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NONEQUILIBRIUM FLUCTUATIONS OF ONE-DIMENSIONAL BOUNDARY DRIVEN WEAKLY ASYMMETRIC EXCLUSION PROCESSES

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We consider one-dimensional, boundary driven, weakly asymmetric exclusion processes in contact with reservoirs at fixed density. For a general set of initial measures and by using a microscopic Cole–Hopf transformation, we derive the nonequilibrium fluctuations which are given by a generalized Ornstein–Uhlenbeck process.

1. Introduction. Nonequilibrium fluctuations of interacting particle systems around the hydrodynamic limit is one of the main open problems in the field of scaling limits of interacting particle systems. It has only been derived for few one-dimensional dynamics and no progress has been made in the last 20 years in Gaussian nonequilibrium fluctuations. We refer to the last section of [13], Chapter 11, for references and an historical account.

We examine in this article the dynamical nonequilibrium fluctuations of one-dimensional weakly asymmetric exclusion processes in contact with reservoirs. In a future work, following the strategy presented in [16] for the symmetric simple exclusion process, we use the results presented here to prove the stationary fluctuations of the density field.

The motivations are twofold. On the one hand, the investigation of the steady states of boundary driven interacting particle systems has attracted a lot of attention in these last fifteen years, mainly after [1, 6]. The density fluctuations at the steady state are an important part of the theory and they can only be seized through the dynamical nonequilibrium fluctuations [16]. On the other hand, several published results [5] still wait for rigorous proofs.

The proof of the nonequilibrium density fluctuations we present here relies on a microscopic Cole–Hopf transformation introduced by Gärtner [11] to investigate the hydrodynamic behavior of weakly asymmetric exclusion processes on $\mathbb{Z}$, and

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used by Dittrich and Gärtner [9] to prove the nonequilibrium fluctuations of the same models.

As in PDE, the microscopic Cole–Hopf transformation turns a nonlinear problem involving local functions into a linear one. For this reason, it permits to avoid proving a nonequilibrium Boltzmann–Gibbs principle [13], Section 11.1, introduced by Rost [3], which is the main technical difficulty in the proof of density fluctuations.

The proof of the nonequilibrium fluctuations relies on sharp estimates of the moments of the microscopic Cole–Hopf variables, and on sharp estimates of the fundamental solution of initial-boundary value semi-discrete linear partial differential equations. These results are presented in the last two sections of this article. The bounds on the fundamental solutions are derived in a similar way as hypercontractivity is proven for ergodic Markov chains.

2. Notation and results.

2.1. *The model.* Fix $E > 0$, α, β in $(0, 1)$ and $N \geq 1$. Denote by $\{\eta_t^N : t \geq 0\}$, the speeded-up, one-dimensional, boundary driven, weakly asymmetric simple exclusion process with state space $\Sigma_N = \{0, 1\}^{\{1,\dots,N-1\}}$. The configurations of the state space are denoted by the symbol η, so that $\eta(j) = 1$ if site j is occupied for the configuration η and $\eta(j) = 0$ if site j is empty. The infinitesimal generator of the Markov process is denoted by $\mathcal{L}_N$ and acts on functions $f : \Sigma_N \to \mathbb{R}$ as

$$(\mathcal{L}_N f)(\eta) = N^2 \sum_{j=0}^{N-1} c_{j,j+1}(\eta)\{f(\sigma^{j,j+1}\eta) - f(\eta)\},$$

where, for $1 \leq j \leq N-2$,

$$\begin{aligned} c_{j,j+1}(\eta) &= \left(1 + \frac{E}{N}\right)\eta(j)[1 - \eta(j+1)] + \eta(j+1)[1 - \eta(j)], \\ c_{0,1}(\eta) &= \left(1 + \frac{E}{N}\right)\eta(0)[1 - \eta(1)] + \eta(1)[1 - \eta(0)], \\ c_{N-1,N}(\eta) &= \left(1 + \frac{E}{N}\right)\eta(N-1)[1 - \eta(N)] + \eta(N)[1 - \eta(N-1)], \end{aligned}$$

with the convention, adopted throughout the article, that

$$\eta(0) = \alpha, \qquad \eta(N) = \beta. \tag{2.1}$$

In these formulas, $\sigma^{j,j+1}\eta$, $1 \leq j \leq N-2$, is the configuration obtained from η by exchanging the occupation variables $\eta(j)$, $\eta(j+1)$,

$$(\sigma^{j,j+1}\eta)(k) = \begin{cases} \eta(j+1), & k = j, \\ \eta(j), & k = j+1, \\ \eta(k), & k \neq j, j+1, \end{cases}$$

while $\sigma^{0,1}\eta = \sigma^1\eta$, $\sigma^{N-1,N}\eta = \sigma^{N-1}\eta$ are the configurations obtained by flipping the occupation variables $\eta(1)$, $\eta(N-1)$, respectively,

$$(\sigma^j\eta)(k) = \begin{cases} \eta(k), & k \neq j, \\ 1-\eta(k), & k = j. \end{cases}$$

2.2. *Hydrodynamic limit.* Let $D(\mathbb{R}_+, \Sigma_N)$ be the space of Σ_N-valued functions which are right continuous with left limits, endowed with the Skorohod topology. For a probability measure μ_N on Σ_N, denote by $\mathbb{P}_{\mu_N}$ the measure on $D(\mathbb{R}_+, \Sigma_N)$ induced by the Markov process η_t^N with initial distribution μ_N. We represent by $\mathbb{E}_{\mu_N}$ the expectation with respect to $\mathbb{P}_{\mu_N}$ and by E_{μ_N} the expectation with respect to μ_N.

Let $\pi_t^N(du)$, $t \geq 0$, be the positive random measure on [0, 1] obtained by rescaling space by N^{-1} and by assigning mass N^{-1} to each particle:

$$\pi_t^N(dx) = \frac{1}{N}\sum_{j=1}^{N-1} \eta_t^N(j)\delta_{j/N}(dx),$$

where $\delta_{j/N}$ is the Dirac mass at j/N.

Fix a measurable density profile $\rho_0 : [0,1] \to [0,1]$ and let $\{\mu_N : N \geq 1\}$ be a sequence of probability measures on Σ_N associated to ρ_0 in the sense that for every continuous function $G : [0,1] \to \mathbb{R}$ and every $\delta > 0$,

$$\lim_{N\to+\infty} \mu_N\left(\left|\frac{1}{N}\sum_{k=1}^{N-1} G(k/N)\eta(k) - \int_0^1 G(x)\rho_0(x)\,dx\right| > \delta\right) = 0.$$

Then, for each $t \geq 0$, π_t^N converges in $\mathbb{P}_{\mu_N}$-probability to a measure which is absolutely continuous with respect to the Lebesgue measure and whose density $\rho(t,x)$ is the unique weak solution of the viscous Burgers equation with Dirichlet's boundary conditions:

$$\begin{cases} \partial_t\rho = \partial_x^2\rho - E\partial_x\sigma(\rho), \\ \rho(t,0) = \alpha, \qquad \rho(t,1) = \beta, \qquad t \geq 0, \\ \rho(0,x) = \rho_0(x), \qquad 0 \leq x \leq 1, \end{cases} \tag{2.2}$$

where $\sigma(\rho)$ represents the mobility and is given by $\sigma(\rho) = \rho(1-\rho)$. We refer to [2, 7, 10, 11, 13] and references therein.

2.3. *Nonequilibrium fluctuations.* To define the space in which the fluctuations take place, denote by $C_0^2([0,1])$ the space of twice continuously differentiable functions on (0, 1) which are continuous on [0, 1] and which vanish at the boundary. Let $-\Delta$ be the positive operator, essentially self-adjoint on $L^2[0,1]$, defined by

$$-\Delta = -\frac{d^2}{dx^2}, \qquad \mathcal{D}(-\Delta) = C_0^2([0,1]).$$

Its eigenvalues and corresponding (normalized) eigenfunctions have the form $\lambda_n = (n\pi)^2$ and $e_n(x) = \sqrt{2}\sin(n\pi x)$, respectively, for any $n \ge 1$. By the Sturm–Liouville theory, $\{e_n, n \ge 1\}$ forms an orthonormal basis of $L^2[0,1]$.

We denote with the same symbol the closure of $-\Delta$ in $L^2[0,1]$. For any nonnegative integer k, we define the Hilbert spaces $\mathcal{H}_k = \mathcal{D}(\{-\Delta\}^{k/2})$, with inner product $(f,g)_k = (\{-\Delta\}^{k/2} f, \{-\Delta\}^{k/2} g)$, where $(\cdot,\cdot)$ is the inner product in $L^2[0,1]$. By the spectral theorem for self-adjoint operators,

$$\mathcal{H}_k = \left\{ f \in L^2[0,1] : \sum_{n=1}^{+\infty} n^{2k} (f, e_n)^2 < \infty \right\},$$

$$(f,g)_k = \sum_{n=1}^{+\infty} (n\pi)^{2k} (f, e_n)(g, e_n).$$

Moreover, if $\mathcal{H}_{-k}$ denotes the topological dual space of $\mathcal{H}_k$,

$$\mathcal{H}_{-k} = \left\{ f \in \mathcal{D}'(0,1) : \sum_{n=1}^{+\infty} n^{-2k} \langle f, e_n\rangle^2 < \infty \right\},$$

$$(f,g)_{-k} = \sum_{n=1}^{+\infty} (n\pi)^{-2k} \langle f, e_n\rangle \langle g, e_n\rangle,$$

where $\mathcal{D}'(0,1)$ represents the space of distributions on $(0,1)$ and $\langle f, \cdot\rangle$ the action of the distribution f on test functions.

Fix a continuous density profile $\rho_0 : [0,1] \to [0,1]$, and denote by $\rho(t,x)$ the unique weak solution of the viscous Burgers equation (2.2). Let Y_t^N represent the density fluctuation field which acts on functions H in $C^1([0,1])$ as

$$Y_t^N(H) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N-1} H(k/N)\{\eta_t(k) - \rho(t, k/N)\}. \tag{2.3}$$

Fix $t > 0$ and a function G in $C_0^2([0,1])$. Recall that we denote by $\rho(s,x) = \rho_s(x)$ the solution of the viscous Burgers equation (2.2). Let $(T_{t,s}G)(x) = G(s,x)$, $0 \le s \le t$, be the solution of the backward linear equation with final condition

$$\begin{cases} -\partial_s G = \partial_x^2 G + E(1 - 2\rho_s)\,\partial_x G, & \\ G(t,x) = G(x), & 0 \le x \le 1, \\ G(s,0) = G(s,1) = 0, & 0 \le s \le t. \end{cases} \tag{2.4}$$

Denote by $D([0,T], \mathcal{H}_{-k})$ the set of trajectories $Y : [0,T] \to \mathcal{H}_{-k}$ which are right continuous and have left limits, endowed with the Skorohod topology.

THEOREM 2.1. *Fix $T > 0$, a positive integer $k > 7/2$, and a density profile $\rho_0 : [0,1] \to [0,1]$ in $C^4([0,1])$ such that $\rho_0(0) = \alpha$, $\rho_0(1) = \beta$. Let $\{\mu_N : N \ge$*

$1\}$ *be a sequence of probability measures on* Σ_N *for which there exists a finite constant* A_2 *such that*

$$\sup_{N\geq 1}\max_{1\leq k\leq N-1} E_{\mu_N}\left[\left(\frac{1}{\sqrt{N}}\sum_{j=1}^{k}\{\eta_0(j)-\rho_0(j/N)\}\right)^4\right]\leq A_2. \tag{2.5}$$

Let Q_N *be the probability measure on* $D([0,T],\mathcal{H}_{-k})$ *induced by the density fluctuation field* Y^N *and the probability measure* μ_N. *Then the sequence* Q_N *is tight and all its limit points* Q^* *are concentrated on paths* Y *such that for all* $t\geq 0$ *and* G *in* $C_0^5([0,1])$,

$$\mathbb{W}(t,G):=Y_t(G)-Y_0(T_{t,0}G)$$

are mean-zero Gaussian random variables with covariances given by

$$\begin{aligned} &E_{Q^*}\big[\mathbb{W}(t,G)\mathbb{W}(s,H)\big] \\ &\quad = 2\int_0^{t\wedge s}\int_0^1 \sigma\big(\rho(r,x)\big)(\partial_x T_{t,r}G)(x)(\partial_x T_{s,r}H)(x)\,dx\,dr, \end{aligned}\tag{2.6}$$

for all $0\leq s,t\leq T$. *Moreover, for all* G *and* H *in* $C_0^5([0,1])$, *and* $t>0$,

$$E_{Q^*}\big[\mathbb{W}(t,G)Y_0(H)\big]=0.$$

REMARK 2.2. Note that in the definition (2.3) of the density fluctuation field $\rho(t,k/N)$ is used instead of $\mathbb{E}_{\mu_N}[\eta_t(k)]$. Part of the proof of Theorem 2.1 consists in justifying this replacement.

