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Overcrowding asymptotics for the Sine$_\beta$ process

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Abstract. We give overcrowding estimates for the Sine$_\beta$ process, the bulk point process limit of the Gaussian β-ensemble. We show that the probability of having exactly n points in a fixed interval is given by $e^{-\frac{\beta}{2}n^2 \log(n) + \mathcal{O}(n^2)}$ as $n \to \infty$. We also identify the next order term in the exponent if the size of the interval goes to zero.

Résumé. Nous obtenons des résultats asymptotiques pour le surpeuplement du processus Sine$_\beta$, le processus ponctuel limite dans le milieu du spectre de l'ensemble β-gaussien. Nous montrons que la probabilité d'observer n points dans un interval fixé est donné par la formule $e^{-\frac{\beta}{2}n^2 \log(n) + \mathcal{O}(n^2)}$ quand $n \to \infty$. Nous obtenons aussi une approximation jusqu'à l'ordre suivant lorsque la longueur de l'interval tend vers 0.

1. Introduction

The Gaussian β-ensemble is a family of point processes on the real line given by the joint density function

$$p_{m,\beta}(\lambda_1, \ldots, \lambda_m) = \frac{1}{Z_{m,\beta}} \prod_{1 \le i < j \le m} |\lambda_i - \lambda_j|^\beta \times \prod_{i=1}^{m} e^{-\frac{\beta}{4}\lambda_i^2}. \tag{1}$$

Here $\beta > 0$ and $Z_{m,\beta}$ is an explicitly computable normalizing constant. For the specific values of $\beta = 1, 2$ and 4 one obtains the joint eigenvalue density of the classical random matrix ensembles GOE, GUE and GSE.

We study the Sine$_\beta$ process, the bulk scaling limit of the Gaussian β-ensemble. Fix $\beta > 0$, and consider a sequence of finite point processes $\Lambda_{m,\beta}$ with distribution given by the density (1). Then as $m \to \infty$ the scaled processes $2\sqrt{m}\Lambda_{m,\beta}$ converge in distribution to a translation invariant point process which we call Sine$_\beta$. The existence and description of the limiting process in the $\beta = 1, 2$ and 4 cases were obtained by Dyson, Gaudin and Mehta (see e.g. the monographs [3,9,13]), while in the general β case this was done in [14]. We note that in our normalization the Sine$_\beta$ process has particle density $\frac{1}{2\pi}$.

We study the problem of overcrowding for the Sine$_\beta$ process. We describe the asymptotic probability of having at least (or exactly) n points in a fixed interval for a large n. Since the process is translation invariant we may assume that the interval is of the form $[0, \lambda]$ with $\lambda > 0$. We will use the notation $N_\beta(\lambda)$ for the counting function of the process, for $\lambda > 0$ this gives the number of points in $[0, \lambda]$. Our main result is as follows.

Theorem 1. *Fix* $\lambda_0 > 0$. *There is a constant* $c > 0$ *depending only on* β *and* λ_0 *so that for any* $n \geq 1$ *and* $0 < \lambda \leq \lambda_0$
we have

$$P\big(N_\beta(\lambda) \geq n\big) \leq e^{-\frac{\beta}{2}n^2 \log(\frac{n}{\lambda}) + cn \log(n+1) \log(\frac{n}{\lambda}) + cn^2}. \tag{2}$$

Moreover, there exists an $n_0 \geq 1$ *so that for any* $n \geq n_0$, $0 < \lambda \leq \lambda_0$ *we also have*

$$P\big(N_\beta(\lambda) = n\big) \geq e^{-\frac{\beta}{2}n^2 \log(\frac{n}{\lambda}) - cn \log(n+1) \log(\frac{n}{\lambda}) - cn^2}. \tag{3}$$

The theorem implies that for a fixed $\lambda > 0$ we have $P(N_\beta(\lambda) = n) = e^{-\frac{\beta}{2}n^2 \log n + \mathcal{O}(n^2)}$ as $n \to \infty$. If we let $n \to \infty$ and $\lambda \to 0$ simultaneously then we get

$$P\big(N_\beta(\lambda) = n\big) = e^{-\frac{\beta}{2}n^2 \log(n/\lambda) + \mathcal{O}(n^2 + n \log n \log(\lambda^{-1}))}.$$

Dyson used Coulomb gas arguments in [6] to predict the asymptotic size of $P(N_\beta(\lambda) = n)$ for a fixed n and large λ. His predictions were made precise in [4] for $\beta = 1, 2$ and 4 using the fact that in these classical cases the probabilities in question have Fredholm determinant representations. For general β the special case of $n = 0$ (i.e. the large gap probability) was treated in [15].

Fogler and Shklovskii in [8] extended Dyson's method to give predictions on the asymptotic size of $P(N_\beta(\lambda) = n)$ in the regime where n and λ are large. Although this does not cover the case we are interested in (λ bounded, n large), our results agree with their prediction in the case when $n \gg \lambda$. (See Equation (25) in [8].) Section 14.6 of [9] gives a nice overview of the Coloumb gas method in the study of the distribution of $N_\beta(\lambda)$.

Our proof uses techniques that we developed in [10], where we proved a large deviation principle for $\frac{1}{\lambda}N_\beta(\lambda)$ with scale λ^2 and a good rate function of the form $\beta I(\rho)$. One of the consequences of this large deviation result is that if $\rho > \frac{1}{2\pi}$ then $P(N_\beta(\lambda) \geq \rho\lambda)$ decays asymptotically as $e^{-\beta\lambda^2 I(\rho)(1+o(1))}$ as $\lambda \to \infty$. It was shown in [10] that $I(\rho)$ grows like $\frac{1}{2}\rho^2 \log \rho$ as $\rho \to \infty$ so the decay of $P(N_\beta(\lambda) \geq \rho\lambda)$ is formally consistent with our Theorem 1, even though it describes a different regime of n, λ.

Rigorous overcrowding estimates have been obtained for certain non-Hermitian random matrix models. Here one is interested in the asymptotic probability of finding at least n points in a fixed disk. In [12] Krishnapour identified the overcrowding asymptotics for the complex Ginibre ensemble and also gave similar estimates for zeros of certain Gaussian analytic functions. In [1] and [2] these results were generalized to point processes obtained by considering products of complex and quaternion Ginibre ensembles. It is interesting to note that in all of these non-Hermitian cases the probability of finding at least n points in a fixed disk decays like $e^{-cn^2 \log n(1+o(1))}$ where c is a parameter depending only on the model and not the size of the disk. This is of the same form as our result for the Sine_β process.

The finite version of our problem would be to study the probability of having a large number of points in a small interval of order $m^{-1/2}$ in the bulk of the support of (1). This problem has been considered for the spectrum of general Wigner matrices in the context of showing the universality of the bulk limit process, see e.g. Theorem 3.5 of [7] or Corollary B.2 of [5]. These results concentrate on giving sufficiently strong (but not necessarily optimal) upper bounds for the overcrowding probabilities.

The starting point of our proof is the following characterization of the counting function of Sine_β, due to [14].

Theorem 2 ([14]). *Let* $\lambda > 0$ *and consider the strong solution of the following diffusion*:

$$d\alpha_\lambda = \lambda\frac{\beta}{4}e^{-\frac{\beta}{4}t}\,dt + 2\sin(\alpha_\lambda/2)\,dB_t, \quad \alpha_\lambda(0) = 0. \tag{4}$$

Then $\lim_{t\to\infty} \frac{1}{2\pi}\alpha_\lambda(t)$ *exists almost surely, and it has the same distribution as* $N_\beta(\lambda)$.

