The Annals of Applied Probability
2017, Vol. 27, No. 3, 1678–1701
DOI: 10.1214/16-AAP1241

ON THE CAPACITY FUNCTIONAL OF THE INFINITE CLUSTER OF A BOOLEAN MODEL

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Consider a Boolean model in $\mathbb{R}^d$ with balls of random, bounded radii with distribution F_0, centered at the points of a Poisson process of intensity $t > 0$. The capacity functional of the infinite cluster Z_∞ is given by $\theta_L(t) = \mathbb{P}\{Z_\infty \cap L \neq \varnothing\}$, defined for each compact $L \subset \mathbb{R}^d$.

We prove for any fixed L and F_0 that $\theta_L(t)$ is infinitely differentiable in t, except at the critical value t_c; we give a Margulis–Russo-type formula for the derivatives. More generally, allowing the distribution F_0 to vary and viewing θ_L as a function of the measure $F := tF_0$, we show that it is infinitely differentiable in all directions with respect to the measure F in the supercritical region of the cone of positive measures on a bounded interval.

We also prove that $\theta_L(\cdot)$ grows at least linearly at the critical value. This implies that the critical exponent known as β is at most 1 (if it exists) for this model. Along the way, we extend a result of Tanemura [*J. Appl. Probab.* **30** (1993) 382–396], on regularity of the supercritical Boolean model in $d \geq 3$ with fixed-radius balls, to the case with bounded random radii.

1. Introduction. The Boolean model is a fundamental model of random sets in stochastic geometry; see Hall (1988), Meester and Roy (1996), Schneider and Weil (2008), Stoyan, Kendall and Mecke (1987). It is obtained by taking the union Z of a collection of (in general, random) compact sets (known as grains) centered on the points of a homogeneous Poisson process of intensity t in d-space. For a large class of grain distributions, it is known that for t above a critical value t_c that is dependent on the grain distribution, the resulting random set, denoted $Z(t)$, includes a unique infinite component, denoted $Z_\infty(t)$.

The random set $Z_\infty = Z_\infty(t)$ is an important and fascinating object of study. One way to investigate its distribution is through its *capacity functional*, defined as the set function $L \mapsto \theta_L(t) := \mathbb{P}\{Z_\infty(t) \cap L \neq \varnothing\}$, defined for compact $L \subset \mathbb{R}^d$. If L is a singleton, then $\theta(t) := \theta_{\{0\}}(t)$ is called the *volume fraction* of $Z_\infty(t)$, and in the case where the grains are all translates of a fixed set K_0 (e.g., a unit ball), $\theta_{K_0}(t)$ is (loosely speaking) the proportion of grains that lie in Z_∞. More generally,

Received January 2016; revised July 2016.

[1]Supported by the German Research Foundation (DFG) through the research unit "Geometry and Physics of Spatial Random Systems" under the Grant HU 1874/3-1.

MSC2010 subject classifications. Primary 60K35; secondary 60D05.

Key words and phrases. Continuum percolation, Boolean model, infinite cluster, capacity functional, percolation function, Reimer inequality, Margulis–Russo-type formula.

the capacity functional of a random set and, in particular, of Z_∞, determines its distribution; see Schneider and Weil (2008).

In this article, we investigate the capacity functional of Z_∞ as a function of the intensity t. We consider the case where the grains are balls with random radii with distribution F_0 for some probability measure F_0 on $\mathbb{R}_+$ with bounded support.

We show for any compact $L \subset \mathbb{R}^d$ that $\theta_L(t)$ is infinitely differentiable in t for $t > t_c$ (it is identically 0 for $t < t_c$), thereby adding to earlier results on continuity of $\theta_L(t)$, $t > t_c$, and give an explicit expression for the derivatives (Theorem 3.2). More generally, allowing F_0 to vary and viewing θ_L as a function of the measure $F := tF_0$, we show (Theorem 3.1) that it is infinitely differentiable in all directions with respect to the measure F in the supercritical region of the cone of positive measures on a bounded interval.

We also prove (in Theorem 3.4) that θ_L grows at least linearly in the right neighbourhood of the threshold t_c. This is similar behaviour to that of the percolation function in discrete percolation models; see Grimmett (1999), Chapter 5, and the references therein. See Duminil-Copin and Tassion (2015) for a recent alternative proof of the discrete result, under the assumption of nonpercolation at the critical point. It would be interesting to try to adapt this to the continuum.

In the course of proving the results mentioned above, we show (in Theorem 3.7) that if our Boolean model with random but bounded radii is supercritical in $\mathbb{R}^d$ for $d \geq 3$, then it is also supercritical in a sufficiently thick slab. Previously, only the case with fixed radii had been considered, although the analogous result in the lattice is well known [Grimmett and Marstrand (1990)]. Also noteworthy is the fact that our proof of Theorem 3.4 requires the continuum Reimer inequality [Gupta and Rao (1999)].

We believe that our methods could also be used to give smoothness of the n-point connectivity function as a function of t for $t > t_c$. We also expect similar methods to be applicable for more general grains. See Section 7 for further discussion.

2. Preliminaries. Let $d \in \mathbb{N}$ with $d \geq 2$. We shall be dealing with a stationary (spherical) Boolean model in $\mathbb{R}^d$ which is described by means of a (marked) point process. Consider the space $\mathbb{X} := \mathbb{R}^d \times \mathbb{R}_+$ (where $\mathbb{R}_+ := [0, \infty)$), equipped with the Borel σ-field $\mathcal{B}(\mathbb{X})$ and the space $\mathbf{N}$ of integer-valued locally finite measures φ on $\mathcal{B}(\mathbb{X})$. For $b \in (0, \infty)$, let $\mathbf{N}^b$ be the space of all $\varphi \in \mathbf{N}$ that are supported by $\mathbb{R}^d \times [0, b]$. Let $\mathcal{N}$ denote the smallest σ-algebra of subsets of $\mathbf{N}$ making the mappings $\varphi \mapsto \varphi(D)$ measurable for all measurable $D \subset \mathbb{X}$. It is often convenient to write $z \in \varphi$ instead of $\varphi(\{z\}) > 0$.

A *point process* on $\mathbb{X}$ is then a measurable mapping Φ from some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ into the measurable space $(\mathbf{N}, \mathcal{N})$. It is convenient to fix the mapping Φ and to consider for any locally finite measure μ on $\mathcal{B}(\mathbb{X})$ a probability measure $\mathbb{P}_\mu$ on $(\Omega, \mathcal{F})$ such that the distribution $\mathbb{P}_\mu\{\Phi \in \cdot\}$ of Φ is that of a *Poisson process* with *intensity measure* μ. This means that under $\mathbb{P}_\mu$ the point process Φ has independent increments, with $\Phi(D)$ Poisson distributed with mean $\mu(D)$, for each bounded $D \in \mathcal{B}(\mathbb{X})$. See, for example, Kallenberg (2002) or Last and Penrose (2017). Expectation under $\mathbb{P}_\mu$ is denoted by $\mathbb{E}_\mu$.

For $r > 0, x \in \mathbb{R}^d$, we denote by $B_r(x)$ the closed Euclidean ball of radius r centered at x. Also we write 0 for the origin of $\mathbb{R}^d$ and B_r for $B_r(0)$.

Any $\varphi \in \mathbf{N}$ is of the form $\sum_i \delta_{z_i} = \sum_i \delta_{(x_i,r_i)}$, where the Dirac measure δ_z at $z \in \mathbb{X}$ is defined by $\delta_z(D) = \mathbf{1}\{z \in D\}$ for every $D \in \mathcal{B}(\mathbb{X})$. We then define $Z(\varphi) := \bigcup_i B_{r_i}(x_i)$. The balls $B_{r_i}(x_i)$ are referred to as *grains*.

Connected components of $Z(\varphi)$ are called *clusters*. Given $\varphi \in \mathbf{N}$, let $Z_\infty(\varphi)$ denote the union of the unbounded connected components of $Z(\varphi)$, that is, of the infinite clusters.

In this paper, we deal with Poisson processes whose intensity measure is of the form $\mu(d(x,r)) := dx\,F(dr)$, where dx is the d-dimensional Lebesgue measure and F is a finite measure on $\mathbb{R}_+$ (not necessarily a probability measure). When μ is of this form, we shall write $\mathbb{P}_F$ for $\mathbb{P}_\mu$ and $\mathbb{E}_F$ for $\mathbb{E}_\mu$. Also let Π_F denote the distribution of Φ under $\mathbb{P}_F$, that is, the probability measure on $(\mathbf{N}, \mathcal{N})$ given by $\Pi_F(\cdot) = \mathbb{P}_F\{\Phi \in \cdot\}$. Set $|F| = F(\mathbb{R}_+)$, the total mass of F. Then $|F|$ is called the *density* (or *intensity*) of the Poisson process under $\mathbb{P}_F$. We shall assume that F has no atom at $\{0\}$; in any case the singletons do not contribute to percolation properties of Z we study here.

Let $\mathbf{M}$ (resp., $\mathbf{M}_1, \mathbf{M}_\pm$) denote the class of finite nonzero Borel measures (resp., probability measures and finite signed measures) F on $\mathbb{R}_+$ satisfying $F(\{0\}) = 0$. Given $b \in (0, \infty)$, we write $\mathbf{M}^b$ (resp., $\mathbf{M}_1^b$, $\mathbf{M}_\pm^b$) for the measures that are supported by $[0, b]$, that is, that satisfy $F((b, \infty)) = 0$. Let $\mathbf{M}^\sharp := \bigcup_{b\in(0,\infty)} \mathbf{M}^b$, the measures with bounded support. Likewise, set $\mathbf{M}_1^\sharp := \bigcup_{b\in(0,\infty)} \mathbf{M}_1^b$ and $\mathbf{M}_\pm^\sharp := \bigcup_{b\in(0,\infty)} \mathbf{M}_\pm^b$.

Let $F \in \mathbf{M}$. Under $\mathbb{P}_F$, the set $Z := Z(\Phi)$ is called a *Boolean model*. It can be constructed, alternatively, by first generating an infinite independent sequence $\{R_i\}$ from the probability distribution $F(\cdot)/|F|$, and then placing balls of the corresponding radii at the points $\{X_i\}$ of a homogeneous Poisson point process with intensity $|F|$ in $\mathbb{R}^d$. This equivalence stems from the independent marking property of a Poisson process; for more details, see, for example, Kallenberg (2002) or Last and Penrose (2017), Chapter 5.

The point process Φ is *stationary* under $\mathbb{P}_F$, which means that for all $x \in \mathbb{R}^d$ we have $\mathbb{P}_F\{T_x\Phi \in \cdot\} = \mathbb{P}_F\{\Phi \in \cdot\}$, where for any $\mu \in \mathbf{N}$, the measure $T_x\mu \in \mathbf{N}$ is defined by $T_x\mu(B \times C) := \mu((B + x) \times C)$, with $B + x := \{y + x : y \in B\}$. Hence, $Z(\Phi)$ is stationary as well, that is, $\mathbb{P}_F\{Z + x \in \cdot\}$ does not depend on $x, x \in \mathbb{R}^d$. Since $Z_\infty(\Phi) + x = Z_\infty(T_{-x}\Phi)$ for all $x \in \mathbb{R}^d$, $Z_\infty(\Phi)$ is also stationary.

The *volume fraction* of the Boolean model is the probability that Z covers a fixed point, for instance, the origin 0, or in other words, the proportion of space covered by grains:

$$\mathbb{P}_F\{0 \in Z\} = 1 - \exp\left[-\kappa_d \int r^d F(dr)\right],$$

where $\kappa_d := \pi^{d/2}/\Gamma(d/2+1)$ stands for the volume of a d-dimensional unit ball.

