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Quenched invariance principle for random walk in time-dependent balanced random environment

Jean-Dominique Deuschel[a], Xiaoqin Guo[b,1] and Alejandro F. Ramírez[c,2]

[a] *Institüt für Mathematik, Technische Universität Berlin, Berlin, Germany. E-mail: deuschel@math.tu-berlin.de*
[b] *Department of Mathematics, Purdue University, West Lafayette, USA. E-mail: guo297@purdue.edu*
[c] *Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Santiago, Chile. E-mail: aramirez@mat.uc.cl*

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Abstract. We prove a quenched central limit theorem for balanced random walks in time-dependent ergodic random environments which is not necessarily nearest-neighbor. We assume that the environment satisfies appropriate ergodicity and ellipticity conditions. The proof is based on the use of a maximum principle for parabolic difference operators.

Résumé. Nous démontrons un théorème de loi limite centrale presque sûr pour des marches aléatoires équilibrées non nécessairement aux plus proches voisins, dans un milieu aléatoire ergodique. Nous supposons que l'environnement satisfait des conditions d'ergodicité et d'ellipicité appropriées. Notre preuve est basée sur un principe du maximum pour des opérateurs aux différences paraboliques.

MSC: 35K10; 60K37; 82D30

Keywords: Random walk in random environment; Maximum principle; Quenched central limit theorem

1. Introduction

We consider random walks in a balanced time-dependent random environment. Under a mild ergodicity assumption on the law of the environment and a moment condition on the jump probabilities, we prove a quenched central limit theorem (QCLT). Our results extend previous results of Lawler [21] and of Guo and Zeitouni [13] and are based on the use of a new maximum principle for parabolic difference operators. Furthermore, they can be considered as a version of discrete homogenization of stochastic parabolic operators in non-divergence form without uniform ellipticity (for homogenization results in a similar PDE settings, we refer to [5,22]).

We state our results in both discrete and continuous time settings.

1.1. *Discrete time RWRE*

Let U be a nonempty finite subset of $\mathbb{Z}^d$ which will be called the *jump range*. Define a set of probability vectors

$$\mathcal{P} = \mathcal{P}(U) := \left\{ v = \left\{ v(e) > 0 : e \in U \right\} : \sum_{e \in U} v(e) = 1 \right\}.$$

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Consider a discrete time stochastic process $\omega := \{\omega_n : n \in \mathbb{N}\}$ with state space $\Omega := \mathcal{P}^{\mathbb{Z}^d}$, so that $\omega_n := \{\omega_n(x) : x \in \mathbb{Z}^d\}$ with $\omega_n(x) := \{\omega_n(x,e) : e \in U\} \in \mathcal{P}$. We call $\Omega^{\mathbb{N}}$ the *environmental space* while an element

$$\omega \in \Omega^{\mathbb{N}} \tag{1.1}$$

a *discrete time environment*. Note that throughout this construction the set U is fixed and is not dependent on ω. Let us denote by $\mathbb{P}$ the law of ω and $\mathbb{E}_{\mathbb{P}}$ its expectation. Given $\omega \in \Omega^{\mathbb{N}}$, $x \in \mathbb{Z}^d$ and $n \in \mathbb{Z}$ consider the random walk $\{X_m : m \geq 0\}$ with a law $P_{x,n,\omega}$ on $(\mathbb{Z}^d)^{\mathbb{N}}$ defined through $P_{x,n,\omega}(X_n = x) = 1$ and the transition probabilities

$$P_{x,n,\omega}(X_{n+k+1} = y + e | X_{n+k} = y) = \omega_{n+k}(y,e),$$

for $k \geq 0$, $y \in \mathbb{Z}^d$ and $e \in U$. We call this process a *discrete time random walk in time-dependent random environment* and call $P_{x,n,\omega}$ the *quenched law* of the random walk starting from x at time n. The expectation of the law $P_{x,n,\omega}$ is denoted by $E_{x,n,\omega}$.

Given any topological space T, we will denote by $\mathcal{B}(T)$ the corresponding Borel sets. For $x \in \mathbb{Z}^d$ and $n \geq 0$ define the space–time shift $\theta_{n,x} : \Omega^{\mathbb{N}} \to \Omega^{\mathbb{N}}$ by $(\theta_{n,x}\omega)_m(y) := \omega_{n+m}(y+x)$ for all $m \geq 0$ and $y \in \mathbb{Z}^d$. Throughout, we will assume that $\mathbb{P}$ is stationary under the action of $\{\theta_{n,x} : n \geq 0, x \in \mathbb{Z}^d\}$. Let now $Z \subset \mathbb{N} \times \mathbb{Z}^d$. We will say that $\{\theta_{n,e} : (n,e) \in Z\}$ is an ergodic family of transformations for the probability space $(\Omega^{\mathbb{N}}, \mathcal{B}(\Omega^{\mathbb{N}}), \mathbb{P})$ if whenever $A \in \mathcal{B}(\Omega^{\mathbb{N}})$ satisfies $\theta_{n,e}^{-1} A = A$ for all $(n,e) \in Z$, then $\mathbb{P}(A) \in \{0,1\}$. Note that $\{\theta_{n,e} : (n,e) \in Z\}$ is an ergodic family of transformations for $\mathbb{P}$ if and only if $\{\theta_{n,e} : (n,e) \in \langle Z \rangle\}$ is an ergodic family for $\mathbb{P}$, where $\langle Z \rangle$ denotes the subset of $\mathbb{N} \times \mathbb{Z}^d$ generated by Z.

When the environment is time-independent, i.e., $\omega_n = \omega_{n+1}$ for all $n \geq 0$, we call ω a *static* environment. In this case, we may drop the time subscripts. E.g., we may write $\omega_n(x,e), \theta_{n,x}$ and $P_{x,n,\omega}$ simply as $\omega(x,e), \theta_x$ and $P_{x,\omega}$.

Define the space–time *environmental process* as the discrete time Markov chain

$$\bar{\omega}_n := \theta_{n,X_n}\omega, \quad n \geq 0,$$

with state space $\Omega^{\mathbb{N}}$. Here the random walk $\{X_n : n \geq 0\}$ has law $P_{0,0,\omega}$. In general, if ω is distributed according to some law μ on Ω, we define $P_\mu := \int P_{0,0,\omega}\, d\mu$. We will say that μ is an invariant distribution for the environmental process if $\bar{\omega}_n$ under P_μ has identical distribution for $n \geq 0$.

Let $D \subset U$. We say that a random environment with law $\mathbb{P}$ is *balanced in D* if for every $x \in \mathbb{Z}^d$ and $n \geq 0$

$$\mathbb{P}\Bigl(\sum_{e \in D} e\omega_n(x,e) = 0\Bigr) = 1.$$

We say that the environment is *uniformly elliptic in D* with ellipticity constant $\kappa > 0$ if

$$\mathbb{P}\bigl(\omega_n(x,e) > \kappa \text{ for all } e \in D \setminus \{0\}, x \in \mathbb{Z}^d \text{ and } n \geq 0\bigr) = 1.$$

When the environment is balanced (resp. uniformly elliptic) in U, we simply say that it is balanced (resp. uniformly elliptic). We call the environment *elliptic* if the jump range U spans $\mathbb{R}^d$.

Let us now recursively define the range of the random walk after n steps as $U_1 := U$, while for $n \geq 1$

$$U_{n+1} := \bigl\{y \in \mathbb{Z}^d : y = x + e \text{ for some } x \in U_n \text{ and } e \in U\bigr\}.$$

Let also for $n \geq 1$ and $x, y \in \mathbb{Z}^d$

$$p_n(x,y) := P_{x,0,\omega}(X_n = y).$$

Now set

$$V_n(x) := \bigl\{p_n(x, x+z)z : z \in U_n\bigr\} \subset \mathbb{R}^d. \tag{1.2}$$

Throughout, given a subset $V \subset \mathbb{R}^d$, we will denote by $\operatorname{conv}(V)$ its convex hull and by $|V|$ its Lebesgue measure. Define for $n \geq 1$

$$\varepsilon_n(x) := \left(p_n(x,x)\left|\operatorname{conv}\left(V_n(x)\right)\right|\right)^{1/(d+1)}. \tag{1.3}$$

Denote also by $\{e_1, \dots, e_d\}$ the canonical basis of $\mathbb{Z}^d$.

We say that the random walk $X_\cdot$ in random environment satisfies the *quenched central limit theorem* (QCLT) with a non-degenerate covariance matrix A if

> For almost all environments ω, under $P_{0,0,\omega}$, the sequence $X_{[n\cdot]}/\sqrt{n}$ converges in law to a Brownian motion with a deterministic non-degenerate covariance matrix A.

Theorem 1.1. *Consider a discrete time random walk in an elliptic balanced time-dependent random environment with law $\mathbb{P}$. Suppose that the family of shifts $\{\theta_{1,e} : e \in U\}$ is ergodic and that*

$$\inf_{n\geq 1} \mathbb{E}_{\mathbb{P}}\left[\varepsilon_n^{-(d+1)}(0)\right] < \infty. \tag{1.4}$$

Then, the following are satisfied.

(i) *The environmental process has a unique invariant probability measure ν which is absolutely continuous with respect to $\mathbb{P}$.*
(ii) *The QCLT holds with a non-degenerate covariance matrix $A = \{a_{i,j} : 1 \leq i, j \leq d\}$, where*

$$a_{i,j} := \sum_{e\in U} (e \cdot e_i)(e \cdot e_j) \int \omega_0(0,e)\,\mathrm{d}\nu.$$

Theorem 1.1 extends the static version of the QCLT proved by Lawler [21] for uniformly elliptic environments and by Guo and Zeitouni [13] for elliptic environments. Other recent related results for random walks in balanced static environments include Berger and Deuschel [9] and Baur [7]. On the other hand, it should be pointed out that several results exist proving QCLT for random walks in time-dependent environments, but in general under mixing condition which are stronger that our ergodicity assumption (see for example [11] or [3]). Recently in [4], the QCLT is obtained for continuous-time random walk in time-dependent ergodic random conductance under similar moment conditions as in (1.8) on the jump rates.

Remark 1.1. For nearest-neighbor random walks in a static random environment, a similar criteria for QCLT as (1.4) for $n = 1$ is obtained by Guo and Zeitouni [13], where $\mathbb{P}$ is ergodic with respect to the spatial shifts $\{\theta_x : x \in \mathbb{Z}^d\}$. Note that in the time-dependent case, neither do we demand the environment to be ergodic under the spatial shifts, nor under the time shifts alone.

One may replace the ergodic family of transformations $\{\theta_{1,e}, e \in U\}$ with $\{\theta_{n,x} : (n,x) \in Z\}$ for some set $Z \subset \mathbb{N} \times \mathbb{Z}^d$. Clearly, the smaller the set Z is, the stronger our ergodicity assumption is. A natural question is, can we weaken our condition by enlarging the ergodic family of transformations? We give a negative answer through a counterexample in Section 7, where the ergodic family of transformations is larger than $\{\theta_{1,e}, e \in U\}$ but the QCLT fails. (In that case, a CLT holds for almost all ω but with a random covariance matrix dependent on ω.)

