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Mean-field interaction of Brownian occupation measures, I: Uniform tube property of the Coulomb functional

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Abstract. We study the transformed path measure arising from the self-interaction of a three-dimensional rownian motion via an exponential tilt with the Coulomb energy of the occupation measures of the motion by time t. The logarithmic asymptotics of the partition function were identified in the 1980s by Donsker and Varadhan (*Comm. Pure Appl. Math.* **505** (1983) 505–528) in terms of a variational formula. Recently (Brownian occupations measures, compactness and large deviations (2014) Preprint) a new technique for studying the path measure itself was introduced, which allows for proving that the normalized occupation measure asymptotically concentrates around the set of all maximizers of the formula. In the present paper, we show that likewise the Coulomb functional of the occupation measure concentrates around the set of corresponding Coulomb functionals of the maximizers in the uniform topology. This is a decisive step on the way to a rigorous proof of the convergence of the normalized occupation measures towards an explicit mixture of the maximizers, derived in (Mean-field interaction of Brownian occupation measures, II: A rigorous construction of the Pekar process. Preprint). Our methods rely on deriving Hölder-continuity of the Coulomb functional of the occupation measure with exponentially small deviation probabilities and invoking the large deviation theory developed in (Brownian occupations measures, compactness and large deviations (2014) Preprint) to a certain shift-invariant functional of the occupation measures.

Résumé. Nous étudions la mesure de trajectoire transformée engendrée par l'auto-interaction d'un mouvement Brownien tridimensionnel en utilisant un biais exponentiel par l'énergie de Coulomb des mesures d'occupation de ce mouvement au temps t. Les asymptotes logarithmiques de la fonction de partition ont été identifiées dans les années 1980 par Donsker et Varadhan [*Comm. Pure Appl. Math.* **505** (1983) 505–528] au moyen d'une formule variationnelle. Récemment, dans (Brownian occupations measures, compactness and large deviations (2014) Preprint), une nouvelle technique pour étudier la mesure de chemins elle-même a été introduite. Elle permet de prouver que la mesure d'occupation normalisée se concentre asymptotiquement autour de l'ensemble des maximums de la formule.

Dans le présent article, nous prouvons que la fonctionnelle de Coulomb de la mesure d'occupation se concentre elle aussi, dans la topologie uniforme, autour de l'ensemble des fonctionnelles de Coulomb correspondant aux maximums. Ceci représente une étape décisive vers une preuve rigoureuse de la convergence des mesures d'occupation normalisées vers un mélange explicite des maximums, dérivée dans (Mean-field interaction of Brownian occupation measures, II: A rigorous construction of the Pekar process. Preprint). Nos méthodes reposent sur l'obtention de la continuité hölderienne de la fonctionnelle de Coulomb de la mesure d'occupation avec des probabilités de déviations exponentiellement petites, en invoquant la théorie des grandes déviations développée dans (Brownian occupations measures, compactness and large deviations (2014) Preprint) pour une certaine fonctionnelle, invariante par dcalage, des mesures d'occupation.

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1. Introduction and main results

In this paper, we study a transformed path measure that arises from a mean-field type interaction of a three dimensional Brownian motion in a Coulomb potential. Under the influence of such a transformed measure, the large-t behavior of the normalized occupation measures, denoted by L_t, is of high interest. This is intimately connected to the well-known polaron problem from statistical mechanics and a full understanding of the behavior of L_t under the aforementioned transformation is crucial for the analysis of the polaron path measure under "strong coupling", its effective mass and justification of mean-field approximations. For physical relevance of this model, we refer to [9]. Some mathematically rigorous research in this direction began in the 1980s with the analysis of the partition function of Donsker and Varadhan [5], but it was not until recently that a new technique was developed [7] for handling the actual path measures, and the main results of the present paper, besides being interesting on their own, make determinant contribution towards a deeper analysis and a full identification of the limiting distribution of L_t under the transformed path measure.

We start with developing the mathematical layout of the model in Section 1.1, remind on earlier results in Section 1.2, present our new progress in Section 1.3 and report on the achievements of [7] in Section 1.4, which plays an important role in the present context.

1.1. *The transformed path measure*

We start with the Wiener measure $\mathbb{P}$ on $\Omega = C([0,\infty),\mathbb{R}^3)$ corresponding to a 3-dimensional Brownian motion $W=(W_t)_{t\geq 0}$ starting from the origin. We are interested in the transformed path measure

$$\widehat{\mathbb{P}}_t(\mathrm{d}\omega) = \frac{1}{Z_t}\exp\left\{\frac{1}{t}\int_0^t\int_0^t \mathrm{d}\sigma\,\mathrm{d}s\,\frac{1}{|\omega_\sigma-\omega_s|}\right\}\mathbb{P}(\mathrm{d}\omega), \quad \omega\in\Omega, \tag{1.1}$$

with the normalizing constant, the *partition function*,

$$Z_t = \mathbb{E}\left[\exp\left\{\frac{1}{t}\int_0^t\int_0^t \mathrm{d}\sigma\,\mathrm{d}s\,\frac{1}{|W_\sigma-W_s|}\right\}\right]. \tag{1.2}$$

We remark that the asymptotic behavior of $\widehat{\mathbb{P}}_t$ is determined by those influential paths which make $|W_\sigma - W_s|$ small, i.e., the interaction is *self-attractive*. We also remark that the factor $\frac{1}{t}$ in the exponent in (1.1) makes the model interesting. Indeed, the double-integral in the exponent is of order t^2 for paths that stay in a compact region, and the entropic cost for this behaviour is $\mathrm{e}^{-O(t)}$; it relatively easy to suspect that such a behaviour is typical under the transformed measure. Hence, it is the factor $\frac{1}{t}$ that makes the energy and the entropy terms run on the same scale and still gives the path enough freedom to fluctuate.

Let

$$L_t = \frac{1}{t}\int_0^t \mathrm{d}s\,\delta_{W_s} \tag{1.3}$$

be the normalized occupation measure of W until time t. This is a random element of $\mathcal{M}_1(\mathbb{R}^3)$, the space of probability measures on $\mathbb{R}^3$. Then the path measure $\widehat{\mathbb{P}}_t$ can be written as

$$\widehat{\mathbb{P}}_t(A) = \frac{1}{Z_t}\mathbb{E}\big[\mathbb{1}_A\exp\{tH(L_t)\}\big], \quad A\subset\Omega,$$

where

$$H(\mu) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\mu(\mathrm{d}x)\mu(\mathrm{d}y)}{|x-y|}, \quad \mu\in\mathcal{M}_1(\mathbb{R}^3), \tag{1.4}$$

denotes the *Coulomb potential energy functional* of μ. Hence, $\widehat{\mathbb{P}}_t$ is an exponential tilt of the Coulomb energy function of L_t with parameter t. It is the goal of this paper to make a contribution to a rigorous understanding of the behavior of L_t under $\widehat{\mathbb{P}}_t$.

For any $\mu \in \mathcal{M}_1(\mathbb{R}^3)$, we define the function

$$(\Lambda\mu)(x) = \left(\mu \star \frac{1}{|\cdot|}\right)(x) = \int_{\mathbb{R}^3} \frac{\mu(\mathrm{d}y)}{|x-y|},$$

which is also sometimes called its *Coulomb potential energy functional*. In order to avoid misunderstandings, we will call $H(\mu)$ the *Coulomb energy* and $\Lambda(\mu)$ the *Coulomb functional* of μ. Note that $H(\mu) = \langle \mu, \Lambda\mu\rangle = \int (\Lambda\mu)(x)\mu(\mathrm{d}x)$. We remark that the Coulomb functional of the Brownian occupation measure,

$$\Lambda_t(x) = (\Lambda L_t)(x) = \int_{\mathbb{R}^3} \frac{L_t(\mathrm{d}y)}{|x-y|} = \frac{1}{t}\int_0^t \frac{\mathrm{d}s}{|W_s - x|}, \tag{1.5}$$

is almost surely finite in $\mathbb{R}^3$.

