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Brownian motion correlation in the peanosphere for $\kappa > 8$

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Abstract. The peanosphere (or "mating of trees") construction of Duplantier, Miller, and Sheffield encodes certain types of γ-Liouville quantum gravity (LQG) surfaces ($\gamma \in (0,2)$) decorated with an independent SLE_κ ($\kappa = 16/\gamma^2 > 4$) in terms of a correlated two-dimensional Brownian motion and provides a framework for showing that random planar maps decorated with statistical physics models converge to LQG decorated with an SLE. Previously, the correlation for the Brownian motion was only explicitly identified as $-\cos(4\pi/\kappa)$ for $\kappa \in (4,8]$ and unknown for $\kappa > 8$. The main result of this work is that this formula holds for all $\kappa > 4$. This supplies the missing ingredient for proving convergence results of the aforementioned type for $\kappa > 8$. Our proof is based on the calculation of a certain tail exponent for SLE_κ on a quantum wedge and then matching it with an exponent which is well-known for Brownian motion.

Résumé. La sphère de Peano (ou «Accouplement d'arbres») construite par Duplantier, Miller, et Sheffield encode certains types de surfaces de γ-gravité quantique de Liouville (LQG) décorées par un SLE_κ (pour $\gamma \in (0,2)$ et $\kappa = 16/\gamma^2 > 4$), en termes d'un mouvement Brownien 2-dimensionnel corrélé et fournit un cadre pour montrer que les cartes planaires décorées par un modèle de physique statistique convergent vers un LQG décoré par un SLE. Précédemment, la corrélation du mouvement Brownien était seulement explicitement identifiée à $-\cos(4\pi/\kappa)$ pour $\kappa \in (4,8]$, mais inconnue pour $\kappa > 8$. Le résultat principal de ce travail est que cette formule reste vraie pour $\kappa > 8$. Cela donne l'ingrédient manquant pour prouver les résultats de convergence mentionnés précédemment pour $\kappa > 8$. Notre preuve est basée sur le calcul d'un exposant de queue pour le SLE_κ sur un coin quantique et sur son identification avec un exposant bien connu pour le mouvement Brownien.

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1. Introduction

Suppose that h is an instance of the Gaussian free field (GFF) on a planar domain D and $\gamma \in (0,2)$. Formally the γ-Liouville quantum gravity (LQG) surface associated with h is the Riemannian manifold with metric tensor given by

$$e^{\gamma h(z)}(dx^2 + dy^2), \tag{1.1}$$

where $dx^2 + dy^2$ denotes the Euclidean metric on D. This expression does not make literal sense since h is a distribution and does not take values at points. However, one can make sense of the volume form associated with (1.1) as a random measure via various regularization procedures, e.g. the ones used in [6]. The metric space structure of LQG has been constructed in the special case $\gamma = \sqrt{8/3}$ in [21] building on [22] and, upon combining with [20], will be identified with the Brownian map in [26,27], but it remains an open problem to construct the metric for other values of $\gamma \in (0, 2)$.

One of the main sources of significance of LQG is that it has been conjectured to describe the scaling limits of random planar maps decorated by statistical physics models. This conjecture can be formulated in several different ways by specifying the topology. For example, one can view random planar maps as metric spaces and endow them with the Gromov–Hausdorff topology. Convergence under this topology has been established in the case of uniformly random quadrangulations to the Brownian map in [18,19]. Combining with the aforementioned works gives the Gromov–Hausdorff convergence to $\sqrt{8/3}$-LQG. An alternative approach is to start off with a random planar map, embed it conformally into $\mathbb{C}$ (e.g. via circle packing, Riemann uniformization, etc…) and show that the random area measure it induces (i.e., the pushforward of the uniform measure on the faces of the map) converges weakly to an LQG measure. Establishing this type of convergence is an open problem for any $\gamma \in (0, 2)$.

The work [5] takes a third approach through its *peanosphere* or *mating of trees* construction. More precisely, let $\gamma \in (0, 2)$, $\kappa' = 16/\gamma^2 > 4$, and $(Z_t)_{t \in \mathbb{R}} = (L_t, R_t)_{t \in \mathbb{R}}$ be a correlated two-dimensional two-sided Brownian motion. Then Z encodes a pair of Brownian continuum random trees [1–3] with L and R as their contour functions. As explained in [5, Section 1.1], one can glue the two trees together to obtain a topological sphere endowed with a measure and the space-filling peano curve which traces the interface between the two trees.[1] In [5] the authors show that there is a canonical way of embedding this measure-endowed topological sphere into $\mathbb{C} \cup \{\infty\}$ such that the pushforward of the measure is a form of γ-LQG and the image of the spacing-filling curve is an independent space-filling form of Schramm's SLE [32] with parameter[2] κ' from ∞ to ∞ as defined in [29]; see also [5]. Moreover, it is shown in [5] that both the field h and the space-filling SLE are a.s. determined by Z. That is, the peanosphere comes equipped with a canonical conformal structure.

It is proved in [5] that for $\gamma \in [\sqrt{2}, 2)$ (equivalently, for $\kappa' \in (4, 8]$) the correlation between L and R is given by $-\cos(4\pi/\kappa') \geq 0$. The correlation between L and R for $\gamma \in (0, \sqrt{2})$ (equivalently, for $\kappa' > 8$) is left as an open problem [5, Question 13.4]. The main result of this paper is that the correlation between L and R is given by $-\cos(4\pi/\kappa')$ for all $\kappa' > 4$ (so that L and R are *negatively* correlated for $\kappa' > 8$).

For $\kappa' \in (4, 8]$, the peanosphere construction can be viewed as a continuum analogue of the bijection introduced by Sheffield in [36, Section 4.1], which encodes a critical Fortuin–Kasteleyn (FK) decorated planar map in terms of a word in a certain alphabet of five letters. Indeed, the manner in which the space-filling $\mathrm{SLE}_{\kappa'}$ path η and the γ-LQG surface are encoded by Z closely parallels the manner in which an FK planar map is described by a word under the bijection of [36] (see [5,11,12] for more details). This correspondence allows one to interpret various scaling limit statements for FK planar maps, as proven in [11,13,14,36], as convergence results for FK decorated random planar maps to SLE decorated LQG with respect to the peanosphere topology; see also [4] for a calculation of some exponents associated with an FK planar map which match the corresponding exponents which can be derived in the continuum using [5]. Under this topology, two spanning tree decorated surfaces are said to be close if the contour functions of the tree/dual tree pairs are close. On the FK planar map side, the tree/dual tree pair is generated using Sheffield's bijection [36] and in the continuum this pair is given by trees of GFF flow lines [29] whose peano curve is space-filling $\mathrm{SLE}_{\kappa'}$. In [12], the authors use peanosphere convergence, plus some additional estimates, to prove convergence of critical FK planar maps toward $\mathrm{CLE}_{\kappa'}$-decorated LQG for $\kappa' \in (4, 8)$ in a stronger topology, which encodes the full topological structure of the collection of loops as well as the areas and boundary lengths of all of their complementary connected components.

Recently the techniques of [36] have been generalized in [10] to the setting of random planar maps decorated with a certain type of spanning tree. It is in particular shown in [10] that for a certain range of parameter values, the contour functions converge in the scaling limit to a *negatively* correlated Brownian motion (which extends [36, Theorem 2.5]). In another work [16], it is shown that the height functions associated with the northwest tree and its dual tree which arise from a so-called bipolar orientation on a random planar map also converge to a certain pair of negatively correlated Brownian motions. The result of [16] is strengthened (for the case of triangulations) in [9], which shows convergence of two pairs of height functions to two pairs of negatively correlated Brownian motions, corresponding to two space-filling SLE curves traveling in a direction perpendicular (in the sense of imaginary geometry) to each other. In all of the above cases the correlation of the Brownian motion is explicit. Our main result allows us to interpret

[1] This is the source of the name peanosphere.

[2] We use the convention of [23–25,29] of writing $\kappa' > 4$ for the SLE parameter and $\kappa = 16/\kappa'$ for the dual parameter.

these limit results as convergence of random planar maps decorated with a statistical physics model to certain γ-LQG surfaces with $\gamma \in (0, \sqrt{2})$ decorated with an $\mathrm{SLE}_{\kappa'}$ with $\kappa' > 8$.

Moreover, knowing the correlation of (L, R) allows us to understand the interplay between two-dimensional Brownian motion and the space-filling SLE on top of the LQG surface at a quantitative level. For example, the KPZ-like formula established in [8] relates the Hausdorff dimension of an arbitrary random Borel set $A \subset \mathbb{C}$ which is determined by the space-filling $\mathrm{SLE}_{\kappa'}$ (viewed modulo monotone reparameterization of time) in the peanosphere construction to the Hausdorff dimension of its pre-image under the Brownian motion (L, R). This reduces the problem of computing the Hausdorff dimension of A to the problem of computing the dimension of an (often much simpler) set defined in terms of (L, R) (many examples of this type are given in [8]). Our result implies that the formula derived in [8] is valid for all $\kappa' > 4$ and not just $\kappa' \in (4, 8]$.

Finally, we remark that our result supplies the missing ingredient in order to identify the correlation of the two-dimensional Brownian excursion appearing in the finite-volume version of the peanosphere construction [22, Theorem 1.1] in the case $\gamma \in (0, \sqrt{2})$.

1.1. *Main result*

Now we give the formal statement of our main result. We will remind the reader of the precise description of the objects involved in Section 1.2.

Given $\gamma \in (0, 2)$ and $\kappa' = 16/\gamma^2$, let η be a whole-plane space-filling $\mathrm{SLE}_{\kappa'}$ from ∞ to ∞ (defined in [29, Sections 1.2.3 and 4.3]; see also Section 1.2.2 of the present paper). Let $\gamma = 4/\sqrt{\kappa'}$ and let $\mathcal{C} = (\mathbb{C}, h, 0, \infty)$ be a γ-quantum cone independent from η, as in [5, Section 4.3] or Section 1.2.3 of the present paper. Let μ_h and ν_h, respectively, be the γ-quantum area measure and γ-quantum boundary measure induced by h. Let $\widetilde{\eta}$ be the curve obtained by parameterizing η by μ_h-mass, so that $\widetilde{\eta}(0) = 0$ and $\mu_h(\widetilde{\eta}([t_1, t_2])) = t_2 - t_1$ for each $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$. Let $Z_t = (L_t, R_t)$ denote the net change in the ν_h-length of the left and right boundaries of $\widetilde{\eta}((-\infty, t])$ relative to time 0. Then Z evolves as a two-sided Brownian motion with some correlation [5, Theorem 1.13] and Z a.s. determines the pair $(\eta, \mathcal{C})$ modulo rotation and scaling [5, Theorem 1.14] (this is the mathematically precise formulation of the mating of trees/peanosphere construction described above). For $\gamma \in [\sqrt{2}, 2)$, by [5, Theorem 1.13] the correlation of Z is $-\cos(4\pi/\kappa')$. Similar results are also proved in the upper half-plane setting. See Section 1.2.3 for the definition of the quantum wedge.

Theorem 1.1. *In the above setting, for $\gamma \in (0, \sqrt{2})$, the correlation of Z is still given by $-\cos(4\pi/\kappa')$. Furthermore, suppose that $(\mathbb{H}, \mathfrak{h}, 0, \infty)$ is a $3\gamma/2$-quantum wedge and η' is a chordal $\mathrm{SLE}_{\kappa'}$ from 0 to ∞ in $\mathbb{H}$ sampled independently of $\mathfrak{h}$ and let $\widetilde{\eta}'$ be the curve which arises by reparameterizing η' by quantum mass with respect to $\mathfrak{h}$. Then the change in the left and right quantum boundary lengths of $\mathbb{H} \setminus \widetilde{\eta}'([0, t])$ with respect to $\mathfrak{h}$ evolve as a two-dimensional correlated Brownian motion with correlation $-\cos(4\pi/\kappa')$.*

We note that in light of Lemma 1.8 below, either of the two statements of Theorem 1.1 implies the other.

1.2. *Preliminaries*

1.2.1. *Basic notation*

Here we record some basic notation which we will use throughout this paper.

Notation 1.2. If a and b are two quantities, we write $a \preceq b$ (resp. $a \succeq b$) if there is a constant C (independent of the parameters of interest) such that $a \leq Cb$ (resp. $a \geq Cb$). We write $a \asymp b$ if $a \preceq b$ and $a \succeq b$.

Notation 1.3. If a and b are two quantities which depend on a parameter x, we write $a = o_x(b)$ (resp. $a = O_x(b)$) if $a/b \to 0$ (resp. a/b remains bounded) as $x \to 0$ or as $x \to \infty$, depending on context. We write $a = o_x^\infty(b)$ if $a = o_x(b^s)$ for each $s > 0$.

Unless otherwise stated, all implicit constants in $\asymp$, $\preceq$, $\succeq$, $O_x(\cdot)$, and $o_x(\cdot)$ which are involved in the proof of a result are allowed to depend only on the extra parameters which the implicit constants in the statement of the result are allowed to depend on.

1.2.2. *Schramm–Loewner evolution*

Schramm–Loewner evolution (SLE_κ) is a one-parameter family of conformally invariant laws on two-dimensional fractal curves indexed by $\kappa > 0$, originally introduced in [32] as a candidate for the scaling limit of various discrete statistical physics models. We refer the reader to [17,38] for an introduction to SLE.

Whole-plane space-filling $\mathrm{SLE}_{\kappa'}$ from ∞ to ∞ for $\kappa' > 4$ is a variant of $\mathrm{SLE}_{\kappa'}$ introduced in [29, Sections 1.2.3 and 4.3] and [5, Footnote 9]. In the case when $\kappa' \geq 8$, so ordinary $\mathrm{SLE}_{\kappa'}$ is space-filling (which is the only case we will use in this paper), space-filling $\mathrm{SLE}_{\kappa'}$ from ∞ to ∞ is a bi-infinite $\mathrm{SLE}_{\kappa'}$ curve which fills in all of $\mathbb{C}$, starting and ending at ∞. It has the property that if one runs it up until any stopping time τ, its complement is an unbounded simply connected domain and the conditional law of the path is given by that of an ordinary chordal $\mathrm{SLE}_{\kappa'}$ in the remaining domain from the tip at time τ to ∞. It can also be constructed directly from ordinary $\mathrm{SLE}_{\kappa'}$ using a limiting procedure as follows (this is not equivalent to but easy to see from the GFF-based construction given in [29]). Suppose that η' is a chordal $\mathrm{SLE}_{\kappa'}$ in $\mathbb{H}$ from 0 to ∞ and that $z_0 \in \mathbb{H}$ is fixed. For each $\varepsilon > 0$ let η'_ε be given by $\varepsilon^{-1}(\eta' - z_0)$ parameterized according to Lebesgue measure and, for each $r > 0$, let $\tau_{\varepsilon,r}$ (resp. $\sigma_{\varepsilon,r}$) be the first time that η'_ε hits $\partial B_r(0)$ (resp. fills $B_r(0)$). Then the law of $\eta'_\varepsilon|_{[\tau_{\varepsilon,r},\sigma_{\varepsilon,r}]}$ converges in total variation as $\varepsilon \to 0$ to the restriction of whole-plane $\mathrm{SLE}_{\kappa'}$ from ∞ to ∞ to the interval of times between when it first hits $B_r(0)$ and fills $B_r(0)$, also parameterized according to Lebesgue measure.