Remark 1.2. The non-degeneracy of the matrix A of part (ii) of Theorem 1.1 follows from the fact that for any vector $u = (u_1, \dots, u_d)$ one has that

$$u \cdot Au = \sum_{e\in U} (e \cdot u)^2 \int \omega_0(0,e)\,\mathrm{d}\nu > 0.$$

Indeed, (1.4) implies that the vectors of U span $\mathbb{R}^d$. On the other hand, ν is absolutely continuous with respect to $\mathbb{P}$, which implies that A is a positive-definite matrix.

Remark 1.3. Condition (1.4) of Theorem 1.1 is always satisfied if the environment is balanced and uniformly elliptic (with constant $\kappa > 0$) in a subset $D \subset U$ such that $|\operatorname{conv}(D)| > 0$. Indeed, since the environment is balanced in D, we see that $0 \in \operatorname{conv}(D)$, so that for some constants $\lambda_e > 0$, $e \in D$, we have $\sum_{e \in D} \lambda_e e = 0$. Moreover, the coefficients λ_e can always be chosen as integer numbers, as all $e \in U$ have integer coordinates. Therefore, the random walk returns to the origin after $N = \sum_{e \in D} \lambda_e$ steps with a probability larger than κ^N, so that $p_N(0) \geq \kappa^N$. On the other hand, the fact that $|\operatorname{conv}(D)| > 0$ implies that $|\operatorname{conv}(V_N)| > 0$, cf. (1.2), is also bounded by some positive constant so that ε_N, cf. (1.3), is bounded from below by some positive constant.

1.2. *Continuous time RWRE*

We can also formulate a continuous time version of Theorem 1.1. Recall that U is a finite subset of $\mathbb{Z}^d$. Define

$$\mathcal{Q} := \{ v = \{ v(e) > 0 : e \in U \} \}.$$

Note that we do not assume any upper bound on the transition rates $v(\cdot) \in \mathcal{Q}$. We call $D([0, \infty); \mathfrak{H})$, where $\mathfrak{H} := \mathcal{Q}^{\mathbb{Z}^d}$, the *environmental space* while an element

$$\omega := \{\omega_t : t \geq 0\} \in D([0, \infty); \mathfrak{H})$$

a *continuous time environment*, so that $\omega_t := \{\omega_t(x) : x \in \mathbb{Z}^d\}$ with $\omega_t(x) := \{\omega_t(x, e) : e \in U\} \in \mathcal{Q}$. Let us denote by $\mathbb{Q}$ the law of the continuous time environment ω. Given an environment ω, for $u : \mathbb{Z}^d \times [0, \infty) \to \mathbb{R}$ bounded and differentiable in time for each $x \in \mathbb{Z}^d$, we define the parabolic difference operator

$$\mathcal{L}_\omega u(x, t) := \sum_{e \in U} \omega_t(x, e)\big[u(x + e, t) - u(x, t)\big] + \partial_t u(x, t).$$

Let $(X_t, t)_{t \geq 0}$ be the Markov process on $\mathbb{Z}^d \times [0, \infty)$ with generator $\mathcal{L}_\omega$ and initial state $(0, 0)$. We call $(X_t)_{t \geq 0}$ a *continuous time random walk in the time-dependent environment* ω and denote for each $x \in \mathbb{Z}^d$ by $P^c_{x,t,\omega}$ the law on $D([0, \infty); \mathbb{Z}^d)$ of this random walk starting from x at time t. We call $P^c_{x,t,\omega}$ the *quenched law* of the random walk.

For each $s \geq 0$ and $x \in \mathbb{Z}^d$ define the transformation

$$\theta_{s,x} : D([0, \infty); \mathfrak{H}) \to D([0, \infty); \mathfrak{H})$$

by $(\theta_{s,x}\omega)_t(y) := \omega_{t+s}(x + y)$. We assume that the law of the environment $\mathbb{Q}$ is stationary under the action of the shifts $\{\theta_{s,x} : s \geq 0, x \in \mathbb{Z}^d\}$.

We say that the random environment ω with law $\mathbb{Q}$ is *balanced* if for every $t \geq 0$, $x \in \mathbb{Z}^d$,

$$\sum_{e \in U} e\omega_t(x, e) = 0 \quad \mathbb{Q}\text{-a.s.} \tag{1.5}$$

As in the discrete-time case, we can also define the *environmental process* as the continuous time Markov process

$$\bar{\omega}_t := \theta_{t, X_t}\omega$$

for $t \geq 0$. Here the process $(X_t)_{t \geq 0}$ is sampled according to $P^c_{0,0,\omega}$. In general, if ω is distributed according to some law μ, we define $P^c_\mu := \int P^c_{0,0,\omega}\, d\mu$. We will say that μ is an invariant distribution for the environmental process if the law of $\bar{\omega}_t$ under P^c_μ is independent of t for $t \geq 0$.

For each $(x, t) \in \mathbb{Z}^d \times [0, \infty)$, let

$$U_{x,t} = \{\omega_t(x, e)e : e \in U\} \tag{1.6}$$

and

$$\begin{aligned} \varepsilon(x, t) = \varepsilon_\omega(x, t) &:= \big|\operatorname{conv}(U_{x,t})\big|^{1/(d+1)}, \\ \upsilon(x, t) = \upsilon_\omega(x, t) &:= \sum_{e \in U} \omega_t(x, e). \end{aligned} \tag{1.7}$$

We will denote by $\mathbb{E}_{\mathbb{Q}}$ the expectation with respect to the law $\mathbb{Q}$ of the environment and write $\varepsilon = \varepsilon(0,0)$ and $\upsilon = \upsilon(0,0)$.

Theorem 1.2. *Consider a continuous time random walk in an elliptic time-dependent balanced random environment with law $\mathbb{Q}$. Suppose that the family of shifts $\{\theta_{s,x} : s > 0, x \in U\}$ is ergodic. Assume that*

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{\upsilon^{d+1}+1}{\varepsilon^{d+1}}\right] < \infty. \tag{1.8}$$

Then, the following are satisfied.

(i) *The environmental process has a unique invariant distribution which is absolutely continuous with respect to $\mathbb{Q}$.*
(ii) *$\mathbb{Q}$-a.s. under $P^c_{0,0,\omega}$ the sequence $X_{t\cdot}/\sqrt{t}$ converges, as $t \to \infty$, in law on the Skorokhod space $D([0,\infty); \mathbb{R}^d)$ to a Brownian motion with a deterministic non-degenerate covariance matrix.*

1.3. *Applications of the QCLTs*

Theorem 1.2 can be applied to derive quenched central limit theorems for balanced environments driven by some interacting particle systems. An example of this situation is a random walk moving among the zero-range process. Given a function $g : \mathbb{N} \to [0,\infty)$ satisfying $g(k) > g(0) = 0$ for all $k > 0$, the zero-range process can be constructed as a Markov process describing the movement of particles on the lattice $\mathbb{Z}^d$, so that if at a site $x \in \mathbb{Z}^d$ and time $t \geq 0$ there are $\eta_t(x)$ particles, a particle jumps uniformly to the nearest neighboring sites of x at a rate $g(\eta_t(x))$. The infinitesimal generator L of this interacting particle system is defined by its action on functions $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$ depending on a finite number of coordinates of $\eta = \{\eta(x) : x \in \mathbb{Z}^d\} \in \mathbb{N}^{\mathbb{Z}^d}$ by

$$Lf(\eta) = \sum_{x,y\in\mathbb{Z}^d:|x-y|_1=1} g\bigl(\eta(x)\bigr)\bigl(f\bigl(\eta^{x,y}\bigr) - f(\eta)\bigr),$$

where

$$\eta^{x,y}(z) := \eta(z) - \mathbb{1}_{z=x} + \mathbb{1}_{z=y}.$$

Under the condition

$$\sup_{k\in\mathbb{N}}\bigl|g(k+1) - g(k)\bigr| < \infty,$$

this process is well defined whenever the initial condition $\eta \in S$, where

$$S := \left\{\eta \in \mathbb{N}^{\mathbb{Z}^d} : \sum_{x\in\mathbb{Z}^d} \eta(x)\alpha(x) < \infty\right\},$$

and

$$\alpha(x) := \sum_{n=0}^{\infty} \frac{1}{2^n} p_n(0,x),$$

with $p_n(0,x)$ the probability that a discrete time simple symmetric random walk starting from 0 is at position x at time n (see [2] for this construction). The above process is called *zero-range process*, and we will denote by P_η its law on $D([0,\infty); S)$ starting from $\eta \in S$. This process has a family of invariant measures (see also [2]) defined through the partition function $Z : [0,\infty) \to [0,\infty)$ by

$$Z(\alpha) = \sum_{k\geq 0} \frac{\alpha^k}{g(1)\cdots g(k)}.$$

Define

$$\alpha^* := \sup\{\alpha \in [0,\infty) : Z(\alpha) < \infty\}.$$

Assume also that

$$\lim_{\alpha\to\alpha^*} Z(\alpha) = \infty.$$

Now, for each $0 \le \alpha < \alpha^*$ define the product probability measure μ_α on the Borel σ-algebra of $\mathbb{N}^{\mathbb{Z}^d}$, with marginals given by

$$\mu_\alpha(k) = \frac{1}{Z(\alpha)} \frac{\alpha^k}{g(1)\cdots g(k)}.$$

As a matter of fact $\mu_\alpha(S) = 1$, so that we can define a probability measure

$$P_\alpha := \int P_\eta \, \mathrm{d}\mu_\alpha(\eta).$$

Let us assume that the function g is non-decreasing, so that for each $0 \le \alpha < \alpha^*$, the invariant measure μ_α is also extremal [2]. Therefore, for $\alpha \in [0, \alpha^*)$, the shifts $\{\theta_{s,x} : s > 0, x \in \mathbb{Z}^d\}$ form an ergodic family of transformations for P_α. For each $e \in \{e_1, \dots, e_d\}$, fix a finite range function $u(e, \cdot) : S \to (0, \infty)$: in other words, there is an $R > 0$ such that for each $\eta \in S$, $u(e, \eta)$ depends only on $\eta(x)$ for $|x|_1 \le R$. Define for $e \in \{e_1, \dots, e_d\}$,

$$u(-e, \cdot) := u(e, \cdot).$$

Now, define the stochastic process for $t \ge 0$, $\omega_t := \{\omega_t(x) : x \in \mathbb{Z}^d\}$ where for $x \in \mathbb{Z}^d$, we define

$$\omega_t(x) := \{u(e, \theta_x \eta_t) : e \text{ such that } |e|_1 = 1\}.$$

Let us call $\omega := \{\omega_t : t \ge 0\}$ a *an environment generated by an attractive zero-range process*. We then have the following immediate corollary to Theorem 1.2.