COROLLARY 2.3. *In addition to the hypotheses of Theorem* 2.1, *assume that* Y_0^N *converges to a zero-mean Gaussian field* Y *with covariance denoted by* $\langle\langle\cdot,\cdot\rangle\rangle$, *so that for all* G,H *in* $C^2([0,1])$,

$$\lim_{N\to\infty} E_{\mu^N}\big[Y_0^N(H)Y_0^N(G)\big]=\langle\langle H,G\rangle\rangle.$$

Then the sequence Q^N *converges to a mean-zero Gaussian measure* Q *whose covariances are given by*

$$\begin{aligned} E_Q\big[Y_t(G)Y_s(H)\big] &= \langle\langle T_{t,0}G, T_{s,0}H\rangle\rangle \\ &\quad + 2\int_0^{t\wedge s}\int_0^1 \sigma\big(\rho(r,x)\big)(\partial_x T_{t,r}G)(x)(\partial_x T_{s,r}H)(x)\,dx\,dr, \end{aligned}$$

for all $0\leq s,t\leq T$, H,G *in* $C_0^5([0,1])$.

This result is an immediate consequence of Theorem 2.1. Under any limit point Q^* of the sequence Q^N, for any function G in $C_0^5([0,1])$, $Y_t(G)$ can be written as the sum of two uncorrelated mean-zero Gaussian variables $\mathbb{W}(t,G)$ and $Y_0(T_{t,0}G)$.

Since under the measure Q, $\mathbb{W}(t, G)$ is a Brownian motion changed in time, the process Y_t may be understood as a generalized Ornstein–Uhlenbeck process described by the formal stochastic partial differential equation

$$dY_t = \mathcal{L}_t Y_t \, dt + \sqrt{2\sigma(\rho_t)} \nabla \, d\mathbb{W}_t,$$

where $\mathcal{L}_t$ is the linear differential operator $\partial_x^2 - E\partial_x[(1-2\rho_t)\cdot]$. The article is organized as follows. In Section 3, we introduce the microscopic Cole–Hopf transformation and we write the density fluctuation field as the sum of a current field and a remainder. In Section 4, we prove Theorem 2.1 and Corollary 2.3, assuming that the density field Y_t^N is tight and that three estimates are in force. In Sections 5–7, we prove these three estimates, and in Section 8 we prove tightness of Y_t^N. All proofs rely on estimates on the moments of the microscopic Cole–Hopf variables, presented in Section 9, and on estimates of the solutions of certain semi-discrete equations, presented in Section 10.

3. A microscopic Cole–Hopf transformation. To keep notation simple, from now on we drop the superscript N on the process η_t^N. Following [9, 11], we define in this section a microscopic Cole–Hopf transformation of the process η_t and an approximate inverse transformation. For $N \geq 1$, let

$$\Lambda_N^- = \{1, \ldots, N-1\}, \qquad \Lambda_N = \{0, \ldots, N-1\}, \qquad \Lambda_N^+ = \{0, \ldots, N\}.$$

For $0 \leq j, k \leq N$ with $|j-k| = 1$, denote by $J_t^{j,k}$, the total number of jumps from j to k in the time interval $[0, t]$, and let $W_t^{j,j+1}$ be the total current over the bond $\{j, j+1\}$, that is,

$$W_t^{j,j+1} = J_t^{j,j+1} - J_t^{j+1,j}.$$

In this formula, $J_t^{0,1}$ (resp., $J_t^{1,0}$) stands for the total number of particles created (resp., removed) at the left boundary, with a similar convention at the right boundary.

For $j \in \Lambda_N$, let $\xi_t(j)$ be the Cole–Hopf transformation of $\eta_t(j)$, which is defined as

$$\xi_t(j) = \exp\left\{(\gamma/N)\left[W_t^{j,j+1} - \sum_{k=1}^{j} \eta_0(k)\right]\right\}. \tag{3.1}$$

Since

$$\xi_t(j) - \xi_0(j) = \int_0^t \xi_{s-}(j)[e^{\gamma/N} - 1] \, dJ_s^{j,j+1} + \int_0^t \xi_{s-}(j)[e^{-\gamma/N} - 1] \, dJ_s^{j+1,j},$$

taking $\gamma = \gamma_N \leq 0$ such that $e^{-\gamma/N} = 1 + E/N$ we see that $\xi_t(j)$ can be written as

$$\xi_t(j) = \xi_0(j) + \int_0^t \xi_s(j) \mathfrak{g}_{j,j+1}(\eta_s) \, ds + \mathcal{M}_t^N(j), \tag{3.2}$$

where $\mathcal{M}_t^N(j)$ is a martingale with quadratic variation given by

$$\langle \mathcal{M}^N(j), \mathcal{M}^N(k)\rangle_t = \delta_{j,k} E^2 \int_0^t \xi_s(j)^2 \mathfrak{h}_j(\eta_s)\, ds. \tag{3.3}$$

Above $\delta_{j,k}$ is the delta of Kroenecker, and recalling the convention (2.1),

$$\begin{aligned} \mathfrak{g}_{j,j+1}(\eta) &= EN[\eta(j+1) - \eta(j)], \\ \mathfrak{h}_j(\eta) &:= e^{\gamma/N}\eta(j)[1-\eta(j+1)] + \eta(j+1)[1-\eta(j)]. \end{aligned} \tag{3.4}$$

By the continuity equation, for $1 \le j \le N-1$,

$$W_t^{j-1,j} - W_t^{j,j+1} = \eta_t(j) - \eta_0(j). \tag{3.5}$$

As a consequence, for $0 \le j \le N-2$, $1 \le k \le N-1$,

$$\begin{aligned} &\xi_t(j+1) - \xi_t(j) = \xi_t(j)\eta_t(j+1)[\exp\{-\gamma/N\} - 1], \\ &\xi_t(k-1) - \xi_t(k) = \xi_t(k)\eta_t(k)[\exp\{\gamma/N\} - 1]. \end{aligned} \tag{3.6}$$

These equations explain the choice of the term $\sum_{1\le k\le j} \eta_0(k)$ in the definition of $\xi_t(j)$. In view of the previous identities, by definition of $\mathfrak{g}_{j,j+1}$, and by the choice of γ,

$$\xi_t(j) = \xi_0(j) + \int_0^t (\Omega\xi_s)(j)\, ds + \mathcal{M}_t^N(j), \tag{3.7}$$

where $\Omega = \Omega^N$ is the linear operator defined on functions $f : \Lambda_N \to \mathbb{R}$ by

$$\begin{cases} (\Omega f)(0) = -\alpha ENf(0) + N(\nabla_N^+ f)(0), \\ (\Omega f)(j) = (\Delta_N f)(j) - E(\nabla_N^- f)(j), \qquad 1 \le j \le N-2, \\ (\Omega f)(N-1) = \beta ENf(N-1) - N\left(1 + \dfrac{E}{N}\right)(\nabla_N^- f)(N-1). \end{cases} \tag{3.8}$$

In this formula,

$$\begin{aligned} (\nabla_N^+ f)(j) &= N[f(j+1) - f(j)], \\ (\nabla_N^- f)(j) &= -N[f(j-1) - f(j)], \end{aligned}$$

and

$$(\Delta_N f)(j) = N^2[f(j+1) + f(j-1) - 2f(j)].$$

The advantage of the process ξ_t compared to the original process η_t is that it evolves according to the linear equation (3.7). Of course, the original process η_t can be recovered from ξ_t, since from (3.1) and by the continuity equation appearing in (3.5), for $1 \le j \le N-1$,

$$\eta_t(j) = -\frac{1}{\gamma}[\nabla_N^- \ln(\xi_t)](j).$$

Now, let $\lambda_t = \lambda_t^N$ be the solution of the linear equation

$$\begin{cases} (\partial_t \lambda_t)(j) = (\Omega \lambda_t)(j), & 0 \le j \le N-1, \\ \lambda_0(j) = \exp\left\{-(\gamma/N)\sum_{k=1}^{j} \rho_0(k/N)\right\}, \end{cases} \tag{3.9}$$

where $\rho_0 : [0,1] \to [0,1]$ is a density profile satisfying the assumptions of Theorem 2.1. For $j \in \Lambda_N^-$, let

$$r_t(j) = -\frac{1}{\gamma}[\nabla_N^- \ln(\lambda_t)](j). \tag{3.10}$$

Denote by $\tilde{Y}_t^N$, $t \ge 0$, the modified density fluctuation field defined on functions G in $C^1([0,1])$ by

$$\tilde{Y}_t^N(G) = \frac{1}{\sqrt{N}} \sum_{j=1}^{N-1} G(j/N)\{\eta_t(j) - r_t(j)\}.$$

Next, result asserts that the original density fluctuation field Y_t^N is close to the modified density field $\tilde{Y}_t^N$.

PROPOSITION 3.1. *For each $T > 0$,*

$$\sup_{N\ge 1} \sup_{0\le t\le T} \max_{1\le j\le N-1} N\left|r_t(j) - \rho(t, j/N)\right| < \infty.$$

Denote by J_t^N, $t \ge 0$, the current fluctuation field defined on functions $G \in C^1([0,1])$ by

$$J_t^N(G) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \frac{(\nabla_N^+ G)(j/N)}{\gamma \lambda_t(j)} \big(\xi_t(j) - \lambda_t(j)\big).$$

By the formula for $\eta_t(j)$ in terms of $\xi_t(j)$, and by (3.10), a summation by parts yields that for functions $G \in C_0^1([0,1])$

$$\tilde{Y}_t^N(G) = J_t^N(G) + R_t^N(G), \tag{3.11}$$

where the remainder $R_t^N(G)$ is given by

$$R_t^N(G) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \frac{1}{\gamma} (\nabla_N^+ G)(j/N)\left[\ln\left(\frac{\xi_t(j)}{\lambda_t(j)}\right) + 1 - \frac{\xi_t(j)}{\lambda_t(j)}\right].$$

Notice that both the current field J_t^N and the remainder R_t^N depend only on the process ξ_t. Sometimes, by abuse of notation, we consider that R_t^N acts on discrete functions $g : \{0, \dots, N\} \to \mathbb{R}$ instead of continuous functions $G : [0,1] \to \mathbb{R}$. This is the case in the next proposition.

The second result of this section asserts that the modified density fluctuation field $\tilde{Y}_t^N$ is close to the current fluctuation field J_t^N.

PROPOSITION 3.2. *Fix $T > 0$ and a function $\phi : [0, T] \times \Lambda_N^+ \to \mathbb{R}$ such that*

$$\sup_{0\le t\le T} \max_{j\in\Lambda_N} \big|(\nabla_N^+\phi_t)(j)\big| < \infty.$$

Then, for any $\delta > 0$,

$$\lim_{N\to+\infty} \mathbb{P}_{\mu_N}\Big[\sup_{0\le t\le T} \big|R_t^N(\phi_t)\big| > \delta\Big] = 0.$$

4. Proof of Theorem 2.1. Fix a density profile ρ_0 satisfying the assumptions of the theorem and denote by $\rho(t, x)$ the solution of the viscous Burgers equation (2.2) with initial condition ρ_0. Let $\{\mu_N : N \ge 1\}$ be a sequence of probability measures on Σ_N for which (2.5) holds.

Let $\phi : \Lambda_N \to \mathbb{R}$ be a strictly positive function. Denote by $\mathcal{A}_\phi = \mathcal{A}_\phi^N$ the difference operator which acts on functions $g : \Lambda_N^+ \to \mathbb{R}$ by

$$\begin{cases} (\mathcal{A}_\phi g)(0) = (\mathcal{A}_\phi g)(N) = 0, \\ (\mathcal{A}_\phi g)(j) = (\Delta_N g)(j) + E\dfrac{[1-\theta_\phi(j)]}{1+(E/N)\theta_\phi(j)}(\nabla_N^+ g)(j) - E\theta_\phi(j)(\nabla_N^- g)(j) \end{cases}$$

for $1 \le j \le N-1$, where

$$\theta_\phi(j) = \frac{(\nabla_N^-\phi)(j)}{E\phi(j-1)}.$$

Denote by λ_s the solution of (3.9). For $s \ge 0$, let $\mathcal{A}_s = \mathcal{A}_{\lambda_s}$, and let

$$\tilde{r}_s(j) := \theta_{\lambda_s}(j) = \frac{(\nabla_N^-\lambda_s)(j)}{E\lambda_s(j-1)}, \qquad 1 \le j \le N-1. \tag{4.1}$$

By Lemma 5.2 below, $|\tilde{r}_t(j) - \rho(t, j/N)| \le C_0/N$ uniformly in $0 \le t \le T$ and $1 \le j \le N-1$. Moreover, as $(\mathcal{A}_s g)(0) = (\mathcal{A}_s g)(N) = 0$, the solution g_s of the semi-discrete equation

$$\begin{cases} -(\partial_s g)(s, j) = (\mathcal{A}_s g)(s, j), & 0 \le j \le N, \\ g(t, j) = G(j/N), & 0 \le j \le N, \end{cases} \tag{4.2}$$

for some $t > 0$ and some G in $C_0^2([0, 1])$, is such that $g_s(0) = g_s(N) = 0$ for all $0 \le s \le t$. Hence, the semi-discrete equation (4.2) has to be understood as a discrete approximation of the differential equation (2.4).

Fix a function G in $C_0^2([0, 1])$ and $t > 0$. Let $g_s(j) = g_s^{N,t}(j)$ be the solution of (4.2). A long computation yields that for $0 \le s \le t$,

$$M_s^N(t, G) := J_s^N(g_s) - J_0^N(g_0) = \frac{1}{\sqrt{N}} \sum_{j\in\Lambda_N} \int_0^s \frac{(\nabla_N^+ g_r)(j)}{\gamma\lambda_r(j)}\, d\mathcal{M}_r^N(j), \tag{4.3}$$

where $\mathcal{M}_s^N(j)$ is the martingale introduced in (3.2). We present some details of this computation below equation (7.2).

