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ASYMPTOTIC BIAS OF STOCHASTIC GRADIENT SEARCH

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The asymptotic behavior of the stochastic gradient algorithm using biased gradient estimates is analyzed. Relying on arguments based on dynamic system theory (chain-recurrence) and differential geometry (Yomdin theorem and Lojasiewicz inequalities), upper bounds on the asymptotic bias of this algorithm are derived. The results hold under mild conditions and cover a broad class of algorithms used in machine learning, signal processing and statistics.

CONTENTS

1. Introduction. Many problems in automatic control, system identification, signal processing, machine learning, operations research and statistics can be posed as a stochastic optimization problem, that is, as a minimization (or maximization) of an unknown objective function whose values are available only through noisy observations. Such a problem can be solved efficiently by stochastic gradient search (also known as the stochastic gradient algorithm). Stochastic gradient search is a procedure of the stochastic approximation type which iteratively approximates the minima of the objective function using a statistical or Monte Carlo estimator of the gradient of the objective function. Often, the estimator is biased, since unbiased gradient estimation is usually either too computationally expensive or not available at all. As a result of using biased gradient estimates, the

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stochastic gradient search is also biased, that is, the algorithm does not converge to the minima, but to their vicinity. In order to interpret the results produced by such an algorithm and to tune the algorithm's parameters (e.g., to achieve a better bias/variance tradeoff and a better convergence rate), it is important to study the asymptotic behavior and the asymptotic bias of the algorithm iterates.

Despite its practical and theoretical importance, the asymptotic behavior of stochastic gradient search using biased gradient estimates (also referred to as biased stochastic gradient search) has not attracted much attention in the literature. To the best of the authors' knowledge, this has only been analyzed in [11, 15, 16] and [14]. Although these results provide a good insight, they hold under restrictive conditions which are very hard to verify for complex stochastic gradient algorithms. Moreover, unless the objective function is of a simple form (e.g., convex or polynomial), none of these papers offers explicit bounds on the asymptotic bias of the algorithm iterates.

In this paper, we provide an original analysis of the asymptotic behavior of biased stochastic gradient search. Using arguments based on dynamic system theory (chain-recurrence) and differential geometry (Yomdin theorem and Lojasiewicz inequalities), we prove that the algorithm iterates converge to a vicinity of the set of minima. Relying on the same arguments, we also derive upper bounds on the radius of the vicinity (i.e., on the asymptotic bias of the algorithm iterates). Our results hold under mild and easily verifiable conditions and cover a broad class of complex stochastic gradient algorithms. We illustrate here how these results can be applied to the asymptotic analysis of a popular policy-gradient (reinforcement) learning algorithm proposed in [2]. In [33] (an extended version of this paper), these results have also been used to evaluate the asymptotic bias of an adaptive population Monte Carlo method and the asymptotic bias of recursive maximum split-likelihood estimation procedure for hidden Markov models.

The rest of this paper is organized as follows. The main results are presented in Section 2, where the biased stochastic gradient search is analyzed. In Section 3, these general results are applied to stochastic gradient algorithms with Markovian dynamics. In Section 4, we apply the results of Sections 2 and 3 to a policy-gradient algorithm. The results presented in Sections 2–4 are proved in Sections 5–8.

2. Main results. In this section, the asymptotic behavior of the following algorithm is analyzed:

$$\theta_{n+1} = \theta_n - \alpha_n\big(\nabla f(\theta_n) + \xi_n\big), \qquad n \geq 0. \tag{2.1}$$

Here, $f : \mathbb{R}^{d_\theta} \to \mathbb{R}$ is a differentiable function, while $\{\alpha_n\}_{n\geq 0}$ is a sequence of positive real numbers. θ_0 is an $\mathbb{R}^{d_\theta}$-valued random variable defined on a probability space $(\Omega, \mathcal{F}, P)$, while $\{\xi_n\}_{n\geq 0}$ is an $\mathbb{R}^{d_\theta}$-valued stochastic process defined on the same probability space. To allow more generality, we assume that for each $n \geq 0$, ξ_n is a random function of $\theta_0, \ldots, \theta_n$. In the area of stochastic optimization, recursion (2.1) is known as a stochastic gradient search or stochastic gradient algorithm.

The recursion minimizes the objective function $f(\cdot)$. The term $\nabla f(\theta_n) + \xi_n$ is interpreted as an estimator of the gradient $\nabla f(\theta_n)$, ξ_n representing the estimator's noise. For further details, see [27, 30] and references given therein.

Throughout the paper, the following notation is used. $\|\cdot\|$ and $d(\cdot,\cdot)$ stand for the Euclidean norm and the distance induced by the Euclidean norm (respectively). For $t \in (0,\infty)$ and $n \geq 0$, $a(n,t)$ is the integer defined as

$$a(n,t) = \max\left\{k \geq n : \sum_{i=n}^{k-1} \alpha_i \leq t\right\}.$$

$\mathcal{S}$ and $f(\mathcal{S})$ denote the sets of stationary points and critical values of $f(\cdot)$, that is,

$$\mathcal{S} = \{\theta \in \mathbb{R}^{d_\theta} : \nabla f(\theta) = 0\}, \qquad f(\mathcal{S}) = \{f(\theta) : \theta \in S\}. \tag{2.2}$$

For $\theta \in \mathbb{R}^{d_\theta}$, $\pi(\cdot;\theta)$ is the solution to the ODE $d\theta/dt = -\nabla f(\theta)$ satisfying $\pi(0;\theta) = \theta$. $\mathcal{R}$ denotes the set of chain-recurrent points of this ODE, that is, $\theta \in \mathcal{R}$ if and only if for any $\delta, t \in (0,\infty)$, there exist an integer $N \geq 1$, real numbers $t_1, \ldots, t_N \in [t,\infty)$ and vectors $\vartheta_1, \ldots, \vartheta_N \in \mathbb{R}^{d_\theta}$ (each of which can depend on θ, δ, t) such that

$$\|\vartheta_1 - \theta\| \leq \delta, \qquad \|\pi(t_N;\vartheta_N) - \theta\| \leq \delta, \qquad \|\vartheta_{k+1} - \pi(t_k;\vartheta_k)\| \leq \delta \tag{2.3}$$

for $1 \leq k < N$.

Elements of $\mathcal{R}$ can be considered as limits to slightly perturbed solutions to the ODE $d\theta/dt = -\nabla f(\theta)$. As the piecewise linear interpolation of sequence $\{\theta_n\}_{n\geq 0}$ falls into the category of such solutions, the concept of chain-recurrence is tightly connected to the asymptotic behavior of stochastic gradient search. In [3, 4], it has been shown that for unbiased gradient estimates, all limit points of $\{\theta_n\}_{n\geq 0}$ belong to $\mathcal{R}$ and that each element of $\mathcal{R}$ can potentially be a limit point of $\{\theta_n\}_{n\geq 0}$ with a nonzero probability.

If $f(\cdot)$ is Lipschitz continuously differentiable, it can be established that $\mathcal{S} \subseteq \mathcal{R}$. If additionally $f(\mathcal{S})$ is of a zero Lebesgue measure (which holds when $f(\mathcal{S})$ is discrete or when $f(\cdot)$ is d_θ-times continuously differentiable), then $\mathcal{S} = \mathcal{R}$. However, if $f(\cdot)$ is only Lipschitz continuously differentiable, then it is possible to have $\mathcal{R} \setminus \mathcal{S} \neq \varnothing$ (see [18], Section 4). Hence, in general, a limit point of $\{\theta_n\}_{n\geq 0}$ is in $\mathcal{R}$ but not necessarily in $\mathcal{S}$. For more details on chain-recurrence, see [3, 4, 11] and references therein. Given these results, it will prove useful to involve both $\mathcal{R}$ and $\mathcal{S}$ in the asymptotic analysis of biased stochastic gradient search.

The algorithm (2.1) is here analyzed under the following assumptions.

ASSUMPTION 2.1. $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

ASSUMPTION 2.2. $\{\xi_n\}_{n\geq 0}$ admits the decomposition $\xi_n = \zeta_n + \eta_n$ for each $n \geq 0$. $\{\zeta_n\}_{n\geq 0}$ and $\{\eta_n\}_{n\geq 0}$ are $\mathbb{R}^{d_\theta}$-valued stochastic processes [defined on

$(\Omega, \mathcal{F}, P)$] which satisfy

$$\lim_{n\to\infty} \max_{n\le k<a(n,t)} \left\| \sum_{i=n}^{k} \alpha_i \zeta_i \right\| = 0, \qquad \limsup_{n\to\infty} \|\eta_n\| < \infty \tag{2.4}$$

almost surely on $\{\sup_{n\ge 0} \|\theta_n\| < \infty\}$ for any $t \in (0, \infty)$.

ASSUMPTION 2.3.a. $\nabla f(\cdot)$ is locally Lipschitz continuous on $\mathbb{R}^{d_\theta}$.

ASSUMPTION 2.3.b. $f(\cdot)$ is p-times differentiable on $\mathbb{R}^{d_\theta}$, where $p > d_\theta$.

ASSUMPTION 2.3.c. $f(\cdot)$ is real-analytic on $\mathbb{R}^{d_\theta}$.

REMARK. Due to Assumption 2.1, $a(n, t)$ is well defined and finite for all $t \in (0, \infty)$, $n \ge 0$.

Assumption 2.1 corresponds to the step-size sequence $\{\alpha_n\}_{n\ge 0}$ and is commonly used in the asymptotic analysis of stochastic gradient algorithms. It is satisfied if $\alpha_n = n^{-a}$ for $n \ge 1$, where $a \in (0, 1]$.

Assumption 2.2 is a noise condition. It can be interpreted as a decomposition of the gradient estimator's noise $\{\xi_n\}_{n\ge 0}$ into a zero-mean sequence $\{\zeta_n\}_{n\ge 0}$ (which is averaged out by step-sizes $\{\alpha_n\}_{n\ge 0}$) and the estimator's bias $\{\eta_n\}_{n\ge 0}$. Assumption 2.2 is satisfied if $\{\zeta_n\}_{n\ge 0}$ is a martingale-difference or mixingale sequence, and if $\{\eta_n\}_{n\ge 0}$ are continuous functions of $\{\theta_n\}_{n\ge 0}$. It also holds for gradient search with Markovian dynamics (see Section 3). If the gradient estimator is asymptotically unbiased (i.e., $\lim_{n\to\infty} \eta_n = 0$ almost surely), Assumption 2.2 reduces to the Kushner–Clark condition, the weakest noise assumption under which the almost sure convergence of (2.1) can be demonstrated.

Assumptions 2.3.a, 2.3.b and 2.3.c are related to the objective function $f(\cdot)$ and its analytical properties. Assumption 2.3.a is involved in practically any asymptotic result for stochastic gradient search (as well as in many other asymptotic and nonasymptotic results for stochastic and deterministic optimization). Although much more restrictive than Assumption 2.3.a, Assumptions 2.3.b and 2.3.c hold for a number of algorithms routinely used in engineering, statistics, machine learning and operations research. In Section 4, Assumptions 2.3.b and 2.3.c are shown to hold for a policy-gradient algorithm. In [33], the same assumptions are verified for an adaptive population Monte Carlo method and for recursive maximum split-likelihood estimation in hidden Markov models. In [31], Assumption 2.3.c (which is a special case of Assumption 2.3.b) has been shown to hold for recursive maximum (full) likelihood estimation in hidden Markov models. In [32], the same assumption has also been verified for supervised and temporal-difference learning, online principal component analysis, Monte Carlo optimization of controlled Markov chains and recursive parameter estimation in linear stochastic systems.

Compared to Assumption 2.3.a, Assumptions 2.3.b and 2.3.c allow some sophisticated results from differential geometry to be applied to the asymptotic analysis of stochastic gradient search. More specifically, Yomdin theorem (a qualitative version of the Morse–Sard theorem; see [34] and Proposition 6.1 in Section 6) can be applied to functions satisfying Assumption 2.3.b, while Lojasiewicz inequalities (see [23, 24]; see also Proposition 6.2 in Section 6) hold for functions verifying Assumption 2.3.c. Using the Yomdin theorem and Lojasiewicz inequalities, a more precise characterization of the asymptotic bias of the stochastic gradient search can be obtained [see Parts (ii) and (iii) of Theorem 2.1].

In order to state the main results of this section, we need some further notation. Let η denote the asymptotic magnitude of the gradient estimator's bias $\{\eta_n\}_{n\geq 0}$, that is,

$$\eta = \limsup_{n\to\infty} \|\eta_n\|. \tag{2.5}$$

Moreover, for a compact set $Q \subset \mathbb{R}^{d_\theta}$, let Λ_Q denote the event

$$\Lambda_Q = \liminf_{n\to\infty} \{\theta_n \in Q\} = \bigcup_{n=0}^{\infty} \bigcap_{k=n}^{\infty} \{\theta_k \in Q\}. \tag{2.6}$$

With this notation, our main result on the asymptotic bias of the recursion (2.1) can be stated as follows.

THEOREM 2.1. *Suppose that Assumptions 2.1 and 2.2 hold. Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set. Then the following are true*:

(i) *If $f(\cdot)$ satisfies Assumption 2.3.a, there exists a (deterministic) nondecreasing function $\psi_Q : [0,\infty) \to [0,\infty)$ [independent of η and depending only on $f(\cdot)$] such that $\lim_{t\to 0} \psi_Q(t) = \psi_Q(0) = 0$ and*

$$\limsup_{n\to\infty} d(\theta_n, \mathcal{R}) \leq \psi_Q(\eta) \tag{2.7}$$

almost surely on Λ_Q.

(ii) *If $f(\cdot)$ satisfies Assumption 2.3.b, there exists a real number $K_Q \in (0,\infty)$ [independent of η and depending only on $f(\cdot)$] such that*

$$\limsup_{n\to\infty} \|\nabla f(\theta_n)\| \leq K_Q \eta^{q/2}, \tag{2.8}$$

$$\limsup_{n\to\infty} f(\theta_n) - \liminf_{n\to\infty} f(\theta_n) \leq K_Q \eta^q \tag{2.9}$$

almost surely on Λ_Q, where $q = (p - d_\theta)/(p-1)$.

(iii) *If $f(\cdot)$ satisfies Assumption 2.3.c, there exist real numbers $r_Q \in (0,1)$, $L_Q \in (0,\infty)$ [independent of η and depending only on $f(\cdot)$] such that*

$$\limsup_{n\to\infty} d(\theta_n, \mathcal{S}) \le L_Q \eta^{r_Q}, \tag{2.10}$$

$$\limsup_{n\to\infty} \|\nabla f(\theta_n)\| \le L_Q \eta^{1/2}, \tag{2.11}$$

$$\limsup_{n\to\infty} d\big(f(\theta_n), f(\mathcal{S})\big) \le L_Q \eta \tag{2.12}$$

almost surely on Λ_Q.

Theorem 2.1 is proved in Sections 5 and 6, while its global version is provided in Appendix A.

REMARK. If Assumption 2.3.b (or Assumption 2.3.c) is satisfied, then $\mathcal{S} = \mathcal{R}$. Hence, under Assumption 2.3.b, (2.7) still holds if $\mathcal{R}$ is replaced with $\mathcal{S}$.

REMARK. Function $\psi_Q(\cdot)$ depends on $f(\cdot)$ in two ways. First, it depends on $f(\cdot)$ through $\mathcal{R}$ and its geometric properties. Second, it depends on $f(\cdot)$ through upper bounds of $\|\nabla f(\cdot)\|$ and Lipschitz constants of $\nabla f(\cdot)$. An explicit construction of $\psi_Q(\cdot)$ is provided in the proof of Part (i) of Theorem 2.1 (Section 5).

REMARK. Like $\psi_Q(\cdot)$, constants K_Q and L_Q depend on $f(\cdot)$ through upper bounds of $\|\nabla f(\cdot)\|$ and Lipschitz constants of $\nabla f(\cdot)$. K_Q and L_Q also depend on $f(\cdot)$ through the Yomdin and Lojasiewicz constants (quantities M_Q, $M_{1,Q}$, $M_{2,Q}$ specified in Propositions 6.1, 6.2). Explicit formulas for K_Q and L_Q are included in the proof of Parts (ii) and (iii) of Theorem 2.1 (Section 6).

According to the literature on stochastic optimization and stochastic approximation, stochastic gradient search with unbiased gradient estimates (the case when $\eta = 0$) exhibits the following asymptotic behavior. Under mild conditions, sequences $\{\theta_n\}_{n\ge 0}$ and $\{f(\theta_n)\}_{n\ge 0}$ converge to $\mathcal{R}$ and $f(\mathcal{R})$ (respectively), that is,

$$\lim_{n\to\infty} d(\theta_n, \mathcal{R}) = 0, \qquad \lim_{n\to\infty} d\big(f(\theta_n), f(\mathcal{R})\big) = 0 \tag{2.13}$$

almost surely on $\{\sup_{n\ge 0}\|\theta_n\| < \infty\}$ (see [4], Proposition 4.1, Theorem 5.7, which hold under Assumptions 2.1, 2.2, 2.3.a). Under more restrictive conditions, sequences $\{\theta_n\}_{n\ge 0}$ and $\{f(\theta_n)\}_{n\ge 0}$ converge to $\mathcal{S}$ and a point in $f(\mathcal{S})$ (respectively), that is,

$$\lim_{n\to\infty} d(\theta_n, \mathcal{S}) = 0, \qquad \lim_{n\to\infty} \nabla f(\theta_n) = 0, \tag{2.14}$$

$$\lim_{n\to\infty} d\big(f(\theta_n), f(\mathcal{S})\big) = 0, \qquad \limsup_{n\to\infty} f(\theta_n) = \liminf_{n\to\infty} f(\theta_n) \tag{2.15}$$

almost surely on $\{\sup_{n\geq 0}\|\theta_n\| < \infty\}$ (see [4], Corollary 6.7, which holds under Assumptions 2.1, 2.2, 2.3.b). The same asymptotic behavior occurs when Assumptions 2.1, 2.3.a hold and $\{\xi_n\}_{n\geq 0}$ is a martingale-difference sequence (see [9], Proposition 1). When the gradient estimator is biased (the case where $\eta > 0$), (2.13)–(2.15) are not true any more. Now, the quantities

$$\limsup_{n\to\infty} d(\theta_n, \mathcal{R}), \qquad \limsup_{n\to\infty} \|\nabla f(\theta_n)\|, \tag{2.16}$$

$$\limsup_{n\to\infty} d\big(f(\theta_n), f(\mathcal{R})\big), \qquad \limsup_{n\to\infty} f(\theta_n) - \liminf_{n\to\infty} f(\theta_n) \tag{2.17}$$

are strictly positive and depend on η (it is reasonable to expect these quantities to decrease in η and to tend to zero as $\eta \to 0$). Hence, the quantities (2.16), (2.17) and their dependence on η can be considered as a sensible characterization of the asymptotic bias of the gradient search with biased gradient estimation. In the case of algorithm (2.1), such a characterization is provided by Theorem 2.1. The theorem includes relatively tight, explicit bounds on the quantities (2.16), (2.17) in the terms of the gradient estimator's bias η and analytical properties of $f(\cdot)$.

The results of Theorem 2.1 are of a local nature. They hold only on the event where algorithm (2.1) is stable (i.e., where sequence $\{\theta_n\}_{n\geq 0}$ belongs to a compact set Q). Stating results on the asymptotic bias of stochastic gradient search in such a local form is quite sensible due to the following reasons. The stability of stochastic gradient search is based on well-understood arguments which are rather different from the arguments used here to analyze the asymptotic bias. Moreover (and more importantly), as demonstrated in Appendix A, it is relatively easy to get a global version of Theorem 2.1 by combining the theorem with stability results for stochastic approximation (e.g., with the results of [12]). It is also worth mentioning that local asymptotic results are quite common in the areas of stochastic optimization and stochastic approximation (e.g., most of the results of [7], Part II, similarly as Theorem 2.1, hold only on set Λ_Q).

Stochastic gradient search with biased gradient estimation has found many applications in areas such as statistical inference, system identification and machine learning (see, e.g., [8, 13, 17, 27–29] and reference cited therein). However, to the best of the authors' knowledge, the asymptotic properties of biased stochastic gradient search and biased stochastic approximation have only been studied in [11], Section 5.3, [15, 16], [14], Section 2.7. The results obtained in these papers provide a good insight into the asymptotic behavior of the biased gradient search but are based on restrictive conditions. They only hold if $f(\cdot)$ is unimodal or if $\{\theta_n\}_{n\geq 0}$ belongs to the domain of an asymptotically stable attractor of $d\theta/dt = -\nabla f(\theta)$. Additionally, they do not provide any explicit bound on the asymptotic bias of the stochastic gradient search unless $f(\cdot)$ is of a simple form (e.g., convex or polynomial). Unfortunately, in the case of complex stochastic gradient algorithms (such as those studied in Section 4 and [33]), $f(\cdot)$ is usually multimodal with lot of unisolated local extrema and saddle points. For such algorithms, not only it is hard

to verify the assumptions adopted in [11], Section 5.3, [15, 16], [14], Section 2.7, but these assumptions are likely not to hold at all.

Relying on the chain-recurrence, Yomdin theorem and Lojasiewicz inequalities, Theorem 2.1 overcomes the described difficulties. The theorem allows the objective function $f(\cdot)$ to be multimodal (with manifolds of unisolated extrema and saddle points) and does not require $d\theta/dt = -\nabla f(\theta)$ to have an asymptotically stable attractor which is infinitely often visited by $\{\theta_n\}_{n\geq 0}$. In addition to this, Theorem 2.1 provides relatively tight, explicit bounds on the asymptotic bias of algorithm (2.1).

3. Stochastic gradient search with Markovian dynamics. In order to illustrate the results of Section 2 and to set up a framework for the analysis carried out in Section 4 and [33], we apply Theorem 2.1 to stochastic gradient algorithms with Markovian dynamics. These algorithms are defined by the following difference equation:

$$\theta_{n+1} = \theta_n - \alpha_n\big(F(\theta_n, Z_{n+1}) + \eta_n\big), \qquad n \geq 0. \tag{3.1}$$

In this recursion, $F : \mathbb{R}^{d_\theta} \times \mathbb{R}^{d_z} \to \mathbb{R}^{d_\theta}$ is a Borel-measurable function, while $\{\alpha_n\}_{n\geq 0}$ is a sequence of positive real numbers. θ_0 is an $\mathbb{R}^{d_\theta}$-valued random variable defined on a probability space $(\Omega, \mathcal{F}, P)$. $\{Z_n\}_{n\geq 0}$ is an $\mathbb{R}^{d_z}$-valued stochastic process defined on $(\Omega, \mathcal{F}, P)$, while $\{\eta_n\}_{n\geq 0}$ is an $\mathbb{R}^{d_\theta}$-valued stochastic process defined on the same probability space. $\{Z_n\}_{n\geq 0}$ is a Markov process controlled by $\{\theta_n\}_{n\geq 0}$, that is, there exists a family of transition probability kernels $\{\Pi_\theta(\cdot,\cdot) : \theta \in \mathbb{R}^{d_\theta}\}$ defined on $\mathbb{R}^{d_z}$ such that

$$P(Z_{n+1} \in B | \theta_0, Z_0, \ldots, \theta_n, Z_n) = \Pi_{\theta_n}(Z_n, B) \tag{3.2}$$

almost surely for any Borel-measurable set $B \subseteq \mathbb{R}^{d_z}$ and $n \geq 0$. $\{\eta_n\}_{n\geq 0}$ are random functions of $\{\theta_n\}_{n\geq 0}$, that is, η_n is a random function of $\theta_0, \ldots, \theta_n$ for each $n \geq 0$. In the context of stochastic gradient search, $F(\theta_n, Z_{n+1}) + \eta_n$ represents an estimator of the gradient $\nabla f(\theta_n)$.