Corollary 1.1. *Consider a continuous time random walk $\{X_t : t \ge 0\}$ in an environment ω generated by an attractive zero-range process, with law P_α for some $\alpha \in [0, \alpha^*)$. Assume that*

$$\int \frac{(\sum_{e\in U} u(e,\eta))^{d+1} + 1}{\prod_{i=1}^d u(e_i, \eta)} \, \mathrm{d}\mu_\alpha < \infty. \tag{1.9}$$

Then P_α-a.s. the random walk $\{X_t : t \ge 0\}$ satisfies the quenched central limit theorem with nondegenerate covariance matrix.

The condition (1.9) is satisfied when the function u is bounded from both above and below. Furthermore, Corollary 1.1 includes the case of a second class particle in the zero-range process, solved by Saada in [26], where nevertheless the main problem we face here, which is the construction of the invariant measure, is not present.

Theorem 1.1 can be applied to derive QCLTs for a certain class of random walks in static random environment. In order to give a simple example, we will consider a random walk on $\mathbb{Z}^{d_1+d_2}$, $d_1, d_2 \in \mathbb{N}$. For $x \in \mathbb{Z}^{d_1+d_2}$, we write $x = (x^{(1)}, x^{(2)})$ so that $x^{(1)} \in \mathbb{Z}^{d_1}$ and $x^{(2)} \in \mathbb{Z}^{d_2}$. We say that a static environment $\omega \in \mathcal{P}^{\mathbb{Z}^{d_1+d_2}}$ is *autonomous in the first coordinates* if $P_{x,\omega}(X_1^{(1)} = e^{(1)})$ depends only on the first d_1 coordinates of x. That is, for $x, z \in \mathbb{Z}^{d_1+d_2}$,

$$P_{x,\omega}(X_1^{(1)} = e^{(1)}) = P_{z,\omega}(X_1^{(1)} = e^{(1)}) \quad \text{if } x^{(1)} = z^{(1)}.$$

In other words, $X_n^{(1)}$ is a Markov chain under $P_{0,\omega}$.

Corollary 1.2. *Consider a random walk $X_n = (X_n^{(1)}, X_n^{(2)})$ on $\mathbb{Z}^{d_1+d_2}$ in a static random environment with jump range $U = \{e \in \mathbb{Z}^{d_1+d_2} : |e| \le 1\}$. Assume that $\mathbb{P}$ is stationary under $\{\theta_x : x \in \mathbb{Z}^{d_1+d_2}\}$ and for all $i = 1, \dots, d_1$, it is ergodic under the shifts $\{\theta_{\pm e_i}\}$. Suppose the following are satisfied.*

(a) (autonomous first coordinates) $\mathbb{P}$-*almost surely, the environment is autonomous in the first coordinates.*
(b) (equivalent ergodic invariant measure for the first coordinates) *There is an invariant measure ν which is equivalent to $\mathbb{P}$ such that the environmental process $(\theta_{(X_n^{(1)},0)}\omega)_{n\in\mathbb{N}}$ is an ergodic sequence under $\nu \times P_{0,\omega}$.*
(c) (QCLT for the first coordinates of the random walk) *There exists a deterministic vector $v_1 \in \mathbb{Z}^{d_1}$ such that $\mathbb{P}$-a.s. under $P_{0,\omega}$, the sequence $(X_{[n\cdot]}^{(1)} - v_1 n\cdot)/\sqrt{n}$ converges in law to a Brownian motion with a deterministic non-degenerate covariance matrix.*
(d) (balanced and uniformly-elliptic in the second coordinates) $\mathbb{P}$-*a.s., ω is uniformly-elliptic and balanced in $\{\pm e_i : i = d_1+1, \dots, d_1+d_2\}$.*

Then $\mathbb{P}$-a.s. under $P_{0,\omega}$, the sequence $(X_{[n\cdot]} - nv)/\sqrt{n}$ converges in law to a Brownian motion with a deterministic non-degenerate covariance matrix, where $v := (v_1, 0) \in \mathbb{Z}^{d_1+d_2}$.

Remark 1.4. As an application of Corollary 1.2, we will obtain QCLT for an environment which is "ballistic and autonomous in the first d_1 directions and balanced in the other d_2 coordinates."

We let $\alpha \in \mathcal{P}^{\mathbb{Z}^{d_1}}$ and $\beta \in \mathcal{P}^{\mathbb{Z}^{d_1+d_2}}$ be static random environments which are independent under their joint law. We construct α, β such that

(i) α is an iid uniformly-elliptic environment on $\mathbb{Z}^{d_1}$, $d_1 \ge 4$, with jump range $\{y \in \mathbb{Z}^{d_1} : |y| \le 1\}$ and it satisfies the ballisticity condition $(\mathscr{P})$, cf. [8, Theorem 1.10].
(ii) β is a stationary (under the shifts $\{\theta_x : x \in \mathbb{Z}^{d_1+d_2}\}$) balanced environment on $\mathbb{Z}^{d_1+d_2}$ which is uniformly-elliptic on its jump range $\{x \in \mathbb{Z}^{d_1+d_2} : x^{(1)} = 0, |x^{(2)}| \le 1\}$.

Define the environment $\omega \in \mathcal{P}^{\mathbb{Z}^{d_1+d_2}}$ such that for $x, e \in \mathbb{Z}^{d_1+d_2}$,

$$\omega(x,e) = \begin{cases} \alpha(x^{(1)}, e^{(1)}) & \text{if } e = \pm e_i, i = 1, \dots, d_1, \\ \alpha(x^{(1)}, 0)\beta(x,e) & \text{if } e = 0 \text{ or } e = \pm e_{d_1+j}, j = 1, \dots, d_2. \end{cases}$$

Then the law of ω satisfies all conditions of Corollary 1.2 and the QCLT holds.

Remark 1.5. One may replace α in the above example with any environment that satisfies conditions (b) and (c) of Corollary 1.2. For instance, let $\xi(y,e)$ be the transition probability of an iid conductance model, we may take a constant $r \in (0,1)$ and let $\alpha(x,e) := (1-r)\xi(x,e) + r\mathbb{1}_{e=0}$. Clearly, the presence of r is to guarantee that $\alpha(x,0) > 0$.

Remark 1.6. Recently Baur [7] proved a QCLT with similar flavor for iid static environment on $\mathbb{Z}^d$, $d \ge 3$, where the environment is a small perturbation of the simple random walk and is balanced in a fixed coordinate direction. The law of the environment is also assumed to be invariant under antipodal reflections.

1.4. *Organization of the article*

Since the proofs of Theorem 1.1 and 1.2 are similar, most of the subsequent sections of this paper will give the details of the proof of the discrete time Theorem 1.1, while an outline of the proof of the continuous time Theorem 1.2 is given in Section 5. In Section 2.1, we state the version of Kozlov's theorem that will be used to construct the absolutely continuous invariant measure for the discrete time random walk. In Section 2.2, the parabolic maximum principle for general meshes is stated while its proof is deferred to Section 4. Both Kozlov's theorem and the parabolic maximum principle are subsequently used in Section 3 to prove Theorem 1.1. Corollary 1.2 is proved in Section 6. In Section 7, we give an example that the ergodicity hypothesis of Theorem 1.1 cannot be weakened by enlarging the ergodic family of transformations.

2. Two preliminary tools

Here we state two theorems that will be used to prove Theorem 1.1. The first is a version of a well known theorem of Kozlov for time dependent random walks, while the second is the parabolic maximum principle for general meshes, whose proof is given in Section 4. The parabolic maximum principle is a crucial tool to construct the absolutely continuous invariant measure of part (i) of Theorem 1.1, while Kozlov's theorem is required to derive part (ii) of the same theorem.

2.1. *Kozlov's theorem*

The proof of Theorem 1.1 will require the following version of Kozlov's theorem [14] for time dependent random walks.

Theorem 2.1. *Consider a random walk in a time-dependent elliptic random environment which has a law $\mathbb{P}$. Assume that $\{\theta_{1,z} : z \in U\}$ is an ergodic family of transformations with respect to $\mathbb{P}$. Assume that there exists an invariant measure ν for the environmental process, which is absolutely continuous with respect to $\mathbb{P}$. Then, the following are satisfied*:

(i) *ν is equivalent to $\mathbb{P}$.*
(ii) *The environment as seen from the random walk with initial law ν is ergodic.*
(iii) *ν is the unique probability measure for the environmental process which is absolutely continuous with respect to $\mathbb{P}$.*

Proof. Since the proof is similar to the case of random walks in static random environments, we will stress the steps which are different (see Theorem 1.2 of Lecture 1 of [10], for example, for a proof of the theorem for static random walk in random environment).

To prove part (i), let f be the Radon-Nikodym derivative of ν with respect to $\mathbb{P}$ and define $E := \{f = 0\}$. We will prove that $\mathbb{P}(E) = 0$. Using the fact that ν is invariant, we can conclude that $\mathbb{P}$-a.s. for every $z \in U$,

$$\mathbb{1}_E(\omega) \geq \sum_{z' \in U} \omega_0\big(0, z'\big) \mathbb{1}_E(\theta_{1,z'}\omega) \geq \omega_0(0, z) \mathbb{1}_E(\theta_{1,z}\omega).$$

From the ellipticity assumption and the fact that $\mathbb{1}_E(\omega)$ and $\mathbb{1}_E(\theta_{1,z}\omega)$ only take the values 0 and 1 we see that for each $z \in U$, $\mathbb{P}$-a.s.

$$\mathbb{1}_E(\omega) \geq \mathbb{1}_E(\theta_{1,z}\omega).$$

Now since $\mathbb{P}(E) = \mathbb{P}(\theta_{1,z}^{-1} E)$, we conclude that for each $z \in U$, $\mathbb{P}$-a.s.

$$\mathbb{1}_E = \mathbb{1}_{\theta_{1,z}^{-1} E}.$$

Thus the function $\mathbb{1}_E : \Omega^{\mathbb{N}} \to \{0, 1\}$ is a.s. shift-invariant under $\{\theta_{1,z} : z \in U\}$. By our ergodicity assumption, we conclude that $\mathbb{1}_E$ is a.s. a constant and so

$$\mathbb{P}(E) \in \{0, 1\}.$$

But $\int_{E^c} f \, d\mathbb{P} = \int f \, d\mathbb{P} = 1$ implies that $\mathbb{P}(E^c) > 0$, so that necessarily $\mathbb{P}(E) = 0$.

