\,ds && (3.25) \\
&= p_0(t,x,y) + \int_0^t \int_{\mathbb{R}^d} p_0(t-s,x,z) \mathcal{S}_z^b q^b(s,z,y)\,dz\,ds. && (3.26)
\end{aligned}
$$

Theorem 3.8. *Suppose that b is a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1.2). Let A_0 be the constant in Lemma 3.4. Then for every $t, s > 0$ with $t + s \le 1 \wedge (A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}$ and $x, y \in \mathbb{R}^d$,*

$$
\int_{\mathbb{R}^d} q^b(t,x,z) q^b(s,z,y)\,dz = q^b(t+s,x,y). \tag{3.27}
$$

Proof. In view of Theorem 3.6, we have

$$
\int_{\mathbb{R}^d} q^b(t,x,z) q^b(s,z,y)\,dz = \sum_{j=0}^{\infty} \sum_{n=0}^{j} \int_{\mathbb{R}^d} q_n^b(t,x,z) q_{j-n}^b(s,z,y)\,dz.
$$

So it suffices to show that for every $j \geq 0$,

$$\sum_{n=0}^{j} \int_{\mathbb{R}^d} q_n^b(t, x, z) q_{j-n}^b(s, z, y)\, dz = q_j^b(t + s, x, y). \tag{3.28}$$

Clearly, (3.28) holds for $j = 0$. Suppose that (3.28) holds for $j = l \geq 1$. Then by (3.3) and Fubini's theorem and using the estimates in (3.1) and Theorem 3.6, one can prove that (3.28) holds for $j = l + 1$; see arXiv version [13] of this paper for details. By induction, (3.28) holds for every $j \geq 0$. □

For notational simplicity, denote $1 \wedge (A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}$ by δ_0. In view of Theorem 3.8, we can uniquely extend the definition of $q^b(t, x, y)$ to $t > \delta_0$ by using the Chapman–Kolmogorov equation recursively as follows.

Suppose that $q^b(t, x, y)$ has been defined and satisfies the Chapman–Kolmogorov equation (3.27) on $(0, k\delta_0] \times \mathbb{R}^d \times \mathbb{R}^d$. Then for $t \in (k\delta_0, (k + 1)\delta_0]$, define

$$q^b(t, x, y) = \int_{\mathbb{R}^d} q^b(s, x, z) q^b(r, z, y)\, dz, \quad x, y \in \mathbb{R}^d \tag{3.29}$$

for any $s, r \in (0, k\delta_0]$ so that $s + r = t$. Such $q^b(t, x, y)$ is well defined on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and satisfies (3.27) for every $s, t > 0$. Moreover, since Chapman–Kolmogorov equation holds for $q^b(t, x, y)$ for all $t, s > 0$, we have by Theorem 3.7 and (1.10) that for every $A \geq A_0$, there are constants $c_i = c_i(d, \alpha, \beta, A, \lambda)$, $i = 1, 2$, so that for every $b(x, z)$ satisfying (1.2) with $\|b\|_\infty \leq A$,

$$\left|q^b(t, x, y)\right| \leq c_1 e^{c_2 t} p_{M_{b,\lambda}}(t, x, y) \quad \text{for every } t > 0 \text{ and } x, y \in \mathbb{R}^d. \tag{3.30}$$

Theorem 3.9. *$q^b(t, x, y)$ satisfies (3.25) and (3.26) for every $t > 0$ and $x, y \in \mathbb{R}^d$.*

Proof. Let $\delta_0 := 1 \wedge (A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}$. It suffices to prove that for every $n \geq 1$, (3.25) and (3.26) hold for all $t \in (0, n\delta_0]$ and $x, y \in \mathbb{R}^d$. Clearly, (3.25) holds for $t \in (0, n\delta_0]$ with $n = 1$. Suppose that (3.25) holds for $t \in (0, n\delta_0]$ with $n = k$. For $t \in (k\delta_0, (k + 1)\delta_0]$, take $l, s \in (0, k\delta_0]$ so that $l + s = t$. Then we can verify by Chapman–Kolmogrov equation of q^b, Fubini's theorem, Lemma 2.5, (3.1) and (3.30) that $q^b(l + s)$ satisfies (3.25) and (3.26). See the arXiv version [13] of this paper for details. By induction, (3.25) and (3.26) hold for every $t > 0$ and $x, y \in \mathbb{R}^d$. □

Theorem 3.10. *Suppose that b is a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1.2). Then $q^b(t, x, y)$ is the unique continuous kernel that satisfies the Chapman–Kolmogorov equation (3.27) on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and that for some $\varepsilon > 0$,*

$$\left|q^b(t, x, y)\right| \leq c p_1(t, x, y) \tag{3.31}$$

and (3.25) hold for $(t, x, y) \in (0, \varepsilon] \times \mathbb{R}^d \times \mathbb{R}^d$. Moreover, (3.30) holds for $q^b(t, x, y)$.

Proof. Suppose that $\overline{q}$ is any continuous kernel that satisfies, for some $\varepsilon > 0$, (3.25) and (3.31) hold for $(t, x, y) \in (0, \varepsilon] \times \mathbb{R}^d \times \mathbb{R}^d$. Without loss of generality, we may and do assume that $\varepsilon < 1 \wedge (A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}$. Using (3.25) recursively, one gets

$$\overline{q}(t, x, y) = \sum_{j=0}^{n} q_j^b(t, x, y) + \int_0^t \int_{\mathbb{R}^d} \overline{q}(t - s, x, z) \left(\mathcal{S}^b p_0\right)_z^{*,n+1}(s, z, y)\, ds\, dz. \tag{3.32}$$

Here $(\mathcal{S}^b p_0)_z^{*,n}(s, z, y)$ denotes the nth convolution operation of the function $\mathcal{S}_z^b p_0(s, z, y)$; that is, $(\mathcal{S}^b p_0)_z^{*,1}(s, z, y) = \mathcal{S}_z^b p_0(s, z, y)$ and

$$\left(\mathcal{S}^b p_0\right)_z^{*,n}(s, z, y) = \int_0^s \int_{\mathbb{R}^d} \mathcal{S}_z^b p_0(r, z, w) \left(\mathcal{S}^b p_0\right)_w^{*,n-1}(s - r, w, y)\, dw\, dr \quad \text{for } n \geq 2. \tag{3.33}$$

It follows from (3.6) that $\mathcal{S}^b_z q^b_n(s,z,y) = (\mathcal{S}^b p_0)^{*,n+1}_z(s,z,y)$. Thus, by (3.32) we have

$$\overline{q}(t,x,y) = \sum_{j=0}^n q^b_j(t,x,y) + \int_0^t \int_{\mathbb{R}^d} \overline{q}(t-s,x,z)\mathcal{S}^b_z q^b_n(s,z,y)\,ds\,dz. \tag{3.34}$$

By the condition (3.31) and (3.21), there is a constant $c_1 > 0$ so that for every $n \geq 1$,

$$\left| \int_0^t \int_{\mathbb{R}^d} \overline{q}(t-s,x,z)\big(\mathcal{S}^b p_0\big)^{*,n}_z(s,z,y)\,ds\,dz \right| \leq c_1 2^{-n} \int_0^t \int_{\mathbb{R}^d} p_1(t-s,x,z) f_0(s,z,y)\,ds\,dz.$$

Noting that $p_1(t,x,y) \asymp g_1(t,x,y)$ on $(0,1] \times \mathbb{R}^d \times \mathbb{R}^d$ and

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^d} f_0(s,z,y)\,dz\,ds &\leq \int_0^t \int_{|y-z|\leq s^{1/\alpha}} s^{-(d+\beta)/\alpha}\,dz\,ds \\ &\quad + \int_0^t \int_{|y-z|>s^{1/\alpha}} \frac{1}{|y-z|^{d+\beta}}\,dz\,ds \\ &= c_2 t^{1-\beta/\alpha}. \end{aligned} \tag{3.35}$$

Then by the similar proof in Lemma 2.5, we can get

$$\int_0^t \int_{\mathbb{R}^d} p_1(t-s,x,z) f_0(s,z,y)\,ds\,dz \leq c_3 p_1(t,x,y).$$

It follows that $\overline{q}(t,x,y) = \sum_{n=0}^\infty q^b_n(t,x,y) = q^b(t,x,y)$ for every $t \in (0,\varepsilon]$ and $x,y \in \mathbb{R}^d$. Since both $\overline{q}$ and q^b satisfy the Chapman–Kolmogorov equation (3.27), $\overline{q} = q^b$ on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$. $\square$

In view of Lemma 3.5 and Chapman–Kolmogorov equation, we have the following.