PROPOSITION 4.1. *Fix a density profile* $\rho_0 : [0,1] \to [0,1]$ *and a sequence* $\{\mu_N : N \geq 1\}$ *of probability measures on* Σ_N *satisfying the assumptions of Theorem* 2.1. *Then, for each function* G *in* $C_0^2([0,1])$ *and* $t > 0$, *there exists a finite constant* C_0, *depending only on* G *and* t, *such that for all* $N \geq 1$,

$$\mathbb{E}_{\mu_N}\Big[\sup_{0\leq s\leq t} M_s^N(t,G)^4\Big] \leq C_0, \qquad \mathbb{E}_{\mu_N}\big[\langle M^N(t,G)\rangle_t^2\big] \leq C_0.$$

If G *belongs to* $C_0^5([0,1])$, *then the sequence of martingales* $M_s^N(t,G)$, $0 \leq s \leq t$, *converges in* $D([0,t],\mathbb{R})$ *to a mean-zero, continuous martingale, denoted by* $W_s(t,G)$. *For* G_1, G_2 *in* $C_0^5([0,1])$, $t_1, t_2 > 0$, *and* $0 \leq s_j \leq t_j$, *the covariances of* $W_{s_1}(t_1,G_1)$ *and* $W_{s_2}(t_2,G_2)$ *are given by*

$$\begin{aligned}&\mathbb{E}\big[W_{s_1}(t_1,G_1)W_{s_2}(t_2,G_2)\big]\\ &\quad = 2\int_0^{s_1\wedge s_2}\int_0^1 \sigma\big(\rho(r,x)\big)(\partial_x T_{t_1,r}G_1)(x)(\partial_x T_{t_2,r}G_2)(x)\,dx\,dr.\end{aligned}$$

REMARK 4.2. We note that since $W_s(t,G)$ is a continuous martingale whose quadratic variation is deterministic, $W_s(t,G)$ is a Brownian motion changed in time. In particular, $W_t(t,G)$ is a mean-zero Gaussian random variable.

PROOF OF THEOREM 2.1. Let Q^* be a limit point of the sequence Q_N, whose existence follows from the estimates of Section 8. Fix a function $G \in C_0^5([0,1])$ and $t > 0$. Let $g_s(j) = g_s^{N,t}(j)$ be the solution of (4.2) with final condition equal to G. By (3.11), Proposition 3.1 and (4.3),

$$Y_t^N(G) - Y_0^N(g_0) = M_t^N(t,G) + R_t^N(G) - R_0^N(g_0) + \frac{C_N}{\sqrt{N}},$$

where C_N is a sequence of numbers uniformly bounded. By Proposition 4.1 and in view of Remark 4.2, $M_s^N(t,G)$, $0 \leq s \leq t$, converges in distribution to a Brownian motion changed in time, denoted by $W_s(t,G)$. In particular, the variance of $W_t(t,G)$ is given by the right-hand side of (2.6), with $H = G$, $s = t$, $\mathbb{W}(r,J) = W_r(r,J)$.

Let $\psi(s,j) = (\nabla_N^+ g_{t-s})(j)/\lambda_{t-s}(j)$, $j \in \Lambda_N$, $0 \leq s \leq t$. By Remark 7.2 and by Proposition 3.2, $R_t^N(G)$ and $R_0^N(g_0)$ converges to 0 in probability. Recall that we denote by $T_{t,s}G$ the solution of equation (2.4). By Lemma 7.4, $|Y_0^N(g_0) - Y_0^N(T_{t,0}G)| \leq C_0/\sqrt{N}$. In conclusion, $Y_t^N(G) - Y_0^N(T_{t,0}G)$ converges in distribution to $W_t(t,G)$.

The covariance between $Y_0(H)$ and $W_t(t,G)$ vanishes because $W_s(t,G)$, $0 \leq s \leq t$ is a martingale which vanishes at $s = 0$.

To complete the proof, it remains to compute the covariance between $W_t(t,G)$ and $W_s(s,H)$. Assume that $s \leq t$. Since $W_r(t,G)$, $0 \leq r \leq t$, is a martingale,

$$E_{Q^*}\big[W_t(t,G)W_s(s,H)\big] = E_{Q^*}\big[W_s(t,G)W_s(s,H)\big].$$

By the polarization identity, we may express the covariance of a pair of random variables (X,Y) in terms of the variances of the variables $X+Y$ and $X-Y$. □

5. Proof of Proposition 3.1. The main result of this section asserts that the solution λ_t of the linear equation (3.9) (satisfied by the expectation of the Cole–Hopf variables ξ_t), is close to the Cole–Hopf transformation of the solution of the viscous Burgers equation (2.2).

Fix a profile $\rho_0 : [0,1] \to [0,1]$ in $C^4([0,1])$, and denote by $\rho(t,x)$ the solution of the hydrodynamic equation (2.2). Let $K(t,x)$ be the Cole–Hopf transformation of $\rho(t,x)$:

$$K(t,x) = \exp\left\{E\left[\int_0^t \{\partial_x \rho(s,x) - E\sigma(\rho(s,x))\}\,ds + \int_0^x \rho_0(y)\,dy\right]\right\}.$$

Since $\partial_t K = KE[\partial_x\rho - E\sigma(\rho)]$ and $\partial_x K = EK\rho$, K satisfies the linear parabolic equation with boundary conditions

$$\begin{cases} \partial_t K = \partial_x^2 K - E\partial_x K, & \\ (\partial_x K)(t,0) = E\alpha K(t,0), \qquad (\partial_x K)(t,1) = E\beta K(t,1), & 0 < t \le T, \\ K(0,x) = \exp\left\{E\int_0^x \rho_0(y)\,dy\right\}, & 0 \le x \le 1. \end{cases} \tag{5.1}$$

As ρ_0 belongs to $C^4([0,1])$, K_0 belongs to $C^5([0,1])$, and, by Lemma 10.1, K belongs to $C^{2,4}(\mathbb{R}_+ \times [0,1])$.

Denote by $\|f\|_M$ the sup norm of a function $f : \Lambda_N, \Lambda_N^\pm \to \mathbb{R}$:

$$\|f\|_M = \max_j |f(j)|,$$

where the maximum is carried over the domain of definition of f. By abuse of notation, if G belongs to $C([0,1])$, $\|G\|_M$ represents $\max_{0\le j\le N} |G(j/N)|$.

LEMMA 5.1. *Let λ_t and K_t be the solutions of* (3.9) *and* (5.1), *respectively. Then, for every $T > 0$,*

$$\sup_{N\ge 1}\sup_{0\le t\le T}\max_{0\le j\le N-1} N|\lambda_t(j) - K_t(j/N)| < +\infty,$$

$$\sup_{N\ge 1}\sup_{0\le t\le T}\max_{1\le j\le N-1} N|(\nabla_N^-\lambda_t)(j) - (\partial_x K_t)(j/N)| < +\infty.$$

PROOF. Fix $T > 0$. In this proof, C_0 represents a finite constant which may depend on the parameters E, β, α, on the initial condition ρ_0, and on T. Let $w_t(j) := \lambda_t(j) - K_t(j/N)$. A simple computation shows that

$$(\partial_t w_t)(j) = (\Omega w_t)(j) + \varphi(t,j), \tag{5.2}$$

where Ω has been introduced in (3.8) and where $\varphi(t,j)$ is given by

$$\begin{cases} N\{(\nabla_N^+ K_t)(j/N) - \alpha E K_t(j/N)\} - (\partial_t K_t)(j/N), & j = 0, \\ [(\Delta_N - \partial_x^2)K_t](j/N) - E[(\nabla_N^- - \partial_x)K_t](j/N), & 1 \le j \le N-2, \\ E\beta N K_t(j/N) - (N+E)(\nabla_N^- K_t)(j/N) - (\partial_t K_t)(j/N), & j = N-1. \end{cases}$$

In view of the boundary conditions satisfied by K_t, we may replace in the previous equation $\alpha E K_t(0)$ by $(\partial_x K_t)(0)$ and $E\beta K_t([N-1]/N)$ by $E\beta\{K_t([N-1]/N) - K_t(1)\} + (\partial_x K_t)(1)$. After these replacements, recalling that K_t and ρ_0 belong to $C^4([0,1])$, we obtain that $\varphi(t,j)$ is absolutely bounded by $C_0 N^{-1}$ for j in $\{1,\dots,N-2\}$ and by C_0 for $j=0$ and for $j=N-1$.

Let $G_t(j) = \varphi_t(j)\mathbf{1}\{1 \le j \le N-2\}$, $U_t(j) = \varphi_t(j) - G_t(j)$ so that $|G_t(j)| \le C_0 N^{-1}$. We may represent the solution w_t of (5.2) as

$$w_t = e^{\Omega t} w_0 + \int_0^t e^{\Omega(t-s)}(G_s + U_s)\,ds.$$

By Lemma 10.4, $\|e^{\Omega t} w_0\|_M$ is bounded by $C_0 e^{C_0 t}\|w_0\|_M \le C_0 N^{-1}$ and $\|e^{\Omega(t-s)} G_s\|_M$ is absolutely bounded by $C_0 e^{C_0(t-s)} N^{-1} \le C_0 N^{-1}$. Furthermore, since U_s vanishes everywhere except at two points, by Corollary 10.7, $\|e^{\Omega(t-s)} U_s\|_M \le C_0 (t-s)^{-1/2} N^{-1}$ for all N large enough. Putting together all the previous estimates, we conclude that $\|w_t\|_M$ is bounded by $C_0 N^{-1}$, proving the first assertion of the lemma.

We turn to the second assertion. Let

$$\gamma_t(j) = \begin{cases} [N/(N+E)]\alpha E \lambda_t(0), & j = 0, \\ (\nabla_N^- \lambda_t)(j), & 1 \le j \le N-1, \\ \beta E \lambda_t(N-1), & j = N. \end{cases}$$

It is not difficult to show that for $1 \le j \le N-1$, γ_t solves the equation

$$\partial_t \gamma_t(j) = (\Delta_N \gamma_t)(j) - E(\nabla_N^- \gamma_t)(j).$$

Clearly, $(\partial_x K)$ satisfies a similar equation where the discrete differential operators are replaced by continuous ones. Therefore, in view of (5.1), $w_t(j) = \{\gamma_t(j) - (\partial_x K)(t, j/N)\}$, $0 \le j \le N$, satisfies

$$\begin{cases} w_t(0) = \alpha E\{[N/(N+E)]\lambda_t(0) - K(t,0)\}, \\ \partial_t w_t(j) = (\Delta_N w_t)(j) - E(\nabla_N^- w_t)(j) + \varphi(t,j), & 1 \le j \le N-1, \\ w_t(N) = \beta E\{\lambda_t(N-1) - K(t,1)\}, \end{cases} \tag{5.3}$$

where $\varphi(t,j)$ accounts for the difference between the discrete and continuous derivatives, namely

$$\varphi(t,j) = (\Delta_N v_t)(j/N) - (\partial_x^2 v)(t, j/N) - E\{(\nabla_N^- v_t)(j/N) - (\partial_x v)(t, j/N)\},$$

where $v(t,j) = (\partial_x K)(t, j/N)$.

Since K_t belongs to $C^4([0,1])$, φ is absolutely bounded by $C_0 N^{-1}$ uniformly in t and j. By the first part of the proof and by Lemma 10.4, $w_t(0)$ and $w_t(N)$ are also absolutely bounded by $C_0 N^{-1}$.

Let $w_t^*(j)$ be the solution of (5.3) with the same initial condition satisfied by $w_t(j)$, but with boundary conditions $w_t^*(0) = C/N$, $w_t^*(N) = C/N$, where C is a finite constant such that $w_t(0) \vee w_t(N) \le C/N$ for all $0 \le t \le T$. By the maximum principle, see (10.3), $w_t(j) \le w_t^*(j)$ for $0 \le t \le T$, $0 \le j \le N$. Denote by $\Omega_\dagger$ the generator of a weakly asymmetric random walk on $\{0, \dots, N\}$ absorbed at 0 and N. We may represent w_t^* as

$$w_t^* = e^{\Omega_\dagger t} w_0^* + \int_0^t e^{\Omega_\dagger (t-s)} \varphi_s \, ds,$$

where above we extend the function φ to the boundaries by setting $\varphi(0) = \varphi(N) = 0$ and we repeat the arguments presented in the first part of the proof to conclude that $\|w_t^*\|_M \le C_0/N$. This provides an upper bound for w_t. A lower bound can be derived along the same lines. □

Recall the definition of $\tilde{r}_t$ given in (4.1).