Let $T : \Omega^{\mathbb{N}\times\mathbb{N}} \to \Omega^{\mathbb{N}\times\mathbb{N}}$ denote the left-shift that maps the sequence $(\bar{\omega}_n)_{n\geq 0}$ to $(\bar{\omega}_{n+1})_{n\geq 0}$. To prove part (ii) as in the static case (see [10]) it is possible to prove that for every $A \in \mathcal{B}((\Omega^{\mathbb{N}})^{\mathbb{N}})$ such that $T^{-1}A = A$, the process

$$\phi(\bar{\omega}_n) := P_{\bar{\omega}_n}(A),$$

is a martingale and also there is a set $B \in \mathcal{B}(\Omega^{\mathbb{N}})$ such that

$$\phi = \mathbb{1}_B.$$

We then show that $\mathbb{P}$-a.s. for each $z \in U$, the inequality

$$\mathbb{1}_B(\omega) \geq \sum_{z' \in U} \omega_0\big(0, z'\big)\mathbb{1}_B(\theta_{1,z'}\omega) \geq \omega_0(0, z)\mathbb{1}_B(\theta_{1,z}\omega)$$

is satisfied. Using an argument similar to the one employed in part (i) we now see that $\nu(B) \in \{0, 1\}$, which proves that $P_\nu(A) = \nu(B) \in \{0, 1\}$.

The uniqueness of ν stated in part (iii) can be obtained following exactly the same argument as in the static case [10]. □

2.2. *A new maximum principle for parabolic difference operators on general meshes*

The quenched central limit theorem for random walks in static balanced random environments [21] can be proved using lattice versions of the maximum principle for elliptic operators of Aleksandrov–Bakel'man–Pucci [1,6,25] for elliptic partial differential equations (see Papanicolaou and Varadhan [24] for an application to prove a QCLT for diffusions with random coefficients). The maximum principle for elliptic difference operators were proved by Kuo and Trudinger in a series of papers (see for example [18]).

Nevertheless, to prove Theorem 1.1, we will need a parabolic maximum principle. Within the context of diffusions, this was first established by Krylov [15], and subsequently a discrete version for general meshes proved by Kuo and Trudinger in [16,19,20]. Here we state a new parabolic maximum principle, Theorem 2.2, for difference operators and prove it in Section 4 using a geometric approach.

We firstly introduce some notation. Given $x \in \mathbb{Z}^d$, we denote by $|x|_2$ its l_2 norm. For $x_0 \in \mathbb{Z}^d$, $R > 0$, let

$$B_R(x_0) := \big\{x \in \mathbb{Z}^d : |x - x_0|_2 \leq R\big\}.$$

Consider a balanced time dependent environment $a = \{a_n : n \geq 0\} \in \Omega^{\mathbb{N}}$ (cf. (1.1)). For any finite subset $\mathcal{D} \subset \mathbb{Z}^d \times \mathbb{Z}$, define its *parabolic boundary* (see Figure 1) by

$$\mathcal{D}^p := \big\{(y, n+1) \notin \mathcal{D} : a_n(x, y - x) > 0 \text{ for some } (x, n) \in \mathcal{D}\big\}.$$

Define the parabolic operator

$$\mathcal{L}_a u(x, n) := \sum_{z \in U} a_n(x, z)u(x + z, n + 1) - u(x, n).$$

For a real function g defined on $\mathcal{D} \subset \mathbb{Z}^d \times \mathbb{Z}$ and $p > 0$ define

$$\|g\|_{\mathcal{D},p} := \Bigg(\sum_{(x,n) \in \mathcal{D}} \big|g(x, n)\big|^p\Bigg)^{1/p}. \tag{2.1}$$

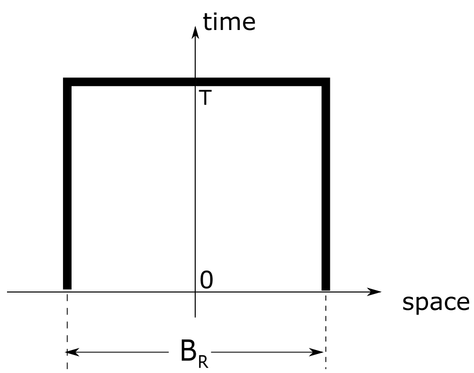

Fig. 1. The thick black lines represent the parabolic boundary of $B_R \times [0, T)$.

Set

$$U_{x,n} := \{a_n(x,z)z : z \in U\}, \qquad v(x,n) := |\mathrm{conv}(U_{x,n})|, \tag{2.2}$$

and define

$$\varepsilon_a(x,n) := \left(a_n(x,0)\frac{v(x,n)}{\#U}\right)^{1/(d+1)},$$

where $\#U$ denotes the cardinality of the discrete set $U \in \mathbb{Z}^d$.

Theorem 2.2. *Assume that $\mathcal{D} \subset \mathbb{Z}^d \times \mathbb{Z}$ is a finite set and $\mathcal{D} \cup \mathcal{D}^p \subset B_R \times [0,T]$ for some $R, T > 0$. Let u be a function on $\mathcal{D} \cup \mathcal{D}^p$ that satisfies*

$$\mathcal{L}_a u \geq -f \quad \text{in } \mathcal{D} \tag{2.3}$$

for some function f on $\mathcal{D}$. Then, if $\varepsilon_a(x,n) > 0$ for all $(x,n) \in \mathcal{D}$, we have

$$\max_{\mathcal{D}} u \leq \max_{\mathcal{D}^p} u + CR^{d/(d+1)}\|f/\varepsilon_a\|_{\mathcal{D},d+1},$$

where $C = C(U,d)$ is a constant.

Remark 2.1. The elliptic version of Theorem 2.2 was implicitly obtained in [17, (44)]. However, there is a minor gap in its proof. That is, [17, Lemma 3] is not true for general non-symmetric convex bodies. This can be fixed by symmetrization and using the balanced assumption (see (4.5)).

3. Proof of the discrete time QCLT (Theorem 1.1)

It is easy to check, as in the case of random walks in static balanced random environments, that part (i) of Theorem 1.1 implies, through Theorem 2.1, part (ii) (see [21]). We therefore will concentrate on the proof of part (i).

Throughout this section, we fix a balanced environment $\omega \in \Omega^{\mathbb{N}}$, so that for all $x \in \mathbb{Z}^d$ and $n \geq 0$,

$$\sum_{e \in U} e\omega_n(x,e) = 0.$$

By (1.4), we know that there is $k \in \mathbb{N}$ such that the random walk returns to its starting point after k steps and such that

$$\mathbb{E}_{\mathbb{P}}\left[\varepsilon_k^{-(d+1)}\right] < \infty. \tag{3.1}$$

We will soon see that the case in which $k > 1$ can be reduced to the case $k = 1$. Therefore, let us first assume that $k = 1$. Let N be an even natural number. We introduce for $(x,n) \in \mathbb{Z}^d \times \mathbb{Z}$ the equivalence classes

$$\overline{(x,n)} := (x,n) + (2N+1)\mathbb{Z}^d \times (N^2+1)\mathbb{Z}. \tag{3.2}$$

In addition we define the periodized version $\omega^{(N)}$ of ω so that $\omega_m^{(N)}(y) = \omega_n(x)$ for $(y,m) \in \mathbb{Z}^d \times \mathbb{Z}$ with $\overline{(y,m)} = \overline{(x,n)}$ and

$$(x,n) \in K_N := \{z \in \mathbb{Z}^d : |z|_\infty \leq N\} \times \{n' : 0 \leq n' \leq N^2\}.$$

Set

$$\Omega_{N,\omega} = \Omega_N := \{\theta_{n,x}\omega^{(N)} : (x,n) \in \mathbb{Z}^d \times \mathbb{Z}\}.$$

It is straightforward to see that the process $(\theta_{n,X_n}\omega^{(N)})_{n\geq 0}$ is a Markov chain with a finite state space Ω_N and has an invariant measure $\nu_N \ll \mathbb{P}_N$, where

$$\mathbb{P}_N := \frac{1}{(N^2+1)}\frac{1}{(2N+1)^d}\sum_{(x,n)\in K_N}\delta_{\theta_{n,x}\omega^{(N)}}.$$

Although it will not be used in this proof, note that the measure ν_N is of the form

$$\nu_N = \sum_{(x,n)\in K_N}\phi_N(x,n)\delta_{\theta_{n,x}\omega^{(N)}},$$

where $\phi_N = \phi_{N,\omega}$ is also the density of an invariant measure of the random walk $\{\overline{(X_n,n)} : n \geq 0\}$, cf. (3.2), on K_N in the environment $\omega^{(N)}$.

Note that since $\mathbb{P}$ is ergodic under the action of $\{\theta_{1,x} : x \in U\}$ which is a subset of the transformations $\{\theta_{n,x} : (n,x) \in \mathbb{N}\times\mathbb{Z}^d\}$, by the multidimensional ergodic theorem (see [12, Theorem VIII.6.9]),

$$\lim_{N\to\infty}\mathbb{P}_N = \mathbb{P}, \quad \mathbb{P}\text{-a.s.}$$

Define the stopping times $\tau_0 = 0$ and

$$\tau_{j+1} = \inf\{i > \tau_j : |X_i - X_{\tau_j}|_\infty > N \text{ or } i - \tau_j > N^2\}, \quad j \geq 0.$$

Lemma 3.1. *There exists a constant $c_2 > 0$ such that for all $c \geq c_2$, there is an N_0 such that for $N \geq N_0$ we have that*

$$\sup_{x\in\mathbb{Z}^d, n\geq 0, \xi\in\Omega_N} E_{x,n,\xi}\left[\left(1-\frac{c}{N^2}\right)^{\tau_1}\right] \leq \frac{1}{2}. \tag{3.3}$$

Proof. The proof follows the lines of [13, Lemma 4]. Since $\{X_n : n \geq 0\}$ is a martingale, by Doob's martingale inequality, for any $1 \leq K < N^2$,

$$\begin{aligned}P_{x,n,\xi}(\tau_1 \leq K) &\leq 2\sum_{i=1}^d \sup_{\xi\in\Omega_N} P_{x,n,\xi}\Big(\max_{0<m\leq K}(X_{n+m}-x)^+(i) > N\Big)\\ &\leq \frac{2}{N}\sum_{i=1}^d \sup_{\xi\in\Omega_N} E_{x,n,\xi}\big[(X_{n+K}-x)^+(i)\big] \leq \frac{2dC_U\sqrt{K}}{N},\end{aligned}$$

where for each $y \in \mathbb{Z}^d$, $y(i)$ denotes its ith coordinate and $C_U = \max\{|e| : e \in U\}$. Hence for every $c > 0$ we have that

$$E_{x,n,\xi}\left[\left(1-\frac{c}{N^2}\right)^{\tau_1}\right] \leq \left(1-\frac{c}{N^2}\right)^K + \frac{2dC_U\sqrt{K}}{N}.$$

Taking $K = (\frac{N}{8dC_U})^2$ it follows that for c large enough whenever N is large enough then inequality (3.3) is satisfied. □