This describes the one-dimensional marginals of the counting function $N_\beta(\lambda)$. One can also describe the distribution of the whole function $N_\beta(\cdot)$ in terms of a one-parameter family of diffusions (see [14]), but we will not need this here. The following proposition summarizes some of the basic properties of the diffusion (4). Most of these properties are straightforward, see Proposition 9 in [14] for some of the proofs.

Proposition 3. *Let $\lambda > 0$.*

(i) *The process $\lfloor \frac{\alpha_\lambda(t)}{2\pi} \rfloor$ is nondecreasing in t. In particular for any integer n we have $\alpha_\lambda(t) > 2\pi n$ if t is larger than the first hitting time of $2\pi n$.*

(ii) *If $0 < \lambda < \lambda'$ and $\alpha_\lambda, \alpha_{\lambda'}$ are strong solutions of (4) with the same Brownian motion B_t then $\alpha_\lambda(t) \leq \alpha_{\lambda'}(t)$ for all $t \geq 0$.*

(iii) *For any integer n the process $\alpha_\lambda(t) + 2n\pi$ solves the same SDE as (4) but with an initial condition of $2n\pi$ instead of 0.*

(iv) *For any $T > 0$ the process $\phi(t) = \alpha_\lambda(T + t)$ solves the same SDE as (4) with $\lambda e^{-\frac{\beta}{4}T}$ in place of λ, B_{t+T} in place of B_t and initial condition $\phi(0) = \alpha_\lambda(T)$.*

An important consequence of the previous proposition is the following observation.

Lemma 4. *Fix $n > 1$ and $\lambda > 0$. Let τ_λ be the first hitting time of 2π of the diffusion α_λ. (Note that τ_λ might be ∞.) Then*

$$P(N_\beta(\lambda) \geq n) = E\big[g(\tau_\lambda)\big] \quad \text{where } g(t) = P\big(N_\beta\big(\lambda e^{-\frac{\beta}{4}t}\big) \geq n - 1\big), \tag{5}$$

and $g(\infty) = 0$.

Proof. According to Proposition 3, if we condition on $\{\tau_\lambda = T\}$ with a $T > 0$ then the process $\{\alpha_\lambda(t + T) - 2\pi, t \geq 0\}$ will have the same distribution as $\{\alpha_{\lambda e^{-\frac{\beta}{4}T}}(t), t \geq 0\}$. Thus

$$P\Big(\lim_{t\to\infty} \alpha_\lambda(t) \geq 2\pi n \big| \tau_\lambda = T\Big) = P\Big(\lim_{t\to\infty} \alpha_{\lambda e^{-\frac{\beta}{4}T}}(t) \geq 2\pi(n - 1)\Big). \tag{6}$$

Identity (5) now follows by taking expectations in τ_λ and using the description of $N_\beta(\lambda)$ from Theorem 2. $\qquad \square$

We record the following simple corollary of Lemma 4.

Proposition 5. *For any $\lambda > 0$ and $n \geq 1$ we have*

$$P\big(N_\beta(\lambda) \geq n\big) \leq \left(\frac{\lambda}{2\pi}\right)^n. \tag{7}$$

Proof. From (4) one gets $E\alpha_\lambda(t) = \lambda(1 - e^{-\frac{\beta}{4}t})$ and using the properties of the diffusion α_λ one can show $E\lim_{t\to\infty} \alpha_\lambda(t) = \lambda$. Markov's inequality now implies the statement of the proposition for $n = 1$. For larger values we can use Lemma 4 and the monotonicity of $g(t)$ from (5) to get

$$P\big(N_\beta(\lambda) \geq n\big) \leq P(\tau_\lambda < \infty) P\big(N_\beta(\lambda) \geq n - 1\big) \leq P\big(N_\beta(\lambda) \geq 1\big) P\big(N_\beta(\lambda) \geq n - 1\big)$$

from which the statement follows by induction. $\qquad \square$

Our proof of Theorem 1 will rely on the recursion Equation (5) and precise estimates on $P(\tau_\lambda < t)$. The next section will state the necessary hitting time estimates while the Sections 3 and 4 will contain the proofs of the lower and upper bounds in Theorem 1. Finally, Section 5 will contain the proofs of the bounds on $P(\tau_\lambda < t)$.

2. Hitting time estimates

We will estimate the hitting time τ_λ by coupling the diffusion $\alpha_\lambda(t)$ of (4) to a similar diffusion with a constant drift term. Let $\tilde{\alpha}_\lambda$ be the strong solution of the SDE

$$d\tilde{\alpha}_\lambda = \lambda\, dt + 2\sin\left(\frac{\tilde{\alpha}_\lambda}{2}\right) dB_t, \quad \tilde{\alpha}_\lambda(0) = 0, \tag{8}$$

using the same Brownian motion as (4). Denote by $\tilde\tau_\lambda$ the first hitting time of 2π by $\tilde\alpha_\lambda$. Simple coupling arguments show that $\alpha_\lambda(t) \le \tilde\alpha_{\lambda'}(t)$ for all $t \ge 0$ with $\lambda' = \frac{\beta}{4}\lambda$, and $\alpha_\lambda(t) \ge \tilde\alpha_{\lambda''}(t)$ for $0 \le t \le T$ if $\lambda'' = \frac{\beta}{4}e^{-\frac{\beta}{4}T}\lambda$. This yields

$$P(\tau_\lambda \le t) \le P(\tilde\tau_{\lambda'} \le t), \quad \text{with } \lambda' = \frac{\beta}{4}\lambda \quad \text{and} \tag{9}$$

$$P(\tau_\lambda \le t) \ge P(\tilde\tau_{\lambda''} \le t), \quad \text{with } \lambda'' = \frac{\beta}{4}e^{-\frac{\beta}{4}t}\lambda. \tag{10}$$

Our bounds will use certain special functions which we now briefly go over. For $a < 1$ we set

$$K(a) = \int_0^{\pi/2} \frac{dx}{\sqrt{1 - a\sin^2 x}}, \qquad E(a) = \int_0^{\pi/2} \sqrt{1 - a\sin^2 x}\,dx. \tag{11}$$

These functions are the complete elliptic integrals of the first and second kind. We note the identity

$$K(-a) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{\sqrt{\cosh^2 z + a}}\,dt, \quad a > -1. \tag{12}$$

We denote by $\mathcal{W}$ the lower branch of the Lambert $\mathcal{W}$ function (or product log function). This is defined as the unique solution of

$$z = \mathcal{W}(z)e^{\mathcal{W}(z)}, \quad \text{with } \mathcal{W}(z) \le -1. \tag{13}$$

Note that $\mathcal{W}$ has domain $[-1/e, 0)$, it is strictly decreasing and satisfies $\mathcal{W}(x\log x) = \log x$ for $0 < x \le 1/e$.

The following proposition summarizes some of the relevant properties of the introduced special functions.

Proposition 6. *There exists a constant $c > 0$, so that the following bounds hold.*

$$\left| K(-a) - \frac{1}{2\sqrt{a}}\log(16a) \right| \le \frac{c}{a^{3/2}}\log(a), \qquad \left| E(-a) - \sqrt{a} \right| \le \frac{c}{a^{1/2}}\log(a), \quad \text{if } a > 2. \tag{14}$$

$$\left| K^{-1}(x) + x^{-2}\mathcal{W}^2(-x/4) \right| \le c, \quad \text{if } 0 < x < 1/2. \tag{15}$$

$$\left| \mathcal{W}(-x) - \mathcal{W}(-y) \right| \le 2\left| \log(x/y) \right|, \qquad \left| \mathcal{W}(-x) \right| \le 2\log(x^{-1}), \quad \text{if } x, y \in (0, 2e^{-2}]. \tag{16}$$

Outline of the proof. The estimates (14) are from the Appendix of [10]. The definition of $\mathcal{W}$ allows us to check that the inverse of the function $\frac{1}{2\sqrt{a}}\log(16a)$ on $[e^2/16, \infty)$ is exactly $x^{-2}\mathcal{W}^2(-x/4)$. From this fact and the first bound in (14) the proof of (15) is a simple exercise. Finally, (16) follows from the observation that $\mathcal{W}'(x) = \frac{1}{x}\cdot\frac{\mathcal{W}(x)}{1+\mathcal{W}(x)}$. $\square$

Now we are ready to state the estimates on the hitting times τ_λ.