In what follows, whenever we refer to whole-plane space-filling $\mathrm{SLE}_{\kappa'}$, we mean whole-plane space-filling $\mathrm{SLE}_{\kappa'}$ from ∞ to ∞. We record the aforementioned fact about the conditional law of whole-plane space-filling $\mathrm{SLE}_{\kappa'}$ for $\kappa' \geq 8$ in the following lemma, which follows from the definition in [5, Footnote 9] together with a basic limiting argument.

Lemma 1.4. *Let $\kappa' \geq 8$ and let η be a whole-plane space-filling* $\mathrm{SLE}_{\kappa'}$ *from ∞ to ∞. Let τ be a stopping time for η. Then $\mathbb{C} \setminus \eta((-\infty, \tau])$ is a.s. simply connected, unbounded, and the conditional law of $\eta|_{[\tau,\infty)}$ given $\eta|_{(-\infty,\tau]}$ is that of a chordal* $\mathrm{SLE}_{\kappa'}$ *from $\eta(\tau)$ to ∞ in $\mathbb{C} \setminus \eta((-\infty, \tau])$.*

In the case when $\kappa' \in (4, 8)$, ordinary $\mathrm{SLE}_{\kappa'}$ does not fill in open sets, but rather forms "bubbles" which it surrounds, but never enters [31]. Space-filling $\mathrm{SLE}_{\kappa'}$ in this case is obtained by continuously filling in these bubbles as they are disconnected from ∞. It is the peano curve associated with the exploration tree in the construction of $\mathrm{CLE}_{\kappa'}$ [34]. We will not need the $\kappa' \in (4, 8)$ case in this paper.

1.2.3. *Quantum surfaces*

Fix $\gamma \in (0, 2)$ (in this paper we will always take $\gamma = 4/\sqrt{\kappa'} \in (0, \sqrt{2})$). Also let k be a non-negative integer. A *γ-LQG surface* with k marked points [5,6,35] is an equivalence class of $k + 2$-tuples $(D, h, z_1, \dots, z_k)$, where $D \subset \mathbb{C}$ is a domain (possibly all of $\mathbb{C}$), h is a distribution on D, and $z_1, \dots, z_k \in \overline{D}$ are marked points. Two such $k + 2$-tuples $(D, h, z_1, \dots, z_k)$ and $(\widetilde{D}, \widetilde{h}, \widetilde{z}_1, \dots, \widetilde{z}_k)$ are declared to be equivalent if there is a conformal map $f : \widetilde{D} \to D$ such that

$$\widetilde{h} = h \circ f + Q \log\left|f'\right| \quad \text{and} \quad f(\widetilde{z}_j) = z_j, \quad \forall j \in \{1, \dots, k\}, \tag{1.2}$$

where

$$Q := \frac{2}{\gamma} + \frac{\gamma}{2}. \tag{1.3}$$

In [6], it is shown that in the case when h is some variant of the GFF on D (which is the only case we will consider in this paper), the corresponding quantum surface has a natural area measure μ_h on D (which is a regularization of "$e^{\gamma h(z)}\, dz$", where dz denotes Lebesgue measure on D) and a natural boundary length measure ν_h on ∂D (which is a regularization of "$e^{\frac{\gamma}{2} h(z)} |dz|$", where $|dz|$ denotes the pushforward of Lebesgue measure on $\partial\mathbb{D}$ under a conformal map $\mathbb{D} \to D$). By [6, Proposition 2.1] and its boundary analogue, these measures are preserved under transformations of the form (1.2). We note that the measure ν_h can be extended to certain curves lying in the interior of the domain D (in particular, this is true for SLE_κ curves with $\kappa = \gamma^2$). See [5,35].

The main types of quantum surfaces which we will be interested in this paper are the so-called quantum wedges and quantum cones, which are defined in [35, Section 1.6] and [5, Sections 4.2 and 4.3]. For $\alpha \in (0, Q)$, an *α-quantum* *wedge* is a doubly marked quantum surface $\mathcal{W} = (\mathbb{H}, \mathring{h}, 0, \infty)$ defined as follows. Let $\mathcal{H}(\mathbb{H})$ be the Hilbert space used to define a free-boundary GFF on $\mathbb{H}$ [35, Section 3] (i.e. the completion of the space of smooth functions on $\mathbb{H}$ with respect to the inner product $(f, g)_\nabla = (2\pi)^{-1} \int_{\mathbb{H}} \nabla f(z) \cdot \nabla g(z)\, dz$). Let $\mathcal{H}^0(\mathbb{H})$ (resp. $\mathcal{H}^\dagger(\mathbb{H})$) be the space of functions in $\mathcal{H}(\mathbb{H})$ which are constant on each semicircle in $\mathbb{H}$ centered at 0 (resp. its orthogonal complement).

Let $\alpha \in (0, Q)$, with Q as in (1.3). Following [5, Definition 4.3], we define an α-quantum wedge to be the doubly marked quantum surface $\mathcal{W} = (\mathbb{H}, \mathring{h}, 0, \infty)$, where $\mathring{h}$ is a random distribution on $\mathbb{H}$ defined as follows. The projection $\mathring{h}^\dagger$ of $\mathring{h}$ onto $\mathcal{H}^\dagger(\mathbb{H})$ agrees in law with the projection onto $\mathcal{H}^\dagger(\mathbb{H})$ of a free-boundary GFF on $\mathbb{H}$. The projection $\mathring{h}^0$ of $\mathring{h}$ onto $\mathcal{H}^0(\mathbb{H})$ is independent of $\mathring{h}^\dagger$ and is defined as follows. For $s \geq 0$, $\mathring{h}^0(e^{-s}) = \mathcal{B}_{2s} + \alpha s$, where $\mathcal{B}$ is a standard linear Brownian motion; and for $s < 0$, $\mathring{h}^0(e^{-s}) = \widehat{\mathcal{B}}_{-2s} + \alpha s$, where $\widehat{\mathcal{B}}$ is independent from $\mathcal{B}$ and has the law of a standard linear Brownian motion conditioned so that $\widehat{\mathcal{B}}_{2s} + (Q - \alpha)s > 0$ for all $s > 0$. Note that a quantum wedge has two marked points, 0 and ∞. Every bounded subset of $\mathbb{H}$ has finite quantum mass a.s. and every neighborhood of ∞ (i.e. any open set which contains $\mathbb{H} \setminus B_r(0)$ for some $r > 0$) has infinite mass a.s.

A quantum wedge is only defined modulo transformations of the form (1.2), so if we continue to parameterize the wedge by $(\mathbb{H}, 0, \infty)$, the distribution $\mathring{h}$ can be replaced with another distribution obtained via (1.2) with f given by a scaling by a positive constant. Different choices of h are referred to as different *embeddings* of the same surface.

Definition 1.5. The distribution $\mathring{h}$ defined just above is called the *circle average embedding* of a quantum wedge.

We will consider several other embeddings of a quantum wedge in Section 4.2.

Remark 1.6. The circle average embedding of a quantum wedge is convenient for the following reason. Suppose that h^F is a free-boundary GFF on $\mathbb{H}$ with additive constant chosen so that its circle average over $\partial B_1(0) \cap \mathbb{H}$ is 0 (which is the main normalization used in [5]) and let $h := h^F - \alpha \log|\cdot|$. Then with $\mathring{h}$ as in Definition 1.5, the restrictions of $\mathring{h}$ and h to $B_1(0) \cap \mathbb{H}$ agree in law. Indeed, if we let h^0 be the projection of h onto $\mathcal{H}^0(\mathbb{H})$ (equivalently the semicircle average process around 0), then $h^0(e^{-s})$ evolves as a two-sided Brownian motion, so $(h^0(e^{-s}))_{s\geq 0} \overset{d}{=} (\mathring{h}^0(e^{-s}))_{s\geq 0}$. Moreover, the projections of $\mathring{h}$ and h onto $\mathcal{H}^\dagger(\mathbb{H})$ agree in law by definition.

For $\alpha \in (0, Q)$, an *α-quantum cone* is a doubly marked quantum surface $\mathcal{C} = (\mathbb{C}, h, 0, \infty)$ which is similar to an α-quantum wedge but is parameterized by the whole plane rather than the half-plane. We will now describe the definition of this object, which first appeared in [5, Definition 4.9]. Let $\mathcal{H}(\mathbb{C})$ be the Hilbert space used to define the whole-plane GFF on $\mathbb{C}$. Let $\mathcal{H}^0(\mathbb{C})$ (resp. $\mathcal{H}^\dagger(\mathbb{C})$) be the space of functions in $\mathcal{H}(\mathbb{C})$ which are constant on each circle centered at 0 (resp. its orthogonal complement). An embedding h of a γ-quantum cone into $\mathbb{C}$ can be constructed as follows. The projection $h^\dagger$ of h onto $\mathcal{H}^\dagger(\mathbb{C})$ agrees in law with the corresponding projection of a whole-plane GFF on $\mathbb{C}$. The projection h^0 of h onto $\mathcal{H}^0(\mathbb{C})$ is independent of $h^\dagger$ and is described as follows. For $s \geq 0$, $h^0(e^{-s}) = \mathcal{B}_s + \alpha s$, where $\mathcal{B}$ is a standard linear Brownian motion; and for $s < 0$, $h^0(e^{-s}) = \widehat{\mathcal{B}}_{-s} + \alpha s$, where $\widehat{\mathcal{B}}$ is a standard linear Brownian motion conditioned so that $\widehat{\mathcal{B}}_s + (Q - \alpha)s > 0$ for all $s > 0$, independent from $\mathcal{B}$.

Remark 1.7. In [5], the sets of quantum cones and quantum wedges are sometimes parameterized by a different parameter, called the *weight*, which is equal to $\gamma(\gamma/2 + Q - \alpha)$ in the wedge case and $2\gamma(Q - \alpha)$ in the cone case, with Q as in (1.2). The reason for this choice of parameter is that it behaves nicely under the various "gluing" and "cutting" operations considered in [5]. In this paper we will not consider the weight parameter and will always identify our wedges and cones by α, the size of the logarithmic singularity at 0.

The main fact which we will use about quantum cones in this paper is the following lemma, which allows us to reduce the problem of studying a space-filling $\mathrm{SLE}_{\kappa'}$ on a γ-quantum cone to the problem of studying an ordinary chordal $\mathrm{SLE}_{\kappa'}$ on a $\frac{3}{2}\gamma$-quantum wedge.

Lemma 1.8. *Let $\kappa' \geq 8$ and $\gamma = 4/\sqrt{\kappa'} \in (0, \sqrt{2}]$. Let $\mathcal{C} = (\mathbb{C}, h, 0, \infty)$ be a γ-quantum cone. Let η be a whole-plane space-filling $\mathrm{SLE}_{\kappa'}$ from ∞ to ∞ independent from $\mathcal{C}$. Let $\widetilde{\eta}$ be the curve obtained by parameterizing η by γ-quantum mass with respect to h so that $\widetilde{\eta}(0) = 0$. Let $\mathcal{W}$ be the quantum surface obtained by restricting h to $\mathbb{C} \setminus \widetilde{\eta}((-\infty, 0])$. Then the pair $(\mathcal{W}, \widetilde{\eta}|_{[0,\infty)})$ has the law of a $\frac{3\gamma}{2}$-quantum wedge together with an independent chordal $\mathrm{SLE}_{\kappa'}$ parameterized by quantum mass with respect to this wedge.*

Proof. This is essentially proven as part of the proof of [5, Lemma 9.2], but we give the details for completeness. Let η_- and η_+ be the left and right boundaries of $\widetilde{\eta}((-\infty, 0])$. Then $\eta_\pm$ are independent of $\mathcal{C}$; the law of η_- is that of a whole-plane $\mathrm{SLE}_\kappa(2-\kappa)$ from 0 to ∞; and the conditional law of η_+ given η_- is that of a chordal $\mathrm{SLE}_\kappa(-\kappa/2; -\kappa/2)$ from 0 to ∞ in $\mathbb{C} \setminus \eta_-$. Indeed, this follows from the construction of [5, Footnote 9] as well as [29, Theorems 1.1 and 1.11]. By [5, Theorem 1.12], the law of the quantum surface $\mathcal{W}'$ obtained by restricting $\mathcal{C}$ to $\mathbb{C} \setminus \eta_-$ is that of a $(2\gamma - 2/\gamma)$-quantum wedge. By [5, Theorem 1.9], the surface $\mathcal{W}$ obtained by cutting $\mathcal{W}'$ by η_+ has the law of a $\frac{3\gamma}{2}$-quantum wedge. The law of $\widetilde{\eta}|_{[0,\infty)}$ is obtained from Lemma 1.4 and the independence of $\mathcal{W}$ and $\widetilde{\eta}|_{[0,\infty)}$ (the latter viewed as a curve modulo monotone reparameterization) follows from independence of η and $\mathcal{C}$ together with Lemma 1.4. □

1.3. *Approximate cone time event*

In this subsection we reduce the proof of Theorem 1.1 to the problem of calculating the tail exponent for the probability of a certain event.

Assume we are in the setting described in Section 1.1. A $\pi/2$-*cone time* of the Brownian motion $Z = (L, R)$ is a time $t \in \mathbb{R}$ for which there exists $t' > t$ such that $L_s \geq L_t$ and $R_s \geq R_t$ for each $s \in [t, t']$. That is, Z stays in the "cone" $\mathbb{R}_+^2 + Z_t$ for some positive amount of time after t. In the case when $\kappa' \in (4, 8)$, the covariance of the peanosphere Brownian motion Z is obtained by computing the Hausdorff dimension of its $\pi/2$-cone times in terms of κ' and comparing the formula thus obtained to the known formula for the Hausdorff dimension of the set of $\pi/2$-cone times in terms of the correlation [7]. The Hausdorff dimension is calculated in terms of κ' by observing that cone times for the Brownian motion correspond to local cut times for η, see [5, Lemma 9.4].

In the case when $\kappa' > 8$, the curve η a.s. does not have any local cut times, so Z a.s. does not have any $\pi/2$-cone times, hence has non-positive correlation [37]. To compute the correlation in this case, we will compute the tail exponent for the probability that 0 is an "approximate $\pi/2$-cone time" for Z, meaning that the event

$$\widetilde{E}_\delta^t := \Big\{ \inf_{s\in[0,t]} L_s \geq -\delta \text{ and } \inf_{s\in[0,t]} R_s \geq -\delta \Big\} \tag{1.4}$$

occurs for t close to 1 and δ close to 0. The tail exponent for the probability of $\widetilde{E}_\delta^t$ is computed in terms of the correlation of L_t and R_t in [37, Equation (4.3)].

Lemma 1.9. *Let $-\alpha = -\alpha(\gamma)$ be the correlation of L and R and let*

$$\sigma(\gamma) := \frac{\pi}{\arccos(\alpha)}. \tag{1.5}$$

There is a constant $c > 0$ depending only on α, such that for $\delta > 0$ and $t \geq \delta^{1/2}$ we have

$$\mathbb{P}\big[\widetilde{E}_\delta^t\big] = \big(c + o_\delta(1)\big) t^{-\sigma(\gamma)/2} \delta^{\sigma(\gamma)},$$

where here the $o_\delta(1)$ is uniformly bounded for $\delta > 0$ and $t \geq \delta^{1/2}$ and tends to 0 as $\delta \to 0$ for each fixed t.