LEMMA 5.2. *For every $T > 0$,*

$$\sup_{N \ge 1} \sup_{0 \le t \le T} \max_{1 \le j \le N-1} N \big| \tilde{r}_t(j) - \rho(t, j/N) \big| < \infty.$$

PROOF. By definition of $\tilde{r}_t$ and by the uniform lower bound for λ_t, proved in Lemma 10.5,

$$\big| \tilde{r}_t(j) - \rho(t, j/N) \big| \le C_0 \big| (\nabla_N^- \lambda_t)(j) - E\lambda_t(j-1)\rho(t, j/N) \big|$$

for some finite constant C_0, whose value may change from line to line. Since $(\partial_x K_t)(j/N) = E\rho(t, j/N) K_t(j/N)$ and since ρ is bounded, the right-hand side of the previous expression is less than or equal to

$$C_0 \big\{ \big| (\nabla_N^- \lambda_t)(j) - (\partial_x K_t)(j/N) \big| + \big| K_t(j/N) - \lambda_t(j-1) \big| \big\}.$$

The result follows from Lemma 5.1 and the smoothness of K. □

LEMMA 5.3. *For every $T > 0$,*

$$\sup_{N \ge 1} \sup_{0 \le t \le T} \max_{1 \le j \le N-2} \big| \nabla_N^+ \tilde{r}_t(j) \big| < \infty.$$

PROOF. Write

$$\begin{aligned} \big| \nabla_N^+ \tilde{r}_t(j) \big| \le N \big| \tilde{r}_t(j+1) - \rho\big(t, [j+1]/N\big) \big| \\ + N \big| \rho\big(t, [j+1]/N\big) - \rho(t, j/N) \big| \\ + N \big| \rho(t, j/N) - \tilde{r}_t(j) \big|. \end{aligned}$$

The first and third terms on the right-hand side of the last expression are bounded by the previous lemma. To complete the proof, it remains to recall that ρ is of class $C^{1,2}$. □

PROOF OF PROPOSITION 3.1. By Lemma 5.2, it is enough to show that

$$\sup_{0\le t\le T}\max_{1\le j\le N-1} N\left|r_t(j)-\tilde r_t(j)\right|\le C_0. \tag{5.4}$$

By definition of r_t and $\tilde r_t$, for $1\le j\le N-1$

$$r_t(j)=\frac{\log(1+[E/N]\tilde r_t(j))}{\log(1+[E/N])}.$$

Since, by Lemma 10.3,

$$0\le \tilde r_t(j)\le 1,$$

for $1\le j\le N-1$, $0\le t\le T$, (5.4) holds, which completes the proof of the proposition. □

6. Proof of Proposition 3.2.

Fix $T>0$ and a sequence of probability measures $\{\mu_N : N\ge 1\}$ fulfilling (2.5).

LEMMA 6.1. *For every $T>0$ and $\delta>0$,*

$$\lim_{N\to\infty}\mathbb{P}_{\mu_N}\left[\sup_{0\le t\le T}\frac{1}{\sqrt N}\sum_{j\in\Lambda_N}[\xi_t(j)-\lambda_t(j)]^2>\delta\right]=0.$$

PROOF. Fix $T>0$. It is enough to show that there exists a sequence $\{\tau_N : N\ge 1\}$ such that for each $\delta>0$

$$\lim_{N\to\infty}\sup_{0\le t\le T}\frac{1}{\tau}\mathbb{P}_{\mu_N}\left[\sup_{t\le s\le t+\tau}\frac{1}{\sqrt N}\sum_{j\in\Lambda_N}\left[\xi_s(j)-\lambda_s(j)\right]^2>\delta\right]=0. \tag{6.1}$$

A long and simple computation shows that for $t\le s$,

$$\begin{aligned}
&\frac{1}{\sqrt N}\sum_{j=0}^{N-1}[\xi_s(j)-\lambda_s(j)]^2-\frac{1}{\sqrt N}\sum_{j=0}^{N-1}[\xi_t(j)-\lambda_t(j)]^2\\
&\quad=\int_t^s\frac{2}{\sqrt N}\sum_{j=0}^{N-1}(\xi_r-\lambda_r)(j)\big[\Omega(\xi_r-\lambda_r)\big](j)\,dr\\
&\qquad+\int_t^s\frac{1}{\sqrt N}\sum_{j=0}^{N-1}\big\{(\Omega_2\xi_r^2)(j)-2\xi_r(j)(\Omega\xi_r)(j)\big\}\,dr\\
&\qquad-\int_t^s\frac{a_N}{\sqrt N}\sum_{j=0}^{N-1}\xi_r^2(j)\eta_r(j)\eta_r(j+1)\,dr+\{M_s-M_t\},
\end{aligned}\tag{6.2}$$

where $a_N=N^2\{e^{\gamma/N}-e^{-\gamma/N}+e^{-2\gamma/N}-1\}$ is a positive constant, M_t is a martingale and the operator Ω_2 is defined in (9.4).

Consider a sequence $\tau = \tau_N$ such that $N^{-1} \ll \tau_N \ll N^{-2/3}$. We show below that with this choice (6.1) holds for each term of the previous decomposition. For instance, by Lemma 9.2 and Chebyshev's inequality,

$$\lim_{N\to\infty} \sup_{0\le t\le T} \frac{1}{\tau}\mathbb{P}_{\mu_N}\left[\frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}[\xi_t(j)-\lambda_t(j)]^2 > \delta\right] = 0.$$

Hence, (6.1) holds for the second term on the left-hand side of (6.2) provided $N^{-1} \ll \tau_N$.

Repeating the arguments presented in the proof of Lemma 10.2, we can show that the expression inside the first integral on the right-hand side of (6.2) is bounded by

$$\frac{C_0}{\sqrt{N}}\sum_{j=0}^{N-1}[\xi_r(j)-\lambda_r(j)]^2$$

for some finite constant C_0. To show that (6.1) holds for this term, it is therefore enough to apply Markov inequality and to recall the statement of Lemma 9.2. No condition on τ_N is needed in this argument due to the time integral.

The expression inside the integral in the second term on the right-hand side of (6.2) is bounded by

$$\frac{C_0}{\sqrt{N}}\left\{\sum_{j=0}^{N-1} N^2[\xi_r(j+1)-\xi_r(j)]^2 + N\xi_r(0)^2 + N\xi_r(N-1)^2\right\}$$

for some finite constant C_0. By (9.1), $\xi_r(0)^2$ and $\xi_r(N-1)^2$ are bounded above by $C_0 N^{-1}\sum_{j\in\Lambda_N}\xi_r(j)^2$, and $|\xi_r(j+1)-\xi_r(j)|$ is less than or equal to $(e^{-\gamma/N}-1)\xi_r(j)$. The previous expression is thus less than or equal to $C_0 N^{-1/2}\sum_{j\in\Lambda_N}\xi_r(j)^2$. By the Chebyshev and Hölder inequalities,

$$\mathbb{P}_{\mu_N}\left[\sup_{t\le s\le t+\tau}\int_t^s \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}\xi_r(j)^2\,dr > \delta\right] \le \frac{N^2\tau^3}{\delta^4}\mathbb{E}_{\mu_N}\left[\int_t^{t+\tau}\frac{1}{N}\sum_{j=0}^{N-1}\xi_r(j)^8\,dr\right].$$

By Lemma 9.1, this expression is bounded above by $C_0 N^2\tau^4\delta^{-4}$. The contribution of the second term on the right-hand side of (6.2) to (6.1) is thus bounded by $C_0 N^2\tau^3\delta^{-4}$, which vanishes, as $N\to\infty$, provided $\tau_N \ll N^{-2/3}$.

Since the third term in (6.2) is negative, it remains to consider the martingale M_t. Its quadratic variation $\langle M\rangle_t$ is such that

$$\langle M\rangle_s - \langle M\rangle_t \le \int_t^s \frac{C_0}{N}\sum_{j=0}^{N-1}\xi_r(j)^2\left\{\frac{1}{N^2}\xi_r(j)^2 + [\xi_r(j)-\lambda_r(j)]^2\right\}dr$$

for some finite constant C_0 and all $t \le s$. Therefore, by Doob's inequality,

$$\mathbb{P}_{\mu_N}\Big[\sup_{t\le s\le t+\tau} |M_s - M_t| > \delta\Big] \le \frac{C_0}{\delta^2}\mathbb{E}_{\mu_N}\left[\int_t^{t+\tau} \frac{1}{N}\sum_{j=0}^{N-1} \xi_r(j)^2\left\{\frac{1}{N^2}\xi_r(j)^2 + [\xi_r(j) - \lambda_r(j)]^2\right\} dr\right].$$

By Lemmas 9.1 and 9.2, this expectation is bounded above by $C_0\tau N^{-1}$. Hence, (6.1) holds for the martingale part in (6.2), which proves the lemma. □

COROLLARY 6.2. *For every* $T > 0$, $\delta > 0$ *and* $a < 1$,

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}\Big[\sup_{0\le t\le T} \big|\xi_t(0) - \lambda_t(0)\big| > \delta\Big] = 0,$$

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}\left[\inf_{0\le t\le T} \frac{\xi_t(0)}{\lambda_t(0)} < a\right] = 0.$$

PROOF. By the triangular inequality, by Lemma 10.3 and by (9.1), $[\xi_t(0) - \lambda_t(0)]^2$ is bounded by

$$C_1\left\{\left(\frac{j}{N}\right)^2 \xi_t(0)^2 + [\xi_t(j) - \lambda_t(j)]^2 + \left(\frac{j}{N}\right)^2 \lambda_t(0)^2\right\},$$

for some finite constant C_1 and all $j \in \Lambda_N$. In view of Lemma 9.1 and Lemma 10.4, averaging over $0 \le j \le \varepsilon N$, the first assertion of the corollary follows from Lemma 6.1.

By Lemma 10.5, there exists a positive constant c_0, depending only on ρ_0, E, α, β and T, such that $\lambda_t(j) \ge c_0$ for all $0 \le t \le T$, $0 \le j \le N-1$. Let $\delta = c_0(1-a) > 0$ so that

$$\mathbb{P}_{\mu_N}\left[\inf_{0\le t\le T} \frac{\xi_t(0)}{\lambda_t(0)} < a\right] \le \mathbb{P}_{\mu_N}\Big[\sup_{0\le t\le T} \big|\xi_t(0) - \lambda_t(0)\big| > \delta\Big].$$

Hence, the second assertion of the corollary follows from the first one. □

PROOF OF PROPOSITION 3.2. By Lemma 10.3 and by (9.1), $\xi_t(j)/\lambda_t(j) \ge e^\gamma \xi_t(0)/\lambda_t(0)$ for all $j \in \Lambda_N$. Therefore, by the second assertion of Corollary 6.2, for every $a < e^\gamma$,

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}\left[\inf_{0\le t\le T} \min_{0\le j\le N-1} \frac{\xi_t(j)}{\lambda_t(j)} < a\right] = 0.$$

Fix $a < e^\gamma$ and denote by Λ_a^c the previous set of trajectories.

For each $0 < \delta < 1$, there exists a finite constant $C(\delta)$ such that

$$\big|\log(z) + 1 - z\big| \le C(\delta)|1 - z|^2, \qquad z \ge \delta.$$

Therefore, on the set Λ_a, by Lemma 10.5 applied to the function λ_t, for every function $\phi : [0, T] \times \Lambda_N^+ \to \mathbb{R}$ satisfying the assumptions of the proposition,

$$|R_t^N(\phi_t)| \le \frac{C_1}{\sqrt{N}} \sum_{j=0}^{N-1} |(\nabla_N^+ \phi_t)(j)| \frac{(\xi_t(j) - \lambda_t(j))^2}{\lambda_t^2(j)} \le \frac{C_1'}{\sqrt{N}} \sum_{j=0}^{N-1} (\xi_t(j) - \lambda_t(j))^2,$$

for some finite constant C_1. Hence, the assertion of the proposition follows from Lemma 6.1. □

7. Proof of Proposition 4.1. Fix a density profile ρ_0 satisfying the assumptions of Theorem 2.1 and denote by $\rho(t, x)$ the solution of the viscous Burgers equation (2.2) with initial condition ρ_0. Let $\{\mu_N : N \ge 1\}$ be a sequence of probability measures on Σ_N for which (2.5) holds.

Denote by Ω^* the adjoint operator of Ω with respect to the counting measure. An elementary computation gives that

$$\begin{cases} (\Omega^* f)(0) = (1-\alpha) ENf(0) + (E + N)(\nabla_N^+ f)(0), \\ (\Omega^* f)(j) = (\Delta_N f)(j) + E(\nabla_N^+ f)(j), \qquad 1 \le j \le N-2, \\ (\Omega^* f)(N-1) = -(1-\beta) ENf(N-1) - N(\nabla_N^- f)(N-1). \end{cases}$$

Note that Ω^* has exactly the same structure as Ω. Fix a function $\psi : \Lambda_N \to \mathbb{R}$, and denote by $\psi(s, j)$, $j \in \Lambda_N$, $s \ge 0$ the solution of

$$\begin{cases} \partial_s \psi_s = \Omega^* \psi_s, \\ \psi_0(j) = \psi(j). \end{cases} \tag{7.1}$$

LEMMA 7.1. *Assume that F belongs to $C^4([0, 1])$ and let $F(t, x)$ be the solution of the linear equation*

$$\begin{cases} \partial_s F = \partial_x^2 F + E\, \partial_x F, \\ F(0, x) = F(x) \qquad , x \in [0, 1], \end{cases}$$

with boundary conditions

$$(\partial_x F)(s, 0) = -(1-\alpha) EF(s, 0),$$
$$(\partial_x F)(s, 1) = -(1-\beta) EF(s, 1), \qquad s \ge 0.$$

Suppose that there exists a finite constant C_0 such that

$$\max_{j \in \Lambda_N} |\psi(j) - F(j/N)| \le C_0/N\cdot$$

Then, for every $T > 0$, there exists a finite constant C_0 such that

$$\sup_{0 \le t \le T} \max_{j \in \Lambda_N} |\psi(t, j) - F(t, j/N)| \le C_0/N.$$