Theorem 3.11. *Suppose that b is a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1.2). $q^b(t,x,y) = \lambda^{d/\alpha} q^{b^{(\lambda)}}(\lambda t, \lambda^{1/\alpha}x, \lambda^{1/\alpha}y)$ on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$, where $b^{(\lambda)}(x,z) := \lambda^{\beta/\alpha-1} b(\lambda^{-1/\alpha}x, \lambda^{-1/\alpha}z)$.*

For a bounded function f on $\mathbb{R}^d$, $t > 0$ and $x \in \mathbb{R}^d$, we define

$$T^b_t f(x) = \int_{\mathbb{R}^d} q^b(t,x,y) f(y)\,dy \quad \text{and} \quad P_t f(x) = \int_{\mathbb{R}^d} p_0(t,x,y) f(y)\,dy.$$

The following lemma follows immediately from (3.27) and (3.29).

Lemma 3.12. *Suppose that b is a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1.2). For all $s,t > 0$, we have $T^b_{t+s} = T^b_t T^b_s$.*

Theorem 3.13. *Let b be a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1.2). Then for every $f \in C^2_b(\mathbb{R}^d)$,*

$$T^b_t f(x) - f(x) = \int_0^t T^b_s \mathcal{L}^b f(x)\,ds \quad \text{for every } t > 0, x \in \mathbb{R}^d.$$

Proof. Note that by Theorem 3.9, for each bounded Borel function f in $\mathbb{R}^d$,

$$T^b_t f(x) = P_t f(x) + \int_0^t T^b_{t-s} \mathcal{S}^b P_s f(x)\,ds = P_t f(x) + \int_0^t T^b_s \mathcal{S}^b P_{t-s} f(x)\,ds. \tag{3.36}$$

Hence, for $f \in C_b^2(\mathbb{R}^d)$,

$$\begin{aligned}
&T_t^b f(x) - f(x) \\
&\quad = P_t f(x) - f(x) + \int_0^t T_s^b \mathcal{S}^b f(x)\,ds + \int_0^t T_s^b \mathcal{S}^b (P_{t-s} f - f)(x)\,ds \\
&\quad = \int_0^t P_s \Delta^{\alpha/2} f(x)\,ds + \int_0^t T_s^b \mathcal{S}^b f(x)\,ds + \int_0^t T_s^b \mathcal{S}^b (P_{t-s} f - f)(x)\,ds \\
&\quad = \int_0^t T_s^b \Delta^{\alpha/2} f(x)\,ds - \int_0^t \left(\int_0^s T_r^b \mathcal{S}^b P_{s-r}\big(\Delta^{\alpha/2} f\big)(x)\,dr \right) ds \\
&\qquad + \int_0^t T_s^b \mathcal{S}^b f(x)\,ds + \int_0^t T_s^b \mathcal{S}^b (P_{t-s} f - f)(x)\,ds \\
&\quad = \int_0^t T_s^b \big(\Delta^{\alpha/2} + \mathcal{S}^b\big) f(x)\,ds - \int_0^t \left(\int_r^t T_r^b \mathcal{S}^b P_{s-r}\big(\Delta^{\alpha/2} f\big)(x)\,ds \right) dr \\
&\qquad + \int_0^t T_s^b \mathcal{S}^b (P_{t-s} f - f)(x)\,ds \\
&\quad = \int_0^t T_s^b \mathcal{L}^b f(x)\,ds - \int_0^t T_r^b \mathcal{S}^b (P_{t-r} f - f)(x)\,dr + \int_0^t T_s^b \mathcal{S}^b (P_{t-s} f - f)(x)\,ds \\
&\quad = \int_0^t T_s^b \mathcal{L}^b f(x)\,ds.
\end{aligned}$$

Here in the third inequality, we used (3.36); while in the fifth inequality we used Lemma 2.2 and (3.30), which allow the interchange of the integral sign $\int_r^t$ with $T_r^b \mathcal{S}^b$, and the fact that

$$\int_r^t P_{s-r}\big(\Delta^{\alpha/2} f\big)(x)\,ds = \int_r^t \left(\frac{d}{ds} P_{s-r} f(x) \right) ds = P_{t-r} f(x) - f(x). \qquad \square$$

Theorem 3.14. *Let b be a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1.2). Then $q^b(t,x,y)$ is jointly continuous in $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and $\int_{\mathbb{R}^d} q^b(t,x,y)\,dy = 1$ for every $x \in \mathbb{R}^d$ and $t > 0$.*

Proof. By Lemma 3.12, we have

$$q^b(t+s,x,y) = \int_{\mathbb{R}^d} q^b(t,x,z) q^b(s,z,y)\,dz, \quad x,y \in \mathbb{R}^d, s,t > 0. \tag{3.37}$$

Continuity of $q^b(t,x,y)$ in $(t,x,y) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ follows from Theorem 3.7, (3.37) and the dominated convergence theorem. For $n \ge 1$ and $t \in (0,T]$, it follows from (3.1), Lemma 2.5, Theorem 3.6 and Fubini's theorem that for every $t \in (0, 1 \wedge (A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}]$,

$$\begin{aligned}
\int_{\mathbb{R}^d} q_n^b(t,x,y)\,dy &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^t q_{n-1}^b(t-s,x,z) \mathcal{S}_z^b p_0(s,z,y)\,ds\,dz\,dy \\
&= \int_{\mathbb{R}^d} \int_0^t q_{n-1}^b(t-s,x,z) \mathcal{S}_z^b \left(\int_{\mathbb{R}^d} p_0(s,z,y)\,dy \right) ds\,dz = 0.
\end{aligned}$$

Hence we have by Lemma 3.4,

$$\int_{\mathbb{R}^d} q^b(t,x,y)\,dy = \int_{\mathbb{R}^d} p_0(t,x,y)\,dy = 1$$

for $t \in (0, 1 \wedge (A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}]$. This conservativeness property extends to all $t > 0$ by (3.37). $\square$

Theorem 1.1 now follows from (1.10), Theorems 3.7, 3.9, 3.10, 3.13 and 3.14.

4. C_∞-Semigroups and positivity

Recall that A_0 is the positive constant in Lemma 3.4.

Lemma 4.1. *Suppose that b is a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying condition (1.2). Then $\{T_t^b, t > 0\}$ is a strongly continuous semigroup in $C_\infty(\mathbb{R}^d)$.*

Proof. Note that $q^b(t,x,y)$ is jointly continuous on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ and there are constants c_1 and c_2 so that

$$\left|q^b(t,x,y)\right| \le c_1 e^{c_2 t} p_1(t,x,y) \quad \text{for every } t>0 \text{ and } x,y\in\mathbb{R}^d. \tag{4.1}$$

Then by a similar argument of [9, Proposition 2.3], we can complete the proof. See arXiv version [13] of this paper for details. □

Lemma 4.2. *Let b be a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1.18). For each $f \in C_\infty^2(\mathbb{R}^d)$, $\mathcal{L}^b f(x)$ exists pointwise and is in $C_\infty(\mathbb{R}^d)$.*

Proof. Suppose that $\gamma\in(0,2)$ and $f\in C_\infty^2(\mathbb{R}^d)$. Denote $\sum_{i,j=1}^d |\partial_{ij}^2 f(x)|$ by $|D^2 f(x)|$. Let $R>1$ to be chosen later. Then for each $x\in\mathbb{R}^d$, we have by Taylor expansion,

$$\begin{aligned}
\Phi_f(x) &:= \int_{\mathbb{R}^d}\left|f(x+z)-f(x)-\nabla f(x)\cdot z\mathbb{1}_{\{|z|\le1\}}\right|\frac{1}{|z|^{d+\gamma}}\,dz\\
&\le \int_{|z|\le1}\left|f(x+z)-f(x)-\nabla f(x)\cdot z\mathbb{1}_{\{|z|\le1\}}\right|\frac{1}{|z|^{d+\gamma}}\,dz\\
&\quad+\int_{1<|z|\le R}\left|f(x+z)-f(x)\right|\frac{1}{|z|^{d+\gamma}}\,dz+\int_{|z|>R}\left|f(x+z)-f(x)\right|\frac{1}{|z|^{d+\gamma}}\,dz\\
&\le c\sup_{|y|\le1}\left|D^2 f(x+y)\right|+\int_{1<|z|\le R}\left|f(x+z)-f(x)\right|\frac{1}{|z|^{d+\gamma}}\,dz+cR^{-\gamma}\|f\|_\infty.
\end{aligned}$$

For any given $\varepsilon>0$, we can take R large so that $cR^{-\gamma}\|f\|_\infty<\varepsilon/2$ to conclude that

$$\lim_{|x|\to\infty}\int_{\mathbb{R}^d}\left|f(x+z)-f(x)-\nabla f(x)\cdot z\mathbb{1}_{\{|z|\le1\}}\right|\frac{1}{|z|^{d+\gamma}}\,dz=0. \tag{4.2}$$

By the same reason, applying the above argument to function $x\mapsto f(x+y)-f(x)$ in place of f yields that for every $\varepsilon>0$ and $x_0\in\mathbb{R}^d$, there is $\delta>0$ so that $\Phi_{f(\cdot+y)-f}(x_0)<\varepsilon$ for every $|y|<\delta$. It follows from the last two displays, the definition of $\mathcal{L}^b$ and (1.4) that $\mathcal{L}^b f(x)$ exists for every $x\in\mathbb{R}^d$ and $\mathcal{L}^b f\in C_\infty(\mathbb{R}^d)$. □