1.2. *Existing results*

Donsker and Varadhan [5] studied the asymptotic behavior of Z_t resulting in the variational formula

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t}\log Z_t &= \sup_{\mu\in\mathcal{M}_1(\mathbb{R}^3)} \big\{H(\mu) - I(\mu)\big\} \\ &= \sup_{\substack{\psi\in H^1(\mathbb{R}^3)\\ \|\psi\|_2=1}} \left\{\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathrm{d}x\,\mathrm{d}y \frac{\psi^2(x)\psi^2(y)}{|x-y|} - \frac{1}{2}\|\nabla\psi\|_2^2\right\} = \rho, \end{aligned} \tag{1.6}$$

with $H^1(\mathbb{R}^3)$ denoting the usual Sobolev space of square integrable functions with square integrable gradient. Furthermore, we put

$$I(\mu) = \frac{1}{2}\|\nabla\psi\|_2^2 \tag{1.7}$$

if μ has a density ψ^2 with $\psi \in H^1(\mathbb{R}^3)$, and $I(\mu) = \infty$ otherwise. Note that both H and I are shift-invariant functionals, i.e., $H(\mu) = H(\mu \star \delta_x)$ and $I(\mu) = I(\mu \star \delta_x)$ for any $x \in \mathbb{R}^3$.

The above result is a consequence of a *large deviation principle* (LDP) for L_t under $\mathbb{P}$ in $\mathcal{M}_1(\mathbb{R}^3)$, developed by Donsker and Varadhan [4]. This means, when $\mathcal{M}_1(\mathbb{R}^3)$ is equipped with the usual weak topology, for every open set $G \subset \mathcal{M}_1(\mathbb{R}^3)$,

$$\liminf_{t\to\infty} \frac{1}{t}\log \mathbb{P}(L_t \in G) \ge -\inf_{\mu\in G} I(\mu), \tag{1.8}$$

and for any compact set $K \subset \mathcal{M}_1(\mathbb{R}^3)$,

$$\limsup_{t\to\infty} \frac{1}{t}\log \mathbb{P}(L_t \in K) \le -\inf_{\mu\in K} I(\mu). \tag{1.9}$$

The above statement is also called a *weak large deviation principle* since the upper bound (1.9) holds only for compact subsets. We say that a family of probability distributions satisfies a *strong large deviation principle* if, along with the lower bound (1.8), the upper bound (1.9) holds also for all closed sets.

The variational formula (1.6) has been analyzed by Lieb [6]. It turns out that there is a smooth, rotationally symmetric and centered maximizer ψ_0 which is unique except for spatial translations. In other words, if $\mathfrak{m}$ denotes the set of maximizing densities, then

$$\mathfrak{m} = \big\{\mu_0 \star \delta_x : x \in \mathbb{R}^3\big\}, \tag{1.10}$$

where μ_0 is a probability measure with a density ψ_0^2 so that ψ_0 maximizes the variational problem (1.6). We will often write $\mu_x = \mu_0 \star \delta_x$ and write ψ_x^2 for its density.

Given (1.6) and (1.10), we expect the distribution of L_t under the transformed measure $\widehat{\mathbb{P}}_t$ to concentrate around $\mathfrak{m}$ and, even more, to converge towards a mixture of spatial shifts of μ_0. Such a precise analysis was carried out by Bolthausen and Schmock [1] for a spatially discrete version of $\widehat{\mathbb{P}}_t$, i.e., for the continuous-time simple random walk on $\mathbb{Z}^d$ instead of Brownian motion and an interaction potential $v\colon \mathbb{Z}^d \to [0,\infty)$ with finite support instead of the singular Coulomb potential $x \mapsto 1/|x|$. A first key step in [1] was to show that, under the transformed measure, the probability of the local times falling outside any neighborhood of the maximizers decays exponentially. For its proof, the lack of a strong LDP for the local times was handled by an extended version of a standard periodization procedure by folding the random walk into some large torus. Combined with this, an explicit tightness property of the distributions of the local times led to an identification of the limiting distribution.

However, in the context of the continuous setting with a singular Coulomb interaction, the aforementioned periodization technique or any standard compactification procedure does not work well to circumvent the lack of a strong LDP. An investigation of $\widehat{\mathbb{P}}_t \circ L_t^{-1}$, the distribution of L_t under $\widehat{\mathbb{P}}_t$, remained open until a recent result [7] rigorously justified the above heuristics, leading to the statement

$$\limsup_{t\to\infty} \frac{1}{t} \log \widehat{\mathbb{P}}_t\big\{L_t \notin U(\mathfrak{m})\big\} < 0, \tag{1.11}$$

where $U(\mathfrak{m})$ is any neighborhood of $\mathfrak{m}$ in the weak topology induced by the Prohorov metric, the metric that is induced by all the integrals against continuous bounded test functions. Hence, (1.11) implies that the distribution of L_t under $\widehat{\mathbb{P}}_t$ is asymptotically concentrated around $\mathfrak{m}$. Since a one-dimensional picture of $\mathfrak{m}$ is an infinite line, its neighborhood resembles an infinite tube. Therefore, assertions similar to (1.11) are sometimes called a *tube property*.

It is worth pointing out that, although (1.11) requires only the weak topology in the statement, its proof is crucially based on a robust theory of a *compactification* $\widetilde{\mathcal{X}}$ of the quotient space

$$\widetilde{\mathcal{M}}_1\big(\mathbb{R}^d\big) \hookrightarrow \widetilde{\mathcal{X}}$$

of orbits $\widetilde{\mu} = \{\mu \star \delta_x \colon x \in \mathbb{R}^d\}$ of probability measures μ on $\mathbb{R}^d$ (for any $d \geq 1$) under translations and a full LDP for the distributions of $\widetilde{L}_t \in \widetilde{\mathcal{M}}_1(\mathbb{R}^d)$ embedded in the compactification. In particular, this is based on a topology induced by a different metric in the compactification $\widetilde{\mathcal{X}}$, see Section 1.4 for details and its consequences in the present context.

1.3. *Our results*: *Uniform tube property and regularity of* $\Lambda(L_t)$

Let us turn to our main results. We write

$$\Lambda\big(\psi^2\big)(x) = \int \mathrm{d}y \frac{\psi^2(y)}{|x-y|}$$

for functions ψ^2, and recall that $\psi_w^2 = \psi_0^2 \star \delta_w$ denotes the shift of the maximizer ψ_0^2 of the second variational formula (1.6) by $w \in \mathbb{R}^3$. Roughly speaking, we will establish that on the large deviations scale, the Coulomb functionals $\Lambda(L_t)$ under the transformed path measures $\widehat{\mathbb{P}}_t$ stay close to the manifold of Coulomb functionals $\Lambda\mathfrak{m} = \{\Lambda\psi_w^2 \colon w \in \mathbb{R}^3\}$ acting on the translations of the Pekar maximizers, see Theorem 1.1. This is a first determinant step towards establishing the full conjecture on the convergence of the distributions $\widehat{\mathbb{P}}_t \circ L_t^{-1}$ towards an explicit spatial mixture of the maximizers $\mathfrak{m}$, see Remark 1. On the way towards proving Theorem 1.1, we also derive some modulus of continuity of $\Lambda(L_t)$, which can be of independent interest in the realm of regularity properties of local times for stochastic processes.

Here is the statement of our first main result.

Theorem 1.1. *For any* $\varepsilon > 0$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \widehat{\mathbb{P}}_t\Big\{\inf_{w\in\mathbb{R}^3} \big\|\Lambda_t - \Lambda\psi_w^2\big\|_\infty > \varepsilon\Big\} < 0. \tag{1.12}$$

This is a tube property for Λ_t in the uniform metric, since the ε-neighbourhood of $\Lambda(\mathfrak{m}) = \{\Lambda(\psi_w^2)\colon w \in \mathbb{R}^3\}$ can be visualized as a tube around the "line" $\mathfrak{m}$. The proof of Theorem 1.1 is given in Section 3.