Proposition 7. *There exist positive constants ε and c (depending only on β) so that if $\lambda t < \varepsilon$ then*

$$P(\tau_\lambda < t) \le \exp\left[-\frac{2}{t}\mathcal{W}^2(-\lambda t) + c\left(1 + \frac{1}{t}\right)\log\left(\frac{1}{\lambda t}\right) \right]. \tag{17}$$

For any $\lambda_0 > 0$ there exist positive constants ε, c (depending only on λ_0, β) so that if $0 < \lambda \le \lambda_0$, $\lambda t < \varepsilon$ and $t \le \varepsilon\log(\frac{1}{\lambda t})$ then

$$P(t/2 \le \tau_\lambda < t) \ge \exp\left[-\frac{2}{t}\mathcal{W}^2(-\lambda t) - c\left(1 + \frac{1}{t}\right)\log\left(\frac{1}{\lambda t}\right) \right]. \tag{18}$$

We defer the proof of the proposition to Section 5.

3. Proof of the lower bound in Theorem 1

Our proof of the lower bound relies on an inductive argument using the recursion identity of Lemma 4 and the bound (18) of Proposition 7.

Proof of the lower bound in Theorem 1. Choose $K > 0$ and then $c_0 > 0$ so that

$$\frac{8}{\beta}\frac{1}{1+K} < \varepsilon, \qquad \log(c_0) < \frac{\beta}{4}\min(\sqrt{c_0}, \varepsilon c_0), \qquad 4\log(2) < \frac{\log(c_0)}{1+K}, \tag{19}$$

where ε is the constant from the conditions of the lower bound (18) of Proposition 7.

We will prove that there is a sequence $f_n > 0$, and a constant $\kappa > 0$ so that for any $0 < \lambda < \lambda_0$ and $n \geq \max(c_0\lambda, 1)$ we have

$$\log P\big(N_\beta(\lambda) = n\big) \geq -f_n \log\left(\frac{n}{\lambda}\right) - \kappa n^2. \tag{20}$$

We will use induction on n. Assume that the statement is true for $n - 1 \geq 1$ for all relevant values of λ and suppose that $n \geq \max(c_0\lambda, 2)$. Consider the process α_λ from (2) and denote the first hitting time of 2π by τ. (Note that this could be infinite.) Considering the identity in Lemma 4 for n and $n + 1$, and then taking the difference we get that for any $T > 0$:

$$P\big(N_\beta(\lambda) = n\big) = E\big[h(\tau)\big] \geq E\big[\mathbf{1}(T/2 \leq \tau < T)h(\tau)\big], \tag{21}$$

where $h(t) = P(N(\lambda e^{-\frac{\beta}{4}t}) = n - 1)$, and $h(\infty) = 0$.

Set $T = \frac{4}{\beta}\frac{\log(n/\lambda)}{n+K}$. From (19) it follows that for $n \geq \max(c_0\lambda, 2)$ we have

$$\frac{n-1}{n}e^{\frac{\beta}{4}\cdot\frac{T}{2}} \geq 1. \tag{22}$$

This implies $n - 1 \geq c_0\lambda e^{-\frac{\beta}{4}t}$ for $T/2 \leq t < T$, and using our induction hypothesis we get that for $T/2 \leq t \leq T$ we have

$$\log h(t) = \log P\big(N\big(\lambda e^{-\frac{\beta}{4}t}\big) = n - 1\big)$$

$$\geq -f_{n-1}\log\left(\frac{n-1}{\lambda e^{-\frac{\beta}{4}t}}\right) - \kappa(n-1)^2 \tag{23}$$

$$\geq -f_{n-1}\log\left(\frac{n}{\lambda}\right) - f_{n-1}\log\left(\frac{n-1}{n}\right) - \frac{\beta}{4}Tf_{n-1} - \kappa(n-1)^2$$

$$\geq -\left(1 + \frac{1}{n+K}\right)f_{n-1}\log\left(\frac{n}{\lambda}\right) - \kappa(n-1)^2. \tag{24}$$

Next we check that T and λ satisfy the conditions of the lower bound (18) of Proposition 7. For this we need to show

$$\lambda T < \varepsilon, \qquad T\frac{1}{\log(\frac{1}{\lambda T})} < \varepsilon, \tag{25}$$

both of which follow from (19). Applying now the lower bound (18) we get

$$\log P(T/2 \leq \tau < T) \geq -\frac{\beta(n+K)}{2\log(\frac{n}{\lambda})}\mathcal{W}^2\left(-\frac{4}{\beta}\cdot\frac{\lambda}{n+K}\log\left(\frac{n}{\lambda}\right)\right)$$

$$-c\left(1 + \frac{\beta(n+K)}{4\log(\frac{n}{\lambda})}\right)\log\left(\frac{\beta n}{4\lambda}\frac{1}{\log(\frac{n}{\lambda})}\right).$$

Assuming that c_0 is large enough we may use (16) to estimate $\mathcal{W}^2(-\frac{4}{\beta} \cdot \frac{\lambda}{n+K} \log(\frac{n}{\lambda}))$ with $\mathcal{W}^2(-\frac{\lambda}{n} \log(\frac{n}{\lambda})) = \log^2(\frac{n}{\lambda})$ plus an error term of size $c' \log(\frac{n}{\lambda})$. This allows us to find positive constants c_1, c_2 (depending on K, c_0) so that

$$\log P(T/2 \leq \tau < T) \geq -\frac{\beta}{2} n \log\left(\frac{n}{\lambda}\right) - c_1 \log\left(\frac{n}{\lambda}\right) - c_2 n. \tag{26}$$

This bound with (21) and (24) gives

$$\log P(N_\beta(\lambda) = n) \geq -\left(\left(1 + \frac{1}{n+K}\right) f_{n-1} + \frac{\beta}{2} n + c_1\right) \log\left(\frac{n}{\lambda}\right) - \kappa(n-1)^2 - c_2 n.$$

If we also assume $c > c_2$ then we can further simplify this to

$$\log P(N_\beta(\lambda) = n) \geq -\left(\left(1 + \frac{1}{n+K}\right) f_{n-1} + \frac{\beta}{2} n + c_1\right) \log\left(\frac{n}{\lambda}\right) - \kappa n^2. \tag{27}$$

Thus we proved that if $f_n > 0$ solves the recursion

$$f_n = \left(1 + \frac{1}{n+K}\right) f_{n-1} + \frac{\beta}{2} n + c_1 \tag{28}$$

and (20) holds with $n - 1 \geq 1$ for all appropriate λ then it holds for n as well (again, for all appropriate λ).

To initialize our induction we show that (20) holds for $n = 1$ with a constant $f_1 > 0$. Note that here we assume that $\lambda < \min(\lambda_0, \frac{1}{c})$. If $\tilde{\lambda} < \pi$ then by Proposition 5 we have

$$P(N_\beta(\tilde{\lambda}) = 0) = 1 - P(N_\beta(\tilde{\lambda}) \geq 1) > \frac{1}{2}.$$

The lower bound (21) holds for $n = 1$. If we choose $c_0 > 1/\pi$ then the bound $0 < \lambda < \min(1/c_0, \lambda_0)$ implies $h(t) > 1/2$ for any $t > 0$ which gives

$$P(N_\beta(\lambda) = 1) \geq \frac{1}{2} P(T/2 \leq \tau \leq T).$$

Choosing the same T as before (with $n = 1$) and using (26) gives

$$P(N(\lambda) = 1) \geq \frac{1}{2} \exp\left[-\left(\frac{\beta}{2} + c_1\right) \log\left(\frac{1}{\lambda}\right) - c_2\right] \geq e^{-f_1 \log(\frac{1}{\lambda}) - \kappa}$$

with a positive f_1 as long as we choose $\kappa > c_2 + \log 2$.