Under $\mathbb{P}_F$, the sets $Z(\Phi)$ and $Z_\infty := Z_\infty(\Phi)$ are almost surely closed. They are *random closed sets*, see Molchanov (2005) or Schneider and Weil (2008). Our primary object of study here is Z_∞. For each compact $L \subset \mathbb{R}^d$, let

$$\theta_L(F) := \mathbb{P}_F\{L \cap Z_\infty \neq \varnothing\},$$

so that $L \mapsto \theta_L(F)$ is the capacity functional of Z_∞ under $\mathbb{P}_F$. As mentioned in Section 1, the capacity functional determines the distribution of Z_∞. In particular, we set

$$\theta(F) := \theta_{\{0\}}(F) = \mathbb{P}_F\{0 \in Z_\infty\} = \mathbb{E}_F\big|Z_\infty \cap [0,1]^d\big|,$$

the volume fraction of Z_∞ under $\mathbb{P}_F$ (also called the *percolation function*).

By ergodicity [see Meester and Roy (1996)], if $\theta(F) > 0$ then $\mathbb{P}_F\{Z_\infty \neq \varnothing\} = 1$, and moreover the infinite cluster is $\mathbb{P}_F$-a.s. unique (i.e., Z_∞ has only one connected component); see Meester and Roy (1996), Theorem 3.6. In this case, we say that *percolation occurs*. Conversely, if $\theta(F) = 0$ then $\mathbb{P}_F\{Z_\infty \neq \varnothing\} = 0$. Setting $\mathbf{U}$ to be the class of $\varphi \in \mathbf{N}$ such that $Z(\varphi)$ has at most one unbounded component, we thus have

$$\mathbb{P}_F\{\Phi \in \mathbf{U}\} = 1, \qquad \text{for any } F \in \mathbf{M}. \tag{2.1}$$

Given $F \in \mathbf{M}$ (not necessarily a probability measure), consider the family of measures of the form $F^* = tF$ with $t > 0$. By a coupling argument, $\theta(tF)$ is nondecreasing in t. The *critical value* (or *percolation threshold*) $t_c(F)$ is the supremum of those t such that $\theta(tF) = 0$. If $\int r^d F(dr) < \infty$ (e.g., if $F \in \mathbf{M}^\sharp$)), then $0 < t_c(F) < \infty$; see Gouéré (2008). If $t_c(F) < 1$, we say that F is *strictly supercritical*.

It is known that $\theta(tF)$ is continuous in t at least for $t \neq t_c(F)$, and right-continuous for all t; see Meester and Roy (1996), Theorem 3.9. For $d = 2$, it is known [Meester and Roy (1996), Theorem 4.5] that $\theta(t_c(F)F) = 0$ [and, therefore, $\theta(tF)$ is continuous for all t], and this is commonly believed to be true for $d \geq 3$ also.

REMARK 2.1. For $r \geq 0$, the quantity $\mathbb{P}_F\{B_r \subset Z_\infty(\Phi + \delta_{(0,r)})\} = \mathbb{P}_F\{B_r \cap Z_\infty \neq \varnothing\} = \theta_{B_r}(F)$ can be interpreted as the conditional probability (under $\mathbb{P}_F$) that B_r belongs to the infinite cluster given that $(0, r)$ belongs to Φ. Therefore, $\int \theta_{B_r}(F)F(dr)/|F|$ is the conditional (Palm) probability that a typical grain (centered at the origin) is a part of the infinite cluster, that is, the proportion of grains belonging to the unbounded connected component.

Next, we describe two important properties of Poisson processes which we use in this paper. One is the *Mecke identity* [see, e.g., Last and Penrose (2017), Chapter 4]:

$$\mathbb{E}_\mu \int f(z, \Phi)\Phi(dz) = \mathbb{E}_\mu \int f(z, \Phi + \delta_z)\mu(dz) \tag{2.2}$$

for any measurable $f : \mathbb{X} \times \mathbf{N} \to \mathbb{R}_+$. This identity characterises the Poisson process.

Another important result is the *perturbation formula* for functionals of Poisson processes, an analogue of the Margulis–Russo formula for Bernoulli fields. For bounded measurable $f : \mathbf{N} \to \mathbb{R}$, and $z \in \mathbb{X}$, define $D_z f(\varphi) := f(\varphi + \delta_z) - f(\varphi)$, for all $\varphi \in \mathbf{N}$. For $n \geq 2$ and $(z_1, \dots, z_n) \in \mathbb{X}^n$ we define a function $D^n_{z_1,\dots,z_n} f : \mathbf{N}(\mathbb{X}) \to \mathbb{R}$ inductively by

$$D^n_{z_1,\dots,z_n} f := D_{z_1} D^{n-1}_{z_2,\dots,z_n} f. \tag{2.3}$$

The operator $D^n_{z_1,\dots,z_n}$ is symmetric in $z_1, \dots, z_n$; indeed, by induction

$$D^n_{z_1,\dots,z_n} f(\varphi) = \sum_{I \subset \{1,\dots,n\}} (-1)^{n-|I|} f\left(\varphi + \sum_{i \in I} \delta_{z_i}\right), \tag{2.4}$$

where $|I|$ denotes the number of elements of I.

PROPOSITION 2.2. *Let μ be a locally finite measure and ν a finite signed measure on $\mathcal{B}(\mathbb{X})$. Let $f : \mathbf{N} \to \mathbb{R}$ be measurable and bounded. If $\mu + a\nu$ is a measure for some $a > 0$, then*

$$\frac{d^+}{ds} \mathbb{E}_{\mu+s\nu} f(\Phi)\bigg|_{s=0} = \int \mathbb{E}_\mu D_z f(\Phi) \nu(dz). \tag{2.5}$$

If also $\mu - a\nu$ is a measure, then $\mathbb{E}_{\mu+s\nu} f(\Phi)$ is differentiable in s at $s = 0$.

The proof of this perturbation formula can be found in Zuev (1992), Theorem 2.1 (for the case $\nu = \mu$), for finite measures in Molchanov and Zuyev (2000), Theorem 2.1, and for locally finite measures and square-integrable functions in Last (2014). It may also be found in Last and Penrose (2017).

3. Main results.

3.1. *Smoothness of the capacity functional.* Our first result concerns differentiating the capacity functional θ_L with respect to the measure F. This can be useful to compare the percolation properties of different radius distributions. For example, in Gouéré and Marchand (2011) and in Meester, Roy and Sarkar (1994) the percolation threshold for F a Dirac measure (i.e., balls of fixed radius) has been compared with the percolation threshold for F the sum of two Dirac measures (i.e., for balls of random radius with just two possible values), or with more general F. Gouéré and Marchand (2011) show that in sufficiently high dimensions the Dirac measure does not minimise the critical volume fraction (as had been previously conjectured) but do not quantify the phrase “sufficiently high” and do not rule out the possibility that the Dirac measure minimises the critical volume fraction in low dimensions. With sufficient analytic tools, it might be possible to compare different radius distributions (perhaps with the same volume fraction) by calculus. For example, we could compare two measures F_1 and F_2 by passing continuously from one to the other.

Our result gives the directional derivative for $\theta_L(F)$ as we vary F. If we wish to keep the total measure (i.e., the density) constant, then we need to add to F a *signed* measure with total measure zero. More generally, we may consider adding an arbitrary signed measure G to F. We use notation for the classes of measures from Section 2 and D^n from (2.3).

THEOREM 3.1. *Suppose that $F \in \mathbf{M}^\sharp$ with $t_c(F) < 1$, and $G \in \mathbf{M}^\sharp_\pm$ is such that $F + aG$ is a measure for some $a > 0$. Let $L \subset \mathbb{R}^d$ be compact. Then*

$$\begin{aligned} &\frac{d^+}{dh}\theta_L(F+hG)\bigg|_{h=0} \\ &\quad = \iint \mathbb{P}_F\{L \cap Z_\infty(\Phi + \delta_{(x,r)}) \neq \varnothing, L \cap Z_\infty(\Phi) = \varnothing\} G(dr)\,dx, \end{aligned} \tag{3.1}$$

and the right-hand side of (3.1) is finite. If also $F - aG$ is a measure, then $\theta_L(F + hG)$ is infinitely differentiable in a neighbourhood of $h = 0$, then setting $\tilde{f}_L(\varphi) = \mathbf{1}\{L \cap Z_\infty(\varphi) \neq \varnothing\}$ for $\varphi \in \mathbf{N}$, for all $n \in \mathbb{N}$ we have

$$\begin{aligned} &\frac{d^n}{dh^n}\theta_L(F+hG)\bigg|_{h=0} \\ &\quad = \int \cdots \int \mathbb{E}_F D^n_{(x_1,r_1),\dots,(x_n,r_n)} \tilde{f}_L(\Phi)\,dx_1 G(dr_1) \cdots dx_n G(dr_n). \end{aligned} \tag{3.2}$$

We shall prove Theorem 3.1 in Section 5. The identity (3.1) tells us that the perturbation formula (2.5) remains valid for $f(\varphi) = \mathbf{1}\{L \cap Z_\infty(\varphi) \neq \varnothing\}$ with $\mu(dx\,dr) = dx\,F(dr)$ and $\nu(dx\,dr) = dx\,G(dr)$, though in this case both μ and ν are *infinite* (but σ-finite).

Our next theorem is a corollary of Theorem 3.1, and significantly adds to the known results mentioned in Section 2 concerning continuity of $\theta_L(tF)$ for fixed F, in the case of deterministically bounded radii. Recall that the *Minkowski difference* $A \ominus B$ of two sets $A, B \subset \mathbb{R}^d$ is defined by $\{x - y : x \in A, y \in B\}$. When $B = \{x\}$ for some $x \in \mathbb{R}^d$, we write simply $A - x$ for $A \ominus \{x\}$.

THEOREM 3.2. *Let $F \in \mathbf{M}^\sharp_1$, and let $L \subset \mathbb{R}^d$ be compact. Then $t \mapsto \theta_L(tF)$ is infinitely differentiable on $(t_c(F), \infty)$ and setting $\tilde{f}_L(\varphi) = \mathbf{1}\{L \cap Z_\infty(\varphi) \neq \varnothing\}$ for all $\varphi \in \mathbf{N}$, we have for $t > t_c(F)$ and $n \in \mathbb{N}$ that*

$$\begin{aligned} &\frac{d^n\theta_L(tF)}{dt^n} \\ &\quad = \int \cdots \int \mathbb{E}_{tF} D^n_{(x_1,r_1),\dots,(x_n,r_n)} \tilde{f}_L(\Phi)\,dx_1 F(dr_1) \cdots dx_n F(dr_n). \end{aligned} \tag{3.3}$$

In particular, for $t > t_c(F)$ we have

$$\frac{d}{dt}\theta_L(tF)$$

$$= \iint \mathbb{P}_{tF}\{L \cap Z_\infty(\Phi + \delta_{(x,r)}) \neq \varnothing, L \cap Z_\infty(\Phi) = \varnothing\}\, dx\, F(dr) \tag{3.4}$$

$$= \mathbb{E}_{tF} \int \big|\big(Z_\infty(\Phi + \delta_{(0,r)}) \ominus L\big) \setminus \big(Z_\infty(\Phi) \ominus L\big)\big| F(dr). \tag{3.5}$$

PROOF. Let $L \subset \mathbb{R}^d$ be compact. The infinite differentiability of $\theta_L(tF)$, and the formula (3.3) for $\frac{d^n}{dt^n}\theta_L(tF)$, follow from applying Theorem 3.1 to the measures F^* and G^* given by $F^* = tF$ and $G^* = F$; also (3.4) follows as a special case of (3.3). It remains to prove (3.5). By stationarity, $T_x\Phi$ has the same distribution as Φ under $\mathbb{P}_{tF}$, so the right-hand side of (3.4) equals

$$\mathbb{E}_{tF} \iint \mathbf{1}\{(L-x) \cap Z_\infty(\Phi + \delta_{(0,r)}) \neq \varnothing, (L-x) \cap Z_\infty(\Phi) = \varnothing\}\, dx\, F(dr)$$
$$= \mathbb{E}_{tF} \iint \mathbf{1}\big\{x \in \big(L \ominus Z_\infty(\Phi + \delta_{(0,r)})\big) \setminus \big(L \ominus Z_\infty(\Phi)\big)\big\}\, dx\, F(dr),$$

and then (3.5) follows from the fact that for any Borel sets A, B, L we have $|(L \ominus A) \setminus (L \ominus B)| = |(A \ominus L) \setminus (B \ominus L)|$. $\square$

REMARK 3.3. Making use of the Mecke identity (2.2), we can also rewrite (3.4) as follows [see also Zuev (1992)]:

$$\frac{d}{dt}\theta_L(tF) = t^{-1}\mathbb{E}_{tF} \int \mathbf{1}\{L \cap Z_\infty(\Phi) \neq \varnothing, L \cap Z_\infty(\Phi - \delta_{(x,r)}) = \varnothing\}\Phi\big(d(x,r)\big).$$

3.2. *Bounds for the capacity functional.* Our next result provides a lower bound for the capacity functional of the infinite cluster. This bound is linear in the right neighbourhood of the critical value.