As a consequence of Theorem 1.1, the Hamiltonian $H(L_t) = \langle L_t, \Lambda L_t\rangle$ converges in distribution towards the common Coulomb energy of any member of $\mathfrak{m}$ and we state this fact as

Corollary 1.2. *Under $\widehat{\mathbb{P}}_t$, the distributions of $H(L_t)$ converge weakly to the Dirac measure at*

$$H\big(\psi_0^2\big) = \int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\psi_0^2(x)\psi_0^2(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y.$$

Let us highlight the core of the proof of Theorem 1.1. An important technical hindrance in the proof of Theorem 1.1 stems from the singularity of the Coulomb potential $x \mapsto 1/|x|$, which does not fit within the set up of standard large deviation theory. This problem was encountered also in [7] for deriving (1.11). As it concerns L_t, this turned out to be a mild technical issue. Indeed, a simple truncation argument with replacing $1/|x|$ by its regularized version $1/\sqrt{|x|^2+\delta^2}$ sufficed to carry over the theory developed in [7] to this singular potential. However, as we need now to work with $\Lambda(L_t)$ in the *uniform* metric, the singularity of $1/|\cdot|$ turns out be a more serious problem, since a standard contraction principle combined with the truncation argument does not work well here. Instead, we need a strategy that shows a strong regularity property of the random map $x \mapsto \Lambda_t(x)$, more precisely, an exponential decay of the probability that its modulus of continuity deviates from zero. This is our second main result.

Theorem 1.3. *For every $b > 0$,*

$$\lim_{\delta\to 0}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\Big\{\sup_{x_1,x_2\in\mathbb{R}^3\colon |x_1-x_2|\le\delta}\big|\Lambda_t(x_1)-\Lambda_t(x_2)\big| \ge b\Big\} = -\infty. \tag{1.13}$$

In Section 2, we prove Theorem 1.3. Let us state the following useful corollary to Theorem 1.3, which is also of independent interest; its proof is also deferred to Section 3.

Corollary 1.4. *For any $b > 0$,*

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\big\{\|\Lambda_t\|_\infty > b\big\} < 0.$$

Concerning the regularity of Λ_t we have a more quantitative result than Theorem 1.3, which we state here because of its own interest. Indeed, one main step in the proof of Theorem 1.3 is the following (stretched) exponential integrability.

Proposition 1.5. *There are constants $\rho > 1$, $a \in (0,1)$ and $\beta \in (0,\infty)$ such that*

$$\sup_{\substack{x_1,x_2\in\mathbb{R}^3\\ |x_1-x_2|\le 1}}\ \sup_{x\in\mathbb{R}^3} \mathbb{E}_x\bigg[\exp\bigg\{\beta\bigg(\frac{|\Lambda_1(x_1)-\Lambda_1(x_2)|}{|x_1-x_2|^a}\bigg)^\rho\bigg\}\bigg] < \infty.$$

This assertion suffices for our purposes, but it is clear that our proof can be extended to prove a number of more refined statements about the regularity of Λ_t, like the identification of the exact index of its Hölder continuity, almost sure limsup and liminf assertions about its modulus of continuity, and local and global laws of iterated logarithms. Let us remark that this might run prallel to the work of Donsker and Varadhan [3] on the law of iterated logarithm for one-dimensional Brownian local times.

Remark 1. Let us remark that Theorem 1.1, in combination with Corollaries 1.2 and 1.4, besides their intrinsic interests on their own right, have proved to be instrumental in proving tightness of the distributions of L_t under $\widehat{\mathbb{P}}_t$ and their convergence towards an explicit (spatially inhomogeneous) mixture of the maximizers $\{\psi_x^2\colon x\in\mathbb{R}^3\}$, which resolves the aforementioned "mean-field approximation" of the polaron problem on the level of path measures. This has been carried out in [2], and in this context, we refer to Section 2.4 in [2] for a heuristic discussion on the relevance of the results derived in the present paper.

1.4. *Review*: *Compactness and large deviations*

We now turn to the second main ingredient for the proof of Theorem 1.1, which is based on the results derived in [7]. Since this will play an important role in our proof, we take the opportunity to introduce the main idea in [7] and review its salient assertions.

Note that the space $\mathcal{M}_1(\mathbb{R}^d)$ of probability measures in $\mathbb{R}^d$ fails to be compact in the weak topology, which is due to several reasons. For instance, the location of the mass can shift away to ∞ as for the sequence $(\mu \star \delta_{a_n})_n$ with $a_n \to \infty$, or the mass can be spread thinly and totally disintegrate into dust, like for a sequence of Gaussians with diverging variance. Similarly, a mixture like $\mu_n = \frac{1}{2}[\mu * \delta_{a_n} + \mu * \delta_{-a_n}]$ can split into two (or more) widely separated pieces if $a_n \to \infty$. To compactify this space one should be allowed to "center" each such piece separately, as well as to allow some mass to be "thinly spread and disappear." Let

$$\widetilde{\mathcal{M}}_1(\mathbb{R}^d) = \{\widetilde{\mu}: \mu \in \mathcal{M}_1(\mathbb{R}^d)\}$$

denote the quotient space of orbits $\widetilde{\mu} = \{\mu \star \delta_x : x \in \mathbb{R}^d\}$ of $\mathcal{M}_1(\mathbb{R}^d)$ under translations. Then intuitively, for any sequence $(\widetilde{\mu}_n)_n$ in $\widetilde{\mathcal{M}}_1(\mathbb{R}^d)$ in the limit, one imagines, an empty, finite or countable collection $\{\alpha_j : j \in J\}$ of sub-probability distributions that are widely separated with total mass $\sum_{j\in J} \alpha_j(\mathbb{R}^d) = p \le 1$ and the remaining mass $1-p$ having totally disintegrated. For example, let μ_n be a mixture of three Gaussians, one with mean 0 and variance 1, one with mean n and variance 1 and one with mean 0 and variance n, each with equal weight $\frac{1}{3}$. Then the limiting object is the collection $\{\widetilde{\alpha}_1, \widetilde{\alpha}_1\}$, where $\widetilde{\alpha}_1$ is the equivalence class of a Gaussian with variance 1 and weight $\frac{1}{3}$.

This intuition naturally inspires the introduction of the space

$$\widetilde{\mathcal{X}} = \{\xi = (\widetilde{\alpha}_j)_{j\in J} : J \text{ at most countable}, \alpha_j \in \mathcal{M}_{\le 1}(\mathbb{R}^d)\ \forall j \in J\}$$

of empty, finite or countable collections of orbits $\{\widetilde{\alpha}_j : j \in J\}$ of sub-probability distributions α_j having masses p_j with $p = \sum_j p_j \le 1$. Note that we have a canonical embedding

$$\widetilde{\mathcal{M}}_1(\mathbb{R}^d) \hookrightarrow \widetilde{\mathcal{X}}.$$

In the proof of Theorem 1.1, the following results will play an important role.

Theorem 1.6 ([7], Theorem 3.2). *There is a metric* $\mathbf{D}$ *on* $\widetilde{\mathcal{X}}$ *so that* $\widetilde{\mathcal{M}}_1(\mathbb{R}^d)$ *is dense in* $(\widetilde{\mathcal{X}}, \mathbf{D})$ *and any sequence* $(\widetilde{\mu}_n)_n$ *in* $\widetilde{\mathcal{M}}_1(\mathbb{R}^d)$ *finds a subsequence which converges in the metric* $\mathbf{D}$ *to some element* $\xi \in \widetilde{\mathcal{X}}$. *In other words,* $\widetilde{\mathcal{X}}$ *is the compactification of* $\widetilde{\mathcal{M}}_1(\mathbb{R}^d)$ *and also the completion under the metric* $\mathbf{D}$ *of the totally bounded space* $\widetilde{\mathcal{M}}_1(\mathbb{R}^d)$.