Proof of Theorem 1.2. Since b satisfies condition (1.18), then $\mathcal{L}^b f\in C_\infty(\mathbb{R}^d)$ for every $f\in C_c^2(\mathbb{R}^d)$ by Lemma 4.2. Let $\widehat{\mathcal{L}}^b$ denote the infinitesimal generator of the strongly continuous semigroup $\{T_t^b; t\ge0\}$ in $C_\infty(\mathbb{R}^d)$, which is a closed linear operator. It follows from Theorem 3.13, Lemmas 4.1 and 4.2 that for every $f\in C_\infty^2(\mathbb{R}^d)$, $(T_t^b f(x)-f(x))/t$ converges uniformly to $\mathcal{L}^b f(x)$ as $t\to0$. So $C_\infty^2(\mathbb{R}^d)\subset D(\widehat{\mathcal{L}}^b)$ and $\widehat{\mathcal{L}}^b f=\mathcal{L}^b f$ for $f\in C_\infty^2(\mathbb{R}^d)$. In view of Theorem 3.7, there are constants $c_1,c_2>0$ so that (4.1) holds. This implies that

$$\sup_{x\in\mathbb{R}^d}\int_0^\infty e^{-\lambda t}\left|T_t^b f\right|(x)\,dt\le c_\lambda\|f\|_\infty,\ f\in C_\infty\left(\mathbb{R}^d\right)$$

for every $\lambda>c_2$. Observe that $e^{-c_2t}T_t^b$ is a strongly continuous semigroup in $C_\infty(\mathbb{R}^d)$ whose infinitesimal generator is $\widehat{\mathcal{L}}^b-c_2$. The above display implies that $(0,\infty)$ is contained in the residual set $\rho(\widehat{\mathcal{L}}^b-c_2)$ of $\widehat{\mathcal{L}}^b-c_2$. Therefore by

Theorem 3.14 and the Hille–Yosida–Ray theorem [17, p. 165], $\{e^{-c_2 t}T_t^b; t \geq 0\}$ is a positive preserving semigroup on $C_\infty(\mathbb{R}^d)$ if and only if $\widehat{\mathcal{L}}^b - c_2$ satisfies the positive maximum principle. On the other hand, Courrége's first theorem (see [1, p. 158]) tells us that $\widehat{\mathcal{L}}^b - c_2$ satisfies the positive maximum principle if and only if for each $x \in \mathbb{R}^d$,

$$\frac{\mathcal{A}(d,-\alpha)}{|z|^{d+\alpha}} + \frac{\mathcal{A}(d,-\beta)b(x,z)}{|z|^{d+\beta}} \geq 0 \quad \text{for a.e. } z \in \mathbb{R}^d.$$

Since $e^{-c_2 t}T_t^b$ has a continuous integral kernel $e^{-c_2 t}q^b(t,x,y)$, it follows that $q^b(t,x,y) \geq 0$ on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ if and only if for each $x \in \mathbb{R}^d$, (1.19) holds. If $b(x,z) = b(x)$ is a function of x only, then by taking $|z| \to \infty$, one concludes that (1.19) holds if and only if $b(x) \geq 0$ on $\mathbb{R}^d$. □

5. Feller process and heat kernel estimates

Throughout this section, b is a bounded function satisfying condition (1.2) and (1.19). We will show that $q^b(t,x,y) > 0$ and so it generates a Feller process X^b that has strong Feller property. We further derive the upper and lower bound estimates on $q^b(t,x,y)$. We will first establish the Feller process X^b and its connection to the martingale problem for $(\mathcal{L}^b, \mathcal{S}(\mathbb{R}^d))$ under an additional assumption (1.18). We will then remove this additional assumption using an approximation method and the uniqueness result on $q^b(t,x,y)$ from Theorem 3.10.

Suppose that b is a bounded function satisfying conditions (1.2), (1.18) and (1.19). Then it follows from Theorem 1.2, Theorem 3.14, Lemma 4.1 and Theorem 3.8, T^b is a Feller semigroup. So it uniquely determines a conservative Feller process $X^b = \{X_t^b, t \geq 0, \mathbb{P}_x, x \in \mathbb{R}^d\}$ having $q^b(t,x,y)$ as its transition density function. Since, by Theorem 3.10, $q^b(t,x,y)$ is continuous and $q^b(t,x,y) \leq c_1 e^{c_2 t} p_{M_{b,\lambda}}(t,x,y)$ for some positive constants c_1 and c_2, X^b enjoys the strong Feller property.

Proposition 5.1. *Suppose that b is a bounded function satisfying conditions* (1.2), (1.18) *and* (1.19). *For each $x \in \mathbb{R}^d$ and $f \in C_b^2(\mathbb{R}^d)$,*

$$M_t^f := f\bigl(X_t^b\bigr) - f\bigl(X_0^b\bigr) - \int_0^t \mathcal{L}^b f\bigl(X_s^b\bigr)\,ds$$

is a martingale under $\mathbb{P}_x$. So in particular, the Feller process $(X^b, \mathbb{P}_x, x \in \mathbb{R}^d)$ solves the martingale problem for $(\mathcal{L}^b, C_\infty^2(\mathbb{R}^d))$.

Proof. This follows immediately from Theorem 3.13 and the Markov property of X^b. □

We next determine the Lévy system of X^b. Recall that

$$J^b(x,y) = \frac{\mathcal{A}(d,-\alpha)}{|x-y|^{d+\alpha}} + \frac{\mathcal{A}(d,-\beta)b(x,y-x)}{|x-y|^{d+\beta}}.$$

By Proposition 5.1 and a similar argument of [9, Theorem 2.6], we have the following proposition. (See arXiv version [13] of this paper for a proof.)

Proposition 5.2. *Suppose that b is a bounded function satisfying conditions* (1.2), (1.18) *and* (1.19). *Let f be a non-negative function on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ vanishing on the diagonal. Then for stopping time T with respect to the minimal admissible filtration generated by X^b,*

$$\mathbb{E}_x\Biggl[\sum_{s \leq T} f\bigl(s, X_{s-}^b, X_s^b\bigr)\Biggr] = \mathbb{E}_x\biggl[\int_0^T \int_{\mathbb{R}^d} f\bigl(s, X_s^b, u\bigr) J^b\bigl(X_s^b, u\bigr)\,du\,ds\biggr].$$

To remove the assumption (1.18) on b, we approximate a general measurable function $b(x,z)$ by continuous $k_n(x,z)$. To show that $q^{k_n}(t,x,y)$ converges to $q^b(t,x,y)$, we establish equi-continuity of $q^b(t,x,y)$ and apply the uniqueness result, Theorem 3.10.

Proposition 5.3. *For each $0 < t_0 < T < \infty$ and $A > 0$, the function $q^b(t,x,y)$ is uniformly continuous in $(t,x) \in (t_0,T) \times \mathbb{R}^d$ for every b with $\|b\|_\infty \le A$ that satisfies (1.2) and for all $y \in \mathbb{R}^d$.*

Proof. In view of Theorem 3.11, it suffices to prove the theorem for $A = A_0$, where A_0 is the constant in Lemma 3.4 (or in Theorem 1.1). Using the Chapman–Kolmogorov equation for $q^b(t,x,y)$ (see Lemma 3.12) and (3.30), it suffices to prove the proposition for $T = 1$.

Noting that by (3.8)

$$q_n^b(t,x,y) = \int_0^t \int_{\mathbb{R}^d} p_0(t-r,x,z) \mathcal{S}_z^b q_{n-1}^b(r,z,y)\, dz\, dr.$$

Hence, for $T > t > s > t_0$, $x_1, x_2 \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$, we have

$$\begin{aligned}
&\left|q_n^b(s,x_1,y) - q_n^b(t,x_2,y)\right| \\
&\quad \le \int_0^s \int_{\mathbb{R}^d} \left|p_0(s-r,x_1,z) - p_0(t-r,x_2,z)\right| \left|\mathcal{S}_z^b q_{n-1}^b(r,z,y)\right| dz\, dr \\
&\qquad + \int_s^t \int_{\mathbb{R}^d} p_0(t-r,x_2,z) \left|\mathcal{S}_z^b q_{n-1}^b(r,z,y)\right| dz\, dr \\
&\quad =: I + II.
\end{aligned}$$