Denote by $S = S_{\omega^{(N)}}$ the transition semigroup of the environment Markov chain $\{\xi_n : n \geq 0\}$ defined for each $\xi \in \Omega_N$ as $\xi_n := \theta_{n,X_n}\xi$ for $n \geq 0$. That is, for every function g on Ω_N and $\xi \in \Omega_N$, for $k \geq 0$ define

$$S^k g(\xi) := E_{0,0,\xi}\big[g(\xi_k)\big], \quad \forall k \in \mathbb{N}.$$

Since ν_N is the invariant distribution for the finite-state Markov chain $(\xi_k)_{k\geq 0}$, we have

$$\int g\,\mathrm{d}\nu_N = \int S^k g\,\mathrm{d}\nu_N, \quad \forall k \in \mathbb{N}.$$

Let c_2, N_0 be the same constants as in Lemma 3.1. For $N \geq N_0$, putting $\rho = \rho(\omega, N) := 1 - \frac{c_2}{N^2} \in (0,1)$, we see that

$$(1-\rho)^{-1} \int g \, \mathrm{d}\nu_N = \sum_{k=0}^{\infty} \rho^k \int S^k g \, \mathrm{d}\nu_N \leq \max_{\xi \in \Omega_N} \sum_{k=0}^{\infty} \rho^k S^k g(\xi) = \max_{\xi \in \Omega_N} E_{0,0,\xi}\left[\sum_{k=0}^{\infty} \rho^k g(\xi_k)\right]. \tag{3.4}$$

On the other hand, we have that

$$\begin{aligned}
\max_{\xi \in \Omega_N} E_{0,0,\xi}\left[\sum_{k=0}^{\infty} \rho^k g(\xi_k)\right] &\leq \sum_{m=0}^{\infty} \max_{\xi \in \Omega_N} E_{0,0,\xi}\left[\rho^{\tau_m} \sum_{k \in [\tau_m, \tau_{m+1})} g(\xi_k)\right] \\
&\leq \sum_{m=0}^{\infty} \Bigl(\max_{\xi \in \Omega_N} E_{0,0,\xi}\left[\rho^{\tau_1}\right]\Bigr)^m \max_{\xi \in \Omega_N} E_{0,0,\xi}\left[\sum_{k=0}^{\tau_1 - 1} g(\xi_k)\right] \\
&\leq 2 \max_{\xi \in \Omega_N} E_{0,0,\xi}\left[\sum_{k=0}^{\tau_1 - 1} g(\xi_k)\right],
\end{aligned} \tag{3.5}$$

where in the first inequality we used the strong Markov property at times τ_m, τ_{m-1} up to τ_1 successively, and in the last inequality we used inequality (3.3) of Lemma 3.1. Recall that K_N^p denotes the parabolic boundary of K_N. Now, for any $(x,n) \in K_N \cup K_N^p$ and $\xi \in \Omega_N$, define

$$f_\xi(x,n) := E_{x,n,\xi}\left[\sum_{k=0}^{\tau - 1} g(\xi_k)\right],$$

where $\tau = \inf\{i \geq 0 : |X_i|_\infty > N \text{ or } i > N^2\}$. Then f_ξ satisfies

$$\begin{cases} \mathcal{L}_a f_\xi(x,n) = -G_\xi(x,n), & \text{if } (x,n) \in K_N, \\ f_\xi(x,n) = 0, & \text{if } (x,n) \in K_N^p, \end{cases}$$

where $G_\xi(x,n) := g(\theta_{x,n}\xi)$. We can now apply Theorem 2.2 to conclude

$$\begin{aligned}
\max_{\xi \in \Omega_N} f_\xi(0,0) &\leq \max_{\xi \in \Omega_N} C N^{d/(d+1)} \|G_\xi / \varepsilon_1\|_{K_N, d+1} \\
&= C N^2 \|g/\varepsilon_1\|_{L^{d+1}(\mathbb{P}_N)},
\end{aligned} \tag{3.6}$$

where the norm $\|\cdot\|_{K_N, d+1}$ is defined in (2.1). Therefore, combining (3.4), (3.5) and (3.6), we conclude that for some constant $C > 0$,

$$\int g \, \mathrm{d}\nu_N \leq C \|g/\varepsilon_1\|_{L^{d+1}(\mathbb{P}_N)}.$$

Since Ω is pre-compact, using Prohorov's theorem, we can extract a subsequence ν_{N_k} of ν_N which converges weakly to some limit ν as $k \to \infty$. Note that by the construction, any limit of ν_N is an invariant distribution for the Markov chain $(\bar{\omega}_n)$, cf. [13] or [21]. Then, by the ergodic theorem and the assumption $E_{\mathbb{P}}[1/\varepsilon_1^{d+1}] < \infty$, cf. (3.1), we would conclude that

$$\int g \, \mathrm{d}\nu \leq C \|g/\varepsilon_1\|_{L^{d+1}(\mathbb{P})} \quad \text{for any continuous function } g \text{ on } \Omega^{\mathbb{N}}.$$

The above inequality implies that ν is absolutely continuous with respect to the probability measure μ defined by

$$\mathrm{d}\mu := \frac{1}{\mathbb{E}_{\mathbb{P}}[\varepsilon_1^{-(d+1)}(0,0)]} \frac{1}{\varepsilon_1^{d+1}(0,0)} \, \mathrm{d}\mathbb{P}.$$

Since μ is by definition absolutely continuous with respect to $\mathbb{P}$, we conclude that $\nu \ll \mathbb{P}$. Now, note also that Theorem 2.1 ensures that ν is unique.

In the case in which (3.1) is satisfied for $k > 1$, by the same argument as in the case $k = 1$, we can construct an invariant measure ν_k which is absolutely continuous with respect to $\mathbb{P}$, for the environmental process looked at times which are multiples of k, defined for $n \geq 0$ by

$$\bar{\omega}_n^{(k)} := \theta_{nk, X_{nk}} \omega.$$

We will now show how to construct from ν_k an invariant measure ν which is absolutely continuous with respect to $\mathbb{P}$, for the environmental process $\{\bar{\omega}_n : n \geq 0\}$. Define for every bounded and continuous function g, the measure ν by

$$\int g \,\mathrm{d}\nu := \frac{1}{k} \int \sum_{i=0}^{k-1} R^i g \,\mathrm{d}\nu_k,$$

where the operator R is defined by

$$Rg(\omega) := E_{0,0,\omega}\big[g(\theta_{1,X_1}\omega)\big] = \sum_{e \in U} \omega_0(0,e) g(\theta_{1,e}\omega)$$

and $R^0 = I$ is the identity map. Then note that

$$\int Rg \,\mathrm{d}\nu = \frac{1}{k} \int \sum_{i=1}^{k} R^i g \,\mathrm{d}\nu_k = \int g \,\mathrm{d}\nu + \frac{1}{k} \int \big(R^k g - g\big) \,\mathrm{d}\nu_k = \int g \,\mathrm{d}\nu,$$

where in the last equality we used the fact that ν_k is an invariant distribution for the kernel R^k. This proves that ν is an invariant measure for the environmental process. To see that ν is absolutely continuous with respect to $\mathbb{P}$ note that for each measurable A in Ω, with $\mathbb{P}(A) = 0$, we have

$$\int R^i \mathbb{1}_A \,\mathrm{d}\nu_k \leq \sum_{z \in U_i} \nu_k\big(\theta_{i,z}^{-1} A\big) = 0 \quad \forall i = 1, \ldots, k,$$

since the stationarity of $\mathbb{P}$ implies that $\mathbb{P}(\theta_{i,z}^{-1} A) = 0$ which in turn implies by the fact that $\nu_k \ll P$ that $\nu_k(\theta_{i,z}^{-1} A) = 0$. Therefore we conclude that $\nu(A) = 0$ and hence ν is absolutely continuous with respect to $\mathbb{P}$.

4. Proof of the maximum principle (Theorem 2.2)

Here we will prove the maximum principle in Theorem 2.2. Define

$$M := \max_{\mathcal{D}} u.$$

Without loss of generality assume that $M > 0$, $\max_{\mathcal{D}^p} u \leq 0$, $\varepsilon_a > 0$ and $f \geq 0$ in $\mathcal{D}$. For each $(x, n) \in \mathcal{D}$ define

$$I_u(x,n) := \big\{ p \in \mathbb{R}^d : u(x,n) - u(y,m) \geq p \cdot (x - y) \text{ for all } (y,m) \in \mathcal{D} \cup \mathcal{D}^p \text{ with } m > n \big\}.$$

Let also

$$\Gamma = \Gamma(u, \mathcal{D}) := \big\{ (x,n) \in \mathcal{D} : I_u(x,n) \neq \varnothing \big\},$$
$$\Gamma^+ = \Gamma^+(u, \mathcal{D}) := \big\{ (x,n) \in \Gamma : R|p|_2 < u(x,n) - p \cdot x \text{ for some } p \in I_u(x,n) \big\}$$

and

$$\Lambda := \left\{ (\xi, h) \in \mathbb{R}^d \times \mathbb{R} : R|\xi|_2 < h < \frac{M}{2} \right\} \subset \mathbb{R}^{d+1}.$$

For $(x,n)\in\mathcal{D}$ define the set

$$\chi(x,n):=\big\{(p,q-x\cdot p):p\in I_u(x,n),q\in\big[u(x,n+1),u(x,n)\big]\big\}\subset\mathbb{R}^{d+1}.$$

Step 1. We will first show that

$$\Lambda\subset\chi\big(\Gamma^+\big)=\bigcup_{(x,n)\in\Gamma^+}\chi(x,n). \tag{4.1}$$

Indeed, let $(\xi,h)\in\Lambda$, and define for $(x,n)\in\mathcal{D}$,

$$\phi(x,n):=u(x,n)-\xi\cdot x-h.$$

Let $(x_0,n_0)\in\mathcal{D}$ be such that $u(x_0,n_0)=M$. Then, by the definition of Λ, we see that $\phi(x_0,n_0)>0$ and

$$\phi(x,n)<0,$$

for $(x,n)\in\mathcal{D}^p$. We now claim that there exists $(x_1,n_1)\in\Gamma^+$ with $n_1\ge n_0$ such that $\phi(x_1,n_1)\ge 0$ and $(\xi,h)\in\chi(x_1,n_1)$. Indeed, for $x\in B_R$, let

$$N_x:=\max\big\{n:(n,x)\in\mathcal{D}\text{ and }\phi(x,n)\ge 0\big\}$$

and

$$n_1:=\max_{x\in B_R}N_x\ge n_0\ge 0,$$

with the convention $\max\varnothing=-\infty$. Let $x_1\in B_R$ be such that $n_1=N_{x_1}$. Thus, for all $(x,n)\in\mathcal{D}\cup\mathcal{D}^p$ with $n>n_1$,

$$u(x,n)-\xi\cdot x<h\le u(x_1,n_1)-\xi\cdot x_1.$$

Hence $\xi\in I_u(x_1,n_1)\ne\varnothing$ and $h+\xi\cdot x_1\in(u(x_1,n_1+1),u(x_1,n_1)]$, which proves the claim and the statement of display (4.1).