The algorithm (3.1) is analyzed under the following assumptions.

ASSUMPTION 3.1. $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$ and $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$.

ASSUMPTION 3.2. There exist a differentiable function $f : \mathbb{R}^{d_\theta} \to \mathbb{R}$ and a Borel-measurable function $\tilde{F} : \mathbb{R}^{d_\theta} \times \mathbb{R}^{d_z} \to \mathbb{R}^{d_\theta}$ such that $\nabla f(\cdot)$ is locally Lipschitz continuous and

$$F(\theta, z) - \nabla f(\theta) = \tilde{F}(\theta, z) - (\Pi\tilde{F})(\theta, z) \tag{3.3}$$

for each $\theta \in \mathbb{R}^{d_\theta}$, $z \in \mathbb{R}^{d_z}$, where $(\Pi\tilde{F})(\theta, z) = \int \tilde{F}(\theta, z')\Pi_\theta(z, dz')$.

ASSUMPTION 3.3. For any compact set $Q \subset \mathbb{R}^{d_\theta}$, there exists a Borel-measurable function $\varphi_Q : \mathbb{R}^{d_z} \to [1, \infty)$ such that

$$\max\{\|F(\theta, z)\|, \|\tilde{F}(\theta, z)\|, \|(\Pi\tilde{F})(\theta, z)\|\} \le \varphi_Q(z),$$
$$\|(\Pi\tilde{F})(\theta', z) - (\Pi\tilde{F})(\theta'', z)\| \le \varphi_Q(z)\|\theta' - \theta''\|$$

for all $\theta, \theta', \theta'' \in Q$, $z \in \mathbb{R}^{d_z}$. Moreover,

$$\sup_{n\ge 0} E\big(\varphi_Q^2(Z_{n+1}) I_{\{\tau_Q > n\}} | \theta_0 = \theta, Z_0 = z\big) < \infty$$

for all $\theta \in \mathbb{R}^{d_\theta}$, $z \in \mathbb{R}^{d_z}$, where τ_Q is the stopping time defined by $\tau_Q = \inf(\{n \ge 0 : \theta_n \notin Q\} \cup \{\infty\})$.

ASSUMPTION 3.4. $\limsup_{n\to\infty} \|\eta_n\| < \infty$ almost surely on $\{\sup_{n\ge 0} \|\theta_n\| < \infty\}$.

Let $\mathcal{R}$, $\mathcal{S}$ and $f(\mathcal{S})$ have the same meaning as in Section 2 for the objective function $f(\cdot)$ now specified in Assumption 3.2. Moreover, let η and Λ_Q have the same meaning as in (2.5), (2.6). Then our results on the asymptotic behavior of the recursion (3.1) read as follows.

THEOREM 3.1. *Suppose that Assumptions 3.1–3.4 hold. Let $f(\cdot)$ be the function specified in Assumption 3.2, and let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set. Then the following are true*:

(i) *If $f(\cdot)$ satisfies Assumption 2.3.a, Part* (i) *of Theorem 2.1 holds.*
(ii) *If $f(\cdot)$ satisfies Assumption 2.3.b, Part* (ii) *of Theorem 2.1 holds.*
(iii) *If $f(\cdot)$ satisfies Assumption 2.3.c, Part* (iii) *of Theorem 2.1 holds.*

Theorem 3.1 is proved in Section 7, while its global version is provided in Appendix B.

Assumption 3.1 is related to the sequence $\{\alpha_n\}_{n\ge 0}$. It is satisfied if $\alpha_n = 1/n^a$ for $n \ge 1$, where $a \in (1/2, 1]$ is a constant. Assumptions 3.2 and 3.3 correspond to the stochastic process $\{Z_n\}_{n\ge 0}$ and are standard for the asymptotic analysis of stochastic approximation algorithms with Markovian dynamics. Basically, Assumptions 3.2 and 3.3 require the Poisson equation associated with algorithm (3.1) to have a solution which is Lipschitz continuous in θ. They hold if the following are satisfied: (i) $\Pi_\theta(\cdot, \cdot)$ is geometrically ergodic for each $\theta \in \mathbb{R}^{d_\theta}$, (ii) the convergence rate of $\Pi_\theta^n(\cdot, \cdot)$ is locally uniform in θ, and (iii) $\Pi_\theta(\cdot, \cdot)$ is locally Lipschitz continuous in θ on $\mathbb{R}^{d_\theta}$ (for further details, see [7], Chapter II.2, [26], Chapter 17, and references cited therein). Assumptions 3.2 and 3.3 have been introduced by Métivier and Priouret in [25] (see also [7], Part II), and later generalized by Kushner and his co-workers (see [22] and references cited therein). However, none of these results cover the scenario where biased gradient estimates are used. Theorem 3.1 fills this gap in the literature on stochastic optimization and stochastic approximation.

4. Application to reinforcement learning. In this section, Theorems 2.1 and 3.1 are applied to the asymptotic analysis of a popular policy-gradient search algorithm for average-cost Markov decision problems introduced in [2]. Policy-gradient search is one of the most important classes of reinforcement learning (for further details see, e.g., [8, 28]).

In order to define controlled Markov chains with parametrized randomized control and to formulate the corresponding average-cost decision problems, we use the following notation. $d_\theta \geq 1$, $N_x > 1$, $N_y > 1$ are integers, while $\mathcal{X} = \{1, \ldots, N_x\}$ and $\mathcal{Y} = \{1, \ldots, N_y\}$. $\phi(x, y)$ is a nonnegative (real-valued) function of $(x, y) \in \mathcal{X} \times \mathcal{Y}$. $p(x'|x, y)$ and $q_\theta(y|x)$ are nonnegative (real-valued) functions of $(\theta, x, x', y) \in \mathbb{R}^{d_\theta} \times \mathcal{X} \times \mathcal{X} \times \mathcal{Y}$ with the following properties: $q_\theta(y|x)$ is differentiable in θ for each $\theta \in \mathbb{R}^{d_\theta}$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$, and

$$\sum_{x' \in \mathcal{X}} p(x'|x, y) = 1, \qquad \sum_{y' \in \mathcal{Y}} q_\theta(y'|x) = 1$$

for the same θ, x, y. For $\theta \in \mathbb{R}^{d_\theta}$, $\{(X_n^\theta, Y_n^\theta)\}_{n \geq 0}$ is an $\mathcal{X} \times \mathcal{Y}$-valued Markov chain which is defined on a (canonical) probability space $(\Omega, \mathcal{F}, P_\theta)$ and satisfies

$$P_\theta\big(X_{n+1}^\theta = x', Y_{n+1}^\theta = y' | X_n^\theta = x, Y_n^\theta = y\big) = q_\theta\big(y'|x'\big) p\big(x'|x, y\big)$$

for each $x, x' \in \mathcal{X}$, $y, y' \in \mathcal{Y}$. $f(\cdot)$ is a function defined by

$$f(\theta) = \lim_{n \to \infty} E_\theta \left(\frac{1}{n} \sum_{i=1}^{n} \phi\big(X_i^\theta, Y_i^\theta\big) \right) \tag{4.1}$$

for $\theta \in \mathbb{R}^{d_\theta}$.[2] With this notation, an average-cost Markov decision problem with parameterized randomized control can be defined as the minimization of $f(\cdot)$. In the literature on reinforcement learning and operations research, $\{X_n^\theta\}_{n \geq 0}$ are referred to as a controlled Markov chain, while $\{Y_n^\theta\}_{n \geq 0}$ are called control actions. $p(x'|x, y)$ is referred to as the (chain) transition probability, while $q_\theta(y|x)$ is called the (control) action probability. For further details on Markov decision processes, see [8, 28] and references cited therein.

Since $f(\cdot)$ and its gradient rarely admit a closed-form expression, $f(\cdot)$ is minimized using methods based on stochastic gradient search and Monte Carlo gradient estimation. Such a method can be derived as follows. Let $s_\theta(x, y)$ be the score function defined by

$$s_\theta(x, y) = \frac{\nabla_\theta q_\theta(y|x)}{q_\theta(y|x)}$$

[2] Notice that $f(\theta)$ is well defined when $\{X_n^\theta\}_{n \geq 0}$ is irreducible.

for $\theta \in \mathbb{R}^{d_\theta}$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$. If $\{(X_n^\theta, Y_n^\theta)\}_{n\geq 0}$ is geometrically ergodic, we have $f(\theta) = \lim_{n\to\infty} E_\theta(\phi(X_n^\theta, Y_n^\theta))$ and

$$\nabla f(\theta) = \lim_{n\to\infty} E_\theta\left(\phi(X_n^\theta, Y_n^\theta)\sum_{i=0}^{n-1} s_\theta(X_{n-i}^\theta, Y_{n-i}^\theta)\right)$$

[see the proof of Lemma 8.2 and in particular (8.4), (8.7)]. Hence, quantity

$$\phi(X_n^\theta, Y_n^\theta)\sum_{i=0}^{n-1} s_\theta(X_{n-i}^\theta, Y_{n-i}^\theta)$$

is an asymptotically unbiased estimator of $\nabla f(\theta)$. However, it can have a very large variance for large n so that the term $s_\theta(X_{n-i}^\theta, Y_{n-i}^\theta)$ is "discounted" by λ^i, where $\lambda \in [0, 1)$ is a constant referred to as the discounting factor. This leads to the following gradient estimator:

$$\phi(X_n^\theta, Y_n^\theta)\sum_{i=0}^{n-1} \lambda^i s_\theta(X_{n-i}^\theta, Y_{n-i}^\theta). \tag{4.2}$$

This gradient estimator (4.2) is biased and its bias is of the order $O(1-\lambda)$ when $\lambda \to 1$ (see Lemma 8.2). Combining gradient search with the estimator (4.2), we get the policy-gradient algorithm proposed in [2]. This algorithm is defined by the following difference equations:

$$\begin{aligned} W_{n+1} &= \lambda W_n + s_{\theta_n}(X_{n+1}, Y_{n+1}),\\ \theta_{n+1} &= \theta_n - \alpha_n \phi(X_{n+1}, Y_{n+1}) W_{n+1}, \qquad n \geq 0. \end{aligned} \tag{4.3}$$

In the recursion (4.3), $\{\alpha_n\}_{n\geq 0}$ is a sequence of positive reals, while θ_0, $W_0 \in \mathbb{R}^{d_\theta}$ are any (deterministic) vectors. $\{X_n\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$ are $\mathcal{X}$ and $\mathcal{Y}$ valued stochastic processes (respectively) generated through the following Monte Carlo simulations:

$$\begin{aligned} &X_{n+1}|\theta_n, X_n, Y_n, \ldots, \theta_0, X_0, Y_0 \sim p(\cdot|X_n, Y_n),\\ &Y_{n+1}|X_{n+1}, \theta_n, X_n, Y_n, \ldots, \theta_0, X_0, Y_0 \sim q_{\theta_n}(\cdot|X_{n+1}), \qquad n \geq 0, \end{aligned} \tag{4.4}$$

where $X_0 \in \mathcal{X}$, $Y_0 \in \mathcal{Y}$ are deterministic quantities. Hence, $\{(X_n, Y_n)\}_{n\geq 1}$ satisfies

$$P(X_{n+1} = x, Y_{n+1} = y|\theta_n, X_n, Y_n, \ldots, \theta_0, X_0, Y_0) = q_{\theta_n}(y|x)p(x|X_n, Y_n)$$

for all $x \in \mathcal{X}$, $y \in \mathcal{Y}$, $n \geq 1$.

Algorithm (4.3) is analyzed under the following assumptions.

ASSUMPTION 4.1. For all $\theta \in \mathbb{R}^{d_\theta}$, $\{X_n^\theta\}_{n\geq 0}$ is irreducible and aperiodic.

ASSUMPTION 4.2. For all $\theta \in \mathbb{R}^{d_\theta}$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$, $s_\theta(x, y)$ is well defined (and finite). Moreover, for each $x \in \mathcal{X}$, $y \in \mathcal{Y}$, $s_\theta(x, y)$ is locally Lipschitz continuous in θ on $\mathbb{R}^{d_\theta}$.

ASSUMPTION 4.3.a. For each $x \in \mathcal{X}$, $y \in \mathcal{Y}$, $q_\theta(y|x)$ is p-times differentiable in θ on $\mathbb{R}^{d_\theta}$, where $p > d_\theta$.

ASSUMPTION 4.3.b. For each $x \in \mathcal{X}$, $y \in \mathcal{Y}$, $q_\theta(y|x)$ is real-analytic in θ on $\mathbb{R}^{d_\theta}$.

Assumption 4.1 is related to the stability of the controlled Markov chain $\{X_n^\theta\}_{n\geq 0}$. In this or similar form, it is often involved in the asymptotic analysis of reinforcement learning algorithms (see, e.g., [8, 28]). Assumptions 4.2, 4.3.a and 4.3.b correspond to the parameterization of the action probabilities $q_\theta(y|x)$. They are satisfied for many commonly used parameterizations (such as natural, exponential and trigonometric).

Let $\mathcal{R}$, $\mathcal{S}$ and $f(\mathcal{S})$ have the same meaning as in Section 2 for the objective function $f(\cdot)$ now defined in (4.1). Moreover, let Λ_Q have the same meaning as in (2.6). Then our results on the asymptotic behavior of the recursion (4.3) read as follows.

THEOREM 4.1. *Suppose that Assumptions 3.1, 4.1 and 4.2 hold. Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set. Then the following are true*:

(i) *There exists a (deterministic) nondecreasing function $\psi_Q : [0,\infty) \to [0,\infty)$ [independent of λ and depending only on $\phi(x,y)$, $p(x'|x,y)$, $q_\theta(y|x)$] such that $\lim_{t\to 0} \psi_Q(t) = \psi_Q(0) = 0$ and*

$$\limsup_{n\to\infty} d(\theta_n, \mathcal{R}) \leq \psi_Q(1-\lambda)$$

almost surely on Λ_Q.

(ii) *If Assumption 4.3.a is additionally satisfied, there exists a real number $K_Q \in (0,\infty)$ [independent of λ and depending only on $\phi(x,y)$, $p(x'|x,y)$, $q_\theta(y|x)$] such that*

$$\limsup_{n\to\infty} \|\nabla f(\theta_n)\| \leq K_Q(1-\lambda)^{q/2},$$

$$\limsup_{n\to\infty} f(\theta_n) - \liminf_{n\to\infty} f(\theta_n) \leq K_Q(1-\lambda)^q$$

almost surely on Λ_Q, where $q = (p-d_\theta)/(p-1)$.

(iii) *If Assumption 4.3.b is additionally satisfied, there exist real numbers $r_Q \in (0,1)$, $L_Q \in (0,\infty)$ [independent of λ and depending only on $\phi(x,y)$, $p(x'|x,y)$, $q_\theta(y|x)$] such that*

$$\limsup_{n\to\infty} d(\theta_n, \mathcal{S}) \leq L_Q(1-\lambda)^{r_Q},$$

$$\limsup_{n\to\infty} \|\nabla f(\theta_n)\| \leq L_Q(1-\lambda)^{1/2},$$

$$\limsup_{n\to\infty} d\big(f(\theta_n), f(\mathcal{S})\big) \leq L_Q(1-\lambda)$$

almost surely on Λ_Q.

Theorem 4.1 is proved in Section 8.

REMARK. Function $\psi_Q(\cdot)$ depends on $\phi(x, y)$, $p(x'|x, y)$, $q_\theta(y|x)$ through function $f(\cdot)$ defined in (4.1) and its properties. It also depends on $p(x'|x, y)$, $q_\theta(y|x)$ through the properties of $\{(X_n^\theta, Y_n^\theta)\}_{n\geq 0}$ (see Lemma 8.1). Additionally, it depends on $\phi(x, y)$, $q_\theta(y|x)$ through upper bounds on $|\phi(x, y)|$, $\|s_\theta(x, y)\|$. Further details can be found in the proofs of Lemmas 8.1, 8.2 and Theorem 4.1 (Section 8).

REMARK. Like $\psi_Q(\cdot)$, constants K_Q and L_Q depend on $\phi(x, y)$, $p(x'|x, y)$, $q_\theta(y|x)$ through function $f(\cdot)$ [defined in (4.1)] and its properties. K_Q and L_Q also depend on $\phi(x, y)$, $p(x'|x, y)$, $q_\theta(y|x)$ through the ergodicity properties of $\{(X_n^\theta, Y_n^\theta)\}_{n\geq 0}$. Moreover, K_Q and L_Q depend on $\phi(x, y)$, $p(x'|x, y)$, $q_\theta(y|x)$ through upper bounds on $|\phi(x, y)|$, $\|s_\theta(x, y)\|$. For further details, see the proofs of Lemmas 8.1, 8.2 and Theorem 4.1 (Section 8).

Although gradient search with "discounted" gradient estimation (4.2) is widely used in reinforcement learning (apart from policy-gradient search, temporal-difference and actor-critic learning also rely on the same approach), the available literature does not give a quite satisfactory answer to the problem of its asymptotic behavior. To the best of the present authors' knowledge, the existing results do not offer even the guarantee that the asymptotic bias of recursion (4.3) goes to zero as $\lambda \to 1$ (i.e., that $\{\theta_n\}_{n\geq 0}$ converges to a vicinity of $\mathcal{S}$ whose radius tends to zero as $\lambda \to 1$). The paper [20] can be considered as the strongest result on the asymptotic behavior of reinforcement learning with "discounted" gradient estimation. However, [20] only claims that a subsequence of $\{\theta_n\}_{n\geq 0}$ converges to a vicinity of $\mathcal{S}$ whose radius goes to zero as $\lambda \to 1$. The main difficulty stems from the fact that reinforcement learning algorithms are so complex that the existing asymptotic results for biased stochastic gradient search and biased stochastic approximation [11], Section 5.3, [15, 16], [14], Section 2.7, cannot be applied. Relying on the results presented in Sections 2 and 3, Theorem 4.1 overcomes these difficulties. Under mild and easily verifiable conditions, Theorem 4.1 guarantees that the asymptotic bias of algorithm (4.3) converges to zero as $\lambda \to 1$ [Part (i)]. Theorem 4.1 also provides relatively tight, polynomial bounds on the rate at which the bias goes to zero [Parts (ii), (iii)].

5. Proof of part (i) of Theorem 2.1. In this section, we rely on the following notation. For a set $A \subseteq \mathbb{R}^{d_\theta}$ and $\varepsilon \in (0, \infty)$, let $V_\varepsilon(A)$ be the ε-vicinity of A, that is, $V_\varepsilon(A) = \{\theta \in \mathbb{R}^{d_\theta} : d(\theta, A) \leq \varepsilon\}$. For $\theta \in \mathbb{R}^{d_\theta}$ and $\gamma \in [0, \infty)$, let $F_\gamma(\theta)$ be the set defined by

$$F_\gamma(\theta) = \{-\nabla f(\theta) + \vartheta : \vartheta \in \mathbb{R}^{d_\theta}, \|\vartheta\| \leq \gamma\}$$

[notice that $F_\gamma(\theta)$ is a set-valued function of θ]. For $\gamma \in [0,\infty)$, let Φ_γ be the family of solutions to the differential inclusion $d\theta/dt \in F_\gamma(\theta)$, that is, Φ_γ is the collection of absolutely continuous functions $\varphi : [0,\infty) \to \mathbb{R}^{d_\theta}$ satisfying $d\varphi(t)/dt \in F_\gamma(\varphi(t))$ almost everywhere (in t) on $[0,\infty)$. For a compact set $Q \subset \mathbb{R}^{d_\theta}$ and $\gamma \in [0,\infty)$, let $\mathcal{H}_{Q,\gamma}$ be the largest invariant set of the differential inclusion $d\theta/dt \in F_\gamma(\theta)$ contained in Q, that is, $\mathcal{H}_{Q,\gamma}$ is the largest set $\mathcal{H}$ with the following property: For any $\theta \in \mathcal{H}$, there exists a solution $\varphi \in \Phi_\gamma$ such that $\varphi(0) = \theta$ and $\varphi(t) \in \mathcal{H}$ for all $t \in [0,\infty)$. For a compact set $Q \subset \mathbb{R}^{d_\theta}$ and $\gamma \in [0,\infty)$, let $\mathcal{R}_{Q,\gamma}$ be the set of chain-recurrent points of the differential inclusion $d\theta/dt \in F_\gamma(\theta)$ contained in Q, that is, $\theta \in \mathcal{R}_{Q,\gamma}$ if and only if for any $\delta, t \in (0,\infty)$, there exist an integer $N \geq 1$, real numbers $t_1,\dots,t_N \in [t,\infty)$ and solutions $\varphi_1,\dots,\varphi_N \in \Phi_\gamma$ (each of which can depend on θ,δ,t) such that $\varphi_k(0) \in \mathcal{H}_{Q,\gamma}$ for $1 \leq k \leq N$ and

$$\|\varphi_1(0) - \theta\| \leq \delta, \qquad \|\varphi_N(t_N) - \theta\| \leq \delta, \qquad \|\varphi_k(t_k) - \varphi_{k+1}(0)\| \leq \delta$$

for $1 \leq k < N$. For more details on differential inclusions and their solutions, invariant sets and chain-recurrent points, see [1, 5] and references cited therein.