It is known for lattice percolation models that the percolation function grows at least linearly in the right neighbourhood of the threshold; see Chayes and Chayes (1986), or Grimmett (1999) and the references therein. Our result shows that this also holds for the spherical Boolean model.

THEOREM 3.4. *Let $b > 0$. Let $F \in \mathbf{M}_1^b$ and let $L \subset \mathbb{R}^d$ be compact. Set $t_c := t_c(F)$ and $\alpha := \mathbb{P}_{t_cF}\{B_b \subset Z(\Phi)\}$. Then*

$$\theta_L(tF) - \theta_L(t_cF) \geq \frac{\alpha(t-t_c)(1-\theta_L(tF))}{t}, \qquad t > t_c. \tag{3.6}$$

Furthermore,

$$\frac{\theta_L(tF) - \theta_L(t_cF)}{t - t_c} \geq \frac{\alpha(1-\theta_L(t_cF))}{t_c} + o(1) \qquad \textit{as } t \downarrow t_c. \tag{3.7}$$

Theorem 3.4 is proved in Section 6.

The bounds (3.6)–(3.7) also hold for the integrated percolation functions $\int \theta_{B_r}(tF)F(dr)$; see Remark 2.1. For a given F, an explicit numerical lower bound for the right-hand side of (3.6) can be established by using the inequality:

$$1 - \theta_L(tF) \geq \mathbb{P}_{tF}\{L \cap Z = \varnothing\} = \exp\left[-t \int |B_r \ominus L| F(dr)\right]$$

and applying a numerical estimation method for t_c such as that in Zuev and Sidorenko (1985), for example. Also, it is not difficult to estimate α (the probability that B_b is fully covered) explicitly from below.

REMARK 3.5. Although the capacity functional $t \mapsto \theta_L(tF)$ is believed to be continuous at the critical value t_c, it is certainly not differentiable there. Indeed, if it is continuous, then $\theta_L(t_c F) = 0$ and the left-hand derivative of $t \mapsto \theta_L(tF)$ at t_c vanishes. But Theorem 3.4 implies that the right-hand derivative, if it exists, is strictly positive.

REMARK 3.6. Given the common belief for discrete percolation [see Grimmett (1999)], one might conjecture that $\theta_L(tF) - \theta_L(t_c F) \sim (t - t_c)^\beta$ as $t \downarrow t_c$ (at least in a logarithmic sense) for some *critical exponent* $\beta > 0$. If this holds, Theorem 3.4 implies $\beta \leq 1$.

3.3. *Percolation in a slab when $d \geq 3$.* Given $K > 0$, let $S(K)$ denote the slab $[0, K] \times \mathbb{R}^{d-1}$. An important result of Grimmett and Marstrand (1990) says that for Bernoulli lattice percolation if the parameter p is supercritical in $\mathbb{Z}^d$, with $d \geq 3$, then for sufficiently large K the parameter p is also supercritical for the model restricted to a sufficiently large slab in $\mathbb{Z}^d$.

To prove our results in the case $d \geq 3$, we need to adapt this result to the Boolean model. In the case where the balls have fixed radius, this was done in Tanemura (1993), and we now describe an extension to balls of random radius. This could potentially be of use elsewhere.

Given $F \in \mathbf{M}$, let us denote by Φ_F a Poisson process in $\mathbb{X}$ with distribution Π_F, that is, with $\mathbb{P}\{\Phi_F \in \cdot\} = \mathbb{P}_F\{\Phi \in \cdot\}$. Given also any measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$, let us denote by $\Phi_{F, f(\rho)}$ the image of Φ_F under the mapping $\sum_i \delta_{(x_i, r_i)} \mapsto \sum_i \delta_{(x_i, f(r_i))}$. Thus, $\Phi_{F, f(\rho)}$ has the same distribution as $\Phi_{F \circ f^{-1}}$; it will be convenient for us to mention ρ in the notation, representing the radius of a ball in the system. For $\varphi \in \mathbf{N}$ and $A \in \mathcal{B}(\mathbb{X})$, let $\varphi|_A$ denote the restriction of φ to A, that is, $\varphi|_A(\cdot) = \varphi(\cdot \cap A)$. Finally, for $B \subset \mathbb{R}^d$ write $[B]$ for $B \times \mathbb{R}_+$.

THEOREM 3.7. *Suppose $d \geq 3$ and let $F \in \mathbf{M}^\sharp$ with $t_c(F) < 1$. Then there exists $K < \infty$ such that $\mathbb{P}_F\{Z_\infty(\Phi|_{[S(K)]}) \neq \varnothing\} = 1$, and*

$$\inf_{x \in S(K)} \mathbb{P}_F\{x \in Z_\infty(\Phi|_{[S(K)]})\} > 0. \tag{3.8}$$

PROOF. By assumption, $F(\{0\}) = 0$. By Meester and Roy (1996), Theorem 3.7, for any $b > 0$ the value of $t_c(F')$ depends continuously (in the weak topology) on $F' \in \mathbf{M}_1^b$, so one can show that there exists $a > 0$ with $t_c(F|_{[a,\infty)}) < 1$, where $F|_{[a,\infty)}$ denotes the restriction of the measure F to the interval $[a, \infty)$. Since there exist coupled Poisson point processes Φ, Φ' having distribution Π_F and $\Pi_{F|_{[a,\infty)}}$, respectively, with $\Phi' \leq \Phi$ almost surely, it suffices to prove (3.8) using the measure $F|_{[a,\infty)}$ rather than F. In other words, we may assume without loss of generality that there exists $a > 0$ with $F([0, a)) = 0$, and then by scaling [see Meester and Roy (1996)] we can (and now do) assume $a = 1$.

For $i = 3, 4, 5$ choose t_i with $t_c(F) < t_3 < t_4 < t_5 < 1$. Then $Z_\infty(\Phi_{t_3 F}) \neq \varnothing$ almost surely, so that by scaling, there exists $\delta > 0$ with $1/\delta \in \mathbb{N}$ such that also $Z_\infty(\Phi_{t_4 F,(1-\delta)\rho}) \neq \varnothing$ almost surely and, therefore, also $Z_\infty(\Phi_{t_4 F,\rho-\delta}) \neq \varnothing$ since almost surely $\rho \geq 1$ so that $(1 - \delta)\rho \leq \rho - \delta$.

Set $\lfloor\rho\rfloor_\delta := \delta\lfloor\rho/\delta\rfloor$, that is, the value of ρ rounded down to the nearest integer multiple of δ. Then $\lfloor\rho\rfloor_\delta \geq \rho - \delta$, so that $Z_\infty(\Phi_{t_4 F,\lfloor\rho\rfloor_\delta}) \neq \varnothing$ almost surely. Note that since $1/\delta \in \mathbb{N}$ we have $\lfloor\rho\rfloor_\delta \geq 1$ almost surely. By further scaling, we can (and do) choose $\varepsilon > 0$ such that $Z_\infty(\Phi_{t_5 F,(1-2\varepsilon)\lfloor\rho\rfloor_\delta}) \neq \varnothing$ almost surely.

Now let $\eta = \varepsilon/(2d)$. Divide $\mathbb{R}^d$ into half-open cubes denoted $Q_z, z \in \mathbb{Z}^d$, where Q_z has side length η and is centered at ηz. For $x \in \mathbb{R}^d$, let $\langle x\rangle_\eta$ denote the point at the center of the cube Q_z containing x. For $\varphi = \sum_i \delta_{(x_i,r_i)} \in \mathbf{N}$, let $\langle\varphi\rangle_\eta := \sum_i \delta_{(\langle x_i\rangle_\eta,r_i)}$ (this counting measure can have multiplicities). Since $|\langle x\rangle_\eta - x| \leq d\eta/2$ for all $x \in \mathbb{R}^d$, and since η is chosen so that $d\eta < \varepsilon$, we have that $Z_\infty(\langle\Phi_{t_5 F,(1-\varepsilon)\lfloor\rho\rfloor_\delta}\rangle_\eta) \neq \varnothing$ almost surely.

For $t > 0$, the occurrence of $Z_\infty(\langle\Phi_{tF,(1-\varepsilon)\lfloor\rho\rfloor_\delta}\rangle_\eta) \neq \varnothing$ is equivalent to existence of an infinite cluster in the following Bernoulli site percolation model on $\mathbb{Z}^d \times \{1, 2, \ldots, \kappa\}$ for some $\kappa \in \mathbb{N}$. Let $r_1, \ldots, r_\kappa$ denote the possible values for $\lfloor\rho\rfloor_\delta$ (where ρ has distribution $F(\cdot)/|F|$), listed in increasing order. For $1 \leq i \leq \kappa$ set $\pi_i := \mathbb{P}\{(1 - \varepsilon)\lfloor\rho\rfloor_\delta = r_i\}$. For $y, z \in \mathbb{Z}^d$ and $i, j \in \{1, \ldots, \kappa\}$, put $(y, i) \sim (z, j)$ if and only if $|\eta y - \eta z| \leq r_i + r_j$. Let each site (z, i) be occupied with probability $p_{t,i}$, where we put $p_{t,i} = 1 - \exp(-t|F|\eta^d \pi_i)$, independent of the other sites. Note that $(p_{1,i})_{i\leq\kappa}$ is supercritical, in that it strictly exceeds (in each entry) the vector $(p_{t_5,i})_{i\leq\kappa}$ which also percolates.

By the result of Grimmett and Marstrand (1990) adapted to this site percolation model, there is a choice of K such that $Z_\infty(\langle\Phi_{F,(1-\varepsilon)\lfloor\rho\rfloor_\delta}|_{[S(K)]}\rangle_\eta) \neq \varnothing$ almost surely. Therefore, since $d\eta \leq \varepsilon$ we have $Z_\infty(\Phi_F|_{[S(K)]}) \neq \varnothing$ almost surely. We may argue similarly to obtain (3.8), following the proof of Lemma 10.8 in Penrose (2003) with obvious modifications.

Let us describe how to adapt some of the steps of Grimmett and Marstrand (1990) to the site percolation model above. In Lemma 2 of Grimmett and Marstrand (1990), we may replace $(1 - p)^t$ by $(1 - \max_i p_{\lambda_5,i})^t$.

In Lemma 3 of Grimmett and Marstrand (1990), instead of the box B_m consider the box $B_{2m\lceil r_\kappa\rceil}$. Also the bound $(1 - p)^{dk}$ would be replaced by $(1 -$

$\max_i p_{\lambda,i})^{-\kappa k B}$ where B denotes the number of sites of $\mathbb{Z}^d$ at distance at most $2r_\kappa$ from the origin.

In Lemma 4 of Grimmett and Marstrand (1990), we let $T(n)$ denote the set of sites (z, i) lying in $B_n \cap \partial((-\infty, n] \times \mathbb{Z}^{d-1}) \times \{1, \dots, \kappa\}$ with all coordinates of z being nonnegative. $\square$

4. A preparatory result. In this section, we present further notation followed by a key lemma (Lemma 4.2) which will be used repeatedly in the proof of Theorems 3.4 and 3.1. For $A \subset \mathbb{R}^d$ and $\varphi \in \mathbf{N}$, let $Z_A(\varphi)$ be the union of all the clusters of $Z(\varphi)$ which have a nonempty intersection with A. In other words, set

$$Z_A(\varphi) := \bigcup_{i : x_i \leftrightarrow_\varphi A} B_{r_i}(x_i), \qquad \text{for } \varphi = \sum_i \delta_{(x_i, r_i)}, \tag{4.1}$$

where $x \leftrightarrow_\varphi A$ means that x lies in a component of $Z(\varphi)$ which intersects A. Also, set $\varphi^{\text{fin}} := \sum_{(x_i,r_i)\in\varphi} \mathbf{1}\{x_i \notin Z_\infty(\varphi)\}\delta_{(x_i,r_i)}$.