It is known (see [11]) that there are positive constants c_1 and θ so that for any $t, s \in [t_0, T]$ and $x_i \in \mathbb{R}^d$ with $i = 1, 2$,

$$\left|p_0(s,x_1,y) - p_0(t,x_2,y)\right| \le c_1 t_0^{-(d+\theta)/\alpha} \left(|t-s|^{1/\alpha} + |x_1 - x_2|\right)^\theta, \ y \in \mathbb{R}^d,$$

we have by (2.1), (3.12) and (3.35), for $\rho \in (0, s/2)$,

$$\begin{aligned}
I &= \int_0^{s-\rho} \int_{\mathbb{R}^d} \left|p_0(s-r,x_1,z) - p_0(t-r,x_2,z)\right| \left|\mathcal{S}_z^b q_{n-1}^b(r,z,y)\right| dz\, dr \\
&\quad + \int_{s-\rho}^s \int_{\mathbb{R}^d} \left|p_0(s-r,x_1,z) - p_0(t-r,x_2,z)\right| \left|\mathcal{S}_z^b q_{n-1}^b(r,z,y)\right| dz\, dr \\
&\le c_2 2^{-(n-1)} \rho^{-(d+\theta)/\alpha} \left(|t-s|^{1/\alpha} + |x_1 - x_2|\right)^\theta \int_0^{s-\rho} \int_{\mathbb{R}^d} f_0(r,z,y)\, dz\, dr \\
&\quad + c_2 2^{-(n-1)} \int_{s-\rho}^s \int_{\mathbb{R}^d} \left(p_0(s-r,x_1,z) + p_0(t-r,x_2,z)\right) f_0(r,z,y)\, dz\, dr \\
&\le c_3 2^{-(n-1)} \rho^{-(d+\theta)/\alpha} \left(|t-s|^{1/\alpha} + |x_1 - x_2|\right)^\theta s^{1-\beta/\alpha} + c_3 2^{-(n-1)} (s-\rho)^{-(d+\beta)/\alpha} \rho.
\end{aligned} \tag{5.1}$$

Moreover, by (2.1) and (3.12),

$$II \le 2^{-(n-1)} \int_s^t \int_{\mathbb{R}^d} p_0(t-r,x_2,z) f_0(r,z,y)\, dz\, dr \le 2^{-(n-1)} s^{-(d+\beta)/\alpha} |t-s|. \tag{5.2}$$

Therefore, noting that

$$\left|q^b(s,x_1,y) - q^b(t,x_2,y)\right| \le \left|p_0(s,x_1,y) - p_0(t,x_2,y)\right| + \sum_{n=1}^\infty \left|q_n^b(s,x_1,y) - q_n^b(t,x_2,y)\right|,$$

then first taking $|t-s|$ and $|x_1 - x_2|$ small, and then making ρ small in (5.1) and (5.2) yields the conclusion of this proposition. $\square$

Proposition 5.4. *For each $0 < t_0 < T < \infty$ and $A > 0$, the function $q^b(t,x,y)$ is uniformly continuous in y for every b with $\|b\|_\infty \le A$ that satisfies (1.2) and for all $(t,x) \in (t_0,T) \times \mathbb{R}^d$.*

Proof. In view of Theorem 3.11, it suffices to prove the theorem for $A = A_0$, where A_0 is the constant in Lemma 3.4 (or in Theorem 1.1). Using the Chapman–Kolmogorov equation for $q^b(t,x,y)$ (see Lemma 3.12) and (3.30), it suffices to prove the proposition for $T = 1$.

Define $P(s,x,y) = p_0(s,x) - p_0(s,y)$. For $s > 0$, we have

$$
\begin{aligned}
&\big|\mathcal{S}^b p_0(s,y_1) - \mathcal{S}^b p_0(s,y_2)\big| \\
&\quad\le c_1 \int_{\mathbb{R}^d} \big|P(s,y_1+h,y_2+h) - P(s,y_1,y_2) - \big\langle \nabla_{(y_1,y_2)} P(s,y_1,y_2), h\mathbb{1}_{|h|\le 1}\big\rangle\big| \frac{dh}{|h|^{d+\beta}} \\
&\quad\le c_1 \int_{|h|\le 1} |h|^2 \sup_{\theta\in(0,1)} \left| \frac{\partial^2}{\partial y_1^2} p_0(s,y_1+\theta h) - \frac{\partial^2}{\partial y_2^2} p_0(s,y_2+\theta h)\right| \frac{dh}{|h|^{d+\beta}} \\
&\qquad + c_1 \int_{|h|>1} \big|p_0(s,y_1+h) - p_0(s,y_2+h) - p_0(s,y_1) + p_0(s,y_2)\big| \frac{dh}{|h|^{d+\beta}} \\
&\quad\le c_2 \sup_y \left|\frac{\partial^3}{\partial y^3} p_0(s,y)\right| |y_1-y_2| \int_{|h|\le 1} |h|^2 \frac{dh}{|h|^{d+\beta}} + c_2 \sup_y \left|\frac{\partial}{\partial y} p_0(s,y)\right| |y_1-y_2| \int_{|h|>1} \frac{dh}{|h|^{d+\beta}} \\
&\quad\le c_3 |y_1-y_2| \big[s^{-(d+3)/\alpha} + s^{-(d+1)/\alpha}\big],
\end{aligned} \tag{5.3}
$$

where in the fourth inequality, $|\frac{\partial^3}{\partial y^3} p_0(s,y)| \le c_3 s^{-(d+3)/\alpha}$ can be proved similarly by the argument in Lemma 2.1. Take $\rho \in (0, t_0/2)$. Then for each $n \ge 1$, we have by (1.9), (3.35), Lemma 2.4, Lemma 3.4 and (5.3), that for $(t,x,y) \in (t_0,1) \times \mathbb{R}^d \times \mathbb{R}^d$,

$$
\begin{aligned}
&\big|q_n^b(t,x,y_1) - q_n^b(t,x,y_2)\big| \\
&\quad\le \int_0^\rho \int_{\mathbb{R}^d} q_{n-1}^b(t-s,x,z) \big|\mathcal{S}_z^b p_0(s,z,y_1) - \mathcal{S}_z^b p_0(s,z,y_2)\big|\, dz\, ds \\
&\qquad + \int_\rho^t \int_{\mathbb{R}^d} q_{n-1}^b(t-s,x,z) \big|\mathcal{S}_z^b p_0(s,z,y_1) - \mathcal{S}_z^b p_0(s,z,y_2)\big|\, dz\, ds \\
&\quad\le c_4 2^{-(n-1)} \int_0^\rho \int_{\mathbb{R}^d} p_1(t-s,x,z) \big|\mathcal{S}_z^b p_0(s,z,y_1) - \mathcal{S}_z^b p_0(s,z,y_2)\big|\, dz\, ds \\
&\qquad + c_4 2^{-(n-1)} \int_\rho^t \int_{\mathbb{R}^d} p_1(t-s,x,z) \big|\mathcal{S}_z^b p_0(s,z-y_1) - \mathcal{S}_z^b p_0(s,z-y_2)\big|\, dz\, ds \\
&\quad\le c_5 2^{-(n-1)} t_0^{-d/\alpha} \int_0^\rho \int_{\mathbb{R}^d} \big(\big|\mathcal{S}_z^b p_0(s,z,y_1)\big| + \big|\mathcal{S}_z^b p_0(s,z,y_2)\big|\big)\, dz\, ds \\
&\qquad + c_5 2^{-(n-1)} \rho^{-(d+3)/\alpha} |y_1-y_2| \int_\rho^t \int_{\mathbb{R}^d} p_1(t-s,x,z)\, dz\, ds \\
&\quad\le c_6 2^{-(n-1)} t_0^{-d/\alpha} \rho^{1-\beta/\alpha} + c_6 2^{-(n-1)} \rho^{-(d+3)/\alpha} |y_1-y_2|.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&\big|q^b(t,x,y_1) - q^b(t,x,y_2)\big| \\
&\quad\le \big|p_0(t,x,y_1) - p_0(t,x,y_2)\big| + \sum_{n=1}^\infty c_6 2^{-(n-1)} t_0^{-d/\alpha} \rho^{1-\beta/\alpha} + \sum_{n=1}^\infty c_6 2^{-(n-1)} \rho^{-(d+3)/\alpha} |y_1-y_2|.
\end{aligned}
$$

By first taking $|y_1 - y_2|$ small and then making ρ small yields the desired uniform continuity of $q^b(t,x,y)$. $\square$

Theorem 5.5. *Suppose b is a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1.2) and (1.19). The kernel $q^b(t,x,y)$ uniquely determines a Feller process $X^b = (X^b_t, t \ge 0, \mathbb{P}_x, x \in \mathbb{R}^d)$ on the canonical Skorokhod space $\mathbb{D}([0,\infty), \mathbb{R}^d)$ such that $\mathbb{E}_x[f(X^b_t)] = \int_{\mathbb{R}^d} q^b(t,x,y) f(y)\,dy$ for every bounded continuous function f on $\mathbb{R}^d$. The Feller process X^b is conservative and has a Lévy system $(J^b(x,y)\,dy, t)$, where J^b is given by (1.25). Moreover, for each $x \in \mathbb{R}^d$, $(X^b, \mathbb{P}_x)$ is the unique solution to the martingale problem $(\mathcal{L}^b, \mathcal{S}(\mathbb{R}^d))$ with initial value x. Here $\mathcal{S}(\mathbb{R}^d)$ denotes the space of tempered functions on $\mathbb{R}^d$.*

Proof. When b is a bounded function satisfying (1.2), (1.18) and (1.19), the theorem has already been established via Propositions 5.1–5.2. We now remove the continuity assumption (1.18). Suppose that $b(x,z)$ is a bounded function that satisfies (1.2) and (1.19). Let φ be a non-negative smooth function with compact support in $\mathbb{R}^d$ so that $\int_{\mathbb{R}^d} \varphi(x)\,dx = 1$. For each $n \ge 1$, define $\varphi_n(x) = n^d \varphi(nx)$ and $k_n(x,z) := \int_{\mathbb{R}^d} \varphi_n(x-y) b(y,z)\,dy$. Then k_n is a function that satisfies (1.2), (1.18) and (1.19) with $\|k_n\|_\infty \le \|b\|_\infty$.