LEMMA 5.1. *Suppose that Assumption 2.3.a holds. Then, given a compact set $Q \subset \mathbb{R}^{d_\theta}$, there exists a nondecreasing function $\phi_Q : [0,\infty) \to [0,\infty)$ such that $\lim_{\gamma\to 0}\phi_Q(\gamma) = \phi_Q(0) = 0$ and $\mathcal{R}_{Q,\gamma} \subseteq V_{\phi_Q(\gamma)}(\mathcal{R})$ for all $\gamma \in [0,\infty)$.*

PROOF. Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set. Moreover, let $\phi_Q : [0,\infty) \to [0,\infty)$ be the function defined by $\phi_Q(0) = 0$ and

$$\phi_Q(\gamma) = \sup\bigl(\{d(\theta,\mathcal{R}) : \theta \in \mathcal{R}_{Q,\gamma}\} \cup \{0\}\bigr)$$

for $\gamma \in (0,\infty)$. Then it is easy to show that $\phi_Q(\cdot)$ is well defined and satisfies $\mathcal{R}_{Q,\gamma} \subseteq V_{\phi_Q(\gamma)}(\mathcal{R})$ for all $\gamma \in [0,\infty)$. It is also easy to check that $F_\gamma(\theta) \subseteq F_\delta(\theta)$ for all $\theta \in \mathbb{R}^{d_\theta}$, $\gamma,\delta \in [0,\infty)$ satisfying $\gamma \leq \delta$. Consequently, $\Phi_\gamma \subseteq \Phi_\delta$, $\mathcal{H}_{Q,\gamma} \subseteq \mathcal{H}_{Q,\delta}$, $\mathcal{R}_{Q,\gamma} \subseteq \mathcal{R}_{Q,\delta}$ for all $\gamma,\delta \in [0,\infty)$ satisfying $\gamma \leq \delta$. Thus, $\phi_Q(\cdot)$ is nondecreasing. Moreover, [6], Theorem 3.1, implies that given $\varepsilon \in (0,\infty)$, there exists a real number $\gamma_Q(\varepsilon) \in (0,\infty)$ such that $\mathcal{R}_{Q,\gamma} \subseteq V_\varepsilon(\mathcal{R})$ for all $\gamma \in [0,\gamma_Q(\varepsilon))$. Therefore, $\phi_Q(\gamma) \leq \varepsilon$ for all $\varepsilon \in (0,\infty)$, $\gamma \in [0,\gamma_Q(\varepsilon))$. Consequently, $\lim_{\gamma\to 0}\phi_Q(\gamma) = \phi_Q(0) = 0$. □

PROOF OF PART (i) OF THEOREM 2.1. Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set and let $\psi_Q : [0,\infty) \to [0,\infty)$ be the function defined by $\psi_Q(t) = \phi_Q(2t)$ for $t \in [0,\infty)$ [$\phi_Q(\cdot)$ is specified in the statement of Lemma 5.1]. Then, due to Lemma 5.1, $\psi_Q(\cdot)$ is nondecreasing and $\lim_{t\to 0}\psi_Q(t) = \psi_Q(0) = 0$. Moreover, owing to Assumption 2.2, there exists an event $N_Q \in \mathcal{F}$ such that the following holds: $P(N_Q) = 0$ and (2.4) is satisfied on $\Lambda_Q \setminus N_Q$ for all $t \in (0,\infty)$. Let ω be an arbitrary sample in $\Lambda_Q \setminus N_Q$. To prove Part (i) of Theorem 2.1, it is sufficient

to show (2.7) for ω. Notice that all formulas that follow in the proof correspond to ω.

If $\eta = 0$, then [4], Proposition 4.1, Theorem 5.7, imply that all limit points of $\{\theta_n\}_{n\geq 0}$ are included in $\mathcal{R}$. Hence, (2.7) holds when $\eta = 0$.

Now, suppose $\eta > 0$. Then there exists $n_0 \geq 0$ (depending on ω) such that $\theta_n \in Q$, $\|\eta_n\| \leq 2\eta$ for $n \geq n_0$. Therefore,

$$\frac{\theta_{n+1} - \theta_n}{\alpha_n} + \zeta_n = -\big(\nabla f(\theta_n) + \eta_n\big) \in F_{2\eta}(\theta_n)$$

for $n \geq n_0$. Consequently, [5], Proposition 1.3, Theorem 3.6, imply that all limit points of $\{\theta_n\}_{n\geq 0}$ are contained in $\mathcal{R}_{Q,2\eta}$. Combining this with Lemma 5.1, we conclude that the limit points of $\{\theta_n\}_{n\geq 0}$ are included in $V_{\phi_Q(2\eta)}(\mathcal{R}) = V_{\psi_Q(\eta)}(\mathcal{R})$. Thus, (2.7) holds when $\eta > 0$. □

6. Proof of parts (ii), (iii) of Theorem 2.1. In this section, the following notation is used. ϕ is the random variable defined by

$$\phi = \limsup_{n\to\infty} \|\nabla f(\theta_n)\|.$$

For $t \in (0, \infty)$ and $n \geq 0$, $\phi_{1,n}(t)$, $\phi_{2,n}(t)$, $\phi_n(t)$ are the random quantities defined as

$$\phi_{1,n}(t) = -\big(\nabla f(\theta_n)\big)^T \sum_{i=n}^{a(n,t)-1} \alpha_i \big(\nabla f(\theta_i) - \nabla f(\theta_n)\big),$$

$$\phi_{2,n}(t) = \int_0^1 \big(\nabla f\big(\theta_n + s(\theta_{a(n,t)} - \theta_n)\big) - \nabla f(\theta_n)\big)^T (\theta_{a(n,t)} - \theta_n)\, ds,$$

$$\phi_n(t) = \phi_{1,n}(t) + \phi_{2,n}(t).$$

Then it is straightforward to show that

$$\begin{aligned} f(\theta_{a(n,t)}) - f(\theta_n) &= -\|\nabla f(\theta_n)\|^2 \sum_{i=n}^{a(n,t)-1} \alpha_i - \big(\nabla f(\theta_n)\big)^T \sum_{i=n}^{a(n,t)-1} \alpha_i \xi_i \\ &\quad + \phi_n(t) \\ &\leq -\|\nabla f(\theta_n)\| \left(\|\nabla f(\theta_n)\| \sum_{i=n}^{a(n,t)-1} \alpha_i - \left\| \sum_{i=n}^{a(n,t)-1} \alpha_i \xi_i \right\| \right) \\ &\quad + |\phi_n(t)| \end{aligned} \tag{6.1}$$

for $t \in (0, \infty)$, $n \geq 0$. Moreover, Assumption 2.1 implies

$$\lim_{n\to\infty} \sum_{i=n}^{a(n,t)-1} \alpha_i = \lim_{n\to\infty} \sum_{i=n}^{a(n,t)} \alpha_i = t \tag{6.2}$$

for $t \in (0, \infty)$.

We also need the following additional notation. The Lebesgue measure is denoted by $m(\cdot)$. For a compact set $Q \subset \mathbb{R}^{d_\theta}$ and $\varepsilon \in (0, \infty)$, $A_{Q,\varepsilon}$ is the set defined by

$$A_{Q,\varepsilon} = \{f(\theta) : \theta \in Q, \|\nabla f(\theta)\| \leq \varepsilon\}. \tag{6.3}$$

In order to treat Assumptions 2.3.b, 2.3.c in a unified way and to provide a unified proof of Parts (ii), (iii) of Theorem 2.1, we introduce the following assumption.

ASSUMPTION 6.1. There exists a real number $s \in (0, 1]$ and for any compact set $Q \subset \mathbb{R}^{d_\theta}$, there exists a real number $M_Q \in [1, \infty)$ such that $m(A_{Q,\varepsilon}) \leq M_Q \varepsilon^s$ for all $\varepsilon \in (0, \infty)$.

PROPOSITION 6.1. *Suppose that Assumption 2.3.b holds. Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set. Then there exists a real number $M_Q \in [1, \infty)$ [depending only on $f(\cdot)$] such that $m(A_{Q,\varepsilon}) \leq M_Q \varepsilon^q$ for all $\varepsilon \in (0, \infty)$ (q is specified in the statement of Theorem 2.1).*

PROOF. The proposition is a particular case of Yomdin theorem [34], Theorem 1.2. □

PROPOSITION 6.2. *Suppose that Assumption 2.3.c holds. Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set. Then the following are true*:

(i) *There exists a real number $M_Q \in [1, \infty)$ [depending only on $f(\cdot)$] such that $m(A_{Q,\varepsilon}) \leq M_Q \varepsilon$ for all $\varepsilon \in (0, \infty)$.*

(ii) *There exist real numbers $r_Q \in (0, 1)$, $M_{1,Q}, M_{2,Q} \in [1, \infty)$ [depending only on $f(\cdot)$] such that*

$$d(\theta, \mathcal{S}) \leq M_{1,Q} \|\nabla f(\theta)\|^{r_Q}, \qquad d\big(f(\theta), f(\mathcal{S})\big) \leq M_{2,Q} \|\nabla f(\theta)\| \tag{6.4}$$

for all $\theta \in Q$ [$\mathcal{S}$ and $f(\mathcal{S})$ are specified in (2.2)].

PROOF. Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set. Owing to Lojasiewicz (ordinary) inequality (see [10], Theorem 6.4, Remark 6.5), there exist real numbers $r_Q \in (0, 1)$, $M_{1,Q} \in [1, \infty)$ such that the first inequality in (6.4) holds for all $\theta \in Q$. Moreover, due to Lojasiewicz gradient inequality (see [21], Theorem ŁI, page 775), we have the following: For any $a \in f(Q) = \{f(\theta) : \theta \in Q\}$, there exist real numbers $\delta_{Q,a} \in (0, 1)$, $\nu_{Q,a} \in (1, 2]$, $N_{Q,a} \in [1, \infty)$ such that

$$|f(\theta) - a| \leq N_{Q,a} \|\nabla f(\theta)\|^{\nu_{Q,a}} \tag{6.5}$$

for all $\theta \in Q$ satisfying $|f(\theta) - a| \leq \delta_{Q,a}$.

Now, we show by contradiction that $f(\mathcal{S} \cap Q) = \{f(\theta) : \theta \in \mathcal{S} \cap Q\}$ has finitely many elements. Suppose the opposite. Then there exists a sequence $\{\vartheta_n\}_{n \geq 0}$ in $\mathcal{S} \cap Q$ such that $\{f(\vartheta_n)\}_{n \geq 0}$ contains infinitely many different elements. Since $\mathcal{S} \cap Q$

is compact, $\{\vartheta_n\}_{n\ge0}$ has a convergent subsequence $\{\tilde\vartheta_n\}_{n\ge0}$ such that $\{f(\tilde\vartheta_n)\}_{n\ge0}$ also contains infinitely many different elements. Let $\vartheta=\lim_{n\to\infty}\tilde\vartheta_n$, $a=f(\vartheta)$. As $\delta_{Q,a}>0$, there exists an integer $n_0\ge0$ such that $|f(\tilde\vartheta_n)-a|\le\delta_{Q,a}$ for $n\ge n_0$. Since $\nabla f(\tilde\vartheta_n)=0$ for $n\ge0$, (6.5) implies $f(\tilde\vartheta_n)=a$ for $n\ge n_0$. However, this is impossible, since $\{f(\tilde\vartheta_n)\}_{n\ge0}$ has infinitely many different elements.

Let n_Q be the number of elements in $f(\mathcal{S}\cap Q)$, while $\{a_i:1\le i\le n_Q\}$ are the elements of $f(\mathcal{S}\cap Q)$. For $1\le i\le n_Q$, let

$$B_{Q,i}=\bigl\{\theta\in Q:\|\nabla f(\theta)\|<1,\,f(\theta)\in(a_i-\delta_{Q,a_i},a_i+\delta_{Q,a_i})\bigr\},$$

while $B_Q=\bigcup_{i=1}^{n_Q}B_{Q,i}$, $\varepsilon_Q=\inf\{\|\nabla f(\theta)\|:\theta\in Q\setminus B_Q\}$. As B_Q is open and $\mathcal{S}\cap Q\subset B_Q$, we have $\varepsilon_Q>0$.

Let $\tilde C_{1,Q}\in[1,\infty)$ be an upper bound of $|f(\cdot)|$ on Q. Moreover, let $\tilde C_{2,Q}=\max_{1\le i\le n_Q}N_{Q,a_i}$, $M_{2,Q}=2\max\{\varepsilon_Q^{-1}\tilde C_{1,Q},\tilde C_{2,Q}\}$. Then, if $\theta\in B_Q$, we have

$$\begin{aligned}d\bigl(f(\theta),f(\mathcal{S})\bigr)&=\min_{1\le i\le n_Q}|f(\theta)-a_i|\le\max_{1\le i\le n_Q}N_{Q,a_i}\|\nabla f(\theta)\|^{\nu_{Q,a_i}}\\&\le\tilde C_{2,Q}\|\nabla f(\theta)\|\le M_{2,Q}\|\nabla f(\theta)\|\end{aligned}$$

[notice that $\|\nabla f(\theta)\|<1$, $\nu_{Q,a_i}>1$]. If $\theta\in Q\setminus B_Q$, we get

$$d\bigl(f(\theta),f(\mathcal{S})\bigr)=\min_{1\le i\le n_Q}|f(\theta)-a_i|\le2\varepsilon_Q^{-1}\tilde C_{1,Q}\|\nabla f(\theta)\|\le M_{2,Q}\|\nabla f(\theta)\|$$

[notice that $|f(\theta)-a_i|\le2\tilde C_{1,Q}$, $\|\nabla f(\theta)\|\ge\varepsilon_Q$]. Hence, the second inequality in (6.4) holds for all $\theta\in Q$.

Let $M_Q=2M_{2,Q}n_Q$. Owing to the second inequality in (6.4), we have

$$A_{Q,\varepsilon}\subseteq\bigcup_{i=1}^{n_Q}\bigl[f(a_i)-M_{2,Q}\varepsilon,\,f(a_i)+M_{2,Q}\varepsilon\bigr]$$

for each $\varepsilon\in(0,\infty)$. Consequently, $m(A_{Q,\varepsilon})\le2M_{2,Q}n_Q\varepsilon=M_Q\varepsilon$ for all $\varepsilon\in(0,\infty)$. □

LEMMA 6.1. *Let Assumptions 2.1 and 2.2 hold. Then there exists an event $N_0\in\mathcal{F}$ such that $P(N_0)=0$ and*

$$\limsup_{n\to\infty}\max_{n\le k<a(n,t)}\left\|\sum_{i=n}^{k}\alpha_i\xi_i\right\|\le\eta t,\tag{6.6}$$

$$\lim_{n\to\infty}|f(\theta_{n+1})-f(\theta_n)|=0\tag{6.7}$$

on $\{\sup_{n\ge0}\|\theta_n\|<\infty\}\setminus N_0$ for all $t\in(0,\infty)$. Moreover, given a compact set $Q\subset\mathbb{R}^{d_\theta}$, there exists a real number $C_{1,Q}\in[1,\infty)$ [independent of η and depending

only on $f(\cdot)$] *such that*

$$\limsup_{n\to\infty} \max_{n\le k\le a(n,t)} \bigl|f(\theta_k) - f(\theta_n)\bigr| \le C_{1,Q}\, t(\phi+\eta), \tag{6.8}$$

$$\limsup_{n\to\infty} \bigl|\phi_n(t)\bigr| \le C_{1,Q}\, t^2(\phi+\eta)^2 \tag{6.9}$$

on $\Lambda_Q \setminus N_0$ *for all* $t \in (0,\infty)$.

PROOF. Owing to Assumption 2.2, there exists $N_0 \in \mathcal{F}$ such that the following holds: $P(N_0) = 0$ and (2.4) is satisfied on $\{\sup_{n\ge 0} \|\theta_n\| < \infty\} \setminus N_0$ for all $t \in (0,\infty)$. Moreover, we have

$$\left\|\sum_{i=n}^{k} \alpha_i \xi_i\right\| \le \left\|\sum_{i=n}^{k} \alpha_i \zeta_i\right\| + \sum_{i=n}^{k} \alpha_i \|\eta_i\| \le \max_{n\le j<a(n,t)} \left\|\sum_{i=n}^{j} \alpha_i \zeta_i\right\| + t \max_{j\ge n} \|\eta_j\|$$

for $0 \le n \le k < a(n,t)$, $t \in (0,\infty)$. Consequently,

$$\begin{aligned}\limsup_{n\to\infty} \max_{n\le k<a(n,t)} \left\|\sum_{i=n}^{k} \alpha_i \xi_i\right\| &\le \limsup_{n\to\infty} \max_{n\le k<a(n,t)} \left\|\sum_{i=n}^{k} \alpha_i \zeta_i\right\| + t \lim_{n\to\infty} \max_{k\ge n} \|\eta_k\| \\ &= \eta t\end{aligned}$$

on $\{\sup_{n\ge 0} \|\theta_n\| < \infty\} \setminus N_0$ for $t \in (0,\infty)$.

Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set, while $\tilde{C}_Q \in [1,\infty)$ stands for a Lipschitz constant of $f(\cdot)$, $\nabla f(\cdot)$ on Q. Moreover, let $C_{1,Q} = 2\tilde{C}_Q$, while ω is an arbitrary sample from $\Lambda_Q \setminus N_0$. In order to prove the lemma, it is sufficient to show that (6.7)–(6.9) hold for ω and any $t \in (0,\infty)$. Notice that all formulas which follow in the proof correspond to ω.

Let $\varepsilon \in (0,\infty)$ be any real number. Then there exists $n_0 \ge 0$ (depending on ω, ε) such that $\theta_n \in Q$, $\|\nabla f(\theta_n)\| \le \phi + \varepsilon$ for $n \ge n_0$ (notice that these relations hold for all but finitely many n). Therefore,

$$\|\theta_k - \theta_n\| \le \sum_{i=n}^{k-1} \alpha_i \bigl\|\nabla f(\theta_i)\bigr\| + \left\|\sum_{i=n}^{k-1} \alpha_i \xi_i\right\| \le t(\phi+\varepsilon) + \max_{n\le j<a(n,t)} \left\|\sum_{i=n}^{j} \alpha_i \xi_i\right\|$$

for $n_0 \le n \le k \le a(n,t)$, $t \in (0,\infty)$. Combining this with (6.6), we get

$$\limsup_{n\to\infty} \max_{n\le k\le a(n,t)} \|\theta_k - \theta_n\| \le t(\phi+\eta+\varepsilon)$$

for $t \in (0,\infty)$. Then the limit process $\varepsilon \to 0$ yields

$$\limsup_{n\to\infty} \max_{n\le k\le a(n,t)} \|\theta_k - \theta_n\| \le t(\phi+\eta)$$

for $t \in (0,\infty)$ [notice that $\varepsilon \in (0,\infty)$ is any real number]. As

$$\bigl|f(\theta_k) - f(\theta_n)\bigr| \le \tilde{C}_Q \|\theta_k - \theta_n\|$$

for $k \geq n \geq n_0$ (notice that $\theta_n \in Q$ for $n \geq n_0$), we have

$$\limsup_{n\to\infty} \max_{n\leq k\leq a(n,t)} \bigl|f(\theta_k) - f(\theta_n)\bigr| \leq \tilde{C}_Q t(\phi+\eta) \leq C_{1,Q} t(\phi+\eta)$$

for $t \in (0,\infty)$. Since

$$\bigl|f(\theta_{n+1}) - f(\theta_n)\bigr| \leq \max_{n\leq k\leq a(n,t)} \bigl|f(\theta_k) - f(\theta_n)\bigr|$$

for $t \in (0,\infty)$ and sufficiently large n [notice that $a(n,t) \geq n+1$ for sufficiently large n], we conclude

$$\limsup_{n\to\infty} \bigl|f(\theta_{n+1}) - f(\theta_n)\bigr| \leq \tilde{C}_Q t(\phi+\eta)$$

for $t \in (0,\infty)$. Then the limit process $t \to 0$ implies (6.7). Moreover, we have

$$\begin{aligned}
\bigl|\phi_{1,n}(t)\bigr| &\leq \tilde{C}_Q \bigl\|\nabla f(\theta_n)\bigr\| \sum_{i=n}^{a(n,t)-1} \alpha_i \|\theta_i - \theta_n\| \\
&\leq \tilde{C}_Q t \bigl\|\nabla f(\theta_n)\bigr\| \max_{n\leq k\leq a(n,t)} \|\theta_k - \theta_n\|, \\
\bigl|\phi_{2,n}(t)\bigr| &\leq \tilde{C}_Q \|\theta_{a(n,t)} - \theta_n\|^2 \leq \tilde{C}_Q \max_{n\leq k\leq a(n,t)} \|\theta_k - \theta_n\|^2
\end{aligned}$$

for $n \geq n_0$, $t \in (0,\infty)$. Therefore,

$$\limsup_{n\to\infty} |\phi_{1,n}(t)| \leq \tilde{C}_Q t^2 \phi(\phi+\eta), \qquad \limsup_{n\to\infty} |\phi_{2,n}(t)| \leq \tilde{C}_Q t^2 (\phi+\eta)^2$$

for $t \in (0,\infty)$. Hence,

$$\limsup_{n\to\infty} |\phi_n(t)| \leq 2\tilde{C}_Q t^2 (\phi+\eta)^2 = C_{1,Q} t^2 (\phi+\eta)^2$$

for $t \in (0,\infty)$. □

LEMMA 6.2. *Let Assumptions 2.1, 2.2 and 6.1 hold. Then, given a compact set $Q \subset \mathbb{R}^{d_\theta}$, there exists a real number $C_{2,Q} \in [1,\infty)$ [independent of η and depending only on $f(\cdot)$] such that*

$$\limsup_{n\to\infty} f(\theta_n) - \liminf_{n\to\infty} f(\theta_n) \leq C_{2,Q}\eta^s \tag{6.10}$$

on $\Lambda_Q \setminus N_0$ (s is specified in Assumption 6.1).

PROOF. Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set, while $\tilde{C}_Q$ stands for an upper bound of $\|\nabla f(\cdot)\|$ on Q. Moreover, let $C_{2,Q} = 4M_Q$. In order to avoid considering separately the cases $\eta = 0$ and $\eta > 0$, we show

$$\limsup_{n\to\infty} f(\theta_n) - \liminf_{n\to\infty} f(\theta_n) \leq C_{2,Q}(\varepsilon+\eta)^s \tag{6.11}$$

on $\Lambda_Q \setminus N_0$ for all $\varepsilon \in (0,\infty)$. Then (6.10) follows directly from (6.11) by letting $\varepsilon \to 0$.

Inequality (6.11) is proved by contradiction: Suppose that there exist a sample $\omega \in \Lambda_Q \setminus N_0$ and a real number $\varepsilon \in (0,\infty)$ such that (6.11) does not hold for them. Notice that all formulas which follow in the proof correspond to ω.

Let $\gamma = 2(\varepsilon+\eta)$, $\delta = M_Q\gamma^s$, while

$$\mu = \delta/\big(C_{1,Q}(\tilde{C}_Q+\eta)\big), \qquad \nu = \gamma^2/\big(4C_{1,Q}(\tilde{C}_Q+\eta)^2\big), \qquad \tau = \min\{\mu,\nu/2\}.$$

Since $\{\theta_n\}_{n\geq 0}$ is bounded and (6.11) is not satisfied, there exist real numbers $a,b \in \mathbb{R}$ (depending on ω,ε) such that $b-a>2\delta$ and such that inequalities $f(\theta_n)<a$, $f(\theta_k)>b$ hold for infinitely many $n,k\geq 0$ [notice that $C_{2,Q}(\varepsilon+\eta)^s \geq 2\delta$]. As $m(A_{Q,\gamma}) \leq M_Q\gamma^s = \delta$, there exists a real number c such that $c \notin A_{Q,\gamma}$ and $a < c < b-\delta$ [otherwise, $(a,b-\delta)\subset A_{Q,\varepsilon}$, which is impossible as $(b-\delta)-a>\delta$].