A straightforward induction argument shows that if f_n solves the recursion (28) with $f_1 > 0$ then there is a $b > 0$ so that

$$f_n \leq \frac{\beta}{2} n^2 + bn \log(n+1)$$

for all $n \geq 1$. This completes the proof of lower bound in Theorem 1 with the choice $n_0 = \lceil c_0 \lambda_0 + 1 \rceil$.   □

4. Proof of the upper bound in Theorem 1

The upper bound will use a two step strategy similar to that of the lower bound, but our recursive step will be a bit more involved.

Proof of the upper bound in Theorem 1. Recall from Proposition 5 that

$$P(N_\beta(\lambda) \geq n) \leq \left(\frac{\lambda}{2\pi}\right)^n \leq \exp(-n \log(1/\lambda)).$$

By choosing $c_1 > 1$ we get $-n \log(1/\lambda) \le -n \log(n/\lambda) + c_1 n^2$ which yields

$$\log P\big(N_\beta(\lambda) \ge n\big) \le -n \log(n/\lambda) + c_1 n^2. \tag{29}$$

Now assume that $n \ge 2$ and there are positive constants $f_{n-1} \le \frac{\beta}{2}(n-1)^2$, c_1 so that for all $0 \le \lambda \le \lambda_0$ we have

$$P\big(N_\beta(\lambda) \ge n-1\big) \le \exp\left[-f_{n-1} \log\left(\frac{n-1}{\lambda}\right) + c_1(n-1)^2 \right]. \tag{30}$$

Our goal is to show that we can choose an n_0 (depending only on λ_0, β) so that if $n \ge n_0$ and c_1 is larger than a fixed constant then we can get an upper bound of a similar form for $P(N_\beta(\lambda) \ge n)$ as well.

Let $T_0 = 0 < T_1 < \cdots < T_M$. Then

$$P\big(N_\beta(\lambda) \ge n\big) = \sum_{i=0}^{M-1} P\big(N_\beta(\lambda) \ge n, T_i \le \tau_\lambda \le T_{i+1}\big) + P\big(N_\beta(\lambda) \ge n, \tau_\lambda \ge T_M\big). \tag{31}$$

Choose real numbers $\gamma_1 < \cdots < \gamma_M$ with $\gamma_1 = 1/3$, $\gamma_M = 3$, $M \le 3n$ and $\gamma_{i+1} - \gamma_i \le 1/n$. We set $T_i = \gamma_i \frac{4 \log(n/\lambda)}{\beta n}$ for $i = 1, \ldots, M$, if $n \ge \lambda_0$ then these are positive numbers. We will estimate the sum (31) term by term.

Using the identity (6) from the proof of Lemma 4 we can show that

$$P\big(N_\beta(\lambda) \ge n, T_i \le \tau \le T_{i+1}\big) \le P\big(N_\beta\big(\lambda e^{-\frac{\beta}{4} T_i}\big) \ge n-1\big) P(\tau < T_{i+1}). \tag{32}$$

Note that $\lambda T_i \le \frac{12}{\beta} \frac{\lambda}{n} \log(n/\lambda)$ which can be made as small we wish by choosing n_0 large enough. Thus we can apply the upper bound in Proposition 7 to get

$$\begin{aligned}
\log P(\tau < T_{i+1}) &\le -\frac{2}{T_{i+1}} \mathcal{W}^2(-\lambda T_{i+1}) + c\left(1 + \frac{1}{T_{i+1}}\right) \log\left(\frac{1}{\lambda T_{i+1}}\right) \\
&\le -\frac{\beta}{2\gamma_{i+1}} \frac{n}{\log(n/\lambda)} \mathcal{W}^2\left(\frac{\lambda}{n} \log\left(\frac{\lambda}{n}\right)\right) + C\left(1 + \frac{\beta}{4\gamma_{i+1}} \frac{n}{\log(n/\lambda)}\right) \log\left(\frac{n}{\lambda}\right) \\
&\le -\frac{\beta}{2\gamma_{i+1}} n \log\left(\frac{n}{\lambda}\right) + C \log\left(\frac{n}{\lambda}\right) + Cn.
\end{aligned} \tag{33}$$

Here we used the bounds (16), $\gamma_{i+1} \ge 1/3$ and the identity $\mathcal{W}(x \log x) = \log x$. The constant C depends only on β and λ_0, but may change from line to line.

Using the assumption (30) for $\lambda e^{-\frac{\beta}{4} T_i} \le \lambda_0$ we get

$$P\big(N_\beta\big(\lambda e^{-\frac{\beta}{4} T_i}\big) \ge n-1\big) \le \exp\left[-f_{n-1} \log\left(\frac{n-1}{\lambda e^{-\frac{\beta}{4} T_i}}\right) + c_1(n-1)^2 \right]. \tag{34}$$

Note that from the definition of T_i (if $n > \max(1, \lambda_0)$) we get

$$\left(1 + \gamma_i \frac{1}{n}\right) \log(n/\lambda) - \frac{1}{n-1} \le \log\left(\frac{n-1}{\lambda e^{-\frac{\beta}{4} T_i}}\right). \tag{35}$$

Using (34) and (35) we can write

$$\begin{aligned}
-f_{n-1} \log\left(\frac{n-1}{\lambda e^{-\frac{\beta}{4} T_i}}\right) &\le -f_{n-1}\left(1 + \frac{\gamma_i}{n}\right) \log(n/\lambda) + \frac{f_{n-1}}{n-1} \\
&\le -\left(f_{n-1}\left(1 + \frac{\gamma_{i+1}}{n}\right) - \frac{\beta}{2}\right) \log(n/\lambda) + \frac{\beta}{2}(n-1),
\end{aligned}$$

where we used $f_{n-1} \leq \frac{\beta}{2}(n-1)^2$ and $\gamma_{i+1} - \gamma_i \leq \frac{1}{n}$, which holds for $i \geq 1$. We also have

$$\log\left(\frac{1}{\lambda T_{i+1}}\right) = \log(n/\lambda) - \log\log(n/\lambda) - \log(4\gamma_{i+1}/\beta) \leq \log(n/\lambda),$$

where we used $1/3 \leq \gamma_i \leq 3$ and that n/λ can be made sufficiently large by choosing n_0 appropriately.

Collecting all our estimates we get that for $1 \leq i \leq M - 1$ we have

$$\log P\big(N_\beta(\lambda) \geq n, T_i \leq \tau \leq T_{i+1}\big)$$
$$\leq -\left(f_{n-1}\left(1 + \frac{\gamma_{i+1}}{n}\right) + \frac{\beta}{2\gamma_{i+1}}n\right)\log(n/\lambda) + c_1(n-1)^2 + C\log\left(\frac{n}{\lambda}\right) + Cn.$$

Assuming $c_1 > C$ this yields

$$\log P\big(N_\beta(\lambda) \geq n, T_i \leq \tau \leq T_{i+1}\big)$$
$$\leq -\left(f_{n-1}\left(1 + \frac{\gamma_{i+1}}{n}\right) + \frac{\beta}{2\gamma_{i+1}}n - C\right)\log(n/\lambda) + c_1 n^2 - c_1 n$$
$$\leq -(f_{n-1} + \sqrt{2\beta f_{n-1}} - C)\log(n/\lambda) + c_1 n^2 - c_1 n. \tag{36}$$

In the last step we used that $x(1 + \frac{\gamma}{n}) + \frac{\beta}{2\gamma}n \geq x + \sqrt{2\beta x}$ if $x > 0, \gamma > 0$.

