

Integration formulas for Brownian motion on classical compact Lie groups

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Abstract. Combinatorial formulas for the moments of the Brownian motion on classical compact Lie groups are obtained. These expressions are deformations of formulas of B. Collins and P. Śniady for moments of the Haar measure and yield a proof of the First Fundamental Theorem of invariant theory and of classical Schur–Weyl dualities based on stochastic calculus.

Résumé. On obtient ici des formules combinatoires pour les moments du mouvement Brownien sur les groupes de Lie classiques compacts. Elles sont des déformations de celles obtenues par B. Collins et P. Śniady pour les moments de la mesure de Haar et permettent de donner une preuve fondée sur le calcul stochastique du premier théorème fondamental de la théorie des invariants et de la dualité de Schur–Weyl.

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1. Introduction

Let G be a compact Lie group belonging to one of the three classical series: the unitary groups $U(N)$, the orthogonal groups $O(N)$ and the compact symplectic groups $Sp(N)$. The aim of this article is to give an explicit expression to

$$\int_G f(g)\mu(dg), \tag{1}$$

where f is a polynomial function of the coefficients of the matrix and their adjoint and μ is a measure associated to the marginal of a Brownian motion on G .

When μ is the Haar measure, such a study was started by the physicist D. Weingarten [22], a complete answer was achieved later on by B. Collins in [5] for the unitary series and by the same author together with P. Śniady in [7] for any of the above series. For a more recent point of view, we refer to [6,24]. These results were obtained using whether representation theory of the three series and the orthogonality of characters [5], or the first theorem of invariant theory or equivalently Schur–Weyl dualities [6,7,24].

When μ is a marginal of a Brownian motion, this very question has been tackled by T. Lévy in [15], following previous works [2,20,23] focusing on polynomials of the form $\text{Tr}(g^n)$, for $n \in \mathbb{N}$.

The starting point of this article is to extend the results and the point of view of [15] in the following way: a first question answered here is to consider, in the unitary case, not only polynomials of coefficients of matrices but also polynomials in the coefficients and their adjoints. A second one is to get formulas that are both well suited for large N and large t asymptotics. Letting t go to infinity, we recover formulas of [7], for the integration against the Haar measure. Another motivation for the present work is to answer the latter questions without using representation

theory or the Theory of invariants. Indeed, our proofs only rely on stochastic calculus and we moreover obtain a new proof of the First Fundamental Theorem of invariants theory (often abbreviated as FFT) and recover in this way the Schur–Weyl duality theorems for the classical compact Lie groups. Another feature of our work is that we express expectations of functions measurable with respect to a Brownian motion on any compact matrix Lie group, in terms of expectations with respect to the Brownian motion on unitary matrices of the same size.

The article is organized as follows. Section 2 gives the definition and the scaling we are using for a Brownian motion on a classical compact Lie group. Section 3 recalls the statements of the first fundamental theorem of invariants and of Schur–Weyl dualities and their equivalence. Section 4 states and proves formulas for the Brownian motions. We prove here the formula for mixed moments of unitary matrices and their adjoint and recall the former results of T. Lévy. Then, we give an expression for expectations of tensors of Brownian motions on any classical group in terms of non mixed moments of the unitary Brownian motion. This is stated in Theorem 4.3. Using the expression of T. Lévy for non-mixed moments, we deduce from it a new combinatorial expression for expectation of entries of a Brownian motion for any of the three series, in terms of functions on the symmetric group. We obtain in this way the Theorem 4.7. Section 5 gives two corollaries of the latter: we obtain combinatorial formulas for integration against the Haar measure, that we compare with the one of [7], and a new proof of the first fundamental theorem of invariants. To show the convergence of our expressions as the time goes to infinity, we have to show that expectations of non-mixed tensors of a Brownian motion vanish. This is simply proved using the factorization of a unitary Brownian motion as the product of a Brownian motion on $SU(N)$ with an Brownian motion on the circle. To conclude, in the Appendix, we prove in our framework, for the sake of completeness, two already known lemmas for the invariants of the symplectic groups.

2. Brownian motion on classical compact Lie groups

We discuss here about the definition of the Brownian motion we use all along the text. We have chosen here to define it as the solution of an SDE in the linear space of matrices.

Classical compact Lie groups: For any positive integer N , we shall consider the following groups of matrices: the orthogonal group $O(N) = \{O \in M_N(\mathbb{R}) : {}^t O O = \text{Id}\}$, the unitary group $U(N) = \{U \in M_N(\mathbb{C}) : U^* U = \text{Id}\}$ and the special unitary group $SU(N) = \{U : U(N) : \det(U) = 1\}$, and as N is even, the unitary symplectic group $Sp(N) = \{S \in U(N) : {}^t S J S = J\}$, where J is the matrix $J = \begin{pmatrix} 0 & I_{N/2} \\ -I_{N/2} & 0 \end{pmatrix}$. Their Lie algebras are denoted by small gothic letters: $\mathfrak{o}(N) = \{X \in M_N(\mathbb{R}) : {}^t X + X = 0\}$, $\mathfrak{u}(N) = \{X \in M_N(\mathbb{C}) : X + X^* = 0\}$ and $\mathfrak{su}(N) = \{X \in \mathfrak{u}(N) : \text{Tr}(X) = 0\}$, and $\mathfrak{sp}(N) = \{X \in \mathfrak{u}(N) : {}^t X J + J X = 0\}$. Let G be one of the above mentioned compact Lie groups with Lie algebra \mathfrak{g} . All along this text, ε shall be 1 if $G = O(N)$, -1 , if $G = Sp(N)$ and 0 otherwise.

Choice of a metric: For any $x, y \in \text{End}(\mathbb{C}^N)$, let us define

$$\langle x, y \rangle = -\frac{N}{2^{|\varepsilon|}} \text{Tr}(xy).$$

The restriction of the real bilinear form $\langle \cdot, \cdot \rangle$ to \mathfrak{g} defines a G -invariant scalar product on the three series of Lie algebras we are considering. The choice of normalization of the latter scalar product is motivated by the following remarks. Let K be the Gaussian vector on \mathfrak{g} with covariance given by $\langle \cdot, \cdot \rangle$. When $\mathfrak{g} = \mathfrak{u}(N)$, the random matrix iK has the same law as a matrix from the Gaussian unitary ensemble.¹ On the other hand, as $N \rightarrow \infty$, $\dim_{\mathbb{R}}(\mathfrak{g}) \sim \frac{N^2}{2^{|\varepsilon|}}$ and according to the law of large numbers, almost surely, $\frac{1}{\dim_{\mathbb{R}}(\mathfrak{g})} \|K\|^2 \rightarrow 1$. Therefore, for any of the above classical series, $\frac{1}{N} \text{Tr}(K K^*) = \frac{2^{|\varepsilon|}}{N^2} \|K\|^2 \rightarrow 1$. Furthermore, the Wigner theorem and its generalization to other invariant ensemble (see [17]) show that the empirical measure of eigenvalues of iK converges weakly towards the semicircle law $\sqrt{|4 - x^2|_+} \frac{dx}{\pi}$.

Wrapping a Brownian motion of \mathfrak{g} on G : Let us denote by $(K_t^{\mathfrak{g}})_{t \geq 0}$ the classical Brownian motion on the Euclidean space² $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$, that is the unique Gaussian process with covariance given for all $x, y \in \mathfrak{g}$ and $t, s \geq 0$, by

$$\mathbb{E}[\langle x, K_t \rangle \langle y, K_s \rangle] = \min(t, s) \langle x, y \rangle.$$

¹Though, when $\mathfrak{g} = \mathfrak{o}(N)$ or $\mathfrak{sp}(N)$, iK is not equal in law to a matrix of the Gaussian orthogonal (resp. symplectic) ensemble.

²We shall drop the symbol \mathfrak{g} when the context is clear.

The matrix of quadratic variations $\langle dK_t, dK_t \rangle$ is equal to $C_{\mathfrak{g}} dt$, where $C_{\mathfrak{g}}$ is a deterministic matrix commuting with G . Let S be an element of G and $(G_t)_{t \geq 0}$ be the $\text{End}(\mathbb{C}^N)$ -valued stochastic process solution to the following stochastic differential equation

$$dG_t = G_t dK_t + \frac{C_{\mathfrak{g}}}{2} G_t dt, \quad G_0 = S. \tag{2}$$

This Markovian process is called the *Brownian motion* on G issued from S . When not precised, we shall assume that $(G_t)_{t \geq 0}$ is issued from Id . The rest of the paper being based on this definition, we shall recall why it is justified. For any $x \in \mathfrak{g}$, let \mathcal{L}_x be the left-invariant first order differential operator defined by setting for any differentiable function $F \in C^\infty(\text{End}(\mathbb{C}^N))$ and any $\phi \in \text{End}(\mathbb{C}^N)$,

$$\mathcal{L}_x F(\phi) = \left. \frac{d}{dt} \right|_{t=0} F(\phi \exp(tx)).$$

Let $\mathcal{U}(\mathfrak{g})$ be the real enveloping algebra of \mathfrak{g} . The application \mathcal{L} extends to a morphism of associative algebra between $\mathcal{U}(\mathfrak{g})$ and the algebra of differential operators on $\text{End}(\mathbb{C}^N)$ commuting with left translations by G . For any orthonormal basis $(x_i)_{1 \leq i \leq d}$ of \mathfrak{g} with respect to the scalar product $\langle \cdot, \cdot \rangle$, let $c_{\mathfrak{g}} = \sum_{i=1}^d x_i^2 \in \mathcal{U}(\mathfrak{g})$. The element $c_{\mathfrak{g}}$ does not depend on the orthonormal basis $(x_i)_{1 \leq i \leq d}$ and is called the *Casimir* element. The image of the Casimir element $c_{\mathfrak{g}}$ is denoted by Δ_G :

$$\Delta_G = \mathcal{L}_{c_{\mathfrak{g}}} = \sum_{i=1}^d \mathcal{L}_{x_i} \circ \mathcal{L}_{x_i}.$$

The image of $c_{\mathfrak{g}}$ by the classical representation ρ on \mathbb{C}^N is $\sum_{i=1}^d \rho(x_i)^2 = C_{\mathfrak{g}} \in \text{End}(\mathbb{C}^N)$. The latter operators can be defined as well on $C^\infty(G)$, we denote them abusively by the same letters. The operator Δ_G is then called the *Laplacian* associated to the metric $\langle \cdot, \cdot \rangle$.

Lemma 2.1. For any $F \in C^\infty(\text{End}(\mathbb{C}^N))$, $(F(G_t))_{t \geq 0}$ is solution to the following SDE

$$d(F(G_t)) = \mathcal{L}_{dK_t} F(G_t) + \frac{1}{2} \Delta_G(F)(G_t) dt.$$

Proof. For any $F \in C^\infty(\text{End}(\mathbb{C}^N))$ and $\phi \in \text{End}(\mathbb{C}^N)$, let $dF_\phi \in \text{End}(\mathbb{C}^N)^*$, $d_\phi^2 F \in (\text{End}(\mathbb{C}^N)^{\otimes 2})^*$ be the one form and two form associated to the first and second derivatives of F . For any $x \in \mathfrak{g}$, $\phi \in \text{End}(\mathbb{C}^N)$,

$$\mathcal{L}_x \circ \mathcal{L}_x(F)(\phi) = d_\phi F(\phi x^2) + d_\phi^2 F(\phi x \otimes \phi x). \tag{*}$$

Let $(x_i)_{i=1}^d$ be an orthogonal basis of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$, the Itô formula combined with the latter equation implies that

$$\begin{aligned} d(F(G_t)) &= d_{G_t} F(G_t) dK_t + \frac{1}{2} d_{G_t} F(G_t C_{\mathfrak{g}}) dt + \frac{1}{2} \langle d_{G_t}^2 F(G_t) dK_t \otimes G_t dK_t \rangle \\ &= \mathcal{L}_{dK_t} F(G_t) + \frac{1}{2} d_{G_t} F(G_t C_{\mathfrak{g}}) dt + \frac{1}{2} \sum_{i=1}^d d_{G_t}^2 F(G_t x_i \otimes G_t x_i) dt. \end{aligned}$$

Combined with (*), the latter equation yields the result. □

It can be proved that the process $(G_t)_{t \geq 0}$ satisfies the following properties (more details can be found in [16]).