Let $n_0 = 0$, while

$$\begin{aligned} l_k &= \min\{n \geq n_{k-1} : f(\theta_n) \leq c\},\\ n_k &= \min\{n \geq l_k : f(\theta_n) \geq b\},\\ m_k &= \max\{n \leq n_k : f(\theta_n) \leq c\} \end{aligned}$$

for $k\geq 1$. It can easily be deduced that sequences $\{l_k\}_{k\geq 1}$, $\{m_k\}_{k\geq 1}$, $\{n_k\}_{k\geq 1}$ are well defined and satisfy $l_k < m_k < n_k < l_{k+1}$ and

$$\begin{aligned} &f(\theta_{m_k}) \leq c < f(\theta_{m_k+1}),\\ &f(\theta_{n_k}) - f(\theta_{m_k}) \geq b-c, \qquad \min_{m_k<n\leq n_k} f(\theta_n) > c \end{aligned} \tag{6.12}$$

for $k\geq 1$. Moreover, Lemma 6.1 implies

$$\lim_{k\to\infty} \big|f(\theta_{m_k+1}) - f(\theta_{m_k})\big| = 0, \tag{6.13}$$

$$\limsup_{k\to\infty} \max_{m_k\leq j\leq a(m_k,\tau)} \big|f(\theta_j) - f(\theta_{m_k})\big| \leq C_{1,Q}\tau(\tilde{C}_Q+\eta) \leq \delta < b-c \tag{6.14}$$

[to get (6.14), notice that $\theta_n \in Q$ for all but finitely many n and that $\phi \leq \tilde{C}_Q$]. Owing to (6.14) and the second inequality in (6.12), there exists $k_0 \geq 1$ such that $a(m_k,\tau) \leq n_k$ for $k\geq k_0$.[3] Then the last inequality in (6.12) implies $f(\theta_{a(m_k,\tau)}) \geq c$ for $k\geq k_0$, while $\lim_{k\to\infty} f(\theta_{m_k}) = c$ follows from (6.13) and the first inequality in (6.12). Since $\|\nabla f(\theta)\| > \gamma$ for any $\theta \in Q$ satisfying $f(\theta) = c$ (due to the way c

[3] If $a(m_k,\tau) > n_k$ for infinitely many k, then (6.14) yields

$$\liminf_{k\to\infty}\big(f(\theta_{n_k}) - f(\theta_{m_k})\big) \leq \delta < b-c.$$

However, this contradicts the second inequality in (6.12).

is selected), we have $\liminf_{k\to\infty}\|\nabla f(\theta_{m_k})\| \geq \gamma$. Consequently, Lemma 6.1 and (6.2) yield

$$\liminf_{k\to\infty}\left(\|\nabla f(\theta_{m_k})\|\sum_{i=m_k}^{a(m_k,\tau)-1}\alpha_i - \left\|\sum_{i=m_k}^{a(m_k,\tau)-1}\alpha_i\xi_i\right\|\right) \geq \tau(\gamma-\eta) \geq \tau\gamma/2 > 0$$

(notice that $\eta < \gamma/2$). Therefore,

$$\liminf_{k\to\infty}\|\nabla f(\theta_{m_k})\|\left(\|\nabla f(\theta_{m_k})\|\sum_{i=m_k}^{a(m_k,\tau)-1}\alpha_i - \left\|\sum_{i=m_k}^{a(m_k,\tau)-1}\alpha_i\xi_i\right\|\right) \geq \tau\gamma^2/2.$$

Combining this with Lemma 6.1 and (6.1), we get

$$\begin{aligned}
&\limsup_{k\to\infty}\bigl(f(\theta_{a(m_k,\tau)}) - f(\theta_{m_k})\bigr)\\
&\quad\leq -\liminf_{k\to\infty}\|\nabla f(\theta_{m_k})\|\left(\|\nabla f(\theta_{m_k})\|\sum_{i=m_k}^{a(m_k,\tau)-1}\alpha_i - \left\|\sum_{i=m_k}^{a(m_k,\tau)-1}\alpha_i\xi_i\right\|\right)\\
&\qquad+\limsup_{k\to\infty}|\phi_{m_k}(\tau)|\\
&\quad\leq -\tau\gamma^2/2 + C_{1,Q}\tau^2(\phi+\eta)^2 < 0
\end{aligned}$$

[notice that $\phi \leq \tilde{C}_Q$, $C_{1,Q}\tau^2(\tilde{C}_Q+\eta) \leq \gamma^2/4$]. However, this is not possible, as $f(\theta_{a(m_k,\tau)}) \geq c \geq f(\theta_{m_k})$ for each $k \geq k_0$. Hence, (6.11) is true. □

LEMMA 6.3. *Let Assumptions 2.1 and 2.2 hold. Then, given a compact set $Q \subset \mathbb{R}^{d_\theta}$, there exists a real number $C_{3,Q} \in (0,1)$ [independent of η and depending only on $f(\cdot)$] such that*

$$\limsup_{n\to\infty} f(\theta_n) - \liminf_{n\to\infty} f(\theta_n) \geq C_{3,Q}\phi^2 \tag{6.15}$$

on $(\Lambda_Q \setminus N_0) \cap \{\phi > 2\eta\}$.

PROOF. Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set, while $C_{3,Q} = 1/(64C_{1,Q})$ and $\tau = 1/(16C_{1,Q})$. Moreover, let ω be an arbitrary sample from $(\Lambda_Q \setminus N_0) \cap \{\phi > 2\eta\}$. In order to prove the lemma's assertion, it is sufficient to show that (6.15) holds for ω. Notice that all formulas which follow in the proof correspond to ω.

Let $n_0 = 0$ and

$$n_k = \min\bigl\{n > n_{k-1} : \|\nabla f(\theta_n)\| \geq \phi - 1/k\bigr\}$$

for $k \geq 1$. Obviously, sequence $\{n_k\}_{k\geq 0}$ is well defined and satisfies

$$\lim_{k\to\infty}\|\nabla f(\theta_{n_k})\| = \phi.$$

Then Lemma 6.1 and (6.2) yield

$$\liminf_{k\to\infty}\|\nabla f(\theta_{n_k})\|\left(\|\nabla f(\theta_{n_k})\|\sum_{i=n_k}^{a(n_k,\tau)-1}\alpha_i-\left\|\sum_{i=n_k}^{a(n_k,\tau)-1}\alpha_i\xi_i\right\|\right)$$
$$\geq \tau\phi(\phi-\eta)\geq \tau\phi^2/2>0.$$

Combining this with Lemma 6.1 and (6.1), we get

$$\begin{aligned}&\limsup_{k\to\infty}\bigl(f(\theta_{a(n_k,\tau)})-f(\theta_{n_k})\bigr)\\&\quad\leq-\liminf_{k\to\infty}\|\nabla f(\theta_{n_k})\|\left(\|\nabla f(\theta_{n_k})\|\sum_{i=n_k}^{a(n_k,\tau)-1}\alpha_i-\left\|\sum_{i=n_k}^{a(n_k,\tau)-1}\alpha_i\xi_i\right\|\right)\\&\qquad+\limsup_{k\to\infty}|\phi_{n_k}(\tau)|\\&\quad\leq-\tau\phi^2/2+C_{1,Q}\tau^2(\phi+\eta)^2\leq-C_{3,Q}\phi^2\end{aligned}$$

(notice that $\eta<\phi$). Consequently,

$$\limsup_{n\to\infty}f(\theta_n)-\liminf_{n\to\infty}f(\theta_n)\geq-\limsup_{k\to\infty}\bigl(f(\theta_{a(n_k,\tau)})-f(\theta_{n_k})\bigr)\geq C_{3,Q}\phi^2.$$

Hence, (6.15) is true. □

PROPOSITION 6.3. *Suppose that Assumptions 2.1, 2.2 and 6.1 hold. Let $Q\subset\mathbb{R}^{d_\theta}$ be any compact set. Then there exists a real number $K_Q\in[1,\infty)$ [independent of η and depending only on $f(\cdot)$] such that*

$$\begin{aligned}\limsup_{n\to\infty}\|\nabla f(\theta_n)\|&\leq K_Q\eta^{s/2},\\\limsup_{n\to\infty}f(\theta_n)-\liminf_{n\to\infty}f(\theta_n)&\leq K_Q\eta^s\end{aligned}\tag{6.16}$$

on $\Lambda_Q\setminus N_0$.

PROOF. Let $Q\subset\mathbb{R}^{d_\theta}$ be any compact set, while $\tilde{C}_Q\in[1,\infty)$ stands for an upper bound of $\|\nabla f(\cdot)\|$ on Q. Moreover, let $K_Q=\max\{2,\tilde{C}_Q,C_{2,Q}\}$. Obviously, it is sufficient to show $\phi\leq K_Q\eta^{s/2}$ on $\Lambda_Q\setminus N_0$ [notice that the second inequality in (6.16) is a direct consequence of Lemma 6.2].

Owing to Lemmas 6.2, 6.3, we have $C_{3,Q}\phi^2\leq C_{2,Q}\eta^s$ on $(\Lambda_Q\setminus N_0)\cap\{\phi>2\eta\}$. Therefore, $\phi\leq(C_{2,Q}/C_{3,Q})^{1/2}\eta^{s/2}\leq K_Q\eta^{s/2}$ on $(\Lambda_Q\setminus N_0)\cap\{\phi>2\eta\}$. Moreover, $\phi\leq 2\eta\leq K_Q\eta^{s/2}$ on $(\Lambda_Q\setminus N_0)\cap\{\phi\leq 2\eta,\eta\leq 1\}$ (notice that $s/2<1$), while $\phi\leq\tilde{C}_Q\leq K_Q\eta^{s/2}$ on $(\Lambda_Q\setminus N_0)\cap\{\phi\leq 2\eta,\eta>1\}$. Thus, $\phi\leq K_Q\eta^{s/2}$ indeed holds on $\Lambda_Q\setminus N_0$. □

PROOF OF PARTS (ii), (iii) OF THEOREM 2.1. Part (ii) of the theorem directly follows from Propositions 6.1, 6.3, while Part (iii) is a direct consequence of Propositions 6.2, 6.3. □

7. Proof of Theorem 3.1. The following notation is used in this section. For $\theta \in \mathbb{R}^{d_\theta}$, $z \in \mathbb{R}^{d_z}$, $E_{\theta,z}(\cdot)$ denotes the conditional expectation given $\theta_0 = \theta$, $Z_0 = z$. For $n \geq 1$, ζ_n, ξ_n are the random variables defined by

$$\zeta_n = F(\theta_n, Z_{n+1}) - \nabla f(\theta_n), \qquad \xi_n = \zeta_n + \eta_n, \tag{7.1}$$

while $\zeta_{1,n}$, $\zeta_{2,n}$, $\zeta_{3,n}$ are random variables defined as

$$\zeta_{1,n} = \tilde{F}(\theta_n, Z_{n+1}) - (\Pi\tilde{F})(\theta_n, Z_n),$$

$$\zeta_{2,n} = (\Pi\tilde{F})(\theta_n, Z_n) - (\Pi\tilde{F})(\theta_{n-1}, Z_n),$$

$$\zeta_{3,n} = -(\Pi\tilde{F})(\theta_n, Z_{n+1}).$$

Then it is straightforward to verify that algorithm (3.1) admits the form (2.1). Moreover, using Assumption 3.2, it is easy to show

$$\begin{aligned}\sum_{i=n}^{k} \alpha_i \zeta_i = {} & \sum_{i=n}^{k} \alpha_i \zeta_{1,i} + \sum_{i=n}^{k} \alpha_i \zeta_{2,i} + \sum_{i=n}^{k} (\alpha_i - \alpha_{i+1})\zeta_{3,i} \\ & + \alpha_{k+1}\zeta_{3,k} - \alpha_n \zeta_{3,n-1}\end{aligned} \tag{7.2}$$

for $1 \leq n \leq k$.

PROOF OF THEOREM 3.1. Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set and $\tilde{\Lambda}_Q$ be the event defined by $\tilde{\Lambda}_Q = \bigcap_{n=0}^{\infty}\{\theta_n \in Q\}$. Then, owing to Assumptions 3.1 and 3.3, we have

$$E_{\theta,z}\left(\sum_{n=0}^{\infty}(\alpha_n^2 + \alpha_{n+1}^2)\varphi_Q^2(Z_{n+1})I_{\{\tau_Q > n\}}\right) < \infty, \tag{7.3}$$

$$E_{\theta,z}\left(\sum_{n=0}^{\infty}|\alpha_n - \alpha_{n+1}|\varphi_Q^2(Z_{n+1})I_{\{\tau_Q > n\}}\right) < \infty \tag{7.4}$$

for all $\theta \in \mathbb{R}^{d_\theta}$, $z \in \mathbb{R}^{d_z}$.

Let $\mathcal{F}_n = \sigma\{\theta_0, Z_0, \ldots, \theta_n, Z_n\}$ for $n \geq 0$. Since $\{\tau_Q > n\} \in \mathcal{F}_n$ for $n \geq 0$, Assumption 3.2 implies

$$\begin{aligned}E_{\theta,z}(\zeta_{1,n}I_{\{\tau_Q>n\}}|\mathcal{F}_n) &= \big(E_{\theta,z}\big(\tilde{F}(\theta_n, Z_{n+1})|\mathcal{F}_n\big) - (\Pi\tilde{F})(\theta_n, Z_n)\big)I_{\{\tau_Q>n\}} \\ &= 0\end{aligned}$$

almost surely for each $\theta \in \mathbb{R}^{d_\theta}$, $z \in \mathbb{R}^{d_z}$, $n \geq 0$. Assumption 3.3 also yields

$$\|\zeta_{1,n}\|I_{\{\tau_Q>n\}} \leq \varphi_Q(Z_n)I_{\{\tau_Q>n-1\}} + \varphi_Q(Z_{n+1})I_{\{\tau_Q>n\}}$$

for $n \geq 0$. Combining this with (7.3), we get

$$\begin{aligned}E_{\theta,z}\left(\sum_{n=0}^{\infty}\alpha_n^2\|\zeta_{1,n}\|^2 I_{\{\tau_Q>n\}}\right) &\leq 2E_{\theta,z}\left(\sum_{n=0}^{\infty}(\alpha_n^2 + \alpha_{n+1}^2)\varphi_Q^2(Z_{n+1})I_{\{\tau_Q>n\}}\right) \\ &< \infty\end{aligned}$$

for all $\theta \in \mathbb{R}^{d_\theta}$, $z \in \mathbb{R}^{d_z}$. Then, using the Doob theorem, we conclude that $\sum_{n=0}^{\infty} \alpha_n \zeta_{1,n} I_{\{\tau_Q > n\}}$ converges almost surely. As $\tilde{\Lambda}_Q \subseteq \{\tau_Q > n\}$ for $n \geq 0$, $\sum_{n=0}^{\infty} \alpha_n \zeta_{1,n}$ converges almost surely on $\tilde{\Lambda}_Q$.[4]

Due to Assumption 3.3, we have

$$\begin{aligned}\|\zeta_{2,n}\| I_{\tilde{\Lambda}_Q} &\leq \varphi_Q(Z_n)\|\theta_n - \theta_{n-1}\| I_{\tilde{\Lambda}_Q} \\ &\leq \alpha_{n-1}\varphi_Q(Z_n)\big(\big\|F(\theta_{n-1}, Z_n)\big\| + \|\eta_{n-1}\|\big) I_{\tilde{\Lambda}_Q} \\ &\leq \alpha_{n-1}\varphi_Q(Z_n)\big(\varphi_Q(Z_n) + \|\eta_{n-1}\|\big) I_{\tilde{\Lambda}_Q} \\ &\leq 2\alpha_{n-1}\big(\varphi_Q^2(Z_n) + \|\eta_{n-1}\|^2\big) I_{\tilde{\Lambda}_Q}\end{aligned}$$

for $n \geq 1$ [notice that $\varphi_Q(z) \geq 1$ for any $z \in \mathbb{R}^{d_z}$]. Thus,

$$\begin{aligned}\sum_{n=1}^{j} \alpha_n \|\zeta_{2,n}\| I_{\tilde{\Lambda}_Q} &\leq 2\sum_{n=0}^{\infty} \alpha_n \alpha_{n+1}\big(\varphi_Q^2(Z_{n+1}) + \|\eta_{n+1}\|^2\big) I_{\tilde{\Lambda}_Q} \\ &\leq \sum_{n=0}^{\infty} (\alpha_n^2 + \alpha_{n+1}^2)\varphi_Q^2(Z_{n+1}) I_{\{\tau_Q > n\}} \\ &\quad + \sup_{n \geq 0} \|\eta_n\|^2 I_{\tilde{\Lambda}_Q} \sum_{n=0}^{\infty} (\alpha_n^2 + \alpha_{n+1}^2)\end{aligned}$$

(notice that $2\alpha_n\alpha_{n+1} \leq \alpha_n^2 + \alpha_{n+1}^2$). Then Assumption 3.4 and (7.3) imply that $\sum_{n=1}^{\infty} \alpha_n \zeta_{2,n}$ converges almost surely on $\tilde{\Lambda}_Q$.

Owing to Assumption 3.3, we have

$$\|\zeta_{3,n}\| I_{\tilde{\Lambda}_Q} \leq \varphi_Q(Z_{n+1}) I_{\tilde{\Lambda}_Q} \leq \varphi_Q^2(Z_{n+1}) I_{\{\tau_Q > n\}}$$

for $n \geq 0$. Hence,

$$\begin{aligned}\sum_{n=0}^{\infty} \alpha_{n+1}^2 \|\zeta_{3,n}\|^2 I_{\tilde{\Lambda}_Q} &\leq \sum_{n=0}^{\infty} \alpha_{n+1}^2 \varphi_Q^2(Z_{n+1}) I_{\{\tau_Q > n\}}, \\ \sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| \|\zeta_{3,n}\| I_{\tilde{\Lambda}_Q} &\leq \sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| \varphi_Q^2(Z_{n+1}) I_{\{\tau_Q > n\}}.\end{aligned}$$

Combining this with (7.3), (7.4), we conclude $\lim_{n\to\infty} \alpha_{n+1}\zeta_{3,n} = 0$ almost surely on $\tilde{\Lambda}_Q$. We also deduce that $\sum_{n=0}^{\infty}(\alpha_n - \alpha_{n+1})\zeta_{3,n}$ converges almost surely on $\tilde{\Lambda}_Q$. Since $\sum_{n=0}^{\infty} \alpha_n \zeta_{1,n}$, $\sum_{n=1}^{\infty} \alpha_n \zeta_{2,n}$ converge almost surely on $\tilde{\Lambda}_Q$, (7.2) implies that $\sum_{n=0}^{\infty} \alpha_n \zeta_n$ also converges almost surely on $\tilde{\Lambda}_Q$. As Q is any compact set in $\mathbb{R}^{d_\theta}$, $\sum_{n=0}^{\infty} \alpha_n \zeta_n$ converges almost surely on $\{\sup_{n\geq 0} \|\theta_n\| < \infty\}$. Consequently,

[4] Notice that $\sum_{n=0}^{\infty} \alpha_n \zeta_{1,n} I_{\{\tau_Q > n\}} = \sum_{n=0}^{\infty} \alpha_n \zeta_{1,n}$ on $\tilde{\Lambda}_Q$.

Assumption 3.4 yields that $\{\xi_n\}_{n\geq 0}$ defined in (7.1) satisfies Assumption 2.2. Then the theorem's assertion directly follows from Theorem 2.1. □

8. Proof of Theorem 4.1. In this section, we use the following notation. $\phi(v)$, $s_\theta(v)$ are the functions defined by

$$\phi(v) = \phi(x, y), \qquad s_\theta(v) = s_\theta(x, y)$$

for $\theta \in \mathbb{R}^{d_\theta}$, $v = (x, y) \in \mathcal{X} \times \mathcal{Y}$. For $\theta \in \mathbb{R}^{d_\theta}$, $\{V_n^\theta\}_{n\geq 0}$, $\{W_n^\theta\}_{n\geq 0}$ and $\{Z_n^\theta\}_{n\geq 0}$ are stochastic processes defined by

$$V_n^\theta = (X_n^\theta, Y_n^\theta), \qquad W_{n+1}^\theta = \lambda W_n^\theta + s_\theta(V_n^\theta), \qquad Z_n^\theta = (V_n^\theta, W_n^\theta)$$

for $n \geq 0$, where $W_0^\theta \in \mathbb{R}^{d_\theta}$ is a (deterministic) vector (notice that $\{V_n^\theta\}_{n\geq 0}$, $\{Z_n^\theta\}_{n\geq 0}$ are Markov chains). Moreover, for $\theta \in \mathbb{R}^{d_\theta}$, $r_\theta(\cdot|\cdot)$ and $\nu_\theta(\cdot)$ are the transition kernel and invariant probability of $\{V_n^\theta\}_{n\geq 0}$,[5] while $\Pi_\theta(\cdot, \cdot)$ is the transition kernel of $\{Z_n^\theta\}_{n\geq 0}$.[6] For $\theta \in \mathbb{R}^{d_\theta}$, $n \geq 0$, $r_\theta^n(\cdot|\cdot)$ is the nth transition probability of $\{V_n^\theta\}_{n\geq 0}$, while

$$\tilde{r}_\theta^n(v'|v) = r_\theta^n(v'|v) - \nu_\theta(v')$$

for $\theta \in \mathbb{R}^{d_\theta}$, $v, v' \in \mathcal{X} \times \mathcal{Y}$, $n \geq 0$. Additionally, the functions $\eta(\cdot)$, $F(\cdot, \cdot)$ are defined by

$$\eta(\theta) = \sum_{n=0}^{\infty} \sum_{v,v' \in \mathcal{X}\times\mathcal{Y}} \lambda^n \phi(v') \tilde{r}_\theta^n(v'|v) s_\theta(v) \nu_\theta(v) - \nabla f(\theta),$$

$$F(\theta, z) = \phi(v)w - \eta(\theta)$$

for $\theta \in R^{d_\theta}$, $z = (v, w) \in (\mathcal{X} \times \mathcal{Y}) \times \mathbb{R}^{d_\theta}$.[7] $\{Z_n\}_{n\geq 0}$, $\{\eta_n\}_{n\geq}$ are the stochastic processes defined as

$$Z_n = (X_n, Y_n, W_n), \qquad \eta_n = \eta(\theta_n)$$

for $n \geq 0$. Then it is straightforward to show that the algorithm (4.3) is of the same form as the recursion studied in Section 3 [i.e., $\{\theta_n\}_{n\geq 0}$, $\{\eta_n\}_{n\geq 0}$, $F(\cdot, \cdot)$, $\Pi_\theta(\cdot, \cdot)$ defined in Section 4 and here admit (3.1), (3.2)].

We will use the following additional notation. N_v is the integer defined by $N_v = N_x N_y$, while $e \in \mathbb{R}^{N_v}$ is the vector whose all components are one. For $v \in \mathcal{X} \times \mathcal{Y}$, $e(v) \in \mathbb{R}^{N_v}$ is the vector representation of $I_v(\cdot)$, while $\phi \in \mathbb{R}^{N_v}$ is the vector

[5] Under Assumption 4.1, $\nu_\theta(\cdot)$ exists and is unique (the details are provided in Lemma 8.1). The transition $r_\theta(\cdot|\cdot)$ can be defined by $r_\theta(v'|v) = q_\theta(y'|x')p(x'|x, y)$ for $v = (x, y) \in \mathcal{X} \times \mathcal{Y}$, $v' = (x', y') \in \mathcal{X} \times \mathcal{Y}$.

[6] $\Pi_\theta(\cdot, \cdot)$ can be defined by $\Pi_\theta(z, \{v'\} \times B) = I_B(\lambda w + s_\theta(v'))r_\theta(v'|v)$ for $z = (v, w) \in (\mathcal{X} \times \mathcal{Y}) \times \mathbb{R}^{d_\theta}$ and a Borel-measurable set $B \subseteq \mathbb{R}^{d_\theta}$.