Proof. Let A be a linear transformation chosen in such a way that $\widetilde{Z} := AZ$ is a standard two-dimensional Brownian motion (variances equal to 1, covariance equal to 0). Then an approximate $\pi/2$-cone time for Z is the same as an approximate $\arccos(\alpha)$-cone time for $\widetilde{Z}$. Hence the statement of the lemma follows from [37, Equation (4.3)]. □

In light of Lemma 1.9, to prove Theorem 1.1 it suffices to show that

$$\sigma(\gamma) = \frac{4}{\gamma^2} = \frac{\kappa'}{4}. \tag{1.6}$$

Remark 1.10. The event $\widetilde{E}_\delta^t$ of (1.4) can equivalently be defined as follows. Let $f : \mathbb{C} \setminus \widetilde{\eta}((-\infty, 0]) \to \mathbb{H}$ be a conformal map which takes 0 to 0 and ∞ to ∞. Let $\hbar := h \circ f^{-1} + Q \log |(f^{-1})'|$ and let $\widetilde{\eta}' := f(\widetilde{\eta}|_{[0,\infty)})$. By Lemma 1.8, the quantum surface $\mathcal{W} = (\mathbb{H}, \hbar, 0, \infty)$ has the law of a $\frac{3}{2}\gamma$-quantum wedge and $\widetilde{\eta}'$ is a chordal $\mathrm{SLE}_{\kappa'}$

from 0 to ∞ in $\mathbb{H}$ which is independent from $\mathcal{W}$ and parameterized by γ-quantum mass with respect to $\mathfrak{h}$. For $\delta > 0$, let $x_{\delta,L}$ and $x_{\delta,R}$ be the unique points respectively in $\mathbb{R}_-$ and $\mathbb{R}_+$ so that $\nu_{\mathfrak{h}}([-x_{\delta,L}, 0]) = \nu_{\mathfrak{h}}([0, x_{\delta,R}]) = \delta$. Then $\widetilde{E}_\delta^t$ is the same as the event that $\widetilde{\eta}'$ does not hit either $(-\infty, -x_{\delta,L}]$ or $[x_{\delta,R}, \infty)$ before time t. Since $\widetilde{\eta}'$ is boundary filling, $\widetilde{E}_\delta^t$ is also the same as the event that $\widetilde{\eta}'$ does not hit either $-x_{\delta,L}$ or $x_{\delta,R}$ before time t.

1.4. *Outline*

In the remainder of this paper, we will prove (1.6), hence Theorem 1.1. For the proof, we will use the alternative description of the event $\widetilde{E}_\delta^t$ given in Remark 1.10. In Section 2, we will use the SLE martingales of [33] to prove an estimate for the probability that a chordal $\mathrm{SLE}_{\kappa'}$ from 0 to ∞ in $\mathbb{H}$ exits the Euclidean ball of fixed radius $r > 0$ before hitting $-z_L$ or z_R, where $z_L, z_R \in (0, \infty)$. In Section 3, we will prove some moment estimates for the quantum boundary measure induced by a GFF which together with the estimates of Section 2 will enable us to prove a variant of (1.6) with $\widetilde{E}_\delta^t$ replaced by the event that the following is true. With $x_{\delta,L}$ and $x_{\delta,R}$ as in Remark 1.10, the curve $\widetilde{\eta}'$ exits the Euclidean ball of radius r before hitting either $-x_{\delta,L}$ or $x_{\delta,R}$. The arguments of this section are similar to those used to estimate the quantum measure in [6, Section 4]. In Section 4, we will extract (1.6) from the estimate of Section 3 using some techniques which are similar to those found in [5, Section 10.4].

2. Euclidean exponent for the SLE event

Recall that η' is an $\mathrm{SLE}_{\kappa'}$ from 0 to ∞ in $\mathbb{H}$, and let $(W_t)_{t\geq 0}$ and $(g_t)_{t\geq 0}$ denote its Loewner driving function and Loewner maps, respectively. Assume throughout this section that η' is parameterized by half-plane capacity, and let

$$\mathcal{F}_t := \sigma\big(\eta'(s) : s \in [0, t]\big). \tag{2.1}$$

The purpose of this section is to prove the following proposition, i.e., we calculate the exponent for a Euclidean analogue of the event $\widetilde{E}_\delta^t$ of (1.4).

Proposition 2.1. *For any $T > 0$ and $z_L, z_R \in (0, 1)$ define the event E_{z_L,z_R}^T by*

$$E_{z_L,z_R}^T := \big\{-z_L, z_R \notin \eta'\big([0, T]\big)\big\}.$$

Then the following estimate holds for $\rho := \kappa' - 4$:

$$\begin{aligned} &\mathbb{P}\big[E_{z_L,z_R}^T\big] = \mathbb{P}\big[E^1_{z_L/\sqrt{T}, z_R/\sqrt{T}}\big], \\ &\mathbb{P}\big[E^1_{z_L,z_R}\big] = (z_L + z_R)^{\rho^2/(2\kappa')} z_L^{\rho/\kappa' + o_{z_L}(1)} z_R^{\rho/\kappa' + o_{z_R}(1)}, \end{aligned} \tag{2.2}$$

where the rates of convergence of $o_{z_L}(1)$ and $o_{z_R}(1)$ depend only on κ'. For any $r > 0$ define the stopping time $T_r := \inf\{t > 0 : |\eta'(t)| \geq r\}$. Then

$$\begin{aligned} &\mathbb{P}\big[E_{z_L,z_R}^{T_r}\big] = \mathbb{P}\big[E^{T_1}_{z_L/r, z_R/r}\big], \\ &\mathbb{P}\big[E^{T_1}_{z_L,z_R}\big] = (z_L + z_R)^{\rho^2/(2\kappa')} z_L^{\rho/\kappa' + o_{z_L}(1)} z_R^{\rho/\kappa' + o_{z_R}(1)}. \end{aligned} \tag{2.3}$$

Both for the upper and the lower bound in Proposition 2.1 we will use the following result from [33, Theorem 6, Remark 7].

Lemma 2.2. *Define $\rho := \kappa' - 4$, and let $\widehat{T}_L$ (resp. $\widehat{T}_R$) denote the first time that η' hits $-z_L$ (resp. z_R). For each $t \in [0, \widehat{T}_L]$ (resp. $t \in [0, \widehat{T}_R]$) define $z_t^L := g_t(-z_L)$ (resp. $z_t^R := g_t(z_R)$) and define the stochastic process $(M_t)_{t\geq 0}$ by*

$$M_t = \begin{cases} |W_t - z_t^R|^{\rho/\kappa'} |W_t - z_t^L|^{\rho/\kappa'} |z_t^R - z_t^L|^{\rho^2/(2\kappa')} & \text{if } t \in [0, \widehat{T}_L \wedge \widehat{T}_R], \\ 0 & \text{if } t \geq \widehat{T}_L \wedge \widehat{T}_R. \end{cases}$$

Then M_t is a local martingale.

It is also proved in [33, Theorem 6] that the law of η' weighted by M_t (run up to an appropriate stopping time) has the law of a chordal $\mathrm{SLE}_{\kappa'}(\rho;\rho)$ with force points at $-z_L$ and z_R, but we will not need this result. Note that the derivative term in [33] vanishes for $\rho = \kappa' - 4$.

For our proof of the lower bound in Proposition 2.1 we will need that $(M_t)_{t\geq 0}$ is a true martingale, not only a local martingale.

Lemma 2.3. *The local martingale $(M_t)_{t\geq 0}$ defined in Lemma 2.2 is a martingale.*

Proof. It is sufficient to prove that for any $t \geq 0$ we have $\mathbb{E}[\sup_{s\in[0,t]} M_s] < \infty$. This is sufficient, since if $(\sigma_k)_{k\in\mathbb{N}}$ are stopping times such that $(M_{\sigma_k\wedge t})_{t\geq 0}$ are martingales and $\sigma_k \to \infty$ a.s., we can use the dominated convergence theorem to argue that $M_{\sigma_k\wedge t} \to M_t$ in L^1 for each $t \geq 0$.

Applying Itô's formula and the Loewner equation, we have that both $z_t^R - W_t$ and $W_t - z_t^L$ are constant multiples of Bessel processes of dimension $1 + \frac{4}{\kappa'} < 2$. Since the law of a Bessel process of dimension δ is stochastically dominated by the law of a Bessel process of dimension δ' provided $0 < \delta < \delta'$, it follows that there exist two stochastic processes $\widehat{B}^R$, $\widehat{B}^L$ which are constant multiples of two-dimensional Bessel processes such that $z_t^R - W_t \leq \widehat{B}_t^R$ and $z_t^L - W_t \leq \widehat{B}_t^L$ for all $t \geq 0$. Since $\widehat{B}^R$ and $\widehat{B}^L$ each have the law of the modulus of a two-dimensional Brownian motion, Doob's maximal inequality implies that

$$\mathbb{P}\Big[\sup_{s\in[0,t]} \big|z_s^q - W_s\big| > x\Big] = o_x^\infty(x) \quad \text{for } q \in \{L, R\}. \tag{2.4}$$

Using that $|z_t^R - z_t^L| \leq 2\max(|z_t^L - W_t|, |z_t^R - W_t|)$, $\rho > 0$ (so that all of the exponents in the definition of M_t are positive), and the sum of the exponents in the definition of M_t is equal to $\frac{\kappa'}{2} - 2$, we have that

$$\mathbb{E}\Big[\sup_{s\in[0,t]} M_s\Big] \leq \mathbb{E}\Big[\sup_{s\in[0,t]} \big(\big|z_s^R - W_s\big| \vee 1\big)^{\kappa'/2-2}\Big] + \mathbb{E}\Big[\sup_{s\in[0,t]} \big(\big|z_s^L - W_s\big| \vee 1\big)^{\kappa'/2-2}\Big] < \infty. \qquad \square$$

We have $M_0 = (z_L + z_R)^{\rho^2/(2\kappa')} z_L^{\rho/\kappa'} z_R^{\rho/\kappa'}$, i.e., M_0 has the same exponents as the probability of the event $E^1_{z_L,z_R}$ in Proposition 2.1. In order to prove the estimate (2.2) for $T = 1$ it is therefore sufficient to prove that $\mathbb{P}[E^1_{z_L,z_R}]$ is approximately equal to the expected value of $(M_t)_{t\geq 0}$ stopped at some appropriate stopping time. This is our strategy for the proof both of the upper bound and of the lower bound in (2.2).

2.1. *Euclidean upper bound*

As indicated above we will establish the upper bound of $\mathbb{P}[E^1_{z_L,z_R}]$ in Proposition 2.1 by defining an appropriate stopping time for $(M_t)_{t\geq 0}$. We will use the following stopping time σ_u for some $u > 0$:

$$\sigma_u = \inf\big\{t \geq 0 : M_t = (z_R z_L)^u \text{ or } M_t = 0\big\}. \tag{2.5}$$

In order to prove that $\mathbb{P}[E^1_{z_L,z_R}]$ is bounded above by $\mathbb{E}[M_{\sigma_u}] = \mathbb{E}[M_0]$ (up to $o(1)$ errors) it is sufficient to prove that $\sigma_u < 1$ with very high probability for small z_L, z_R. This is sufficient since $\mathbb{E}[M_{\sigma_u}]$ is approximately equal to $\mathbb{P}[M_{\sigma_u} = (z_L z_R)^u]$ for small u. The following two technical lemmas will help us establish that $\sigma_u < 1$ with very high probability. In Lemma 2.4 we prove that with very high probability $\operatorname{Im}\eta'(t)$ does not stay close to the real line for all $t \in [0, 1]$. Lemma 2.5 implies a lower bound for M_t in terms of $\operatorname{Im}\eta'(t)$.

Lemma 2.4. *For each $\varepsilon > 0$ we let*

$$F_\varepsilon := \Big\{\sup_{t\in[0,1]} \operatorname{Im}\big(\eta'(t)\big) \leq \varepsilon\Big\}.$$

Then $\mathbb{P}[F_\varepsilon] = o_\varepsilon^\infty(\varepsilon)$.

Proof. By scale invariance of SLE the statement of the lemma is equivalent to the statement that if $F_n' := \{\sup_{t\in[0,n]} \operatorname{Im}(\eta'(t)) \le 1\}$ for $n \in \mathbb{N}$ then we have $\mathbb{P}[F_n'] = o_n^\infty(1/n)$. Define the stopping time $\widetilde{T}$ by

$$\widetilde{T} := \inf\{t \ge 0 : \operatorname{Im}(\eta'(t)) \ge 1\}. \tag{2.6}$$

It is sufficient to prove that there is a constant $p > 0$ s.t. for all $n \in \mathbb{N}$

$$\mathbb{P}[\widetilde{T} < (n+1) | \mathcal{F}_n] \ge p, \tag{2.7}$$

since this bound implies $\mathbb{P}[F_n'] \le (1-p)^n = o_n^\infty(1/n)$. Here $\mathcal{F}_n$ is as in (2.1).

Define $p := \mathbb{P}[\widetilde{T} < 1]$. Since (2.7) clearly holds on the event that $\widetilde{T} \le n$ we assume $\widetilde{T} > n$. Define $l := \{x + i : x \in \mathbb{R}\}$, and for each $n \in \mathbb{N}$ define $l_n' := \{g_n(z) : z \in l\}$. By [17, Equation (4.5)] each $z \in l_n'$ satisfies $\operatorname{Im}(z) \le 1$, and l_n' is a connected set dividing the upper half-plane into an upper and a lower part, hence the SLE $g_n(\eta')$ hits l_n' before it hits l. The estimate (2.7) follows by the conformal Markov property of SLE. □

Lemma 2.5. *Consider the stopping time $\widetilde{T}$ defined by (2.6). Let $S \subset \mathbb{H}$ denote the right boundary of $\eta'([0,\widetilde{T}])$, and let λ denote Lebesgue measure on $\mathbb{R}$. Then there is a universal constant $c > 0$ such that $\lambda(g_{\widetilde{T}}(S)) > c$. The same likewise holds if we instead take S to be the left boundary of $\eta'([0,\widetilde{T}])$.*

Proof. We assume without loss of generality that S is equal to the right boundary of $\eta'([0,\widetilde{T}])$. Let $(\mathcal{B}_t)_{t\ge0}$ be a Brownian motion in $\mathbb{C}$ independent of η', and for each $z \in \mathbb{H}$ let $\mathbb{P}^z[\cdot]$ be the law under which $\mathcal{B}_0 = z$. For each $z \in \mathbb{H}$ let I_z be the horizontal line segment from $i + z - 1$ to $i + z + 1$, and define the two stopping times $\overline{\tau}$ and $\widehat{\tau}$ for $(\mathcal{B}_t)_{t\ge0}$ (conditioned on $\mathcal{F}_{\widetilde{T}}$) as follows.

$$\overline{\tau} = \inf\{t \ge 0 : \mathcal{B}_t \notin \mathbb{H} \setminus \eta'([0,\widetilde{T}])\}, \qquad \widehat{\tau} = \inf\{t \ge 0 : \operatorname{Im} \mathcal{B}_t = 2\}.$$

By conformal invariance of Brownian motion and the explicit expression for the Poisson kernel of $\mathbb{H}$, see [17, Exercise 2.23], we have

$$\begin{aligned} \lambda(g_t(S)) &= \lim_{y\to\infty} \pi y \mathbb{P}^{iy}[\mathcal{B}_{\overline{\tau}} \in S | \mathcal{F}_{\widetilde{T}}] \\ &\ge \lim_{y\to\infty} \pi y \mathbb{P}^{iy}[\mathcal{B}_{\widehat{\tau}} \in I_{\eta'(\widetilde{T})} | \mathcal{F}_{\widetilde{T}}] \times \inf_{z \in I_{\eta'(\widetilde{T})}} \mathbb{P}^z[\mathcal{B}_{\overline{\tau}} \in S | \mathcal{F}_{\widetilde{T}}]. \end{aligned} \tag{2.8}$$