Lemma 2.2.

- (i) *Almost surely, for any $t \geq 0$, $G_t \in G$.*

- (ii) For all $T \geq 0$, $(G_T^{-1}G_{T+t})_{t \geq 0}$ is independent of $\sigma(G_s, s \leq T)$ and has the same law as $(gG_tg^{-1})_{t \geq 0}$ for any $g \in G$.
- (iii) Almost surely, the mapping $t \in \mathbb{R}_+ \mapsto G_t \in G$ is continuous.
- (iv) The generator of the Markov process $(G_t)_{t \geq 0}$ on G is $\frac{1}{2}\Delta_G$.

Proof. We only sketch the first point and leave the others to the Reader (see [16] for proofs). For any of the above groups, there is a function $F \in C^\infty(\text{End}(\mathbb{C}^N), \mathbb{R}^n)$ such that $G = F^{-1}(\{1\})$. For any $x \in \mathfrak{g}$, $\mathcal{L}_x F = \Delta_{\mathfrak{g}} F = 0$. Hence, Lemma 2.1 yields that almost surely for any $t \geq 0$, $F(G_t) = 1$. □

For $G = U(N)$, the Brownian motion satisfies the following splitting property.

Lemma 2.3. *Let $(U_t)_{t \geq 0}$ be a Brownian motion on $U(N)$. Then $(U_t)_{t \geq 0}$ has the same law as $(e^{\frac{iB_t}{N}} S_t)_{t \geq 0}$, where $(B_t)_{t \geq 0}$ is a standard real Brownian motion and $(S_t)_{t \geq 0}$ an independent Brownian motion on $SU(N)$.*

Proof. Let us set for any $t \geq 0$, $\tilde{K}_t = K_t^{\text{su}(N)} + \frac{1}{N}iB_t$, where $(B_t)_{t \geq 0}$ is a standard real Brownian motion independent from $(K_t^{\text{su}(N)})_{t \geq 0}$. On the one hand, $(\tilde{K}_t)_{t \geq 0}$ has the same law as $(K_t^{\text{u}(N)})_{t \geq 0}$. On the other hand, since $(S_t)_{t \geq 0}$ satisfies (2) with driving noise $(K_t^{\text{su}(N)})_{t \geq 0}$, so does $(e^{\frac{iB_t}{N}} S_t)_{t \geq 0}$ but with driving noise $(\tilde{K}_t)_{t \geq 0}$. □

Our aim is to get a formula for $\mathbb{E}[P(G_t)]$, for any homogeneous polynomial P in coefficients of matrices and their adjoint, that is well suited to let $t \rightarrow \infty$ and to get another proof of the formulas of [7] for $\mathbb{E}[P(H)]$, where H is the Haar measure on G . We are going to see that for any series, these expectations can be expressed in terms of expectations of coefficients of unitary Brownian motions (and not their adjoint). For a fix degree of homogeneity, we shall gather these expectations in a single one, namely $\mathbb{E}[G_t^{\otimes n} \otimes \overline{G_t}^{\otimes m}]$, that will be expressed in terms of its Schur–Weyl dual. The latter expression will imply the FFT theorem and therefore the Schur–Weyl duality. Before proceeding, we recall the statement of FFT and its relation to Schur–Weyl duality.

3. First Fundamental Theorem of invariants and Schur–Weyl duality

For any vector space W on which G acts, we consider the set W^G of points fixed by G . Let us write V for the canonical representation \mathbb{C}^N . The fundamental theorem of invariant theory gives a spanning family for the vector space W^G , when W is a space of tensors of the natural representation V or its dual V^* . We refer to [10,19] for a nice exposition and a proof of this theorem relying on algebraic geometry. Let us recall its statement. We will restrict to the three series of groups $O(N)$, $U(N)$ and $\text{Sp}(N)$, discarding here the special unitary groups $SU(N)$. For any $n, m \in \mathbb{N}$, we shall consider the representation $(V^{\otimes n} \otimes V^{*\otimes m}, \rho_{n,m})$ of G defined by

$$\rho_{n,m}(g).v_1 \otimes \cdots \otimes v_n \otimes \varphi_1 \otimes \cdots \otimes \varphi_m = (g.v_1) \otimes \cdots \otimes (g.v_n) \otimes \varphi_1(g^{-1}.) \otimes \cdots \otimes \varphi_m(g^{-1}.),$$

for all $g \in G, v_1, \dots, v_n \in V, \varphi_1, \dots, \varphi_m \in V^*$. Let us highlight that when $G = U(N)$, V^* is isomorphic as a representation to \overline{V} , that is, the real vector space V , endowed with the actions λ and ρ of \mathbb{C} and $U(N)$, defined respectively by $\lambda(z, v) = \bar{z}v$ and $\rho(U, v) = \overline{U}v$, for any $z \in \mathbb{C}, U \in U(N)$ and $v \in V$. When G is respectively $O(N)$, $\text{Sp}(N)$ and $U(N)$, we denote by B the canonical G -invariant \mathbb{C} -bilinear form on $V \oplus V^*$, that is skew-symmetric as $G = \text{Sp}(N)$ and symmetric when G is $O(N)$ or $U(N)$. Let $\mathcal{M}(n)$ denote the set of *matchings* of $\{1, \dots, 2n\}$, that is, partitions of $\{1, \dots, 2n\}$ whose blocks are all of size 2.

Theorem 3.1 (FFT). *The vector space $(V^{\otimes n} \otimes V^{*\otimes m})^G$ is trivial if $n + m$ is odd, or $G = U(N)$ and $n \neq m$. Moreover, as $n + m$ is even, it is spanned by the linear maps*

$$V^{*\otimes n} \otimes V^{\otimes m} \longrightarrow \mathbb{C},$$

$$(\phi_1, \dots, \phi_{n+m}) \longmapsto \prod_{\{a,b\} \in \pi} B(\phi_a, \phi_b), \quad \pi \in \mathcal{M}\left(\frac{n+m}{2}\right).$$

Remark 3.2. When $n = dN$ and $U(N)$ is replaced by $SU(N)$, or when N is even, $n = \frac{N}{2}$ and $O(N)$ is replaced by $SO(N)$, the Theorem 3.3 does not hold anymore, invariants associated to the determinant have to be added (see [3,19]).

The first part of the Theorem can be proved with the following remarks. Let us consider $w \in (V^{\otimes n} \otimes V^{*\otimes m})^G$. When $n + m$ is odd, then $-\text{Id} \in G$ and $w = -w$. When $G = U(N)$ and $n \neq m$, $z \text{Id} \in U(N)$ for any $z \in U(1)$, and $w = z^{n-m}w$.

In the latter theorem, when the invariant space is non trivial, the vector space $V^{\otimes n} \otimes V^{*\otimes m}$ can be identified with a space of endomorphisms, and invariant vectors with endomorphisms commuting with G . Indeed, when G is $O(N)$ or $Sp(N)$, the map

$$\begin{aligned} \theta : V &\longrightarrow V^*, \\ v &\longmapsto B(v, \cdot) \end{aligned}$$

is a linear isomorphism of representation, and if $n + m$ is even, $V^{\otimes n} \otimes V^{*\otimes m}$ is isomorphic to $\text{End}(V^{\otimes \frac{n+m}{2}})$. For any vector space W endowed with an action of G , let us denote the commutant of G in $\text{End}(W)$ by

$$\text{End}_G(W) = \{f \in \text{End}(W) : g.f(w) = f(g.w), \text{ for any } w \in W, g \in G\}.$$

Using the application $\theta : V \rightarrow V^*$, the first theorem of invariant theory yields a spanning family of the vector space $\text{End}_G(V^{\otimes \frac{n+m}{2}})$. These statements are known as *Schur–Weyl dualities*. The latter space is furthermore an algebra that we shall describe in the next paragraph. For the symplectic groups, we need there to fix an appropriate sign for each invariant. A natural one is given for invariants in $(V^{\otimes 2n})^{\text{Sp}(N)}$. Let us therefore spell out first the statement of Theorem 3.1 for $(V^{\otimes 2n})^{O(N)}$, $(V^{\otimes n} \otimes \overline{V}^{\otimes n})^{U(N)}$ and $(V^{\otimes 2n})^{Sp(N)}$.

Indexing invariants by $\mathfrak{S}_{2n}/\mathcal{H}_n$: Let $(e_i)_{1 \leq i \leq N}$ be the canonical basis of V . Two combinatorial sets allow to describe the latter invariants: the set $\mathcal{M}(n)$ of matchings of $\{1, \dots, 2n\}$ and its subset $\mathcal{M}_+(n)$, consisting of elements matching $\{1, \dots, n\}$ with $\{n + 1, \dots, 2n\}$. Let us set $\pi_0 = \{\{k, 2n + 1 - k\}, k \in \{1, \dots, n\}\} \in \mathcal{M}_+(n)$. The following element is a G -invariant

$$w_{\pi_0} = \sum_{1 \leq i_1, \dots, i_{2n} \leq N} \prod_{k=1}^n B(e_k, e_{2n+1-k}) e_{i_1} \otimes \dots \otimes e_{i_{2n}}.$$

Theorem 3.1 states that all invariants are obtained by permutations of the tensors. Let us highlight a choice of equivalence class representatives. Both $\mathcal{M}(n)$ and $V^{\otimes 2n}$ (resp. $\mathcal{M}_+(n)$ and $V^{\otimes n} \otimes \overline{V}^{\otimes n}$) are endowed with a left action of the permutation group \mathfrak{S}_{2n} (resp. $\mathfrak{S}_n \times \mathfrak{S}_n$). Let us recall that the second one is defined by setting for any $\sigma \in \mathfrak{S}_{2n}$, and $v \in V^{\otimes 2n}$,

$$\sigma.v = v_{\sigma^{-1}(1)} \otimes \dots \otimes v_{\sigma^{-1}(2n)}. \tag{3}$$

Let \mathcal{H}_n (resp. \mathcal{D}_n) be the stabiliser of π_0 in \mathfrak{S}_{2n} (resp. $\mathfrak{S}_n \times \mathfrak{S}_n$), so that $\mathcal{M}(n) = \mathfrak{S}_{2n}/\mathcal{H}_n$ (resp. $\mathcal{M}_+(n) = \mathfrak{S}_n \times \mathfrak{S}_n/\mathcal{D}_n$). It is isomorphic to the *hyperoctahedral group* (resp. \mathfrak{S}_n). What is more, for any $h \in \mathcal{H}_n$, $h.w_{\pi_0} = w_{\pi_0}$, when G is $U(N)$ or $O(N)$, whereas $h.w_{\pi_0} = \varepsilon(h)w_{\pi_0}$, when $G = \text{Sp}(N)$, where $\varepsilon : \mathfrak{S}_{2n} \rightarrow \{-1, 1\}$ denotes the signature morphism. This consideration shows that the following elements are well defined: for any $\sigma \in \mathfrak{S}_{2n}$, if $G = O(N)$, or $G = U(N)$ and $\sigma \in \mathfrak{S}_n \times \mathfrak{S}_n$, let us set

$$w_{\sigma.\pi_0} = \sigma.w_{\pi_0}, \tag{4}$$

and if $G = \text{Sp}(N)$, set

$$w_{\sigma.\pi_0} = \varepsilon(\sigma)\sigma.w_{\pi_0}. \tag{5}$$

Brauer algebras: Let \mathcal{B}_n be the vector space of functions on $\mathcal{M}(n)$. We write $\rho : \mathfrak{S}_n \rightarrow \text{End}(V^{\otimes n})$ the representation of the symmetric group given by permutation of tensors, as defined in (3), and $\rho_+, \rho_- : \mathcal{B}_n \rightarrow \text{End}(V^{\otimes n})$ the

linear maps such that

$$\rho_\varepsilon(\pi) = \text{Id}^{\otimes n} \otimes \theta^{\otimes n}(w_\pi),$$

where we have identified $V^{\otimes n} \otimes V^{*\otimes n}$ with $\text{End}(V^{\otimes n})$ and the right-hand side is resp. given by (4) and (5), when the symbol ε is resp. $+$ and $-$. For $n, m \in \mathbb{N}^*$, let us define $s : \{1, \dots, 2(n + m)\} \rightarrow \{-1, 1\}$, with $s(k) = -1$ if $n < k \leq n + 2m$ and 1 otherwise. We shall also consider the vector space $\mathcal{B}_{n,m}$ with basis indexed by matchings π of $\{1, \dots, 2(n + m)\}$ with $s(a) \neq s(b)$ for any block $\{a, b\}$ of π . The Schur–Weyl duality theorem is an easy consequence of Theorem 3.1, we leave this proof as an exercise for the Reader (see for example Section 10 of [10]).