Given $b \in (0, \infty)$, for $\varphi, \varphi' \in \mathbf{N}^b$ and for compact $K \subset \mathbb{R}^d$, define $Z_K(\varphi, \varphi')$ to be the union of all the clusters (connected components) of $Z(\varphi^{\text{fin}} + \varphi')$ that intersect K, but do not intersect $Z_\infty(\varphi)$. In particular, with 0 denoting the zero measure we have

$$Z_K(\varphi, 0) = \bigcup_{i : x_i \notin Z_\infty(\varphi), x_i \leftrightarrow_\varphi K} B_{r_i}(x_i), \qquad \text{for } \varphi = \sum_i \delta_{(x_i, r_i)}. \tag{4.2}$$

Recall that $\varphi|_A$ denotes the restriction of $\varphi \in \mathbf{N}$ to $A \in \mathcal{B}(\mathbb{X})$ and $[B] = B \times \mathbb{R}_+$. Define the following "radius of stabilization":

$$R_{K,b}(\varphi, \varphi') := \inf\{nb : n \in \mathbb{N}, K \cup Z_K(\varphi, \varphi') \subset B_{(n-1)b}\}, \tag{4.3}$$

or $R_{K,b}(\varphi, \varphi') := +\infty$ if there is no such n. Write simply R_K for $R_{K,1}$. Note that if $R_{K,b}(\varphi, \varphi') = nb$ for some $n \in \mathbb{N}$, then for any $\psi \in \mathbf{N}^b$ we have

$$Z_K(\varphi, \varphi'|_{[B_{nb}]} + \psi|_{[\mathbb{R}^d \setminus B_{nb}]}) = Z_K(\varphi, \varphi'|_{B_{[nb]}}), \tag{4.4}$$

which is the stabilization property in the present context, and also

$$R_{K,b}(\varphi, \varphi'|_{[B_{nb}]} + \psi|_{[\mathbb{R}^d \setminus B_{nb}]}) = R_{K,b}(\varphi, \varphi'|_{[B_{nb}]}), \tag{4.5}$$

which is the *stopping radius* property of $R_{K,b}$. The notions of stabilization and of stopping radius have proved fruitful in many other stochastic-geometrical contexts; see, for example, Penrose (2007).

For $F, G \in \mathbf{M}$, we denote by Φ_F, Φ'_G a pair of independent Poisson process with distribution Π_F and Π_G, respectively, that is, with $\mathbb{P}\{\Phi_F \in \cdot\} = \mathbb{P}_F\{\Phi \in \cdot\}$ and $\mathbb{P}\{\Phi'_G \in \cdot\} = \mathbb{P}_G\{\Phi \in \cdot\}$.

LEMMA 4.1. *Suppose* $d \geq 3$, $b > 0$ *and* $F \in \mathbf{M}^b$, $G \in \mathbf{M}^b_{\pm}$ *and* $\varepsilon \in (0,1)$ *are such that* $F + \varepsilon G \in \mathbf{M}$, *and* $1 - \varepsilon > t_c(F)$. *Then there exists* $K \in (2b, \infty)$ *and* $\gamma \in (0, 1/2)$ *such that for all Borel* $A \subset \mathbb{R}^d$, *and* $h \in (0, \varepsilon^2/2]$,

$$\begin{aligned} &\max\bigl(\mathbb{P}\bigl\{Z_A(\Phi_{F+hG}|_{[S(K)]}) = \varnothing\bigr\}, \\ &\qquad \mathbb{P}\bigl\{Z_\infty(\Phi_{(1-\varepsilon)F}|_{[S(K)]}) \cap Z_A\bigl(\Phi'_{\varepsilon F+hG}|_{[S(K)]}\bigr) \neq \varnothing\bigr\}\bigr) \geq \gamma. \end{aligned} \tag{4.6}$$

PROOF. Choose K as in Theorem 3.7. Assume without loss of generality that $K \geq 2b$.

Suppose $\mathbb{P}\{Z_A(\Phi_{F+hG}|_{[S(K)]}) \neq \varnothing\} \geq 1/2$ [otherwise (4.6) is immediate]. Since $\mathbb{P}\{Y > 0\} \geq p\mathbb{P}\{X > 0\}$ for any $p \in (0,1)$ and Poisson variables X, Y with $\mathbb{E}Y = p\mathbb{E}X$ (by Bernoulli's inequality), also $\mathbb{P}\{Z_A(\Phi_{(\varepsilon/2)(F+hG)}|_{[S(K)]}) \neq \varnothing\} \geq \varepsilon/4$. For $0 \leq h \leq \varepsilon^2/2$, we have

$$\varepsilon F + hG - (\varepsilon/2)(F + hG) = (\varepsilon/2)\bigl[F + (2h/\varepsilon)(1 - \varepsilon/2)G\bigr] \in \mathbf{M}.$$

Hence, also $\mathbb{P}\{Z_A(\Phi'_{\varepsilon F+hG}|_{[S(K)]}) \neq \varnothing\} \geq \varepsilon/4$. Given $Z_A(\Phi'_{\varepsilon F+hG}|_{[S(K)]}) \neq \varnothing$, the set $Z_A(\Phi'_{\varepsilon F+hG}|_{[S(K)]})$ has a nonempty intersection with $S(K)$, and therefore by Theorem 3.7 and our choice of K, the conditional probability that this set intersects with $Z_\infty(\Phi_{(1-\varepsilon)F}|_{[S(K)]})$ is bounded below by a strictly positive constant γ_1. Hence, we have (4.6) with $\gamma = \gamma_1\varepsilon/4$. □

Recall from Section 2 that $\mathbf{M}^\sharp$ denotes the class of measures on $(0, \infty)$ with bounded support. We now give the main result of this section.

LEMMA 4.2. *Suppose that* $F \in \mathbf{M}^\sharp$, $G \in \mathbf{M}^\sharp_{\pm}$, *and* $\varepsilon \in (0,1)$ *are such that* $t_c(F) < 1 - \varepsilon$ *and* $F + \varepsilon G$ *is a measure. Then for any compact* $L \subset \mathbb{R}^d$, *we have*

$$\limsup_{n\to\infty} \sup_{0 \leq h \leq \varepsilon^2/2} n^{-1} \log \mathbb{P}\bigl\{R_{L,b}\bigl(\Phi_{(1-\varepsilon)F}, \Phi'_{\varepsilon F+hG}\bigr) > n\bigr\} < 0. \tag{4.7}$$

Also, with 0 *denoting the zero measure*,

$$\limsup_{n\to\infty} n^{-1} \log \mathbb{P}\bigl\{R_{L,b}(\Phi_F, 0) > n\bigr\} < 0. \tag{4.8}$$

The ε^2 in the range of h in (4.7) arises because we need $h \leq \varepsilon^2$ to guarantee that $\varepsilon F + hG$ is a measure. The fact that it is $\varepsilon^2/2$ rather than ε^2 in (4.7) is an artefact of the proof.

PROOF OF LEMMA 4.2. Let F, G, ε be as in the statement of Lemma 4.2. First suppose $d = 2$. Using Corollary 4.1 of Meester and Roy (1996) as a starting-point, we can adapt the proof of Lemma 10.5 of Penrose (2003) to random radius balls, thereby showing that the probability that $Z(\Phi_{(1-\varepsilon)F})$ fails to cross the rectangle $[0, 3a] \times [0, a]$ decays exponentially in a.

Given $a > 0$, specify a sequence $D_1(a), D_2(a), \dots$ of rectangles of aspect ratio 3, alternating between horizontal and vertical rectangles, with $D_1(a) = [0, 3a] \times [0, a]$, and with $D_n(a)$ crossing $D_{n+1}(a)$ the short way for each n. By the union bound and the exponential decay just mentioned, the probability that for some n there is no long-way crossing of $D_n(a)$ in $Z(\Phi_{(1-\varepsilon)F})$ decays exponentially in a. Hence, the probability that $Z_\infty(\Phi_{(1-\varepsilon)F})$ fails to include a long-way crossing of $[0, 3a] \times [0, a]$ is exponentially decaying in a. Likewise for the vertical rectangle $[0, a] \times [0, 3a]$.

Let E_a denote the event that $Z_\infty(\Phi_{(1-\varepsilon)F})$ includes long-way crossings of each of the rectangles $[-3a/2, -a/2] \times [-3a/2, 3a/2]$, $[a/2, 3a/2] \times [-3a/2, 3a/2]$, $[-3a/2, 3a/2] \times [-3a/2, -a/2]$ and $[-3a/2, 3a/2] \times [a/2, 3a/2]$ (whose union is the annulus $[-3a/2, 3a/2]^2 \setminus (-a/2, a/2)^2$). By the preceding discussion, $1 - \mathbb{P}[E_a]$ decays exponentially in a.

If E_a occurs for some a large enough so that $L \subset [-a/2, a/2]^2$, then for any $h \ge 0$ and $u > 6a + b$ the set $Z_L(\Phi_{(1-\varepsilon)F}, \Phi'_{\varepsilon F+hG}|_{[B_u]})$ is necessarily contained in the square $[-3a, 3a]^2$. The case $d = 2$ of (4.7) follows.

Now consider $d \ge 3$. Suppose $0 \le h \le \varepsilon^2/2$. Choose $b \in (0, \infty)$ with $F \in \mathbf{M}^b$ and $G \in \mathbf{M}^b_\pm$. Set $\Phi''_{F+hG} := \Phi_{(1-\varepsilon)F} + \Phi'_{\varepsilon F+hG}$. Recall the definition (4.3) of $R_{K,b}$. For $u > 0$, set

$$N_u = \int_{[L\ominus B_b]} \mathbf{1}\{R_{\{x\},b}(\Phi_{(1-\varepsilon)F}, \Phi'_{\varepsilon F+hG}) > u\}\Phi''_{F+hG}(d(x, r)).$$

If $R_{L,b}(\Phi_{(1-\varepsilon)F}, \Phi'_{\varepsilon F+hG}) > u$, then $N_u \ge 1$, so by Markov's inequality

$$\mathbb{P}\{R_{L,b}(\Phi_{(1-\varepsilon)F}, \Phi'_{\varepsilon F+hG}) > u\} \le \mathbb{E}N_u.$$

Define the event $A_{x,r,u} := \{R_{\{x\},b}(\Phi_{(1-\varepsilon)F}, \Phi'_{\varepsilon F+hG} + \delta_{(x,r)}) > u\}$, for $(x, r) \in \mathbb{X}$ and $u > 0$. We assert that for $u > 1$ we have

$$\mathbb{E}N_u \le \iint \mathbb{P}(A_{x,r,u})(F + hG)(dr)\,dx.$$

To see this, let us write simply Φ for $\Phi_{(1-\varepsilon)F}$, Φ' for $\Phi'_{\varepsilon F+hG}$ and Φ'' for Φ''_{F+hG} (so $\Phi'' = \Phi + \Phi'$). Suppose (x, r) is a point of Φ'' that contributes to N_u. Then $B_r(x) \cap Z_\infty(\Phi) = \varnothing$ [otherwise $R_{\{x\},b}(\Phi, \Phi')$ would be zero]. Moreover, $B_r(x)$ lies in a component of $Z(\Phi^{\mathrm{fin}} + \Phi')$ that avoids $Z_\infty(\Phi)$ and is not contained in $B_{ub}(x)$. Let Ψ (resp., Ψ') be the point process Φ (resp., Φ') with the point (x, r) removed (if it is a part of the point process in the first place). Then $Z_\infty(\Psi) = Z_\infty(\Phi)$, and there exists a component of $Z(\Psi^{\mathrm{fin}} \cup \Psi')$ that avoids $Z_\infty(\Psi)$, intersects $B_r(x)$, and is not contained in $B_{ub}(x)$ (at least if $u > 1$). But this conclusion just says that event $A_{x,r,u}$ occurs if we identify Ψ, Ψ' with Φ, Φ', respectively. Therefore, using the Mecke formula gives us the asserted inequality.