By using the explicit formula for the Poisson kernel of $\mathbb{H}$ it holds a.s. that

$$\lim_{y\to\infty} \pi y \mathbb{P}^{iy}[\mathcal{B}_{\widehat{\tau}} \in I_{\eta'(\widetilde{T})} | \mathcal{F}_{\widetilde{T}}] = 2. \tag{2.9}$$

For each $z \in \mathbb{H}$ let K_z' be the half-line $K_z' := z + \mathbb{R}^-$. Let S_z' be the subset of the boundary of $\mathbb{H} \setminus K_z'$ corresponding to the lower part of K_z' (viewing the boundary of $\mathbb{H} \setminus K_z'$ as a collection of prime ends), and let $\tau' := \inf\{t \ge 0 : \mathcal{B}_t \notin \mathbb{H} \setminus K_{\eta'(\widetilde{T})}'\}$. If $\operatorname{Im} \mathcal{B}_0 \ge 2$ it holds by a geometric argument that $\{\mathcal{B}_{\tau'} \in S_{\eta'(\widetilde{T})}'\} \subset \{\mathcal{B}_{\overline{\tau}} \in S\}$. Therefore

$$\inf_{z \in I_{\eta'(\widetilde{T})}} \mathbb{P}^z[\mathcal{B}_{\overline{\tau}} \in S | \mathcal{F}_{\widetilde{T}}] \ge \inf_{z \in I_{\eta'(\widetilde{T})}} \mathbb{P}^z[\mathcal{B}_{\tau'} \in S_{\eta'(\widetilde{T})}' | \mathcal{F}_{\widetilde{T}}] = \inf_{z \in I_i} \mathbb{P}^z[\mathcal{B}_{\tau'} \in S_i'] \succeq 1.$$

This estimate combined with (2.8) and (2.9) implies the assertion of the lemma. □

Proof of upper bound in (2.2) for $T = 1$. It is sufficient to prove that for small enough $u > 0$,

$$\mathbb{P}[E_{z_L,z_R}^1] \preceq (z_R z_L)^{-u} z_R^{\rho/\kappa'} z_L^{\rho/\kappa'} (z_R + z_L)^{\rho^2/(2\kappa')}. \tag{2.10}$$

Recall the definition (2.5) of σ_u. The process $(M_{\sigma_u \wedge t})_{t\ge0}$ is a bounded martingale if u is chosen small enough that $M_0 \le (z_L z_R)^u$, hence the optional stopping theorem implies $\mathbb{E}[M_{\sigma_u}] = M_0$. This implies further that

$$\begin{aligned} \mathbb{P}[M_{\sigma_u} = (z_R z_L)^u] &= (z_R z_L)^{-u} \mathbb{E}[M_{\sigma_u}] \\ &= (z_R z_L)^{-u} \mathbb{E}[M_0] = (z_R z_L)^{-u} z_R^{\rho/\kappa'} z_L^{\rho/\kappa'} (z_R + z_L)^{\rho^2/(2\kappa')}. \end{aligned} \tag{2.11}$$

We claim that for each $u > 0$ there exists some sufficiently small $s > 0$ only depending on u such that $\{\sigma_u \geq 1\} \subset F_{(z_R z_L)^s}$ for sufficiently small z_L, z_R, with the latter event defined as in Lemma 2.4 with $\varepsilon = (z_R z_L)^s$. Define the stopping time $\widetilde{T}_s$ by $\widetilde{T}_s := \inf\{t \geq 0 : \operatorname{Im}(\eta'(t)) \geq (z_R z_L)^s\}$. If $\widetilde{T}_s \leq 1$, i.e. $F_{(z_R z_L)^s}$ does not occur, and $M_1 \neq 0$, Lemma 2.5 implies that $z^L_{\widetilde{T}_s} - W_{\widetilde{T}_s} > c(z_R z_L)^s$ and $W_{\widetilde{T}_s} - z^R_{\widetilde{T}_s} > c(z_R z_L)^s$, where c is the constant in the statement of the lemma. Hence $M_{\widetilde{T}_s} > (z_R z_L)^u$ for sufficiently small s, z_L, z_R, so $\sigma_u < 1$ and the claim follows. We have

$$E^1_{z_L,z_R} \cap \{M_{\sigma_u} = 0\} \subset \{\sigma_u \geq 1\} \subset F_{(z_R z_L)^s}.$$

By Lemma 2.4 and (2.11) we have

$$\begin{aligned}\mathbb{P}\big[E^1_{z_L,z_R}\big] &\leq \mathbb{P}[F_{(z_R z_L)^s}] + \mathbb{P}\big[E^1_{z_L,z_R}; M_{\sigma_u} = (z_R z_L)^u\big] \\ &\preceq z_R^{\rho/\kappa' + o_u(1)} z_L^{\rho/\kappa' + o_u(1)} (z_R + z_L)^{\rho^2/(2\kappa')}.\end{aligned}$$

The result follows since $u > 0$ was arbitrary. □

2.2. *Euclidean lower bound*

Recall the definition of $E^1_{z_L,z_R}$ and $(M_t)_{t\geq 0}$ in Proposition 2.1 and Lemma 2.2, respectively. In this section we will prove that $\mathbb{P}[E^1_{z_L,z_R}]$ is bounded below by $\mathbb{E}[M_1] = M_0$ up to $o(1)$ errors in the exponents. In order to prove this estimate we need to show that the contribution to $\mathbb{E}[M_1]$ of large values of M_1 is very small.

Proof of lower bound in (2.2) for $T = 1$. Since $(M_t)_{t\geq 0}$ is a martingale by Lemma 2.3, we have that

$$M_0 = \mathbb{E}[M_1] = \mathbb{E}\big[M_1; M_1 > (z_R z_L)^{-u}\big] + \mathbb{E}\big[M_1; 0 < M_1 \leq (z_R z_L)^{-u}\big].$$

Since $|z^q_t - W_t| \leq |z^R_t - z^L_t|$ for $q \in \{L, R\}$ and all of the exponents in the definition of M_t are positive and sum to $\frac{\kappa'}{2} - 2$, we have that $M_t \leq |z^R_t - z^L_t|^{\kappa'/2-2}$. Therefore,

$$\begin{aligned}\mathbb{E}\big[M_1; M_1 > (z_R z_L)^{-u}\big] &\leq \mathbb{E}\big[\big|z^R_1 - z^L_1\big|^{\kappa'/2-2}; \big|z^R_1 - z^L_1\big|^{\kappa'/2-2} > (z_R z_L)^{-u}\big] \\ &= o^{\infty}_{z_R z_L}(z_R z_L),\end{aligned} \tag{2.12}$$

where the last equality follows from large deviation estimates for Bessel processes as in the proof of Lemma 2.3. It follows that $\mathbb{E}[M_1; M_1 > (z_R z_L)^{-u}] < \frac{1}{2} M_0$ if either z_R or z_L is sufficiently small, and therefore

$$M_0 \preceq \mathbb{E}\big[M_1; 0 < M_1 \leq (z_R z_L)^{-u}\big] \leq (z_R z_L)^{-u} \mathbb{P}(M_1 > 0),$$

which implies that

$$\mathbb{P}\big[E^1_{z_L,z_R}\big] = \mathbb{P}[M_1 > 0] \succeq z_R^{\rho/\kappa' + o_{z_R}(1)} z_L^{\rho/\kappa' + o_{z_L}(1)} (z_R + z_L)^{\rho^2/(2\kappa')}.$$ □

2.3. *Proof of Proposition 2.1*

By the scaling property of SLE it is sufficient to prove the estimates (2.2) and (2.3) for $T = 1$ and $r = 1$, respectively. Combining the results of Sections 2.1 and 2.2 we have proved (2.2) for $T = 1$. To prove the estimate (2.3) and hence complete the proof of Proposition 2.1, it suffices to show that T_1 is of order 1 with high probability.

Proof of Proposition 2.1. The lower bound of (2.3) is immediate from (2.2), since the half-plane capacity of η' stopped upon hitting $\partial\mathbb{D} \cap \mathbb{H}$ is bounded above by the half-plane capacity of $\mathbb{D} \cap \mathbb{H}$, which implies that we a.s. have $T_1 \preceq 1$. To complete the proof of the proposition we need to prove the upper bound of (2.3). Let λ denote Lebesgue measure on $\mathbb{R}$. By [17, Equation (3.14)] and the surrounding text we have

$$\lambda\big(g_{T_1}\big(\eta'\big([0, T_1]\big)\big)\big) \geq c > 0,$$

where the constant c is universal. Conditioned on $E^{T_1}_{z_L,z_R}$ we have

$$\left(z^R_{T_1} - W_{T_1}\right) + \left(W_{T_1} - z^L_{T_1}\right) \geq \lambda\left(g_{T_1}\left(\eta'\left([0, T_1]\right)\right)\right),$$

hence at least one of the following inequalities holds on $E^{T_1}_{z_L,z_R}$: $z^R_{T_1} - W_{T_1} \geq c/2$ or $W_{T_1} - z^L_{T_1} \geq c/2$. By large deviation estimates for Bessel processes as in the proof of Lemma 2.3 we have that for any $u > 0$

$$\mathbb{P}\left[E^{T_1}_{z_L,z_R}; T_1 < (z_L z_R)^u\right] \preceq \sum_{q\in\{L,R\}} \mathbb{P}\left[\sup_{t\in[0,(z_L z_R)^u]} \left|z^q_t - W_t\right| \geq c/2\right] = o^\infty_{z_L z_R}(z_L z_R). \tag{2.13}$$

We conclude the proof of the proposition by observing that

$$\mathbb{P}\left[E^{T_1}_{z_L,z_R}\right] \leq \mathbb{P}\left[E^{(z_L z_R)^u}_{z_L,z_R}\right] + \mathbb{P}\left[E^{T_1}_{z_L,z_R}; T_1 < (z_L z_R)^u\right],$$

which by (2.10) and (2.13) implies that

$$\mathbb{P}\left[E^{T_1}_{z_L,z_R}\right] \leq (z_L + z_R)^{\rho^2/(2\kappa')} z_L^{\rho/\kappa' + o_{z_L}(1)} z_R^{\rho/\kappa' + o_{z_R}(1)}. \qquad \square$$

3. The quantum exponent

In this section, we will calculate the probability of a certain event associated with the γ-quantum boundary measure of a $\frac{3}{2}\gamma$-quantum wedge. Throughout this section we will make use of the convention introduced in Section 1.2.3, namely we fix $\kappa' > 8$ and $\gamma = 4/\sqrt{\kappa'} \in (0, \sqrt{2})$ and do not make dependence on κ', γ explicit. The main result of this section is the following proposition.

Proposition 3.1. *Let $\mathfrak{h}$ be the circle average embedding of a $\frac{3}{2}\gamma$-quantum wedge in $(\mathbb{H}, 0, \infty)$, as in Definition 1.5, let $\nu_{\mathfrak{h}}$ be the γ-quantum boundary measure induced by $\mathfrak{h}$, and let η' be a chordal $\mathrm{SLE}_{\kappa'}$ in $\mathbb{H}$ from 0 to ∞ independent of $\mathfrak{h}$. For $r > 0$, let*

$$T_r := \inf\left\{t > 0 : \left|\eta'(t)\right| = r\right\} \tag{3.1}$$

and for $\delta > 0$ let

$$E^{T_r}_\delta := \left\{\nu_{\mathfrak{h}}\left(\eta'\left([0, T_r]\right) \cap \mathbb{R}_-\right) \leq \delta \text{ and } \nu_{\mathfrak{h}}\left(\eta'\left([0, T_r]\right) \cap \mathbb{R}_+\right) \leq \delta\right\}. \tag{3.2}$$

For each fixed $r \in (0, 1]$, we have

$$\mathbb{P}\left[E^{T_r}_\delta\right] = \delta^{4/\gamma^2 + o_\delta(1)}.$$

3.1. *Moment estimates for the quantum boundary measure*

In this subsection we will state some estimates for the moments of a certain quantity associated with the quantum boundary measure induced by a free-boundary GFF on $\mathbb{H}$, which will be proven in the next two subsections. Let Q be as in (1.3) and fix $\alpha \in [0, Q)$. For the proof of Proposition 3.1 we only need the case where $\alpha = \frac{3}{2}\gamma$ (note that $\alpha < Q$ for $\gamma \in (0, \sqrt{2})$), but it is no more difficult to treat the general case. Also let

$$a := Q - \alpha. \tag{3.3}$$

Let h^F be a free boundary GFF on $\mathbb{H}$, normalized so that its semicircle average over $\partial B_1(0) \cap \mathbb{H}$ is 0. Let $h := h^F - \alpha \log|\cdot|$, so that h is an unscaled α-quantum wedge as defined in [5, Section 1.4]. Let ν_h be the γ-quantum boundary measure induced by h.

Fix $r \in (0, 1]$. For $\delta > 0$, let $x_{\delta,L}$ and $x_{\delta,R}$ be the non-negative random variables such that $\nu_h([-x_{\delta,L}, 0]) = \nu_h([0, x_{\delta,R}]) = \delta$. Let $\overline{x}_{\delta,L} = x_{\delta,L} \wedge r$ and $\overline{x}_{\delta,R} = x_{\delta,R} \wedge r$. In this subsection we will compute the joint moments of $\overline{x}_{\delta,L}$ and $\overline{x}_{\delta,R}$. This calculation, together with the estimate (2.3) of Section 2, will be used to compute the probability in Proposition 3.1.

Proposition 3.2. *Let $\overline{x}_{\delta,L}$ and $\overline{x}_{\delta,R}$ for $\delta > 0$ be as above. For $\lambda_1, \lambda_2 > 0$ we have*

$$\lim_{\delta \to 0} \frac{\log \mathbb{E}[\overline{x}_{\delta,L}^{\lambda_1} \overline{x}_{\delta,R}^{\lambda_2}]}{\log \delta^{-1}} = \frac{a - \sqrt{a^2 + 4(\lambda_1 + \lambda_2)}}{\gamma}, \tag{3.4}$$

with a as in (3.3).

We will deduce Proposition 3.2 from two similar propositions which concern moments of only a single random variable (rather than joint moments) and imply the upper and lower bounds in Proposition 3.2, respectively.

Proposition 3.3. *Suppose we are in the setting of Proposition 3.2. For each $\lambda > 0$,*

$$\lim_{\delta \to 0} \frac{\log \mathbb{E}[\overline{x}_{\delta,L}^{\lambda}]}{\log \delta^{-1}} = \frac{a - \sqrt{a^2 + 4\lambda}}{\gamma}, \tag{3.5}$$

with a as in (3.3).

Proposition 3.4. *Let h be an unscaled α-quantum wedge as above. For $\delta > 0$, let x_δ be such that $\nu_h([-x_\delta, x_\delta]) = \delta$. Also fix $r > 0$ and let $\overline{x}_\delta := x_\delta \wedge r$. Then*

$$\lim_{\delta \to 0} \frac{\log \mathbb{E}[\overline{x}_{\delta}^{\lambda}]}{\log \delta^{-1}} = \frac{a - \sqrt{a^2 + 4\lambda}}{\gamma},$$

with a as in (3.3).

The following lemma tells us that in order to prove Propositions 3.2, 3.3, and 3.4, we need only prove the upper bound for the limit in Proposition 3.3 and a lower bound for the limit in Proposition 3.4.