PROOF. By the note following Lemma 10.1, F belongs to $C^{1,3}(\mathbb{R}_+ \times [0,1])$. As in the proof of Lemma 5.1, let $w_t(j) := \psi(t,j) - F(t, j/N)$. As F belongs to $C^{1,3}(\mathbb{R}_+ \times [0,1])$, equation (5.2) holds with Ω replaced by Ω^* for some function $\varphi(t,j)$ which is absolutely bounded by $C_0 N^{-1}$ for j in $\{1,\dots,N-2\}$ and by C_0 for $j=0$ and for $j=N-1$. Since, by assumption, the initial condition w_0 is also uniformly bounded by C_0/N, the arguments presented in the proof of the first assertion of Lemma 5.1 yield that let $w_t(j) := \psi(t,j) - F(t,j/N)$ is uniformly bounded by C_0/N. $\square$

Recall the definition of the operator $\mathcal{A}_\phi$ introduced at the beginning of Section 4. The proof of Proposition 4.1 relies on the following remarkable identity, derived from a long, but elementary, computation. For every pair of functions $g : \Lambda_N^+ \to \mathbb{R}$, $\phi : \Lambda_N \to \mathbb{R}$,

$$\Omega^*\left(\frac{\nabla^+ g}{\phi}\right)(j) - (\nabla^+ g)(j)\frac{(\Omega\phi)(j)}{\phi(j)^2} = \frac{[\nabla^+(\mathcal{A}_\phi g)](j)}{\phi(j)}, \qquad j \in \Lambda_N. \tag{7.2}$$

Identity (7.2) explains the second identity in (4.3). Indeed, for a time-independent function $g : \Lambda_N^+ \to \mathbb{R}$, since $\partial_s \lambda_s^{-1} = -\lambda_s^{-2}\Omega\lambda_s$, due to (3.9), (3.7) and an integration by parts,

$$\begin{aligned} &J_s^N(g) - J_0^N(g)\\ &\quad = \frac{1}{\sqrt{N}} \sum_{j\in\Lambda_N} \int_0^s \frac{(\nabla^+ g)(j)}{\gamma\lambda_r(j)}\, d\mathcal{M}_r^N(j) \\ &\qquad + \frac{1}{\gamma\sqrt{N}} \sum_{j\in\Lambda_N} \int_0^s \left\{\Omega^*\left(\frac{\nabla^+ g}{\lambda_s}\right)(j) - (\nabla^+ g)(j)\frac{(\Omega\lambda_s)(j)}{\lambda_s(j)^2}\right\}\xi_r(j)\, dr. \end{aligned} \tag{7.3}$$

By (7.2), the expression inside braces in the previous equation is equal to $[\nabla^+(\mathcal{A}_s g)](j)/\lambda_s(j)$, where $\mathcal{A}_s = \mathcal{A}_{\lambda_s}$. Hence, if we consider a time-dependent function g_s which solves (4.2), the additive part in the previous decomposition of $J_s^N(g_s) - J_0^N(g_0)$ vanishes, yielding (4.3).

REMARK 7.2. Fix a function G in $C_0^2([0,1])$ and $t > 0$. Let g_s be the solution of (4.2) with final condition equal to G, $g(t,j) = G(j/N)$, and let $\psi(s,j) = (\nabla_N^+ g_{t-s})(j)/\lambda_{t-s}(j)$, $j \in \Lambda_N$, $0 \le s \le t$. By (4.2) and (7.2), in the time interval $[0,t]$, $\psi(s,j)$ solves the equation (7.1) with initial condition

$$\psi(0,j) = (\nabla_N^+ G)(j/N)/\lambda_t(j).$$

In particular, by Lemmas 10.5 and 10.4, there exists a finite constant C_0 such that for all $N \ge 1$,

$$\sup_{0\le s\le t} \|\psi_s\|_M \le C_0. \tag{7.4}$$

REMARK 7.3. Similarly, let $G(s,x)$ be the solution of (2.4) with final condition $G(t,x) = G(x)$. A computation, based on a continuous version of equation (7.2), shows that in the time interval $[0,t]$, the function $F_s = \partial_x G_{t-s}/K_{t-s}$ solves the equation appearing in the statement of Lemma 7.1 with initial condition $F(0,x) = (\partial_x G)(x)/K(t,x)$.

Therefore, if G belongs to $C_0^5([0,1])$, since K belongs to $C^{2,4}(\mathbb{R}_+ \times [0,1])$, $F(0,x) = (\partial_x G)(x)/K(t,x)$ belongs to $C^4([0,1])$. Moreover, for ψ given in Remark 7.2, we conclude by Lemmas 10.5 and 5.1 that $\psi(0,j) - F(0,j/N)$ is uniformly bounded by C_0/N. Therefore, by Lemma 7.1, there exists a finite constant C_0 for which for all $N \ge 1$,

$$\sup_{0\le s\le t} \max_{j\in\Lambda_N} \left|\psi(s,j) - F(s,j/N)\right| \le C_0/N. \tag{7.5}$$

LEMMA 7.4. *Fix G in $C_0^5([0,1])$ and $t > 0$. Denote by $G(s,x)$ the solution of* (2.4) *with final condition equal to G, and by g the solution of* (4.2) *with the same final condition. Then there exists a finite constant C_0 such that for all $N \ge 1$,*

$$\left\|G(0,\cdot) - g(0,\cdot)\right\|_M \le C_0/N.$$

PROOF. Since $G(s,0) = g_s(0) = 0$ for $0 \le s \le t$, for every $j \in \Lambda_N$, by Remarks 7.2 and 7.3,

$$\begin{aligned}
&\left|G(0,j/N) - g_0(j)\right| \\
&\quad \le \frac{1}{N}\sum_{k=0}^{j-1}\left|(\nabla_N^+ G)(0,k/N) - (\nabla_N^+ g_0)(k/N)\right| \\
&\quad = \frac{1}{N}\sum_{k=0}^{j-1}\left|N\int_{k/N}^{(k+1)/N} F(t,y)K(0,y)\,dy - \psi(t,k/N)\lambda_0(k)\right|.
\end{aligned}$$

We have seen just before the statement of the lemma, that under the assumptions that G belongs to $C_0^5([0,1])$, $F(0,\cdot)$ belongs to $C^4([0,1])$. Therefore, by the proof of Lemma 7.1, F belongs to $C^{1,3}([0,t]\times[0,1])$. The assertion of the lemma follows from this remark, from the fact that ρ_0 belongs to $C^1([0,1])$ and from (7.5). □

LEMMA 7.5. *For each function G in $C_0^5([0,1])$ and $t > 0$, the quadratic variation $\langle M^N(t,G)\rangle_s$ of the martingale $M_s^N(t,G)$ converges in $L^1(\mathbb{P}_{\mu_N})$ to*

$$2\int_0^s\int_0^1 \sigma\big(\rho(r,x)\big)\big[(\partial_x T_{t,r}G)(x)\big]^2\,dx\,dr,$$

where $T_{t,r}G$ is the solution of (2.4).

PROOF. With the notation introduced in (3.4) and (3.7), the quadratic variation of the martingale $M_s^N(t,G)$ can be written as

$$\langle M^N(t,G)\rangle_s = \int_0^s \frac{E^2}{\gamma^2 N} \sum_{j\in\Lambda_N} \xi_r(j)^2 \mathfrak{h}_j(\eta_r)\psi_{t-r}(j)^2\,dr. \tag{7.6}$$

By (7.4), ψ is uniformly bounded in the time interval $[0,t]$. Since the cylinder functions $\mathfrak{h}_j$ are also bounded, by Lemma 9.2, we may replace $\xi_r(j)^2$ by $\lambda_r(j)^2$ in the previous formula paying the price of an error which converges to 0 in $L^1(\mathbb{P}_{\mu_N})$.

For two functions f, $g:\Lambda_N\to\mathbb{R}$, and $1\le \ell\le N/2$, since $b^2-a^2=(b-a)(b+a)$,

$$\frac{1}{N}\sum_{j=\ell}^{N-1-\ell}\frac{1}{2\ell+1}\sum_{k=-\ell}^{\ell}[f(j+k)^2-f(j)^2]g(j)$$
$$\le \frac{4\ell\|f\|_M\|g\|_M}{N}\sum_{j=0}^{N-2}|f(j+1)-f(j)|.$$

Applying this identity to $\ell=\varepsilon N$, $f=\lambda_r\psi_{t-r}$ and $g(j)=\mathfrak{h}_j$, by Lemma 10.2, we may replace in the quadratic variation of $M_s^N(t,G)$ the term $\lambda_r(j)^2\psi(t-r,j)^2$ by an average of these quantities over a macroscopic interval of length εN, paying the price of an error which vanishes in $L^1(\mathbb{P}_{\mu_N})$, as $N\uparrow\infty$ and then $\varepsilon\downarrow 0$. A summation by parts yields that

$$\langle M^N(t,G)\rangle_s = \int_0^s \frac{E^2}{\gamma^2 N}\sum_{j=\varepsilon N}^{(1-\varepsilon)N}\lambda_r(j)^2\psi(t-r,j)^2 V_{j,\varepsilon N}(\eta_r)\,dr + O(\varepsilon),$$

where $V_{j,\varepsilon N}(\eta)=(2\varepsilon N+1)^{-1}\sum_{|k|\le\varepsilon N}\mathfrak{h}_{j+k}(\eta)$. By Lemma 7.8 below, we may replace $V_{j,\varepsilon N}(\eta_r)$ by $2\rho_r(j/N)[1-\rho_r(j/N)]=2\sigma(\rho_r(j/N))$ with an error of the same type.

Up to this point, we proved that

$$\begin{aligned}\langle M^N(t,G)\rangle_s &\\ &= 2\int_0^s \frac{E^2}{\gamma^2 N}\sum_{j=\varepsilon N}^{(1-\varepsilon)N}\lambda_r(j)^2\psi(t-r,j)^2\sigma\big(\rho_r(j/N)\big)\,dr + O(\varepsilon)+R_{N,\varepsilon},\end{aligned}$$

where $R_{N,\varepsilon}$ is an error which vanishes in $L^1(\mathbb{P}_{\mu_N})$, as $N\uparrow\infty$ and then $\varepsilon\downarrow 0$. Note that the first term on the right-hand side is deterministic.

By Lemma 5.1, λ_s converges to K_s, and, by (7.5), ψ_s converges to $F_s=\partial_x G_{t-s}/K_{t-s}$ uniformly in time and space. Since $K_r^2F_{t-r}^2=(\partial_x G_r)^2$ and since γ converges to E, the lemma is proved. $\square$

LEMMA 7.6. *For each function G in $C_0^2([0,1])$ and $t > 0$, there exists a finite constant C_0, depending only on G and t, such that for all $N \geq 1$,*

$$\mathbb{E}_{\mu_N}\big[\langle M^N(t,G)\rangle_t^2\big] \leq C_0, \qquad \mathbb{E}_{\mu_N}\Big[\sup_{0\leq s\leq t} M_s^N(t,G)^4\Big] \leq C_0.$$

PROOF. We first estimate the quadratic variation $\langle M^N(t,G)\rangle_s$, given by (7.6). By (7.4), the solution ψ_s of equation (7.1) is uniformly bounded. As the cylinder function $\mathfrak{h}_j$ is also bounded, $\langle M^N(t,G)\rangle_s$ is less than or equal to

$$C_0 \int_0^s \frac{1}{N} \sum_{j\in\Lambda_N} \xi_r(j)^2\, dr.$$

The first assertion of the lemma follows therefore from Lemma 9.1 with $n = 2$.

We turn to the second assertion of the lemma. By the Burkholder–Davis–Gundy inequality and by [9], Lemma 3, the second expectation appearing in the statement of the lemma is bounded above by

$$C_0\Big\{\mathbb{E}_{\mu_N}\big[\langle M^N(t,G)\rangle_t^2\big] + \mathbb{E}_{\mu_N}\Big[\sup_{0\leq s\leq t} \big|M_s^N(t,G) - M_{s-}^N(t,G)\big|^4\Big]\Big\}$$

for some finite constant C_0. In view of the first part of the proof, it remains to estimate the fourth moment of the jumps. Clearly, $|M_s^N(t,G) - M_{s-}^N(t,G)| = |J_s^N(g_s) - J_{s_-}^N(g_s)|$. By the definition of J_s^N and of ψ_s, since $|\xi_{s-}(j)/\xi_s(j)| \leq e^{-\gamma/N}$, and since ψ_s is uniformly bounded, this latter quantity is less than or equal to

$$\frac{1}{\gamma\sqrt{N}} \sum_{j=0}^{N-1} \big|(\psi_{t-s})(j)\big|\big|\xi_s(j) - \xi_{s_-}(j)\big| \leq \frac{C_0}{N^{3/2}} \sum_{j=0}^{N-1} \xi_s(j).$$

The second assertion of the lemma follows from Schwarz inequality and from Lemma 9.1. □

LEMMA 7.7. *Fix G in $C_0^5([0,1])$ and $t > 0$. The sequence of martingales $M_s^N(t,G)$ introduced in (4.3) converges in $D([0,t],\mathbb{R})$ to a mean-zero, continuous martingale, denoted by $W_s(t,G)$. For G_1, G_2 in $C_0^5([0,1])$, $t_1, t_2 > 0$, and $0 \leq s_j \leq t_j$, the covariations of $W_{s_1}(t_1,G_1)$ and $W_{s_2}(t_2,G_2)$ are given by*

$$\mathbb{E}\big[W_{s_1}(t_1,G_1)W_{s_2}(t_2,G_2)\big] = 2\int_0^{s_1\wedge s_2}\int_0^1 \sigma\big(\rho(r,x)\big)(\partial_x T_{t_1,r}G_1)(x)(\partial_x T_{t_2,r}G_2)(x)\,dx\,dr.$$

PROOF. The proof of the convergence in $D([0,t],\mathbb{R})$ of the martingales $M_s^N(t,G)$ to a mean-zero, continuous martingale, whose quadratic variation is given by the right-hand side of the displayed equation appearing in the statement of the lemma with $G_j = G$ and $t_j = t$, relies on [12], Theorem VIII.3.12. We claim that conditions (3.14) and b-(iv) are fulfilled. Condition [γ_5-D] (defined in 3.3, page 470 of [12]) follows from Lemma 7.5. By Assertion VIII.3.5 in [12], condition [$\hat{\delta}_5$-D] and condition (3.14) are a consequence of

$$\lim_{N\to\infty} \mathbb{E}_{\mu_N}\Big[\sup_{s\le t}\big|M_s^N(t,G) - M_{s-}^N(t,G)\big|\Big] = 0,$$

an assertion which has been proved in the previous lemma.

