*The Annals of Applied Probability*
2017, Vol. 27, No. 2, 1296–1298
DOI: 10.1214/16-AAP1242


# ADDENDUM AND CORRIGENDUM TO "RANDOMIZED URN MODELS REVISITED USING STOCHASTIC APPROXIMATION"

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This is a short addendum and corrigendum to the paper "Randomized Urn Models revisited using Stochastic Approximation" published in *Annals of Applied Probability*.

The conclusions of items (b) and (c) of Theorem A.2 in Appendix A of [1] require slightly more stringent assumptions to be true. We thank L.-X. Zhang for pointing out this fact. We provide below an appropriate statement—based on recent results from his paper [3]—for our use in the core of the paper. Then we briefly inspect the (limited) consequences on our main results.

First, we introduce the *η-differentiability* of the vector field $h$ at $\theta^*$:

$$h(\theta) = h(\theta^*) + Dh(\theta^*)(\theta - \theta^*) + o(\|\theta - \theta^*\|^{1+\eta}) \quad \text{as } \theta \to \theta^* \text{ for some } \eta > 0. \tag{0.1}$$

Then we add in Assumption (A.3) from [1] a positive sequence $(v_n)_{n\geq 1}$ to be specified:

$$(n+1)v_n \mathbb{E}[\|r_{n+1}\|^2 \mathbb{1}_{\{\|\theta_n-\theta^*\|\leq\epsilon\}}] \longrightarrow 0 \qquad \text{as } n \to +\infty. \tag{0.2}$$

THEOREM A.2. *With the notations and under the hypothesis of Theorem* A.2 *of* [1], *assume furthermore that* $Dh(\theta^*)$ *diagonalizes and, for claims* (b) *and* (c), *that* $h$ *is* $\eta$*-differentiable at* $\theta^*$. *Let* $\lambda_{\min}$ *denote its eigenvalue with the lowest real part.*

(a) *If* $\Re e(\lambda_{\min}) > \frac{1}{2}$ *and and* (0.2) *holds with* $v_n = 1$, $n \geq 1$, *then, on the convergence set* $\{\theta_n \to \theta^*\}$,

$$\sqrt{n}(\theta_n - \theta^*) \xrightarrow[n\to+\infty]{\mathcal{L}} \mathcal{N}(0, \Sigma)$$
$$\text{where } \Sigma = \int_0^{+\infty} e^{-u(Dh(\theta^*)-\frac{\mathrm{Id}}{2})^t}\Gamma e^{-u(Dh(\theta^*)-\frac{\mathrm{Id}}{2})}\,du.$$

Received January 2016; revised July 2016.

(b) *If* $\Re e(\lambda_{\min}) = \frac{1}{2}$ *and* (0.2) *holds with* $v_n = \log n$, $n \geq 1$, *then, on the convergence set* $\{\theta_n \to \theta^*\}$,

$$\sqrt{\frac{n}{\log n}}(\theta_n - \theta^*) \xrightarrow[n\to+\infty]{\mathcal{L}} \mathcal{N}(0, \Sigma)$$

$$\textit{where } \Sigma = \lim_n \frac{1}{\log n} \int_0^{\log n} e^{-u(Dh(\theta^*)-\frac{\mathrm{Id}}{2})^t} \Gamma e^{-u(Dh(\theta^*)-\frac{\mathrm{Id}}{2})}\, du.$$

(c) *If* $\lambda_{\min} \in (0, \frac{1}{2})$ *and* (0.2) *holds with* $v_n = n^{2\lambda_{\min}-1+\varepsilon}$, $n \geq 1$, $(\varepsilon > 0)$, *then, on the convergence event* $\{\theta_n \to \theta^*\}$, $n^{\lambda_{\min}}(\theta_n - \theta^*)$ *a.s. converges towards a finite random variable as* $n \to +\infty$.

REMARK. The above assumption on $Dh(\theta^*)$ can been relaxed: to get (b) and (c), it suffices that all its Jordan blocks of $\lambda_{\min}$ have order 1. When all these orders are not equal to 1 or $\Lambda_{\min}$ is complex in (c), new rates are obtained (even in situations where $H$ is is itself random, see Theorem 2.1 in [3]). Thus, in item (b), if $\nu$ denotes the maximum size of Jordan blocks of $\lambda_{\min}$ then $\sqrt{\frac{n}{\log n}}$ should be replaced by $\frac{\sqrt{n}}{(\log n)^{\nu-\frac{1}{2}}}$ and $\frac{1}{\log n}$ by $\frac{1}{(\log n)^{2\nu-1}}$ in the definition of $\Sigma$.

As a consequence, in Theorem 2.2 from [1], the assumption that the limiting generating matrix $H$ diagonalizes should be added [both Assumption (0.2) and the $\eta$-differentiability are satisfied]. In fact, this property is satisfied by the randomized urn models investigated in [1], mainly because the transpose $H^t$ of the limiting generating matrix of interest is always reversible with respect to its invariant measure (its "first" left eigenvector $v^*$). Hence, our main results and their proofs remain true as stated (up to this additional condition in Theorem 2.2). For more details, we refer to [2].

In [2] (extended version of this note), we also prove the following precise and new results:

- *Spectrum of* $Dh(\theta^*)_{|\mathcal{V}_0^2}$ *in Theorem* 2.2: If we assume the limiting generating matrix $H$ diagonalizes (resp., in $\mathbb{R}$), so is the case of $Dh(\theta^*)_{|\mathcal{V}_0^2}$ and Theorem A.2 applies.
- *Spectrum of* $D\widetilde{h}(\widetilde{\theta}^*)_{|\mathcal{V}_0^2}$ *in Theorem* 3.1: The differential $D\widetilde{h}(\widetilde{\theta}^*)_{|\mathcal{V}_0^2\times\mathbb{R}^d}$ diagonalizes in $\mathbb{R}$.
- *Bai–Hu–Sen model in Section* 3.3: For this model, the limiting generating matrix reads

$$H = \left(p^i\delta_{ij} + \frac{p^i(1-p^j)}{\pi - p^j}(1-\delta_{ij})\right)_{1\leq i,j\leq d} \qquad \text{where } \pi = \sum_{i=1}^d p^i$$

  always diagonalizes in $\mathbb{R}$ since its transpose is reversible with respect to its invariant measure.

A more computational proof is possible when the $p^i$ are pairwise distinct which provides bounds for the eigenvalues. Thus, we can give a sufficient condition to get a standard $CLT$ for this randomized urn dynamics.

THEOREM 0.1. *Let $d \geq 2$ be an integer. The characteristic polynomial of the above $BHS$ generating matrix $H$ is given by*

$$\det(H - \lambda I_d) = \prod_{i=1}^{d} \big(p^i(1-a^i) - \lambda\big) + \sum_{i=1}^{d} p^i a^i \prod_{i \neq j} \big(p^j(1-a^j) - \lambda\big),$$

*where $a^i = \frac{1-p^i}{\pi - p^i}$, $i \in \{1, \dots, d\}$. In particular, if for every $i \neq j$, $p^i \neq p^j$, then $H$ has pairwise distinct real eigenvalues hence it is diagonalizable with a real-valued spectrum. Furthermore, the second highest eigenvalue $\lambda^H_{\max_2}$ of $H$ satisfies*

$$\lambda^H_{\max_2} < \max_{1 \leq i \leq d} \frac{p^i(1-p^i)}{\pi - p^i}.$$

A criterion for standard CLT follows from the condition $\lambda_{\min} = 1 - \lambda^H_{\max_2} > \frac{1}{2}$.

## REFERENCES

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