Thus, to prove (4.7) it suffices to prove that $\mathbb{P}(A_{x,r,u})$ decays exponentially in u, uniformly over $x \in L \ominus B_b$, $h \in [0, \varepsilon^2/2]$ and $r \in (0, b]$. We now fix such x, h, r. Let K be as in Lemma 4.1 and choose $n_0 \in \mathbb{N}$ with $L \ominus B_{2b} \subset [-n_0K, n_0K]^d$.

For $n \in \mathbb{Z}$ let S_n denote the slab $((n-1)K, nK] \times \mathbb{R}^{d-1}$ and let H_n denote the half-space $\bigcup_{-\infty<m\le n} S_m$. Given u with $L \subset B_{u-b}$ and $n \ge n_0$, for $\varphi, \varphi' \in \mathbf{N}^b$ set

$$W_{u,n}(\varphi, \varphi') = Z_{\{x\}}(\varphi|_{[H_n]}, \varphi'|_{[H_n \cap B_u]})$$

and define the indicator functions:

$$f_{u,n}(\varphi, \varphi') := \mathbf{1}\{Z((\varphi+\varphi')|_{[S_{n+1}]} \cap W_{u,n}(\varphi,\varphi')) \neq \varnothing\};$$
$$g_{u,n}(\varphi, \varphi') := \mathbf{1}\{Z_\infty(\varphi|_{[S_{n+1}]}) \cap W_{u,n}(\varphi,\varphi') = \varnothing\};$$
$$h_{u,n}(\varphi, \varphi') := \mathbf{1}\{Z_{\{x\}}(\varphi, \varphi'|_{[B_u]}) \setminus H_n \neq \varnothing\}.$$

Then we claim that $f_{u,n+1}(\varphi,\varphi') \le f_{u,n}(\varphi,\varphi') g_{u,n}(\varphi,\varphi')$. Indeed,

$$\text{if } f_{u,n}(\varphi,\varphi') = 0, \quad \text{then } W_{u,n+1}(\varphi,\varphi') = W_{u,n}(\varphi,\varphi') \subset (-\infty, nK+b] \times \mathbb{R}^{d-1}, \tag{4.9}$$

while if $g_{u,n}(\varphi,\varphi') = 0$ then $W_{u,n+1}(\varphi,\varphi') = \varnothing$, and in both cases it is not possible for $Z((\varphi+\varphi')|_{[S_{n+2}]})$ to intersect with $W_{u,n+1}$.

Also $h_{u,n+1}(\varphi,\varphi') \le f_{u,n}(\varphi,\varphi')$ by (4.9). Therefore, for $n = n_0+2, n_0+3, \dots$ we have

$$h_{u,n}(\varphi,\varphi') \le \prod_{m=n_0}^{n-1} f_{u,m}(\varphi,\varphi') \le \prod_{m=n_0}^{n-2} f_{u,m}(\varphi,\varphi') g_{u,m}(\varphi,\varphi'). \tag{4.10}$$

Denote by $\mathcal{F}_n$ the σ-field generated by $(\Phi_{(1-\varepsilon)F}|_{[H_n]}, \Phi'_{\varepsilon F+hG}|_{[H_n]})$. If the conditional expectation of $f_{u,n}(\Phi_{(1-\varepsilon)F}, \Phi'_{\varepsilon F+hG} + \delta_{(x,r)})$ with respect to $\mathcal{F}_n$ is at least 1/2, then by Lemma 4.1 with A taken to be $W_{u,n}(\Phi_{(1-\varepsilon)F}, \Phi'_{\varepsilon F+hG} + \delta_{(x,r)})$, the conditional expectation of $1 - g_{u,n}(\Phi_{(1-\varepsilon)F}, \Phi'_{\varepsilon F+hG} + \delta_{(x,r)})$ is at least γ. Hence, setting

$$V_{u,n} := f_{u,n}(\Phi_{(1-\varepsilon)F}, \Phi'_{\varepsilon F+hG} + \delta_{(x,r)}) g_{u,n}(\Phi_{(1-\varepsilon)F}, \Phi'_{\varepsilon F+hG} + \delta_{(x,r)}),$$

we have

$$\mathbb{E}[V_{u,n}|\mathcal{F}_n] \le \max(1-\gamma, 1/2) = 1-\gamma.$$

Also, for each n, $V_{u,n+1}$ is $\mathcal{F}_n$-measurable and by (4.10) we have

$$\begin{aligned}
&\mathbb{E}[h_{u,n+2}(\Phi_{(1-\varepsilon)F}, \Phi'_{\varepsilon F+hG} + \delta_{(x,r)})] \\
&\quad \le \mathbb{E}\left[\prod_{i=n_0}^{n} V_{u,i}\right] = \mathbb{E}\left[\mathbb{E}[V_{u,n}|\mathcal{F}_n] \times \prod_{i=n_0}^{n-1} V_{u,i}\right] \\
&\quad \le (1-\gamma)\mathbb{E}\prod_{i=n_0}^{n-1} V_{u,i} \le \cdots \le (1-\gamma)^{n-n_0} \le \exp(-\gamma(n-n_0)).
\end{aligned}$$

Arguing similarly in each of the $2d$ positive or negative coordinate directions shows that for u a multiple of b we have

$$\begin{aligned}\mathbb{P}(A_{x,r,u}) &= \mathbb{P}\{Z_{\{x\}}(\Phi_{(1-\varepsilon)F}, \Phi'_{\varepsilon F+hG}|_{[B_u]} + \delta_{(x,r)}) \setminus B_{u-b} \neq \varnothing\}\\ &\leq \mathbb{P}\{Z_{\{x\}}(\Phi_{(1-\varepsilon)F}, \Phi'_{\varepsilon F+hG}|_{[B_u]}) \setminus [-(u-b)/d, (u-b)/d]^d) \neq \varnothing\}\\ &\leq 2d\exp(-\gamma(\lfloor (u-b)/(dK)\rfloor - n_0 - 2)),\end{aligned}$$

which gives us the result (4.7) for $d \geq 3$.

To deduce (4.8), set $G = 0$ in (4.7), and use the fact that

$$Z_L(\Phi_{(1-\varepsilon)F} + \Phi'_{\varepsilon F}, 0) \subset Z_L(\Phi_{(1-\varepsilon)F}, \Phi'_{\varepsilon F}). \qquad \square$$

5. Proof of Theorem 3.1. Suppose that $b \in \mathbb{R}_+$ and $F \in \mathbf{M}^b$ and $G \in \mathbf{M}^b_{\pm}$, with $t_c(F) < 1$ and $F + aG$ a measure for some $a > 0$.

To ease notation, we shall assume additionally that $b = 1$; the result for a general b can be obtained by using the scaling property of the Boolean model; see, for example, Meester and Roy (1996).

Choose $\varepsilon \in (0, 1)$ with $1 - \varepsilon > t_c(F)$ and with $F + \varepsilon G$ a measure. Keep F, G and ε fixed for the rest of this section.

Let G_+ and G_- be the positive and negative parts in the Hahn–Jordan decomposition of G (so that G_+ and G_- are mutually singular measures and $G = G_+ - G_-$). Let $h \in [0, \varepsilon^2]$. Then $\varepsilon F - hG_-$ is a measure. Recall from Section 2 that Π_F denotes the distribution of a Poisson process on $\mathbb{R}^d \times \mathbb{R}_+$ with intensity measure $\mu(d(x,r)) = dx\,F(dr)$. Let $\Phi_{(1-\varepsilon)F}$, $\Psi_{\varepsilon F-hG_-}$, Ψ_{hG_-} and Ψ_{hG_+} be independent Poisson processes in $\mathbb{R}^d \times \mathbb{R}_+$ with respective distributions $\Pi_{(1-\varepsilon)F}$, $\Pi_{\varepsilon F-hG_-}$, Π_{hG_-} and Π_{hG_+}. Set

$$\begin{aligned}\Phi'_{\varepsilon F+hG} &:= \Psi_{\varepsilon F-hG_-} + \Psi_{hG_+};\\ \Phi_F &:= \Phi_{(1-\varepsilon)F} + \Psi_{\varepsilon F-hG_-} + \Psi_{hG_-};\\ \Phi_{F+hG} &:= \Phi_{(1-\varepsilon)F} + \Phi'_{\varepsilon F+hG},\end{aligned}$$

so that $\Phi'_{\varepsilon F+hG}$, Φ_F and Φ_{F+hG} are Poisson processes with distribution $\Pi_{\varepsilon F+hG}$, Π_F, and Π_{F+hG}, respectively. Also, for $n \in \mathbb{N}$ define

$$\Phi'_{h,n} := \Psi_{\varepsilon F-hG_-} + \Psi_{hG_+}|_{[B_n]} + \Psi_{hG_-}|_{[\mathbb{R}^d \setminus B_n]},$$

which is a Poisson process with intensity $dx \times (\varepsilon F + hG)(dr)$ in $[B_n]$, and with intensity $dx \times \varepsilon F(dr)$ in $[\mathbb{R}^d \setminus B_n]$. Since F and G are supported by $[0, 1]$, for $\psi \in \mathbf{N}^1$ we have

$$Z(\Phi_{(1-\varepsilon)F} + \Phi'_{h,n} + \psi) \cap B_{n-1} = Z(\Phi_{F+hG} + \psi) \cap B_{n-1}. \tag{5.1}$$

Our next lemma gives us the first part (3.1) of Theorem 3.1, among other things.

LEMMA 5.1. *Let $L \subset \mathbb{R}^d$ be compact, and let $\psi \in \mathbf{N}^1$ with $\psi(\mathbb{X}) < \infty$. For $\varphi \in \mathbf{N}$, set $\tilde{f}_{L,\psi}(\varphi) := \mathbf{1}\{L \cap Z_\infty(\varphi + \psi) \neq \varnothing\}$. Then*

$$\frac{d^+}{dh} \mathbb{E}_{F+hG} \tilde{f}_{L,\psi}(\Phi) \bigg|_{h=0} = \iint \mathbb{E}_F D_{(x,r)} \tilde{f}_{L,\psi}(\Phi) G(dr)\, dx, \tag{5.2}$$

and the right-hand side of (5.2) is finite. Also, given $h \in [0, \varepsilon^2]$ we have almost surely

$$\tilde{f}_{L,\psi}(\Phi_{F+hG}) = \lim_{n\to\infty} \tilde{f}_{L,\psi}(\Phi_{(1-\varepsilon)F} + \Phi'_{h,n}). \tag{5.3}$$

PROOF. To see (5.3), first suppose $\tilde{f}_{L,\psi}(\Phi_{F+hG}) = 1$. If also $\Phi_{F+hG} \in \mathbf{U}$ [which is the case almost surely by (2.1)], there must be a path from L through $Z(\Phi'_{\varepsilon F+hG} + \psi + \Phi^{\mathrm{fin}}_{(1-\varepsilon)F})$ to $Z_\infty(\Phi_{(1-\varepsilon)F})$ [if $L \cap Z_\infty(\Phi_{(1-\varepsilon)F}) \neq \varnothing$ we interpret this path as being empty]. Choose such a path, and choose $m \in \mathbb{N}$ such that this path is contained in B_{m-1}. Then for $n \geq m$, by (5.1) we have $\tilde{f}_{L,\psi}(\Phi_{(1-\varepsilon)F} + \Phi'_{h,n}) = 1$.