Lemma 3.5. *Let $\overline{x}_{\delta,L}, \overline{x}_{\delta,R}$ be defined as in the beginning of this subsection and let $\overline{x}_\delta$ be as in Proposition* 3.4 (*with the same choice of r*). *For each $\lambda_1, \lambda_2 > 0$, we have*

$$\lim_{\delta \to 0} \frac{\log \mathbb{E}[\overline{x}_{\delta}^{\lambda_1+\lambda_2}]}{\log \delta^{-1}} \le \lim_{\delta \to 0} \frac{\log \mathbb{E}[\overline{x}_{\delta,L}^{\lambda_1} \overline{x}_{\delta,R}^{\lambda_2}]}{\log \delta^{-1}} \le \lim_{\delta \to 0} \frac{\log \mathbb{E}[\overline{x}_{\delta,L}^{\lambda_1+\lambda_2}]}{\log \delta^{-1}}. \tag{3.6}$$

Proof. By definition, we have $\overline{x}_\delta \le \overline{x}_{\delta,L} \wedge \overline{x}_{\delta,R}$, which gives the first inequality in (3.6). The second inequality follows from

$$\begin{aligned} \mathbb{E}\big[\overline{x}_{\delta,L}^{\lambda_1} \overline{x}_{\delta,R}^{\lambda_2}\big] &\le \mathbb{E}\big[(\overline{x}_{\delta,L} + \overline{x}_{\delta,R})^{\lambda_1+\lambda_2}\big] \\ &\le 2^{\lambda_1+\lambda_2} \mathbb{E}\big[\overline{x}_{\delta,L}^{\lambda_1+\lambda_2} + \overline{x}_{\delta,R}^{\lambda_1+\lambda_2}\big] \\ &= 2^{\lambda_1+\lambda_2+1} \mathbb{E}\big[\overline{x}_{\delta,L}^{\lambda_1+\lambda_2}\big]. \end{aligned}$$ □

The proofs of the lower bound in Proposition 3.3 and the upper bound in Proposition 3.4 will be completed in the next two subsections. Both proofs use arguments similar to those found in [6, Section 4]. In particular, both estimates are established by first proving the semicircle average version of the estimate and then showing that the exponential of γ times the semicircle average is in some sense a good approximation for the quantum measure.

3.2. *Circle average KPZ and tail estimates*

In this subsection we will establish several lemmas which are similar to various results in [6, Section 4] and which are needed for the proofs of the results in Section 3.1. Throughout this subsection and the next, we assume we are in the setting of Section 3.1, and we use the notation introduced there plus the following additional notation. For $\varepsilon > 0$, let $h_\varepsilon(z)$ be the semicircle average of h about $\partial B_\varepsilon(z) \cap \mathbb{H}$. For $t \in \mathbb{R}$, let

$$V_t := -h_{e^{-t}}(0) + Qt, \tag{3.7}$$

with Q as in (1.3). As explained in [6, Section 6.1], V_t is distributed as $\mathcal{B}_{2t} + at$ where $\mathcal{B}$ is a standard linear two-sided Brownian motion and a is as in (3.3) (here we recall that h has a $-\alpha$-log singularity at 0).

Let

$$A_\delta := \frac{2}{\gamma} \log \delta^{-1} \quad \text{and} \quad \tau_\delta := \inf\{t \geq 0 : V_t = A_\delta\}. \tag{3.8}$$

As we will see, $\exp(-\tau_\delta)$ is a good estimator of $\overline{x}_{\delta,L}$. The semicircle average version of Proposition 3.3 is the following simple fact regarding Brownian motion.

Lemma 3.6. *For $\lambda > 0$ we have*

$$\lim_{\delta \to 0} \frac{\log \mathbb{E}[e^{-\lambda \tau_\delta}]}{\log \delta^{-1}} = \frac{a - \sqrt{a^2 + 4\lambda}}{\gamma}. \tag{3.9}$$

Proof. Write $V_t = \mathcal{B}_{2t} + at$, with $\mathcal{B}$ a standard linear Brownian motion. Let

$$\beta := \frac{\sqrt{a^2 + 4\lambda} - a}{2}$$

so that $\beta^2 + a\beta = \lambda$. We observe that $t \mapsto \exp(\beta \mathcal{B}_{2t} - \beta^2 t)$ is a non-negative martingale. Furthermore, using that $\alpha < Q$ so that $a > 0$, for $t \leq \tau_\delta$ we have $\mathcal{B}_{2t} \leq A_\delta$. In particular, $t \mapsto \exp(\beta \mathcal{B}_{2\tau_\delta \wedge t} - \beta^2 \tau_\delta \wedge t)$ is bounded and $\mathbb{P}[\tau_\delta < \infty] = 1$. By the optional stopping theorem,

$$\mathbb{E}\big[\exp\big(\beta \mathcal{B}_{2\tau_\delta} - \beta^2 \tau_\delta\big)\big] = 1.$$

Since $\mathcal{B}_{2\tau_\delta} = A_\delta - a\tau_\delta$, we have

$$\mathbb{E}\big[e^{-\lambda \tau_\delta}\big] = e^{-\beta A_\delta} = \delta^{2\beta/\gamma} = \delta^{\frac{\sqrt{a^2+4\lambda}-a}{\gamma}}$$

which implies the statement of the lemma. $\square$

To deduce Proposition 3.3 from Lemma 3.6, we first need a lower bound for the γ-quantum boundary length of an interval. The needed estimate can be deduced in a similar manner to [6, Lemma 4.5], but for brevity we give an alternative argument based on the theory of Gaussian multiplicative chaos [15,30].

Lemma 3.7. *Let h^F be as in Section 3.1 and let ν_{h^F} be its associated γ-quantum boundary measure. Let $I \subset \partial\mathbb{H}$ be a bounded open interval. Then $\nu_{h^F}(I)$ has finite moments of all negative orders.*

Proof. This follows from general Gaussian multiplicative chaos theory applied to ν_{h^F}. See, e.g. [30, Theorems 2.11 and 2.12]. See also [28, Section 4.4] for an approximation scheme for ν_{h^F} to which Gaussian multiplicative chaos theory applies (the approximation scheme is stated in the context of the unit disk $\mathbb{D}$, but a similar formula works for the upper half-plane). $\square$

Lemma 3.8. *Let τ be a stopping time for the filtration $\mathcal{F}_t = \sigma(V_s : s \in (-\infty, t])$. Also fix $u > 0$. We have*

$$\mathbb{P}\left[\nu_h\big([0, e^{-\tau}]\big) < \delta^u \exp\left(-\frac{\gamma}{2} V_\tau\right) \Big| \mathcal{F}_\tau\right] = o_\delta^\infty(\delta) \tag{3.10}$$

as $\delta \to 0^+$, at a deterministic rate which does not depend on τ.

Proof. Fix a stopping time τ as in the statement of the lemma. The restriction of h to $B_{e^{-\tau}}(0) \cap \mathbb{H}$ is determined by the orthogonal projection of h onto the set of functions with mean zero on all semicircles centered at 0 together with the values of V_t for $t \geq \tau$ (cf. [5, Lemma 4.1]). Since $t \mapsto V_t$ has the law of a two-sided drifted Brownian motion normalized to vanish at 0, it follows from the strong Markov property of Brownian motion that the conditional law given $\mathcal{F}_\tau$ of the restriction of $h - h_{e^{-\tau}}(0)$ to $B_{e^{-\tau}}(0) \cap \mathbb{H}$ is that of a free boundary GFF restricted to $B_{e^{-\tau}}(0) \cap \mathbb{H}$ and normalized so that its semicircle average vanishes on $\partial B_{e^{-\tau}}(0) \cap \mathbb{H}$. It follows from the construction of ν_h via semicircle averages (see [6, Section 6]) that $e^{-\frac{\gamma}{2} h_{e^{-\tau}}(0)} \nu_h([0, e^{-\tau}])$ is determined by the restriction of $h - h_{e^{-\tau}}(0)$ to $B_{e^{-\tau}}(0) \cap \mathbb{H}$.

Let $\phi(z) := e^{-\tau} z$. Let $\widetilde{h} = h \circ \phi + Q \log e^{-\tau}$, with Q as in (1.3). By the boundary analogue of [6, Proposition 2.1], we have $\nu_h([0, e^{-\tau}]) = \nu_{\widetilde{h}}([0, 1])$. Let $\widetilde{h}_*$ be the restriction to $B_1(0) \cap \mathbb{H}$ of the field $h \circ \phi - h_{e^{-\tau}}(0)$. By conformal invariance of the free boundary GFF and the discussion above, it follows that the conditional law given $\mathcal{F}_\tau$ of the restriction of $h \circ \phi$ to $B_1(0) \cap \mathbb{H}$ is the same as the law of $h|_{B_1(0) \cap \mathbb{H}}$, modulo a global additive constant. The semicircle average of $h \circ \phi$ over $\partial B_1(0) \cap \mathbb{H}$ is given by $h_{e^{-\tau}}(0)$. It therefore follows that the conditional law of $\widetilde{h}_*$ given $\mathcal{F}_\tau$ is the same as the law of $h|_{B_1(0) \cap \mathbb{H}}$.

By the definition of the γ-quantum boundary measure we have

$$\nu_{\widetilde{h}}\big([0, 1]\big) = \exp\left(\frac{\gamma}{2} h_{e^{-\tau}}(0) - \frac{\gamma}{2} Q \tau\right) \nu_{\widetilde{h}_*}\big([0, 1]\big) = \exp\left(-\frac{\gamma}{2} V_\tau\right) \nu_{\widetilde{h}_*}\big([0, 1]\big). \tag{3.11}$$

By Lemma 3.7, the conditional law given $\mathcal{F}_\tau$ of $\nu_{\widetilde{h}_*}([0, 1])$ has moments of all negative orders, so by Chebyshev's inequality, for each $\delta > 0$ we have that $\mathbb{P}[\nu_{\widetilde{h}_*}([0, 1]) \leq \delta^u | \mathcal{F}_\tau]$ decays faster than any power of δ. We thus obtain the statement of the lemma. □

3.3. *Proof of the moment estimates*

In this subsection we will prove the upper bound in Proposition 3.3 and the lower bound in Proposition 3.4, thereby completing the proof of the propositions in Section 3.3. Our first proof is similar to the argument given in [6, Section 4.4].

Proof of Proposition 3.3, upper bound. Fix $s \in (0, 1)$. For $\delta > 0$, let τ_{δ^s} be as in (3.8) with δ^s in place of δ, so that with A_δ as in (3.8) we have

$$\tau_{\delta^s} = \inf\{t \geq 0 : V_t = s A_\delta\}.$$

Let $\widehat{x}_{\delta,s} := \exp(-\tau_{\delta^s})$. For $\lambda > 0$, we have

$$\mathbb{E}\big[\overline{x}_{\delta,L}^\lambda\big] \leq \mathbb{E}\big[\widehat{x}_{\delta,s}^\lambda\big] + \mathbb{E}\big[\overline{x}_{\delta,L}^\lambda ; \widehat{x}_{\delta,s} \leq \overline{x}_{\delta,L}\big]. \tag{3.12}$$

By Lemma 3.6 (applied with δ^s in place of δ) we have

$$\lim_{\delta \to 0} \frac{\log \mathbb{E}[\widehat{x}_{\delta,s}^\lambda]}{\log \delta^{-1}} = s \frac{a - \sqrt{a^2 + 4\lambda}}{\gamma}. \tag{3.13}$$

On the event $\{\widehat{x}_{\delta,s} \leq \overline{x}_{\delta,L}\}$ we have

$$\nu_h\big([-\widehat{x}_{\delta,s}, 0]\big) \leq \delta = \delta^{1-s} \exp\left(-\frac{\gamma}{2} V_{\tau_{\delta^s}}\right).$$

By Lemma 3.8, $\mathbb{P}[\widehat{x}_{\delta,s} \leq \overline{x}_{\delta,L}] = o_\delta^\infty(\delta)$. By definition, we have $\overline{x}_{\delta,L} \leq r$, so $\mathbb{E}[\overline{x}_{\delta,L}^\lambda ; \widehat{x}_{\delta,s} \leq \overline{x}_{\delta,L}]$ decays faster than any power of δ. Since s is arbitrary, the desired upper bound now follows from (3.12) and (3.13). □

Finally we prove the lower bound in Proposition 3.4.

Proof of Proposition 3.4, lower bound. For $\delta > 0$ let τ_δ be as in (3.8) and let $\widehat{x}_\delta := e^{-\tau_\delta}$. We note that $\tau_\delta \geq 0$, so $\widehat{x}_\delta \leq 1$. Also let $\mathcal{F}_{\tau_\delta} := \sigma(V_t : t \leq \tau_\delta)$. We claim there exists a constant $c > 0$ (independent of δ) such that

$$\mathbb{P}[\widehat{x}_\delta \leq \overline{x}_\delta | \mathcal{F}_{\tau_\delta}] = \mathbb{P}\big[\nu_h\big([-\widehat{x}_\delta, \widehat{x}_\delta]\big) \leq \delta | \mathcal{F}_{\tau_\delta}\big] \geq c \quad \text{a.s.} \tag{3.14}$$

Assuming that (3.14) holds, we get that for $\lambda > 0$,

$$\mathbb{E}\big[\overline{x}_\delta^\lambda\big] \geq \mathbb{E}\big[\widehat{x}_\delta^\lambda \mathbb{P}[\widehat{x}_\delta \leq \overline{x}_\delta | \mathcal{F}_{\tau_\delta}]\big] \geq c\mathbb{E}\big[\widehat{x}_\delta^\lambda\big],$$

which implies the desired lower bound.

It remains only to prove (3.14). To this end, we define ϕ, $\widetilde{h}$, and $\widetilde{h}_*$ as in the proof of Lemma 3.8 with $\tau = \tau_\delta$, so that the conditional law of $\widetilde{h}_*$ given $\mathcal{F}_{\tau_\delta}$ is the same as the law of $h|_{B_1(0)\cap\mathbb{H}}$; and (as in (3.11)) we have

$$\nu_h\big([-\widehat{x}_\delta, \widehat{x}_\delta]\big) = \exp\left(-\frac{\gamma}{2} V_{\tau_\delta}\right) \nu_{\widetilde{h}_*}\big([-1,1]\big) = \delta \nu_{\widetilde{h}_*}\big([-1,1]\big).$$

It is easy to see that $\mathbb{P}[\nu_{\widetilde{h}_*}([-1,1]) \leq 1] > 0$, and (3.14) follows. $\square$

3.4. *Proof of Proposition 3.1*

Since $\mathfrak{h}$ is the circle average embedding of a quantum wedge and h is a free-boundary GFF normalized so that its semicircle average over $\partial B_1(0) \cap \mathbb{H}$ vanishes, Remark 1.6 implies that we can couple h and $\mathfrak{h}$ such that $h \equiv \mathfrak{h}$ on $\mathbb{D} \cap \mathbb{H}$. For $\delta > 0$, let $\mathfrak{x}_{\delta,L}$ and $\mathfrak{x}_{\delta,R}$ be chosen so that $\nu_{\mathfrak{h}}([-\mathfrak{x}_{\delta,L}, 0]) = \nu_{\mathfrak{h}}([0, \mathfrak{x}_{\delta,R}]) = \delta$ (as in Remark 1.10). By our choice of coupling we have $\mathfrak{x}_{\delta,L} \wedge r = \overline{x}_{\delta,L}$ and $\overline{\mathfrak{x}}_{\delta,R} \wedge r = \overline{x}_{\delta,R}$.