For the $i = 0$ case we can use a similar argument (and the fact that $\gamma_1 = \frac{1}{3}$) to show that

$$\log P\big(N_\beta(\lambda) \geq n, T_0 \leq \tau \leq T_1\big) \leq -\left(f_{n-1} + \frac{\beta}{2}3n - C\right)\log(n/\lambda) + c_1 n^2 - c_1 n.$$

Since $f_{n-1} \leq \frac{\beta}{2}(n-1)^2$ we have $\frac{\beta}{2}3n \geq \sqrt{2\beta f_{n-1}}$ and so inequality (36) holds for $i = 0$ as well.

To bound the last term in (31) we start with the bound

$$P\big(N_\beta(\lambda) \geq n, T_M \leq \tau\big) \leq P\big(N_\beta(\lambda e^{-\frac{\beta}{4}T_M}) \geq n - 1\big).$$

Recalling that $\gamma_M = 3$ and using arguments similar to what we used to obtain (36) we get

$$P\big(N_\beta(\lambda) \geq n, T_M \leq \tau\big) \leq -f_{n-1}\left(1 + \frac{3}{n}\right)\log(n/\lambda) + c_1 n^2 - c_1 n. \tag{37}$$

Applying the upper bounds (36) and (37) on the respective terms of the sum in (31) yields

$$\log P\big(N_\beta(\lambda) \geq n\big) \leq -\min\left(f_{n-1} + \sqrt{2\beta f_{n-1}} - c_2, f_{n-1}\left(1 + \frac{3}{n}\right)\right)\log(n/\lambda)$$
$$+ c_1 n^2 - c_1 n + \log(3n + 1),$$

with constants c_1, c_2 that only depend on λ_0, β. By choosing $c_1 > 2$ we can simplify this as

$$\log P\big(N_\beta(\lambda) \geq n\big) \leq -\min\left(f_{n-1} + \sqrt{2\beta f_{n-1}} - c_2, f_{n-1}\left(1 + \frac{3}{n}\right)\right)\log(n/\lambda) + c_1 n^2.$$

Combining this result with (29) we get the following statement. There is a positive integer n_0', and positive constants c, c_1 (all depending only on λ_0, β) so that if we set $f_{n_0} = n_0$ for an $n_0 > n_0'$ and

$$f_n = \min\left(f_{n-1} + \sqrt{2\beta f_{n-1}} - c_2, f_{n-1}\left(1 + \frac{3}{n}\right), \frac{\beta}{2}n^2\right) \quad \text{for } n > n_0, \tag{38}$$

then

$$\log P\big(N_\beta(\lambda) \geq n\big) \leq -f_n \log\left(\frac{n}{\lambda}\right) + c_1 n^2$$

for all $n \geq n_0$, $\lambda \leq \lambda_0$.

To finish our proof we just have to show that we can choose a large enough n_0 as our starting point so that the solution of the recursion (38) satisfies $f_n \geq \frac{\beta}{2}n^2 - cn\log(n+1)$ with a constant c.

This is fairly straightforward although somewhat technical. A simple computation shows that if we choose n_0 large enough then the following statements hold:

  (i) f_n is strictly increasing,
 (ii) $f_n = f_{n-1}(1 + \frac{3}{n})$ for $n_0 < n < n_1$ for a certain n_1,
(iii) $f_n = \min(f_{n-1} + \sqrt{2\beta f_{n-1}} - c_2, \frac{\beta}{2}n^2)$ for $n \geq n_1$, and
(iv) if n is large enough and $f_{n-1} \geq \frac{\beta}{2}(n-1)^2 - c(n-1)\log n$ with a large enough c then we also have

$$f_{n-1} + \sqrt{2\beta f_{n-1}} - c \geq \frac{\beta}{2}n^2 - cn\log(n+1).$$

Putting these statements together we get that $f_n \geq \frac{\beta}{2}n^2 - cn\log(n+1)$ for a large enough n which gives

$$\log P\big(N_\beta(\lambda) \geq n\big) \leq -\left(\frac{\beta}{2}n^2 - cn\log(n+1)\right)\log\left(\frac{n}{\lambda}\right) + c_1 n^2 \tag{39}$$

for a large enough n. Using (29) we can now modify c, c_1 so that (39) holds for all n and $\lambda \leq \lambda_0$, which finishes the proof. $\qquad\square$

5. Proof of Proposition 7

We prove the upper and lower bounds separately. In both cases we will prove the estimate for $\tilde{\tau}_\lambda$ first, then use (9) and (10) to get the corresponding statement for τ_λ.

By introducing the new variable $X_\lambda = \log(\tan(\tilde{\alpha}_\lambda/4))$ we can transform (8) into the SDE

$$dX_\lambda = \frac{\lambda}{2}\cosh X_\lambda \, dt + \frac{1}{2}\tanh X_t \, dt + dB_t, \quad X_\lambda(0) = -\infty. \tag{40}$$

Under this transformation $\tilde{\tau}_\lambda$ becomes the hitting time of ∞ for the process X_λ. Following [10] we compare X_λ with another family of diffusions. For $a \geq -1$ consider the SDE

$$dY_{\lambda,a} = \frac{\lambda}{2}\sqrt{\cosh^2 Y_{\lambda,a} + a} \, dt + \frac{1}{2}\tanh Y_{\lambda,a} \, dt + dB_t, \quad Y_{\lambda,a}(0) = -\infty, \tag{41}$$

and denote by $\tau_{Y,\lambda,a}$ the hitting time of ∞. In [10] the Cameron–Martin–Girsanov formula was used for diffusions with explosions to compare X_λ and $Y_{\lambda,a}$ and the following results were obtained.

Proposition 8 ([10]). *For $a > -1$ and $s_1 < s_2$ we have*

$$P\big(\tilde{\tau}_\lambda \in [s_1, s_2]\big) = E\big[\mathbf{1}\big(\tau_{Y,\lambda,a} \in [s_1, s_2]\big)e^{-G_{\tau_{Y,\lambda,a}}(Y)}\big]. \tag{42}$$

Here $G_{\tau_{Y,\lambda,a}}(Y)$ is the Girsanov factor for which the following estimate holds:

$$\left| G_{\tau_{Y,\lambda,a}}(Y) + \frac{\lambda^2 a \tau_{Y,\lambda,a}}{8} - \lambda\big((1+a)K(-a) - E(-a)\big)\right| \leq \frac{\lambda\sqrt{|a|}}{4}\tau_{Y,\lambda,a}. \tag{43}$$

Moreover,

$$Ee^{-\frac{\lambda^2 a}{8}\tilde{\tau}_\lambda - \frac{\lambda\sqrt{|a|}}{4}\tilde{\tau}_\lambda} \leq e^{-\lambda((1+a)K(-a) - E(-a))}. \tag{44}$$

The proof of these statements are contained in Section 3 of [10]. Note the slight notational inconsistency, in the statements in [10] $-a$ is used in place of a.

The motivation behind introducing the diffusions $Y_{\lambda,a}$ can be explained as follows. The expected value of the blow-up time $\tilde{\tau}_\lambda$ is $\frac{2\pi}{\lambda}$. If we want to estimate the probability that the diffusion X_λ blows up a lot faster (in time $T \ll \frac{2\pi}{\lambda}$) then we can first try to find the "cheapest" (or most likely) way for X_λ to do that. This leads to a variational problem whose solution gives the modified diffusions $Y_{\lambda,a}$ (with a as a function of T).

5.1. *Proof of the upper bound in Proposition* 7

The proof of the upper bound will rely on the estimate (44) on the moment generating function of $\tilde{\tau}_\lambda$.