By Theorem 1.1, Proposition 5.3 and Proposition 5.4, $q^{k_n}(t,x,y)$ is uniformly bounded and equi-continuous on $[1/M, M] \times \mathbb{R}^d \times \mathbb{R}^d$ for each $M \ge 1$, then there is a subsequence $\{n_j\}$ of $\{n\}$ so that $q^{k_{n_j}}(t,x,y)$ converges boundedly and uniformly on compacts of $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$, to some continuous function $\overline{q}(t,x,y)$, which again satisfies (1.16). Obviously, $\overline{q}(t,x,y)$ also satisfies the Chapman–Kolmogorov equation and $\int_{\mathbb{R}^d} \overline{q}(t,x,y)\,dy = 1$. By Theorem 3.7, $q^{k_{n_j}}(t,x,y)$ satisfies (3.23) and (3.25). By (3.1), Lemma 2.5 and the dominated convergence theorem, we have that

$$\overline{q}(t,x,y) = p_0(t,x,y) + \int_0^t \int_{\mathbb{R}^d} \overline{q}(t-s,x,z) \mathcal{S}^b_z p_0(s,z,y)\,dy\,ds$$

and $\overline{q}(t,x,y) \le c p_{M_{b,\lambda}}(t,x,y)$ for every $0 < t \le 1 \wedge (A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}$ and $x, y \in \mathbb{R}^d$. Hence we conclude from Theorem 3.10 that $\overline{q}(t,x,y) = q^b(t,x,y)$. This in particular implies that $q^b(t,x,y) \ge 0$. So there is a Feller process X^b having $q^b(t,x,y)$ as its transition density function. The proof of Propositions 5.1–5.2 only uses the condition (1.18) through its implication that $q^b(t,x,y) \ge 0$. So in view of what we just established, Propositions 5.1–5.2 continue to hold for X^b under the current setting without the additional assumption (1.18). The non-local operator $\mathcal{L}^b$ satisfies the assumptions $[A_1]$ and $[A_2]$ of [23]. So by [23, Theorem 3], solution to the martingale problem $(\mathcal{L}^b, \mathcal{S}(\mathbb{R}^d))$ is unique. Since $\mathcal{S}(\mathbb{R}^d) \subset C^2_\infty(\mathbb{R}^d)$, the proof of the theorem is now complete. □

For each $\lambda > 0$, define

$$\widehat{b_\lambda}(x,z) = b(x,z) 1_{\{|z| \le \lambda\}}(z) + b^+(x,z) 1_{\{|z| > \lambda\}}(z). \tag{5.4}$$

In the following, we use a method of Meyer [24] to construct from X^b, by adding suitable jumps, a strong Markov process Y corresponding to the jumping kernel $J^{\widehat{b_\lambda}}$ defined by (1.25) but with $\widehat{b_\lambda}$ in place of b.

Define $\mathcal{J}(x) = \int_{\mathbb{R}^d} (J^{\widehat{b_\lambda}}(x,y) - J^b(x,y))\,dy$. Then there exists a positive constant c_1 so that $0 \le \mathcal{J}(x) \le c_1$ for all $x \in \mathbb{R}^d$. Let $q(x,y) = \frac{J^{\widehat{b_\lambda}}(x,y) - J^b(x,y)}{\mathcal{J}(x)}$. Let S_1 be an exponential random variable of parameter 1 independent of X^b. Set

$$C_t = \int_0^t \mathcal{J}\big(X^b_s\big)\,ds, \qquad U_1 = \inf\{t \ge 0 : C_t \ge S_1\}. \tag{5.5}$$

We let $Y_t = X^b_t$ for $0 \le t < U_1$ and define Y_{U_1} with law $q(Y_{U_1-}, \cdot) = q(X^b_{U_1-}, \cdot)$, and then repeat using an independent exponential random variable S_2 to define U_2, etc. So the construction proceeds now in the same way from the new starting point (U_1, Y_{U_1}). Since $\mathcal{J}(x)$ is bounded, only finitely many new jumps are introduced in any bounded time interval. In [24], it is proved that the resulting process Y is a strong Markov process. By slightly abusing the notation, we still use $\mathbb{P}_x$ and $\mathbb{E}_x$ to denote the above constructed probability law and expectation induced on such enlarged probability space under which $Y_0 = x$.

Lemma 5.6. *For each $x \in \mathbb{R}^d$ and $f \in C^2_b(\mathbb{R}^d)$,*

$$\mathbb{E}_x\big[f(Y_t); t < U_1\big] = f(x) + \mathbb{E}_x\bigg[\int_0^t \big(\mathcal{L}^b - \mathcal{J}(Y_s)\big) f(Y_s) \mathbb{1}_{\{s < U_1\}}\,ds\bigg].$$

Proof. By the definition of U_1 and Ito's formula, for each function $f \in C_b^2(\mathbb{R}^d)$,

$$\begin{aligned}
\mathbb{E}_x\big[f(Y_t); t < U_1\big] &= \mathbb{E}_x\big[f\big(X_t^b\big)\mathbb{1}_{\{U_1>t\}}\big] = \mathbb{E}_x\big[f\big(X_t^b\big)e^{-C_t}\big] \\
&= f(x) + \mathbb{E}_x\left[\int_0^t \big(\mathcal{L}^b - \mathcal{J}\big(X_s^b\big)\big) f\big(X_s^b\big)e^{-C_s}\,ds\right] \\
&= f(x) + \mathbb{E}_x\left[\int_0^t \big(\mathcal{L}^b - \mathcal{J}(Y_s)\big) f(Y_s)\mathbb{1}_{\{s<U_1\}}\,ds\right].
\end{aligned}$$

□

Proposition 5.7. *For each $x \in \mathbb{R}^d$ and $f \in C_b^2(\mathbb{R}^d)$,*

$$M_t^f := f(Y_t) - f(Y_0) - \int_0^t \mathcal{L}^{\widehat{b}_\lambda} f(Y_s)\,ds$$

is a martingale under $\mathbb{P}_x$. So in particular, the strongly Markov process $(Y, \mathbb{P}_x, x \in \mathbb{R}^d)$ solves the martingale problem for $(\mathcal{L}^{\widehat{b}_\lambda}, C_\infty^2(\mathbb{R}^d))$.

Proof. Note that M_t^f is an additive function of Y. So by the Markov property of Y, it suffices to show that $\mathbb{E}_x[M_t^f] = 0$ for every $x \in \mathbb{R}^d$ and $t > 0$. Recall that U_1 is defined in (5.5), and denote by $\{U_n, n \geq 2\}$ the subsequent jump adding times inductively defined according to the construction of Meyer [24]. For every $\alpha > 0$, set $u_\alpha(x) = \mathbb{E}_x[\int_0^{U_1} e^{-\alpha t} f(Y_t)\,dt]$. We have by Lemma 5.6 and Fubini theorem that

$$u_\alpha(x) = \frac{f(x)}{\alpha} + \frac{1}{\alpha}\mathbb{E}_x\left[\int_0^{U_1} e^{-\alpha s}\big(\mathcal{L}^b - \mathcal{J}(Y_s)\big) f(Y_s)\,ds\right].$$

Observe that in view of [28, p. 286] (see, for example, the proof of [15, Proposition 2.2]), for any non-negative function φ on $\mathbb{R}^d$ and $x \in \mathbb{R}^d$,

$$\mathbb{E}_x\big[e^{-\alpha U_1}\varphi(Y_{U_1-})\big] = \mathbb{E}_x\left[\int_0^{U_1} e^{-\alpha s}\mathcal{J}(Y_s)\varphi(Y_s)\,ds\right].$$