Theorem 3.3 (Schur–Weyl duality). *The action of G on $V^{\otimes n}$ has the following commutant:*

$$\text{End}_G(V^{\otimes n}) = \begin{cases} \rho(\mathbb{C}[\mathfrak{S}_n]), & \text{if } G = \text{U}(N), \\ \rho_+(\mathcal{B}_n), & \text{if } G = \text{O}(N), \\ \rho_-(\mathcal{B}_n), & \text{if } G = \text{Sp}(N). \end{cases}$$

Furthermore, for $n, m \in \mathbb{N}^*$,

$$\text{End}_{\text{U}(N)}(V^{\otimes n} \otimes \overline{V}^{\otimes m}) = \rho_+(\mathcal{B}_{n,m}).$$

A nice property of the applications ρ, ρ_+ and ρ_- is that there exists structures of algebra on \mathcal{B}_n , such that they are algebra morphism. Let us fix a complex number $\zeta \in \mathbb{C}$. Let us represent matchings in $\mathcal{M}(n)$ as simple curves in the plane with endpoints $(-\frac{n-1}{2}, \frac{1}{2}), (-\frac{n-1}{2} + 1, \frac{1}{2}), \dots, (\frac{n-1}{2}, \frac{1}{2})$ and $(\frac{n-1}{2}, -\frac{1}{2}), (\frac{n-1}{2} - 1, -\frac{1}{2}), \dots, (-\frac{n-1}{2}, -\frac{1}{2})$, by labelling endpoints in this order from 1 to $2n$. Given two matchings π and $\nu \in \mathcal{M}(n)$, the concatenation of the curves of π translated by $(0, 1)$ with the curves of ν yields curves with endpoints $(-\frac{n-1}{2}, \frac{3}{2}), (-\frac{n-1}{2} + 1, \frac{3}{2}), \dots, (\frac{n-1}{2}, \frac{3}{2}), (\frac{n-1}{2}, -\frac{1}{2}), \dots, (-\frac{n-1}{2}, -\frac{1}{2})$, which represents a matching $\pi \circ \nu \in \mathcal{M}(n)$, and $b(\pi, \nu)$ loops. We set³

$$\pi \cdot \nu = \zeta^{b(\pi, \nu)} \pi \circ \nu.$$

This is a simple exercise to check that the extension of this operation by linearity defines a multiplication on \mathcal{B}_n . This algebra is denoted by $\mathcal{B}_n(\zeta)$ and called the *Brauer algebra* (see [3] or [10], Section 10.1). For any $n, m \in \mathbb{N}^*$, a matching of $\mathcal{M}(2(n + m))$ belongs to $\mathcal{B}_{n,m}$ if and only if it can be represented as a collection of simple curves that cross exactly once the lines $\mathbb{R} \times \{0\}$ or $\{\frac{n-m}{2}\} \times \mathbb{R}$ (see Figure 2). Let us remark that for all $x, y \in \mathcal{B}_{n,m}$, their product in $\mathcal{B}_{n+m}(\zeta)$ satisfies $x \cdot y \in \mathcal{B}_{n,m}$. Therefore, this operation defines an algebra structure, we then denote this algebra by $\mathcal{B}_{n,m}(\zeta)$. It is called the *walled Brauer algebra* (it has been originally considered in [13,21], see also [11]). Note that the element π_0 is the identity of these algebras, we shall denote it alternatively by 1.

³See Figure 1 for an example.

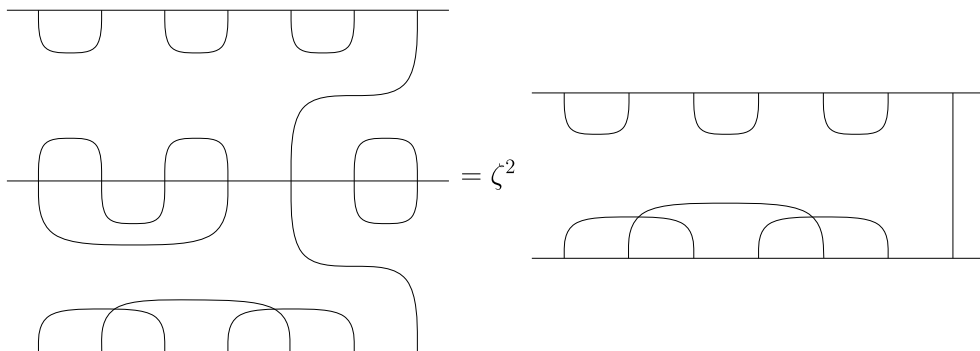


Fig. 1. Multiplication $\pi \cdot \eta$, with $\pi = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 10\}, \{8, 9\}, \{11, 12\}, \{13, 14\}\}$ and $\eta = \{\{1, 4\}, \{2, 3\}, \{5, 8\}, \{6, 7\}, \{9, 11\}, \{10, 13\}, \{12, 14\}\}$.

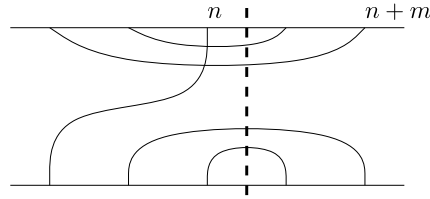


Fig. 2. An element of $\mathcal{B}_{n,m}(\zeta)$.

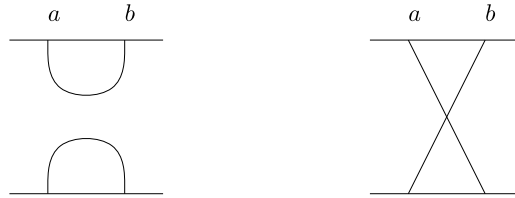


Fig. 3. The two elements $\langle a \ b \rangle$ and $\langle a \ b \rangle$ of $\mathcal{B}_n(\zeta)$.

Lemma 3.4. For any $n, m \in \mathbb{N}^*$, the applications $\rho_+ : \mathcal{B}_n(N) \rightarrow \text{End}_{\text{O}(N)}(V^{\otimes n})$, $\rho_- : \mathcal{B}_n(-N) \rightarrow \text{End}_{\text{Sp}(N)}(V^{\otimes n})$ and $\rho_+ : \mathcal{B}_{n,m}(N) \rightarrow \text{End}_{\text{U}(N)}(V^{\otimes n} \otimes \bar{V}^{\otimes m})$ are onto algebra morphisms.

We shall leave the proof of this fact as an easy exercise for the unitary and orthogonal case and give one in the Appendix for the symplectic case, as we have not found a direct proof in the literature. For any $1 \leq i, j \leq \dim(V)$, let $E_{i,j}$ denote the endomorphism of $\text{End}(V)$, such that $E_{i,j}(e_k) = \delta_{j,k}e_i$, for any $1 \leq k \leq \dim(V)$. Then, for any $\pi \in \mathcal{M}(n)$,

$$\rho_+(\pi) = \sum_{1 \leq i_1, \dots, i_{2n} \leq N} \left(\prod_{\{a,b\} \in \pi} \delta_{i_a, i_b} \right) E_{i_1, i_{2n}} \otimes E_{i_2, i_{2n-1}} \otimes \dots \otimes E_{i_n, i_{n+1}}, \tag{6}$$

whereas if $\pi = \sigma \cdot \pi_0$, with $\sigma \in \mathfrak{S}_{2n}$,

$$\rho_-(\pi) = \sum_{1 \leq i_1, \dots, i_{2n} \leq 2N} \varepsilon(\sigma) \left(\prod_{\{a,b\} \in \pi_0} J_{i_a, i_b} \right) E_{i_{\sigma^{-1}(1)}, i_{\sigma^{-1}(2n)}} \otimes \dots \otimes E_{i_{\sigma^{-1}(n)}, i_{\sigma^{-1}(n+1)}} J^{-1 \otimes n}. \tag{7}$$

To close this section, let us highlight two kinds of matchings that will play thereafter a prominent role. For any integers $1 \leq a < b \leq n$, we set

$$\langle a \ b \rangle = \{ \{a, b\}, \{2n+1-a, 2n+1-b\} \} \cup \{ \{k, 2n+1-k\}, 1 \leq k \leq n, k \notin \{a, b\} \}$$

and the usual transposition is identified as

$$\langle a \ b \rangle = \{ \{a, 2n+1-b\}, \{b, 2n+1-a\} \} \cup \{ \{k, 2n+1-k\}, 1 \leq k \leq n, k \notin \{a, b\} \},$$

see Figure 3. When $n = 2$, they have the following linear representations on $V^{\otimes 2}$,

$$\rho_+(\langle 1 \ 2 \rangle) = \sum_{1 \leq i, j \leq N} E_{i,j} \otimes E_{i,j} \quad \text{and} \quad \rho_+(\langle 1 \ 2 \rangle) = \sum_{1 \leq i, j \leq N} E_{i,j} \otimes E_{j,i}, \tag{8}$$

whereas

$$\rho_-(\langle 1 \ 2 \rangle) = - \sum_{1 \leq a, b, c, d \leq N} J_{a,c} J_{b,d} E_{a,b} \otimes E_{c,d} \quad \text{and} \quad \rho_-(\langle 1 \ 2 \rangle) = - \sum_{1 \leq i, j \leq N} E_{i,j} \otimes E_{j,i}. \tag{9}$$

For $n \geq 2$ and any $1 \leq a < b \leq n$, $\rho_\varepsilon(\langle a \ b \rangle)$ acts like $\rho_\varepsilon(\langle 1 \ 2 \rangle)$ on the a th and b th factors of $V^{\otimes n}$ and trivially elsewhere. Let us lastly remark that the family $\{\langle a \ b \rangle, \langle a \ b \rangle : 1 \leq a < b \leq n\}$ generates $\mathcal{B}_n(\zeta)$ as an algebra.

4. Integration formulas for the Brownian motion

We shall here recall the result of [15] about integration against a Brownian motion and give a slight extension of it for the unitary case. Starting from these formulas, we show that any moment (possibly mixed) for any series can be expressed in terms of non-mixed moments for the unitary groups. The constant (in time) part of this expression corresponds to the integration against the Haar measure. We then check that these expressions do match the one of [7].

Proposition 4.1 ([15], Propositions 2.2, 2.6 and 2.8). *For any $t \geq 0$, $\mathbb{E}[G_t^{\otimes n}] = e^{t\Delta_G(n)}$, where*

$$-\Delta_G(n) = \begin{cases} \rho(\frac{n}{2} + \frac{1}{N} \sum_{1 \leq i < j \leq n} \langle i \ j \rangle), & \text{if } G = \text{U}(N), \\ \rho_\varepsilon(\frac{n}{2}(1 - \frac{1}{\varepsilon N}) + \frac{1}{\varepsilon N} \sum_{1 \leq i < j \leq n} (\langle i \ j \rangle - \langle i \ j \rangle)), & \text{if } G = \text{O}(N) \text{ or } \text{Sp}(N). \end{cases}$$

Our notations being different from the one of [15], we recall for completeness a proof for the symplectic case in the [Appendix](#).