Assume that the SLE curve η' is sampled independently from h and $\mathfrak{h}$. Then $E_\delta^{T_r}$ is the event that η' reaches $\partial B_r(0)$ before hitting either $-\overline{\mathfrak{x}}_{\delta,L}$ or $\overline{\mathfrak{x}}_{\delta,R}$ (in particular, $E_\delta^{T_r}$ occurs a.s. if $\overline{\mathfrak{x}}_{\delta,L} > r$ and $\overline{\mathfrak{x}}_{\delta,R} > r$). By Proposition 2.1, for each $u > 0$ we have

$$\overline{x}_{\delta,L}^{\rho/\kappa'+u} \overline{x}_{\delta,R}^{\rho/\kappa'+u} (\overline{x}_{\delta,L} + \overline{x}_{\delta,R})^{\rho^2/(2\kappa')} \preceq \mathbb{P}\big[E_\delta^{T_r} | \mathfrak{h}\big] \preceq \overline{x}_{\delta,L}^{\rho/\kappa'-u} \overline{x}_{\delta,R}^{\rho/\kappa'-u} (\overline{x}_{\delta,L} + \overline{x}_{\delta,R})^{\rho^2/(2\kappa')} \tag{3.15}$$

with $\rho = \kappa' - 4$, as in Section 2, and the implicit constants deterministic and independent of δ (but possibly depending on r and u). From the inequality

$$\frac{1}{2}\big(\varepsilon_1^p + \varepsilon_2^p\big) \leq (\varepsilon_1 + \varepsilon_2)^p \leq 2^p \big(\varepsilon_1^p + \varepsilon_2^p\big) \quad \text{for all } p, \varepsilon_1, \varepsilon_2 > 0$$

and symmetry between $\overline{x}_{\delta,L}$ and $\overline{x}_{\delta,R}$, we infer that the expectations of the left and right sides of (3.15) are bounded above and below by constants (depending only on κ') times

$$\mathbb{E}\big[\overline{x}_{\delta,L}^{\rho/\kappa'+o_u(1)} \overline{x}_{\delta,R}^{(\rho^2+2\rho)/(2\kappa')+o_u(1)}\big],$$

where here the $o_u(1)$ is deterministic and independent of δ. By Proposition 3.2 applied with $\alpha = \frac{3}{2}\gamma$, this latter quantity is of order $\delta^{4/\gamma^2+o_u(1)}$. Since u is arbitrary, we obtain the proposition. $\square$

4. Conclusion of the proof

In this section we will deduce (1.6) from Proposition 3.1 and thereby complete the proof of Theorem 1.1. Suppose we are in the setting of Section 1.1. Define the $\frac{3}{2}\gamma$-quantum wedge $\mathcal{W} = (\mathbb{H}, \mathfrak{h}, 0, \infty)$ and the $\mathrm{SLE}_{\kappa'}$ curve $\widetilde{\eta}'$ as in Remark 1.10. For $t > 0$ and $\delta > 0$, let $\widetilde{E}_\delta^t$ be as in (1.4). Equivalently, by Remark 1.10,

$$\widetilde{E}_\delta^t := \big\{\nu_{\mathfrak{h}}\big(\widetilde{\eta}'\big([0,t]\big) \cap \mathbb{R}_-\big) \leq \delta \text{ and } \nu_{\mathfrak{h}}\big(\widetilde{\eta}'\big([0,t]\big) \cap \mathbb{R}_+\big) \leq \delta\big\}. \tag{4.1}$$

For $r > 0$ let T_r be as in (3.1) and let $E_\delta^{T_r}$ be as in (3.2).

Roughly speaking, we will show that $\mathbb{P}[\widetilde{E}_\delta^1]$ is a good approximation of $\mathbb{P}[E_\delta^{T_1}]$. Combining this with Proposition 3.1 will complete the proof of (1.6). Showing that $\mathbb{P}[E_\delta^{T_1}]$ is less than or equal to $\mathbb{P}[\widetilde{E}_\delta^1]$ (up to $o(1)$ error in the exponent) is relatively simple. One just needs to notice that when $\widetilde{\eta}'$ exits the Euclidean unit ball, with overwhelmingly high probability it will contain a Euclidean ball of radius $\delta^{o_\delta(1)}$, and will therefore have quantum mass at least $\delta^{o_\delta(1)}$ with overwhelmingly high probability. Therefore $\widetilde{E}_\delta^t$ occurs for some $t \geq \delta^{o_\delta(1)}$. The details are provided in Section 4.1.

The upper bound of $\mathbb{P}[\widetilde{E}_\delta^1]$ in terms of $\mathbb{P}[E_\delta^{T_1}]$ is more difficult. One could worry that if $\widetilde{E}_\delta^1$ occurs, then the Euclidean size of $\widetilde{\eta}'([0,1])$ under the circle average embedding is very small with high probability. This scenario cannot be ruled out directly by using the quantum mass tail estimate because the upper tail only has a power law decay (see [30, Theorems 2.11 and 2.12]). In Section 4.2, by exploring the relationship between various embeddings of a $\frac{3}{2}\gamma$-quantum wedge, we will show that conditioned on $\widetilde{E}_\delta^1$, there is a uniformly positive probability that $E_\delta^{T_r}$ occurs for some δ-independent constant $r > 0$.

4.1. *Upper bound for $\sigma(\gamma)$*

We first prove an analogue of [5, Proposition 10.13], which in turn is an analogue of [6, Lemma 4.5].

Proposition 4.1. *Fix $\gamma \in (0, \sqrt{2})$ and let $(\mathbb{H}, \hbar, 0, \infty)$ be a $\frac{3}{2}\gamma$-quantum wedge under the circle average embedding (Definition 1.5). Let η' be an independent chordal $\mathrm{SLE}_{\kappa'}$ in $\mathbb{H}$ from 0 to ∞ parametrized by capacity. Then with T_1 as in (3.1), we have*

$$\mathbb{P}\big[\mu_\hbar\big(\eta'\big([0,T_1]\big)\big) \leq \delta\big] = o_\delta^\infty(\delta). \tag{4.2}$$

Proof. Let r denote $1/2$ times the radius of the largest Euclidean ball contained in $\eta'([0,T_1])$ and let z be the center of this ball. Then it suffices to show that

$$\mathbb{P}\big[\mu_{h^F}\big(B_r(z)\big) \leq \delta\big] = o_\delta^\infty(\delta),$$

where h^F has the law of a free boundary GFF with the additive constant fixed so that the average of h^F on $\mathbb{H} \cap \partial\mathbb{D}$ is equal to 0. The reason why we can replace $\hbar$ by h^F is that $\hbar|_{\mathbb{D}\cap\mathbb{H}}$ agrees in law with the restriction of $h^F - \frac{3}{2}\gamma\log|\cdot|$ to $\mathbb{D}\cap\mathbb{H}$ and $B_r(z) \subset \mathbb{D}\cap\mathbb{H}$, so $-\frac{3}{2}\gamma\log|\cdot|$ is positive on $B_r(z)$. As argued in the proof of [5, Proposition 10.13], the law of the random variable r^{-1} has an exponential tail at ∞ (although the argument there is for a whole plane, the same argument works for $\mathbb{H}$). In particular, for $\delta > 0$ we have

$$\mathbb{P}\big[r \leq \big(\log\delta^{-1}\big)^{-2}\big] = o_\delta^\infty(\delta).$$

Conditioned on $\eta'([0,T_1])$ (which is independent from h^F) the regular conditional law of the circle average $h_r^F(z)$ is that of a Gaussian with variance at most $-2\log r$ (see [6, Section 3.1]). Here we use the fact that $B_r(z)$ lies at distance at least r from $\mathbb{R}$. By the Gaussian tail bound, for each fixed $s \in (0,1)$ we have

$$\mathbb{P}\big[e^{\gamma h_r^F(z)} \leq \delta^s | r \geq \big(\log\delta^{-1}\big)^{-2}\big] = o_\delta^\infty(\delta).$$

On the other hand, by [6, Lemma 4.6] we have that

$$\mathbb{P}\big[\mu_{h^F}\big(B_r(z)\big) \leq \delta | r \geq \big(\log\delta^{-1}\big)^{-2}, e^{\gamma h_r^F(z)} \geq \delta^s\big] = o_\delta^\infty(\delta).$$

The proof concludes. □

For fixed $s > 0$, we have

$$\mathbb{P}\big[E_\delta^{T_1}\big] \leq \mathbb{P}\big[E_\delta^{\delta^s}\big] + \mathbb{P}\big[T_1 \leq \delta^s\big]. \tag{4.3}$$

By Proposition 4.1, for each $s > 0$,

$$\lim_{\delta \to 0} \frac{\log \mathbb{P}[T_1 \le \delta^s]}{\log \delta^{-1}} = -\infty. \tag{4.4}$$

By Lemma 1.9,

$$\lim_{\delta \to 0} \frac{\log \mathbb{P}[\widetilde{E}_\delta^{\delta^s}]}{\log \delta^{-1}} = -(1 - s/2)\sigma(\gamma), \tag{4.5}$$

with $\sigma(\gamma)$ as in (1.5). By Proposition 3.1,

$$\lim_{\delta \to 0} \frac{\log \mathbb{P}[E_\delta^{T_1}]}{\log \delta^{-1}} = -\frac{4}{\gamma^2}. \tag{4.6}$$

Since s is arbitrary, we can combine (4.3), (4.4), (4.5), and (4.6) to obtain $\sigma(\gamma) \le 4/\gamma^2$, which is the upper bound in (1.6).

4.2. *Lower bound for $\sigma(\gamma)$*

4.2.1. *Notation for quantum surfaces*

Let $\mathcal{W}$ be the $\frac{3}{2}\gamma$-quantum wedge in Section 1. In the remainder of this section, we will consider several different parameterizations of $\mathcal{W}$. Recall that $\mathcal{W}$ is an equivalence class of 4-tuples (D, h, a, b), with $D \subset \mathbb{C}$, h a distribution on D, and $a, b \in \partial D$ where the equivalence relation is defined in terms of transformations on the form (1.2).

We will consider two coordinate systems: $(\mathbb{H}, 0, \infty)$ and $(\mathcal{S}, +\infty, -\infty)$ where $\mathcal{S} := \mathbb{R} \times (0, \pi)$ is a horizontal strip. Whenever we switch between the two coordinate systems, we assume that the corresponding objects are related as in (1.2) with f the *canonical coordinate transformation* between the two systems

$$z \mapsto -e^{-z}, \quad z \in \mathcal{S}. \tag{4.7}$$

Since a wedge only has two marked boundary points, knowing the coordinate system is not sufficient to determine the embedding of the surface, i.e. there is one free parameter corresponding to scaling $\mathbb{H}$ or horizontally translating $\mathcal{S}$. We will consider several different embeddings of $\mathcal{W}$ into each of $\mathbb{H}$ and $\mathcal{S}$. We slightly abuse notation by using the same symbols for embeddings into $\mathbb{H}$ and $\mathcal{S}$, always keeping in mind that the corresponding fields are related via the map (4.7). Hereafter, we will denote an embedding of h in a given coordinate system by $h^\bullet$, where $\bullet$ indicates the particular choice of embedding. After we define $h^\bullet$ in one coordinate system, we simultaneously define $h^\bullet$ in the other coordinate system by applying the coordinate change formula (1.2) with the mapping (4.7).

In $(\mathcal{S}, +\infty, -\infty)$, we let $X_t^\bullet$ be the average process of $h^\bullet$ along the vertical line segment $\{t\} \times (0, \pi)$, where $\bullet$ is the symbol representing the embedding. Before fixing the embedding of $\mathcal{W}$ into $(\mathcal{S}, +\infty, -\infty)$, the average process is defined up to a horizontal translation. Therefore we can fix the embedding of $\mathcal{W}$ on $(\mathcal{S}, +\infty, -\infty)$ by specifying the translation of the average process. We define the *circle average embedding* of $\mathcal{W}$ into $(\mathcal{S}, +\infty, -\infty)$ by requiring $\inf\{t \in \mathbb{R} : X_t^\bullet = 0\} = 0$ and denote the field (resp. average process) on $\mathcal{S}$ by h^C (resp. X^C). Note that the circle average embedding into $\mathcal{S}$ is the image of the circle average embedding into $\mathbb{H}$ (Definition 1.5) under the coordinate change (4.7), and in keeping with our convention the latter embedding will also be denoted by h^C in the remainder of this section. By the definition of a $\frac{3}{2}\gamma$-quantum wedge and by (1.2), under the circle average embedding in $\mathcal{S}$ there are standard Brownian motions $\mathcal{B}, \widehat{\mathcal{B}}$ such that $X_t^C = \mathcal{B}_{2t} - at$ for $t \ge 0$ and $X_t^C = \widehat{\mathcal{B}}_{-2t} - at$ for $t < 0$, where $a = Q - \frac{3}{2}\gamma$ and $\widehat{\mathcal{B}}$ is conditioned such that $X_t^C \ge 0$ for $t < 0$.

We will also consider the so-called *smooth centering embedding*, which we denote by h^S. It is introduced in the context of quantum cones in [5, Section 10.4.2]. Let ϕ be a fixed positive smooth function supported on $[0, 1]$ with integral 1. The smooth centering embedding of $\mathcal{W}$ into $(\mathcal{S}, +\infty, -\infty)$ is such that

$$\inf\left\{t \in \mathbb{R} : \int_{-\infty}^{\infty} X_s \phi(s - t)\, ds \le 0\right\} = 0.$$

Since $\lim_{t\to\infty} X_t = -\infty$ and $\lim_{t\to-\infty} X_t = +\infty$, h^S is well-defined almost surely. Heuristically, the smooth centering embedding is close to the circle average embedding, but is sometimes easier to work with since it involves the integral of the field against a smooth function, rather than the integral against a distribution (namely the uniform measure on a circle).

Different embeddings of $\mathcal{W}$ into $(\mathbb{H}, 0, \infty)$ (resp. $(\mathcal{S}, +\infty, -\infty)$) differ by a scaling (resp. horizontal translation). We let $\sigma^{\mathbb{H}}_{\bullet,\diamond}(\mathcal{W})$ (resp. $\sigma^{\mathcal{S}}_{\bullet,\diamond}(\mathcal{W})$) be the possibly random constant c such that $h^\diamond(\cdot) = h^\bullet(c\cdot) + Q\log c$ (resp. $h^\diamond(\cdot) = h^\bullet(\cdot + c)$). Note that the form of the transformation (4.7) implies that $\sigma^{\mathbb{H}}_{\bullet,\diamond}(\mathcal{W}) = e^{-\sigma^{\mathcal{S}}_{\bullet,\diamond}(\mathcal{W})}$.

Other embeddings of the $\frac{3}{2}\gamma$-quantum wedge are defined using the chordal $\mathrm{SLE}_{\kappa'}$ curve $\widetilde{\eta}'$, which we recall is first sampled independently from $\mathcal{W}$ and then parametrized by γ-quantum mass with respect to $\mathcal{W}$. One such embedding is the so-called *unit radius embedding* of $\mathcal{W}$, which we denote by h^U. On $(\mathbb{H}, 0, \infty)$ it is defined such that

$$1 = \inf\{r \geq 0 : \widetilde{\eta}'([0,1]) \subset B_r(0) \cap \mathbb{H}\}.$$

We will also have the occasion to consider the quantum surface

$$\mathcal{W}_* := (\widetilde{\eta}'([1,\infty)), h_*, \widetilde{\eta}'(1), \widetilde{\eta}'(\infty))$$

obtained by restricting $\mathcal{W}$ to $\widetilde{\eta}'([1,\infty))$. By [5, Lemma 9.3] (see also the proof of [5, Lemma 9.2]), $\mathcal{W}_*$ is a $\frac{3}{2}\gamma$-quantum wedge independent of $(Z_t)_{t\in[0,1]}$. We will mainly be interested in $\mathcal{W}_*$ embedded in $(\mathbb{H}, 0, \infty)$ in two ways. One is the circle average embedding h^C_*. The other one is defined as follows. Consider $\mathcal{W}$ embedded in $(\mathbb{H}, 0, \infty)$ under the unit radius embedding h^U. Let $\Psi : \mathbb{H} \setminus \widetilde{\eta}'([0,1]) \to \mathbb{H}$ be the conformal map such that $\Psi(\widetilde{\eta}'(1)) = 0$, $\Psi(\infty) = \infty$, and $\lim_{z\to\infty} \Psi(z)/z = 1$. Then $h^\Psi_* := h^U \circ \Psi^{-1} + Q\log|(\Psi^{-1})'|$ gives an embedding of $\mathcal{W}_*$ into $(\mathbb{H}, 0, \infty)$, which we will call the Ψ-*embedding* of $\mathcal{W}_*$.