Theorem 1.7 ([7], Theorem 4.1). *The distribution of the orbits* $\widetilde{L}_t$ *of the Brownian occupation measures embedded in the compact metric space* $(\widetilde{\mathcal{X}}, \mathbf{D})$ *satisfy a strong LDP with the rate function*

$$\widetilde{J}(\xi) = \sum_{j\in J} \widetilde{I}(\widetilde{\alpha}_j) = \sum_{j\in J} I(\alpha_j), \quad \xi = (\widetilde{\alpha}_j)_{j\in J} \in \widetilde{\mathcal{X}},$$

where we recall that $I(\cdot)$ *is defined in* (1.7) *and is shift-invariant and for any* $\alpha \in \mathcal{M}_{\le 1}(\mathbb{R}^d)$, $I(\alpha)$ *is a function only of the orbit* $\widetilde{\alpha}$, *which we call* $\widetilde{I}(\widetilde{\alpha})$.

Let us now choose $d = 3$ and recall the transformed path measure $\widehat{\mathbb{P}}_t$ from (1.1).

Theorem 1.8 ([7], Theorem 5.3). *The family of distributions of* $\widetilde{L}_t$ *under* $\widehat{\mathbb{P}}_t$ *satisfies a strong LDP in* $\widetilde{\mathcal{X}}$ *with rate function*

$$\widehat{J}(\xi) = \widehat{\rho} - \sum_j \left\{ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{|x-y|} \alpha_j(\mathrm{d}x)\alpha_j(\mathrm{d}y) - \widetilde{I}(\widetilde{\alpha}_j) \right\}, \quad \xi = \{\widetilde{\alpha}_j\} \in \widetilde{\mathcal{X}},$$

and $\widehat{\rho}$ *is given by*

$$\widehat{\rho} = \sup_{\xi\in\widetilde{\mathcal{X}}} \sum_j \left\{ \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\psi_j^2(x)\psi_j^2(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y - \frac{1}{2}\sum_j \|\nabla\psi_j\|_2^2 \right\} \tag{1.14}$$

and $\alpha_j(\mathrm{d}x) = \psi_j^2(x)\,\mathrm{d}x$ *with* $\sum_j \int_{\mathbb{R}^3}\psi_j^2(x)\,\mathrm{d}x \le 1$.

Let us finally remark that the above theory applies to any shift-invariant functional f of L_t, since $f(L_t) = \widetilde{f}(\widetilde{L}_t)$ for an obviously defined lifting $\widetilde{f}$ of f to the space of orbits. For example, in Theorem 1.8 the theory was applied to $H(L_t)$, recall (1.4). In the present paper, such shift-invariant dependence of $\|\Lambda_t\|_\infty$ on L_t is exhibited by the simple identity

$$\|\Lambda_t\|_\infty = \sup_{y\in\mathbb{R}^3}\left(\int_{\mathbb{R}^3}\frac{L_t(\mathrm{d}z)}{|z-y|}\right) = \sup_{y\in\mathbb{R}^3}\left(\int_{\mathbb{R}^3}\frac{(L_t\star\delta_x)(\mathrm{d}z)}{|z-y|}\right) = \|\Lambda_t\star\delta_x\|_\infty \quad \forall x\in\mathbb{R}^3,$$

and is of crucial importance in the context of deriving Theorem 1.1 from the above theory, see the proof of (3.9) in Section 3.

2. Super-exponential estimate: Proof of Theorem 1.3

For any $x\in\mathbb{R}^3$ we will denote by $\mathbb{P}_x$ the Wiener measure for the Brownian motion $W=(W_t)_{t\ge 0}$ starting at x and by $\mathbb{E}_x$ the corresponding expectation and we continue to write $\mathbb{P}_0=\mathbb{P}$ and $\mathbb{E}_0=\mathbb{E}$. First we turn to the proof of Proposition 1.5, which follows from the following lemma.

Lemma 2.1. *For any* $\varepsilon\in(0,1/2)$, *if* $a=1-2\varepsilon$ *and* $\rho=\frac{1}{1-\varepsilon}$, *then, for some* $\beta\in(0,\infty)$,

$$\sup_{\substack{x_1,x_2\in\mathbb{R}^3\\ |x_1-x_2|\le 1}}\ \sup_{x\in\mathbb{R}^3} \mathbb{E}_x\left[\exp\left\{\beta\left(\frac{|\Lambda_1(x_1)-\Lambda_1(x_2)|}{|x_1-x_2|^a}\right)^\rho\right\}\right] < \infty. \tag{2.1}$$

Proof. We fix $x_1,x_2\in\mathbb{R}^3$ with $|x_1-x_2|\le 1$ and denote

$$V(y) = V_{x_1,x_2}(y) = \frac{1}{|y-x_1|} - \frac{1}{|y-x_2|}, \quad y\in\mathbb{R}^3,$$

so that $\Lambda_1(x_1)-\Lambda_1(x_2) = \int_0^1 V(W_s)\,\mathrm{d}s$. Then by Jensen's inequality,

$$\left|\int_0^1 V(W_s)\,\mathrm{d}s\right|^\rho \le \int_0^1 \left|V(W_s)\right|^\rho\,\mathrm{d}s.$$

Let us now recall Khas'minski's lemma (see [11, p. 8], [8]), which states that, if for a function $\widetilde{V}\ge 0$,

$$\sup_{x\in\mathbb{R}^d}\mathbb{E}_x\left\{\int_0^1 \widetilde{V}(W_s)\,\mathrm{d}s\right\} \le \eta < 1,$$

then

$$\sup_{x\in\mathbb{R}^d}\mathbb{E}_x\left\{\exp\left\{\int_0^1 \widetilde{V}(W_s)\,\mathrm{d}s\right\}\right\} \le \frac{\eta}{1-\eta} < \infty.$$

Hence, (2.1) follows for some $\beta \in (0,\infty)$ if we show that

$$\sup_{\substack{x_1,x_2\in\mathbb{R}^3:\\ |x_1-x_2|\le 1}} |x_1-x_2|^{-a\rho} \sup_{x\in\mathbb{R}^3} \mathbb{E}_x\left[\int_0^1 \left|V(W_s)\right|^\rho \mathrm{d}s\right] < \infty. \tag{2.2}$$

Now let us introduce a generic constant C that does not depend on x, x_1, x_2, y, nor on any integration variable, and may change its value from line to line.

We estimate, for any x_1, x_2 satisfying $|x_1-x_2|\le 1$, and $a = 1-2\varepsilon$,

$$\begin{aligned}\left|V(y)\right| &= \frac{||y-x_2|-|y-x_1||}{|y-x_1||y-x_2|} \le \frac{|x_1-x_2|}{|y-x_1||y-x_2|}\\ &\le |x_1-x_2|^a \frac{[|y-x_2|^{1-a}+|y-x_1|^{1-a}]}{|y-x_1||y-x_2|}.\end{aligned} \tag{2.3}$$

The latter inequality follows from $(r+s)^{1-a} \le r^{1-a}+s^{1-a}$ for any $r,s\ge 0$. Furthermore, let us estimate the integral

$$h(y) = h_x(y) = \int_0^1 \mathrm{d}t \frac{\mathrm{e}^{-|x-y|^2/2t}}{t^{3/2}}$$

as follows. Since for any $b>0$, the map $[1,\infty)\ni z\mapsto z^{3/2-b}\mathrm{e}^{-z}$ is bounded, we can estimate

$$\begin{aligned}\int_0^{|y-x|^2\wedge 1} \mathrm{d}t \frac{\mathrm{e}^{-|y-x|^2/2t}}{t^{3/2}} &\le C|y-x|^{-3-2b}\int_0^{|y-x|^2\wedge 1} \mathrm{d}t\, \mathrm{e}^{-|y-x|^2/2t}\left(\frac{|y-x|^2}{2t}\right)^{3/2+b} t^b\\ &\le C|y-x|^{-3-2b}\int_0^{|y-x|^2\wedge 1}\mathrm{d}t\, t^b\\ &\le C|y-x|^{-3-2b}\big(|y-x|^2\wedge 1\big)^{1+b}, \quad x,y\in\mathbb{R}^3.\end{aligned}$$