Lemma 4.2. *For $G = \text{U}(N)$ and any $n, m \in \mathbb{N}^*$, $\mathbb{E}[G_t^{\otimes n} \otimes \overline{G}_t^{\otimes m}] = e^{t\Delta_U(n,m)}$, with*

$$-\Delta_U(n, m) = \rho_+ \left(\frac{n}{2} + \frac{1}{N} \sum_{1 \leq i < j \leq n+m} (\mathbf{1}_{\{i, j \leq n \text{ or } i, j > n\}} \langle i \ j \rangle - \mathbf{1}_{\{i \leq n < j\}} \langle i \ j \rangle) \right).$$

Proof. Let us recall that for any finite dimensional representation (V_f, f) , setting for all $x \in \mathfrak{g}$, $f(x) = \mathcal{L}_x(f)(\text{Id})$, for any $a, b \in \mathfrak{g}$, $\mathcal{L}_a \circ \mathcal{L}_b f(\text{Id}) = f(a)f(b) \in \text{End}(V_f)$. For example, if $(x_i)_{1 \leq i \leq \dim(\mathfrak{g})}$ is an orthogonal basis of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$, $\Delta_G(f)(\text{Id}) = \sum_{i=1}^d f(x_i)^2 = f(c_{\mathfrak{g}})$. Therefore, for any $t \geq 0$,

$$\frac{d}{dt} \mathbb{E}[f(G_t)] = \frac{1}{2} \mathbb{E}[\Delta_G(f)(G_t)] = \frac{1}{2} \mathbb{E}[f(G_t)] \Delta_G(f) \in \text{End}(V_f),$$

and

$$\mathbb{E}[f(G_t)] = \exp\left(\frac{t}{2} f(c_{\mathfrak{g}})\right).$$

We need here to consider the representation $(V^n \otimes V^{*\otimes m}, \rho_{n,m})$, identifying V^* with \overline{V} . Let us choose the orthonormal basis $\{\frac{1}{\sqrt{2N}}(E_{k,l} - E_{l,k}), \frac{i}{\sqrt{2N}}(E_{k,l} + E_{l,k}) : 1 \leq k < l \leq N\} \cup \{\frac{i}{\sqrt{N}}E_{k,k} : 1 \leq k \leq N\}$ of $(\mathfrak{u}(N), \langle \cdot, \cdot \rangle)$ and compute the Casimir $c_{\mathfrak{u}(N)}$ as an element of the real algebra $\mathcal{U}(\mathfrak{gl}_N(\mathbb{C}))$:

$$2Nc_{\mathfrak{u}(N)} = \sum_{1 \leq k, l \leq N} (1 + i \otimes i) E_{k,l} \otimes E_{k,l} + (i \otimes i - 1) E_{k,l} \otimes E_{l,k} \in \mathcal{U}(\mathfrak{gl}_N(\mathbb{C})).$$

Considering $f = \rho_{n,m}$ as a representation of the real enveloping algebra $\mathcal{U}(\mathfrak{gl}_N(\mathbb{C}))$ implies that

$$2N\rho_{n,m}(c_{\mathfrak{u}(N)}) = (n + m)N \text{Id}_{V^{\otimes n+m}} + 2 \sum_{1 \leq a < b \leq n+m} \langle a \ b \rangle - 2 \sum_{1 \leq a < b \leq n, \text{ or } n < a < b \leq n+m} \langle a \ b \rangle. \quad \square$$

Note that at first sight, these combinatorial expressions do not shed any light about the integration against the Haar measure. We give in what follows a reformulation of the latter that covers up this point. Our formula uses the following notations.

Partial matchings: We shall consider the sets $\mathcal{PM}(n)$ of partitions of $\{1, \dots, n\}$, with blocks of size 1 and 2. For $n, m \in \mathbb{N}^*$, we denote by $\mathcal{PM}(n, m)$ the subset of $\mathcal{PM}(n + m)$ of elements without 2-blocks connecting $\{1, \dots, n\}$ with itself. For any Π in $\mathcal{PM}(n)$, we denote by $|\Pi|$ its number of 2-blocks.

Matched tensors: For $a, b, c \in \{1, \dots, n\}$, with $b < c$, let us define two linear maps from respectively $\text{End}(V)$ and $\text{End}(V)^{\otimes 2}$ towards $\text{End}(V^{\otimes n})$, by setting for any $A, B, C \in \text{End}(V)$,

$$(A)_a = \text{Id}^{\otimes a-1} \otimes A \otimes \text{Id}^{\otimes n-a}$$

and

$$(B \otimes C)_{b,c} = \text{Id}^{\otimes b-1} \otimes B \otimes \text{Id}^{\otimes c-b-1} \otimes C \otimes \text{Id}^{\otimes n-c}.$$

Given a partial matching $\pi \in \mathcal{M}(n)$, let us define a map from $\text{End}(V) \times \text{End}(V)^{\otimes 2}$ to $\text{End}(V^{\otimes n})$, setting for any $A \in \text{End}(V)$ and $T \in \text{End}(V)^{\otimes 2}$,

$$(A, T)^{\otimes \pi} = \prod_{\{i\} \in \pi} (A)_i. \prod_{\{a,b\} \in \pi: a \neq b} (T)_{a,b} \in \text{End}(V^{\otimes n}).$$

Given a partial matching $\pi \in \mathcal{PM}(n, m)$, $A, B \in \text{End}(V)$ and $T \in \text{End}(V)^{\otimes 2}$, we set

$$((A, B), T)^{\otimes \pi} = \prod_{\{i\} \in \pi: i \leq n} (A)_i. \prod_{\{i\} \in \pi: i > n} (B)_i. \prod_{\{a,b\} \in \pi: a \neq b} (T)_{a,b} \in \text{End}(V^{\otimes n}).$$

Let $(U_t)_{t \geq 0}$ and $(V_t)_{t \geq 0}$ be two independent $U(N)$ -Brownian motions.

Theorem 4.3. For any $t \geq 0$, when G is $O(N)$ or $\text{Sp}(N)$,

$$\mathbb{E}[G_t^{\otimes n}] = \sum_{\pi \in \mathcal{PM}(n)} \mathbb{E} \left[\left(e^{\frac{\varepsilon t}{2N}} U_t, \int_0^t e^{\frac{\varepsilon s}{N}} U_s^{\otimes 2} ds \right)^{\otimes \pi} \right] \prod_{\{a,b\} \in \pi} \frac{\rho_\varepsilon(\langle a, b \rangle)}{\varepsilon N}.$$

When $G = U(N)$,

$$\mathbb{E}[U_t^{\otimes n} \otimes \bar{U}_t^{\otimes m}] = \sum_{\pi \in \mathcal{PM}(n, m)} \mathbb{E} \left[\left((U_t, V_t), \int_0^t U_s \otimes V_s ds \right)^{\otimes \pi} \right] \prod_{\{a,b\} \in \pi} \frac{\rho_+(\langle a, b \rangle)}{N}.$$

As we will see in the next section, these formulas easily yield an expression for the integration against the Haar measure. As $t \rightarrow \infty$, the only terms remaining are the one indexed by partial matchings with no block of size 1, except in the orthogonal case.

Proof. We shall consider the case $G \in \{O(N), \text{Sp}(N)\}$, the third one being very similar. We will argue that the function R on the right-hand side satisfies $\frac{d}{dt} R_t = R_t \Delta_G(n) \in \text{End}(V^{\otimes n})$ and $R_0 = \text{Id}$. According to Proposition 4.1, these equations are true for the left-hand side and this therefore yields the announced equality. First note that when $t = 0$, the matched tensors indexed by $\pi \in \mathcal{PM}(n)$ appearing in the right-hand side vanish if $|\pi| \neq 0$ and equal Id otherwise. Hence, the initial condition holds true. For any partial matching $\pi \in \mathcal{PM}(n)$, $\{a, b\} \in \pi$ and $\{c\}, \{d\} \in \pi$, with $c \neq d$, we shall denote respectively by $\hat{\pi}_{a,b} \in \mathcal{PM}(n)$ and $\check{\pi}^{c,d}$, the partial matchings with $\{a, b\}$, resp. $\{c\}, \{d\}$, replaced by $\{a\}, \{b\}$, resp. $\{c, d\}$. On the one hand, Itô's formula implies

$$\begin{aligned} & \frac{d}{dt} \mathbb{E} \left[\left(e^{\frac{\varepsilon t}{2N}} U_t, \int_0^t e^{\frac{\varepsilon s}{N}} U_s^{\otimes 2} ds \right)^{\otimes \pi} \right] \\ &= \sum_{\{a,b\} \in \pi} \mathbb{E} \left[\left(e^{\frac{\varepsilon t}{2N}} U_t, \int_0^t e^{\frac{\varepsilon s}{N}} U_s^{\otimes 2} ds \right)^{\otimes \hat{\pi}_{a,b}} \right] \\ & \quad - \mathbb{E} \left[\left(e^{\frac{\varepsilon t}{2N}} U_t, \int_0^t e^{\frac{\varepsilon s}{N}} U_s^{\otimes 2} ds \right)^{\otimes \pi} \right] \left(\frac{n}{2} \left(1 - \frac{\varepsilon}{N} \right) + \frac{1}{N} \sum_{\{a\}, \{b\} \in \pi: a < b} (a, b) \right). \end{aligned} \tag{10}$$

On the other hand, for any integer $1 \leq h \leq \frac{n}{2}$ and any $t \geq 0$, $\prod_{k=0}^{h-1} \langle n - 2k \ n - 2k - 1 \rangle \mathbb{E}[G_t^{\otimes n}] = \prod_{k=0}^{h-1} \langle n - 2k \ n - 2k - 1 \rangle \mathbb{E}[G_t^{\otimes n-2h} \otimes \text{Id}^{\otimes 2h}]$. Differentiating this equality yields

$$\prod_{k=0}^{h-1} \langle n - 2k \ n - 2k - 1 \rangle \Delta_G(n) = \prod_{k=0}^{h-1} \langle n - 2k \ n - 2k - 1 \rangle \Delta_G(n - 2h) \otimes \text{Id}^{\otimes 2h}, \tag{11}$$

whereas conjugating by elements of \mathfrak{S}_n and using the expressions of Proposition 4.1 implies that for any partial matching $\pi \in \mathcal{PM}(n)$,

$$\begin{aligned} \prod_{\{a,b\} \in \pi} \frac{\rho_\varepsilon(\langle a \ b \rangle)}{\varepsilon N} \Delta_G(n) &= \sum_{\{c\}, \{d\} \in \pi: c < d} \prod_{\{a,b\} \in \check{\pi}^{c,d}} \frac{\rho_\varepsilon(\langle a \ b \rangle)}{\varepsilon N} \\ &- \prod_{\{a,b\} \in \pi} \frac{\rho_\varepsilon(\langle a \ b \rangle)}{\varepsilon N} \rho_\varepsilon \left(\left(\frac{n}{2} - |\pi| \right) \left(1 - \frac{\varepsilon}{N} \right) + \frac{1}{N} \sum_{\{c\}, \{d\} \in \pi: c < d} (c \ d) \right). \end{aligned} \tag{12}$$

Let us denote for any partial matching $\pi \in \mathcal{PM}(n)$,

$$-\Delta(\pi) = \left(\frac{n}{2} - |\pi| \right) \left(1 - \frac{\varepsilon}{N} \right) + \frac{1}{N} \sum_{\{c\}, \{d\} \in \pi: c < d} (c \ d). \tag{13}$$