4.2.2. *Smooth centering embedding and conclusion of the proof*

In Section 4.2.3, we will prove the following proposition, which is a variant of a result proved in [5, Section 10.4.2] for quantum cones. See Figure 1 for an illustration of objects involved in the statement. The proof will use similar techniques as the proof in [5].

Proposition 4.2. *Let $\mathcal{W} = (\mathbb{H}, h^S, 0, \infty)$ be a $\frac{3}{2}\gamma$-quantum wedge with the smooth centering embedding and let $\widetilde{\eta}'$ be an independent chordal $\mathrm{SLE}_{\kappa'}$ in $\mathbb{H}$ from 0 to ∞ parameterized by quantum mass with respect to h^S. There are deterministic constants $c, r > 0$ and an event G such that the following is true for all $\delta \in (0, \frac{1}{2})$.*

(i) $\mathbb{P}[G | \widetilde{E}^1_\delta] \geq c$.
(ii) *On $G \cap \widetilde{E}^1_\delta$, we have $\widetilde{\eta}'([0,1]) \not\subset \overline{B_r(0)}$.*

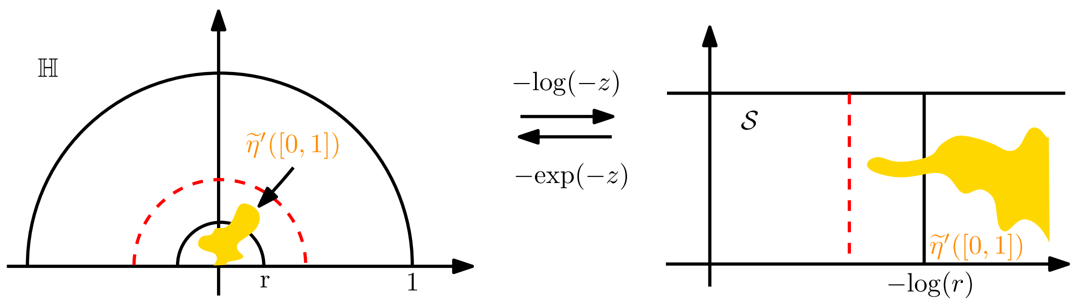

Fig. 1. Smooth centering embedding of $\mathcal{W}$ into $(\mathbb{H}, 0, \infty)$ (left) and $(\mathcal{S}, +\infty, -\infty)$ (right). On the left (resp. right), the dotted red semi-circle (resp. line segment) corresponds to the unit radius (resp. intersection with the imaginary axis) under the circle average embedding of the modified field h_M. Note that this semi-circle (resp. line segment) is contained in (resp. to the right of) the unit circle (resp. imaginary axis) for the smooth centering embedding on the event A^c_M in the proof of the lower bound of $\sigma(\gamma)$. The event G of Proposition 4.2 is such that if $G \cap \widetilde{E}^1_\delta$ occurs we have $\widetilde{\eta}'([0,1]) \not\subset \overline{B_r(0)}$ on the left (resp. $\widetilde{\eta}'([0,1])$ is not contained in $[-\log r, \infty) \times [0, \pi]$ on the right).

Remark 4.3. The condition that $\widetilde{\eta}'([0,1]) \not\subset \overline{B_r(0)}$ in (ii) above can also be written as $\sigma^{\mathbb{H}}_{S,U}(\mathcal{W}) > r$, in the notation of Section 4.2.1.

It is convenient to use the smooth centering embedding rather than the circle average embedding in Proposition 4.2 for the following reason. Our proof of the proposition will involve comparing certain embeddings of the quantum wedge $\mathcal{W}$ with those of the quantum wedge $\mathcal{W}_*$ obtained by restricting the field to $\mathbb{H} \setminus \widetilde{\eta}'([0,1])$. To do this, we will need to consider how various embeddings transform under a conformal map $\mathbb{H} \setminus \widetilde{\eta}'([0,1]) \to \mathbb{H}$. The behavior of the smooth centering embedding under such a map is easier to control than that of the circle average embedding, since controlling the former amounts to estimating the integral of the field against a smooth test function, whereas controlling the latter amounts to estimating the average of the field over some distorted circle. See in particular Lemma 4.6 below.

Before we prove Proposition 4.2 in Section 4.2.3, we first explain why it almost implies the lower bound for $\sigma(\gamma)$ in (1.6). If Proposition 4.2 were true with the circle average embedding $\mathfrak{h}^C$ in place of $\mathfrak{h}^S$, then the corresponding event G would satisfy $G \cap \widetilde{E}^1_\delta \subset E^{T_r}_\delta$. By condition (i) in Proposition 4.2, $\mathbb{P}[\widetilde{E}^1_\delta] \preceq \mathbb{P}[E^{T_r}_\delta]$. Combined with Proposition 3.1, this implies $\sigma(\gamma) \geq 4/\gamma^2$.

The following simple fact bridges the gap between the smooth centering embedding and the circle average embedding.

Lemma 4.4. *Let $a > 0$ and let $\mathcal{B}_t$ be a standard linear Brownian motion starting from 0. For $M > 0$, let $\tau_M = \inf\{t \geq 0 : \mathcal{B}_t - at = -M\}$. Let F_M be the event that*

$$\int_0^\infty (\mathcal{B}_s - as)\phi(s-t)\,ds \geq 0, \quad \forall t \in [0, \tau_M].$$

There are deterministic, M-independent constants $c, C > 0$ such that $\mathbb{P}[F_M] \leq Ce^{-cM^2}$ for each $M > 0$. The same holds if we replace $\mathcal{B}_t$ by $\mathcal{B}_{2t}$.

Proof. By the reflection principle for Brownian motion,

$$\mathbb{P}[\tau_{M-1} \leq 2] \leq \mathbb{P}\Big[\inf_{t\in[0,2]} \mathcal{B}_t \leq 1 + 2a - M\Big] \leq Ce^{-cM^2}$$

for some $c, C > 0$ as in the statement of the lemma.

It remains to control the probability of $F_M \cap \{\tau_{M-1} > 2\}$. We assume that $M > 1$ (so that the time τ'_{M-1} we define next is well-defined and satisfies $\tau'_{M-1} < \tau_M$ almost surely). Let τ'_{M-1} be the last time t before τ_M such that $\mathcal{B}_t - at = 1 - M$. Since ϕ is supported on $[0,1]$, on the event $F_M \cap \{\tau_{M-1} > 2\}$ there must be a time $t \in [\tau'_{M-1} - 1, \tau'_{M-1}]$ such that $\mathcal{B}_t - at \geq 0$. Note that the time reversal of $\{\mathcal{B}_t - at : t \in [\tau'_{M-1} - 1, \tau'_{M-1}]\}$ is a Brownian motion with drift starting from $1 - M$ conditioned on the uniformly positive probability event that it does not reach $-M$ before time 1. Hence Doob's maximal inequality implies $\mathbb{P}[F_M, \tau_{M-1} > 2] \leq Ce^{-cM^2}$.

By scaling, the statement still holds if we replace $\mathcal{B}_t$ by $\mathcal{B}_{2t}$. □

Proof of the lower bound of $\sigma(\gamma)$ given Proposition 4.2. Given $\delta > 0$, set $M = |\log\delta|^{\frac{2}{3}}$. Let A_M be the event that

$$\inf\left\{t \in \mathbb{R} : \int_{-\infty}^\infty X^C_s \phi(s-t)\,ds = 0\right\} > \inf\left\{t \in \mathbb{R} : X^C_t = -M\right\},$$

where X^C_t is the average process of $\mathfrak{h}^C$ in $(\mathcal{S}, +\infty, -\infty)$. In this case, $X^C_t = \mathcal{B}_{2t} - at$ for $t \geq 0$, where $\mathcal{B}$ is a standard linear Brownian motion and $a = Q - \frac{3}{2}\gamma > 0$. Furthermore, $A_M \subset F_M$ where F_M is as in Lemma 4.4 for this Brownian motion with drift. Therefore, $\mathbb{P}[A_M] \leq C\exp(-c|\log\delta|^{\frac{4}{3}}) = o^\infty_\delta(\delta)$.

Let G be as in Proposition 4.2. On the event $G \cap \widetilde{E}^1_\delta \cap A^c_M$, under $(\mathcal{S}, +\infty, -\infty)$ coordinates and the smooth centering embedding of $\mathcal{W}$, the following are true:

(i) $\widetilde{\eta}'([0,1]) \not\subset [-\log r, \infty) \times [0, \pi]$.
(ii) $\inf\{t \in \mathbb{R} : X^S_t = -M\} > 0$.

(iii) $\nu_{\mathfrak{h}^S}(\widetilde{\eta}'([0,1]) \cap (\mathbb{R} \times \{0\})) \le \delta$ and $\nu_{\mathfrak{h}^S}(\widetilde{\eta}'([0,1]) \cap (\mathbb{R} \times \{\pi\})) \le \delta$.

Let $\mathfrak{h}_M := \mathfrak{h} + M$. Since the law of a quantum wedge is invariant under multiplying its area by a constant [5, Proposition 4.6], $(\mathcal{S}, \mathfrak{h}_M, +\infty, -\infty)$ has the law of a $\frac{3}{2}\gamma$-quantum wedge. Let $\mathfrak{h}_M^C$ be the circle-average embedding of this wedge into $\mathcal{S}$. Let $\widetilde{\eta}'_M$ be given by $\widetilde{\eta}'$ in $(\mathcal{S}, +\infty, -\infty)$-coordinates parameterized by quantum mass with respect to $\mathfrak{h}_M^C$. If $G \cap \widetilde{E}_\delta^1 \cap A_M^c$ occurs, then if we consider $\mathfrak{h}_M^C$ under $(\mathcal{S}, +\infty, -\infty)$ coordinates, the above conditions (i) through (iii) imply that the following are true.

(i) $\widetilde{\eta}'_M([0, e^{\gamma M}]) \not\subset [-\log r, \infty) \times [0, \pi]$.
(ii) $\nu_{\mathfrak{h}_M^C}(\widetilde{\eta}'_M([0, e^{\gamma M}]) \cap (\mathbb{R} \times \{0\})) \le \delta e^{\gamma M/2}$ and $\nu_{\mathfrak{h}_M^C}(\widetilde{\eta}'_M([0, e^{\gamma M}]) \cap (\mathbb{R} \times \{\pi\})) \le \delta e^{\gamma M/2}$.

In particular, if we switch back to $(\mathbb{H}, 0, \infty)$ coordinates, the event $E_{\delta e^{\gamma M/2}}^{T_r}$ as defined in Proposition 3.1 with $(\mathfrak{h}_M^C, \widetilde{\eta}'_M)$ in place of $(\mathfrak{h}^C, \widetilde{\eta}')$ occurs. Since $(\mathfrak{h}_M^C, \widetilde{\eta}'_M) \overset{d}{=} (\mathfrak{h}^C, \widetilde{\eta}')$,

$$\mathbb{P}\big[G \cap \widetilde{E}_\delta^1 \cap A_M^c\big] \le \mathbb{P}\big[E_{\delta e^{\gamma M/2}}^{T_r}\big].$$

By Proposition 3.1 (recall that $M = |\log\delta|^{\frac{2}{3}}$) we have

$$\lim_{\delta\to 0} \frac{\log \mathbb{P}[E_{\delta e^{\gamma M/2}}^{T_r}]}{\log \delta^{-1}} = -\frac{4}{\gamma^2}.$$

By condition (i) in Proposition 4.2,

$$-\sigma(\gamma) = \lim_{\delta\to 0} \frac{\log \mathbb{P}[G, \widetilde{E}_\delta^1]}{\log \delta^{-1}} \le \lim_{\delta\to 0} \frac{\log \mathbb{P}[E_{\delta e^{\gamma M/2}}^{T_r}]}{\log \delta^{-1}} \vee \lim_{\delta\to 0} \frac{\log \mathbb{P}[A_M]}{\log \delta^{-1}} = -\frac{4}{\gamma^2}.$$ □

4.2.3. *Proof of Proposition* 4.2

In light of the preceding two subsections, to complete the proof of (1.6) and hence of Theorem 1.1, it remains only to prove Proposition 4.2. The proof will proceed as follows. Recall the definition of the wedge $\mathcal{W}_*$ from Section 4.2.1. In Lemma 4.5, we will construct events G_1 and G_2 such that $G_1 \cap G_2$ has uniformly positive conditional probability given $\widetilde{E}_\delta^1$ and on $G_1 \cap G_2 \cap \widetilde{E}_\delta^1$, the Ψ-embedding and circle average embedding of $\mathcal{W}_*$ differ by a bounded scaling factor. Heuristically, this means that the $\frac{3}{2}\gamma$-quantum wedges $\mathcal{W}$ and $\mathcal{W}_*$ are comparable to one another, up to a δ-independent constant. Then, in Lemma 4.6, we will define an event $F(t_0)$ of probability close to 1 which is independent from $\widetilde{E}_\delta^1$ such that on $F(t_0)$, the smoothed averages over large semicircles of the embedding $\mathfrak{h}_*^\Psi$ of $\mathcal{W}_*$ (expressed as perturbed smoothed semicircle averages of $\mathfrak{h}^C$) are under control. These lemmas together will enable us to bound the scaling factor between $\mathfrak{h}^U$ and $\mathfrak{h}^S$ on an event of positive conditional probability given $\widetilde{E}_\delta^1$, which by Remark 4.3 will prove Proposition 4.2. Also see Figure 2 for an illustration of the structure of the argument.

We first construct an event where the scaling factor $\sigma_{\Psi,C}^{\mathbb{H}}(\mathcal{W}_*)$ (as defined in Section 4.2.1) is bounded from above and below.

Lemma 4.5. *There is a deterministic constant $c_0 \in (0,1)$, an event G_1 which is measurable with respect to $(Z_t)_{t\in[0,1]}$, and an event G_2 which is independent of $(Z_t)_{t\in[0,1]}$ such that the following holds for each $\delta \in (0, \frac{1}{2})$:*

(i) $\mathbb{P}[G_1 | \widetilde{E}_\delta^1] \ge c_0$ *and* $\mathbb{P}[G_2] \ge c_0$;
(ii) *On the event* $\widetilde{E}_\delta^1 \cap G_1 \cap G_2$, $\sigma_{\Psi,C}^{\mathbb{H}}(\mathcal{W}_*) \in [10^{-3}, 10^3]$.

Proof. Let $G_1 := \{1 \le L_1 \le 2\} \cap \{1 \le R_1 \le 2\}$. By [37, Theorem 2], $\mathbb{P}[G_1 | \widetilde{E}_\delta^1] \ge c_0$ for some $c_0 > 0$ independent of δ.