For the remaining integral, we have the upper bound

$$\int_{|y-x|^2\wedge 1}^1 \mathrm{d}t \frac{\mathrm{e}^{-|y-x|^2/2t}}{t^{3/2}} \le \int_{|y-x|^2\wedge 1}^1 \mathrm{d}t\, t^{-3/2} \le \big[|y-x|^2\wedge 1\big]^{-1/2} - 1.$$

Combining the preceding two estimates, we obtain that

$$h(y) = \int_0^1 \mathrm{d}t \frac{\mathrm{e}^{-|y-x|^2/2t}}{t^{3/2}} \le C\frac{1}{|y-x|(1+|y-x|)^b}. \tag{2.4}$$

Let us now combine (2.3) and (2.4), to get

$$\begin{aligned}|x_1-x_2|^{-a\rho}\mathbb{E}_x\left[\int_0^1 \left|V(W_s)\right|^\rho \mathrm{d}s\right] &= (2\pi)^{-3/2}\int_{\mathbb{R}^3}\mathrm{d}y |x_1-x_2|^{-a\rho}\left|\widehat{V}(y)\right|^\rho h(y)\\ &\le C\int_{\mathbb{R}^3}\mathrm{d}y \frac{|y-x_2|^{\rho(1-a)}+|y-x_1|^{\rho(1-a)}}{|y-x_1|^\rho|y-x_2|^\rho}\frac{1}{|y-x|(1+|y-x|)^b}.\end{aligned}$$

Taking the symmetry in x_1 and x_2 into account, we see that (2.2) follows once we have

$$\sup_{\substack{x_1,x_2\in\mathbb{R}^3:\\ |x_1-x_2|\le 1}}\sup_{x\in\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\mathrm{d}y}{(1+|y-x|)^b}\frac{1}{|y-x_1|^\rho}\times\frac{1}{|y-x|}\times\frac{1}{|y-x_2|^{\rho a}} < \infty.$$

For this, we apply Hölder's inequality to the measure $\frac{\mathrm{d}y}{(1+|y-x|)^b}$ and the other three functions with parameters $p_1, p_2, p_3 > 1$ satisfying $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$. Hence, it suffices to show that all the integrals

$$\int_{\mathbb{R}^3} \frac{\mathrm{d}y}{(1+|y-x|)^b} \frac{1}{|y-x_1|^{\rho p_1}}, \qquad \int_{\mathbb{R}^3} \frac{\mathrm{d}y}{(1+|y-x|)^b} \frac{1}{|y-x|^{p_2}}, \qquad \int_{\mathbb{R}^3} \frac{\mathrm{d}y}{(1+|y-x|)^b} \frac{1}{|y-x_2|^{\rho a p_3}},$$

are bounded in x, x_1, x_2 for proper choices of p_1, p_2, p_3 and b. But this is ensured by requiring $b > 3$ and $p_1 < 3/\rho$ and $p_2 = \rho p_1$ (enforcing that $p_3 = p_1\rho/(p_1\rho - \rho + 1)$) and $p_3 < 3/a\rho$. The latter mean that $\frac{3(\rho-1)}{\rho(3-\rho a)} < p_1 < \frac{3}{\rho}$ and are possible as soon as $4 > \rho(1+a)$. But this is satisfied for our choices $\rho = \frac{1}{1-\varepsilon}$ and $a = 1 - 2\varepsilon$, for any $\varepsilon \in (0, 1)$.

This finishes the proof of Lemma 2.1. □

Lemma 2.2. *Fix $\varepsilon \in (\frac{1}{3}, \frac{1}{2})$ and choose $a = 1 - 2\varepsilon$ and $\rho = \frac{1}{1-\varepsilon}$ as in Lemma 2.1. Then there exists a constant $\beta_1 = \beta_1(\varepsilon) > 0$ such that the random variable*

$$M = \int_{\mathbb{R}^3} \mathrm{d}x_1 \int_{\mathbb{R}^3} \mathrm{d}x_2\, \mathbb{1}\big\{|x_1 - x_2| \le 1\big\}\left[\exp\left\{\beta_1\left(\frac{|\Lambda_1(x_1) - \Lambda_1(x_2)|}{|x_1 - x_2|^a}\right)^\rho\right\} - 1\right] \tag{2.5}$$

has a finite expectation under $\mathbb{P}_0$.

Proof. By Lemma 2.1 and Fubini's theorem, it suffices to show that

$$\iint_{|x_1-x_2|\le 1} \mathrm{d}x_1\, \mathrm{d}x_2\, \mathbb{E}\left[\exp\left\{\beta_1\left(\frac{|\Lambda_1(x_1) - \Lambda_1(x_2)|}{|x_1 - x_2|^a}\right)^\rho\right\} - 1\right] < \infty. \tag{2.6}$$

We decompose $\mathbb{R}^3 \subset \bigcup_{n=0}^\infty \{x \in \mathbb{R}^3 : n \le |x| < n+1\}$ and put $\tau_n = \inf\{t > 0 : |W_t| > n - n^\alpha\}$ for some $\alpha \in (0, 1)$. For any $r > 0$ and $x \in \mathbb{R}^3$, we also denote by $B_r(x)$ the open Euclidean ball of radius r around x. Then

$$\begin{aligned} &\iint_{|x_1-x_2|\le 1} \mathrm{d}x_1\, \mathrm{d}x_2\, \mathbb{E}\left[\exp\left\{\beta_1\left(\frac{|\Lambda_1(x_1) - \Lambda_1(x_2)|}{|x_1 - x_2|^a}\right)^\rho\right\} - 1\right] \\ &\quad \le \sum_{n=0}^\infty \int_{|x_1|\in[n,n+1)} \mathrm{d}x_1 \int_{B_1(x_1)} \mathrm{d}x_2 \left[\mathbb{E}\left\{\mathbb{1}_{\{\tau_n>1\}}\left(\exp\left\{\beta_1\left(\frac{|\Lambda_1(x_1) - \Lambda_1(x_2)|}{|x_1 - x_2|^a}\right)^\rho\right\} - 1\right)\right\}\right. \\ &\qquad \left. + \mathbb{E}\left\{\mathbb{1}_{\{\tau_n\le 1\}}\left(\exp\left\{\beta_1\left(\frac{|\Lambda_1(x_1) - \Lambda_1(x_2)|}{|x_1 - x_2|^a}\right)^\rho\right\} - 1\right\}\right)\right]. \end{aligned} \tag{2.7}$$

The first expectation inside the integrals is handled as follows. We note that, with $|x_1| \in [n, n+1)$ and $x_2 \in B_1(x_1)$, if $\tau_n > 1$, then $|W_s - x_1| > n^\alpha$ and $|W_s - x_2| > n^\alpha - 1$ for any $s \in [0, 1]$. Hence, for any $n \in \mathbb{N}$, on the event $\{\tau_n > 1\}$,

$$\frac{|\Lambda_1(x_1) - \Lambda_1(x_2)|}{|x_1 - x_2|^a} \le \frac{|x_1 - x_2|}{|x_1 - x_2|^{1-2\varepsilon}} \int_0^1 \frac{\mathrm{d}s}{|W_s - x_1||W_s - x_2|} \le c_1 |x_1 - x_2|^{2\varepsilon} n^{-2\alpha} \le c_1 n^{-2\alpha}.$$

Hence,

$$\begin{aligned} &\sum_{n=0}^\infty \int_{|x_1|\in[n,n+1)} \mathrm{d}x_1 \int_{B_1(x_1)} \mathrm{d}x_2\, \mathbb{E}\left\{\mathbb{1}_{\{\tau_n>1\}}\left(\exp\left\{\beta_1\left(\frac{|\Lambda_1(x_1) - \Lambda_1(x_2)|}{|x_1 - x_2|^a}\right)^\rho\right\} - 1\right)\right\} \\ &\quad \le \sum_{n=0}^\infty \big(\mathrm{e}^{\beta_1 c_1^\rho n^{-2\alpha\rho}} - 1\big) \mathrm{Leb}\big\{x_1 \in \mathbb{R}^3 : |x_1| \in [n, n+1)\big\} \mathrm{Leb}\big(B_1(0)\big). \end{aligned} \tag{2.8}$$

Since the first term is of size $O(n^{-2\alpha\rho})$ and the first Lebesgue measure is of size $O(n^2)$, the above sum is finite for $\alpha > \frac{3}{2\rho}$. Since we chose $\varepsilon > \frac{1}{3}$ and hence $\rho = \frac{1}{1-\varepsilon} > \frac{3}{2}$, we can choose some $\alpha \in (0, 1)$ so that $\alpha > \frac{3}{2\rho}$, as desired.