Set $U_0 = 0$ and let θ_t to denote the time shift operator for the Markov process Y. Then we have from above and the strong Markov property of Y that

$$\begin{aligned}
\mathbb{E}_x\left[\int_0^\infty e^{-\alpha t} f(Y_t)\,dt\right] &= \sum_{j=0}^\infty \mathbb{E}_x\left[\int_{U_j}^{U_{j+1}} e^{-\alpha t} f(Y_t)\,dt\right] = \sum_{j=0}^\infty \mathbb{E}_x\big[e^{-\alpha U_j} u_\alpha(Y_{U_j})\big] \\
&= \frac{f(x)}{\alpha} + \frac{1}{\alpha}\sum_{j=1}^\infty \mathbb{E}_x\big[e^{-\alpha U_j} f(Y_{U_j})\big] \\
&\quad + \frac{1}{\alpha}\sum_{j=0}^\infty \mathbb{E}_x\left[\int_{U_j}^{U_{j+1}} e^{-\alpha s}\big(\mathcal{L}^b - \mathcal{J}(Y_s)\big) f(Y_s)\,ds\right] \\
&= \frac{f(x)}{\alpha} + \frac{1}{\alpha}\sum_{j=1}^\infty \mathbb{E}_x\left[e^{-\alpha U_j}\int_{\mathbb{R}^d} f(y) q(Y_{U_j-}, y)\,dy\right] \\
&\quad + \frac{1}{\alpha}\mathbb{E}_x\left[\int_0^\infty e^{-\alpha s}\big(\mathcal{L}^b - \mathcal{J}(Y_s)\big) f(Y_s)\,ds\right] \\
&= \frac{f(x)}{\alpha} + \frac{1}{\alpha}\sum_{j=1}^\infty \mathbb{E}_x\left[e^{-\alpha U_{j-1}}\int_{\mathbb{R}^d} f(y)\big(e^{-\alpha U_1} q(Y_{U_1-}, y)\big)\circ\theta_{U_{j-1}}\,dy\right]
\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{1}{\alpha}\mathbb{E}_x\left[\int_0^\infty e^{-\alpha s}\big(\mathcal{L}^b - \mathcal{J}(Y_s)\big) f(Y_s)\,ds\right] \\
&= \frac{f(x)}{\alpha} + \frac{1}{\alpha}\sum_{j=1}^\infty \mathbb{E}_x\left[e^{-\alpha U_{j-1}}\int_{\mathbb{R}^d} f(y)\left(\int_0^{U_1} e^{-\alpha s}\mathcal{J}(Y_s)q(Y_s,y)\,ds\right)\circ\theta_{U_{j-1}}\,dy\right] \\
&\quad + \frac{1}{\alpha}\mathbb{E}_x\left[\int_0^\infty e^{-\alpha s}\big(\mathcal{L}^b - \mathcal{J}(Y_s)\big) f(Y_s)\,ds\right] \\
&= \frac{f(x)}{\alpha} + \frac{1}{\alpha}\mathbb{E}_x\left[\int_{\mathbb{R}^d} f(y)\left(\int_0^\infty e^{-\alpha s}\mathcal{J}(Y_s)q(Y_s,y)\,ds\right)dy\right] \\
&\quad + \frac{1}{\alpha}\mathbb{E}_x\left[\int_0^\infty e^{-\alpha s}\big(\mathcal{L}^b - \mathcal{J}(Y_s)\big) f(Y_s)\,ds\right] \\
&= \frac{f(x)}{\alpha} + \frac{1}{\alpha}\mathbb{E}_x\left[\int_0^\infty e^{-\alpha s}\left(\mathcal{L}^b f(Y_s) + \int_{\mathbb{R}^d}\mathcal{J}(Y_s)q(Y_s,y)\big(f(y)-f(Y_s)\big)\,dy\right)ds\right] \\
&= \frac{f(x)}{\alpha} + \frac{1}{\alpha}\mathbb{E}_x\left[\int_0^\infty e^{-\alpha s}\mathcal{L}^{\widehat{b}_\lambda} f(Y_s)\,ds\right].
\end{aligned}
$$

By the uniqueness of the Laplace transform, we conclude from above that $\mathbb{E}_x[M_t^f] = 0$ for all $t \ge 0$ and $x \in \mathbb{R}^d$. □

Note that $\widehat{b}_\lambda$ defined by (5.4) is a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1.2) and (1.19). By Theorem 5.5, the kernel $q^{\widehat{b}_\lambda}(t,x,y)$ uniquely determines a Feller process $X^{\widehat{b}_\lambda} = (X_t^{\widehat{b}_\lambda}, t \ge 0, \mathbb{P}_x, x \in \mathbb{R}^d)$ on the canonical Skorokhod space $\mathbb{D}([0,\infty),\mathbb{R}^d)$, and $(X^{\widehat{b}_\lambda}, \mathbb{P}_x)$ is the unique solution to the martingale problem for $(\mathcal{L}^{\widehat{b}_\lambda}, \mathcal{S}(\mathbb{R}^d))$ with initial value x. This, together with Proposition 5.7 implies that the process Y coincides with $X^{\widehat{b}_\lambda}$ in the sense of distribution.

Theorem 5.8. *For every $\lambda > 0$ and $A > 0$, there is a positive constant $C_{15} = C_{15}(d,\alpha,\beta,A,\lambda)$ such that for any bounded b satisfying (1.2) and (1.19) with $\|b\|_\infty \le A$,*

$$
q^b(t,x,y) \le C_{15}\, p_{M_{b^+,\lambda}}(t,x,y) \quad \textit{for } t \in (0,1] \textit{ and } x,y \in \mathbb{R}^d.
$$

Proof. Noting that $\widehat{b}_\lambda$ is a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ with $\|\widehat{b}_\lambda\|_\infty \le \|b\|_\infty$ satisfying (1.2) and (1.19), then by Theorem 1.1, there is a positive constant $C = C(d,\alpha,\beta,A,\lambda)$ so that

$$
q^{\widehat{b}_\lambda}(t,x,y) \le C p_{M_{b^+,\lambda}}(t,x,y) \quad \text{for } t \in (0,1] \text{ and } x,y \in \mathbb{R}^d. \tag{5.6}
$$

Let $\{\mathcal{M}_t\}_{t\ge 0}$ be the filtration generated by X^b. Note that $X^{\widehat{b}_\lambda}$ has the same distribution as Y. Then by Lemma 3.6 in [2], for any $A \in \mathcal{M}_t$,

$$
\mathbb{P}^x\big(X_t^{\widehat{b}_\lambda} \in A\big) = \mathbb{P}^x(Y_t \in A) \ge \mathbb{P}^x\big(\big\{Y_s = X_s^b \text{ for all } 0 \le s \le t\big\} \cap A\big) \ge e^{-t\|\mathcal{J}\|_\infty}\mathbb{P}^x\big(X_t^b \in A\big).
$$

Hence by (5.6), $q^b(t,x,y) \le e^{\|\mathcal{J}\|_\infty} q^{\widehat{b}_\lambda}(t,x,y) \le C e^{\|\mathcal{J}\|_\infty} p_{M_{b^+,\lambda}}(t,x,y)$ for $t \in (0,1]$ and $x,y \in \mathbb{R}^d$. □

For a Borel set $B \subset \mathbb{R}^d$, we define $\tau_B^b = \inf\{t > 0 : X_t^b \notin B\}$ and $\sigma_B^b := \inf\{t \ge 0 : X_t^b \in B\}$.

Proposition 5.9. *For each $A > 0$ and $R_0 > 0$, there is a positive constant $\kappa = \kappa(d,\alpha,\beta,A,R_0) < 2^\alpha(1-(1/3)^\alpha)$ so that for every b satisfying (1.2) and (1.19) with $\|b\|_\infty \le A$, $r \in (0,R_0]$ and $x \in \mathbb{R}^d$,*

$$
\mathbb{P}_x\big(\tau^b_{B(x,r)} \le \kappa r^\alpha\big) \le \frac{1}{2}.
$$

Proof. Let f be a C^2 function taking values in $[0,1]$ such that $f(0)=0$ and $f(u)=1$ if $|u|\geq 1$. Set $f_{x,r}(y)=f(\frac{y-x}{r})$. Note that $f_{x,r}$ is a C^2 function taking values in $[0,1]$ such that $f_{x,r}(x)=0$ and $f_{x,r}(y)=1$ if $y\notin B(x,r)$. Moreover, $\sup_{y\in\mathbb{R}^d}|\frac{\partial^2 f_{x,r}(y)}{\partial y_i\,\partial y_j}|\leq r^{-2}\sup_{y\in\mathbb{R}^d}|\frac{\partial^2 f(y)}{\partial y_i\,\partial y_j}|$. Denote $\sum_{i,j=1}^d|\partial^2_{ij}f(x)|$ by $|D^2f(x)|$. By Taylor's formula, it follows that

$$\begin{aligned}\big|\mathcal{L}^b f_{x,r}(u)\big| &\leq c_1\int\Big|f_{x,r}(u+h)-f_{x,r}(u)-\big\langle\nabla f_{x,r}(u),h\big\rangle\mathbb{1}_{\{|h|\leq r\}}\Big|\bigg(\frac{1}{|h|^{d+\alpha}}+\frac{1}{|h|^{d+\beta}}\bigg)dh\\ &\leq c_2\|D^2f\|_\infty r^{-2}\int_{|h|\leq r}|h|^2\bigg(\frac{1}{|h|^{d+\alpha}}+\frac{1}{|h|^{d+\beta}}\bigg)dh\\ &\quad+c_2\|f\|_\infty\int_{\{|h|>r\}}\bigg(\frac{1}{|h|^{d+\alpha}}+\frac{1}{|h|^{d+\beta}}\bigg)dh\\ &\leq c_3\big(r^{-\alpha}+r^{-\beta}\big)\leq c_3\big(1+R_0^{\alpha-\beta}\big)r^{-\alpha},\end{aligned}$$

where $c_i=c_i(d,\alpha,\beta,A)$, $i=1,2,3$ are positive constants. Therefore, for each $t>0$,

$$\mathbb{P}_x\big(\tau^b_{B(x,r)}\leq t\big)\leq\mathbb{E}_x\big[f_{x,r}\big(X^b_{\tau^b_{B(x,r)}\wedge t}\big)\big]-f_{x,r}(x)=\mathbb{E}_x\bigg[\int_0^{\tau^b_{B(x,r)}\wedge t}\mathcal{L}^b f_{x,r}\big(X^b_s\big)\,ds\bigg]\leq c_3\big(1+R_0^{\alpha-\beta}\big)\frac{t}{r^\alpha}.$$

Set $\kappa=(2^\alpha(1-(1/3)^\alpha))\wedge(2c_3(1+R_0^{\alpha-\beta}))^{-1}$, then $\mathbb{P}_x(\tau^b_{B(x,r)}\leq\kappa r^\alpha)\leq 1/2$. □

Recall that $m_{b,\lambda}=\operatorname{essinf}_{x,z\in\mathbb{R}^d,|z|>\lambda}b(x,z)$.