Step 2. We will now show that for each $(x,n)\in\Gamma^+$,

$$\big|I_u(x,n)\big|\le C(\#U)^d\frac{(L_a^*u(x,n))^d}{|\operatorname{conv}(U_{x,n})|}, \tag{4.2}$$

where for every $(x,n)\in\Gamma^+$ and function $h(x,n):\Gamma^+\to\mathbb{R}$ we define

$$L_a^*h(x,n):=\sum_{z\ne 0}a_n(x,z)\big(h(x,n)-h(x+z,n+1)\big).$$

Fix $p\in I_u(x,n)$ and set

$$w(y,m):=u(y,m)-p\cdot y.$$

Then $I_w(x,n)=I_u(x,n)-p$. In particular, $0\in I_w(x,n)$ and we have $w(x,n)-w(x+e,n+1)\ge 0$ for all $e\in U$. Furthermore, if $q\in I_w(x,n)$ and $e\in U$ then

$$w(x,n)-w(x+e,n+1)\ge -e\cdot q.$$

Hence, for each $q\in I_w(x,n)$ and $z\in U_{x,n}$ we have

$$L_a^*u(x,n)=L_a^*w(x,n)=\sum_{e\ne 0}a_n(x,e)\big(w(x,n)-w(x+e,n+1)\big)\ge -z\cdot q. \tag{4.3}$$

Recall the definition of the $U_{x,n}$ in (2.2). Let now $V_{x,n} := \operatorname{conv}(U_{x,n})$ and consider the polar body of $V_{x,n}$, given by $V^o_{x,n} := \{z \in \mathbb{R}^d : z \cdot y \le 1 \text{for all} y \in V_{x,n}\}$. Display (4.3) implies that

$$-I_w(x,n) \subset L_a^* u(x,n) V^o_{x,n}. \tag{4.4}$$

Using the fact that $\sum_{l\in U_{x,n}} l = 0$, note that if $z \in V^o_{x,n}$, then for each $y \in U_{x,n}$,

$$z \cdot (-y) = z \cdot \sum_{l \in U_{x,n} \setminus \{y\}} l \le C(\#U).$$

Hence, setting $\tilde{U}_{x,n} := \{\pm y : y \in U_{x,n}\}$ and $\tilde{V}_{x,n} := \operatorname{conv}(\tilde{U}_{x,n})$ we see that

$$V^o_{x,n} \subset \left\{ z : z \cdot y \le (\#U) \text{ for all } y \in \tilde{U}_{x,n} \right\} = (\#U) \tilde{V}^o_{x,n}. \tag{4.5}$$

Combining (4.4) with (4.5) we conclude that

$$-I_w(x,n) \subset (\#U) L_a^* u(x,n) \tilde{V}^o_{x,n}.$$

Now, since $\tilde{V}^o_{x,n}$ is a symmetric convex body, by Mahler's inequality [23], we see that

$$\left|\tilde{V}^o_{x,n}\right| \le \frac{4^d}{|\tilde{V}_{x,n}|},$$

which finishes the proof of (4.2).

Step 3. Here we derive the maximum inequality from steps 1 and 2. Set

$$\chi\left(\Gamma^+, x\right) := \bigcup_{m:(x,m)\in\Gamma^+} \chi(x,m).$$

For each $x \in \mathcal{D}$, define $\rho_x : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\rho_x(y,m) = (y, m + y \cdot x)$$

and let

$$\tilde{\chi}(x,n) := \rho_x \circ \chi(x,n) = I_u(x,n) \times \left[u(x,n+1), u(x,n)\right] \subset \mathbb{R}^{d+1}.$$

Then, using the inequality $a^{\frac{1}{d+1}} b^{\frac{d}{d+1}} \le \frac{a+db}{d+1}$, valid for $a \ge 0$, $b \ge 0$, and the notation $\sum'$ for the sum running from $n = 1$ to $n = T$ with $u(x,n) - u(x,n+1)$ and $L_a^* u(x,n)$ positive we see that

$$\begin{aligned}
\left|\chi\left(\Gamma^+, x\right)\right| &= \left|\tilde{\chi}\left(\Gamma^+, x\right)\right| \\
&\le \sum{}' \left(u(x,n) - u(x,n+1)\right) \left|I_u(x,n)\right| \mathbb{1}_{(x,n)\in\Gamma^+} \\
&\le (\#U)^d \sum{}' \left(u(x,n) - u(x,n+1)\right) \frac{(L_a^* u(x,n))^d}{v(x,n)} \mathbb{1}_{(x,n)\in\Gamma^+} \\
&\le C(\#U)^d \sum{}' \left(\frac{a_n(x,0)(u(x,n) - u(x,n+1)) + L_a^* u(x,n)}{(d+1)\varepsilon(x,n)} \right)^{d+1} \mathbb{1}_{(x,n)\in\Gamma^+} \\
&= C(\#U)^d \sum{}' \left(\frac{-\mathcal{L}_a u(x,n)}{\varepsilon(x,n)} \right)^{d+1} \mathbb{1}_{(x,n)\in\Gamma^+}.
\end{aligned} \tag{4.6}$$

Now, note that

$$|\Lambda| = C\frac{M^{d+1}}{R^d}.$$

Combining this with inequalities (4.1), (4.6) and using the hypothesis (2.3), we see that

$$C\frac{M^{d+1}}{R^d} \le \sum_{(x,n)\in\mathcal{D}} \frac{1}{\varepsilon^{d+1}}|f|^{d+1}\mathbb{1}_{(x,n)\in\Gamma^+}.$$

Therefore,

$$\max_{(x,n)\in\mathcal{D}} u(x,n) \le C R^{\frac{d}{d+1}} \left\| \frac{f}{\varepsilon} \right\|_{\mathcal{D},d+1}.$$

5. Proof of the continuous time QCLT (Theorem 1.2)

The proof of Theorem 1.2 follows a strategy similar to that of Theorem 1.1. In other words, since the continuous time random walk is also $\mathbb{Q}$-a.s. a martingale, it suffices to construct an invariant measure for the environmental process which is absolutely continuous with respect to the initial law $\mathbb{Q}$ of the environment. However, unlike the discrete time case, the continuous time process is allowed to jump at unbounded rates. To obtain a QCLT, we need not only to deal with the degeneracy of the ellipticity, but also to control the jump rates. This is achieved by first performing a time change to “slow-down” the original RWRE, and then applying a maximum principle for (continuous-time) parabolic difference operators to construct the invariant measure.

Let us state the version of the parabolic maximum principle that we use. Consider a balanced continuous time-dependent environment $\{a_t : t \ge 0\}$, cf. (1.5), with $a_t := \{a_t(x) : x \in \mathbb{Z}^d\}$ and $a_t(x) := \{a_t(x,e) : e \in U\} \in \mathcal{Q}$. Given any finite set $D \subset \mathbb{Z}^d$ and $T > 0$, we define

$$\mathcal{D} := D \times [0, T).$$

Define the *parabolic boundary* of $\mathcal{D}$ by

$$\mathcal{D}^p := \mathcal{D}^\ell \cup \mathcal{D}^T,$$

where $\mathcal{D}^T = D \times \{T\}$ denotes its *time boundary* and

$$\mathcal{D}^\ell := \{(x,t) \notin \mathcal{D} : a_t(y, x-y) > 0 \text{ for some } (y,t) \in \mathcal{D}\}$$

is the *lateral boundary* of $\mathcal{D}$.

For $p > 0$ and any real-valued function g that is summable on $\mathcal{D}$, define

$$\|g\|_{\mathcal{D},p} := \left(\int_0^T \sum_{x\in D} |g(x,t)|^p \,\mathrm{d}t\right)^{1/p}.$$

We can now state the maximum principle.

Theorem 5.1. *Assume that a is a balanced environment. Let u be a function on $\mathcal{D} \cup \mathcal{D}^p$ which is differentiable with respect to t in $(0,T)$. Let f be an integrable function in $\mathcal{D}$. Assume that u satisfies*

$$\mathcal{L}_a u \ge f \quad \text{in } \mathcal{D}.$$

Then, there is a constant $C = C(U,d) > 0$ such that

$$\sup_{\mathcal{D}} u \le \sup_{\mathcal{D}^p} u + C R^{d/(d+1)} \|f/\varepsilon\|_{\mathcal{D},d+1},$$

where $R := \operatorname{diam}(D)$ and ε is as defined in (1.7).

Recall that the space–time process $(X_t, t)_{t\ge 0}$ is a Markov process on $\mathbb{Z}^d \times \mathbb{R}$ with generator $\mathcal{L}_\omega$. To show that $(X_t)_{t\ge 0}$ does not explode, i.e., there are only finitely many jumps within finite time, we will first consider a slowed-down process. Recall the definition of υ_ω in (1.7). Let

$$(Y_t, T_t)_{t\ge 0}$$

be the Markov process on $\mathbb{Z}^d \times \mathbb{R}$ with generator $(\upsilon_\omega + 1)^{-1}\mathcal{L}_\omega$ and initial state $(Y_0, T_0) = (0, 0)$. Note that the process $(Y_\cdot, T_\cdot)$ has slower transition rates on both the $\mathbb{Z}^d$-coordinate and the $\mathbb{R}$-coordinate, compared to (X_t, t). Note also that

$$T_t = \int_0^t \frac{1}{\upsilon_\omega(Y_s, T_s) + 1}\,\mathrm{d}s \tag{5.1}$$

and

$$X_{T_t} \overset{d}{=} Y_t. \tag{5.2}$$

Define the stopping times $\tau_0 = \tau_0(Y_\cdot, T_\cdot) = 0$, and

$$\tau_{j+1} = \tau_{j+1}(Y_\cdot, T_\cdot) = \inf\bigl\{t > \tau_j : |Y_t - Y_{\tau_j}|_\infty > N \text{ or } T_t - \tau_j > N^2\bigr\}.$$

With abuse of notation, we enlarge the probability space and still use $P^c_{0,0,\omega}$ to denote the joint law of $X_\cdot$, the environmental process $\bar{\omega}_\cdot$ and the process $(Y_\cdot, T_\cdot)$ with initial state $(0, 0)$. We let $E^c_{0,0,\omega}$ denote the expectation under $P^c_{0,0,\omega}$. We have the following analogue of Lemma 3.1.