Let us now handle the second expectation in (2.7). By the Cauchy–Schwarz inequality and Proposition 1.5, if β_1 is small enough, for any $x_1, x_2 \in \mathbb{R}^3$ such that $|x_1 - x_2| \le 1$,

$$\mathbb{E}\bigg[\mathbb{1}_{\{\tau_n \le 1\}}\bigg\{\exp\bigg\{\beta_1\bigg(\frac{|\Lambda_1(x_1) - \Lambda_1(x_2)|}{|x_1 - x_2|^a}\bigg)^\rho\bigg\} - 1\bigg\}\bigg]$$
$$\le \mathbb{P}(\tau_n \le 1)^{\frac{1}{2}}\mathbb{E}\bigg[\exp\bigg\{2\beta_1\bigg(\frac{|\Lambda_1(x_1) - \Lambda_1(x_2)|}{|x_1 - x_2|^a}\bigg)^\rho\bigg\}\bigg]^{\frac{1}{2}}$$
$$\le C\mathbb{P}\Big(\max_{[0,1]} W > n - n^\alpha\Big)^{\frac{1}{2}},$$

where C does not depend on x_1, x_2. Since the last probability is of order e^{-cn^2}, the second sum on n in (2.7) is obviously finite. This, combined with the finiteness of the sum in (2.8), proves (2.6) and hence finishes the proof of Lemma 2.2. □

For the proof of Theorem 1.3 we will use the following (multidimensional) estimate of Garsia–Rodemich–Rumsey [10, p. 60].

Lemma 2.3. *Let $p(\cdot)$ and $\Psi(\cdot)$ be strictly increasing continuous functions on $[0,\infty)$ so that $p(0) = \Psi(0) = 0$ and $\lim_{t\uparrow\infty}\Psi(t) = \infty$. If $f : \mathbb{R}^d \to \mathbb{R}$ is continuous on the closure of the ball $B_{2r}(z)$ for some $z \in \mathbb{R}^d$ and $r > 0$, then the bound*

$$\int_{B_r(z)} \mathrm{d}x \int_{B_r(z)} \mathrm{d}y\, \Psi\bigg(\frac{|f(x) - f(y)|}{p(|x-y|)}\bigg) \le M < \infty, \tag{2.9}$$

implies that

$$\big|f(x) - f(y)\big| \le 8\int_0^{2|x-y|} \Psi^{-1}\bigg(\frac{M}{\gamma u^{2d}}\bigg) p(\mathrm{d}u), \qquad x, y \in B_r(z), \tag{2.10}$$

for some constant γ that depends only on d.

Finally we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. The Brownian scaling property implies that

$$\Lambda_t(x) = \frac{1}{t}\int_0^t \frac{1}{|W_s - x|}\,\mathrm{d}s = \int_0^1 \frac{1}{|W(ts) - x|}\,\mathrm{d}s \stackrel{\mathcal{D}}{=} \int_0^1 \frac{1}{|\sqrt{t}W(s) - x|}\,\mathrm{d}s = t^{-1/2}\Lambda_1\big(xt^{-1/2}\big),$$

where $\stackrel{\mathcal{D}}{=}$ denotes equality in distribution. Hence, the claim of Theorem 1.3 is equivalent to

$$\lim_{\delta\to 0}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\Big\{\sup_{x_1,x_2\in\mathbb{R}^3:\,|x_1-x_2|\le\delta t^{-1/2}}\big|\Lambda_1(x_1) - \Lambda_1(x_2)\big| \ge bt^{1/2}\Big\} = -\infty, \qquad b > 0. \tag{2.11}$$

Now we would like to apply Lemma 2.3. We pick $\varepsilon \in (\frac{1}{3}, \frac{1}{2})$ and $a = 1 - 2\varepsilon$ and $\rho = \frac{1}{1-\varepsilon}$ and $\beta = \beta_1$ as in Lemma 2.2 and choose

$$\Psi(x) = \mathrm{e}^{\beta|x|^\rho} - 1, \qquad p(x) = |x|^a = |x|^{1-2\varepsilon}, \qquad f(x) = \Lambda_1(x). \tag{2.12}$$

Then $\Psi(\cdot)$, $p(\cdot)$ and $f(\cdot)$ all satisfy the requirements of Lemma 2.3. Furthermore, Lemma 2.2 implies that hypothesis (2.9) is satisfied if $|x_1 - x_2| \le \delta$ and $\delta > 0$ is chosen small enough, where the random variable M is given in (2.5). Hence, (2.10) implies that for $|x_1 - x_2| \le \delta t^{-1/2}$ and all $t \ge 1$,

$$\big|\Lambda_1(x_1) - \Lambda_1(x_2)\big| \le 8\int_0^{\delta t^{-1/2}} \Psi^{-1}\bigg(\frac{M}{\gamma u^6}\bigg)p(\mathrm{d}u) = 8\frac{1-2\varepsilon}{\beta^{1/\rho}}\int_0^{\delta t^{-1/2}} \log\bigg(1 + \frac{M}{\gamma u^6}\bigg)^{1/\rho} u^{-2\varepsilon}\,\mathrm{d}u. \tag{2.13}$$

For $u \in (0, \delta t^{-1/2}]$ and all sufficiently large t, we estimate

$$8\frac{1-2\varepsilon}{\beta^{1/\rho}}\log\left(1+\frac{M}{\gamma u^6}\right)^{1/\rho} \le C\left(\left(\log(M\vee 1)\right)^{1/\rho}+\left(\log\frac{1}{u}\right)^{1/\rho}\right),$$

for some constant C that does not depend on t if t is sufficiently large. Hence, the right-hand side of (2.13) is not larger than

$$C_\delta\left(\log(M\vee 1)\right)^{1/\rho}t^{\varepsilon-1/2}+C_\delta(\log t)^c t^{\varepsilon-1/2}$$

for some C_δ, c, not depending on t, and $C_\delta \to 0$ as $\delta \to 0$. Substituting this in (2.13) and recalling that $\rho = \frac{1}{1-\varepsilon}$, we obtain

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{x_1,x_2\in\mathbb{R}^3:\,|x_1-x_2|\le\delta t^{-1/2}}\left|\Lambda_1(x_1)-\Lambda_1(x_2)\right|\ge bt^{1/2}\Big\} \\
&\quad\le \mathbb{P}\left\{\left(\log(M\vee 1)\right)^{1/\rho}+(\log t)^c \ge \frac{b}{C_\delta}t^{1-\varepsilon}\right\} \\
&\quad\le \mathbb{P}\left\{\log(M\vee 1)\ge \frac{b^\rho}{C_\delta^\rho}t - C_2(\log t)^{c\rho}\right\} \le \mathbb{E}(M\vee 1)\exp\left\{-\frac{b^\rho}{C_\delta^\rho}t + C_2(\log t)^{c\rho}\right\}.
\end{aligned} \tag{2.14}$$

Recall that by Lemma 2.2, $\mathbb{E}(M\vee 1)<\infty$. If we now let $t\to\infty$, followed by $\delta\to 0$, the above estimate now implies (2.11) and therefore Theorem 1.3. □

Corollary 2.4. *For any $b>0$,*

$$\lim_{\delta\to 0}\limsup_{t\to\infty}\frac{1}{t}\log\widehat{\mathbb{P}}_t\Big\{\sup_{x_1,x_2\in\mathbb{R}^3:\,|x_1-x_2|\le\delta}\left|\Lambda_t(x_1)-\Lambda_t(x_2)\right|\ge b\Big\}=-\infty.$$