For any $\pi \in \mathcal{PM}(n)$, $\Delta(\pi)$ commutes with $\prod_{\{a,b\} \in \pi} \rho_\varepsilon(\langle a \ b \rangle)$, so that using (10) and (12) implies

$$\begin{aligned} \frac{d}{dt} R_t &= \sum_{\pi \in \mathcal{PM}(n)} \sum_{\{c,d\} \in \pi} \mathbb{E} \left[\left(e^{\frac{\varepsilon t}{2N}} U_t, \int_0^t e^{\frac{s\varepsilon}{N}} U_s^{\otimes 2} ds \right)^{\otimes \check{\pi}^{c,d}} \right] \prod_{\{a,b\} \in \pi} \frac{\rho_\varepsilon(\langle a \ b \rangle)}{\varepsilon N} \\ &- \sum_{\pi \in \mathcal{PM}(n)} \mathbb{E} \left[\left(e^{\frac{\varepsilon t}{2N}} U_t, \int_0^t e^{\frac{s\varepsilon}{N}} U_s^{\otimes 2} ds \right)^{\otimes \pi} \right] \prod_{\{a,b\} \in \pi} \frac{\rho_\varepsilon(\langle a \ b \rangle)}{\varepsilon N} \Delta(\pi). \end{aligned}$$

Permuting its two sums, the first line of the right-hand side can be rewritten as

$$\sum_{1 \leq c < d \leq n} \sum_{\substack{\pi \in \mathcal{PM}(n) \\ \{c\}, \{d\} \in \pi}} \mathbb{E} \left[\left(e^{\frac{\varepsilon t}{2N}} U_t, \int_0^t e^{\frac{s\varepsilon}{N}} U_s^{\otimes 2} ds \right)^{\otimes \pi} \right] \prod_{\{i,j\} \in \check{\pi}^{c,d}} \frac{\rho_\varepsilon(\langle i \ j \rangle)}{\varepsilon N}.$$

Permuting again these sums and using (12), we find that $\frac{d}{dt} R_t = R_t \Delta_G(n)$. □

The expectations on the right-hand side of the above formulas are given by the Proposition 4.1, yielding the Lemma 4.6 below. Let us define for any $1 \leq k \leq n$,

$$\Delta_\varepsilon^n(k) = \frac{k}{\varepsilon N} + \Delta_U(k) \otimes \text{Id}^{n-k} \in \text{End}(V^{\otimes n})$$

and for any $1 \leq k \leq \min(n, m)$,

$$\Delta_\varepsilon^{n,m}(\min(n, m) - k) = \Delta_U(n - k) \otimes \text{Id}^{\otimes n+m-k} + \text{Id}^{n+m-k} \otimes \Delta_U(m - k) \in \text{End}(V^n \otimes \overline{V}^{\otimes m}).$$

We set $\Delta_\varepsilon^n(0) = 0$ and $\Delta_\varepsilon^{n,m}(0) = 0$. We shall consider the set $\text{Sym}[X_0, \dots, X_a]$ of symmetric polynomials in $a + 1$ variables X_0, \dots, X_a . For any $d \geq 1$, we denote by $H_d(X_0, \dots, X_a) = \sum_{0 \leq i_1 \leq \dots \leq i_d \leq a} X_{i_1} \cdots X_{i_d} = \sum_{\lambda_0, \dots, \lambda_a \geq 0: \lambda_0 + \dots + \lambda_a = d} X_0^{\lambda_0} \cdots X_a^{\lambda_a} \in \text{Sym}[X_0, \dots, X_a]$ the complete symmetric polynomial of degree d with the convention $H_0 = 1$, and we define an analytic function valued in symmetric polynomials setting for all $t \in \mathbb{R}$,

$$\mathcal{E}_t^{a,H} = \sum_{d \geq 0} \frac{t^{d+a}}{(d+a)!} H_d(X_0, \dots, X_a).$$

When the context is clear, we shall drop the first upper index and denote this function by \mathcal{E}_t^H . Let us stress that these functions are not compatible: for all $a \geq 0$, $\mathcal{E}_t^{a+1,H}(X_0, \dots, X_a, 0) \neq \mathcal{E}_t^{a,H}(X_0, \dots, X_a)$. In return, they satisfy the following

Lemma 4.4. For all $t \in \mathbb{R}_+$,

$$\mathcal{E}_t^{a,H} = \sum_{b=0}^a \frac{e^{tX_b}}{\prod_{c \neq b} (X_b - X_c)} = \int_{\Delta_a(t)} e^{t_0 X_0 + \dots + t_a X_a} d\mathbf{t},$$

where $\Delta_a(t) = \{(t_0, \dots, t_a) \in \mathbb{R}_+^{a+1} : t_0 + \dots + t_a = t\}$ and $d\mathbf{t}$ denotes the Lebesgue measure on this simplex.

Proof. Indeed, one easily checks that these three functions are the unique solution to the following problem: $(S^a)_{a \geq 0}$ are analytic functions, such that $S_t^0(X_0) = e^{tX_0}$ and for any $a \geq 1$, $S^a \in \text{Sym}[X_0, \dots, X_a]$ satisfy for any $t \geq 0$,

$$\frac{d}{dt} S_t^a(X_0, \dots, X_a) = X_a S_t^a(X_0, \dots, X_a) + S_t^{a-1}(X_0, \dots, X_{a-1}). \quad (14)$$

□

Remark 4.5. Note that expanding the last column of the following, one can also rewrite

$$\mathcal{E}_t^H = \prod_{0 \leq i < j \leq a} (X_j - X_i)^{-1} \det \begin{pmatrix} e^{tX_0} & X_0^{a-1} & X_0^{a-2} & \dots & 1 \\ e^{tX_1} & X_1^{a-1} & X_1^{a-2} & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e^{tX_a} & X_a^{a-1} & X_a^{a-2} & \dots & 1 \end{pmatrix}.$$

For any $1 \leq a \leq \frac{n}{2}$ and $1 \leq a' \leq \min(n, m)$, let $\mu_a \in \mathcal{PM}(n)$ and $\nu_{a'} \in \mathcal{PM}(n, m)$ be the partial matchings with respective 2-blocks $\{n, n-1\}, \dots, \{n-2a+2, n-2a+1\}$ and $\{n, n+1\}, \dots, \{n-a'+1, n+a'\}$, whereas $\mu_0 \in \mathcal{PM}(n)$ and $\nu_0 \in \mathcal{PM}(n, m)$ are the partial matchings with only blocks of size 1. When π is a partial matching in respectively $\mathcal{PM}(n)$ and $\mathcal{PM}(n, n)$, we set $\mathcal{H}(\pi) = \{\sigma \in \mathfrak{S}_n : \sigma(\mu_{|\pi|}) = \pi\}$ and $\mathcal{D}(\pi) = \{\sigma \in \mathfrak{S}_n \times \mathfrak{S}_n : \sigma(\nu_{|\pi|}) = \pi\}$.

Lemma 4.6. The families $(\Delta_\varepsilon^n(k))_{1 \leq k \leq n}$ and $(\Delta^{n,m}(k))_{1 \leq k \leq \min(n,m)}$ are commutative. For any $t \geq 0$ and partial matching $\pi \in \mathcal{PM}(n)$,

$$\mathbb{E} \left[\left(e^{\frac{st}{2N}} U_t, \int_0^t e^{\frac{ss}{N}} U_s^{\otimes 2} ds \right)^{\otimes \pi} \right] = \frac{2^{-|\pi|}}{(n-2|\pi|)!} \sum_{h \in \mathcal{H}(\pi)} h \mathcal{E}_t^H(\Delta_\varepsilon^n(n), \Delta_\varepsilon^n(n-2), \dots, \Delta_\varepsilon^n(n-2|\pi|)) h^{-1}$$

and for any $\pi \in \mathcal{PM}(n, m)$,

$$\begin{aligned} & \mathbb{E} \left[\left((U_t, V_t), \int_0^t U_s \otimes V_s ds \right)^{\otimes \pi} \right] \\ &= \frac{1}{(n-|\pi|)!(m-|\pi|)!} \sum_{h \in \mathcal{D}(\pi)} h \mathcal{E}_t^H(\Delta^{n,m}(n), \Delta^{n,m}(n-1), \dots, \Delta^{n,m}(n-|\pi|)) h^{-1}. \end{aligned}$$

Proof. The first property can be seen directly or using the fact (ii) of Lemma 2.2 stating that the law of the Brownian motion is invariant by conjugation. Let us prove the formulas of the Lemma by induction on $|\pi|$. Again, we consider only the case $G \in \{\text{O}(N), \text{Sp}(N)\}$. When $|\pi| = 0$, according to Proposition 4.1, the left-hand side is $\mathbb{E}[(e^{\frac{st}{2N}} U_t)^{\otimes n}] = \exp[t(\Delta_U(n) + \frac{n\varepsilon}{2N})] = \exp(t\Delta_\varepsilon^n(n)) = \mathcal{E}_t^H(\Delta_\varepsilon^n(n))$. For any $\mu \in \mathcal{PM}(n)$, $t \geq 0$, let us write $R_t(\pi)$ for the right-hand

side of the formula. Let us use once more the notation $\Delta(\pi) \in \text{End}(V^{\otimes n})$ that has been defined in (13). Note that for any $h \in \mathcal{H}(\pi)$, $\Delta(\pi) = h\Delta_\varepsilon^n(n - 2|\pi|)$. Using (14), we get

$$\frac{d}{dt}R_t(\pi) = R_t(\pi)\Delta(\pi) + \sum_{\{a,b\} \in \pi} R_t(\hat{\pi}_{a,b}),$$

where, as above, $\hat{\pi}_{a,b}$ denotes the matching with $\{a, b\}$ replaced by $\{a\}, \{b\}$, when $\{a, b\} \in \pi$.

Let us recall that according to the equation (10), the family $\{\mathbb{E}[(e^{\frac{\varepsilon t}{2N}}U_t, \int_0^t e^{\frac{\varepsilon s}{N}}U_s^{\otimes 2}ds)^{\otimes \pi}] : \pi \in \mathcal{PM}(n)\}$ satisfies the same system of ordinary differential equations. Using the induction assumption yields then the announced equality. \square

We want now to deduce from the Theorem 4.3 an explicit expression for the expectations of monomials in entries of G_t in terms of some functions on $\mathcal{B}_n(\varepsilon N)$ or $\mathcal{B}_{n,m}(N)$.