Proof of the upper bound in Proposition 7. By Markov's inequality we have

$$P(\tilde{\tau}_\lambda \le t) \le \left(Ee^{A\tilde{\tau}_\lambda}\right)e^{-At} \quad \text{for } A < 0.$$

We will choose $A = -\frac{\lambda^2 a}{8} - \frac{\lambda\sqrt{a}}{4}$ with $a = -K^{(-1)}(t\lambda/4) \ge -K^{(-1)}(\varepsilon/4) > 0$. Using the estimate (44) of Proposition 8 we get

$$Ee^{A\tilde{\tau}_\lambda} \le e^{-\lambda((1+a)K(-a)-E(-a))}.$$

This gives the upper bound

$$P(\tilde{\tau}_\lambda \le t) \le e^{-\lambda((1+a)K(-a)-E(-a))+\frac{\lambda^2 a}{8}t+\frac{\lambda\sqrt{a}}{4}t} \le e^{K^{(-1)}(t\lambda/4)\frac{\lambda^2 t}{8}+\lambda E(-a)+\frac{\lambda\sqrt{a}t}{4}},$$

where we used our specific choice of a. If ε is small enough we will have $a \ge -K^{(-1)}(\varepsilon/4) > 2$ which allows us to use the asymptotics of Lemma 6. This gives the bound $a \le \frac{C}{\lambda^2 t^2}\log^2(\frac{1}{\lambda t})$ with a positive constant C, and also

$$\frac{K^{(-1)}(t\lambda/4)\lambda^2 t}{8} + \lambda E(-a) + \frac{\lambda\sqrt{a}t}{4} \le -\frac{2}{t}\mathcal{W}(-\lambda t/4)^2 + \frac{C}{t} + C\left(1 + \frac{1}{t}\right)\log\left(\frac{1}{\lambda t}\right)$$

(with a possibly different C). Using the bounds in (16) and changing the value of C again we get

$$\frac{K^{(-1)}(t\lambda/4)\lambda^2 t}{8} + \lambda E(-a) + \frac{\lambda\sqrt{a}t}{4} \le -\frac{2}{t}\mathcal{W}(-\lambda t)^2 + C\left(1 + \frac{1}{t}\right)\log\left(\frac{1}{\lambda t}\right),$$

which finishes the proof for $\tilde{\tau}_\lambda$.

To get the similar statement for τ_λ we use (9) to obtain

$$P(\tau_\lambda \le t) \le P(\tilde{\tau}_{\frac{\beta}{4}\lambda} \le t) \le \frac{2}{t}\mathcal{W}^2\left(-\frac{\lambda\beta}{4}t\right) + c\left(1 + \frac{1}{t}\right)\log\left(\frac{1}{\frac{\lambda\beta}{4}t}\right). \tag{45}$$

Note that we are allowed to use the just proved bound on $\tilde{\tau}$ if we choose the upper bound ε on λt small enough. To complete the proof we use (16) to replace $\frac{\lambda\beta}{4}$ with λ everywhere by choosing ε small enough and modifying the constant in front of the last term of (45). □

5.2. *Proof of the lower bounds in Proposition* 7

Our lower bound proof will rely on the following lemma, which will be proved at the end of this subsection.

Lemma 9. *There is a constant $c > 0$ so that for any a, λ with $\lambda\sqrt{a} \ge 2$ and $a > 2$ we have*

$$P\left(\lambda\tau_{Y,\lambda,a} \in \left[4K(-a)\left(1 - \frac{5}{\lambda\sqrt{a}}\right), 4K(-a)\left(1 + \frac{5}{\lambda\sqrt{a}}\right)\right]\right) \ge c\sqrt{K(-a)}. \tag{46}$$

Now we are ready to prove the lower bounds in Proposition 7.

Proof of the lower bounds in Proposition 7. We first prove that the following (weaker) lower bound holds if t and λ satisfy the listed conditions

$$P(\tau_\lambda < t) \geq \exp\left[-\frac{2}{t}\mathcal{W}^2(-\lambda t) - c\left(1 + \frac{1}{t}\right)\log\left(\frac{1}{\lambda t}\right)\right]. \tag{47}$$

We first show the inequality for $\tilde{\tau}_\lambda$ instead of τ_λ.

We use (42) and (43) with an $a > 0$ to get

$$
\begin{aligned}
P(\lambda\tilde{\tau}_\lambda \leq s_2) &\geq P\left(\lambda\tilde{\tau}_\lambda \in [s_1, s_2]\right) \\
&= E\left[\mathbf{1}\left(\lambda\tau_{Y,\lambda,a} \in [s_1, s_2]\right)e^{-G_{\tau_{Y,\lambda,a}}(Y)}\right] \\
&\geq E\left[\mathbf{1}\left(\lambda\tau_{Y,\lambda,a} \in [s_1, s_2]\right)e^{\frac{\lambda^2 a\tau_{Y,\lambda,a}}{8} - \lambda((1+a)K(-a) - E(-a)) - \frac{\lambda\sqrt{a}}{4}\tau_{Y,\lambda,a}}\right].
\end{aligned}
$$

Assuming $\lambda\sqrt{a} \geq 2$ we see that the exponential term under the last expectation is nondecreasing in $\tau_{Y,\lambda,a}$ and because of the indicator function we get the bound

$$P(\lambda\tilde{\tau}_\lambda \leq s_2) \geq P\left(\lambda\tau_{Y,\lambda,a} \in [s_1, s_2]\right)e^{\frac{\lambda a s_1}{8} - \lambda((1+a)K(-a) - E(-a)) - \frac{\sqrt{a}}{4}s_1}.$$

Now choose $[s_1, s_2] = [4K(-a)(1 - \frac{5}{\lambda\sqrt{a}}), 4K(-a)(1 + \frac{5}{\lambda\sqrt{a}})]$, and assume $a > 2$. With Lemma 9 we can estimate the probability on the right and we can use the asymptotics (14) of Proposition 6 to estimate the exponential term. Collecting all the terms gives

$$P\left(\lambda\tilde{\tau}_\lambda \leq 4K(-a)\left(1 + \frac{5}{\lambda\sqrt{a}}\right)\right) \geq \exp\left[-\frac{\lambda a}{2}K(-a) - C(1 + \lambda)\log(a)\right] \tag{48}$$

with an absolute constant C.

Now assume that $\lambda_0 > 0$ is fixed, choose a small $\hat{\varepsilon} > 0$ and assume that $0 < \lambda \leq \lambda_0$, $\lambda T \leq \hat{\varepsilon}$ and $T \leq \hat{\varepsilon}\log(\frac{1}{\lambda T})$. Set $a = -K^{(-1)}(\frac{\lambda T}{4})$. If $\hat{\varepsilon}$ is chosen small enough then using the bounds of Proposition 6 one can readily show that

$$\frac{1}{2}\frac{\log(\frac{1}{\lambda T})}{\lambda T} \leq \sqrt{a} \leq 10\frac{\log(\frac{1}{\lambda T})}{\lambda T}, \tag{49}$$

which yields both $\lambda\sqrt{a} \geq 2$ and $a > 2$ (using $\lambda \leq \lambda_0$) for small enough $\hat{\varepsilon}$. Thus we can use (48) to get

$$
\begin{aligned}
\log P\left(\tilde{\tau}_\lambda \leq T\left(1 + \frac{10T}{\log(1/(\lambda T))}\right)\right) &\geq -\frac{\lambda a}{2}\cdot\frac{\lambda T}{4} - C(1 + \lambda)\log a \\
&\geq -\frac{2}{T}\mathcal{W}^2\left(-\frac{\lambda T}{16}\right) - C\log\left(\frac{1}{\lambda T}\right) \\
&\geq -\frac{2}{T}\mathcal{W}^2(-\lambda T) - C\log\left(\frac{1}{\lambda T}\right), \tag{50}
\end{aligned}
$$

where we again used Proposition 6 and the constant C can change from line to line (but only depends on λ_0 and $\hat{\varepsilon}$). Now set $t = T(1 + \frac{10T}{\log(1/(\lambda T))})$, we can see that $T \leq t \leq T(1 + 10\hat{\varepsilon})$. The bound (50) gives

$$
\begin{aligned}
\log P(\tilde{\tau}_\lambda \leq t) &\geq -\frac{2}{t}\mathcal{W}^2(-\lambda T)\left(1 + \frac{10T}{\log(1/(\lambda T))}\right) - C\log\left(\frac{1}{\lambda T}\right) \\
&= -\frac{2}{t}\mathcal{W}^2(-\lambda T) - \frac{20T}{t}\mathcal{W}^2(-\lambda T)\frac{1}{\log(1/(\lambda T))} - C\log\left(\frac{1}{\lambda T}\right).
\end{aligned}
$$