Proof. Let us denote by $A_{t,\delta}$ the above event inside the probability. Then the Cauchy–Schwarz inequality gives that

$$\frac{1}{t}\log\widehat{\mathbb{P}}_t\{A_{t,\delta}\}\le\frac{1}{2t}\log\mathbb{E}\left\{e^{2tH(L_t)}\right\}-\frac{1}{t}\log\mathbb{E}\left\{e^{tH(L_t)}\right\}+\frac{1}{2}\frac{1}{t}\log\mathbb{P}\{A_{t,\delta}\}.$$

While the first two terms have finite large-t limits, by Theorem 1.3 the large-t limit of the third term tends to $-\infty$ as $\delta\to 0$. This proves the corollary. □

3. LDP for Λ_t in the uniform metric: Proof of Theorem 1.1

Recall that we need to show, for any $\varepsilon>0$,

$$\limsup_{t\to\infty}\frac{1}{t}\log\widehat{\mathbb{P}}_t\Big\{\inf_{w\in\mathbb{R}^3}\left\|\Lambda_t-\Lambda\psi_w^2\right\|_\infty\ge\varepsilon\Big\}<0. \tag{3.1}$$

We approximate the sup-norm inside the probability via a coarse graining argument as follows. For any $\delta\in(0,1)$, we can estimate

$$\begin{aligned}
\inf_{w\in\mathbb{R}^3}\left\|\Lambda_t-\Lambda\psi_w^2\right\|_\infty &= \inf_{w\in\mathbb{R}^3}\sup_{x\in\mathbb{R}^3}\left|\Lambda_t(x)-\left(\Lambda\psi_w^2\right)(x)\right| \\
&\le \sup_{x_1,x_2\in\mathbb{R}^3:\,|x_1-x_2|\le\delta}\left|\Lambda_t(x_1)-\Lambda_t(x_2)\right| \\
&\quad + \inf_{w\in\mathbb{R}^3}\sup_{z\in\delta\mathbb{Z}^3}\Big[\left|\Lambda_t(z)-\left(\Lambda\psi_w^2\right)(z)\right|+\sup_{\tilde z\in B_\delta(z)}\left|\left(\Lambda\psi_w^2\right)(\tilde z)-\left(\Lambda\psi_w^2\right)(z)\right|\Big].
\end{aligned} \tag{3.2}$$

Note that, for any $w \in \mathbb{R}^3$ the deterministic function $\Lambda\psi_w^2$ is uniformly continuous on $\mathbb{R}^3$ and hence

$$\lim_{\delta\downarrow 0} \sup_{z\in\delta\mathbb{Z}^3} \sup_{\tilde z\in B_\delta(z)} \left|\left(\Lambda\psi_w^2\right)(\tilde z) - \left(\Lambda\psi_w^2\right)(z)\right| = 0.$$

Since $\varepsilon > 0$ is arbitrary, the above fact and Corollary 2.4 imply that, to deduce (3.1), it suffices to prove, for any $\varepsilon, \delta > 0$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \widehat{\mathbb{P}}_t \left\{ \inf_{w\in\mathbb{R}^3} \sup_{z\in\delta\mathbb{Z}^3} \left|\Lambda_t(z) - \left(\Lambda\psi_w^2\right)(z)\right| \ge \varepsilon \right\} < 0. \tag{3.3}$$

For any $z \in \delta\mathbb{Z}^3$, $w \in \mathbb{R}^3$ and any $\eta > 0$, we can estimate

$$\left|\Lambda_t(z) - \left(\Lambda\psi_w^2\right)(z)\right| \le \int_{B_\eta(z)} \frac{\psi_w^2(y)}{|y-z|}\,\mathrm{d}y + \int_{B_\eta(z)} \frac{L_t(\mathrm{d}y)}{|y-z|} + \left|\int_{\mathbb{R}^3} \frac{\mathbb{1}\{|y-z| \ge \eta\}}{|y-z|} \left(L_t(\mathrm{d}y) - \psi_w^2(y)\,\mathrm{d}y\right)\right|. \tag{3.4}$$

The first term can be handled easily. Note that, for any $w \in \mathbb{R}^3$, ψ_w is radially symmetric and $\|\psi_w\|_2 = 1$. Hence using polar coordinates and invoking the dominated convergence theorem we can argue that

$$\lim_{\eta\to 0} \sup_{z\in\delta\mathbb{Z}^3} \int_{B_\eta(z)} \frac{\psi_w^2(y)}{|y-z|}\,\mathrm{d}y = 0. \tag{3.5}$$

Let us turn to the second term in (3.4). We claim that, for any $\delta > 0$ and $\eta > 0$ small enough,

$$\limsup_{t\to\infty} \frac{1}{t} \log \widehat{\mathbb{P}}_t \left\{ \sup_{z\in\delta\mathbb{Z}^3} \int_{B_\eta(z)} \frac{L_t(\mathrm{d}y)}{|y-z|} \ge \varepsilon \right\} < 0. \tag{3.6}$$

Let us first handle the above event with the Wiener measure $\mathbb{P}$ replacing $\widehat{\mathbb{P}}_t$. Then we can estimate

$$\mathbb{P}\left\{ \sup_{z\in\delta\mathbb{Z}^3} \int_{B_\eta(z)} \frac{L_t(\mathrm{d}y)}{|y-z|} > \varepsilon \right\} \le \sum_{\substack{z\in\delta\mathbb{Z}^3 \\ |z|\le t^2}} \mathbb{P}\left\{ \int_{B_\eta(z)} \frac{L_t(\mathrm{d}y)}{|y-z|} \ge \varepsilon/2 \right\} + \mathbb{P}\left\{ \sup_{\substack{z\in\delta\mathbb{Z}^3 \\ |z|> t^2}} \int_{B_\eta(z)} \frac{L_t(\mathrm{d}y)}{|y-z|} \ge \varepsilon/2 \right\}. \tag{3.7}$$

The second term can be estimated by the probability that the Brownian path, starting at origin, travels a distance $t^2 - \varepsilon$ by time t. This probability is of order $\exp\{-ct^3\}$ and can be ignored. For the first term we note that a box of size t^2 in $\mathbb{R}^3$ can be covered by $O(t^6)$ sub-boxes of side length δ and that the probability is maximal for $z = 0$. Hence, we can estimate, with the help of Markov's inequality, for any $\beta > 0$,

$$\sum_{\substack{z\in\delta\mathbb{Z}^3 \\ |z|\le t^2}} \mathbb{P}\left\{ \int_{B_\eta(z)} \frac{L_t(\mathrm{d}y)}{|y-z|} > \varepsilon/2 \right\} \le Ct^6 \mathbb{P}\left\{ \beta \int_0^t V_\eta(W_s)\,\mathrm{d}s > t\beta\varepsilon/2 \right\} \le Ct^6 \mathrm{e}^{-\frac{\varepsilon}{2}t\beta} \mathbb{E}\left\{\mathrm{e}^{\beta\int_0^t V_\eta(W_s)\,\mathrm{d}s}\right\}, \tag{3.8}$$

where $V_\eta(x) = \mathbb{1}_{\{|x|\le\eta\}} \frac{1}{|x|}$. Note that, for any $\beta > 0$ and some constants c_1, c_2 independent of η,

$$\sup_{y\in\mathbb{R}^3} \mathbb{E}_y \left\{ \beta \int_0^1 V_\eta(W_s)\,\mathrm{d}s \right\} \le \beta \int_{B_\eta(0)} \frac{\mathrm{d}x}{|x|} \int_0^1 p_s(0,x)\,\mathrm{d}s \le \beta c_1 \int_{B_\eta(0)} \frac{\mathrm{d}x}{|x|^2} = c_2\eta\beta.$$

For any fixed $\beta > 0$ and η small enough, this is not larger than $1/2$, and by Khas'minskii's lemma [11, p. 8], successive conditioning and the Markov property,

$$\mathbb{E}\left\{\mathrm{e}^{\beta\int_0^t V_\eta(W_s)\,\mathrm{d}s}\right\} \le 2^{\lceil t\rceil}.$$

Then (3.8) and (3.7) imply, for any $\beta > 0$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\bigg\{ \sup_{z\in\delta\mathbb{Z}^3} \int_{B_\eta(z)} \frac{L_t(\mathrm{d}y)}{|y-z|} \geq \varepsilon \bigg\} \leq -\varepsilon\beta/2 + \log 2.$$

From this we can deduce (3.6) by choosing $\beta > 0$ large enough and invoking Hölder's inequality as in the proof of Corollary 2.4. We drop the details to avoid repetition.