Orbits under \mathfrak{S}_n left and right actions on \mathcal{B}_n : Let us consider the action of $\mathfrak{S}_n \times \mathfrak{S}_n$ on $\mathcal{M}(n)$ given for any $\alpha, \beta \in \mathfrak{S}_n$ and $\pi \in \mathcal{M}(n)$ by $(\alpha, \beta).\pi = \alpha\pi\beta^{-1}$. For any element $\pi \in \mathcal{B}_n$, let us write $h(\pi)$ for half the number of horizontal curves of the diagram of π . Two matchings $\mu, \nu \in \mathcal{M}(n)$ are in the same orbit if and only if $h(\mu) = h(\nu)$. Let us highlight representatives of these orbits setting for any integer $1 \leq a \leq \frac{n}{2}$,

$$\pi_a = \prod_{k=0}^{a-1} \langle n - 2k - 1n - 2k \rangle.$$

Let us recall that π_0 also denotes the identity of $\mathcal{B}_n(z)$. Similarly $(\mathfrak{S}_n \times \mathfrak{S}_m)^{\times 2}$ acts on $\mathcal{M}(n, m)$, the function h characterizes its orbits and we set for any $1 \leq a \leq \min(n, m)$,

$$\tilde{\pi}_a = \prod_{k=0}^{a-1} \langle n - kn + k + 1 \rangle.$$

Deformed Weingarten function: Let us fix a notation for the sum of permutation like elements appearing in Proposition 4.1, setting for any integer $l \in \{1, \dots, n\}$,

$$-Z_l = \frac{l(1 - \varepsilon/N)}{2} + \frac{1}{N} \sum_{1 \leq i < j \leq l} (i j) \in \mathbb{C}[\mathfrak{S}_n]$$

and $Z_0 = 0$. We shall also set for $l \leq \min(n, m)$,

$$-Y_{\min(n,m)-l} = \min(n, m) - l + \frac{1}{N} \sum_{1 \leq i < j \leq n-l} (i j) + \frac{1}{N} \sum_{n+l \leq i < j \leq n+m} (i j) \in \mathbb{C}[\mathfrak{S}_n \times \mathfrak{S}_m].$$

Notice that these two families of operators are commutative. We set for all $t \geq 0$, and all integer $0 \leq a \leq \frac{n}{2}$,

$$W_t^a = \frac{1}{(2\varepsilon N)^a(n - 2a)!} \mathcal{E}_t^H(Z_n, Z_{n-2}, \dots, Z_{n-2a})$$

and for $a \leq \min(n, m)$,

$$W_t^a = \frac{1}{N^a(n - a)!(m - a)!} \mathcal{E}_t^H(Y_n, Y_{n-1}, \dots, Y_{n-a}).$$

Kernel and sign of indices functions: For any $I = (i_k)_{1 \leq k \leq 2n} \in \{1, \dots, N\}^{2n}$, let $\ker_1(I)$ (or simply $\ker(I)$) and $\ker_{-1}(I)$ be the set of matchings $\pi \in \mathcal{M}(n)$, such that for all blocks $\{a, b\}$ of π , we have respectively $i_a = i_b$ and $i_a = i_b + \frac{N}{2} \pmod{N}$. For any permutation $\sigma \in \mathfrak{S}_{2n}$, let us consider

$$\text{inv}_2(\sigma, I) = \#\{k \in \{1, \dots, p\} : i_{\sigma(2n-k+1)} < i_{\sigma(k)}\}.$$

If $\pi \in \ker_{-1}(I)$, the quantity $\varepsilon(\sigma)(-1)^{\text{inv}_2(\sigma^{-1}, I)}$ does not depend on the choice of $\sigma \in \mathfrak{S}_{2n}$ such that $\sigma(\pi_0) = \pi$. For any indices $(i_k)_{1 \leq k \leq 2n} \in \{1, \dots, N\}^{2n}$, let us define $I^\pi = (i'_k)_{1 \leq k \leq 2n} \in \{1, \dots, N\}^{2n}$ setting $i'_k = i_k + \frac{N}{2} \pmod{N}$, for all $n < k \leq 2n$ that is matched by π with an element of $\{1, \dots, n\}$, and $i'_k = i_k$ otherwise. We then consider $\ker_{-1}(\pi) = \{I^\pi : I \in \ker_{-1}(\pi)\}$ and set for any $I \in \ker_{-1}(\pi)$,

$$\varepsilon_I(\pi) = \varepsilon(\sigma)(-1)^{\text{inv}_2(\sigma, I^\pi)},$$

for any $\sigma \in \mathfrak{S}_{2n}$ such that $\sigma(\pi_0) = \pi$. Let us also denote by $\langle \cdot, \cdot \rangle$ the inner product on $V^{\otimes n}$ such that for all $v_1, \dots, v_n \in V$,

$$\|v_1 \otimes \dots \otimes v_n\|^2 = \|v_1\|^2 \dots \|v_n\|^2. \tag{15}$$

Formulas (6) and (7) show that the above definitions are tailored so that for any $\pi \in \mathcal{M}(n)$ and $I \in \{1, \dots, N\}^{2n}$,

$$\langle e_{i_1} \otimes \dots \otimes e_{i_n}, \rho_\varepsilon(\pi) e_{i_{2n}} \otimes \dots \otimes e_{i_{n+1}} \rangle = \varepsilon_I(\pi)^{(1-\varepsilon)/2},$$

if $I \in \ker_\varepsilon(\pi)$ and 0 otherwise. We can now give an equivalent statement to Theorem 4.3.

Theorem 4.7. For any $I = (i_k)_{1 \leq k \leq 2n} \in \{1, \dots, N\}^{2n}$,

$$\mathbb{E}[(G_t)_{i_1, i_{2n}} \dots (G_t)_{i_{n-1}, i_{n+2}} (G_t)_{i_n, i_{n+1}}] = \sum_{\mu \in \ker_\varepsilon(I)} \varepsilon_I(\pi)^{(1-\varepsilon)/2} \sum_{\alpha, \beta \in \mathfrak{S}_n : \alpha \pi_n(\mu) \beta = \mu} W_t^{h(\mu)}(\beta \alpha),$$

if $G = O(N)$ or $Sp(N)$. When $G = U(N)$,

$$\begin{aligned} & \mathbb{E}[(U_t)_{i_1, i_{2(n+m)}} \dots (U_t)_{i_n, i_{n+2m+1}} (\bar{U}_t)_{i_{n+1}, i_{n+2m}} \dots (\bar{U}_t)_{i_{n+m}, i_{n+m+1}}] \\ &= \sum_{\mu \in \ker(I)} \sum_{\alpha, \beta \in \mathfrak{S}_n \times \mathfrak{S}_m : \alpha \tilde{\pi}_h(\mu) \beta = \mu} \mathcal{W}_t^{h(\mu)}(\beta \alpha). \end{aligned}$$

Proof. This is a direct application of Theorem 4.3 together with Lemma 4.6. □

Remark 4.8. For any $\zeta \in \mathbb{C}$, $\alpha \in \{0, 1\}$, let us consider

$$-\Delta_\zeta^\alpha(n) = \frac{n}{2}(\zeta - 1) + \sum_{1 \leq i < j \leq n} ((i \ j) - \alpha \langle i \ j \rangle) \in \mathcal{B}_n(\zeta)$$

and

$$-\Delta_\zeta^\alpha(n, m) = \frac{n\zeta}{2} + \sum_{1 \leq i < j \leq n+m} (\mathbf{1}_{\{i, j \leq n \text{ or } i, j > n\}}(i \ j) - \alpha \mathbf{1}_{\{i \leq n < j\}} \langle i \ j \rangle) \in \mathcal{B}_{n, m}(\zeta).$$

For any $n' \geq n$, $m' \geq m$, let us consider the embedding of the algebras $\mathcal{B}_n(\zeta)$, $\mathcal{B}_{n, m}(\zeta)$, respectively into $\mathcal{B}_{n'}(\zeta)$ and $\mathcal{B}_{n', m'}(\zeta)$, such that any diagram $\pi \in \mathcal{B}_n$ (resp. $\mathcal{B}_{n, m}$) is sent to a diagram in resp. $\mathcal{B}_{n'}$ and $\mathcal{B}_{n', m'}$, having vertical lines going through points labeled by $\{n+1, \dots, n'\}$ (resp. $\{n+1, \dots, n'\} \times \{n'+1, \dots, n'+m'-m\}$). Note that

$$\langle n+1n+2 \rangle \Delta_\zeta^1(n+2) = \langle n+1n+2 \rangle \Delta_\zeta^1(n) \in \mathcal{B}_{n+1}(\zeta)$$

and

$$\langle nn+1 \rangle \Delta_\zeta^1(n+1, m+1) = \langle nn+1 \rangle \Delta_\zeta^1(n, m) \in \mathcal{B}_{n+1, m+1}(\zeta).$$

These equalities are generalization of (11). Following the above proof, it can also be shown that for any $z, \zeta \in \mathbb{C}$,

$$\exp(z \Delta_\zeta^1(n)) = \sum_{\pi \in \mathcal{P}\mathcal{M}(n)} W_\zeta^\pi(\pi) \prod_{\{a, b\} \in \pi} \langle a \ b \rangle \in \mathcal{B}_n(\zeta)$$

and

$$\exp(z\Delta_\zeta^1(n, m)) = \sum_{\pi \in \mathcal{PM}(n, m)} \mathcal{W}_z^\zeta(\pi) \prod_{\{a, b\} \in \pi} \langle a \ b \rangle \in \mathcal{B}_{n, m}(\zeta),$$

where

$$\mathcal{W}_z^\zeta(\pi) = \frac{|\pi|!}{|\mathcal{H}(\pi)|} \sum_{h \in \mathcal{H}(\pi)} h \mathcal{E}_z^H(\Delta_\zeta^0(n), \Delta_\zeta^0(n-2), \dots, \Delta_\zeta^0(n-2|\pi|)) h^{-1} \in \mathbb{C}[\mathfrak{S}_n]$$

and $\mathcal{W}_z^\zeta(\pi)$ is

$$\frac{|\pi|!}{|\mathcal{D}(\pi)|} \sum_{h \in \mathcal{D}(\pi)} h \mathcal{E}_z^H(\Delta_\zeta^0(n, m), \Delta_\zeta^0(n-1, m-1), \dots, \Delta_\zeta^0(n-|\pi|, m-|\pi|)) h^{-1} \in \mathbb{C}[\mathfrak{S}_{n+m}].$$

5. Integration against the Haar measure and FFT

Let us refer here the Reader to [5–7,24] where he can find a classical approach to integration over classical groups. These approaches use orthogonality of irreducible characters together with the decomposition of the tensor space into irreducible representations or the first fundamental theorem of invariants. The latter is used in the following way (see [6,24]): by left-invariance of the Haar measure, the expectation of the representation of a Haar distributed random variable is equal to the projection on the invariant space. Besides, the FFT theorem gives a generating family of this space. To compute the projection, and thereby the mean of the representation, one way is therefore to find a pseudo inverse for the Gram matrix of this family.

Let us give here another approach considering the behavior of the expectation with respect to Brownian motions as $t \rightarrow \infty$. Let $(U_t)_{t \geq 0}, (V_t)_{t \geq 0}$ be two $U(N)$ -Brownian motions and H be a Haar distributed random variable on G . Let $\varepsilon_N = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \varepsilon(\sigma) \sigma$ and recall that $\langle \cdot, \cdot \rangle$ denotes the inner product on $V^{\otimes n}$ such that (15) holds.

Lemma 5.1. *The endomorphisms $(\Delta_\varepsilon^n(k))_{1 \leq k \leq n}, (\Delta^{n, m}(k))_{1 \leq k \leq \min(n, m)}$ are self-adjoint operators with nonpositive spectrum and are all invertible but $\Delta_1^n(N)$, that satisfies $\ker(\Delta_1^n(N)) = \text{Im}(\rho(\varepsilon_N))$. The following holds as $t \rightarrow \infty$. Let $\pi \in \mathcal{PM}(n)$ be a partial matching with $|\pi| < \frac{n}{2}$. If $n \neq 2|\pi| + N$ or $\varepsilon = -1$, then*

$$\mathbb{E} \left[\left(e^{\frac{\varepsilon t}{2N}} U_t, \int_0^t e^{\frac{\varepsilon s}{N}} U_s^{\otimes 2} ds \right)^{\otimes \pi} \right] \rightarrow 0.$$

If $n = 2|\pi| + N$, then for any $\sigma \in \mathfrak{S}_n$ sending the 2-blocks of π to $\{N+1, N+2\}, \dots, \{n-1, n\}$,

$$\mathbb{E} \left[\left(e^{\frac{t}{2N}} U_t, \int_0^t e^{\frac{s}{N}} U_s^{\otimes 2} ds \right)^{\otimes \pi} \right] \rho(1 - \sigma^{-1} \varepsilon_N \sigma) \rightarrow 0.$$

If $\pi \in \mathcal{PM}(n, m)$, with $|\pi| < \min(n, m)$, then

$$\mathbb{E} \left[\left((U_t, V_t), \int_0^t U_s \otimes V_s ds \right)^{\otimes \pi} \right] \rightarrow 0.$$