It remains to prove the formula for the covariances. Fix G_1, G_2 in $C_0^5([0,1])$, $t_1, t_2 > 0$, $0 \le s_j \le t_j$, and let $s = s_1 \wedge s_2$. Since $M_s^N(t_j, G_j)$, $0 \le s \le t_j$, are martingales in $L^2(\mathbb{P}_{\mu_N})$, $\mathbb{E}_{\mu_N}[M_{s_1}^N(t_1, G_1) M_{s_2}^N(t_2, G_2)] = \mathbb{E}_{\mu_N}[M_s^N(t_1, G_1) \times M_s^N(t_2, G_2)]$. By the polarization identity, the computation of the covariance is reduced to the computation of the variance of the martingales $M_s^N(t_1, G_1) \pm M_s^N(t_2, G_2)$. In view of (4.3), the martingale $M_s^N(t_1, G_1) \pm M_s^N(t_2, G_2)$ can be represented as a martingale $M_s^N(t_1, t_2, G_1, G_2)$. The proof of Lemma 7.5 shows that the quadratic variation of this martingale converges in $L^1(\mathbb{P}_{\mu_N})$ to

$$2\int_0^s \int_0^1 \sigma\big(\rho(r,x)\big)\big[(\partial_x T_{t_1,r} G_1 \pm T_{t_2,r} G_2)(x)\big]^2 \, dx \, dr. \tag{7.7}$$

By the first part of the proof, the martingale $M_s^N(t_1, G_1) \pm M_s^N(t_2, G_2)$ converges in distribution to the martingale $W_s(t_1, G_1) \pm W_s(t_2, G_2)$. As the limit is continuous, the convergence in the Skorohod topology entails convergence in distribution at fixed times. Since, by Lemma 7.6, $M_s^N(t_1, G_1) \pm M_s^N(t_2, G_2)$ is bounded in $L^4(\mathbb{P}_{\mu_N})$,

$$\mathbb{E}\big[\{W_s(t_1, G_1) \pm W_s(t_2, G_2)\}^2\big] = \lim_{N\to\infty} \mathbb{E}_{\mu_N}\big[\{M_s^N(t_1, G_1) \pm M_s^N(t_2, G_2)\}^2\big]$$

which completes the proof of the lemma since the right-hand side converges to (7.7). □

We conclude this section stating a result which permits to replace cylinder functions by functions of the empirical measure. Denote by ν_ρ, $0 \le \rho \le 1$, the Bernoulli product measure on $\{0,1\}^{\mathbb{Z}}$ with density ρ. For a function $\mathfrak{h} : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ which depends only on a finite number of sites, let $\widehat{\mathfrak{h}}(\rho) = E_{\nu_\rho}[\mathfrak{h}(\eta)]$. Denote by $\tau_j \eta$, $j \in \mathbb{Z}$, $\eta \in \{0,1\}^{\mathbb{Z}}$, the configuration η translated by j: $(\tau_j \eta)(k) = \eta(j+k)$, $k \in \mathbb{Z}$. For a cylinder function $\mathfrak{h}$, whose support is represented by $\Lambda \subset \mathbb{Z}$, and for a configuration $\eta \in \Sigma_N$ the meaning of $\mathfrak{h}(\tau_j \eta)$ is clear provided $j + \Lambda \subset \{1, \dots, N\}$.

LEMMA 7.8. *Let $\{\mu_N : N \ge 1\}$ be a sequence of probability measures in Σ_N. For every continuous function $G : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ and every cylinder function $\mathfrak{h}$,*

$$\limsup_{N\to+\infty} \mathbb{E}_{\mu_N}\bigg[\int_0^t \bigg|\frac{1}{N}\sum_j G(s, j/N)\mathfrak{h}(\tau_j \eta_s) - \int_0^1 G(s,x)\widehat{\mathfrak{h}}\big(\rho(s,x)\big)\,dx\bigg|\,ds\bigg] = 0,$$

where $\rho(s,x)$ is the solution of the hydrodynamic equation (2.2) *and where the sum over j is carried over all j's for which the support of $\mathfrak{h}$ is contained in $\Sigma_N - j$.*

The proof of this result is similar to the one presented in [13], given the estimate presented in [2], Lemma 3.1.

8. Tightness of the density field. We prove in this section that the sequence $\{Y_t^N : N \ge 1\}$ is tight in $D(\mathbb{R}_+, \mathcal{H}_{-k})$ for $k > 7/2$. Recall from Section 2.3 the definition of the eigenfunctions $\{e_n : n \ge 1\}$ and of the eigenvalues $\{\lambda_n : n \ge 1\}$ of the operator $-\Delta$ defined on $C_0^2([0,1])$. Denote by $\|\cdot\|_{-k}$ the norm of $\mathcal{H}_{-k}$, defined as

$$\|f\|_{-k}^2 = \sum_{n\ge 1} \lambda_n^{-2k} \langle f, e_n\rangle^2.$$

By Propositions 3.1, 3.2 and by (3.11), to prove that the sequence $\{Y_t^N : N \ge 1\}$ is tight it is enough to show that the sequence $\{J_t^N : N \ge 1\}$ is tight: We claim that for every $k > 7/2$, $T > 0$, $\varepsilon > 0$,

$$\lim_{A\to\infty} \limsup_{N\to\infty} \mathbb{P}_{\mu_N}\Big[\sup_{0\le t\le T} \|J_t^N\|_{-k} > A\Big] = 0,$$

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \mathbb{P}_{\mu_N}\big[\omega_\delta(J_t^N) \ge \varepsilon\big] = 0,$$

where, for $\delta > 0$,

$$\omega_\delta(J_t^N) = \sup_{\substack{|s-t|<\delta \\ 0\le s,t\le T}} \|J_t^N - J_s^N\|_{-k}.$$

The first condition in the penultimate displayed equation is a consequence of part (a) of Corollary 8.2. The second condition follows from part (b) of that corollary and from Lemma 8.3.

LEMMA 8.1. *There exists a finite constant C_0, such that for every $n \ge 1$,*

$$\mathbb{E}_{\mu_N}\Big[\sup_{0\le t\le T} \langle J_t^N, e_n\rangle^2\Big] \le C_0 n^6.$$

PROOF. By (7.2) and (7.3),

$$J_t^N(e_n) = J_0^N(e_n) + \int_0^t J_s^N(\mathcal{A}_s e_n)\,ds + \mathcal{M}_t^N(e_n), \tag{8.1}$$

where $\mathcal{M}_t^N(e_n)$ is the martingale appearing on the right-hand side of (7.3) with $g = e_n$. We estimate separately each term of the previous expression. By Schwarz's inequality,

$$\mathbb{E}_{\mu_N}[J_0^N(e_n)^2] \le \frac{1}{\gamma^2} \sum_{j=0}^{N-1} \frac{(\nabla_N^+ e_n)(j/N)^2}{\lambda_0(j)^2} \mathbb{E}_{\mu_N}[\{\xi_0(j) - \lambda_0(j)\}^2].$$

By assumption (2.5), the expectation is bounded by C_0/N. Hence, since λ_0 is bounded below by a strictly positive constant, the previous sum is less than or equal to $C_0 n^2$.

We turn to the time integral term in the decomposition of $J_t^N(e_n)$. By Schwarz's inequality, and by the definition of J_t^N,

$$\mathbb{E}_{\mu_N}\left[\sup_{0\le t\le T}\left(\int_0^t J_s^N(\mathcal{A}_s^N e_n)\,ds\right)^2\right]$$
$$\le T\int_0^T \frac{1}{\gamma^2 N}\sum_{j,k=0}^{N-1}\frac{[\nabla_N^+(\mathcal{A}_s e_n)](j)[\nabla_N^+(\mathcal{A}_s e_n)](k)}{\lambda_s(j)\lambda_s(k)}\varphi_s(j,k)\,ds,$$

where $\varphi_s(j,k)=\mathbb{E}_{\mu_N}[\{\xi_s(j)-\lambda_s(j)\}\{\xi_s(k)-\lambda_s(k)\}]$. Recall from Lemma 10.5 that $\lambda_s(j)$ is bounded below by a strictly positive constant. By Lemma 9.2, $\sup_{0\le s\le T}\max_{j,k}|\varphi_s(j,k)|\le C_0/N$. On the other hand, in view of Lemma 5.3, by a Taylor expansion and since $(\mathcal{A}_s e_n)(0)=(\mathcal{A}_s e_n)(N)=0$,

$$\sup_{0\le s\le T}\max_{1\le j\le N-2}\bigl|[\nabla_N^+(\mathcal{A}_s e_n)](j)\bigr|\le C_0 n^3,$$
$$\sup_{0\le s\le T}\max_{k=0,N-1}\bigl|[\nabla_N^+(\mathcal{A}_s e_n)](k)\bigr|\le C_0 n^2 N.$$

It follows from these bounds that the penultimate displayed equation is bounded by $C_0 n^6$.

It remains to examine the martingale term in the decomposition of $J_t^N(e_n)$. By definition (7.3) of the martingale $\mathcal{M}_t^N(e_n)$, by Doob's inequality and by (3.3),

$$\mathbb{E}_{\mu_N}\Bigl[\sup_{0\le t\le T}\mathcal{M}_t^N(e_n)^2\Bigr]\le \mathbb{E}_{\mu_N}\left[\int_0^T\frac{4E^2}{\gamma^2 N}\sum_{j=0}^{N-1}\frac{(\nabla_N^+ e_n)(j)^2}{\lambda_s(j)^2}\xi_s(j)^2\mathfrak{h}_j(\eta_s)\,ds\right].$$

Since the cylinder functions $\mathfrak{h}_j$ are bounded and since, by Lemma 10.5, λ_s is uniformly bounded from below, by Lemma 9.1 this expression is less than or equal to $C_0 n^2$. This completes the proof of the lemma. □

COROLLARY 8.2. *For each $k>7/2$,*

(a) $$\limsup_{N\to+\infty}\mathbb{E}_{\mu_N}\Bigl[\sup_{0\le t\le T}\|J_t^N\|_{-k}^2\Bigr]<\infty,$$

(b) $$\lim_{m\to+\infty}\limsup_{N\to+\infty}\mathbb{E}_{\mu_N}\left[\sup_{0\le t\le T}\sum_{n\ge m}\langle J_t^N,e_n\rangle^2\lambda_n^{-2k}\right]=0.$$

PROOF. This result is a consequence of the previous lemma and of the observation that

$$\sup_{0\le t\le T}\|J_t^N\|_{-k}^2\le\sum_{n\ge1}\lambda_n^{-2k}\sup_{0\le t\le T}|J_t^N(e_n)|^2.$$

□

LEMMA 8.3. *For every $n \ge 1$ and every $\varepsilon > 0$,*

$$\lim_{\delta\to 0}\limsup_{N\to+\infty}\mathbb{P}_{\mu_N}\Big[\sup_{\substack{|s-t|<\delta\\ 0\le s,t\le T}}\big[J_t^N(e_n)-J_s^N(e_n)\big]^2>\varepsilon\Big]=0.$$

PROOF. Recall the decomposition of $J_t^N(e_n)$ presented at the beginning of the proof of Lemma 8.1. We first claim that for every $\varepsilon > 0$,

$$\lim_{\delta\to 0}\limsup_{N\to+\infty}\mathbb{P}_{\mu_N}\Big[\sup_{\substack{|s-t|<\delta\\ 0\le s,t\le T}}\big|\mathcal{M}_t^N(e_n)-\mathcal{M}_s^N(e_n)\big|>\varepsilon\Big]=0. \tag{8.2}$$

Denote by $\omega'_\delta(\mathbf{x})$ the modified modulus of continuity of a path $\mathbf{x}$ in $D([0,T],\mathbb{R})$ defined by

$$\omega'_\delta(\mathbf{x})=\inf_{\{t_i\}}\max_{0\le i\le r}\sup_{t_i\le s<t\le t_{i+1}}|\mathbf{x}_t-\mathbf{x}_s|,$$

where the infimum is taken over all partitions of $[0,T]$ such that $0\le i\le r$, $0=t_0<t_1<\cdots<t_r=T$ with $t_{i+1}-t_i>\delta$. Since $\omega_\delta(\mathbf{x})\le 2\omega'_\delta(\mathbf{x})+\sup_{t\le T}|\mathbf{x}_t-\mathbf{x}_{t-}|$, to prove (8.2) it is enough to show that for every $\varepsilon>0$

$$\begin{aligned}&\lim_{\delta\to 0}\limsup_{N\to+\infty}\mathbb{P}_{\mu_N}\big[\omega'_\delta(\mathcal{M}_t^N(e_n))>\varepsilon\big]=0,\\ &\limsup_{N\to+\infty}\mathbb{P}_{\mu_N}\Big[\sup_{t\le T}\big|\mathcal{M}_t^N(e_n)-\mathcal{M}_{t_-}^N(e_n)\big|>\varepsilon\Big]=0.\end{aligned}\tag{8.3}$$

Clearly, $|\mathcal{M}_t^N(e_n)-\mathcal{M}_{t_-}^N(e_n)|=|J_t^N(e_n)-J_{t_-}^N(e_n)|$. By definition of J_t^N and since $|\xi_{t-}(j)/\xi_t(j)|\le e^{-\gamma/N}$ this latter quantity is less than or equal to

$$\frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}\frac{|(\nabla_N^+e_n)(j)|}{\lambda_t(j)}\big|\xi_t(j)-\xi_{t_-}(j)\big|\le\frac{C_0 n}{N^{3/2}}\sum_{j=0}^{N-1}\xi_t(j).$$

The second condition of (8.3) follows from the previous estimate, from Markov inequality and from the fact that the expectation of $\xi_t(j)$ [which is equal to $\lambda_t(j)$] is uniformly bounded.