Lemma 5.1. *There exists a constant $c > 0$ such that for all N large and any $\omega \in \Omega$,*

$$E^c_{0,0,\omega}\biggl[\biggl(1 - \frac{c}{N^2}\biggr)^{\tau_1(Y_\cdot, T_\cdot)}\biggr] \le \frac{1}{2}.$$

Proof. The proof follows similar argument as in Lemma 3.1. Recall that $C_U = \max\{|e| : e \in U\}$. Note that $(Y_t)_{t\ge 0}$ is a martingale and $(|Y_t|^2 - C_U t)_{t\ge 0}$ is a super-martingale. Let $K = \frac{N^2}{2C_U}$. Then, by Doob's L^2-martingale inequality,

$$\begin{aligned} P^c_{0,0,\omega}(\tau_1 \le K) &= P^c_{0,0,\omega}\Bigl(\max_{0<t\le K} |Y_t| > N\Bigr) \\ &\le \frac{1}{N^2} E^c_{0,0,\omega}\bigl[|Y_K|^2\bigr] \le \frac{C_U K}{N^2} = \frac{1}{2}, \end{aligned}$$

where in the first equality we used the fact that $T_K \le K < N^2$. □

Theorem 5.2. *Assume the same conditions as in Theorem 1.2. Then the environmental process $(\theta_{T_t, Y_t}\omega)_{t\ge 0}$ has a unique invariant probability measure $\bar{\nu}$ which is equivalent to $\mathbb{Q}$.*

Proof. Let $Q_N = \{z \in \mathbb{Z}^d : |z|_\infty \le N\} \times [0, N^2)$. We introduce on $\mathbb{Z}^d$ the equivalent classes

$$\overline{(x,t)} := (x, t) + (2N + 1)\mathbb{Z}^d \times N^2\mathbb{Z}.$$

Fix a balanced environment $\omega \in \mathcal{Q}$, and define its periodized environment $\omega^{(N)}$ so that for any $(x, t) \in Q_N$,

$$\omega^{(N)}_s(y) = \omega_t(x)$$

whenever $\overline{(y,s)} = \overline{(x,t)}$.

Set

$$\Omega_{N,\omega} = \Omega_N := \bigl\{\theta_{t,x}\omega^{(N)} : (x, t) \in Q_N\bigr\}$$

and let $\mathbb{P}_N = \mathbb{P}_{N,\omega}$ denote the probability measure

$$\mathbb{P}_N(\mathrm{d}\xi) = \frac{1}{N^{d+2}} \sum_{x:(x,t)\in Q_N} \mathbb{1}_{\theta_{t,x}\omega^{(N)}=\xi}\,\mathrm{d}t.$$

Under the environment $\omega^{(N)}$, recall that Y_t is the slowed-down process. Since under $P^c_{0,0,\omega^{(N)}}$, the process $\overline{(Y_t, T_t)}_{t\ge 0}$ is a Markov process on the compact set Q_N, it has an invariant distribution whose density we denote by $\phi_N(x,t)\delta_x\,\mathrm{d}t$, with $(x,t)\in Q_N$ and δ_x denotes the Dirac mass. As in the proof of Theorem 1.1, the probability measure $\nu_N \ll \mathbb{P}_N$ defined by

$$\nu_N(\mathrm{d}\xi) = \sum_{x:(x,t)\in Q_N} \phi_N(x,t)\mathbb{1}_{\theta_{t,x}\omega^{(N)}=\xi}\,\mathrm{d}t,$$

is an invariant distribution of the Markov process $(\theta_{T_t,Y_t}\omega^{(N)})_{t\ge 0}$.

For $\xi \in \Omega_N$, let $\xi_t := \theta_{T_t,Y_t}\xi$ denote the environmental process. By similar arguments as in Section 3, Lemma 5.1 implies that for any bounded continuous function g on $\Omega^{\mathbb{N}}$,

$$\int g\,\mathrm{d}\nu_N \le CN^{-2}\max_{\xi\in\Omega_N} E^c_{0,0,\xi}\left[\int_0^{\tau_1} g(\xi_t)\,\mathrm{d}t\right].$$

Letting

$$u(x,t) = E^c_{0,0,\theta_{t,x}\xi}\left[\int_0^{\tau_1} g(\xi_s)\,\mathrm{d}s\right],$$

we have

$$\begin{cases} (\upsilon_\omega+1)^{-1}\mathcal{L}_\xi u = -g(\theta_{t,x}\xi) & \text{in } Q_N,\\ u=0 & \text{in } Q_N^p. \end{cases}$$

Then, applying Theorem 5.1 to the operator $\mathcal{L}_\xi$, we get

$$\max_{Q_N} u \le CN^2\big\|(\upsilon+1)g/\varepsilon\big\|_{L^{d+1}(\mathbb{P}_N)}$$

and so

$$\int g\,\mathrm{d}\nu_N \le C\big\|(\upsilon+1)g/\varepsilon\big\|_{L^{d+1}(\mathbb{P}_N)}.$$

Since, $\lim_{N\to\infty}\mathbb{P}_{N,\omega} = \mathbb{Q}$, $\mathbb{Q}$-a.s. and

$$E_{\mathbb{Q}}\big[(\upsilon+1)^{d+1}/\varepsilon_\omega^{d+1}\big] \le 2^d E_{\mathbb{Q}}\big[\big(\upsilon^{d+1}+1\big)/\varepsilon_\omega^{d+1}\big] < \infty,$$

using the ergodic theorem and Kozlov's argument, the conclusion follows. □

Corollary 5.1. *Assume the same conditions as in Theorem 1.2. For $\mathbb{Q}$-almost all ω, $P^c_{0,0,\omega}$-almost surely the process $(X_t)_{t\ge 0}$ does not explode. Moreover, the environmental process $(\theta_{t,X_t}\omega)_{t\ge 0}$ has a unique invariant probability measure ν which is equivalent to $\mathbb{Q}$.*

Proof. Set $\bar{\omega}_t := \theta_{t,X_t}\omega$. By (5.1), Theorem 5.2 and the ergodic theorem,

$$\lim_{t\to\infty}\frac{T_t}{t} = E_{\bar{\nu}}\left[\frac{1}{\upsilon+1}\right] \in (0,1) \quad \mathbb{Q}\otimes P_{0,0,\omega}\text{-a.s.}$$

Hence, by (5.2), $(X_t)_{t\geq 0}$ is not explosive. Furthermore, let

$$\mathrm{d}\nu := \frac{N}{\upsilon+1}\,\mathrm{d}\bar{\nu},$$

where $N = (E_{\bar{\nu}}[\frac{1}{\upsilon+1}])^{-1}$ is a normalization constant. Then ν is the invariant measure of $(\bar{\omega}_t)_{t\geq 0}$ and it is equivalent to $\mathbb{Q}$. □

Theorem 1.2(i) is proved in the above corollary. As in Theorem 1.1, this implies the invariance principle Theorem 1.2(ii).

6. Proof of Corollary 1.2

Recall the definitions of $x^{(1)}, x^{(2)}$ before Corollary 1.2. We set

$$Y_n := \big(X_n^{(1)}, 0\big) \in \mathbb{Z}^{d_1+d_2} \quad \text{and} \quad Z_n := \big(0, X_n^{(2)}\big) \in \mathbb{Z}^{d_1+d_2},$$

so that $X_n = Y_n + Z_n$. For each $n \geq 0$, denote by $\mathcal{F}^Y$ the σ-algebra generated by $\{Y_0, Y_1, \ldots\}$. Furthermore, we define a time-dependent environment ω^Y on $\mathcal{P}^{\mathbb{Z}^{d_2}}$ by

$$\omega_n^Y(z,e) := \frac{\omega(Y_n + (0,z), Y_{n+1} - Y_n + (0,e))}{\omega(Y_n, Y_{n+1} - Y_n)} \quad \text{for } z, e \in \mathbb{Z}^{d_2}, n \in \mathbb{N}.$$

Lemma 6.1. *$\mathbb{P} \times P_{0,\omega}$-a.s. under the law $P_{0,\omega}(\cdot|\mathcal{F}^Y)$, $\{X_n^{(2)} : n \geq 0\}$ is a random walk on the lattice $\mathbb{Z}^{d_2}$ in the time dependent environment ω^Y.*

Proof. For $0 \leq m < n-1$ and any two sequences $z_1, \ldots, z_{m+1} \in \mathbb{Z}^{d_2}$ and $y_1, \ldots, y_n \in \mathbb{Z}^{d_1}$, we let $x_i = (y_i, z_i)$ for $1 \leq i \leq m+1$. By the Markov property,

$$\begin{aligned}
&P_{0,\omega}\big(X_{m+1}^{(2)} = z_{m+1} | X_1^{(2)} = z_1, \ldots, X_m^{(2)} = z_m, X_1^{(1)} = y_1, \ldots, X_n^{(1)} = y_n\big)\\
&\quad = \frac{P_{0,\omega}(X_1 = x_1, \ldots, X_{m+1} = x_{m+1}, X_{m+2}^{(1)} = y_{m+2}, \ldots, X_n^{(1)} = y_n)}{P_{0,\omega}(X_1 = x_1, \ldots, X_m = x_m, X_{m+1}^{(1)} = y_{m+1}, \ldots, X_n^{(1)} = y_n)}\\
&\quad = \frac{P_{0,\omega}(X_{m+2}^{(1)} = y_{m+2}, \ldots, X_n^{(1)} = y_n | X_{m+1} = x_{m+1}) P_{0,\omega}(X_{m+1} = x_{m+1} | X_m = x_m)}{P_{0,\omega}(X_{m+1}^{(1)} = y_{m+1}, \ldots, X_n^{(1)} = y_n | X_m = x_m)}\\
&\quad = \frac{\omega(x_m, x_{m+1} - x_m)}{P_{x_m,\omega}(X_1^{(1)} = y_{m+1})},
\end{aligned}$$

where in the last equality we used condition (a) which says that $X_n^{(1)}$ is a Markov chain. We finish the proof by observing that

$$\frac{\omega(x_m, x_{m+1} - x_m)}{P_{x_m,\omega}(X_1^{(1)} = y_{m+1})} = \omega_m^Y(z_m, z_{m+1} - z_m)|_{Y_i = y_i, i=1,\ldots,n}.$$

□

Proof of Corollary 1.2. Let $\Omega = \mathcal{P}^{\mathbb{Z}^{d_1+d_2}}$. We will only consider the non-trivial case where $\mathbb{P}$-almost surely, $\theta_{e_i}\omega \neq \omega$ for all $i = 1, \ldots, d_1$. (Otherwise, since $\{\omega : \theta_{e_i}\omega = \omega\}$ is shift-invariant under all shifts $\{\theta_x : x \in \mathbb{Z}^{d_1+d_2}\}$, it follows by ergodicity that $\mathbb{P}(\theta_{e_i}\omega = \omega) = 1$ for some $i \in \{1, \ldots, d_1\}$, which then implies that every measurable set $A \subset \Omega$ is shift-invariant under the shift θ_{e_i}. By our ergodicity assumption, we conclude that $\mathbb{P}$ is a singleton, i.e., $\mathbb{P}(\omega = \xi) = 1$ for some $\xi \in \Omega$. In this case, the RWRE is a simple random walk and the QCLT is trivial.)

For any $z \in \mathbb{Z}^{d_2}$, we denote by $\hat{z} := (0, z) \in \mathbb{Z}^{d_1+d_2}$ so that $\hat{z}^{(2)} = z$. Our proof contains several steps.