[7] Under Assumptions 4.1, 4.2, $f(\cdot)$ is differentiable (the details are provided in Lemma 8.2).

representation of $\phi(\cdot)$.[8] For $\theta \in \mathbb{R}^{d_\theta}$, $R_\theta \in \mathbb{R}^{N_v \times N_v}$ and $\nu_\theta \in \mathbb{R}^{N_v}$ are the transition matrix and the invariant probability vector of $\{V_n^\theta\}_{n\geq 0}$,[9] while $\tilde{R}_\theta = R_\theta - e\nu_\theta^T$. For $\theta \in \mathbb{R}^{d_\theta}$, $1 \leq j \leq d_\theta$, $s_{\theta,j}(\cdot)$ is the jth component of $s_\theta(\cdot)$, while $S_{\theta,j} \in \mathbb{R}^{N_v \times N_v}$ is the diagonal matrix representation of $s_{\theta,j}(\cdot)$.[10]

LEMMA 8.1. *Suppose that Assumptions 4.1 and 4.2 hold. Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set. Then the following are true*:

(i) *$\{V_n^\theta\}_{n\geq 0}$ is geometrically ergodic for each $\theta \in \mathbb{R}^{d_\theta}$. Moreover, there exist real numbers $\varepsilon_Q \in (0,1)$, $C_{1,Q} \in [1,\infty)$ (independent of λ) such $\|\tilde{R}_\theta^n\| \leq C_{1,Q}\varepsilon_Q^n$ for all $\theta \in Q$, $n \geq 0$.*

(ii) *There exists a real number $C_{2,Q} \in [1,\infty)$ (independent of λ) such that*

$$\max\{\|\nu_{\theta'} - \nu_{\theta''}\|, \|R_{\theta'}^n - R_{\theta''}^n\|\} \leq C_{2,Q}\|\theta' - \theta''\|, \tag{8.1}$$

$$\|\tilde{R}_{\theta'}^n - \tilde{R}_{\theta''}^n\| \leq C_{2,Q}\varepsilon_Q^n\|\theta' - \theta''\| \tag{8.2}$$

for all $\theta', \theta'' \in Q$, $n \geq 0$.

(iii) *ν_θ is differentiable on $\mathbb{R}^{d_\theta}$. Moreover, $\nabla_\theta \nu_\theta$ is locally Lipschitz continuous on $\mathbb{R}^{d_\theta}$.*

(iv) *If Assumption 4.3.a is satisfied, ν_θ is p times differentiable on $\mathbb{R}^{d_\theta}$.*

(v) *If Assumption 4.3.b is satisfied, ν_θ is real-analytic on $\mathbb{R}^{d_\theta}$.*

PROOF. (i) For $\theta \in \mathbb{R}^{d_\theta}$, $n \geq 0$, let $p_\theta^n(\cdot|\cdot)$ and $\mu_\theta(\cdot)$ be the nth transition probability and the invariant probability of $\{X_n^\theta\}_{n\geq 0}$. Moreover, for $\theta \in \mathbb{R}^{d_\theta}$, $v = (x,y) \in \mathcal{X} \times \mathcal{Y}$, let $\tilde{\nu}_\theta(v) = q_\theta(y|x)\mu_\theta(x)$. Then it is straightforward to verify

$$r_\theta^{n+1}(v'|v) - \tilde{\nu}_\theta(v') = \sum_{x'' \in \mathcal{X}} q_\theta(y'|x')\big(p_\theta^n(x'|x'') - \mu_\theta(x')\big)p(x''|x,y)$$

for $\theta \in \mathbb{R}^{d_\theta}$, $v = (x,y) \in \mathcal{X} \times \mathcal{Y}$, $v' = (x',y') \in \mathcal{X} \times \mathcal{Y}$, $n \geq 0$. Therefore,

$$\begin{aligned} |r_\theta^{n+1}(v'|v) - \tilde{\nu}_\theta(v')| &\leq \sum_{x'' \in \mathcal{X}} q_\theta(y'|x')\big|p_\theta^n(x'|x'') - \mu_\theta(x')\big|p(x''|x,y) \\ &\leq N_x \max_{x'' \in \mathcal{X}} \big|p_\theta^n(x'|x'') - \mu_\theta(x')\big| \end{aligned}$$

for all $\theta \in \mathbb{R}^{d_\theta}$, $v = (x,y) \in \mathcal{X} \times \mathcal{Y}$, $v' = (x',y') \in \mathcal{X} \times \mathcal{Y}$, $n \geq 0$. Combining this with Assumption 4.1, we conclude that $\{V_n^\theta\}_{n\geq 0}$ is geometrically ergodic for each $\theta \in \mathbb{R}^{d_\theta}$. We also conclude that $\tilde{\nu}_\theta(\cdot)$ is the invariant probability of $\{V_n^\theta\}_{n\geq 0}$

[8]For $v = (x,y) \in \mathcal{X} \times \mathcal{Y}$, element i of $e(v)$ is one if $i = (x-1)N_y + y$ and zero otherwise. For the same v, $\phi(v)$ is element $(x-1)N_y + y$ of ϕ.

[9]For $v = (x,y) \in \mathcal{X} \times \mathcal{Y}$, $v' = (x',y') \in \mathcal{X} \times \mathcal{Y}$, $r_\theta(v'|v)$ is entry $((x-1)N_y + y, (x'-1)N_y + y')$ of R_θ, while $\nu_\theta(v)$ is element $(x-1)N_y + y$ of ν_θ.

[10]For $v = (x,y) \in \mathcal{X} \times \mathcal{Y}$, $s_{\theta,j}(v)$ is entry $((x-1)N_y + y, (x-1)N_y + y)$ of $S_{\theta,j}$. The off-diagonal elements of $S_{\theta,j}$ are zero.

for each $\theta \in \mathbb{R}^{d_\theta}$, that is, $\nu_\theta(v) = \tilde{\nu}_\theta(v) = q_\theta(y|x)\mu_\theta(x)$ for $\theta \in \mathbb{R}^{d_\theta}$, $v = (x, y) \in \mathcal{X} \times \mathcal{Y}$.

For $\theta \in \mathbb{R}^{d_\theta}$, let $\rho_\theta = \min_{v\in\mathcal{V}} \nu_\theta(x)/3$. Then we have $0 < \rho_\theta \le 1/(3N_v)$, $\rho_\theta \le \nu_\theta(v)/3$ for all $\theta \in \mathbb{R}^{d_\theta}$, $v \in \mathcal{V}$. Moreover, for any $\theta \in \mathbb{R}^{d_\theta}$, there exists an integer $n_\theta \ge 0$ such that $|r_\theta^n(v'|v) - \nu_\theta(v')| \le \rho_\theta$ for each $v, v' \in \mathcal{V}$, $n \ge n_\theta$. Hence, $r_\theta^n(v'|v) \ge \nu_\theta(v') - \rho_\theta \ge 2\rho_\theta$ for all $\theta \in \mathbb{R}^{d_\theta}$, $v, v' \in \mathcal{V}$, $n \ge n_\theta$. Additionally, Assumption 4.2 implies that for each $v, v' \in \mathcal{V}$, $n \ge 0$, $r_\theta^n(v'|v)$ is locally Lipschitz continuous in θ on $\mathbb{R}^{d_\theta}$.[11] Consequently, for any $\theta \in \mathbb{R}^{d_\theta}$, there exists a real number $\delta_\theta \in (0, 1)$ such that $|r_\vartheta^{n_\theta}(v'|v) - r_\theta^{n_\theta}(v'|v)| \le \rho_\theta$ for all $\vartheta \in \mathbb{R}^{d_\theta}$, $v, v' \in \mathcal{V}$ satisfying $\|\vartheta - \theta\| \le \delta_\theta$. Thus, $r_\vartheta^{n_\theta}(v'|v) \ge r_\theta^{n_\theta}(v'|v) - \rho_\theta \ge \rho_\theta$ for each $\vartheta \in \mathbb{R}^{d_\theta}$, $v, v' \in \mathcal{V}$ satisfying $\|\vartheta - \theta\| \le \delta_\theta$. Since

$$r_\vartheta^n(v'|v) = \sum_{v''\in\mathcal{V}} r_\vartheta^{n_\theta}(v'|v'')r_\vartheta^{n-n_\theta}(v''|v) \ge \rho_\theta \sum_{v''\in\mathcal{V}} r_\vartheta^{n-n_\theta}(v''|v) = \rho_\theta$$

for any $\vartheta \in \mathbb{R}^{d_\theta}$, $v, v' \in \mathcal{V}$, $n \ge n_\theta$ satisfying $\|\vartheta - \theta\| \le \delta_\theta$, we conclude $r_\vartheta^n(v'|v) \ge \rho_\theta$ for the same ϑ, v, v', n.

Let $B_\theta = \{\vartheta \in \mathbb{R}^{d_\theta} : \|\vartheta - \theta\| < \delta_\theta\}$ for $\theta \in \mathbb{R}^{d_\theta}$. As $\{B_\theta\}_{\theta\in Q}$ is an open covering of Q, there exists a finite set $\tilde{Q} \subseteq Q$ such that $\bigcup_{\theta\in\tilde{Q}} B_\theta \supset Q$. Let $\tilde{n}_Q = \max_{\theta\in\tilde{Q}} n_\theta$, $\tilde{\rho}_Q = \min_{\theta\in\tilde{Q}} \rho_\theta$, $\tilde{\varepsilon}_Q = (1 - \tilde{\rho}_Q)^{1/\tilde{n}_Q}$. Since each element of Q is also an element of one of $\{B_\theta\}_{\theta\in\tilde{Q}}$, we have $r_\theta^n(v'|v) \ge \tilde{\rho}_Q$ for all $\theta \in Q$, $v, v' \in \mathcal{V}$, $n \ge \tilde{n}_Q$.[12] Then standard results of Markov chain theory (see, e.g., [26], Theorem 16.0.2) imply

$$|r_\theta^n(v'|v) - \nu_\theta(v')| \le (1 - \tilde{\rho}_Q)^{n/\tilde{n}_Q} \le \tilde{\varepsilon}_Q^n$$

for all $\theta \in Q$, $v, v' \in \mathcal{V}$, $n \ge 0$.

Let $\varepsilon_Q = \tilde{\varepsilon}_Q^{1/2}$, $C_{1,Q} = N_v$. Then we have

$$\|\tilde{R}_\theta^n\| \le N_v \max_{v,v'\in\mathcal{X}\times\mathcal{Y}} |\tilde{r}_\theta^n(v'|v)| \le N_v\tilde{\varepsilon}_Q^n = C_{1,Q}\varepsilon_Q^{2n} \tag{8.3}$$

for all $\theta \in Q$, $n \ge 0$.

(ii) Let g be the N_vth standard unit vector in $\mathbb{R}^{N_v}$ (i.e., the first $N_v - 1$ elements of g are zero, while the last element of g is one) and, for $A \in \mathbb{R}^{N_v\times N_v}$, let $G(A)$ be the $N_v \times N_v$ matrix obtained when the last row of $I - A^T$ is replaced by e^T (here, I is the $N_v \times N_v$ unit matrix). Additionally, let $\mathcal{Q}_0^{N_v\times N_v} = \{A \in \mathbb{R}^{N_v\times N_v} : \det(G(A)) \ne 0\}$ and, for $A \in \mathcal{Q}_0^{N_v\times N_v}$, let $h(A) = (G(A))^{-1}g$. Then it is easy to conclude that $\mathcal{Q}_0^{N_v\times N_v}$ is an open set [notice that $\det(G(A))$ is a polynomial function of the entries of A]. It is also easy to deduce that $h(\cdot)$ is well defined and

[11] Notice that, due to Assumption 4.2, $q_\theta(y|x)$ is locally Lipschitz continuous in θ for each $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and that $r_\theta^n(\cdot|\cdot)$ is a polynomial function of $p(\cdot|\cdot,\cdot)$, $q_\theta(\cdot|\cdot)$.

[12] If $\theta \in B_\vartheta$ and $\vartheta \in \tilde{Q}$, then $n_\vartheta \le \tilde{n}_Q$ and $r_\theta^n(v'|v) \ge \rho_\vartheta \ge \tilde{\rho}_Q$ for $n \ge n_\vartheta$.

real-analytic on $\mathcal{Q}_0^{N_v \times N_v}$ [notice that due to the Cramer's rule, all elements of $h(A)$ are rational functions of the entries of A].

Let $\mathcal{P}_0^{N_v \times N_v}$ be the set of $N_v \times N_v$ geometrically ergodic stochastic matrices. Then each $P \in \mathcal{P}_0^{N_v \times N_v}$ has a unique invariant probability vector. Moreover, the invariant probability vector of $P \in \mathcal{P}_0^{N_v \times N_v}$ is the unique solution to the linear system of equations $G(P)x = g$, where $x \in \mathbb{R}^{N_v}$ is the unknown. Hence, $\det(G(P)) \neq 0$ for each $P \in \mathcal{P}_0^{N_v \times N_v}$ so $\mathcal{P}_0^{N_v \times N_v} \subset \mathcal{Q}_0^{N_v \times N_v}$.

Owing to (i), $R_\theta \in \mathcal{P}_0^{N_v \times N_v}$ for each $\theta \in \mathbb{R}^{d_\theta}$. Thus, $\nu_\theta = h(R_\theta)$ for all $\theta \in \mathbb{R}^{d_\theta}$. Moreover, due to Assumption 4.2, R_θ is locally Lipschitz continuous on $\mathbb{R}^{d_\theta}$.[13] Since $h(\cdot)$ is real-analytic on $\mathcal{Q}_0^{N_v \times N_v}$ and $\mathcal{P}_0^{N_v \times N_v} \subset \mathcal{Q}_0^{N_v \times N_v}$, ν_θ is locally Lipschitz continuous on $\mathbb{R}^{d_\theta}$.

Let $\tilde{C}_{1,Q} \in [1, \infty)$ be a Lipschitz constant of R_θ, ν_θ on Q, while $\tilde{C}_{2,Q} \in [1, \infty)$ is an upper bound of the sequence $\{n\varepsilon_Q^n\}_{n \geq 1}$. Moreover, let $C_{2,Q} = 3\varepsilon_Q^{-1} C_{1,Q}^2 \tilde{C}_{1,Q} \tilde{C}_{2,Q}$. It is straightforward to verify

$$\tilde{R}_{\theta'}^{n+1} - \tilde{R}_{\theta''}^{n+1} = \sum_{i=0}^{n} \tilde{R}_{\theta'}^{i} \big(R_{\theta'} - R_{\theta''} - e(\nu_{\theta'} - \nu_{\theta''})^T\big) \tilde{R}_{\theta''}^{n-i}$$

for $\theta', \theta'' \in \mathbb{R}^{d_\theta}$, $n \geq 0$. Combining this with (8.3), we get

$$\begin{aligned} \big\| \tilde{R}_{\theta'}^{n+1} - \tilde{R}_{\theta''}^{n+1} \big\| &\leq \sum_{i=0}^{n} \big\| \tilde{R}_{\theta'}^{i} \big\| \big\| \tilde{R}_{\theta''}^{n-i} \big\| \big(\| R_{\theta'} - R_{\theta''} \| + \| \nu_{\theta'} - \nu_{\theta''} \| \big) \\ &\leq 2C_{1,Q}^2 \tilde{C}_{1,Q} (n+1) \varepsilon_Q^{2n} \| \theta' - \theta'' \| \\ &\leq C_{2,Q} \varepsilon_Q^n \| \theta' - \theta'' \| \end{aligned}$$

for each $\theta', \theta'' \in Q$, $n \geq 0$. Therefore,

$$\begin{aligned} \big\| R_{\theta'}^n - R_{\theta''}^n \big\| &\leq \big\| \tilde{R}_{\theta'}^n - \tilde{R}_{\theta''}^n \big\| + \| \nu_{\theta'} - \nu_{\theta''} \| \\ &\leq \tilde{C}_{1,Q} \big(2C_{1,Q}^2 \tilde{C}_{2,Q} n \varepsilon_Q^{n-1} + 1 \big) \| \theta' - \theta'' \| \\ &\leq C_{2,Q} \| \theta' - \theta'' \| \end{aligned}$$

for all $\theta', \theta'' \in Q$, $n \geq 0$ (notice that $\tilde{R}_\theta^k = R_\theta^k - e\nu_\theta^T$).

(iii) Due to (i), $R_\theta \in \mathcal{P}_0^{N_v \times N_v}$ for each $\theta \in \mathbb{R}^{d_\theta}$. Hence, $\nu_\theta = h(R_\theta)$ for all $\theta \in \mathbb{R}^{d_\theta}$. Moreover, owing to Assumption 4.2, R_θ is differentiable on $\mathbb{R}^{d_\theta}$ and its first-order derivatives are locally Lipschitz continuous on the same space.[14] As $h(\cdot)$ is real-analytic on $\mathcal{Q}_0^{N_v \times N_v}$ and $\mathcal{P}_0^{N_v \times N_v} \subset \mathcal{Q}_0^{N_v \times N_v}$, ν_θ is differentiable on

[13] Notice that $r_\theta(v'|v) = q_\theta(y'|x') p(x'|x, y)$ for $v = (x, y)$, $v' = (x', y')$ and that $q_\theta(y|x)$ is locally Lipschitz continuous in θ.

[14] Notice that $\nabla_\theta r_\theta(v'|v) = s_\theta(x', y') q_\theta(y'|x') p(x'|x, y)$ for $v = (x, y)$, $v' = (x', y')$.

$\mathbb{R}^{d_\theta}$. The same arguments also imply that $\nabla_\theta \nu_\theta$ is locally Lipschitz continuous on $\mathbb{R}^{d_\theta}$.

(iv), (v) If Assumption 4.3.a is satisfied, then R_θ is p times differentiable on $\mathbb{R}^{d_\theta}$, and consequently, ν_θ is p times differentiable on $\mathbb{R}^{d_\theta}$, too.[15] Similarly, if Assumption 4.3.b is satisfied, then R_θ is real-analytic on $\mathbb{R}^{d_\theta}$, and therefore, ν_θ is also real-analytic on $\mathbb{R}^{d_\theta}$. □

LEMMA 8.2. *Suppose that Assumptions 4.1 and 4.2 hold. Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set. Then the following are true*:

(i) *$f(\cdot)$ is differentiable and $\nabla f(\cdot)$ is locally Lipschitz continuous.*
(ii) *There exists a real number $C_{3,Q} \in [1, \infty)$ (independent of λ) such that $\|\eta(\theta)\| \le C_{3,Q}(1-\lambda)$ for all $\theta \in Q$.*
(iii) *If Assumption 4.3.a is satisfied, $f(\cdot)$ is p times differentiable.*
(iv) *If Assumption 4.3.b is satisfied, $f(\cdot)$ is real-analytic.*

PROOF. (i), (iii), (iv) Owing to Lemma 8.1, we have

$$f(\theta) = \lim_{n\to\infty} E_\theta\big(\phi(V_n^\theta)\big) = \sum_{v\in\mathcal{X}\times\mathcal{Y}} \phi(v)\nu_\theta(v) = \phi^T \nu_\theta \tag{8.4}$$

for all $\theta \in \mathbb{R}^{d_\theta}$. Then these parts of the lemma directly follow from Lemma 8.1:

(ii) For each $1 \le j \le d_\theta$, let $\tilde{C}_Q \in [1, \infty)$ be an upper bound of $\|S_{\theta,j}\|$ on Q. For $\theta \in \mathbb{R}^{d_\theta}$, $v \in \mathcal{X}\times\mathcal{Y}$, $n \ge 0$, let us also define

$$f_n(\theta, v) = \sum_{v'\in\mathcal{X}\times\mathcal{Y}} \phi(v') r_\theta^n(v'|v), \tag{8.5}$$

$$h(\theta) = \sum_{n=0}^{\infty} \sum_{v,v'\in\mathcal{X}\times\mathcal{Y}} \phi(v')\tilde{r}_\theta^n(v'|v) s_\theta(v)\nu_\theta(v). \tag{8.6}$$

Owing to Lemma 8.1, $f_n(\theta, v)$ converges to $f(\theta)$ as $n \to \infty$ uniformly in (θ, v) on $Q \times (\mathcal{X}\times\mathcal{Y})$. Due to the same lemma, $h(\cdot)$ is well defined on Q [notice that when $\theta \in Q$, each term in the sums in (8.5), (8.6) tends to zero at the rate ε_Q^n]. Moreover, it is straightforward to show

$$\begin{aligned}
\nabla_\theta f_n(\theta, v_0) &= \nabla_\theta \left(\sum_{v_1,\dots,v_n\in\mathcal{X}\times\mathcal{Y}} \phi(v_n) \left(\prod_{i=1}^n r_\theta(v_i|v_{i-1}) \right) \right) \\
&= \sum_{v_1,\dots,v_n\in\mathcal{X}\times\mathcal{Y}} \phi(v_n) \left(\sum_{i=1}^n \frac{\nabla_\theta r_\theta(v_i|v_{i-1})}{r_\theta(v_i|v_{i-1})} \right) \left(\prod_{i=1}^n r_\theta(v_i|v_{i-1}) \right) \qquad (8.7) \\
&= \sum_{i=1}^n \sum_{v',v''\in\mathcal{X}\times\mathcal{Y}} \phi(v'') r_\theta^{n-i}(v''|v') s_\theta(v') r_\theta^i(v'|v_0)
\end{aligned}$$

[15] Notice that $R_\theta \in \mathcal{P}_0^{N_v\times N_v} \subset \mathcal{Q}_0^{N_v\times N_v}$, $\nu_\theta = h(R_\theta)$ for all $\theta \in \mathbb{R}^{d_\theta}$. Notice also that $h(\cdot)$ is real-analytic on $\mathcal{Q}_0^{N_v\times N_v}$.