We turn to the first condition of (8.3). By Aldous's criterium, it is enough to show that for every $\varepsilon>0$

$$\lim_{\delta\to 0}\limsup_{N\to+\infty}\sup_{\substack{\tau\in\mathfrak{T}_\tau\\ 0\le\theta\le\delta}}\mathbb{P}_{\mu_N}\big[\big|\mathcal{M}_{\tau+\theta}^N(e_n)-\mathcal{M}_\tau^N(e_n)\big|>\varepsilon\big]=0,$$

where $\mathfrak{T}_\tau$ represents the set of stopping times bounded by T. By Chebyshev inequality and by the explicit expression for the quadratic variation of $\mathcal{M}_t^N(e_n)$ given in (7.6), the previous probability is bounded by

$$\mathbb{E}_{\mu_N}\Big[\int_\tau^{\tau+\theta}\frac{E^2}{\gamma^2\varepsilon^2 N}\sum_{j=0}^{N-1}\xi_s(j)^2\mathfrak{h}_j(\eta_s)\frac{(\nabla_N^+e_n)(j/N)^2}{\lambda_s(j)^2}\,ds\Big].$$

By Lemma 9.1 and Lemma 10.5, the previous expectation is bounded above by $C_0 n^2\delta/\varepsilon^2$, proving the first assertion of (8.3). This proves (8.2).

We claim that for every $\varepsilon > 0$

$$\lim_{\delta\to 0}\limsup_{N\to+\infty}\mathbb{P}_{\mu_N}\Big[\sup_{\substack{|s-t|<\delta\\ 0\le s,t\le T}}\Big|\int_s^t J_r^N(\mathcal{A}_r e_n)\,dr\Big| > \varepsilon\Big] = 0. \tag{8.4}$$

By Chebyshev's inequality, the previous probability is bounded by

$$\frac{\delta}{\varepsilon^2}\mathbb{E}_{\mu_N}\Big[\int_0^T\Big(\frac{1}{\gamma\sqrt{N}}\sum_{j=0}^{N-1}\frac{\nabla_N^+(\mathcal{A}_r e_n)(j/N)}{\lambda_r(j)}[\xi_r(j)-\lambda_r(j)]\Big)^2\,dr\Big].$$

The computations performed in the proof of Lemma 8.1 yield that the previous expression is bounded by $C_0 n^6\delta/\varepsilon^2$. This proves (8.4).

The assertion of the lemma is a consequence of (8.2), (8.4). $\square$

9. Exponential estimates. We present in this section some bounds on the process ξ_t. By (3.6) and by the definition of the variables $\xi_t(j)$, for $0\le j\le N-2$,

$$\xi_t(j)\le\xi_t(j+1)\le e^{-\gamma/N}\xi_t(j). \tag{9.1}$$

LEMMA 9.1. *Fix $n\ge 1$, $T>0$ and a sequence of probability measures $\{\mu_N : N\ge 1\}$ on Σ_N. There exists a finite constant C_1 and $N_0\ge 1$, depending only on n, β, E and T, such that for all $0\le j\le N-1$ and all $N\ge N_0$,*

$$\mathbb{E}_{\mu_N}\Big[\sup_{0\le t\le T}\xi_t(j)^n\Big]\le C_1.$$

PROOF. Fix $n\ge 1$ and $T>0$. In the proof C_1 represents a finite constant which depends only on n, β, T and E and which may change from line to line. We first claim that

$$\sup_{0\le t\le T}\max_{0\le j\le N-1}\mathbb{E}_{\mu_N}[\xi_t(j)^n]\le C_1. \tag{9.2}$$

A similar computation to the one performed just after (3.1) shows that for each $0\le j\le N-1$

$$\xi_t(j)^n = \xi_0(j)^n + \int_0^t\{[\Omega_n\xi_s^n](j) + A_n(s,j)\}\,ds + \mathcal{M}_n^N(t,j). \tag{9.3}$$

In this formula, $\mathcal{M}_n^N(\cdot,j)$ is a zero-mean martingale; Ω_n is the linear operator equal to Ω in the interior of Λ_N and given at the boundary by

$$\begin{cases}(\Omega_n f)(0) = -\alpha N R_n f(0) + N(\nabla_N^+ f)(0),\\ (\Omega_n f)(N-1) = \beta N S_n f(N-1) - N\Big(1+\dfrac{E}{N}\Big)(\nabla_N^- f)(N-1),\end{cases} \tag{9.4}$$

where

$$R_n = N\Big(1+\frac{E}{N}\Big)(1-e^{n\gamma/N}), \qquad S_n = N\big(e^{-n\gamma/N}-1\big)$$

and

$$A_n(t,j) = -N^2\Big\{\Big(1+\frac{E}{N}\Big)(e^{\gamma n/N}-1)+\big(e^{-\gamma n/N}-1\big)\Big\}\xi_t(j)^n\eta_t(j)\eta_t(j+1).$$

Notice that $A_1(t,j)=0$ and that $R_1=S_1=E$ so that $\Omega_1=\Omega$.

It follows from the previous computations that $f_n(t,j)=\mathbb{E}_{\mu_N}[\xi_t(j)^n]$ satisfies the differential inequality

$$\partial_t f(t,j) \le (\Omega_n f)(t,j).$$

Let $F_n(t,\cdot)$ be the solution of equation (3.9), with Ω_n instead of Ω and initial condition $F_n(0,j)=f_n(0,j)$. By the maximum principle, see (10.3), $f_n(t,\cdot)\le F_n(t,\cdot)$ for all $t\ge 0$. Claim (9.2) follows from Lemma 10.4 and the bound $F_n(0,j)\le \exp\{-\gamma n\}$.

It remains to bring the supremum inside the expectation. Since, by (9.1), $\xi_t(j)$ is increasing in j, it is enough to prove the lemma for $j=N-1$. However, by (9.1), $\xi_t(N-1)\le e^{-\gamma}\xi_t(j)$ so that

$$\mathbb{E}_{\mu_N}\Big[\sup_{0\le t\le T}\xi_t(N-1)^n\Big] \le e^{-\gamma n}\mathbb{E}_{\mu_N}\Bigg[\sup_{0\le t\le T}\frac{1}{N}\sum_{j=0}^{N-1}\xi_t(j)^n\Bigg].$$

By (9.3),

$$\xi_t(j)^n \le \xi_0(j)^n + \int_0^t [\Omega_n\xi_s^n](j)\,ds + \mathcal{M}_n^N(t,j).$$

We need therefore to estimate three terms. The first one is given by

$$\frac{1}{N}\sum_{j=0}^{N-1}\xi_0(j)^n \le e^{-\gamma n}.$$

The second one is also simple to handle. Since

$$\frac{1}{N}\sum_{j=0}^{N-1}[\Omega_n\xi^n](j) \le E\xi(0)^n + \beta N\big(e^{-\gamma n/N}-1\big)\xi(N-1)^n,$$

we have that

$$\mathbb{E}_{\mu_N}\Bigg[\sup_{0\le t\le T}\int_0^t\frac{1}{N}\sum_{j=0}^{N-1}[\Omega_n\xi_s^n](j)\,ds\Bigg] \le C_1\mathbb{E}_{\mu_N}\Bigg[\int_0^T\{\xi_s(0)^n+\xi_s(N-1)^n\}\,ds\Bigg].$$

By (9.2), this expression is bounded by a constant independent of N. To estimate the martingale term, apply Doob's inequality and use the fact that the martingales $\mathcal{M}_n^N(t,\cdot)$ are orthogonal to get that

$$\mathbb{E}_{\mu_N}\Bigg[\Bigg(\sup_{0\le t\le T}\frac{1}{N}\sum_{j=0}^{N-1}\mathcal{M}_n^N(t,j)\Bigg)^2\Bigg]\le \mathbb{E}_{\mu_N}\Bigg[\int_0^T\frac{C_1}{N^2}\sum_{j=0}^{N-1}\xi_t(j)^{2n}\,dt\Bigg].$$

By (9.2), this expression is bounded by C_1N^{-1}, which completes the proof of the lemma. □

LEMMA 9.2. *Let $\{\mu_N : N\ge 1\}$ be a sequence of measures on Σ_N satisfying* (2.5). *Then, for each fixed $T>0$, there exist finite constants C_1 and $N_0\ge 1$, depending only on E, β, T and A_2 such that*

$$\sup_{0\le t\le T}\max_{j\in\Lambda_N}\mathbb{E}_{\mu_N}\big[\big(\xi_t(j)-\lambda_t(j)\big)^4\big]\le\frac{C_1}{N^2}.$$

PROOF. For $0\le k\le N-1$ and $t\ge 0$, let $q_t(k,\cdot)$ be the solution of equation (3.9) with initial condition $q_0(k,j)=\delta_{k,j}$. By (3.7),

$$\xi_t(j)=\sum_{k=0}^{N-1}\xi_0(k)q_t(k,j)+\sum_{k=0}^{N-1}\int_0^t q_{t-s}(k,j)\,d\mathcal{M}_s^N(k),$$

so that

$$\xi_t(j)-\lambda_t(j)=\sum_{k=0}^{N-1}\big(\xi_0(k)-\lambda_0(k)\big)q_t(k,j)+\sum_{k=0}^{N-1}\int_0^t q_{t-s}(k,j)\,d\mathcal{M}_s^N(k).\tag{9.5}$$

To prove the lemma, we need to estimate the fourth moments of the terms on the right-hand side of (9.5).

By Hölder's inequality,

$$\begin{aligned}&E_{\mu_N}\Bigg[\Bigg(\sum_{k=0}^{N-1}\big(\xi_0(k)-\lambda_0(k)\big)q_t(k,j)\Bigg)^4\Bigg]\\&\qquad\le E_{\mu_N}\Bigg[\sum_{k=0}^{N-1}\big(\xi_0(k)-\lambda_0(k)\big)^4q_t(k,j)\Bigg]\Bigg(\sum_{k=0}^{N-1}q_t(k,j)\Bigg)^3.\end{aligned}$$

Notice that

$$\big|\xi_0(k)-\lambda_0(k)\big|\le\frac{C_1}{N}\Bigg|\sum_{j=1}^{k}\Big\{\eta_0(j)-\rho_0\Big(\frac{j}{N}\Big)\Big\}\Bigg|$$

for some finite constant C_1 which depends only on E, β, T, A_2, and whose value may change from line to line. Therefore, by assumption (2.5) and since, by (10.11),

$\sum_{k=0}^{N-1} q_s(k,j)$ is uniformly bounded in j and $0 \le s \le T$, the fourth moment of the first term on the right-hand side of (9.5) is bounded by C_1/N^2.

We turn to the martingale term in (9.5). For $0 \le r \le t$, let $\mathcal{M}_{j,t}^N(r)$ be the martingale defined by

$$\mathcal{M}_{j,t}^N(r) = \sum_{k=0}^{N-1} \int_0^r q_{t-s}(k,j)\, d\mathcal{M}_s^N(k).$$

By the Burkholder–Davis–Gundy inequality and [9], Lemma 3, there exists a finite constant C_0 such that

$$\mathbb{E}_{\mu_N}[\mathcal{M}_{j,t}^N(t)^4] \le C_0\Big\{\mathbb{E}_{\mu_N}[\langle\mathcal{M}_{j,t}^N\rangle_t^2] + \mathbb{E}_{\mu_N}\Big[\sup_{0\le s\le t} |\mathcal{M}_{j,t}^N(s) - \mathcal{M}_{j,t}^N(s-)|^4\Big]\Big\},$$

where $\langle\mathcal{M}_{j,t}^N\rangle_r$ stands for the quadratic variation of the martingale $\mathcal{M}_{j,t}^N$.

We first estimate the jump term. By (9.5) and by definition of ξ_s, $|\mathcal{M}_{j,t}^N(s) - \mathcal{M}_{j,t}^N(s-)| = |\xi_s(j) - \xi_{s-}(j)| \le (C_0/N)\xi_s(j)$. Hence, by Lemma 9.1, the second expectation on the right-hand side of the previous formula is bounded above by C_0/N^4.