Step 1. Set $\tilde{\omega}_n := \theta_{Y_n}\omega$. By the ergodic theorem, the measure ν that satisfies the properties in condition (b) of Corollary 1.2 is unique. Let us denote by Q_ν the law of $(\tilde{\omega}_n)_{n\geq 0}$ starting from ν.

Step 2. We will show that the law Q_ν of the space–time environment $\{\tilde{\omega}_n(x) : (x,n) \in \mathbb{Z}^{d_1+d_2} \times \mathbb{N}\}$ is translation invariant under the spatial shifts $\{\theta_{0,\hat{z}} : z \in \mathbb{Z}^{d_2}\}$. To this end, it is enough to prove that ν, as a measure on the static environments Ω, is translation invariant under these spatial shifts. Indeed, since by condition (a), the law of $\{X_n^{(1)} : n \in \mathbb{N}\}$ under $P_{x,\omega}$ depends only on the first d_1 coordinates of x, we conclude that for any $z \in \mathbb{Z}^{d_2}$, the law $\hat{\nu}_z$ defined by $\hat{\nu}_z(A) := \nu(\theta_{\hat{z}}A)$ is still an invariant measure for the Markov chain $(\theta_{Y_n}\omega)_{n\in\mathbb{N}}$. Furthermore, by the stationarity of $\mathbb{P}$ under the spatial shifts, $\hat{\nu}_z$ is also equivalent to $\mathbb{P}$. Therefore by the uniqueness of ν, we have $\hat{\nu}_z = \nu$ for any $z \in \mathbb{Z}^{d_2}$ and so Q_ν is translation invariant under $\{\theta_{0,\hat{z}} : z \in \mathbb{Z}^{d_2}\}$.

Step 3. Next, we claim that $\mathbb{P} \times P_{0,\omega}$-almost surely, ω_n^Y can be written as a function of $\tilde{\omega}_n$ and $\tilde{\omega}_{n+1}$, $n \geq 0$. Indeed, for any $n \geq 0$,

$$\begin{aligned} Q_\nu(\tilde{\omega}_n = \tilde{\omega}_{n+1}, Y_n - Y_{n+1} \neq 0) &\leq Q_\nu\big(\theta_{e_i}\tilde{\omega}_n = \tilde{\omega}_n \text{ for some } i \in \{1,\dots,d_1\}\big) \\ &= \nu\big(\theta_{e_i}\omega = \omega \text{ for some } i \in \{1,\dots,d_1\}\big) \\ &= 0, \end{aligned}$$

where in the first equality we used that $(\tilde{\omega}_n)_{n\geq 0}$ is a stationary sequence under Q_ν and in the last equality we used $\nu \approx \mathbb{P}$ and the assumption at the beginning of the proof. Hence, the events $\{Y_n = Y_{n+1}\}$ and $\{\tilde{\omega}_n = \tilde{\omega}_{n+1}\}$ are equivalent. In particular, we write for $z, e \in \mathbb{Z}^{d_2}$,

$$\omega_n^Y(z,e) = \frac{\tilde{\omega}_n(\hat{z},\hat{e})}{\tilde{\omega}_n(0,0)} \mathbb{1}_{\tilde{\omega}_n = \tilde{\omega}_{n+1}} + \mathbb{1}_{e=0, \tilde{\omega}_n \neq \tilde{\omega}_{n+1}}. \tag{6.1}$$

Step 4. Notice that ω^Y is not an elliptic environment. However, we will show that under a time change, $X_n^{(2)}$ is a random walk in an ergodic uniformly-elliptic random environment (conditioning on $Y_\cdot$). Indeed, let $D = \{\xi = (\xi_i)_{i\geq 0} \in \Omega^{\mathbb{N}} : \xi_0 = \xi_1\}$ and

$$\phi_D(\tilde{\omega}) = \inf\{n \geq 0 : \theta_{n,0}\tilde{\omega} \in D\} = \inf\{n \geq 0 : \tilde{\omega}_n = \tilde{\omega}_{n+1}\}.$$

Since $Q_\nu(\tilde{\omega} \in D) = E_{Q_\nu}[P_{0,\omega}(Y_1 = 0)] > 0$ and the law Q_ν of $\tilde{\omega}$ is ergodic under the time shift $\theta_{1,0}$, by ergodicity, $\phi_D < \infty$ almost surely. Moreover, defining the induced shift $T_D : \Omega^{\mathbb{N}} \to \Omega^{\mathbb{N}}$ as

$$T_D\tilde{\omega} := \theta_{\phi_D(\tilde{\omega}),0}\tilde{\omega},$$

then $(T_D^k\tilde{\omega})_{k\geq 1}$ (under law Q_ν) is still an ergodic sequence, cf. [27, Theorem 1.6]. Recall that by (6.1), ω_n^Y is a function of $\tilde{\omega}_n$ and $\tilde{\omega}_{n+1}$. Hence, by ergodicity of the sequence $(\tilde{\omega}_n)_{n\geq 0}$, the time-dependent environment $\zeta^Y \in \mathcal{P}^{\mathbb{Z}^{d_2}\times\mathbb{N}}$ defined by

$$\zeta_n^Y(x,e) := \big(T_D^{n+1}\omega^Y\big)_0(x,e), \quad x \in \mathbb{Z}^{d_2}, n \geq 0$$

is ergodic under time-shifts and stationary (by Step 2) under the spatial shifts. We thus conclude that ζ^Y is a uniformly-elliptic balanced time-dependent random environment which (conditioning on $Y_\cdot$ and under Q_ν) is ergodic with respect to the space–time shifts $\{\theta_{1,z} : |z| \leq 1, z \in \mathbb{Z}^{d_2}\}$. Furthermore, define recursively random times $\phi_0 := \phi_D$ and

$$\phi_{i+1}(\tilde{\omega}) := \inf\{n > \phi_i : \tilde{\omega}_n = \tilde{\omega}_{n+1}\}.$$

Then, conditioning on $Y_\cdot$,

$$W_n' := X_{\phi_n(\tilde{\omega})}^{(2)}, \quad n \geq 0 \tag{6.2}$$

is a random walk in the time-dependent environment ζ^Y defined above.

Step 5. Now, we will prove a QCLT for $X_n^{(2)}$. First, since for $\tilde{\omega}$ sampled according to Q_ν, the sequence W_n' is a random walk in a uniformly elliptic and ergodic (with respect to $\{\theta_{1,z} : |z| \le 1, z \in \mathbb{Z}^{d_2}\}$) balanced environment with jump range $\{z \in \mathbb{Z}^{d_2} : |z| \le 1\}$, by Theorem 1.1, we obtain for W_n' a QCLT with non-degenerate $d_2 \times d_2$ covariance matrix. Then, noticing that by Kac's formula, for Q_ν-almost every $\tilde{\omega}$,

$$\lim_{n\to\infty} \frac{\phi_n(\tilde{\omega})}{n} = \frac{1}{Q_\nu(\tilde{\omega} \in D)} = \frac{1}{E_{Q_\nu}[P_{0,\omega}(Y_1 = 0)]} < \infty,$$

with a standard time-change argument, we conclude a QCLT for $X_n^{(2)}$ (conditioning on $Y_\cdot$). That is, $\mathbb{P}$-a.s., for almost all trajectories $\{Y_0, Y_1, \dots\}$ and any open $B \in C([0,\infty); \mathbb{Z}^{d_2})$,

$$\lim_{n\to\infty} P_{0,\omega}\left(\frac{X_{[n\cdot]}^{(2)}}{\sqrt{n}} \in B \,\middle|\, \mathcal{F}^Y\right) = Q(B), \tag{6.3}$$

where Q denotes the law of a Brownian motion on $\mathbb{R}^{d_2}$ with a deterministic non-degenerate covariance matrix.

Step 6. With condition (c), we can now conclude that for any pair of open sets $A \in C([0,\infty); \mathbb{Z}^{d_1})$ and $B \in C([0,\infty); \mathbb{Z}^{d_2})$,

$$P_{0,\omega}\left(\frac{X_{[n\cdot]}^{(1)} - v_1 n\cdot}{\sqrt{n}} \in A, \frac{X_{[n\cdot]}^{(2)}}{\sqrt{n}} \in B\right) = E_{0,\omega}\left[\mathbb{1}_{X_{[n\cdot]}^{(1)} - v_1 n\cdot/\sqrt{n} \in A} P_{0,\omega}\left(\frac{X_{[n\cdot]}^{(2)}}{\sqrt{n}} \in B \,\middle|\, \mathcal{F}^Y\right)\right],$$

which by (6.3) converges to the probability that a Brownian motion (with a deterministic non-degenerate covariance matrix) in $\mathbb{R}^{d_1+d_2}$ belongs to $A \times B$. Using the fact that any open set in $C([0,\infty) : \mathbb{Z}^{d_1+d_2})$ is a countable union of sets of the form $A \times B$, we conclude the proof. $\square$

7. An example that QCLT fails when the environment is not ergodic enough

Here we show that the QCLT could fail if the ergodicity hypothesis of Theorem 1.1 is weakened.

Consider a discrete time-dependent balanced random environment on $\mathbb{Z}^2$. Let $p = \{p(e) : |e|_1 = 1\}, q = \{q(e) : |e|_1 = 1\}$ be two probability vectors such that $p(e) = \frac{1}{4}$ for all $e \in \mathbb{Z}^2$ with $|e|_1 = 1$, and $q(\pm e_1) = \frac{1}{6}, q(\pm e_2) = \frac{1}{3}$. For any $(x,n) \in \mathbb{Z}^2 \times \mathbb{Z}$, define two space–time environments ξ, ξ' such that

$$\xi_n(x,e) = \begin{cases} p(e) & \text{if } |x|_1 + n \text{ is even,} \\ q(e) & \text{if } |x|_1 + n \text{ is odd,} \end{cases}$$

$$\xi_n'(x,e) = \begin{cases} p(e) & \text{if } |x|_1 + n \text{ is odd,} \\ q(e) & \text{if } |x|_1 + n \text{ is even.} \end{cases}$$

Define the environment measure $\mathbb{P}$ to be

$$\mathbb{P}(\omega = \xi) = \mathbb{P}(\omega = \xi') = \frac{1}{2}.$$

Noting that $\theta_{1,0}\xi = \xi'$ and the jump range $U = \{e \in \mathbb{Z}^d : |e| = 1\}$. The measure $\mathbb{P}$ is ergodic under the shifts $\{\theta_{1,e} : |e| \le 1\}$. However, $\mathbb{P}$ is not ergodic under the set of shifts $\{\theta_{1,e} : e \in U\}$ and the QCLT fails, since $X_{n\cdot}/\sqrt{n}$ converges to a Brownian motion with a random covariance matrix

$$\Sigma(\omega) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \mathbb{1}_{\omega=\xi} + \begin{pmatrix} 1/3 & 0 \\ 0 & 2/3 \end{pmatrix} \mathbb{1}_{\omega=\xi'}.$$

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