Conversely, suppose $\tilde{f}_{L,\psi}(\Phi_{F+hG}) = 0$. Then, recalling the definition (4.1) of $Z_L(\varphi)$, we have that $Z_L(\Phi_{F+hG} + \psi)$ is bounded so we can choose m such that $Z_L(\Phi_{F+hG} + \psi) \subset B_{m-1}$. Then for $n \geq m$, by (5.1) we have $\tilde{f}_{L,\psi}(\Phi_{(1-\varepsilon)F} + \Phi'_{h,n}) = 0$. Thus, we have demonstrated (5.3).

For $h \in [0, \varepsilon]^2$ and $n \in \mathbb{N}$ set,

$$U_{h,n} = h^{-1}\big(\tilde{f}_{L,\psi}(\Phi_{(1-\varepsilon)F} + \Phi'_{h,n}) - \tilde{f}_{L,\psi}(\Phi_{(1-\varepsilon)F} + \Phi'_{h,n-1})\big).$$

By (5.3) and dominated convergence,

$$\mathbb{E}\tilde{f}_{L,\psi}(\Phi_{F+hG}) = \lim_{n\to\infty} \mathbb{E}\tilde{f}_{L,\psi}(\Phi_{(1-\varepsilon)F} + \Phi'_{h,n}).$$

Also $\Phi_{(1-\varepsilon)F} + \Phi'_{h,0} = \Phi_F$ almost surely. Thus,

$$h^{-1}\big(\mathbb{E}\tilde{f}_{L,\psi}(\Phi_{F+hG}) - \mathbb{E}\tilde{f}_{L,\psi}(\Phi_F)\big) = \sum_{n=1}^{\infty} \mathbb{E}U_{h,n}. \tag{5.4}$$

By Proposition 2.2, for each n we have

$$\begin{aligned} \lim_{h\to 0+} \mathbb{E}U_{h,n} &= \lim_{h\to 0+} h^{-1}\mathbb{E}\big[\tilde{f}_{L,\psi}(\Phi_{(1-\varepsilon)F} + \Phi'_{h,n}) - \tilde{f}_{L,\psi}(\Phi_F)\big] \\ &\quad - \lim_{h\to 0+} h^{-1}\mathbb{E}\big[\tilde{f}_{L,\psi}(\Phi_{(1-\varepsilon)F} + \Phi'_{h,n-1}) - \tilde{f}_{L,\psi}(\Phi_F)\big] \\ &= \mathbb{E}\int_{\mathbb{R}_+}\int_{B_n\setminus B_{n-1}} D_{(x,r)}\tilde{f}_{L,\psi}(\Phi_F)\, dx\, G(dr). \end{aligned} \tag{5.5}$$

If we can take this limit through the sum on the right-hand side of (5.4), then we have the desired result (5.2). To justify this interchange, we seek to dominate the terms of (5.4) by those of a summable sequence, independently of h.

Note that $|U_{h,n}| \le h^{-1}$. Also if $\Phi'_{h,n} = \Phi'_{h,n-1}$ then clearly $U_{h,n} = 0$, so that

$$\{U_{h,n} \neq 0\} \subset \{\Phi'_{h,n} \neq \Phi'_{h,n-1}\}. \tag{5.6}$$

Recall that we write R_K for the radius of stabilization $R_{K,1}$ defined at (4.3). We assert the further event inclusion

$$\{U_{h,n} \neq 0\} \subset \{R_L(\Phi_{(1-\varepsilon)F}, \Phi'_{\varepsilon F+hG} + \psi) > n-1\} \cup \{\Phi_F \notin \mathbf{U}\}. \tag{5.7}$$

To see this, suppose $R_L(\Phi_{(1-\varepsilon)F}, \Phi'_{\varepsilon F+hG} + \psi) \le n-1$. Then by the stabilization property (4.4), since $\Phi'_{h,n-1}|_{[B_{n-1}]} = \Phi'_{h,n}|_{[B_{n-1}]} = \Phi'_{\varepsilon F+hG}|_{[B_{n-1}]}$ we have $R_L(\Phi_{(1-\varepsilon)F}, \Phi'_{h,n} + \psi) = R_L(\Phi_{(1-\varepsilon)F}, \Phi'_{h,n-1} + \psi) \le n-1$. If $\tilde{f}_{L,\psi}(\Phi_{(1-\varepsilon)F} + \Phi'_{h,n}) = 1$ and $\Phi_F \in \mathbf{U}$, then either $Z_\infty(\Phi_{(1-\varepsilon)F}) \cap L \neq \varnothing$ or there exists a component of $Z(\Phi'_{h,n} + \psi + \Phi^{\mathrm{fin}}_{(1-\varepsilon)F})$ which meets both L and $Z_\infty(\Phi_{(1-\varepsilon)F})$. This component is contained in B_{n-1} by (4.3), so then by (4.4) there must be a component of $Z((\Phi'_{h,n} + \psi)|_{[B_{n-1}]} + \Phi^{\mathrm{fin}}_{(1-\varepsilon)F})$ that meets both L and $Z_\infty(\Phi_{(1-\varepsilon)F})$, and hence $\tilde{f}_{L,\psi}(\Phi_{(1-\varepsilon)F} + \Phi'_{h,n-1}) = 1$. Similarly, if $\tilde{f}_{L,\psi}(\Phi_{(1-\varepsilon)F} + \Phi'_{h,n-1}) = 1$ and $\Phi_F \notin \mathbf{U}$, then $\tilde{f}_{L,\psi}(\Phi_{(1-\varepsilon)F} + \Phi'_{h,n}) = 1$. This justifies (5.7).

The event $\{\Phi'_{h,n} \neq \Phi'_{h,n-1}\} = \{(\Psi_{hG_+} + \Psi_{hG_-})(B_n \setminus B_{n-1}) = 0\}$ is independent of the event $\{R_L(\Phi_{(1-\varepsilon)F}, \Phi'_{\varepsilon F+hG} + \psi) > n-1\}$ by the stopping radius property (4.5). Also we have the event inclusion:

$$\begin{aligned}&\{R_L(\Phi_{(1-\varepsilon)F}, \Phi'_{\varepsilon F+hG} + \psi) > n-1\}\\ &\qquad \subset \{R_{L\cup Z(\psi)}(\Phi_{(1-\varepsilon)F}, \Phi'_{\varepsilon F+hG}) > n-1\}.\end{aligned}$$

Therefore, by (5.6), (5.7) and (2.1), we have

$$\mathbb{E}|U_{h,n}| \le h^{-1}\mathbb{P}\{\Phi'_{h,n} \neq \Phi'_{h,n-1}\}\mathbb{P}\{R_{L\cup Z(\psi)}(\Phi_{(1-\varepsilon)F}, \Phi'_{\varepsilon F+hG}) > n-1\}.$$

Also there is a constant $c' \in (0,\infty)$ such that $\mathbb{P}\{\Phi'_{h,n} \neq \Phi'_{h,n-1}\} \le n^{d-1}hc'$. Hence by Lemma 4.2, there is a constant $c \in (0,\infty)$ (independent of n and h, provided $0 \le h \le \varepsilon^2/2$) such that

$$\mathbb{E}|U_{h,n}| \le cn^{d-1} \times \exp(-c^{-1}n)$$

which is summable in n. Hence, by (5.4), (5.5) and dominated convergence we have (5.2). This also shows that the right-hand side of (5.2) is finite. □

PROOF OF THEOREM 3.1. We prove the result just for $b = 1$. The first part (3.1) holds by Lemma 5.1. To prove the second part, we take F, G and ε as before but now assume additionally that $F - \varepsilon G$ is a measure. Let $L \subset \mathbb{R}^d$ be compact. As above, for each $\varphi \in \mathbf{N}$ and each $\psi \in \mathbf{N}^1$ with $\psi(\mathbb{X}) < \infty$, set $\tilde{f}_{L,\psi}(\varphi) := \mathbf{1}\{L \cap Z_\infty(\varphi + \psi) \neq \varnothing\}$, and set $\tilde{f}_L(\varphi) := \tilde{f}_{L,0}(\varphi) = \mathbf{1}\{L \cap Z_\infty(\varphi) \neq \varnothing\}$. We shall prove by induction that for $n \in \mathbb{N}$ and $h \in (-\varepsilon^2, \varepsilon^2)$, we have

$$\begin{aligned}&\frac{d^n}{dh^n}\theta_L(F + hG)\\ &\quad = \int \cdots \int \mathbb{E}_{F+hG} D^n_{(x_1,r_1),\dots,(x_n,r_n)} \tilde{f}_L(\Phi)\, dx_1 G(dr_1) \cdots dx_n G(dr_n),\end{aligned} \tag{5.8}$$

which implies (3.2).

First, consider $n = 1$. Then (5.8) holds for the right derivative at $h = 0$ by Lemma 5.1. Also, by applying this fact to $-G$ instead of G we have that (5.8) holds for the left derivative at $h = 0$ too, so (5.8) holds at $h = 0$. Therefore, (5.8) also holds at other $h \in (-\varepsilon^2, \varepsilon^2)$ because we can apply the case $h = 0$ of (5.8) to F^* and G^*, given by $F^* = F + hG$, and $G^* = G$. Note that F^* is strictly supercritical because $F^* = (1-\varepsilon)F + \varepsilon(F + (h/\varepsilon)G)$ and $\varepsilon(F + (h/\varepsilon)G)$ is a measure since $|h| < \varepsilon^2$.

Now we perform the inductive step. Let $n \in \mathbb{N}$, and suppose (5.8) holds for all $h \in (-\varepsilon^2, \varepsilon^2)$. Then for $0 < h < \varepsilon^2$,

$$\begin{aligned} &h^{-1}\left(\frac{d^n}{ds^n}\theta_L(F+sG)\bigg|_{s=h} - \frac{d^n}{ds^n}\theta_L(F+sG)\bigg|_{s=0}\right) \\ &\quad = \int\cdots\int u_{x_1,r_1,\dots,x_n,r_n}(h)\,dx_1 G(dr_1)\cdots dx_n G(dr_n), \end{aligned} \tag{5.9}$$

where we set

$$\begin{aligned} u_{x_1,r_1,\dots,x_n,r_n}(h) := h^{-1}\big(&\mathbb{E}D^n_{(x_1,r_1),\dots,(x_n,r_n)}\tilde{f}_L(\Phi_{F+hG}) \\ &- \mathbb{E}D^n_{(x_1,r_1),\dots,(x_n,r_n)}\tilde{f}_L(\Phi_F)\big). \end{aligned}$$

Applying Lemma 5.1 to the function $D^n_{(x_1,r_1),\dots,(x_n,r_n)}\tilde{f}_L$ [expressed as a sum as in (2.4)] gives us as $h \to 0$ that

$$u_{x_1,r_1,\dots,x_n,r_n}(h) \to \iint \mathbb{E}D^{n+1}_{(x,r),(x_1,r_1),\dots,(x_n,r_n)}\tilde{f}_L(\Phi_F)\,dx\,G(dr). \tag{5.10}$$

For $1 \le i \le n$, write z_i for (x_i, r_i). By (5.3) applied to $\tilde{f}_{L,\psi}$ for each $\psi \in \mathbf{N}^1$ with $\psi \le \sum_{i=1}^n \delta_{z_i}$, and dominated convergence, we have for $(z_1, \dots, z_n) \in \mathbb{X}^n$ and $|h| < \varepsilon^2$ that

$$\begin{aligned} &u_{z_1,\dots,z_n}(h) \\ &\quad = h^{-1}\Big(\lim_{m\to\infty}\mathbb{E}D^n_{z_1,\dots,z_n}\tilde{f}_L\big(\Phi_{(1-\varepsilon)F} + \Phi'_{h,m}\big) - \mathbb{E}D^n_{z_1,\dots,z_n}\tilde{f}_L(\Phi_F)\Big) \\ &\quad = \sum_{m=1}^{\infty}\mathbb{E}V(h, m, z_1, \dots, z_n), \end{aligned} \tag{5.11}$$

where we set

$$\begin{aligned} V(h, m, z_1, \dots, z_n) := h^{-1}\big(&D^n_{z_1,\dots,z_n}\tilde{f}_L\big(\Phi_{(1-\varepsilon)F} + \Phi'_{h,m}\big) \\ &- D^n_{z_1,\dots,z_n}\tilde{f}_L\big(\Phi_{(1-\varepsilon)F} + \Phi'_{h,m-1}\big)\big). \end{aligned} \tag{5.12}$$