Proof. Any endomorphism $\rho_\varepsilon(\tau) \in \text{End}(V^{\otimes n})$, with $\tau \in \mathfrak{S}_n$ a transposition, is self-adjoint. In particular, all the operators considered are self-adjoint. For any $\lambda_0, \dots, \lambda_a \in (\mathbb{R}_-^*)^{a+1}$, $\mathcal{E}_t^H(\lambda_0, \dots, \lambda_a) = \int_{\Delta_a(t)} \prod_{k=0}^a e^{\lambda_k k} dt \rightarrow 0$ and using Lemma 4.6 and 4.4, it is enough to prove that all the operators considered have a negative spectrum. Using the expression of Proposition 4.1 and bounding the operator norm of $\rho_\varepsilon(\tau)$ by 1 for any transposition τ yield that the operators $(\Delta_\varepsilon^n(k))_{k < N}, (\Delta^{n, m}(k))_{1 \leq k \leq N}$ and $\Delta_{-1}^n(N)$ have negative spectrum, whereas $\Delta_1^n(N)$ have non-positive spectrum. An elementary inspection leads then to $\ker(\Delta_1^n(N)) = \text{Im}(\rho(\varepsilon_N))$. Let $(G_t)_{t \geq 0}$ be a Brownian motion on $SU(N)$ and let $\Delta_{SU}(k) \in \text{End}(V^{\otimes k})$ be such that $\mathbb{E}[S_t^{\otimes k}] = \exp(t \Delta_{SU}(k))$. According to Lemma 2.3, $\Delta_U(k) =$

$\Delta_{\text{SU}}(k) - \frac{k^2}{2N^2}$. The operator $\Delta_{\text{SU}}(k)$ is self-adjoint and as $\mathbb{E}[S_t^{\otimes k}]$ is a contraction, it has nonpositive spectrum. Therefore, when $k > N$,

$$\Delta_\varepsilon^n(k) = -\frac{k^2 - \varepsilon k N}{2N^2} + \Delta_{\text{SU}}(k) \otimes \text{Id}^{n-k}$$

and setting $k' = \min(n, m) - k$,

$$\Delta_\varepsilon^{n,m}(k) = -\frac{(n-k')^2 + (m-k')^2}{2N^2} + \Delta_{\text{SU}}(n-k') \otimes \text{Id}^{\otimes n+m-k'} + \text{Id}^{n+m-k'} \otimes \Delta_{\text{SU}}(m-k')$$

have negative spectrum. □

Proof of Theorem 3.1. Let us recall from Section 3 that it is equivalent to prove that the vector spaces $(V^{\otimes 2n})^{\text{O}(N)}$, $(V^{\otimes 2n})^{\text{Sp}(N)}$ and $(V^{\otimes n} \otimes \overline{V}^{\otimes n})^{\text{U}(N)}$ are spanned respectively by $(w_\pi)_{\pi \in \mathcal{M}(n)}$, respectively for $\varepsilon \in \{1, -1\}$, and $(w_\pi)_{\pi \in \mathcal{M}_+(n)}$, as defined in (4) and (5). We denote in any each case by \mathcal{I}_G the linear span of the latter families. For any unitary representation (ρ, W) of G , denoting by H a Haar distributed random variable on G , $\mathbb{E}[\rho(H)]$ is equal to the Hermitian projection on W^G . Let us consider a G -Brownian motion $(G_t)_{t \geq 0}$, respectively issued from Id , when G is $\text{U}(N)$ or $\text{Sp}(N)$ and from a random variable S such that $\mathbb{E}[\det(S)] = 0$, when $G = \text{O}(N)$. When W is $V^{\otimes 2n}$ or $V^{\otimes n} \otimes \overline{V}^{\otimes n}$ and G is $\text{O}(N)$, $\text{Sp}(N)$ or $\text{U}(N)$, according to Theorem 4.3 and Lemma 5.1, for any $t \geq 0$, $\mathbb{E}[\rho(G_t)] = e^{t\Delta}$, where Δ is Hermitian and nonpositive and as $t \rightarrow \infty$, $\mathbb{E}[\rho(G_t)] \rightarrow P$, where $\text{Im } P \subset \mathcal{I}_G$. The endomorphism P is an Hermitian projection. Moreover, by left-invariance of the Haar measure, $P\mathbb{E}[\rho(H)] = \mathbb{E}[\rho(H)]$. Hence, $W^G \subset \mathcal{I}_G$ (and $P = \mathbb{E}[\rho(H)]$). □

Note that the latter argument gives an explicit way to prove that $\mathbb{E}[G_t^{\otimes n}] \rightarrow \mathbb{E}[H^{\otimes n}]$, as $t \rightarrow \infty$, where H is distributed according to the Haar measure on G , and $(G_t)_{t \geq 0}$ is a G -Brownian motion with a proper initial condition. It therefore leads to a new way to compute moments for the Haar measure. For any partial matching in respectively $\mathcal{PM}(n)$ and $\mathcal{PM}(n, n)$, let us set $\mathcal{H}(\pi) = \{\sigma \in \mathfrak{S}_n : \sigma(\mu_n) = \pi\}$ and $\mathcal{D}(\pi) = \{\sigma \in \mathfrak{S}_n \times \mathfrak{S}_n : \sigma(\nu_n) = \pi\}$.

Lemma 5.2. *Let us denote by H a Haar distributed random variable on G and by $(G_t)_{t \geq 0}$ a Brownian motion on G , respectively issued from Id , when G is $\text{Sp}(N)$ and from a random variable S such that $\mathbb{E}[\det(S)] = 0$, when $G = \text{O}(N)$. Then,*

$$\begin{aligned} \mathbb{E}[G_t^{\otimes n}] &\rightarrow \mathbb{E}[H^{\otimes n}] \\ &= \sum_{\pi \in \mathcal{PM}(n): 2|\pi|=n} \left(\frac{|\pi|!}{|\mathcal{H}(\pi)|} \sum_{h \in \mathcal{H}(\pi)} h(\Delta_\varepsilon^n(n) \Delta_\varepsilon^n(n-2) \cdots \Delta_\varepsilon^n(2))^{-1} h^{-1} \right) \prod_{\{a,b\} \in \pi} \frac{\rho_\varepsilon(\langle a \ b \rangle)}{-\varepsilon N}. \end{aligned}$$

When $G = \text{U}(N)$,

$$\begin{aligned} \mathbb{E}[U_t^{\otimes n} \otimes \overline{U}_t^{\otimes n}] &\rightarrow \mathbb{E}[H^{\otimes n} \otimes \overline{H}^{\otimes n}] \\ &= \sum_{\pi \in \mathcal{PM}(n,n): |\pi|=n} \left(\frac{|\pi|!}{|\mathcal{D}(\pi)|} \sum_{h \in \mathcal{D}(\pi)} h(\Delta^{n,n}(n) \Delta^{n,n}(n-1) \cdots \Delta^{n,n}(1))^{-1} h^{-1} \right) \prod_{\{a,b\} \in \pi} \frac{\rho_+(\langle a \ b \rangle)}{-N}. \end{aligned}$$

Proof. As argued in the above proof of Theorem 3.1, as $t \rightarrow \infty$, $\mathbb{E}[G_t^{\otimes n}] \rightarrow \mathbb{E}[H^{\otimes n}]$. Using Theorem 4.3, Lemma 4.6 and the first expression of Lemma 4.4 together with the first part of Lemma 5.1 yields the statement. □

The latter expressions can be simplified as follows. When W is a subgroup of \mathfrak{S}_n , we shall consider the idempotent $\Pi_W = \frac{1}{|W|} \sum_{w \in W} w$ and set $\Pi_W^{\varepsilon N} = \rho_\varepsilon(\Pi_W)$. Let us introduce for any integer $1 \leq a \leq n$,

$$X_a = \sum_{b < a} (ba) \in \mathbb{C}[\mathfrak{S}_n].$$

These elements form a commutative family and are called *Jucys–Murphy elements*. These elements enjoy a lot of properties (see [4,12,18]), we shall recall two of them (see Propositions 1 and 3 of [24] for a proof). For any integer $k \in \mathbb{N}^*$, let us consider the *lattice of partitions* of $\{1, \dots, k\}$ given by inclusion. For any pair of partitions π and ν , $\pi \vee \nu$ is the finest partition coarser than π and ν . When π is respectively a partition or a permutation, we denote by $\#\pi$ its *number of blocks* or *its number of cycle*. This function is related to Jucys–Murphy elements as follows: for any $\zeta \in \mathbb{C}$,

$$\prod_{k=1}^n (\zeta + X_{2k-1})\pi_0 = \sum_{\mu \in \mathcal{M}(n)} \zeta^{\#\mu \vee \mu_n} \mu \tag{16}$$

and

$$\prod_{k=1}^n (\zeta + X_k) = \sum_{\sigma \in \mathfrak{S}_n} \zeta^{\#\sigma} \sigma. \tag{17}$$

Note that these formulas imply that $\prod_{k=1}^n (\zeta + X_{2k-1})$ commutes with $\Pi_{\mathcal{H}_n}$ and that $\prod_{k=1}^n (\zeta + X_k)$ is in the center of $\mathbb{C}[\mathfrak{S}_n]$.

Lemma 5.3. *For any integer $n \geq 1$,*

$$(-\varepsilon N)^n n! \rho_\varepsilon \left(\prod_{1 \leq a \leq n} (\varepsilon N + X_{2a-1}) \right) \Pi_{\mathcal{H}_n}^{\varepsilon N} (\Delta_\varepsilon^{2n}(2n) \Delta_\varepsilon^{2n}(2n-2) \cdots \Delta_\varepsilon^{2n}(2))^{-1} \Pi_{\mathcal{H}_n}^{\varepsilon N} = \Pi_{\mathcal{H}_n}^{\varepsilon N}$$

and

$$(-N)^n n! \rho \left(\prod_{1 \leq a \leq n} (N + X_a) \right) \Pi_{\mathcal{D}_n}^N (\Delta^{n,n}(n) \Delta^{n,n}(n-1) \cdots \Delta^{n,n}(1))^{-1} \Pi_{\mathcal{D}_n}^N = \Pi_{\mathcal{D}_n}^N.$$

Proof. First notice that

$$\sum_{1 \leq a < b \leq 2n} (a \ b) \Pi_{\mathcal{H}_n} = \Pi_{\mathcal{H}_n} n(1 + X_{2n-1}) \Pi_{\mathcal{H}_n} \tag{*}$$

and

$$\sum_{\substack{1 \leq a < b \leq n \\ \text{or } n < a < b \leq 2n}} (a \ b) \Pi_{\mathcal{D}_n} = \Pi_{\mathcal{D}_n} n X_n \Pi_{\mathcal{D}_n}.$$

Indeed, on the right-hand side $\Pi_{\mathcal{H}_n} X_{2n-1} \Pi_{\mathcal{H}_n} = 2(n-1) \Pi_{\mathcal{H}_n} (2n-2 \ 2n-1) \Pi_{\mathcal{H}_n}$ and on the left-hand side, each transposition $(2k-1 \ 2k)$ with $1 \leq k \leq n$ belongs to \mathcal{H}_n , whereas the $2(n-1)$ others are conjugated to $(2n-2 \ 2n-1)$ by an element of \mathcal{H}_n . As $\sum_{1 \leq a < b \leq 2n} (a \ b)$ is central in \mathfrak{S}_{2n} the first formula holds true. A similar argument with respectively \mathcal{D}_n and $\mathfrak{S}_n \times \mathfrak{S}_n$ in place of \mathcal{H}_n and \mathfrak{S}_{2n} yields the second one. Let us set $\Pi_n = \Pi_{\mathcal{H}_n}^{\varepsilon N}$, and for any $1 \leq k \leq n$, $F_k = -\varepsilon N \Delta_\varepsilon^{2n}(2k)$ and $L_k = k \rho_\varepsilon(\varepsilon N + X_{2k-1})$ (resp. $\rho(\Pi_{\mathcal{D}_n}^N)$, $N \Delta^{n,n}(k)$ and $k \rho(N + X_k)$). We consider $\mathcal{R}_n = \prod_{k=1}^n F_k^{-1}$, and $\Omega_n = \prod_{k=1}^n L_k$. The two equalities above imply that

$$F_n = \Pi_n L_n \Pi_n, \tag{**}$$

for any integer $n \geq 1$. The claim is equivalent to the following equality

$$\Pi_n \Omega_n \mathcal{R}_n \Pi_n = \Pi_n, \tag{18}$$

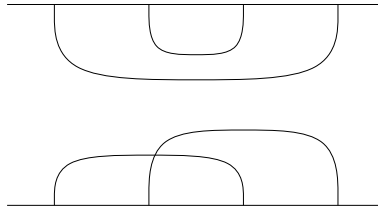


Fig. 4. Diagram $\pi_{\mu, \mu'} = \{\{1, 4\}, \{2, 3\}, \{5, 7\}, \{6, 8\}\}$, where $\mu = \{\{1, 4\}, \{2, 3\}\}$ and $\mu' = \{\{1, 3\}, \{2, 4\}\}$.