Using the second bound of (16) (which is allowed since $\lambda T \leq \hat{\varepsilon}$) together with $1 \leq \frac{t}{T} \leq 1 + 10\hat{\varepsilon}$ we get that

$$\frac{20T}{t}\mathcal{W}^2(-\lambda T)\frac{1}{\log(1/(\lambda T))} + C\log\left(\frac{1}{\lambda T}\right) \leq C'\log\left(\frac{1}{\lambda t}\right).$$

Using the first and then the second bound of (16) we get

$$-\frac{2}{t}\mathcal{W}^2(-\lambda T) \geq -\frac{2}{t}\left(\mathcal{W}^2(-\lambda t) + C\log\left|\frac{T}{t}\right| \cdot \mathcal{W}(-\lambda t)\right) \geq -\frac{2}{t}\mathcal{W}^2(-\lambda t) - \frac{C'}{t}\log\left(\frac{1}{\lambda t}\right).$$

Putting everything together gives

$$\log P(\tilde{\tau}_\lambda \leq t) \geq -\frac{2}{t}\mathcal{W}^2(-\lambda t) - c\left(1 + \frac{1}{t}\right)\log\left(\frac{1}{\lambda t}\right)$$

with a constant c. The only thing left to show is that for small enough $\varepsilon' > 0$ if $\lambda t \leq \varepsilon'$, $t \leq \varepsilon'\log(1/(\lambda t))$ then we can find T with $t = T(1 + \frac{10T}{\log(1/(\lambda T))})$ which satisfies $\lambda T \leq \hat{\varepsilon}$ and $T \leq \hat{\varepsilon}\log(\frac{1}{\lambda T})$. This is fairly straightforward, one can check that if ε' is small enough then we can find a T with $(1 - 10\varepsilon')t \leq T \leq t$ satisfying all the listed conditions.

Finally, to prove (47) for τ_λ we use the comparison (10). Note that if we have the bound $\lambda \leq \lambda_0$ then $\lambda'' = \frac{\beta}{4}e^{-\frac{\beta}{4}t}\lambda \leq \lambda_0'' = \frac{\beta}{4}\lambda_0$. By choosing a small enough ε in the bounds $\lambda t \leq \varepsilon$ and $t \leq \varepsilon\log(\frac{1}{\lambda t})$ we can use the just proved lower bound for $\tilde{\tau}_{\lambda''}$ to give

$$\log P(\tau_\lambda \leq t) \geq -\frac{2}{t}\mathcal{W}^2\left(-\lambda\frac{\beta}{4}e^{-\frac{\beta}{4}t}\right) - c\left(1 + \frac{1}{t}\right)\left(\log\left(\frac{1}{\lambda t}\right) + \frac{\beta}{4}t + \log\frac{4}{\beta}\right).$$

The fact that the right-hand side is bounded from below by $-\frac{2}{t}\mathcal{W}^2(-\lambda t) - \frac{c'}{t}\log(\frac{1}{\lambda t})$ with a new constant c' follows along the line of the already seen arguments using Proposition 6 and our assumptions on λ, t.

To prove (18) we use (47) together with the upper bound from Proposition 7 to get the final lower bound.

$$P(t/2 \leq \tau_\lambda < t) = P(\tau_\lambda < t) - P(\tau_\lambda < t/2)$$
$$\geq \exp\left[-\frac{2}{t}\mathcal{W}^2(-\lambda t) - c\left(1 + \frac{1}{t}\right)\log\left(\frac{1}{\lambda t}\right)\right](1 - \exp[\mathcal{E}(t)]),$$

where

$$\mathcal{E}(t) = \frac{2}{t}\mathcal{W}^2(-\lambda t) - \frac{4}{t}\mathcal{W}^2(-\lambda T/2) + c\left(1 + \frac{2}{t}\right)\log\left(\frac{2}{\lambda t}\right) + c\left(1 + \frac{1}{t}\right)\log\left(\frac{1}{\lambda t}\right)$$
$$\leq -\frac{2}{T}\mathcal{W}^2(-\lambda t/2) + \tilde{c}\left(1 + \frac{1}{t}\right)\log\left(\frac{1}{\lambda t}\right)$$
$$\leq -\frac{c}{t}\log^2\left(\frac{1}{\lambda t}\right) + \tilde{c}\left(1 + \frac{1}{t}\right)\log\left(\frac{1}{\lambda t}\right).$$

In the last two steps we used the bounds from Proposition 6. By choosing $\varepsilon > 0$ small enough our conditions on λ and t together with the just proved inequality gives $\mathcal{E}(t) \leq -1$. Hence

$$P(t/2 \leq \tau_\lambda < t) \geq \exp\left[-\frac{2}{t}\mathcal{W}^2(-\lambda t) - c\left(1 + \frac{1}{t}\right)\log\left(\frac{1}{\lambda t}\right)\right](1 - e^{-1})$$
$$\geq \exp\left[-\frac{2}{t}\mathcal{W}^2(-\lambda t) - c'\left(1 + \frac{1}{t}\right)\log\left(\frac{1}{\lambda t}\right)\right]$$

with an appropriate $c' > 0$.   □

Now we return to the proof of Lemma 9.

Proof of Lemma 9. Our main tool is again a coupling argument. We introduce the diffusions

$$dY_- = \frac{\lambda}{2}\sqrt{\cosh^2 Y_- + a}\, dt - \frac{1}{2}\, dt + dB,$$

$$dY_+ = \frac{\lambda}{2}\sqrt{\cosh^2 Y_+ + a}\, dt + \frac{1}{2}\, dt + dB$$

driven by the same Brownian motion as Y. Comparing the drifts of Y_-, Y_+ to that of Y we see that $Y_- \leq Y$ and $Y \leq Y_+$ until the blowup of the larger diffusions.

We also introduce $Z_- = Y_- + t/2 - B_t$ and $Z_+ = Y_+ - t/2 - B_t$ which satisfy the differential equations

$$Z'_- = \frac{\lambda}{2}\sqrt{\cosh^2(Z_- - t/2 + B_t) + a} > 0 \quad \text{and}$$

$$Z'_+ = \frac{\lambda}{2}\sqrt{\cosh^2(Z_+ + t/2 + B_t) + a} > 0$$

until their blowup times. Note that Z_- blows up exactly when Y_- blows up, and Z_+ blows up when Y_+ does. Thus Z_+ will blow up before Y does and Z_- will blow up after.