Now, $|V(h, m, z_1, \dots, z_n)| \le 2^{n+1}h^{-1}$ and clearly we have

$$\big\{V(h, m, z_1, \dots, z_n) \ne 0\big\} \subset \big\{\Phi'_{h,m} \ne \Phi'_{h,m-1}\big\}. \tag{5.13}$$

Set $M = \max(m, |x_1|, \dots, |x_n|)$. Suppose $M \ge (2n+4)(\operatorname{diam}(L \cup \{0\}) + 4)$. Choose $I \in \{1, \dots, 2n+3\}$ such that the annulus $B_{(I+1)M/(2n+4)} \setminus B_{IM/(2n+4)}$ intersects none of the balls $B_{r_1}(x_1), \dots, B_{r_n}(x_n)$ and also does not intersect the annulus $B_{m+1} \setminus B_{m-2}$; to be definite, choose the smallest such I. Define the event

$$A'_{h,m,z_1,\dots,z_n} := \{Z_{B_{IM/(2n+4)}}(\Phi_{(1-\varepsilon)F}, \Phi'_{h,m-1}) \setminus B_{(I+1)M/(2n+4)} \neq \varnothing\}.$$

Write just Φ for $\Phi_{(1-\varepsilon)F}$. Event $A'_{h,m,z_1,\dots,z_n}$ says that there is a crossing of the annulus $B_{(I+1)M/(2n+4)} \setminus B_{IM/(2n+4)}$ (which we shall call the "moat") by a component of $Z_\Phi^{\text{fin}} \cup Z_{\Phi'_{h,m}}$, and hence also by a component of $Z_\Phi^{\text{fin}} \cup Z_{\Phi'_{h,m-1}}$. Note that the events $\{\Phi'_{h,m} \neq \Phi'_{h,m-1}\}$ and $A'_{h,m,z_1,\dots,z_n}$ are independent. We assert that

$$\{V(h, m, z_1, \dots, z_n) \neq 0\} \subset A'_{h,m,z_1,\dots,z_n}. \tag{5.14}$$

To justify this, observe first that by the definition of M, at least one of the sets $B_{r_1}(x_1), \dots, B_{r_n}(x_n), B_{m+2} \setminus B_{m-1}$ is *exterior to the moat*, that is, has empty intersection with $B_{(I+1)M/(2n+4)}$.

Suppose that no crossing of the moat by a component of $Z_\Phi^{\text{fin}} \cup Z_{\Phi'_{h,m}}$ occurs. Suppose also that one of the balls $B_{r_i}(x_i)$ [say the ball $B_{r_1}(x_1)$] is exterior to the moat; then for any $\psi \in \mathbf{N}$ with $\psi \le \sum_{i=2}^n \delta_{z_i}$ we have

$$\tilde{f}_{L,\psi}(\Phi + \Phi'_{h,m}) = \tilde{f}_{L,\psi}(\Phi + \Phi'_{h,m} + \delta_{z_1})$$

so that $D^n_{x_1,\dots,x_n} \tilde{f}_L(\Phi + \Phi'_{h,m}) = 0$, and similarly $D^n_{x_1,\dots,x_n} \tilde{f}_L(\Phi + \Phi'_{h,m-1}) = 0$, so that $V(h, m, z_1, \dots, z_n) = 0$.

Now suppose instead that the annulus $B_{m+1} \setminus B_{m-2}$ is exterior to the moat, and as before that no crossing of the moat occurs. Then for any $\psi \in \mathbf{N}$ with $\psi \le \sum_{i=1}^n \delta_{z_i}$ we have that $\tilde{f}_{L,\psi}(\Phi + \Phi'_{h,m}) = \tilde{f}_{L,\psi}(\Phi + \Phi'_{h,m-1})$, so that $V(h, m, z_1, \dots, z_n) = 0$. Together with the previous paragraph, this implies the assertion (5.14).

We show next that for n and L fixed there is a constant c such that for all m, all $z_1, \dots, z_n \in \mathbb{R}^d \times [0, 1]$ and all $h \in (-\varepsilon^2/2, \varepsilon^2/2)$ we have

$$\mathbb{P}[A'_{h,m,z_1,\dots,z_n}] \le c \exp(-c^{-1} M). \tag{5.15}$$

Assume again that $M \ge (2n+4)(\operatorname{diam}(L \cup \{0\}) + 4)$. Cover the boundary $\partial B_{(I+0.5)M/(2n+4)}$ of the ball $B_{(I+0.5)M/(2n+4)}$ (i.e., the "middle of the moat") with a deterministic collection of unit balls $C_1, \dots, C_{k(M)}$, each with center in $\partial B_{(I+0.5)M/(2n+4)}$, with $k(M) = O(M^{d-1})$. If there is a crossing of the moat, there must be a crossing from one or more of the balls $C_1, \dots, C_{k(M)}$ to a boundary of the moat. Therefore,

$$A'_{h,m,z_1,\dots,z_n} \subset \bigcup_{j=1}^{k(M)} \{\operatorname{diam} Z_{C_j}(\Phi_{(1-\varepsilon)F}, \Phi'_{h,m}) \ge (M/(4n+8) - 2)\}.$$

For each j, let C_j' denote the ball with the same center as C_j and with radius $M/(4n+8)$. Since the restriction of Poison process $\Phi'_{h,m}$ to $[C_j']$ has intensity of product form [either $dx \times (\varepsilon F + hG)(dr)$ or $dx \times \varepsilon F(dr)$, depending on whether or not the annulus $B_m \setminus B_{m-1}$ is exterior to the moat], we can use the union bound and Lemma 4.2 to obtain (5.15).

Using (5.13), (5.14), (2.1) and (5.15), we obtain that there is a finite constant (again denoted c, and depending on n) such that

$$\mathbb{E}\big|V(h, m, z_1, \dots, z_n)\big| \le c m^{d-1} \exp\bigl(-\bigl(m + \max\bigl(|x_1|, \dots, |x_n|\bigr)\bigr)/c\bigr),$$

which is summable in m with the sum being integrable in $(z_1, \dots, z_m)$. Then using (5.11) and dominated convergence we can take the limit (5.10) inside the integral (5.9), so that

$$\frac{d^+}{dh}\frac{d^n}{dh^n}\theta_L(F + hG)\bigg|_{h=0} = \mathbb{E}_F \int \cdots \int D^{n+1}_{(x,r),(x_1,r_1),\dots,(x_n,r_n)} \tilde{f}_L(\Phi) \\ \times dx G(dr)\, dx_1 G(dr_1) \cdots dx_n G(dr_n).$$

Also we can repeat this argument using $-G$ instead of G to get the same value for the left derivative at $h = 0$ leading to (5.8) for $n+1$ with $h = 0$. Then for $n+1$ and for a general $h \in (-\varepsilon^2, \varepsilon^2)$, we have (5.8) by applying the $h = 0$ result and using the measure $F + hG$ instead of F. This completes the induction. □

6. Proof of Theorem 3.4. Given a graph $\mathbf{G} = (V, E)$, and given $v \in V$, let us denote by $\mathbf{G} \setminus v$ the graph $\mathbf{G}$ with v and all edges incident to v removed. If u, v, w are distinct vertices of $\mathbf{G}$, let us say vertex w is (u, v)-pivotal if u and v lie in the same component of $\mathbf{G}$ but different components of $\mathbf{G} \setminus w$.

LEMMA 6.1. *Suppose $\mathbf{G} = (V, E)$ is a finite connected graph, and $u, v \in E$ with $u \neq v$. Then either $\mathbf{G}$ has at least one (u, v)-pivotal vertex, or there exist at least two vertex-disjoint paths in $\mathbf{G}$ from u to v. Also, in the first case, every path from u to v in $\mathbf{G}$ passes through the (u, v)-pivotal vertices in the same order.*

PROOF. The first assertion is an immediate consequence of Menger's theorem [see, e.g., Bollobás (1979)].

To see the second assertion, suppose w, w' are distinct (u, v)-pivotal vertices, and there is a path from u to v passing through w before w', and another such path passing through w' before w. Then following the first path from u as far as w, and then the second path from w to v, we obtain a path from u to v avoiding w'; hence w' is not (u, v)-pivotal, which is a contradiction. □

PROOF OF THEOREM 3.4. Our proof uses ideas from Chayes and Chayes (1986). We start by introducing some notation. Fix $b > 0$, $F \in \mathbf{M}_1^b$ and compact $L \subset \mathbb{R}^d$. Since F is fixed, we write $\theta_L(t)$ for $\theta_L(tF)$ in this proof. By a *path* in a configuration $\varphi \in \mathbf{N}$ we mean a finite or infinite sequence $K_1, K_2, \dots$ of distinct grains such that $K_i = B_{r_i}(x_i)$ for some $(x_i, r_i) \in \varphi$ and $K_i \cap K_{i+1} \neq \varnothing$ for all $i \geq 1$ with K_{i+1} part of the sequence. A path *intersects* a subset of $\mathbb{R}^d$, if one of its constituent grains intersects this set. If A, A' are disjoint subsets of $\mathbb{R}^d$ and a path intersects both A and A', then we say the path *joins* A to A'. We shall say that two paths $(K_1, K_2, \dots)$ and $(K_1', K_2', \dots)$ in φ are *disjoint* if $K_i \neq K_j'$ for all i, j.

For $n \in \mathbb{N}$, introduce events that there is a path joining L to the complement of B_n or to infinity,

$$J_L^n := \{\varphi \in \mathbf{N} : Z_L(\varphi) \setminus B_n \neq \varnothing\}; \qquad J_L := \{\varphi \in \mathbf{N} : L \cap Z_\infty(\varphi) \neq \varnothing\},$$

where the notation Z_L is as defined in (4.1). Assume from now on that n is so large that $L \subset B_{n-2b}$. If $\varphi \in J_L^n$, but $(x, r) \in \varphi$ is such that $\varphi - \delta_{(x,r)} \notin J_L^n$, we say that the grain $B(x, r)$ is *pivotal* for J_L^n in the configuration φ.