Proposition 5.10. *For every $A>0$, $\lambda>0$, $0<\varepsilon<1$ and $R_0>0$, there exists a constant $C_{16}=C_{16}(d,\alpha,\beta,A,\lambda,\varepsilon,R_0)>0$ so that for every b satisfying* (1.2) *and* (1.23) *with $\|b\|_\infty\leq A$, $r\in(0,R_0]$ and $x,y\in\mathbb{R}^d$ with $|x-y|\geq 3r$,*

$$\mathbb{P}_x\big(\sigma^b_{B(y,r)}<\kappa r^\alpha\big)\geq C_{16}r^{d+\alpha}\bigg(\frac{1}{|x-y|^{d+\alpha}}+\frac{m_{b^+,\lambda}}{|x-y|^{d+\beta}}\bigg).$$

Proof. By Proposition 5.9,

$$\mathbb{E}_x\big[\kappa r^\alpha\wedge\tau^b_{B(x,r)}\big]\geq\kappa r^\alpha\mathbb{P}_x\big(\tau^b_{B(x,r)}\geq\kappa r^\alpha\big)\geq\frac{1}{2}\kappa r^\alpha.$$

Note that $J^b(x,y)\geq m_{b^+,\lambda}\mathcal{A}(d,-\beta)|x-y|^{-d-\beta}\mathbb{1}_{\{|x-y|>\lambda\}}$. Thus by (1.23) and Proposition 5.2, there are positive constants $c_1=c_1(d,\alpha,\beta)$ and $c_2=c_2(d,\alpha,\beta,A,\lambda,\varepsilon,R_0)$ so that

$$\begin{aligned}\mathbb{P}_x\big(\sigma^b_{B(y,r)}<\kappa r^\alpha\big)&\geq\mathbb{P}_x\big(X^b_{\kappa r^\alpha\wedge\tau^b_{B(x,r)}}\in B(y,r)\big)\\ &=\mathbb{E}_x\int_0^{\kappa r^\alpha\wedge\tau^b_{B(x,r)}}\int_{B(y,r)}J^b\big(X^b_s,u\big)\,du\,ds\\ &\geq c_1\mathbb{E}_x\big[\kappa r^\alpha\wedge\tau^b_{B(x,r)}\big]\int_{B(y,r)}\bigg(\frac{\varepsilon}{|x-y|^{d+\alpha}}+\frac{m_{b^+,\lambda}}{|x-y|^{d+\beta}}\mathbb{1}_{\{|x-y|>\lambda\}}\bigg)du\\ &\geq c_2\varepsilon\kappa r^{d+\alpha}\bigg(\frac{1}{|x-y|^{d+\alpha}}+\frac{m_{b^+,\lambda}}{|x-y|^{d+\beta}}\bigg).\end{aligned}$$

Here in the last inequality, we used the fact that $|x-y|^{-(d+\alpha)}\geq(1+\lambda^{\alpha-\beta}A)^{-1}(|x-y|^{-(d+\alpha)}+m_{b,\lambda}\cdot|x-y|^{-(d+\beta)})$ for $|x-y|\leq\lambda$. □

Proposition 5.11. *For every $A>0$, there exists a constant $C_{17}=C_{17}(d,\alpha,\beta,A)>0$ so that for every bounded b that satisfies* (1.2) *and* (1.19) *with $\|b\|_\infty\le A$, and $3r\le|x-y|\le R_*:=\frac13(2A\frac{\mathcal{A}(d,-\beta)}{\mathcal{A}(d,-\alpha)})^{1/(\beta-\alpha)}$,*

$$\mathbb{P}_x\big(\sigma^b_{B(y,r)}<\kappa r^\alpha\big)\ge C_{17}\frac{r^{d+\alpha}}{|x-y|^{d+\alpha}}.$$

Proof. Note that for $w\in B(x,r)$ and $u\in B(y,r)$, we have $|w-u|\le 3R_*$, thus

$$J^b(w,u)=\frac{\mathcal{A}(d,-\alpha)}{|w-u|^{d+\alpha}}+\frac{\mathcal{A}(d,-\beta)b(x,u-x)}{|w-u|^{d+\beta}}\ge\frac{\mathcal{A}(d,-\alpha)}{|w-u|^{d+\alpha}}-A\frac{\mathcal{A}(d,-\beta)}{|w-u|^{d+\beta}}\ge\frac12\frac{\mathcal{A}(d,-\alpha)}{|w-u|^{d+\alpha}}.$$

By a similar argument as that for Proposition 5.10, one can obtain the desired conclusion. See the arXiv version [13] of this paper for details. □

Theorem 5.12. *For every $\lambda>0$, $\varepsilon\in(0,1)$ and $A>0$, there are positive constants $C_{18}=C_{18}(d,\alpha,\beta,A,\lambda,\varepsilon)$ and $C_{19}=C_{19}(d,\alpha,\beta,A,\lambda)$ such that for any b with $\|b\|_\infty\le A$ that satisfies* (1.2) *and* (1.23),

$$C_{18}p_{m_{b^+,\lambda}}(t,x,y)\le q^b(t,x,y)\le C_{19}p_{M_{b^+,\lambda}}(t,x,y),\quad t\in(0,1],x,y\in\mathbb{R}^d. \tag{5.7}$$

Proof. Noting that the condition (1.23) in particular implies (1.19), so the upper bound estimate follows immediately from Theorem 5.8. We only need to prove the lower bound. Let $\delta_0:=1\wedge(A_0/A)^{\alpha/(\alpha-\beta)}$. (3.24) together with (1.8) also yields that for any $\|b\|_\infty\le A$,

$$q^b(t,x,y)\ge c_0t^{-d/\alpha}\quad\text{for } t\in(0,\delta_0]\text{ and }|x-y|\le 3t^{1/\alpha}. \tag{5.8}$$

Here $c_0=c_0(d,\alpha,\beta)$ is a positive constant. For every $t\in(0,\delta_0]$, by Proposition 5.9 and Proposition 5.10 with $R_0=1$, $r=t^{1/\alpha}/2$ and the strong Markov property of the process X^b, we get for $|x-y|>3t^{1/\alpha}$,

$$\begin{aligned}
&\mathbb{P}_x\big(X^b_{2^{-\alpha}\kappa t}\in B\big(y,t^{1/\alpha}\big)\big)\\
&\quad\ge\mathbb{P}_x\big(X^b\text{ hits }B\big(y,t^{1/\alpha}/2\big)\text{ before }2^{-\alpha}\kappa t\text{ and stays there for at least }2^{-\alpha}\kappa t\text{ units of time}\big)\\
&\quad\ge\mathbb{P}_x\big(\sigma^b_{B(y,t^{1/\alpha}/2)}<2^{-\alpha}\kappa t\big)\inf_{z\in B(y,t^{1/\alpha}/2)}\mathbb{P}_z\big(\tau^b_{B(z,t^{1/\alpha}/2)}\ge 2^{-\alpha}\kappa t\big)\\
&\quad\ge c_1t^{(d+\alpha)/\alpha}\bigg(\frac{1}{|x-y|^{d+\alpha}}+\frac{m_{b^+,\lambda}}{|x-y|^{d+\beta}}\bigg).
\end{aligned}\tag{5.9}$$