for all $n \geq 0$, that we shall prove by induction. Consider as initial step $n = 0$, where factors are empty and equal to 1, and the statement is trivial. Assuming the equality for $n \geq 0$ and then using (***) yields

$$\begin{aligned} \Pi_{n+1} \Omega_{n+1} \mathcal{R}_{n+1} \Pi_{n+1} &= \Pi_{n+1} L_{n+1} \Omega_n \mathcal{R}_n F_{n+1}^{-1} \Pi_{n+1} \\ &= \Pi_{n+1} L_{n+1} \Pi_n \Omega_n \mathcal{R}_n \Pi_n F_{n+1}^{-1} \Pi_{n+1} \\ &= \Pi_{n+1} L_{n+1} \Pi_n F_{n+1}^{-1} \Pi_{n+1} = \Pi_{n+1} L_{n+1} \Pi_{n+1} F_{n+1}^{-1} = \Pi_{n+1}, \end{aligned}$$

where we have furthermore used that F_{n+1} commutes with the action of \mathfrak{S}_{2n+2} (resp. $\mathfrak{S}_{n+1} \times \mathfrak{S}_{n+1}$), L_{n+1} with Π_n . □

Let us consider respectively an element W_n^ε of $\mathbb{C}[\mathfrak{S}_{2n}]$ invariant by left or right translation by \mathcal{H}_n and a central element $W_{\mathfrak{g}_n}$ of $\mathbb{C}[\mathfrak{S}_{2n}]$, such that

$$\rho_\varepsilon(W_n^\varepsilon) = (-\varepsilon N)^n n! \Pi_{\mathcal{H}_n}^{\varepsilon N} (\Delta_\varepsilon^{2n}(2n) \Delta_\varepsilon^{2n}(2n-2) \cdots \Delta_\varepsilon^n(2))^{-1} \Pi_{\mathcal{H}_n}^{\varepsilon N}$$

and

$$\sum_{\alpha, \beta} W_{\mathfrak{g}_n}(\alpha\beta^{-1}) \rho(\alpha) \otimes \rho(\beta) = (-N)^n n! \Pi_{\mathcal{D}_n}^N (\Delta^{n,n}(n) \Delta^{n,n}(n-1) \cdots \Delta^{n,n}(1))^{-1} \Pi_{\mathcal{D}_n}^N.$$

The Lemma 5.3 yields that $\rho_\varepsilon(W_n)$ and $\sum_{\alpha, \beta} W_{\mathfrak{g}_n}(\alpha\beta^{-1}) \rho(\alpha) \otimes \rho(\beta)$ are respectively the *pseudo-inverses* of the two self-adjoint operators $\sum_{\sigma \in \mathfrak{S}_{2n}} N^{\#\sigma} \mu_n \vee \mu_n \rho_\varepsilon(\sigma)$ and $\sum_{\alpha, \beta \in \mathfrak{S}_n} N^{\#\alpha\beta^{-1}} \rho(\alpha \times \beta)$. We can reformulate Lemma 5.2 as follows. For any pair of matchings $(\mu, \mu') \in \mathcal{M}(n)$, let us consider the matching $\pi_{\mu, \mu'} \in \mathcal{M}(2n)$ whose diagram is given by μ in the positive half plane and by μ' in the negative half plane (see Figure 4). Then, Lemma 5.2 can be reformulated into

Theorem 5.4 ([6,7,24]). *If G is $O(N)$ or $Sp(N)$, then*

$$\mathbb{E}[H^{\otimes 2n}] = \sum_{\alpha, \beta \in \mathfrak{S}_{2n}} W_n^\varepsilon(\alpha\beta^{-1}) \rho_\varepsilon(\pi_{\alpha\mu_n, \beta\mu_n}).$$

If $G = U(N)$,

$$\mathbb{E}[H^{\otimes n} \otimes \overline{H}^{\otimes n}] = \sum_{\alpha, \beta \in \mathfrak{S}_n} W_{\mathfrak{g}_n}(\alpha\beta^{-1}) \rho_+(\pi_{\alpha\nu_n, \beta\nu_n}).$$

The two pseudo-inverses appearing in these formulae can be re-expressed using representation theory. For example, it can be shown that for any $k \in \{1, \dots, 2n\}$, $\rho_\varepsilon(\varepsilon N + X_k)$ is invertible in $\text{End}(V^{\otimes 2n})$, so that

$$\rho_\varepsilon(W_n^\varepsilon) = \rho \left(\prod_{k=1}^n (\varepsilon N + X_{2k-1}) \right)^{-1} \rho_\varepsilon(\Pi_{\mathcal{H}_n})$$

and

$$\sum_{\alpha, \beta \in \mathfrak{S}_n} W_{\mathfrak{g}_n}(\alpha\beta^{-1})\rho(\alpha) \otimes \rho(\beta) = \rho\left(\prod_{k=1}^n (N + X_k)\right)^{-1} \rho(\Pi_{\mathcal{D}_n}).$$

Using these two equalities together with (16) and (17), leads to the exact statement of the formulae of [24], that is equivalent to the one of [7].

Remark 5.5. The above way to get formulae for the Haar measure can also be applied to $SU(N)$. This case has not been explored as much as the three series $O(N)$, $Sp(N)$ and $U(N)$. Let us consider an example. If S is Haar distributed on $SU(N)$, then multiplying on the left by rotation matrices leads to

$$\mathbb{E}\left[\prod_{1 \leq i, j \leq N} S_{i,j}\right] = 0,$$

when N is odd. On the other hand, the following is a conjecture [14]: that for all N even,

$$\mathbb{E}\left[\prod_{1 \leq i, j \leq N} S_{i,j}\right] \neq 0. \tag{19}$$

This conjecture has been shown in [14] to be equivalent to the Alon–Tarsi conjecture ([1]) about even and odd Latin squares, as well as Hadamard–Howe and Foulkes conjectures (see [14] for references). In [8,9], it has been proved to hold true for any even number such that $N \pm 1$ is a prime number. Though, the general case is still open. Using $SU(N)$ -Brownian motion, it can be reformulated as follows. Let $(S_t)_{t \geq 0}$ be a $SU(N)$ -Brownian motion issued from identity. As $\mathbb{E}[S_t^{\otimes N^2}] \rightarrow \mathbb{E}[S^{\otimes N^2}]$, when $t \rightarrow \infty$, according to Lemma 2.3 and Proposition 4.1, the Alon–Tarsi conjecture is equivalent to the following: for any even integer N ,

$$\left\langle e_1^{\otimes N} \otimes \dots \otimes e_N^{\otimes N}, \exp\left(-t \sum_{1 \leq i < j \leq N^2} (i j)\right) (e_1 \otimes \dots \otimes e_N)^{\otimes N} \right\rangle \rightarrow x_N \neq 0, \tag{20}$$

as $t \rightarrow \infty$.

Appendix: Symplectic invariants

Proof of Lemma 3.4 for $\mathcal{B}_n(-N)$. As $\{(a b), \langle a b \rangle : 1 \leq a < b \leq n\}$ generates the algebra $\mathcal{B}_n(-N)$, it is sufficient to prove that for any $\pi \in \mathcal{M}(n)$, $1 \leq a < b \leq n$, $\rho_-(\pi(a b)) = \rho_-(\pi)\rho_-(\langle a b \rangle)$ and $\rho_-(\pi \langle a b \rangle) = \rho_-(\pi)\rho_-(\langle a b \rangle)$. On the one hand, $\pi.(a b) = (2n + 1 - a \ 2n + 1 - b)(\pi)$, where the left-hand side is a product in $\mathcal{B}_n(-N)$ and the right-hand side an action of \mathfrak{S}_n on $\mathcal{M}(n)$. On the other hand, if π matches $2n + 1 - a$ and $2n + 1 - b$ with respectively a' and b' , then whether $a' = 2n + 1 - b$ and $b' = 2n + 1 - a$, in which case, $\pi \langle a b \rangle = -N\pi$, or $\pi \langle a b \rangle = (2n + 1 - a b')(\pi) = (2n + 1 - b a')(\pi)$. Using (7) and (9), it is now elementary to check the multiplicativity in both cases. □

Proof of Proposition 4.1 for $G = Sp(N)$. Let us define $\iota : M_{N/2}(\mathbb{C}) \times M_{N/2}(\mathbb{C}) \rightarrow M_N(\mathbb{C}) : (A, B) \mapsto \begin{pmatrix} A & -\bar{B} \\ B & A \end{pmatrix}$ and recall that

$$\mathfrak{sp}(N) = \{\iota(A, B) : A \in \mathfrak{u}(N), B \in M_N(\mathbb{C}), {}^t B = B\}.$$

Let us choose the basis of $\mathfrak{sp}(N)$ formed by the unions of the following families:

$$\frac{1}{\sqrt{2N}} \left\{ \iota(E_{a,b} - E_{b,a}, 0), \iota(0, E_{a,b} - E_{b,a}), \iota(i(E_{a,b} + E_{b,a}), 0), \iota(0, i(E_{a,b} + E_{b,a})) : 1 \leq a < b \leq \frac{N}{2} \right\}$$

and

$$\frac{1}{\sqrt{N}} \left\{ \iota(iE_{a,a}, 0), \iota(0, E_{a,a}), \iota(0, iE_{a,a}) : 1 \leq a \leq \frac{N}{2} \right\}.$$

The Casimir element of $\mathfrak{sp}(N)$, that is $c_{\mathfrak{sp}(N)} = \sum_{1 \leq i \leq \frac{N^2}{2}} x_i \otimes x_i$, where $(x_i)_{1 \leq i \leq \frac{N^2}{2}}$ is the above orthogonal basis, viewed as an element of the complex enveloping algebra $\mathcal{U}(\mathfrak{gl}_N(\mathbb{C}))$, has the following expression:

$$-Nc_{\mathfrak{sp}(N)} = - \sum_{1 \leq a, b, c, d \leq N} J_{a,c} J_{b,d} E_{a,b} \otimes E_{c,d} + \sum_{1 \leq a, b \leq N} E_{a,b} \otimes E_{b,a} \in \mathcal{U}(\mathfrak{gl}_N(\mathbb{C})).$$

Considering $\rho_{V^{\otimes n}}$ as a representation of the algebra $\mathcal{U}(\mathfrak{gl}_N(\mathbb{C}))$,

$$\begin{aligned} -N\rho_{V^{\otimes n}}(\Delta_{\mathfrak{sp}(N)}) &= 2(N+1)n + 2 \sum_{1 \leq a < b \leq n} \rho_+((a \ b) - \langle a \ b \rangle) \\ &= 2\rho_- \left((N+1)n + \sum_{1 \leq a < b \leq n} (\langle a \ b \rangle - (a \ b)) \right). \end{aligned}$$

Dividing by $2N$ gives the announced formula. □

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