Let $\theta_L^n(t) := \mathbb{P}_{tF}\{\Phi \in J_L^n\}$. We claim that when $\varphi \in J_L^n$, either there are at least two disjoint paths from L to $\mathbb{R}^d \setminus B_n$ in φ or there is at least one pivotal grain for J_L^n and there is a unique *last pivotal grain* for J_L^n when counting from L. To see this claim, apply Lemma 6.1 to the intersection graph of the set:

$$\mathbf{B}_n = \{B_{r_i}(x_i) : (x_i, r_i) \in \varphi \text{ such that } B_{r_i}(x_i) \cap B_n \neq \varnothing\} \cup \{L, \mathbb{R}^d \setminus B_n\}. \tag{6.1}$$

Let $\varphi \in \mathbf{N}$ and $n \in \mathbb{N}$. Suppose there is a unique last pivotal grain for J_L^n and denote this last pivotal grain by $K := B_r(x)$. If $K \subset B_n$, then there exist three disjoint paths in $\varphi - \delta_{(x,r)}$: one which joins L to K and two which join K to $\mathbb{R}^d \setminus B_n$. If not (i.e., if $K \setminus B_n \neq \varnothing$), there is still a path joining K to L. Even in this case, we say that there are two disjoint paths joining K to $\mathbb{R}^d \setminus \partial B_n$ which are just both empty; see Figure 1. We then have

$$\begin{aligned}
\theta_L^n(t) &= \mathbb{P}_{tF}\{\text{there are two disjoint paths in } \Phi \text{ joining } L \text{ to } \mathbb{R}^d \setminus B_n\} \\
&\qquad + \mathbb{E}_{tF} \int \mathbf{1}\{B_r(x) \text{ is the last pivotal grain for } J_L^n \text{ in } \Phi\}\Phi(d(x,r)) \\
&\leq (\theta_L^n(t))^2 \\
&\qquad + t \iint \mathbb{P}_{tF}\{B_r(x) \text{ is the last pivotal grain for } J_L^n \text{ in } \Phi + \delta_{(x,r)}\}\, dx\, F(dr),
\end{aligned}$$

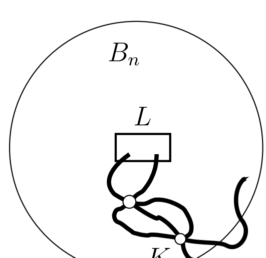

FIG. 1. *Geometry of the paths (depicted as black curves) connecting L to $\mathbb{R}^d \setminus B_n$. Pivotal grains for J_L^n are coloured white. The last pivotal grain starting from L is denoted K.*

where we have used the B-K inequality [Meester and Roy (1996), Theorem 2.3], to bound the first term from above and the Mecke identity (2.2) for the second term. Now $B_r(x)$ is the last pivotal grain for J_L^n in $\Phi + \delta_{(x,r)}$ if and only if there are two disjoint paths in the configuration $\Phi + \delta_{(x,r)}$, one of them joining $B_r(x)$ and $\mathbb{R}^d \setminus B_n$ [possibly empty, if $B_r(x) \setminus B_n \neq \varnothing$], and the other one joining L and $\mathbb{R}^d \setminus B_n$ and using $B_r(x)$, and all paths joining L to $\mathbb{R}^d \setminus B_n$ use $B_r(x)$. We claim that this is the same as saying the events

$$E_{n,x,r} := \{\varphi : \varphi + \delta_{(x,r)} \in J_L^n, \varphi \notin J_L^n\}$$

and $J^n_{B_r(x)}$ occur disjointly in the sense of Gupta and Rao (1999), so that by the continuum Reimer inequality in that paper, we get

$$\theta_L^n(t) \le (\theta_L^n(t))^2 + t \iint \mathbb{P}_{tF}\{\Phi \in E_{n,x,r}\}\mathbb{P}_{tF}\{\Phi \in J^n_{B_r(x)}\}\, dx\, F(dr). \tag{6.2}$$

Let us justify our claim. With probability 1, there exists a finite (random) ε such that displacement of the locations x_i with $(x_i, r_i) \in \Phi|_{[B_{n+2b}]}$ by at most ε, and modification of the corresponding r_i by at most ε, would not affect the intersection graph on $\mathbf{B}_n \cup B_r(x)$. Suppose Φ is such that there are disjoint paths P, P' in configuration $\Phi + \delta_{(x,r)}$, with P joining $B_r(x)$ and $\mathbb{R}^d \setminus B_n$ and P' joining L and $\mathbb{R}^d \setminus B_n$ and using $B_r(x)$, and suppose also all paths joining L to $\mathbb{R}^d \setminus B_n$ use $B_r(x)$. Let ε be as defined above. Let $\mathcal{K}$ be a union of rational $(d+1)$-cubes of side less than $\varepsilon/(d+1)$ centered on the points (x_i, r_i) such that $B_{r_i}(x_i) \in P$. Let $\mathcal{L}$ be the complement of $\mathcal{K}$ in $\mathbb{R}_d \times \mathbb{R}_+$.

If we modify our configuration Φ arbitrarily in $\mathcal{L}$, then we are still in $J^n_{B_r(x)}$, since the points of Φ inside $\mathcal{K}$ guarantee occurrence of $J^n_{B_r(x)}$.

On the other hand, if we modify Φ arbitrarily in $\mathcal{K}$ then we still have a path joining L to $\mathbb{R}^d \setminus B_n$ using $B_r(x)$ (because our configuration in $\mathcal{L}$ contains such a path) but every such path uses $B_r(x)$ [because in Φ our path P did not intersect with any path joining L to $B_r(x)$, and hence, by the choice of ε, neither does any modification of P by moving points a distance at most $\varepsilon/(d+1)$ in each coordinate, and the rest of Φ is unchanged].

Note that our regions $\mathcal{K}, \mathcal{L}$ are unions of rational rectangles in $(d+1)$-space, not in d-space as in Gupta and Rao (1999). To see that Gupta and Rao (1999) is applicable, note that we can generate our Poisson process $\Phi = \sum_i \delta_{(x_i,r_i)}$ in $\mathbb{R}^d \times \mathbb{R}_+$ from a homogeneous point process $\sum_i \delta_{y_i}$ of intensity t in $\mathbb{R}^d \times [0, 1]$, with the spatial locations x_i generated by projecting y_i onto the first d coordinates and the random radii r_i generated as a suitable increasing function (namely, the quantile function of F) of the final coordinate of y_i.

Now let $n \to \infty$. Since $(J_L^n)_{n\ge1}$ is a decreasing sequence of events and $\bigcap_n J_L^n = J_L$, we have $\theta_L^n(t) \to \theta_L(t)$, and for every $\varphi \in \mathbf{N}$ we have

$$\mathbf{1}\{\varphi \in E_{n,x,r}\} \to \mathbf{1}\{\varphi + \delta_{(x,r)} \in J_L\}\mathbf{1}\{\varphi \notin J_L\}.$$

Also $\mathbb{P}_{tF}\{\Phi \in J^n_{B_r(x)}\} \to \mathbb{P}_{tF}\{\Phi \in J_{B_r(x)}\} = \theta_{B_r}(t)$ by stationarity.

By the definition (4.2), the first factor of the integrand in (6.2) satisfies

$$\mathbb{P}_{tF}\{\Phi \in E_{n,x,r}\} \le \mathbb{P}_{tF}\{Z_L(\Phi, 0) \cap B_r(x) \neq \varnothing\},$$

since if $Z_L(\Phi, 0) \cap B_r(x) = \varnothing$, then $B_r(x)$ cannot be pivotal for J_L^n in $\Phi + \delta_{(x,r)}$. Indeed, there must be a path joining L to $B_r(x)$ to give $B_r(x)$ a chance of being pivotal, but if this path is part of $Z_\infty(\Phi)$ then $B_r(x)$ is not pivotal.

Recall that we are assuming $F((b, \infty)) = 0$. By (4.3), we have $Z_L(\Phi, 0) \subset B_{R_{L,b}(\Phi,0)}$, $\mathbb{P}_{tF}$-almost surely. Hence, by (4.8) from Lemma 4.2 the integrand in (6.2) is bounded by an integrable function of (x, r). Hence, by (6.2) and dominated convergence, setting $\theta'_L(t) := \frac{d}{dt}\theta_L(t)$ we have

$$\begin{aligned} \theta_L(t) &\le (\theta_L(t))^2 + t \iint \mathbb{P}_{tF}\{\Phi + \delta_{(x,r)} \in J_L, \Phi \notin J_L\}\theta_{B_r}(t)\, dx\, F(dr) \\ &\le (\theta_L(t))^2 + t\theta_{B_b}(t)\theta'_L(t), \end{aligned} \tag{6.3}$$

where for the last line we have used Theorem 3.2 and the monotonicity of $\theta_{B_r}(t)$ in r.

Next, we bound $\theta_L(t)/\theta_{B_b}(t)$ from below. Pick $x \in L$. If $B_b(x) \cap Z_\infty(\Phi) \neq \varnothing$ and also $B_b(x) \subset Z(\Phi)$, then $L \cap Z_\infty(\Phi) \neq \varnothing$. Hence, by the Harris-FKG inequality for Poisson processes [see, e.g., Last and Penrose (2017), Chapter 20], and translation-invariance, we have for $t \ge t_c$ that

$$\theta_L(t) \ge \theta_{B_b(x)}(t)\mathbb{P}_{tF}\{B_b(x) \subset Z\} = \theta_{B_b}(t)\mathbb{P}_{t_cF}\{B_b \subset Z\}. \tag{6.4}$$

Setting $\alpha = \mathbb{P}_{t_cF}\{B_b \subset Z\}$ and substituting (6.4) into (6.3), we get to

$$\theta'_L(t) \ge \frac{(1 - \theta_L(t))\theta_L(t)}{t\theta_{B_b}(t)} \ge \frac{\alpha(1 - \theta_L(t))}{t}.$$

Integrating over t, using the continuity of $\theta'_L(\cdot)$ on (t_c, ∞) and the monotonicity of $\theta_L(\cdot)$, we therefore have that

$$\theta_L(t) - \theta_L(t_c) \ge \alpha(t - t_c)\frac{1 - \theta_L(t)}{t},$$

which is (3.6). Since $\theta_L(t)$ is continuous from the right, $\theta_L(t) = \theta_L(t_c) + o(1)$ as $t \downarrow t_c$, giving (3.7). $\square$

7. Final remarks. In this paper we have studied the capacity functional of the infinite cluster of a spherical Boolean model. Our main results (Theorems 3.2 and 3.4) require the radii to be deterministically bounded. It can be expected that these results also hold for more general Boolean models with connected grains having a deterministically bounded circumradius. It can also be conjectured that good moment properties of the circumradius should suffice to imply the result for unbounded radii. The proof of this latter extension, however, does not seem to be straightforward.

The methods of this paper can probably be used to derive differentiability properties of the expectations of other functionals of the Boolean model. A whole family of such functionals in the subcritical regime can be defined in terms of the number N_r, $r > 0$, of grains in the cluster of $Z(\Phi + \delta_{(0,r)})$, intersecting the ball B_r. Given $F \in \mathbf{M}_1^\sharp$ and $m \in \mathbb{Z}$, it is then of interest to study the functional $\int \mathbb{E}_{tF}[N_r^m]F(dr)$ as a function of $t < t_c$. In the case $m = -1$, this is the mean number of clusters per Poisson point. Preliminary results in the latter case can be found in Jiang, Zhang and Guo (2011).

A natural step after the infinite differentiability would be to show that the capacity is an analytic function of intensity in the supercritical phase. It might be possible to use (3.3) to show that for fixed supercritical t, the Taylor series for $\theta_L((t+h)F)$ as a function of h has positive radius of convergence; however, this seems to need tighter bounds than those used here, and hence, new ideas.

Also of interest is the *n-point connectivity function* of the Boolean model. Given $x_1, \dots, x_n \in \mathbb{R}^d$, and given $F \in \mathbf{M}_1^\sharp$, for $t > 0$ let $\tau_{x_1,\dots,x_n}(t)$ denote the $\mathbb{P}_{tF}$-probability that the points $x_1, \dots, x_n$ all lie in the same component of $Z(\Phi)$. It is not hard to prove that $\tau_{x_1,\dots,x_n}(t)$ is continuous in t [see, e.g., Jiang, Zhang and Guo (2011)]. Using the method of proof of Theorem 3.1, it should be possible to show further that $\tau_{x_1,\dots,x_n}(t)$ is infinitely differentiable in t on the interval $(t_c(F), \infty)$. Moreover, the n-point connectivity function of $Z_\infty(\Phi)$ [as opposed to that of $Z(\Phi)$] is certainly infinitely differentiable, since by the inclusion-exclusion formula the probability $\mathbb{P}_{tF}[\bigcap_{i=1}^n \{x_i \in Z_\infty(\Phi)\}]$ can be expressed as a linear combination of the capacity functionals of the subsets of $\{x_1, \dots, x_n\}$, plus a constant.

It may be possible to generalize Theorem 3.4 as follows. Let $F_0 \in \mathbf{M}^\sharp$ and $F \in \mathbf{M}_1^\sharp$. Let $t_c(F_0, F)$ be the supremum of those t such that $\theta(F_0 + tF) = 0$ and assume F_0, F are such that $t_c(F_0, F) > 0$. Then we might expect that similar results to (3.6) and (3.7) would hold with tF replaced by $F_0 + tF$ (and t_cF replaced by $F_0 + t_cF$ and t_1F replaced by $F_0 + t_1F$) wherever they appear.

We have shown that θ_L is (under the assumptions of Theorem 3.2) infinitely differentiable on (t_c, ∞). It would be extremely interesting to understand the behaviour of the second derivative near the critical value. We leave this as a challenging problem for future research.

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