Here $c_1=c_1(d,\alpha,\beta,A,\lambda,\varepsilon)$ is a positive constant. Hence, by (5.8) and (5.9), for $|x-y|>3t^{1/\alpha}$ and $t\in(0,\delta_0]$,

$$\begin{aligned}
q^b(t,x,y)&\ge\int_{B(y,t^{1/\alpha})}q^b\big(2^{-\alpha}\kappa t,x,z\big)q^b\big(\big(1-2^{-\alpha}\kappa\big)t,z,y\big)\,dz\\
&\ge\inf_{z\in B(y,t^{1/\alpha})}q^b\big(\big(1-2^{-\alpha}\kappa\big)t,z,y\big)\mathbb{P}_x\big(X^b_{2^{-\alpha}\kappa t}\in B\big(y,t^{1/\alpha}\big)\big)\\
&\ge c_2t^{-d/\alpha}t^{(d+\alpha)/\alpha}\bigg(\frac{1}{|x-y|^{d+\alpha}}+\frac{m_{b^+,\lambda}}{|x-y|^{d+\beta}}\bigg)\\
&\ge c_2\bigg(\frac{t}{|x-y|^{d+\alpha}}+\frac{tm_{b^+,\lambda}}{|x-y|^{d+\beta}}\bigg),
\end{aligned}\tag{5.10}$$

where $c_2=c_2(d,\alpha,\beta,A,\lambda,\varepsilon)>0$, the third inequality holds due to $|z-y|\le t^{1/\alpha}\le 3((1-2^{-\alpha}\kappa)t)^{1/\alpha}$ when $\kappa\le 2^\alpha(1-3^{-\alpha})$ and (5.8)–(5.9). Finally, (5.8), (5.10) together with (1.10) and the Chapman–Kolmogorov equation yields the desired lower bound estimate. □

Theorem 5.13. *For every $\lambda > 0$ and $A > 0$, there are positive constants $C_k = C_k(d, \alpha, \beta, A)$, $k = 20, 21$ and $C_{22} = C_{22}(d, \alpha, \beta, A, \lambda)$ such that for any bounded b satisfying (1.2) and (1.19) with $\|b\|_\infty \le A$,*

$$C_{20}\overline{p}_0(t, C_{21}x, C_{21}y) \le q^b(t, x, y) \le C_{22}p_{M_{b^+,\lambda}}(t, x, y) \quad \text{for } t \in (0, 1] \text{ and } x, y \in \mathbb{R}^d. \tag{5.11}$$

Proof. By Theorem 5.8, it suffices to prove the lower bound of q^b. Let $\delta_0 := 1 \wedge (A_0/A)^{\alpha/(\alpha-\beta)}$. By Chapman–Kolmogorov equation, we only need to consider (5.11) for $t \in (0, \delta_0]$. By (1.20), (1.21) and (3.24), it suffices to prove (5.11) when $|x - y| > 3t^{1/\alpha}$ and $t \in (0, \delta_0]$. Let R_* be the constant defined in Proposition 5.11.

First, by a similar argument as in the proof of Theorem 5.12, there exists $c_2 = c_2(d, \alpha, \beta, A) > 0$ such that for $R_* \ge |x - y| > 3t^{1/\alpha}$,

$$q^b(t, x, y) \ge c_2 \frac{t}{|x - y|^{d+\alpha}}. \tag{5.12}$$

Next we consider the case $|x - y| > R_* > 3t^{1/\alpha}$. Take $C_* = R_*^{-1}$. Then $|x - y| > R_* = C_*^{-1} \ge t/C_*$ for $t \in (0, \delta_0]$. Set $R := |x - y|$ and $c_+ = R_*^{-1} \vee 1$. Let $l \ge 2$ be an integer so that $c_+R \le l \le c_+R + 1$, and $x = x_0, x_1, \dots, x_l = y$ be such that $|x_i - x_{i-1}| \asymp R/l \asymp 1/c_+$ for $1 \le i \le 1l - 1$. Since $t/l \le C_*R/l \le C_*/c_+ \le 1$ and $R/l \le 1/c_+ \le R_*$, we have by (5.8) and (5.12),

$$q^b(t/l, x_i, x_{i+1}) \ge c_2\left((t/l)^{-d/\alpha} \wedge \frac{t/l}{(R/l)^{d+\alpha}}\right) \ge c_2\big((t/l)^{-d/\alpha} \wedge (t/l)\big) \ge c_3t/l. \tag{5.13}$$

Then by (5.13) and a similar chain argument as in [8, Theorem 3.6], we have

$$q^b(t, x, y) \ge c_3\left(\frac{t}{|x - y|}\right)^{c_4|x-y|}, \quad |x - y| > R_* > 3t^{1/\alpha}. \tag{5.14}$$

By (5.12), (5.14) and together with the estimates of $\overline{p}_0$ in (1.20)–(1.21), we get the desired conclusion. □

Proof of Theorem 1.3. Theorem 1.3 now follows from Theorems 5.5, 5.12 and 5.13. □

To prove Theorem 1.5, we use the main result in [10] of the heat kernel estimates for non-local operators under the non-local Feynman–Kac perturbation. For each Borel function $q(x)$ on $\mathbb{R}^d$ and Borel function $F(x, y)$ on $\mathbb{R}^d \times \mathbb{R}^d$ that vanishes along the diagonal, we define a non-local Feynman–Kac transform for the process X^b as follows:

$$T_t^{b,F} f(x) = \mathbb{E}_x\left[\exp\left(\int_0^t q\big(X_s^b\big)\,ds + \sum_{s\le t} F\big(X_{s-}^b, X_s^b\big)\right) f\big(X_t^b\big)\right]. \tag{5.15}$$

By Ito's formula, Proposition 5.1, the Stieljes exponential expression for the Feynman–Kac transform, and the Lévy system formula in Proposition 5.2, we have the following proposition. See the arXiv version [13] of this paper for the details of its proof.

Proposition 5.14. *Suppose b is a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1.2) and (1.19), q is a bounded function on $\mathbb{R}^d$ and $|F(x, y)| \le c(|x - y|^2 \wedge 1)$ for some constant c. Then for each f in $C_b^2(\mathbb{R}^d)$,*

$$T_t^{b,F} f(x) = f(x) + \int_0^t T_s^{b,F}\mathcal{L}^{b,F} f(x)\,ds,$$

where $\mathcal{L}^{b,F} f(x) = \mathcal{L}^b f(x) + \int_{\mathbb{R}^d}(e^{F(x,y)} - 1) f(y)J^b(x, y)\,dy + q(x)f(x)$.

Proof of Theorem 1.5. Let $b_0(x, z) = b(x, z)\mathbb{1}_{|z|\le 1}(z)$, which is a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1.2) and (1.19). By Theorem 1.3, $q^{b_0}(t, x, y)$ is continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and

$$C_4p_0(t, x, y) \le q^{b_0}(t, x, y) \le C_3p_0(t, x, y) \tag{5.16}$$

for all $(t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$.

Set $F(x,y)=\log\frac{J^b(x,y)}{J^{b_0}(x,y)}$ and $q(x)=\int_{\mathbb{R}^d}(J^{b_0}(x,y)-J^b(x,y))\,dy$. It is easy to see that q is a bounded function on $\mathbb{R}^d$ and $J^b(x,y)=J^{b_0}(x,y)$ for $|x-y|\le 1$. Moreover, there exist two positive constants c_3 and c_4 so that $c_3\le\frac{J^b(x,y)}{J^{b_0}(x,y)}\le c_4$ for all $|x-y|>1$ and any bounded b with $\|b\|_\infty\le A$. Hence, there is a positive constant c_5 so that $|F(x,y)|\le c_5(|x-y|^2\wedge 1)$. Let $T_t^{b_0,F}$ be the semigroup $T_t^{b,F}$ defined by (5.15) but with b_0 in place of b. By (5.16) and [10, Theorem 1.3], the non-local Feynman–Kac semigroup $(T_t^{b_0,F}, t\ge 0)$ has a continuous density $\widetilde{q}(t,x,y)$ and there is a positive constant c_6 so that for all $(t,x,y)\in(0,1]\times\mathbb{R}^d\times\mathbb{R}^d$,

$$c_6^{-1}p_0(t,x,y)\le\widetilde{q}(t,x,y)\le c_6 p_0(t,x,y). \tag{5.17}$$

On the other hand, by the definition of $F(x,y)$ and $q(x)$, for each f in $C_b^2(\mathbb{R}^d)$, $\mathcal{L}^{b_0,F}f(x)=\mathcal{L}^b f(x)$. By taking $f=1$ in Proposition 5.14, we get $T_t^{b_0,F}1=1$. Hence $\widetilde{q}(t,x,y)$ uniquely determines a conservative Feller process $\widetilde{Y}$ with $\{T_t^{b_0,F}; t\ge 0\}$ as its transition semigroup. Proposition 5.14 implies that the distribution of $\widetilde{Y}$ on the canonical Skorokhod space $\mathbb{D}([0,\infty),\mathbb{R}^d)$ is a solution to the martingale problem $(\mathcal{L}^b, C_b^2(\mathbb{R}^d))$ and in particular to the martingale problem $(\mathcal{L}^b,\mathcal{S}(\mathbb{R}^d))$. However by Theorem 1.3, martingale solution to the operator $(\mathcal{L}^b,\mathcal{S}(\mathbb{R}^d))$ is unique. This yields that $\widetilde{q}=q^b$ and so we get the desired conclusion from (5.17). □

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