We introduce the constants

$$\varepsilon_1 = \frac{10K(-a)}{\lambda\sqrt{a}}, \qquad \varepsilon_2 = \frac{K(-a)}{8\lambda}, \qquad \theta = \frac{4K(-a) + \varepsilon_1}{\lambda},$$

and we record for further use that since $\lambda\sqrt{a} \geq 2$ we have

$$\varepsilon_1 > \frac{4\varepsilon_2 + \theta}{\sqrt{a}}. \tag{51}$$

We will show that on the event $H = \{\sup_{t \leq \theta} |B_t| \leq \varepsilon_2\}$ the process Z_- will blow up before θ and the process Z_+ will blow up after $\lambda^{-1}(4K(-a) - 2\varepsilon_1)$. The final bound will come from an estimate on the probability of the event H.

Note that $Z'_- > \frac{2\sqrt{a}}{\lambda} > 0$ until its blowup, in particular Z_- is strictly increasing. Moreover, on the event H for $t \leq \theta$ we have

$$Z_-(t) - \theta/2 - \varepsilon_2 \leq Z_-(t) - t/2 + B_t \leq Z_-(t) + \varepsilon_2.$$

Let η_1 be the hitting time of $-\varepsilon_2$ by Z_-. Then for $t \leq \eta_1 \wedge \theta$ on the event H we have

$$Z'_-(t) = \frac{\lambda}{2}\sqrt{\cosh^2\big(Z_-(t) - t/2 + B_t\big) + a} \geq \frac{\lambda}{2}\sqrt{\cosh^2\big(Z_-(t) + \varepsilon_2\big) + a}.$$

Rearranging, integrating and using identity (12):

$$\frac{\lambda}{2}(\eta_1 \wedge \theta) \leq \int_0^{\eta_1 \wedge \theta} \frac{Z'_-(t)}{\sqrt{\cosh^2(Z_-(t) + \varepsilon_2) + a}}\, dt \leq \int_{-\infty}^{-\varepsilon_2} \frac{1}{\sqrt{\cosh^2(z + \varepsilon_2) + a}}\, dz = K(-a),$$

which gives

$$\lambda(\eta_1 \wedge \theta) \leq 2K(-a) \leq \lambda\theta \quad \text{and} \quad \lambda\eta_1 \leq 2K(-a).$$

Next we bound the travel time of Z_- from $-\varepsilon_2$ to $\theta/2 + \varepsilon_2$ which we denote by η_2. Since $Z'_- \geq \frac{\lambda}{2}\sqrt{a}$ we have the bound

$$\lambda\eta_2 \leq \frac{4\varepsilon_2 + \theta}{\sqrt{a}} \leq \varepsilon_1.$$

Note that we have $\eta_1 + \eta_2 \le \lambda^{-1}(2K(-a) + \varepsilon_1) < \theta$ on H. Consider now the travel time η_3 the from $\theta/2 + \varepsilon_2$ to ∞. Denote by $\hat{\eta}_3$ the minimum of η_3 and $\theta - \eta_1 - \eta_2 > 0$. On the event H if $Z_- \ge \theta/2 + \varepsilon_2$, $\eta_1 + \eta_2 \le t \le \theta$ then we have the lower bound

$$Z'_-(t) = \frac{\lambda}{2}\sqrt{\cosh^2\big(Z_-(t) - t/2 + B_t\big) + a} \ge \frac{\lambda}{2}\sqrt{\cosh^2\big(Z_-(t) - \theta/2 - \varepsilon_2\big) + a}.$$

Rearranging the inequality and integrating from $\eta_1 + \eta_2$ to $\eta_1 + \eta_2 + \hat{\eta}_3$ we get

$$\lambda\hat{\eta}_3 \le 2K(-a).$$

Collecting our estimates we see that on the event H we have

$$\lambda(\eta_1 + \eta_2 + \hat{\eta}_3) \le 4K(-a) + \varepsilon_1 = \lambda\theta.$$

This immediately implies $\hat{\eta}_3 = \eta_3$ and since $\eta_1 + \eta_2 + \eta_3$ is the blowup time of Y_- we also get that $\tau_{Y,\lambda,a} \le \theta$ on H.

To show that the blow up time of Z_+ is greater than $\lambda^{-1}(4K(-a) - 2\varepsilon_1)$ on the event H we carry out a similar calculation. The process Z_+ is strictly increasing and on the event H for $t \le \theta$ we have

$$Z_+(t) - \varepsilon_2 \le Z_+(t) + t/2 + B_t \le Z_-(t) + \theta/2 + \varepsilon_2.$$

Let ζ_1 be the hitting time of $-\theta/2 - \varepsilon_2$ for process Z_+. Since the blowup time of Z_+ is smaller than that of Z_- we get that $\zeta_1 < \theta$ on the event H. For $Z_+ < -\theta/2 - \varepsilon_2$ on H we get

$$Z'_+(t) = \frac{\lambda}{2}\sqrt{\cosh^2(Z_+ + t/2 + B_t) + a} \le \frac{\lambda}{2}\sqrt{\cosh^2\big(Z_+(t) - \varepsilon_2\big) + a},$$

which gives us

$$\frac{\lambda}{2}\zeta_1 \ge \int_0^{\zeta_1} \frac{Z'_+(t)}{\sqrt{\cosh^2(Z_+(t) - \varepsilon_2) + a}}\, dt = K(-a) - \int_{-\theta/2 - 2\varepsilon_2}^0 \frac{1}{\sqrt{\cosh^2(z) + a}}\, dz$$

$$\ge K(-a) - \frac{\theta + 4\varepsilon_2}{2\sqrt{a}} \ge K(-a) - \varepsilon_1/2.$$

In the last step we used (51). Now take ζ_3 to be the time spent by Z_+ traveling from ε_2 to ∞. Again note that the blow up time of Z_+ is bounded by that of Z_- which is bounded by θ on H. Therefore on H for $\varepsilon_2 \le Z_+$ we have

$$Z'_+(t) = \frac{\lambda}{2}\sqrt{\cosh^2\big(Z_+(t) + t/2 + B_t\big) + a} \le \frac{\lambda}{2}\sqrt{\cosh^2\big(Z_+(t) + \theta/2 + \varepsilon_2\big) + a}.$$

This yields

$$\frac{\lambda}{2}\zeta_3 \ge K(-a) - \int_0^{\theta/2 + 2\varepsilon_2} \frac{1}{\sqrt{\cosh^2(z) + a}}\, dz \ge K(-a) - \varepsilon_1/2.$$

These estimates give

$$\lambda(\zeta_1 + \zeta_3) \ge 4K(-a) - 2\varepsilon_1.$$

Our last bound shows that on H the process Y blows up after $\lambda^{-1}(4K(-a) - 2\varepsilon_1)$. Putting everything together yields

$$P\left(\lambda\tau_{Y,\lambda,a} \in \left[4K(-a)\left(1 - \frac{5}{\lambda\sqrt{a}}\right), 4K(-a)\left(1 + \frac{5}{\lambda\sqrt{a}}\right)\right]\right) \ge P(H).$$

On the other hand we have

$$P(H) = P\left(\sup_{t \le \lambda^{-1}\theta} |B_t| \le \varepsilon_2\right) = P\left(\sup_{t \le 1} |B_t| \le \lambda^{1/2}\theta^{-1/2}\varepsilon_2\right)$$

and

$$\lambda^{1/2}\theta^{-1/2}\varepsilon_2 = \frac{K(-a)}{8\sqrt{4K(-a)(1 + \frac{10}{\lambda\sqrt{a}})}} \geq \frac{1}{40}\sqrt{K(-a)}.$$

It is known that $P(\sup_{t\leq 1}|B_t| \leq \delta) \geq 2(2\Phi(\delta/4) - 1)$ for $\delta > 0$ (see e.g. Proposition 2.8.10 in [11]) which can be bounded from below by $c\delta$ if we have an upper bound on δ. Since $a > 2$ we have an upper bound on $\frac{1}{40}\sqrt{K(-a)}$ which finishes the proof. $\qquad\square$

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