for all $\theta \in \mathbb{R}^{d_\theta}$, $v_0 \in \mathcal{X} \times \mathcal{Y}$, $n \geq 1$. Therefore,

$$\partial_\theta^j f_n(\theta, v) = \sum_{i=1}^{n} e^T(v) R_\theta^i S_{\theta,j} R_\theta^{n-i} \phi = \sum_{i=0}^{n-1} e^T(v) R_\theta^{n-i} S_{\theta,j} R_\theta^i \phi \tag{8.8}$$

for $\theta \in \mathbb{R}^{d_\theta}$, $v \in \mathcal{X} \times \mathcal{Y}$, $1 \leq j \leq d_\theta$, $n \geq 1$, where $\partial_\theta^j f_n(\theta, v)$ is the jth component of $\nabla_\theta f_n(\theta, v)$. We also have

$$\sum_{i=0}^{n-1} e^T(v) R_\theta^{n-i} S_{\theta,j} R_\theta^i e = 0 \tag{8.9}$$

for $\theta \in \mathbb{R}^{d_\theta}$, $v \in \mathcal{X} \times \mathcal{Y}$, $1 \leq j \leq d_\theta$, $n \geq 1$.[16] Hence,

$$\sum_{i=0}^{n-1} e^T(v) R_\theta^{n-i} S_{\theta,j} e \nu_\theta^T \phi = \nu_\theta^T \phi \sum_{i=0}^{n-1} e^T(v) R_\theta^{n-i} S_{\theta,j} R_\theta^i e = 0$$

for $\theta \in \mathbb{R}^{d_\theta}$, $v \in \mathcal{X} \times \mathcal{Y}$, $1 \leq j \leq d_\theta$, $n \geq 1$ (notice that $R_\theta^i e = e$). Therefore,

$$\partial_\theta^j f_n(\theta, v) = \sum_{i=0}^{n-1} e^T(v) R_\theta^{n-i} S_{\theta,j} \tilde{R}_\theta^i \phi$$

for $\theta \in \mathbb{R}^{d_\theta}$, $v \in \mathcal{X} \times \mathcal{Y}$, $1 \leq j \leq d_\theta$, $n \geq 1$. Additionally, we have

$$h_j(\theta) = \sum_{n=0}^{\infty} \nu_\theta^T S_{\theta,j} \tilde{R}_\theta^n \phi = \sum_{i=0}^{n-1} e^T(v) e \nu_\theta^T S_{\theta,j} \tilde{R}_\theta^i \phi + \sum_{i=n}^{\infty} \nu_\theta^T S_{\theta,j} \tilde{R}_\theta^i \phi$$

for $\theta \in \mathbb{R}^{d_\theta}$, $v \in \mathcal{X} \times \mathcal{Y}$, $1 \leq j \leq d_\theta$, $n \geq 1$ [notice that $e^T(v)e = 1$], where $h_j(\theta)$ is the jth component of $h(\theta)$. Thus,

$$\partial_\theta^j f_n(\theta, v) - h_j(\theta) = \sum_{i=0}^{n-1} e^T(v) \tilde{R}_\theta^{n-i} S_{\theta,j} \tilde{R}_\theta^i \phi - \sum_{i=n}^{\infty} \nu_\theta^T S_{\theta,j} \tilde{R}_\theta^i \phi$$

for $\theta \in \mathbb{R}^{d_\theta}$, $v \in \mathcal{X} \times \mathcal{Y}$, $1 \leq j \leq d_\theta$, $n \geq 1$. Then Lemma 8.1 implies

$$\begin{aligned} \left|\partial_\theta^j f_n(\theta, v) - h_j(\theta)\right| &\leq \|\phi\| \left\|e(v)\right\| \|S_{\theta,j}\| \sum_{i=0}^{n-1} \left\|\tilde{R}_\theta^i\right\| \left\|\tilde{R}_\theta^{n-i}\right\| \\ &\quad + \|\phi\| \|\nu_\theta\| \|S_{\theta,j}\| \sum_{i=n}^{\infty} \left\|\tilde{R}_\theta^i\right\| \\ &\leq \tilde{C}_Q C_{1,Q}^2 \|\phi\| n \varepsilon_Q^n + \frac{\tilde{C}_Q C_{1,Q} \|\phi\| \varepsilon_Q^n}{1 - \varepsilon_Q} \end{aligned}$$

[16] If $\phi = e$, then $f_n(\theta, v)$ is identically one, while $\nabla_\theta f_n(\theta, v)$ is identically zero. Hence, (8.8) reduces to (8.9) when $\phi = e$.

for all $\theta \in Q$, $v \in \mathcal{X} \times \mathcal{Y}$, $1 \le j \le d_\theta$, $n \ge 1$. Hence, $\nabla_\theta f_n(\theta, v)$ converges to $h(\theta)$ as $n \to \infty$ uniformly in (θ, v) on $Q \times (\mathcal{X} \times \mathcal{Y})$. Therefore, $\nabla f(\theta) = h(\theta)$ for all $\theta \in \mathbb{R}^{d_\theta}$ (notice that Q is any compact set). Consequently,

$$\eta_j(\theta) = \sum_{n=0}^{\infty} \lambda^n v_\theta^T S_{\theta,j} \tilde{R}_\theta^n \phi - h_j(\theta) = -\sum_{n=0}^{\infty} (1-\lambda^n) v_\theta^T S_{\theta,j} \tilde{R}_\theta^n \phi$$

for $\theta \in \mathbb{R}^{d_\theta}$, $1 \le j \le d_\theta$, where $\eta_j(\theta)$ is the jth component of $\eta(\theta)$. Combining this with Lemma 8.1, we get

$$\begin{aligned} \bigl|\eta_j(\theta)\bigr| &\le \|\phi\| \|\nu_\theta\| \|S_{\theta,j}\| \sum_{n=0}^{\infty} (1-\lambda^n) \bigl\|\tilde{R}_\theta^n\bigr\| \\ &\le \tilde{C}_Q C_{1,Q} \|\phi\| \sum_{n=0}^{\infty} (1-\lambda^n) \varepsilon_Q^n \\ &\le \frac{\tilde{C}_Q C_{1,Q} \|\phi\| (1-\lambda)}{(1-\varepsilon_Q)^2} \end{aligned}$$

for all $\theta \in Q$, $1 \le j \le d_\theta$. Then we conclude that there exists a real number $C_{3,Q} \in [1, \infty)$ with the properties specified in (ii). □

LEMMA 8.3. *Suppose that Assumptions 4.1 and 4.2 hold. Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set. Then the following are true*:

(i) *There exist real numbers $\delta_Q \in (0,1)$, $C_{4,Q} \in [1, \infty)$ (possibly depending on λ) such that*

$$\begin{aligned} &\bigl\|(\Pi^n F)(\theta, z) - \nabla f(\theta)\bigr\| \le C_{4,Q} n \delta_Q^n (1 + \|w\|), \\ &\bigl\|\bigl((\Pi^n F)(\theta', z) - \nabla f(\theta')\bigr) - \bigl((\Pi^n F)(\theta'', z) - \nabla f(\theta'')\bigr)\bigr\| \\ &\qquad \le C_{4,Q} n \delta_Q^n \|\theta' - \theta''\| (1 + \|w\|), \end{aligned}$$

for all $\theta, \theta', \theta'' \in Q$, $z = (x, y, w) \in \mathcal{X} \times \mathcal{Y} \times \mathbb{R}^{d_\theta}$, $n \ge 0$.

(ii) *There exits a real number $C_{5,Q} \in [1, \infty)$ (possibly depending on λ) such that*

$$\|W_{n+1}\| I_{\{\tau_Q > n\}} \le C_{5,Q} (1 + \|W_0\|)$$

for all $n \ge 0$ (τ_Q is specified in Assumption 3.3).

PROOF. (i) For each $1 \le j \le d_\theta$, let $\tilde{C}_{1,Q} \in [1, \infty)$ be an upper bound of $\|S_{\theta,j}\|$ on Q and a Lipschitz constant of $S_{\theta,j}$ on the same set. Moreover, let $\tilde{C}_{2,Q} = 3\tilde{C}_{1,Q} C_{1,Q} C_{2,Q} N_v$, $\tilde{C}_{3,Q} = 2\tilde{C}_{2,Q}(1-\varepsilon_Q)^{-1}$, while $\delta_Q = \max\{\lambda, \varepsilon_Q\}$.

Owing to Lemma 8.1, we have

$$\|\tilde{R}_\theta^k S_{\theta,j} R_\theta^l\| \le \|\tilde{R}_\theta^k\| \|S_{\theta,j}\| \|R_\theta^l\| \le \tilde{C}_{2,Q} \varepsilon_Q^k, \tag{8.10}$$

$$\|v_\theta^T S_{\theta,j} R_\theta^l\| \le \|v_\theta^T\| \|S_{\theta,j}\| \|R_\theta^l\| \le \tilde{C}_{2,Q} \tag{8.11}$$

for all $\theta \in Q$, $1 \le j \le d_\theta$, $k, l \ge 1$. Due to the same lemma, we also have

$$\begin{aligned} \|\tilde{R}_{\theta'}^k S_{\theta',j} R_{\theta'}^l - \tilde{R}_{\theta''}^k S_{\theta'',j} R_{\theta''}^l\| &\le \|\tilde{R}_{\theta'}^k - \tilde{R}_{\theta''}^k\| \|S_{\theta',j}\| \|R_{\theta'}^l\| \\ &\quad + \|\tilde{R}_{\theta''}^k\| \|S_{\theta',j} - S_{\theta'',j}\| \|R_{\theta'}^l\| \\ &\quad + \|\tilde{R}_{\theta''}^k\| \|S_{\theta'',j}\| \|R_{\theta'}^l - R_{\theta''}^l\| \\ &\le \tilde{C}_{2,Q} \varepsilon_Q^k \|\theta' - \theta''\| \end{aligned} \tag{8.12}$$

for all $\theta', \theta'' \in Q$, $1 \le j \le d_\theta$, $k, l \ge 1$. In addition to this, Lemma 8.1 implies

$$\begin{aligned} \|v_{\theta'}^T S_{\theta',j} R_{\theta'}^l - v_{\theta''}^T S_{\theta'',j} R_{\theta''}^l\| &\le \|v_{\theta'}^T - v_{\theta''}^T\| \|S_{\theta',j}\| \|R_{\theta'}^l\| \\ &\quad + \|v_{\theta''}^T\| \|S_{\theta',j} - S_{\theta'',j}\| \|R_{\theta'}^l\| \\ &\quad + \|v_{\theta''}^T\| \|S_{\theta'',j}\| \|R_{\theta'}^l - R_{\theta''}^l\| \\ &\le \tilde{C}_{2,Q} \|\theta' - \theta''\| \end{aligned} \tag{8.13}$$

for each $\theta', \theta'' \in Q$, $1 \le j \le d_\theta$, $l \ge 1$. Moreover, it is straightforward to show

$$\begin{aligned} (\Pi^n F)(\theta, z) &= -\eta(\theta) + E_\theta\big(\phi(V_n^\theta) W_n^\theta | V_0^\theta = v, W_0^\theta = w\big) \\ &= -\eta(\theta) + E_\theta\left(\phi(V_n^\theta)\left(\lambda^n w + \sum_{i=0}^{n-1} \lambda^i s_\theta(V_{n-i}^\theta)\right)\Big| V_0^\theta = v\right) \\ &= -\eta(\theta) + \sum_{i=0}^{n-1} \sum_{v', v'' \in \mathcal{X} \times \mathcal{Y}} \lambda^i \phi(v'') r_\theta^i(v''|v') s_\theta(v') r_\theta^{n-i}(v'|v) \\ &\quad + \lambda^n w \sum_{v' \in \mathcal{X} \times \mathcal{Y}} \phi(v') r_\theta^n(v'|v) \end{aligned}$$

for $\theta \in \mathbb{R}^{d_\theta}$, $z = (v, w) \in (\mathcal{X} \times \mathcal{Y}) \times \mathbb{R}^{d_\theta}$, $n \ge 1$. Therefore,

$$(\Pi^n F_j)(\theta, z) = -\eta_j(\theta) + \sum_{i=0}^{n-1} \lambda^i e^T(v) R_\theta^{n-i} S_{\theta,j} R_\theta^i \phi + \lambda^n e_j^T w e^T(v) R_\theta^n \phi$$

for $\theta \in \mathbb{R}^{d_\theta}$, $z = (v, w) \in (\mathcal{X} \times \mathcal{Y}) \times \mathbb{R}^{d_\theta}$, $1 \le j \le d_\theta$, $n \ge 1$. Here, $F_j(\theta, z)$, $\eta_j(\theta)$ are the jth components of $F(\theta, z)$, $\eta(\theta)$, while e_j is the jth standard unit vector in $\mathbb{R}^{d_\theta}$. Moreover, we have

$$\partial^j f(\theta) = -\eta_j(\theta) + \sum_{n=0}^{\infty} \lambda^n v_\theta^T S_{\theta,j} \tilde{R}_\theta^n \phi$$

for $\theta \in \mathbb{R}^{d_\theta}$, $1 \le j \le d_\theta$, where $\partial^j f(\theta)$ is the jth component of $\nabla f(\theta)$. Since $e^T(v)e = 1$, $\tilde{R}_\theta^n = R_\theta^n - e\nu_\theta^T$ and

$$\nu_\theta^T S_{\theta,j} e = \sum_{v \in \mathcal{X}\times\mathcal{Y}} \nu_\theta(v) s_{\theta,j}(v) = \sum_{x\in\mathcal{X}} \Biggl(\sum_{y\in\mathcal{Y}} \partial_\theta^j q_\theta(y|x) \Biggr) \mu_\theta(x) = 0$$

for $\theta \in \mathbb{R}^{d_\theta}$, $v \in \mathcal{X}\times\mathcal{Y}$, $1 \le j \le d_\theta$, $n \ge 0$,[17] we get

$$\begin{aligned}\partial^j f(\theta) &= -\eta_j(\theta) + \sum_{i=0}^{n-1} \lambda^i \nu_\theta^T S_{\theta,j} R_\theta^i \phi + \sum_{i=0}^{n-1} \lambda^i \nu_\theta^T S_{\theta,j} e \nu_\theta^T \phi + \sum_{i=n}^{\infty} \lambda^i \nu_\theta^T S_{\theta,j} \tilde{R}_\theta^i \phi \\ &= -\eta_j(\theta) + \sum_{i=0}^{n-1} \lambda^i e^T(v) e \nu_\theta^T S_{\theta,j} R_\theta^i \phi + \sum_{i=n}^{\infty} \lambda^i \nu_\theta^T S_{\theta,j} \tilde{R}_\theta^i \phi\end{aligned}$$

for the same θ, v, j, n. Consequently,

$$\begin{aligned}\bigl(\Pi^n F_j\bigr)(\theta,z) - \partial^j f(\theta) = &\sum_{i=0}^{n-1} \lambda^i e^T(v) \tilde{R}^{n-i} S_{\theta,j} R_\theta^i \phi - \sum_{i=n}^{\infty} \lambda^i \nu_\theta^T S_{\theta,j} \tilde{R}_\theta^i \phi \\ &+ \lambda^n e_j^T w e^T(v) R_\theta^n \phi\end{aligned}$$

for $\theta \in \mathbb{R}^{d_\theta}$, $z = (v,w) \in (\mathcal{X}\times\mathcal{Y}) \times \mathbb{R}^{d_\theta}$, $1 \le j \le d_\theta$, $n \ge 1$. Then (8.10), (8.11) imply

$$\begin{aligned}&\bigl|\bigl(\Pi^n F_j\bigr)(\theta,z) - \partial^j f(\theta)\bigr| \\ &\quad\le \|\phi\| \bigl\|e(v)\bigr\| \sum_{i=0}^{n-1} \lambda^i \bigl\|\tilde{R}_\theta^{n-i} S_{\theta,j} R_\theta^i\bigr\| \\ &\qquad + \|\phi\| \sum_{i=n}^{\infty} \lambda^i \bigl\|\nu_\theta^T S_{\theta,j} \tilde{R}_\theta^i\bigr\| + \lambda^n \|\phi\| \bigl\|e(v)\bigr\| \bigl\|R_\theta^n\bigr\| \|w\| \\ &\quad\le \tilde{C}_{2,Q} \Biggl(\sum_{i=1}^{n} \lambda^i \varepsilon_Q^{n-i} + \sum_{i=n}^{\infty} \lambda^i \varepsilon_Q^i + \lambda^n \|w\| \Biggr) \\ &\quad\le C_{3,Q} n \delta_Q^n \bigl(1 + \|w\|\bigr)\end{aligned} \tag{8.14}$$

[17] Notice that $\sum_{y\in\mathcal{Y}} \partial_\theta^j q_\theta(y|x) = \partial_\theta^j \bigl(\sum_{y\in\mathcal{Y}} q_\theta(y|x)\bigr) = 0$. Notice also that $\nu_\theta(v) = q_\theta(y|x)\mu_\theta(x)$ for $v = (x,y) \in \mathcal{X}\times\mathcal{Y}$, where $\mu_\theta(x)$ is the invariant probability of $\{X_n^\theta\}_{n\ge 0}$ [see the proof of Part (i) of Lemma 8.1].

for all $\theta \in Q$, $z = (v, w) \in (\mathcal{X} \times \mathcal{Y}) \times \mathbb{R}^{d_\theta}$, $1 \le j \le d_\theta$, $n \ge 1$. Similarly, (8.12), (8.13) yield

$$
\begin{aligned}
&\big|\big((\Pi^n F_j)(\theta', z) - \partial^j f(\theta')\big) - \big((\Pi^n F_j)(\theta'', z) - \partial^j f(\theta'')\big)\big| \\
&\quad \le \|\phi\| \big\|e(v)\big\| \sum_{i=0}^{n-1} \lambda^i \big\| \tilde{R}_{\theta'}^{n-i} S_{\theta', j} R_{\theta'}^i - \tilde{R}_{\theta''}^{n-i} S_{\theta'', j} R_{\theta''}^i \big\| \\
&\qquad + \|\phi\| \sum_{i=n}^{\infty} \lambda^i \big\| v_{\theta'}^T S_{\theta', j} \tilde{R}_{\theta'}^i - v_{\theta''}^T S_{\theta'', j} \tilde{R}_{\theta''}^i \big\| \\
&\qquad + \lambda^n \|\phi\| \big\|e(v)\big\| \|w\| \big\| R_{\theta'}^n - R_{\theta''}^n \big\| \\
&\quad \le \tilde{C}_{2, Q} \|\theta' - \theta''\| \left(\sum_{i=1}^{n} \lambda^i \varepsilon_Q^{n-i} + \sum_{i=n}^{\infty} \lambda^i \varepsilon_Q^i + \lambda^n \|w\| \right) \\
&\quad \le C_{3, Q} n \delta_Q^n \|\theta' - \theta''\| (1 + \|w\|)
\end{aligned}
\tag{8.15}
$$

for all $\theta', \theta'' \in Q$, $z = (v, w) \in (\mathcal{X} \times \mathcal{Y}) \times \mathbb{R}^{d_\theta}$, $1 \le j \le d_\theta$, $n \ge 1$. Using (8.14), (8.15), we conclude that there exist real numbers δ_Q, $C_{4, Q}$ with properties specified in (i).

(ii) Let $C_{5, Q} = \tilde{C}_{1, Q} (1 - \lambda)^{-1}$ [$\tilde{C}_{1, Q}$ is specified in the proof of (i)]. Then, due to Assumption 4.2, we have

$$
\begin{aligned}
\|W_{n+1}\| I_{\{\tau_Q > n\}} &= \left\| \lambda^{n+1} W_0 + \sum_{i=0}^{n} \lambda^{n-i} s_{\theta_i}(X_{i+1}, Y_{i+1}) \right\| I_{\{\tau_Q > n\}} \\
&\le \lambda^{n+1} \|W_0\| + \tilde{C}_{1, Q} \sum_{i=0}^{n} \lambda^{n-i} \\
&\le C_{5, Q} (1 + \|W_0\|)
\end{aligned}
$$

for $n \ge 0$. $\square$

PROOF OF THEOREM 4.1. For $\theta \in \mathbb{R}^{d_\theta}$, $z = (v, w) \in (\mathcal{X} \times \mathcal{Y}) \times \mathbb{R}^{d_\theta}$, let

$$
\tilde{F}(\theta, z) = \sum_{n=0}^{\infty} \big((\Pi^n F)(\theta, z) - \nabla f(\theta) \big), \qquad \varphi(z) = 1 + \|w\|.
$$

Then, using Lemma 8.3, we conclude that for each $\theta \in \mathbb{R}^{d_\theta}$, $z \in \mathcal{X} \times \mathcal{Y} \times \mathbb{R}^{d_\theta}$, $\tilde{F}(\theta, z)$ is well defined and satisfies $(\Pi \tilde{F})(\theta, z) = \sum_{n=1}^{\infty} ((\Pi^n F)(\theta, z) - \nabla f(\theta))$. Thus, Assumption 3.2 holds. Relying on Lemma 8.3, we also deduce that for any compact set $Q \subset \mathbb{R}^{d_\theta}$, there exists a real number $\tilde{C}_Q \in [1, \infty)$ (possibly depending

on λ) such that

$$\begin{aligned}\max\{\|F(\theta,z)\|,\|\tilde{F}(\theta,z)\|,\|(\Pi\tilde{F})(\theta,z)\|\} &\le \tilde{C}_Q\varphi(z),\\ \|(\Pi\tilde{F})(\theta',z)-(\Pi\tilde{F})(\theta'',z)\| &\le \tilde{C}_Q\varphi(z)\|\theta'-\theta''\|,\\ E(\varphi^2(Z_{n+1})I_{\{\tau_Q>n\}}|\theta_0=\theta,Z_0=z) &\le \tilde{C}_Q\varphi^2(z)\end{aligned}$$

for all $\theta,\theta',\theta''\in Q$, $z\in\mathcal{X}\times\mathcal{Y}\times\mathbb{R}^{d_\theta}$. Hence, Assumptions 3.3 is satisfied, too. Moreover, Lemma 8.2 yields

$$\eta=\limsup_{n\to\infty}\|\eta_n\|\le C_{3,Q}(1-\lambda)$$

on Λ_Q (notice that $C_{3,Q}$ does not depend on λ). Then the theorem's assertion directly follows from Theorem 3.1 and Parts (i), (iii), (iv) of Lemma 8.2. □

APPENDIX A

In this section, a global version of Theorem 2.1 is presented. This result is based the following assumptions.

ASSUMPTION A.1. $f(\cdot)$ is uniformly lower bounded [i.e., $\inf_{\theta\in\mathbb{R}^{d_\theta}} f(\theta)>-\infty$], and $\nabla f(\cdot)$ is (globally) Lipschitz continuous. Moreover, there exist real numbers $c\in(0,1)$, $\rho\in[1,\infty)$ such that $\|\nabla f(\theta)\|\ge c$ for all $\theta\in\mathbb{R}^{d_\theta}$ satisfying $\|\theta\|\ge\rho$.

ASSUMPTION A.2. $\{\xi_n\}_{n\ge0}$ admits the decomposition $\xi_n=\zeta_n+\eta_n$ for each $n\ge0$, where $\{\zeta_n\}_{n\ge}$ and $\{\eta_n\}_{n\ge0}$ are $\mathbb{R}^{d_\theta}$-valued stochastic processes satisfying

$$\text{(A.1)}\qquad \lim_{n\to\infty}g(\theta_n)\max_{n\le j<a(n,t)}\left\|\sum_{i=n}^{j}\alpha_i\zeta_i\right\|=0,\qquad \limsup_{n\to\infty}g(\theta_n)\|\eta_n\|<\infty$$

almost surely for any $t\in(0,\infty)$. In addition, there exists a real number $\delta\in(0,1)$ such that

$$\text{(A.2)}\qquad \lim_{n\to\infty}h(\theta_n)\|\eta_n\|<\delta$$

almost surely. Here, $g,h:\mathbb{R}^{d_\theta}\to(0,\infty)$ are the (scaling) functions defined by

$$g(\theta)=\bigl(\|\nabla f(\theta)\|+1\bigr)^{-1},\qquad h(\theta)=\begin{cases}\|\nabla f(\theta)\|^{-1} & \text{if } \|\theta\|\ge\rho,\\ 0 & \text{otherwise}\end{cases}$$

for $\theta\in\mathbb{R}^{d_\theta}$ (ρ is specified in Assumption A.1).

Assumption A.1 is a stability condition. In this or a similar form, it is involved in practically any stability analysis of stochastic gradient search and stochastic approximation (see, e.g., [7, 11, 14] and references cited therein). This assumption is restrictive, as it requires $\nabla^2 f(\cdot)$ to be uniformly bounded. Assumption A.1 also requires $\nabla f(\cdot)$ to grow at most linearly as $\theta \to \infty$. Using random projections, these restrictive conditions can considerably be relaxed (see [14, 33]).