Let us turn to the third term on the right hand side of (3.4). Then by (3.5) and (3.6), it suffices to prove that, for every $\eta, \varepsilon > 0$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \widehat{\mathbb{P}}_t\{L_t \in F_\eta\} < 0, \tag{3.9}$$

where

$$F_\eta = \Big\{ \mu \in \mathcal{M}_1(\mathbb{R}^3) \colon \forall w \in \mathbb{R}^3 \sup_{z\in\mathbb{R}^3} \big|\langle f_{z,\eta}, \mu - \psi_w^2 \rangle\big| \geq \varepsilon \Big\},$$

where we put $f_{z,\eta}(y) = \frac{1}{|y-z|} \wedge \frac{1}{\eta}$. We claim that for each $\eta > 0$, F_η is a closed set in the weak topology in $\mathcal{M}_1(\mathbb{R}^3)$. First note that the family $\mathcal{A}_\eta = \{f_{z,\eta} \colon z \in \mathbb{R}^d\}$ is equicontinuous and uniformly bounded. Hence, for any $\eta > 0$, the set

$$G_{\eta,w} = \Big\{ \mu \in \mathcal{M}_1(\mathbb{R}^3) \colon \sup_{f\in\mathcal{A}_\eta} \big|\langle f, \mu - \psi_w^2 \rangle\big| < \varepsilon \Big\}$$

is weakly open and hence

$$F_\eta = \bigcap_{w\in\mathbb{R}^d} G_{\eta,w}^{\mathrm{c}}$$

is weakly closed. Furthermore, we note that F_η is shift-invariant, i.e., if $\mu \in F_\eta$, then $\mu \star \delta_x \in F_\eta$ for any $x \in \mathbb{R}^3$. In other words,

$$\widehat{\mathbb{P}}_t\{L_t \in F_\eta\} = \widehat{\mathbb{P}}_t\{\widetilde{L}_t \in \widetilde{F}_\eta\},$$

where $\widetilde{F}_\eta = \{\widetilde{\mu} \colon \mu \in F_\eta\}$, the set of orbits $\widetilde{\mu} = \{\mu \star \delta_x : x \in \mathbb{R}^3\}$ of members of F_η, is a closed set in $\widetilde{\mathcal{M}}_1(\mathbb{R}^3) \hookrightarrow \widetilde{\mathcal{X}}$, and by Theorem 1.6, $\widetilde{F}_\eta$ is also compact in $(\widetilde{\mathcal{X}}, \mathbf{D})$. Then by Theorem 1.8,

$$\limsup_{t\to\infty} \frac{1}{t} \log \widehat{\mathbb{P}}_t\{\widetilde{L}_t \in \widetilde{F}_\eta\} \leq -\inf_{\xi\in\widetilde{F}_\eta} \widehat{J}(\xi).$$

According to [7, Lemma 5.4], the variational formula in (1.14) attains its maximum only in trivial sequences ξ consisting of just one single orbit of a probability measure $\mu(\mathrm{d}x) = \psi^2(x)\,\mathrm{d}x$ with ψ a rotationally symmetric, L^2-normalized function, which, by the uniqueness of the variational problem (1.6) (recall (1.10)), must be one of the maximizers of the formula in (1.6), and $\rho = \widehat{\rho}$. Since $\widetilde{F}_\eta$ is in particular compact and does not contain such an element ξ, we have that $\inf_{\xi\in\widetilde{F}_\eta} \widehat{J}(\xi) > 0$. These two facts imply (3.9) and hence Theorem 1.1.

We end this section with the proof of Corollary 1.4.

Proof of Corollary 1.4. The proof is straightforward and similar to the last line of arguments. Indeed, we note that for any $\delta > 0$,

$$\begin{aligned}
\mathbb{P}\big\{\|\Lambda_t\|_\infty > b\big\} &\leq \mathbb{P}\Big\{ \sup_{|x_1-x_2|\leq\delta} \big|\Lambda_t(x_1) - \Lambda_t(x_2)\big| \geq b/2 \Big\} + \mathbb{P}\Big\{ \sup_{x\in\delta\mathbb{Z}^3} \Lambda_t(x) \geq b/2 \Big\} \\
&\leq \mathbb{P}\Big\{ \sup_{|x_1-x_2|\leq\delta} \big|\Lambda_t(x_1) - \Lambda_t(x_2)\big| \geq b/2 \Big\} + \mathbb{P}\Big\{ \sup_{\substack{x\in\delta\mathbb{Z}^3 \\ |x|\leq t^2}} \Lambda_t(x) \geq b/2 \Big\} + \mathbb{P}\Big\{ \sup_{\substack{x\in\delta\mathbb{Z}^3 \\ |x|> t^2}} \Lambda_t(x) \geq b/2 \Big\}.
\end{aligned}$$

By Theorem 1.3, the first term has a strictly negative exponential rate. The third term can again be neglected since this is of order $\exp\{-ct^3\}$. Also for the second term, the box of size t^2 can be covered by $O(t^6)$ sub-boxes of side length δ. Therefore,

$$\mathbb{P}\Big\{\sup_{\substack{x\in\delta\mathbb{Z}^3\\|x|\le t^2}}\Lambda_t(x)\ge b/2\Big\}\le Ct^6\mathbb{P}\big\{\Lambda_t(0)>b/2\big\}.$$

For any $\kappa>0$,

$$\mathbb{P}\big\{\Lambda_t(0)>b/2\big\}\le \mathrm{e}^{-\kappa bt/2}\mathbb{E}\Big\{\exp\Big\{\kappa\int_0^t\frac{\mathrm{d}s}{|W_s|}\Big\}\Big\}.$$

We choose $t>u\gg 1$ and $\kappa>0$ small enough so that $\sqrt{u}\kappa\ll 1$ and

$$\alpha=\sup_{x\in\mathbb{R}^3}\mathbb{E}_x\Big\{\kappa\int_0^u\frac{\mathrm{d}s}{|W_s|}\Big\}=\mathbb{E}_0\Big\{\kappa\int_0^u\frac{\mathrm{d}s}{|W_s|}\Big\}=2\kappa\sqrt{u}\mathbb{E}\Big(\frac{1}{|W_1|}\Big)\ll 1.$$

Then by Khas'minskii's lemma [11, p. 8],

$$\sup_{x\in\mathbb{R}^3}\mathbb{E}_x\Big\{\exp\Big\{\kappa\int_0^u\frac{\mathrm{d}s}{|W_s|}\Big\}\Big\}\le\frac{1}{1-\alpha},$$

and by successive conditioning and the Markov property,

$$\mathbb{E}\Big\{\exp\Big\{\kappa\int_0^t\frac{\mathrm{d}s}{|W_s|}\Big\}\Big\}\le\Big(\frac{1}{1-\alpha}\Big)^{t/u}.$$

Since $\log(1+\alpha)\approx\alpha$ as $\alpha\to 0$, for any $b>0$ and $\kappa>0$ suitably chosen and u large enough,

$$\begin{aligned}\mathbb{P}\big\{\Lambda_t(0)>b/2\big\}&\le\exp\Big\{-\frac{\kappa bt}{2}+\frac{t}{u}\log(1-\alpha)\Big\}\\&\le\exp\Big[-t\kappa\Big\{\frac{b}{2}-\frac{1}{\sqrt{u}}c\Big\}\Big]\\&\le\exp\{-t\kappa\widetilde{C}\}\end{aligned}$$

for some $\widetilde{C}=\widetilde{C}(u,a,c)>0$. This proves the corollary. □

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