Suppose $\mathcal{W}$ has the unit radius embedding into $(\mathbb{H}, 0, \infty)$. Let x^- and x^+ be defined such that $\widetilde{\eta}'([0,1]) \cap \mathbb{R} = [x^-, x^+]$. By [17, Equation (3.14)] and the definitions of the unit radius embedding and the Ψ-embedding, we have $|\Psi(x^+) - \Psi(x^-)| \in [10^{-2}, 10^2]$. On the other hand, if $\delta \in (0, \frac{1}{2})$ then on $G_1 \cap \widetilde{E}_\delta^1$, we have

$$\nu_{\mathfrak{h}_*^\Psi}\big(\big[\Psi\big(x^-\big), 0\big]\big) = L_1 + \nu_{\mathfrak{h}^U}\big(\big[x^-, 0\big]\big) \in [1, 3]$$

and similarly for $\nu_{\mathfrak{h}_*^\Psi}([0, \Psi(x^+)])$.

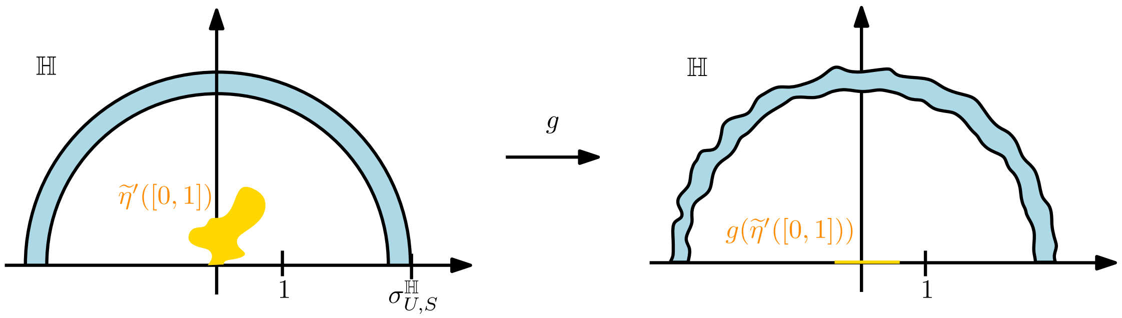

Fig. 2. Unit radius embedding of $\mathcal{W}$ (left) and circle average embedding of $\mathcal{W}_*$ (right). The boundary of $\widetilde{\eta}'([0,1])$ on the left divides $\mathcal{W}$ into two independent quantum surfaces: $\mathcal{U} = (\hbar|_{\widetilde{\eta}'([0,1])}, \widetilde{\eta}'([0,1]), 0, \widetilde{\eta}'(1), \inf(\widetilde{\eta}'([0,1]) \cap \mathbb{R}), \sup(\widetilde{\eta}'([0,1]) \cap \mathbb{R}))$ and $\mathcal{W}_* = (\hbar|_{\widetilde{\eta}'([1,\infty))}, \widetilde{\eta}'([1,\infty)), \widetilde{\eta}'(1), \infty)$. The occurrence of $\widetilde{E}^1_\delta$ depends only on $\mathcal{U}$, while the diameter under the smooth centering embedding of $\widetilde{\eta}'([0,1])$ depends mainly on $\mathcal{W}_*$. We use the independence of $\mathcal{U}, \mathcal{W}_*$ to establish Lemma 4.5, which implies that $g \in \mathcal{G}$ on the event $G_1 \cap G_2$ (see the statement of Lemmas 4.5 and 4.6 for the notation). Then we approximate the smoothed drifted circle average for large radii on the left figure (in blue) by a "distorted" average over the corresponding region on the right figure, and use Lemma 4.6 to conclude that for sufficiently large radii this is positive with uniformly positive probability conditioned on $\widetilde{E}^1_\delta$. This result is the content of Proposition 4.2.

Let

$$G_2 := \{\nu_{\hbar^C_*}([-1,1]) < 1/2\} \cap \{\nu_{\hbar^C_*}([0,2]) > 3\} \cap \{\nu_{\hbar^C_*}([-2,0]) > 3\}.$$

Then G_2 is independent of $(Z_t)_{t\in[0,1]}$ and $\mathbb{P}[G_2] \geq c_0$ for some (possibly smaller) $c_0 > 0$ independent of δ. On G_2, the interval $[\Psi(x^-), \Psi(x^+)]$ after mapping to the circle average embedding of $\mathcal{W}_*$ will contain $[-1,1]$ and be contained in $[-2,2]$. Therefore, on $\widetilde{E}^1_\delta \cap G_1 \cap G_2$, the scaling factor between $\hbar^\Psi_*$ and $\hbar^C_*$ lies in $[10^{-3}, 10^3]$. □

Our next lemma is a variant of [5, Proposition 10.19]. The proof follows from essentially the same argument, so we will make it brief. In the statement of the lemma, we let $\mathcal{G}$ be the collection of conformal maps of the following form: $g : \mathbb{H} \setminus A \to \mathbb{H}$, where A ranges over all hulls with a tip $p \in \partial A \setminus \mathbb{R}$ in $\overline{\mathbb{H}}$ such that $0 \in A \subset \overline{\mathbb{D}}$ and g satisfies $g(p) = 0$, $g(\infty) = \infty$, and $\lim_{z\to\infty} g(z)/z \in [10^{-3}, 10^3]$.

Lemma 4.6. *Let K be a fixed constant and $\widetilde{\phi} : \mathbb{H} \to [0,\infty)$ be a radially symmetric smooth function supported on $B_1(0) \setminus B_{e^{-1}}(0)$. Suppose $\hbar^C$ is the circle average embedding of a $\frac{3}{2}\gamma$-quantum wedge into $(\mathbb{H}, 0, \infty)$. For $t \in \mathbb{R}$, let $\hbar^C_t = \hbar^C(e^t\cdot) + Q\log|e^t\cdot|$. For $t_0 \in \mathbb{R}$, let $F(t_0)$ be the event that the inner product $(\hbar^C_t, |(g^{-1})'|^2\widetilde{\phi} \circ g^{-1})$ is bigger than K for all $t \geq t_0$ and $g \in \mathcal{G}$. Then $\lim_{t_0\to\infty} \mathbb{P}[F(t_0)] = 1$.*

Proof. Let h be the whole plane Gaussian free field plus $-\frac{3}{2}\gamma\log|z|$ normalized so that its circle average over $\partial\mathbb{D}$ is 0. Let $h_t = h(e^t\cdot) + Qt$ and

$$m_n := \inf_{t\in[n,n+1], g\in\mathcal{G}} \left(h_t, \left|(g^{-1})'\right|^2 \widetilde{\phi} \circ g^{-1}\right) \quad \forall n = 0, 1, 2, \ldots.$$

Write $h = h^0 + h^\dagger$ where h^0 is the radially symmetric part of h and $h^\dagger = h - h^0$ (recall Section 1.2.3). Define m^0_n and $m^\dagger_n$ in the same manner as m_n but with h^0 and $h^\dagger$, respectively, in place of h, so that m^0_n and $m^\dagger_n$ are independent and $m^0_n + m^\dagger_n \leq m_n$. Since the law of $h^\dagger$ is scale invariant in law, $\{m^\dagger_n\}_{n\in\mathbb{N}}$ is a stationary sequence, hence it has stationary increments. Since the circle average process of h is a drifted Brownian motion, and we observe that $(1, |(g^{-1})'|^2\widetilde{\phi} \circ g^{-1})$ is independent of g, $\{m^0_n\}_{n\in\mathbb{N}}$ also has stationary increments. By results from Gaussian analysis (see [5, Proposition 10.18] and the discussion afterwards), both m^0_n and $m^\dagger_n$ are finite and have finite variance. The Birkhoff ergodic theorem now implies that both $M^0 := \lim_{n\to\infty} n^{-1}m^0_n$ and $M^\dagger := \lim_{n\to\infty} n^{-1}m^\dagger_n$ exist a.s. Since $\{m^\dagger_n\}_{n\in\mathbb{N}}$ is stationary, $M^\dagger = 0$ a.s. Since $Q - \frac{3}{2}\gamma > 0$, we find that $M^0 > 0$ a.s. Hence $\lim_{n\to\infty} n^{-1}m_n > 0$, which implies the statement of the lemma with h_t in place of $\hbar^C_t$.

Since $\hbar^C$ and h are absolutely continuous with respect to each other on $\mathbb{C} \setminus B(0,10)$, if we define m_n with $\hbar^C$ in place of h, we still have $\lim_{n\to\infty} m_n = \infty$ a.s. By straightforward distortion estimates and since $\widetilde{\phi}$ is compactly supported, $(Q \log|\cdot|, |(g^{-1})'|^2 \widetilde{\phi} \circ g^{-1})$ is bounded from above and below uniformly over all $g \in \mathcal{G}$. This concludes the proof. □

Proof of Proposition 4.2. Let G_1, G_2, and c_0 be chosen so that the conclusion of Lemma 4.5 holds. Throughout the proof we will assume $G_1 \cap G_2 \cap \widetilde{E}^1_\delta$ occurs.

Suppose $(\hbar^U, \widetilde{\eta}')$ is the unit radius embedding of $(\mathcal{W}, \widetilde{\eta}')$ into $(\mathbb{H}, 0, \infty)$. Note that $\widetilde{\eta}'([0,1]) \subset \overline{B_1(0)}$ in this embedding. Let $g : \mathbb{H} \setminus \widetilde{\eta}'([0,1]) \to \mathbb{H}$ be such that $g(\widetilde{\eta}'(1)) = 0$, $g(\infty) = \infty$ and

$$\hbar^U \circ g^{-1} + Q \log\big|\big(g^{-1}\big)'\big| = \hbar^C_*, \tag{4.8}$$

where $\hbar^C_*$ is the circle average embedding of $\mathcal{W}_*$ into $(\mathbb{H}, 0, \infty)$. By condition (4.5) in Lemma 4.5, we have $g \in \mathcal{G}$ where $\mathcal{G}$ is defined as in Lemma 4.6.

Let ϕ be as in the definition of the smooth centering embedding. Let $\widetilde{\phi} : \mathbb{H} \mapsto [0,\infty)$ be defined by

$$\widetilde{\phi}(z) := \frac{1}{\pi}|z|^{-2}\phi\big(-\log|z|\big),$$

so that $\widetilde{\phi}$ is a radially symmetric bump function on $\mathbb{H}$ and $\pi\widetilde{\phi}(e^{-t})e^{-2t} = \phi(t)$ for each $t \in \mathbb{R}$. Also define $\widetilde{\phi}_t(\cdot) := e^{-2t}\widetilde{\phi}(e^{-t}\cdot)$ for $t \in \mathbb{R}$.

Letting $(X^U_s)_{s\in\mathbb{R}}$ denote the average process of $\hbar^U$ over vertical lines in $(\mathcal{S}, +\infty, -\infty)$ coordinates the following holds

$$\begin{aligned}\int_{-\infty}^{\infty} X^U_s \phi(s+t)\,ds &= \big(\hbar^U\big(e^t\cdot\big) + Q\log\big|e^t\cdot\big|, \widetilde{\phi}\big)\\ &= \big(\hbar^U\big(g^{-1}(\cdot)\big) + Q\log\big|g^{-1}(\cdot)\big|, \big|\big(g^{-1}\big)'(\cdot)\big|^2\widetilde{\phi}_t \circ g^{-1}\big)\\ &= \big(\hbar^C_* + Q\log|\cdot|, \big|\big(g^{-1}\big)'(\cdot)\big|^2\widetilde{\phi}_t \circ g^{-1}\big) - Q\big(\log\big|\big(g^{-1}\big)'(\cdot)\big|, \big|\big(g^{-1}\big)'(\cdot)\big|^2\widetilde{\phi}_t \circ g^{-1}\big)\\ &\quad + Q\big(\log\big|g^{-1}(\cdot)\big| - \log|\cdot|, \big|\big(g^{-1}\big)'(\cdot)\big|^2\widetilde{\phi}_t \circ g^{-1}\big),\end{aligned} \tag{4.9}$$

where the first identity follows by using polar coordinates and the canonical transformation between $\mathbb{H}$ and $\mathcal{S}$, the second identity follows by using the coordinate change formula for the inner product $(\cdot,\cdot)$, and the third identity follows from (4.8).

We will bound for each term on the right-hand side of (4.9). By the Taylor expansion of g near ∞, $|g(z)/z|$ is bounded from above and below outside of $\mathbb{H} \setminus B_{e^2}(0)$. Therefore there exists a constant $K_1 > 0$ such that for all $t \geq 2$ we have

$$\big|\big(\log\big|g^{-1}(\cdot)\big| - \log|\cdot|, \big|\big(g^{-1}\big)'(\cdot)\big|^2\widetilde{\phi}_t \circ g^{-1}\big)\big| = \big|\big(\log|\cdot| - \log\big|g(\cdot)\big|, \widetilde{\phi}_t\big)\big| \leq K_1,$$

which bounds the second term on the right-hand side of (4.9). Since $g \in \mathcal{G}$, distortion estimates imply that $|g'(\cdot)|$ has universal upper and lower bounds on $\mathbb{H} \setminus B_{e^2}(0)$. Since $|(g^{-1})'(g(\cdot))| = 1/|g'(\cdot)|$, there exists a constant $K_2 > 0$ such that for all $t \geq 2$ we have

$$\big|\big(\log\big|\big(g^{-1}\big)'(\cdot)\big|, \big|\big(g^{-1}\big)'(\cdot)\big|^2\widetilde{\phi}_t \circ g^{-1}\big)\big| = \big|\big(\log\big|\big(g^{-1}\big)' \circ g(\cdot)\big|, \widetilde{\phi}_t\big)\big| \leq K_2(1, \widetilde{\phi}_t) = K_2,$$

which bounds the third term on the right-hand side of (4.9). Define $K := Q(K_1 + K_2)$.

Let $F(t_0)$ be defined as in Lemma 4.6 for the quantum wedge $\mathcal{W}_*$ and this choice of constant K. Then we can pick a deterministic $t_0 \geq 2$ large enough such that $\mathbb{P}[F(t_0)] \geq 1 - c_0/2$.

Let $G := G_1 \cap G_2 \cap F(t_0)$. By independence of $\{G_1, \widetilde{E}^1_\delta\}$ and $\{G_2, F(t_0)\}$ together with condition (4.5) in Lemma 4.5, we have

$$\mathbb{P}\big[G|\widetilde{E}^1_\delta\big] \geq c_0^2/2.$$

For $t \geq 0$, let $g_t(z) := e^{-t} g(e^t z)$ so that $g_t : \mathbb{H} \setminus e^{-t}\eta'([0,1]) \to \mathbb{H}$. Then for $t \geq 0$, we have $g_t \in \mathcal{G}$ and

$$\big(\mathfrak{h}_*^C + Q \log|\cdot|, \big|(g^{-1})'(\cdot)\big|^2 \widetilde{\phi}_t \circ g^{-1}\big) = \big(\mathfrak{h}_*^C(e^t \cdot) + Q \log\big|e^t \cdot\big|, \big|(g_t^{-1})'(\cdot)\big|^2 \widetilde{\phi} \circ g_t^{-1}\big).$$

By definition of $F(t_0)$, on G this latter quantity is at least K for each $t \geq t_0$. By (4.9) and our bounds for the second and third term on the right-hand side, we obtain

$$\int_{-\infty}^{\infty} X_s^U \phi(s+t)\, ds \geq 0 \quad \forall t \geq t_0.$$

Therefore $\sigma_{S,U}^{\mathcal{S}}(\mathcal{W}) \leq t_0$, which means $\sigma_{S,U}^{\mathbb{H}}(\mathcal{W}) \geq e^{-t_0}$. In light of Remark 4.3, the constants $c = c_0^2/2$ and $r = e^{-t_0}$ and the event G meet the requirements in Proposition 4.2. □

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