Assumption A.2 is a noise condition and can be considered as a global version of Assumption 2.2. Assumption A.2 requires the gradient of the objective function $f(\cdot)$ (asymptotically) to cancel the effect of the gradient estimator's error $\{\xi_n\}_{n\geq 0}$. Assumption A.2 is true whenever (2.4) holds almost surely. It is also satisfied for stochastic gradient search with Markovian dynamics (see Theorem B.1, Appendix B). Assumption A.2 and the results based on it (Theorem A.1, below) are motivated by the scaled ODE approach to the stability analysis of stochastic approximation [12].[18]

Our results on the stability and asymptotic bias of algorithm (2.1) are provided in the next theorem.

THEOREM A.1. *Suppose that Assumptions* 2.1, A.1 *and* A.2 *hold. Then the following are true*:

(i) *There exists a compact* (*deterministic*) *set* $Q \subset \mathbb{R}^{d_\theta}$ *such that* $P(\Lambda_Q) = 1$ [Λ_Q *is specified in* (2.6)].

(ii) *There exists a* (*deterministic*) *nondecreasing function* $\psi : [0, \infty) \to [0, \infty)$ [*independent of* η *and depending only on* $f(\cdot)$] *such that* $\lim_{t\to 0} \psi(t) = \psi(0) = 0$ *and*

$$\limsup_{n\to\infty} d(\theta_n, \mathcal{R}) \leq \psi(\eta)$$

almost surely.

(iii) *If* $f(\cdot)$ *satisfies Assumption* 2.3.b, *there exists a real number* $K \in (0, \infty)$ [*independent of* η *and depending only on* $f(\cdot)$] *such that*

$$\limsup_{n\to\infty} \|\nabla f(\theta_n)\| \leq K\eta^{q/2}, \qquad \limsup_{n\to\infty} f(\theta_n) - \liminf_{n\to\infty} f(\theta_n) \leq K\eta^q$$

almost surely (q *is specified in the statement of Theorem* 2.1).

[18] The main difference between [12] and the results presented here is the choice of the scaling functions. The scaling adopted in [12] is (asymptotically) proportional to $\|\theta\|$. In this paper, the scaling is (asymptotically) proportional to $\|\nabla f(\theta)\|$.

(iv) *If $f(\cdot)$ satisfies Assumption 2.3.c, there exist real numbers $r \in (0,1)$, $L \in (0,\infty)$ [independent of η and depending only on $f(\cdot)$] such that*

$$\limsup_{n\to\infty} \|\nabla f(\theta_n)\| \le L\eta^{1/2},$$

$$\limsup_{n\to\infty} d\big(f(\theta_n), f(\mathcal{S})\big) \le L\eta,$$

$$\limsup_{n\to\infty} d(\theta_n, \mathcal{S}) \le L\eta^{r}$$

almost surely.

PROOF. Owing to Assumption A.1, there exists a real number $\tilde{C}_1 \in [1,\infty)$ such that the following are true: (i) $f(\theta) > -\tilde{C}_1$ for all $\theta \in \mathbb{R}^{d_\theta}$, and (ii) $f(\theta) \le \tilde{C}_1$ for any $\theta \in \mathbb{R}^{d_\theta}$ satisfying $\|\theta\| \le \rho + 1$. Moreover, due to Assumption A.2, there also exists an event $N_0 \in \mathcal{F}$ with the following properties: (i) $P(N_0) = 0$, and (ii) (A.1), (A.2) hold on N_0^c for all $t \in (0,\infty)$.

Let $\varepsilon = (1-\delta)/6$, $T = 2\tilde{C}_1\varepsilon^{-1}c^{-2}$ and let $\phi : [0,\infty) \to [0,\infty)$ be the function defined by

$$\phi(z) = \sup\{\|\nabla f(\theta)\| : \theta \in \mathbb{R}^{d_\theta}, \|\theta\| \le z\}$$

for $z \in [0,\infty)$. As $\nabla f(\cdot)$ is locally Lipschitz continuous, $\phi(\cdot)$ is locally Lipschitz continuous, too. $\phi(\cdot)$ is also nonnegative and satisfies $\|\nabla f(\theta)\| \le \phi(\|\theta\|)$ for all $\theta \in \mathbb{R}^{d_\theta}$.

For $z \in [0,\infty)$, let $\lambda(\cdot; z)$ be the solution to the ODE $dz/dt = 2\phi(z)$ satisfying $\lambda(0; z) = z$. As $2\phi(\cdot)$ is nonnegative and locally Lipschitz continuous, $\lambda(\cdot;\cdot)$ is well defined and locally Lipschitz continuous (in both arguments) on $[0,\infty) \times [0,\infty)$. We also have

$$\lambda(t; z) = z + 2\int_0^t \phi\big(\lambda(s; z)\big)\, ds \tag{A.3}$$

for all $t, z \in [0,\infty)$. Then there exists $\rho_1 \in [1,\infty)$ such that $\rho_1 \ge \rho + 1$ and such that $|\lambda(t; z)| \le \rho_1$ for all $t \in [0, T]$, $z \in [0, \rho + 1]$.

Let $\rho_2 = \rho_1 + 1$, $Q = \{\theta \in \mathbb{R}^{d_\theta} : \|\theta\| \le \rho_2\}$, while Λ is the event defined by

$$\Lambda = \limsup_{n\to\infty}\{\|\theta_n\| < \rho\} = \bigcap_{m=0}^{\infty}\bigcup_{n=m}^{\infty}\{\|\theta_n\| < \rho\}.$$

Let also $\tilde{C}_2 \in [1,\infty)$ stand for a (global) Lipschitz constant of $\nabla f(\cdot)$ and for an upper bound of $\|\nabla f(\cdot)\|$ on Q. Finally, let $\tilde{C}_3 = 2\tilde{C}_2\exp(2\tilde{C}_2)$, $\tilde{C}_4 = 12\tilde{C}_1\tilde{C}_2\tilde{C}_3$, while $\tau = 4^{-1}\tilde{C}_4^{-1}\varepsilon c^2$.

In order to prove the theorem's assertion, it is sufficient to show $N_0^c \subseteq \Lambda$ (i.e., to establish that on N_0^c, $\|\theta_n\| \le \rho_2$ for all, but finitely many n).[19] To prove this,

[19] Assumption 2.2 is a consequence of Assumption A.2 and, therefore, Parts (ii)–(iv) directly follow from Part (i) and Theorem 2.1.

we use contradiction. We assume that $\|\theta_n\| > \rho_2$ for infinitely many n and some $\omega \in N_0^c$. Notice that all formulas which follow in the proof correspond to ω.

Owing to (A.1), (A.2), there exists an integer $k_1 \geq 0$ (depending on ω) such that

$$g(\theta_n) \max_{n \leq j < a(n,T)} \left\| \sum_{i=n}^{j} \alpha_i \zeta_i \right\| \leq \tau^2, \qquad h(\theta_n)\|\eta_n\| \leq \delta \tag{A.4}$$

for $n \geq k_1$. Due to Assumption 2.1 and (A.1), we also have

$$\lim_{n\to\infty} g(\theta_n)\|\alpha_n \zeta_n\| = \lim_{n\to\infty} g(\theta_n)\|\alpha_n \eta_n\| = 0. \tag{A.5}$$

Since

$$g(\theta_n)\|\theta_{n+1} - \theta_n\| \leq \alpha_n + g(\theta_n)\|\alpha_n \zeta_n\| + g(\theta_n)\|\alpha_n \eta_n\|$$

for $n \geq 0$, Assumption 2.1 and (A.5) imply $\lim_{n\to\infty} g(\theta_n)\|\theta_{n+1} - \theta_n\| = 0$. Then (6.2) implies that there exists an integer $k_2 \geq 0$ (depending on ω) such that

$$\sum_{i=n}^{a(n,\tau)-1} \alpha_i \geq (1-\varepsilon)\tau, \qquad g(\theta_n)\|\theta_{n+1} - \theta_n\| \leq \tau \tag{A.6}$$

for $n \geq k_2$.

Let $k_0 = \max\{k_1, k_2\}$. Moreover, let l_0, m_0, n_0 be the integers defined as follows. If $\omega \in \Lambda$ (i.e., if $\|\theta_n\| < \rho$ for infinitely many n), let

$$l_0 = \min\{n > k_0 : \|\theta_{n-1}\| < \rho\}, \tag{A.7}$$

$$m_0 = \min\{n > l_0 : \|\theta_n\| > \rho_2\}, \tag{A.8}$$

$$n_0 = \max\{n \leq m_0 : \|\theta_{n-1}\| < \rho\}. \tag{A.9}$$

Otherwise, if $\omega \in \Lambda^c$ (i.e., if $\|\theta_n\| < \rho$ for finitely many n), let

$$l_0 = \max\{n > 0 : \|\theta_{n-1}\| < \rho\}, \qquad m_0 = \infty, \qquad n_0 = \max\{k_0, l_0\}.$$

Then we have $k_0 < n_0 \leq m_0$ and $\|\theta_n\| \geq \rho$ for $n_0 \leq n < m_0$.

Let $\phi_n(\tau), \phi_{1,n}(\tau), \phi_{2,n}(\tau)$ have the same meaning as in Section 6. Now, the asymptotic properties of $\phi_n(\tau)$ are analyzed. As $\|\theta_n\| \geq \rho$ for $n_0 \leq n < m_0$, (A.4) implies

$$\left\| \sum_{i=n}^{j} \alpha_i \xi_i \right\| \leq \left\| \sum_{i=n}^{j} \alpha_i \zeta_i \right\| + \sum_{i=n}^{j} \alpha_i \|\eta_i\| \leq \tau^2 g^{-1}(\theta_n) + \delta \sum_{i=n}^{j} \alpha_i \|\nabla f(\theta_i)\| \tag{A.10}$$

for $n_0 \leq n \leq j < \min\{m_0, a(n,T)\}$ [notice that $\|\eta_i\| \leq \delta\|\nabla f(\theta_i)\|$ when $\|\theta_i\| \geq \rho$]. Therefore,

$$\begin{aligned} \|\nabla f(\theta_j)\| &\leq \|\nabla f(\theta_n)\| + \|\nabla f(\theta_j) - \nabla f(\theta_n)\| \\ &\leq \|\nabla f(\theta_n)\| + \tilde{C}_2\|\theta_j - \theta_n\| \end{aligned}$$

$$\le \|\nabla f(\theta_n)\| + \tilde{C}_2 \sum_{i=n}^{j-1} \alpha_i \|\nabla f(\theta_i)\| + \tilde{C}_2 \left\| \sum_{i=n}^{j-1} \alpha_i \xi_i \right\|$$

$$\le \|\nabla f(\theta_n)\| + \tilde{C}_2 \tau^2 g^{-1}(\theta_n) + 2\tilde{C}_2 \sum_{i=n}^{j-1} \alpha_i \|\nabla f(\theta_i)\|$$

for $n_0 \le n < j \le \min\{m_0 - 1, a(n,\tau)\}$.[20] Combining this with the Bellman–Gronwall inequality (see, e.g., [11], Appendix B), we deduce

$$\begin{aligned}
\|\nabla f(\theta_j)\| &\le \left(\|\nabla f(\theta_n)\| + \tilde{C}_2 \tau^2 g^{-1}(\theta_n)\right) \exp\left(2\tilde{C}_2 \sum_{i=n}^{j-1} \alpha_i\right) \\
&\le \left(\|\nabla f(\theta_n)\| + \tilde{C}_2 \tau^2 g^{-1}(\theta_n)\right)(1 + \tilde{C}_3 \tau) \\
&\le \|\nabla f(\theta_n)\| + \left(\tilde{C}_3 \tau + \tilde{C}_2 \tau^2 + \tilde{C}_2 \tilde{C}_3 \tau^3\right) g^{-1}(\theta_n) \\
&\le \|\nabla f(\theta_n)\| + \tilde{C}_4 \tau g^{-1}(\theta_n)
\end{aligned}$$

for $n_0 \le n \le j \le \min\{m_0 - 1, a(n,\tau)\}$.[21] Then (A.10) implies

$$\begin{aligned}
\left\| \sum_{i=n}^{j} \alpha_i \xi_i \right\| &\le \tau^2 g^{-1}(\theta_n) + \delta\left(\|\nabla f(\theta_n)\| + \tilde{C}_4 \tau g^{-1}(\theta_n)\right) \sum_{i=n}^{j} \alpha_i \\
&\le \delta\tau \|\nabla f(\theta_n)\| + 2\tilde{C}_4 \tau^2 g^{-1}(\theta_n)
\end{aligned} \tag{A.11}$$

for $n_0 \le n \le j < \min\{m_0, a(n,\tau)\}$. Consequently,

$$\begin{aligned}
\|\theta_j - \theta_n\| &\le \sum_{i=n}^{j-1} \alpha_i \|\nabla f(\theta_i)\| + \left\| \sum_{i=n}^{j-1} \alpha_i \xi_i \right\| \\
&\le \left(\|\nabla f(\theta_n)\| + \tilde{C}_4 \tau g^{-1}(\theta_n)\right)\left(\sum_{i=n}^{j-1} \alpha_i + \delta\tau\right) + 2\tilde{C}_4 \tau^2 g^{-1}(\theta_n) \\
&\le 3\tau g^{-1}(\theta_n)
\end{aligned} \tag{A.12}$$

[20] Notice that τ, T are defined as $\tau = 4^{-1}\tilde{C}_4^{-1}\varepsilon c^2$, $T = 2\tilde{C}_1 \varepsilon^{-1} c^{-2}$. Notice also $\tau < 1 < T$ since $\tilde{C}_1, \tilde{C}_4 \in [1, \infty)$, $\varepsilon, c \in (0, 1)$.

[21] Notice that $\sum_{i=n}^{j-1} \alpha_i \le \tau < 1$ when $n \le j \le a(n,\tau)$. Notice also $g^{-1}(\theta_n) > \|\nabla f(\theta_n)\|$ and $\exp(2\tilde{C}_2\tau) \le 2\tilde{C}_2\tau \exp(2\tilde{C}_2) = \tilde{C}_3\tau$.

for $n_0 \le n \le j \le \min\{m_0 - 1, a(n,\tau)\}$ (notice that $\delta < 1$, $\tilde{C}_4\tau \le 1/4$). Therefore,

$$\begin{aligned}\left|\phi_{1,n}(\tau)\right| &\le \tilde{C}_2\left\|\nabla f(\theta_n)\right\| \sum_{i=n}^{a(n,\tau)-1} \alpha_i \|\theta_i - \theta_n\| \\ &\le 3\tilde{C}_2\tau g^{-1}(\theta_n)\left\|\nabla f(\theta_n)\right\| \sum_{i=n}^{a(n,\tau)-1} \alpha_i \le 3\tilde{C}_2\tau^2 g^{-2}(\theta_n)\end{aligned}$$

for $n \ge n_0$ satisfying $a(n,\tau) < m_0$. We also have

$$\left|\phi_{2,n}(\tau)\right| \le \tilde{C}_2\|\theta_{a(n,\tau)} - \theta_n\|^2 \le 9\tilde{C}_2\tau^2 g^{-2}(\theta_n)$$

for $n \ge n_0$ satisfying $a(n,\tau) < m_0$. Thus,

$$\left|\phi_n(\tau)\right| \le \tilde{C}_4\tau^2 g^{-2}(\theta_n) \tag{A.13}$$

when $n \ge n_0$, $a(n,\tau) < m_0$. Additionally, as a result of (A.6), (A.11), we get

$$\begin{aligned}\left\|\nabla f(\theta_n)\right\| \sum_{i=n}^{a(n,\tau)-1} \alpha_i - \left\| \sum_{i=n}^{a(n,\tau)-1} \alpha_i \xi_i \right\| &\ge (1-\delta-\varepsilon)\tau\left\|\nabla f(\theta_n)\right\| \\ &\quad - 2\tilde{C}_4\tau^2 g^{-1}(\theta_n) \\ &= 5\varepsilon\tau\left\|\nabla f(\theta_n)\right\| - 2\tilde{C}_4\tau^2 g^{-1}(\theta_n) \\ &\ge 3\varepsilon\tau\left\|\nabla f(\theta_n)\right\|\end{aligned}$$

when $n \ge n_0$, $a(n,\tau) < m_0$.[22] Then (6.1), (A.13) imply

$$\begin{aligned}f(\theta_{a(n,\tau)}) - f(\theta_n) &\le -3\varepsilon\tau\left\|\nabla f(\theta_n)\right\|^2 + \tilde{C}_4\tau^2 g^{-2}(\theta_n) \\ &\le -\varepsilon\tau\left\|\nabla f(\theta_n)\right\|^2 \le -\varepsilon\tau c^2\end{aligned} \tag{A.14}$$

for $n \ge n_0$ satisfying $a(n,\tau) < m_0$.[23]

Let $\{n_k\}_{k\ge 0}$ be the sequence recursively defined by $n_{k+1} = a(n_k,\tau)$ for $k \ge 0$. Now, we show by contradiction $\omega \in \Lambda$ (i.e., $\|\theta_n\| < \rho$ for infinitely many n). We assume the opposite. Then $m_0 = \infty$ and $\|\theta_n\| \ge \rho$ for $n \ge n_0$, while (A.14) implies $f(\theta_{n_{k+1}}) - f(\theta_{n_k}) \le -\varepsilon\tau c^2$ for $k \ge 0$. Hence, $\lim_{k\to\infty} f(\theta_{n_k}) = -\infty$. However, this is impossible due to Assumption A.1. Thus, $\omega \in \Lambda$ (i.e., $\|\theta_n\| < \rho$ for infinitely many n). Therefore, m_0, n_0 are defined through (A.7), and thus, $\|\theta_{n_0-1}\| < \rho$, $\|\theta_{m_0}\| > \rho_2$. Combining this with (A.6), we conclude

$$\|\theta_{n_0} - \theta_{n_0-1}\| \le \tau g^{-1}(\theta_{n_0-1}) \le \tau(\tilde{C}_2 + 1) \le 1/2$$

[22] Notice that $1 - \delta = 6\varepsilon$, $\varepsilon \ge \varepsilon c \ge 2\tilde{C}_4\tau$. Notice also that $2\varepsilon\tau\|\nabla f(\theta_n)\| \ge \varepsilon\tau\|\nabla f(\theta_n)\| + \varepsilon\tau c \ge 2\tilde{C}_4\tau^2 g^{-1}(\theta_n)$ for $n_0 \le n < m_0$.

[23] Notice that $2\varepsilon\|\nabla f(\theta_n)\|^2 \ge \varepsilon\|\nabla f(\theta_n)\|^2 + \varepsilon c^2 \ge \tilde{C}_4\tau g^{-2}(\theta_n)$ for $n_0 \le n < m_0$.

[notice that $\|\nabla f(\theta_{n_0-1})\| \le \tilde{C}_2$, $\tilde{C}_2\tau \le 1/4$]. Consequently,

$$\|\theta_{n_0}\| \le \|\theta_{n_0-1}\| + \|\theta_{n_0} - \theta_{n_0-1}\| \le \rho + 1/2 < \rho_2. \tag{A.15}$$

Hence, $n_0 < m_0$, $f(\theta_{n_0}) \le \tilde{C}_1$.

Let i_0, j_0 be the integers defined by $j_0 = \max\{j \ge 0 : n_j < m_0\}$, $i_0 = n_{j_0}$. Then we have $n_0 \le i_0 = n_{j_0} < n_{j_0+1} = m_0 \le a(i_0, \tau)$. As a result of this and (A.12), we get

$$\|\theta_{m_0} - \theta_{i_0}\| \le 3\tau g^{-1}(\theta_{i_0}) \le 3\tau(\tilde{C}_2 + 1) \le 1/2$$

[notice that $\|\nabla f(\theta_{i_0})\| \le \tilde{C}_2$, $\tilde{C}_2\tau \le 1/12$]. Consequently,

$$\|\theta_{i_0}\| \ge \|\theta_{m_0}\| - \|\theta_{m_0} - \theta_{i_0}\| \ge \rho_2 - 1/2 > \rho_1. \tag{A.16}$$

Let $\{\gamma_n\}_{n\ge 0}$, $\theta_0(\cdot)$ have the same meaning as in Section 5. Now, we show by contradiction that $\gamma_{i_0} - \gamma_{n_0} \ge T$. We assume the opposite. Then (A.10), (A.15) imply

$$\begin{aligned}
\|\theta_0(t)\| = \|\theta_j\| &\le \|\theta_{n_0}\| + \sum_{i=n_0}^{j-1} \alpha_i \|\nabla f(\theta_i)\| + \left\| \sum_{i=n_0}^{j-1} \alpha_i \xi_i \right\| \\
&\le \|\theta_{n_0}\| + \tau^2 g^{-1}(\theta_{n_0}) + 2\sum_{i=n_0}^{j-1} \alpha_i \|\nabla f(\theta_i)\| \\
&\le \rho + 1 + 2\sum_{i=n_0}^{j-1} \alpha_i \phi(\|\theta_i\|) \\
&\le \rho + 1 + 2\int_{\gamma_{n_0}}^{t} \phi(\|\theta_0(s)\|)\, ds
\end{aligned} \tag{A.17}$$

for $t \in [\gamma_j, \gamma_{j+1})$, $n_0 \le j \le i_0$.[24] Applying the comparison principle (see [19], Section 3.4) to (A.3), (A.17), we conclude $\|\theta_0(t)\| \le \lambda(t - \gamma_{n_0}; \rho + 1) \le \rho_1$ for all $t \in [\gamma_{n_0}, \gamma_{i_0}]$. Thus, $\|\theta_{i_0}\| = \|\theta_0(\gamma_{i_0})\| \le \rho_1$. However, this is impossible, due to (A.16). Hence, $\gamma_{i_0} - \gamma_{n_0} \ge T$. Consequently,

$$T \le \gamma_{i_0} - \gamma_{n_0} = \sum_{j=0}^{j_0-1} (\gamma_{n_{j+1}} - \gamma_{n_j}) \le j_0\tau \tag{A.18}$$

(notice that $n_{j_0} = i_0$, $\gamma_{n_{j+1}} - \gamma_{n_j} = \sum_{i=n_j}^{n_{j+1}-1} \alpha_i \le \tau$).

[24] As $j \le i_0 < m_0$, we have $\gamma_j - \gamma_{n_0} \le \gamma_{i_0} - \gamma_{n_0} \le T$ and $j \le \min\{m_0 - 1, a(n_0, T)\}$. We also have $\tau^2 g^{-1}(\theta_{n_0}) \le \tau^2(\tilde{C}_2 + 1) \le 1/2$.

Owing to (A.14), we have $f(\theta_{n_{j+1}}) - f(\theta_{n_j}) \le -\varepsilon\tau c^2$ for $0 \le j \le j_0$. Combining this with (A.18), we get

$$f(\theta_{i_0}) = f(\theta_{n_{j_0}}) \le f(\theta_{n_0}) - j_0\varepsilon\tau c^2 \le \tilde{C}_1 - \varepsilon c^2 T \le -\tilde{C}_1.$$

However, this is impossible, since $f(\theta) > -\tilde{C}_1$ for all $\theta \in \mathbb{R}^{d_\theta}$. Hence, $\|\theta_n\| > \rho_2$ for finitely many n. □

APPENDIX B

In this section, a global version of Theorem 3.1 is presented. This result is based the following assumptions.

ASSUMPTION B.1. There exists a Borel-measurable function $\varphi : \mathbb{R}^{d_z} \to [1, \infty)$ such that

$$\max\{\|F(\theta, z)\|, \|\tilde{F}(\theta, z)\|, \|(\Pi\tilde{F})(\theta, z)\|\} \le \varphi(z)(\|\nabla f(\theta)\| + 1),$$
$$\|(\Pi\tilde{F})(\theta', z) - (\Pi\tilde{F})(\theta'', z)\| \le \varphi(z)\|\theta' - \theta''\|$$

for all $\theta, \theta', \theta'' \in \mathbb{R}^{d_\theta}$, $z \in \mathbb{R}^{d_z}$. In addition, one has

$$\sup_{n\ge 0} E(\varphi^2(Z_n)|\theta_0 = \theta, Z_0 = z) < \infty$$

for all $\theta \in \mathbb{R}^{d_\theta}$, $z \in \mathbb{R}^{d_z}$.

ASSUMPTION B.2. $\eta_n = \eta(\theta_n)$ for $n \ge 0$, where $\eta : \mathbb{R}^{d_\theta} \to \mathbb{R}^{d_\theta}$ is a continuous function. Moreover, there exists a real number $\delta \in (0, 1)$ such that $\|\eta(\theta)\| \le \delta\|\nabla f(\theta)\|$ for all $\theta \in \mathbb{R}^{d_\theta}$ satisfying $\|\theta\| \ge \rho$ (ρ is specified in Assumption A.1).

Assumption B.1 is a global version of Assumption 3.3. In a similar form, it is involved in the stability analysis of stochastic approximation carried out in [7], Section II.1.9. Assumption B.2 is related to the bias of the gradient estimator. It requires the bias $\{\eta_n\}_{n\ge 0}$ to be a deterministic function of the algorithm iterates $\{\theta_n\}_{n\ge 0}$. As demonstrated in Section 4 and [33], this is often satisfied in practice. Assumption B.2 can be considered as one of the weakest conditions under which the stability of the perturbed ODE $d\theta/dt = -(\nabla f(\theta) + \eta(\theta))$ can be shown.

Our results on the stability and asymptotic bias of algorithm (3.1) are provided in the next theorem.

THEOREM B.1. *Suppose that Assumptions* 3.1, 3.2, A.1, B.1 *and* B.2 *hold. Let* $f(\cdot)$ *be the function specified in Assumption* 3.2. *Then the following are true*:

(i) *If* $f(\cdot)$ *satisfies Assumption* 2.3.a, *Part* (i) *of Theorem* A.1 *holds*.
(ii) *If* $f(\cdot)$ *satisfies Assumption* 2.3.b, *Part* (ii) *of Theorem* A.1 *holds*.
(iii) *If* $f(\cdot)$ *satisfies Assumption* 2.3.c, *Part* (iii) *of Theorem* A.1 *holds*.

PROOF. Let $g(\cdot)$, $h(\cdot)$ be the functions defined in Assumption A.2. Then, due to Assumption B.2, $g(\theta)\eta(\theta)$ is uniformly bounded in $\theta \in \mathbb{R}^{d_\theta}$, while $h(\theta)\eta(\theta) \le \delta$ for all $\theta \in \mathbb{R}^{d_\theta}$ satisfying $\|\theta\| \ge \rho$. Let $C \in [1, \infty)$ stand for a (global) Lipschitz constant of $\nabla f(\cdot)$ and for an (global) upper bound of $g(\cdot)\eta(\cdot)$. Define $\tau = 1/(18C^2)$ and let $\{\zeta_n\}_{n\ge 0}$, $\{\zeta_{1,n}\}_{n\ge 0}$, $\{\zeta_{2,n}\}_{n\ge 0}$, $\{\zeta_{3,n}\}_{n\ge 0}$ have the same meaning as in the proof of Theorem 3.1, while τ_n is the stopping time defined by

$$\tau_n = \min\bigl(\bigl\{j \ge n : g(\theta_n)g^{-1}(\theta_j) > 3\bigr\} \cup \{\infty\}\bigr)$$

for $n \ge 0$. Finally, for $\theta \in \mathbb{R}^{d_\theta}$, $z \in \mathbb{R}^{d_z}$, let $E_{\theta,z}(\cdot)$ denote the conditional mean given $\theta_0 = \theta$, $Z_0 = z$.

As a direct consequence of Assumptions 3.1, B.1, we get

$$E_{\theta,z}\left(\sum_{n=0}^{\infty} \alpha_n^2 \varphi^2(Z_{n+1})\right) < \infty$$

for all $\theta \in \mathbb{R}^{d_\theta}$, $z \in \mathbb{R}^{d_z}$. We also have

$$g(\theta_n)\|\zeta_n\| \le \varphi(Z_{n+1}) + 1 \le 2\varphi(Z_{n+1})$$

for $n \ge 0$. Consequently,

$$\lim_{n\to\infty} \alpha_n \varphi(Z_{n+1}) = \lim_{n\to\infty} \alpha_n g(\theta_n)\|\zeta_n\| = 0 \tag{B.1}$$

almost surely.

Let $\{m_k\}_{k\ge 0}$ be the sequence recursively defined by $m_0 = 0$ and $m_{k+1} = a(m_k, \tau)$ for $k \ge 0$. Moreover, let $\mathcal{F}_n = \sigma\{\theta_0, Z_0, \ldots, \theta_n, Z_n\}$ for $n \ge 0$. Due to Assumption 3.2, we have

$$\begin{aligned} &E_{\theta,z}\bigl(g(\theta_n)\zeta_{1,j} I_{\{\tau_n > j\}} | \mathcal{F}_j\bigr) \\ &\quad = g(\theta_n)\bigl(E_{\theta,z}\bigl(\tilde{F}(\theta_j, Z_{j+1}) | \mathcal{F}_j\bigr) - (\Pi\tilde{F})(\theta_j, Z_j)\bigr) I_{\{\tau_n > j\}} = 0 \end{aligned}$$

almost surely for each $\theta \in \mathbb{R}^{d_\theta}$, $z \in \mathbb{R}^{d_z}$, $0 \le n \le j$ (notice that $\{\tau_n > j\}$ is measurable with respect to $\mathcal{F}_j$). Moreover, Assumption B.1 implies

$$\begin{aligned} g(\theta_n)\|\zeta_{1,j}\| I_{\{\tau_n > j\}} &\le g(\theta_n)g^{-1}(\theta_j)\bigl(\varphi(Z_j) + \varphi(Z_{j+1})\bigr) I_{\{\tau_n > j\}} \\ &\le 3\bigl(\varphi(Z_j) + \varphi(Z_{j+1})\bigr) \end{aligned}$$

for $0 \le n \le j$. Then, as a result of Doob inequality, we get

$$\begin{aligned}
&E_{\theta,z}\left(\max_{n<j<a(n,\tau)}\left\|\sum_{i=n+1}^{j}\alpha_i g(\theta_n)\zeta_{1,i}\right\|^2 I_{\{\tau_n>j\}}\right)\\
&\quad\le E_{\theta,z}\left(\max_{n<j<a(n,\tau)}\left\|\sum_{i=n+1}^{j}\alpha_i g(\theta_n)\zeta_{1,i} I_{\{\tau_n>i\}}\right\|^2\right)\\
&\quad\le 4E_{\theta,z}\left(\sum_{i=n+1}^{a(n,\tau)-1}\alpha_i^2 g^2(\theta_n)\|\zeta_{1,i}\|^2 I_{\{\tau_n>i\}}\right)\\
&\quad\le 72E_{\theta,z}\left(\sum_{i=n+1}^{a(n,\tau)}\alpha_i^2\big(\varphi^2(Z_i)+\varphi^2(Z_{i+1})\big)\right)
\end{aligned}$$

for all $\theta \in \mathbb{R}^{d_\theta}$, $z \in \mathbb{R}^{d_z}$, $n \ge 0$. Combining this with Assumptions 3.1, B.1, we deduce

$$\begin{aligned}
&E_{\theta,z}\left(\sum_{k=0}^{\infty} g^2(\theta_{m_k})\max_{m_k<j<m_{k+1}}\left\|\sum_{i=m_k}^{j}\alpha_i\zeta_{1,i}\right\|^2 I_{\{\tau_{m_k}>j\}}\right)\\
&\quad\le 72E_{\theta,z}\left(\sum_{n=0}^{\infty}\big(\alpha_i^2+\alpha_{i+1}^2\big)\varphi^2(Z_{i+1})\right)<\infty
\end{aligned}$$

for each $\theta \in \mathbb{R}^{d_\theta}$, $z \in \mathbb{R}^{d_z}$, $n \ge 0$. Therefore,

$$\lim_{k\to\infty} g(\theta_{m_k})\max_{m_k<j<m_{k+1}}\left\|\sum_{i=m_k}^{j}\alpha_i\zeta_{1,i}\right\| I_{\{\tau_{m_k}>j\}} = 0 \tag{B.2}$$

almost surely.

Since $\alpha_n\alpha_{n+1} = O(\alpha_n^2)$, $\alpha_n - \alpha_{n+1} = O(\alpha_n^2)$ for $n \to \infty$ (see the proof of Theorem 3.1), Assumptions 3.1, B.1 yield

$$\begin{aligned}
E_{\theta,z}\left(\sum_{n=0}^{\infty}\alpha_n\alpha_{n+1}\varphi^2(Z_{n+1})\right)<\infty,\\
E_{\theta,z}\left(\sum_{n=0}^{\infty}|\alpha_n-\alpha_{n+1}|\varphi^2(Z_{n+1})\right)<\infty
\end{aligned}$$

for all $\theta \in \mathbb{R}^{d_\theta}$, $z \in \mathbb{R}^{d_z}$. Additionally, due to Assumptions B.1, B.2, we have

$$\begin{aligned}
g(\theta_n)\|\zeta_{2,j}\| I_{\{\tau_n>j\}} &\le g(\theta_n)\varphi(Z_j)\|\theta_j-\theta_{j-1}\| I_{\{\tau_n>j-1\}}\\
&\le \alpha_{j-1}g(\theta_n)\varphi(Z_j)\big(\|F(\theta_{j-1},Z_j)\|+\|\eta_{j-1}\|\big)I_{\{\tau_n>j\}}\\
&\le \alpha_{j-1}g(\theta_n)g^{-1}(\theta_{j-1})\varphi(Z_j)\big(\varphi(Z_j)+C\big)I_{\{\tau_n>j\}}\\
&\le 6C\alpha_{j-1}\varphi^2(Z_j)
\end{aligned}$$

for $0 \le n < j$ [notice that $\varphi(z) \ge 1$ for any $z \in \mathbb{R}^{d_z}$]. We also have

$$g(\theta_n)\|\zeta_{3,j}\| I_{\{\tau_n > j\}} \le g(\theta_n) g^{-1}(\theta_j)\varphi(Z_{j+1}) I_{\{\tau_n > j\}} \le 3\varphi(Z_{j+1}) \le 3\varphi^2(Z_{j+1})$$

for $0 \le n \le j$. Hence,

$$\begin{aligned} g(\theta_n)\left\| \sum_{i=n+1}^{j} \alpha_i \zeta_{2,i} \right\| I_{\{\tau_n > j\}} &\le \sum_{i=n+1}^{j} \alpha_i g(\theta_n)\|\zeta_{2,i}\| I_{\{\tau_n > i\}} \\ &\le 6C \sum_{i=n}^{j} \alpha_i \alpha_{i+1} \varphi^2(Z_{i+1}), \\ g(\theta_n)\left\| \sum_{i=n+1}^{j} (\alpha_i - \alpha_{i+1})\zeta_{3,i} \right\| I_{\{\tau_n > j\}} &\le \sum_{i=n+1}^{j} |\alpha_i - \alpha_{i+1}| g(\theta_n)\|\zeta_{3,i}\| I_{\{\tau_n > i\}} \\ &\le 3 \sum_{i=n+1}^{j} |\alpha_i - \alpha_{i+1}| \varphi^2(Z_{i+1}) \end{aligned}$$

for $0 \le n < j$. Consequently,

$$\begin{aligned} &\lim_{n\to\infty} g(\theta_n) \max_{j>n} \left\| \sum_{i=n+1}^{j} \alpha_i \zeta_{2,i} \right\| I_{\{\tau_n > j\}} \\ &\quad = \lim_{n\to\infty} g(\theta_n) \max_{j>n} \left\| \sum_{i=n+1}^{j} (\alpha_i - \alpha_{i+1})\zeta_{3,i} \right\| I_{\{\tau_n > j\}} = 0 \end{aligned} \tag{B.3}$$

almost surely [notice that $\alpha_{j+1}/\alpha_j = O(1)$ for $j \to \infty$]. Moreover, (B.1) yields

$$\lim_{n\to\infty} g(\theta_n) \max_{j\ge n} \alpha_{j+1}\|\zeta_{3,j}\| I_{\{\tau_n > j\}} = 0 \tag{B.4}$$

almost surely. Combining (B.1)–(B.4) with (7.2), we deduce

$$\lim_{k\to\infty} g(\theta_{n_k}) \max_{m_k \le j < m_{k+1}} \left\| \sum_{i=m_k}^{j} \alpha_i \zeta_i \right\| I_{\{\tau_{m_k} > j\}} = 0 \tag{B.5}$$

almost surely.

Owing to Assumptions A.1, B.2, we have

$$\begin{aligned} g^{-1}(\theta_{j+1})I_{\{\tau_n>j\}} &\le g^{-1}(\theta_n) + \|\nabla f(\theta_{j+1}) - \nabla f(\theta_n)\| I_{\{\tau_n>j\}} \\ &\le g^{-1}(\theta_n) + C\|\theta_{j+1} - \theta_n\| I_{\{\tau_n>j\}} \\ &\le g^{-1}(\theta_n) + C\sum_{i=n}^{j} \alpha_i \|\nabla f(\theta_i)\| I_{\{\tau_n>j\}} + C\left\|\sum_{i=n}^{j} \alpha_i \zeta_i\right\| I_{\{\tau_n>j\}} \\ &\quad + C\sum_{i=n}^{j} \alpha_i \|\eta_i\| I_{\{\tau_n>j\}} \\ &\le g^{-1}(\theta_n) + C\left\|\sum_{i=n}^{j} \alpha_i \zeta_i\right\| I_{\{\tau_n>j\}} + 2C^2 \sum_{i=n}^{j} \alpha_i g^{-1}(\theta_i) I_{\{\tau_n>j\}} \end{aligned}$$

for $0 \le n \le j$ [notice that $\|\eta(\theta)\| \le Cg^{-1}(\theta)$ for each $\theta \in \mathbb{R}^{d_\theta}$]. Combining this with the Bellman–Gronwall inequality (see, e.g., [11], Appendix B), we conclude

$$\begin{aligned} g^{-1}(\theta_{j+1})I_{\{\tau_n>j\}} &\le \left(g^{-1}(\theta_n) + C\max_{n\le j<a(n,\tau)} \left\|\sum_{i=n}^{j} \alpha_i \zeta_i\right\| I_{\{\tau_n>j\}}\right) \\ &\quad \cdot \exp\left(2C^2 \sum_{i=n}^{j-1} \alpha_i\right) \\ &\le 2g^{-1}(\theta_n)\left(1 + Cg(\theta_n) \max_{n\le j<a(n,\tau)} \left\|\sum_{i=n}^{j} \alpha_i \zeta_i\right\| I_{\{\tau_n>j\}}\right) \end{aligned}$$

for $0 \le n \le j \le a(n,\tau)$.[25] Then (B.5) yields

$$\limsup_{k\to\infty} g(\theta_{m_k}) \max_{m_k\le j<m_{k+1}} g^{-1}(\theta_{j+1}) I_{\{\tau_{m_k}>j\}} \le 2 \tag{B.6}$$

almost surely.

Let N_0 be the event where (B.5) or (B.6) does not hold. Then, in order to prove the theorem's assertion, it is sufficient to show that (A.1), (A.2) are satisfied on N_0^c for any $t \in (0,\infty)$. Let ω be any sample in N_0^c, while $t \in (0,\infty)$ is any real number. Notice that all formula which follow in the proof correspond to ω.

Due to Assumption B.2, we have

$$\limsup_{n\to\infty} g(\theta_n)\|\eta_n\| \le C < \infty, \qquad \limsup_{n\to\infty} h(\theta_n)\|\eta_n\| \le \delta < 1.$$

[25] Notice that $\sum_{i=n}^{j-1} \alpha_i \le \tau$ for $n \le j \le a(n,\tau)$. Notice also that $\exp(2C^2\tau) \le \exp(1/2) \le 2$.

Moreover, Assumption 3.1 and (6.2), (B.6) imply that there exists an integer $k_0 \geq 0$ (depending on ω) such that $\sum_{i=m_k}^{m_{k+1}-1} \alpha_i \geq \tau/2$ and

$$g(\theta_{m_k})\Biggl\|\sum_{i=m_k}^{j} \alpha_i \zeta_i\Biggr\| I_{\{\tau_{m_k} > j\}} \leq \tau, \qquad g(\theta_{m_k}) g^{-1}(\theta_{j+1}) I_{\{\tau_{m_k} > j\}} \leq 3 \tag{B.7}$$

for $k \geq k_0$, $m_k \leq j < m_{k+1}$. As $\tau_n > n$ for $n \geq 0$, we conclude $\tau_{m_k} > m_{k+1}$ for $k \geq k_0$.[26] Consequently, $I_{\{\tau_{m_k} > j\}} = 1$ for $k \geq k_0$, $m_k \leq j \leq m_{k+1}$. Combining this with (B.7), we get $g(\theta_{m_k}) \leq 3g(\theta_{j+1})$ and

$$\begin{aligned} g^{-1}(\theta_{j+1}) &\geq g^{-1}(\theta_{m_k}) - \bigl\|\nabla f(\theta_{j+1}) - \nabla f(\theta_n)\bigr\| \\ &\geq g^{-1}(\theta_{m_k}) - C\|\theta_{j+1} - \theta_n\| \\ &\geq g^{-1}(\theta_{m_k}) - C\sum_{i=m_k}^{j} \alpha_i \bigl\|\nabla f(\theta_i)\bigr\| \\ &\quad - C\Biggl\|\sum_{i=m_k}^{j} \alpha_i \zeta_i\Biggr\| - C\sum_{i=m_k}^{j} \alpha_i \|\eta_i\| \\ &\geq g^{-1}(\theta_{m_k}) - 2C^2 \sum_{i=m_k}^{j} \alpha_i g^{-1}(\theta_i) - C\Biggl\|\sum_{i=m_k}^{j} \alpha_i \zeta_i\Biggr\| \\ &\geq g^{-1}(\theta_{m_k})\bigl(1 - 6C^2\tau - C\tau\bigr) \\ &\geq 3^{-1} g^{-1}(\theta_{m_k}) \end{aligned} \tag{B.8}$$

for $k \geq k_0$, $m_k \leq j < m_{k+1}$.[27] Hence, $3^{-1} g(\theta_{m_k}) \leq g(\theta_j) \leq 3g(\theta_{m_k})$ for $k \geq k_0$, $m_k \leq j \leq m_{k+1}$.

Let $n_0 = m_{k_0}$, while $k(n) = \max\{k \geq 0 : m_k \leq n\}$, $m(n) = m_{k(n)}$ for $n \geq 0$. Then (B.8) implies $g(\theta_n) \leq 3g(\theta_{m(n)})$, $g(\theta_{m_k}) \leq 3g(\theta_{m_{k+1}})$ for $n \geq n_0$, $k \geq k_0$ [notice that $k(n) \geq k_0$, $m_{k(n)} \leq n < m_{k(n)+1}$ when $n \geq n_0$]. Hence, $g(\theta_n) \leq C_{n,k} g(\theta_{m_k})$ for $n \geq n_0$, $k \geq m(n)$, where $C_{n,k} = 3^{k-k(n)+1}$.[28] Since

$$2^{-1}\bigl(k(j) - k(n)\bigr)\tau \leq \sum_{k=k(n)+1}^{k(j)} \sum_{i=m_k}^{m_{k+1}-1} \alpha_i \leq \sum_{i=n}^{j} \alpha_i \leq t$$

[26] If $\tau_{m_k} \leq m_{k+1}$, then $\tau_{m_k} = j$ and $g(\theta_{m_k}) g^{-1}(\theta_j) I_{\{\tau_{m_k} > j-1\}} = g(\theta_{m_k}) g^{-1}(\theta_j) > 3$ for some j satisfying $m_k < j \leq m_{k+1}$.

[27] Notice that $g^{-1}(\theta_i) \leq 3g^{-1}(\theta_{m_k})$, $\sum_{m_k}^{m_{k+1}-1} \alpha_i \leq \tau$ when $k \geq k_0$, $m_k \leq i < m_{k+1}$. Notice also that $6C^2\tau = 1/3$, $C\tau \leq 1/3$.

[28] Notice that $g(\theta_n) g^{-1}(\theta_{m(n)}) \leq 3$, $g(\theta_{m(n)}) g^{-1}(\theta_{m_k}) \leq 3^{k-k(n)}$ when $n \geq n_0$, $k \geq m(n)$. Notice also $g(\theta_n) = (g(\theta_n) g^{-1}(\theta_{m(n)}))(g(\theta_{m(n)}) g^{-1}(\theta_{m_k})) g(\theta_{m_k})$.

for $n_0 \le n \le j \le a(n,\tau)$, we conclude $k(j) - k(n) \le 2t/\tau$ for the same n, j. Consequently,

$$
\begin{aligned}
g(\theta_n)\left\|\sum_{i=n}^{j}\alpha_i\zeta_i\right\| &= g(\theta_n)\left\|\sum_{k=k(n)}^{k(j)}\sum_{i=m_k}^{m_{k+1}-1}\alpha_i\zeta_i - \sum_{i=m(n)}^{n-1}\alpha_i\zeta_i + \sum_{i=m(j)}^{j}\alpha_i\zeta_i\right\| \\
&\le \sum_{k=k(n)}^{k(j)-1} C_{n,k}\, g(\theta_{m_k})\left\|\sum_{i=m_k}^{m_{k+1}-1}\alpha_i\zeta_i\right\| \\
&\quad + C_{n,k(n)}\, g(\theta_{m(n)})\left\|\sum_{i=m(n)}^{n-1}\alpha_i\zeta_i\right\| \\
&\quad + C_{n,k(j)}\, g(\theta_{m(j)})\left\|\sum_{i=m(j)}^{j}\alpha_i\zeta_i\right\| \\
&\le C(t)\max_{\substack{m_k\le l<m_{k+1}\\ k(n)\le k}} g(\theta_{m_k})\left\|\sum_{i=m_k}^{l}\alpha_i\zeta_i\right\|
\end{aligned}
$$

for $n_0 \le n \le j \le a(n,t)$,[29] where $C(t) = (2t/\tau + 3)3^{2t/\tau+3}$. Since $\tau_{m_k} > m_{k+1}$ for $k \ge k_0$ (i.e., $I_{\{\tau_{m_k} > j\}} = 1$ for $k \ge k_0$, $m_k \le j \le m_{k+1}$), (B.5) implies

$$
\lim_{n\to\infty} g(\theta_n) \max_{n\le j<a(n,t)}\left\|\sum_{i=n}^{j}\alpha_i\zeta_i\right\| = 0
$$

[notice that $\lim_{n\to\infty} k(n) = \infty$]. Hence, (A.1), (A.2) hold. □

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