It remains to examine the quadratic variation. By (3.3), the quadratic variation of the martingale $\mathcal{M}_{j,t}^N(r)$ is bounded above by

$$\begin{aligned} C_1 &\int_0^r \sum_{k=0}^{N-1} q_{t-s}(k,j)^2 \xi_s(k)^2\, ds \\ &\le C_1 \int_0^r \max_{0\le k\le N-1} q_{t-s}(k,j) \sum_{k=0}^{N-1} q_{t-s}(k,j)\xi_s(k)^2\, ds. \end{aligned}$$

By remark (10.11), $\sum_{k=0}^{N-1} q_s(k,j)$ is uniformly bounded in j and $0 \le s \le T$, and by Corollary 10.7, $\max_{0\le k\le N-1} q_{t-s}(k,j)$ is bounded above by $C_1\{N^2(t-s)\}^{-1/2}$ for all N large enough and all j. Since, by (9.1), $\xi_s(k) \le \xi_s(N-1)$, $0 \le k \le N-1$, the previous expression is less than or equal to

$$C_1 \int_0^r \frac{1}{N\sqrt{t-s}} \xi_s(N-1)^2\, ds.$$

Hence, by the Cauchy–Schwarz inequality,

$$\mathbb{E}_{\mu_N}[\langle\mathcal{M}_{j,t}^N\rangle_t^2] \le \frac{C_1}{N^2}\mathbb{E}_{\mu_N}\Big[\int_0^t \frac{1}{\sqrt{t-s}} \xi_s(N-1)^4\, ds\Big],$$

which completes the proof of the lemma in view of Lemma 9.1. □

10. The operators $\boldsymbol{\Omega_n}$. We prove in this section some properties of the solutions of the differential equation $\partial_t f_t = \Omega_n f_t$, where Ω_n is the linear operator defined by (3.8) and (9.4). We start with a result on classical solutions of the viscous Burgers equation (2.2).

LEMMA 10.1. *Let ρ_0 be a density profile in $C^4([0,1])$. Then the solution of the viscous Burgers equation (2.2) belongs to $C^{2,3}([0,\infty)\times[0,1])$ and the solution of the linear equation (5.1) belongs to $C^{2,4}([0,\infty)\times[0,1])$.*

PROOF. Since ρ_0 belongs to $C^4([0,1])$, K_0 defined by (5.1) belongs to $C^{2m+1}([0,1])$ with $m=2$. Therefore, the (generalized) Fourier series expansion of the solution K of (5.1) with initial condition K_0, provided by the method of separation of variables, yields that $K\in C^{m,2m}([0,\infty)\times[0,1])$. Moreover, since the semigroup corresponding to (5.1) is positivity improving and since K_0 is bounded below by a positive constant, so is K_t. Thus, $\rho(t,x)=\partial_x K/EK$, which solves the viscous Burgers equation, is well defined and belongs to $C^{2,3}([0,\infty)\times[0,1])$. Uniqueness of classical solutions of (2.2) completes the proof. □

Note: With the same notation as in the previous lemma, assume that K_0 belongs to $C^{2m+2}([0,1])$, $m\geq 0$, so that $\partial_x K_0\in C^{2m+1}([0,1])$. Since $\partial_x K$ satisfies the same equation as K, one obtains by the previous argument that $\partial_x K\in C^{m,2m}([0,\infty)\times[0,1])$, so that $K\in C^{m,2m+1}([0,\infty)\times[0,1])$.

We turn to the operator Ω_n, which should be understood as a small perturbation of Ω_0, obtained from Ω_n by setting $\alpha=\beta=0$, and which represents the generator of a weakly asymmetric random walk on Λ_N with reflection at the boundary.

Let m_N be the measure given by

$$m_N(k)=\left(1+\frac{E}{N}\right)^{-k},\qquad 0\leq k\leq N-1.$$

Denote by $\langle\cdot,\cdot\rangle_{m_N}$ the scalar product in $L^2(m_N)$. A calculation shows that for each $n\geq 0$, Ω_n is self-adjoint in $L^2(m_N)$, that is,

$$\langle g,\Omega_n f\rangle_{m_N}=\langle\Omega_n g,f\rangle_{m_N},\qquad f,g\in L^2(m_N).$$

For $p\geq 0$, denote by $\|\cdot\|_p$, the L^p norm with respect to m_N and by D_N the Dirichlet form associated to Ω_0 and m_N:

$$D_N(f)=\langle f,-\Omega_0 f\rangle_{m_N}=N^2\sum_{k=0}^{N-2}[f(k+1)-f(k)]^2 m_N(k).$$

The logarithmic Sobolev inequality for the weakly asymmetric random walk on Λ_N with reflection at the boundary [8], Example 3.6, states that there exists a finite constant A_0, depending only on E, such that

$$\sum_{k=0}^{N-1} f(k)^2\log f(k)^2 m_N(k)\leq A_0 D_N(f)\tag{10.1}$$

for all functions f such that $\|f\|_2 = 1$ and all $N \ge 2$.

Fix $n \ge 1$, an initial condition $f : \Lambda_N \to \mathbb{R}$ and denote by $f^{(n)}$ the solution of the linear differential equation

$$\partial_t f_t^{(n)} = \Omega_n f_t^{(n)}, \qquad f_0^{(n)} = f. \tag{10.2}$$

It is not difficult to prove a maximum principle for the solution of this linear equation,

$$f_t^{(n)} \ge 0 \qquad \text{for all } t \ge 0 \text{ if } f \ge 0, \tag{10.3}$$

and to deduce the existence of a unique solution.

LEMMA 10.2. *Fix $n \ge 1$ and let $f_t = f_t^{(n)}$ be the solution of* (10.2). *There exists a finite constant C_0, depending only on E, β and n, such that for any $t \ge 0$*

$$\|f_t\|_2^2 + \int_0^t D_N(f_s)\, ds \le e^{C_0 t} \|f_0\|_2^2.$$

PROOF. Fix $n \ge 1$. Differentiating $\|f_t\|_2^2$ yields

$$\begin{aligned} \frac{1}{2}\frac{d}{ds}\|f_t\|_2^2 &= -\alpha N R_n f_s(0)^2 m_N(0) \\ &\quad + \beta N S_n f_s(N-1)^2 m_N(N-1) - D_N(f_s). \end{aligned} \tag{10.4}$$

For every $1 \le m \le N$ and every $s \ge 0$,

$$f_s(N-1)^2 \le 2e^{-\gamma}\left(\frac{m}{N^2} D_N(f_s) + \frac{1}{m}\langle f_s, f_s\rangle_{m_N}\right). \tag{10.5}$$

Indeed, fix $1 \le m \le N$. By Young's inequality,

$$f_s(N-1)^2 \le 2\left(f_s(N-1) - \frac{1}{m}\sum_{k=N-m}^{N-1} f_s(k)\right)^2 + 2\left(\frac{1}{m}\sum_{k=N-m}^{N-1} f_s(k)\right)^2.$$

By Schwarz's inequality and since $m_N(k) \ge e^{\gamma}$ for $0 \le k \le N-1$, the second term on the right-hand side is less than or equal to

$$\frac{2}{m}\sum_{k=N-m}^{N-1} f_s(k)^2 \le \frac{2e^{-\gamma}}{m}\sum_{k=0}^{N-1} f_s(k)^2 m_N(k) = \frac{2e^{-\gamma}}{m}\langle f_s, f_s\rangle_{m_N}.$$

The first term on the right-hand side can be rewritten as

$$2\left(\frac{1}{m}\sum_{k=N-m}^{N-1}\sum_{j=k}^{N-2}[f_s(j+1) - f_s(j)]\right)^2 \le 2\sum_{k=N-m}^{N-1}\sum_{j=k}^{N-2}[f_s(j+1) - f_s(j)]^2.$$

Since $m_N(k) \geq e^{\gamma}$, this sum is bounded above by

$$2me^{-\gamma} \sum_{j=0}^{N-2} [f_s(j+1) - f_s(j)]^2 m_N(j) = 2e^{-\gamma} \frac{m}{N^2} D_N(f_s),$$

which proves (10.5).

Set $m = [Ne^{\gamma}/4\beta S_n] \wedge N$, where $[a]$ represents the integer part of a. Putting together (10.4) and (10.5) yields

$$\frac{d}{ds} \langle f_s, f_s \rangle_{m_N} \leq -D_N(f_s) + C_0 \langle f_s, f_s \rangle_{m_N}.$$

To conclude the proof it remains to apply Gronwall's inequality. □

Next result shows that the solutions of (10.2) are monotone.

LEMMA 10.3. *Fix $n \geq 1$ and a nonnegative initial condition $f_0 : \Lambda_N \to \mathbb{R}$ such that $f_0(j) \leq f_0(j+1)$, $0 \leq j < N-1$. Then the solution $f_t = f_t^{(n)}$ of* (10.2) *conserves the monotonicity*:

$$f_t(j) \leq f_t(j+1)$$

for all $t \geq 0$ and $0 \leq j < N-1$. Conversely, if the nonnegative initial condition is such that $f_0(j+1) \leq e^{-\gamma n/N} f_0(j)$, $0 \leq j < N-1$, the same property holds at later times:

$$f_t(j+1) \leq e^{-\gamma n/N} f_t(j)$$

for all $t \geq 0$ and $0 \leq j < N-1$.

PROOF. For $t > 0$, $j \in \{1, \dots, N-1\}$, let $g_t(j) = f_t(j) - f_t(j-1)$. It is easy to show that g_t satisfies an equation of the form

$$\frac{d}{dt} g_t = \widetilde{\Omega}_n g_t + \psi_t, \tag{10.6}$$

where all the entries in ψ_t are null except for the first and the last which are equal to $\alpha N R_n f_t(0)$ and $\beta N S_n f_t(N-1)$, respectively.

Moreover, $\widetilde{\Omega}_n$ is a tridiagonal matrix whose diagonal elements are equal to $-N^2(2 + E/N)$, upper off-diagonal elements equal N^2 and lower off-diagonal elements are equal to $N^2(1 + E/N)$.

We may now apply the maximum principle, see (10.3), to conclude the proof of the first assertion of the lemma because, as already seen, the solution f_t is nonnegative. Alternatively, we can recall the observation (see [17], Exercise 97, page 375) that for any $t > 0$ the exponential e^{At} of a matrix A has all its entries positive if and only if all the off-diagonal elements of A are nonnegative. Since that holds for $\widetilde{\Omega}_n$ and Ω_n, then g_t, which can be written as

$$g_t = e^{\widetilde{\Omega}_n t} g_0 + \int_0^t e^{\widetilde{\Omega}_n (t-s)} \psi_s \, ds,$$

is nonnegative.

The same argument applies to the second assertion. For $t > 0$, $j \in \{1, \dots, N-1\}$, let $g_t(j) = e^{-\gamma n/N} f_t(j-1) - f_t(j)$. Then, g_t satisfies the equation (10.6) where all the entries in ψ_t are null except for the first and the last which are equal to $N(N+E)(1-\alpha)(e^{-\gamma n/N} - 1) f_t(0)$ and $N^2(1-\beta)(e^{-\gamma n/N} - 1) f_t(N-1)$, respectively. □

LEMMA 10.4. *Fix $n \ge 1$ and let $f_t = f_t^{(n)}$ be the solution of* (10.2). *There exists a finite constant C_0, depending only on E, β and n, such that for any $t \ge 0$*

$$\|f_t\|_M \le C_0 e^{C_0 t} \|f_0\|_M,$$

for all $t \ge 0$.

PROOF. Let g_0 be the function which is constant and equal to $\|f_0\|_M$ and denote by g_t the solution of (10.2) with initial condition g_0. By the maximum principle, see (10.3) $f_t(j)^2 \le g_t(j)^2$, for all $1 \le j \le N$, $t \ge 0$.

By Lemma 10.3, $e^{n\gamma} g_t(k) \le g_t(j) \le e^{-n\gamma} g_t(k)$ for all $0 \le j, k \le N-1$, $t \ge 0$, which together with $m_N(j) \ge e^{\gamma}$, $0 \le j \le N-1$, gives that $\|g_t\|_M^2 \le e^{-(2n+1)\gamma} N^{-1} \|g_t\|_2^2$. By Lemma 10.2, $\|g_t\|_2^2 \le e^{C_0 t} \|g_0\|_2^2$. In conclusion,

$$f_t(j)^2 \le C_0 e^{C_0 t} N^{-1} \|g_0\|_2^2 \le C_0 e^{C_0 t} \|g_0\|_M^2 = C_0 e^{C_0 t} \|f_0\|_M^2,$$

which proves the lemma. □

Fix $n \ge 1$ and denote by $q_t(j, \cdot) = q_t^{(n)}(j, \cdot)$ the solution of the linear equation (10.2) with initial condition $q_0(j,k) = \delta_{j,k}$. Fix a function $f : \Lambda_N \to \mathbb{R}$. We may represent the solution f_t of (10.2) with initial condition f as $f_t(k) = \sum_{j \in \Lambda_N} f(j) q_t(j,k)$. In the particular case where $f(k) = 1$ for all $k \in \Lambda_N$, by Lemma 10.4,

$$\max_{k \in \Lambda_N} \sum_{j \in \Lambda_N} q_t(j,k) = \max_{k \in \Lambda_N} f_t(k) \le C_0 e^{C_0 t}.$$

LEMMA 10.5. *Fix $n \ge 1$, a strictly positive initial condition $f_0 : \Lambda_N \to \mathbb{R}$ and let f_t be the solution of* (10.2). *For every $T > 0$, there exists a positive constant c_0, depending only on f_0, E, α, β and T, such that*

$$c_0 \le f_t(j)$$

for all $0 \le t \le T$, $j \in \Lambda_N$.

PROOF. By the maximum principle [see (10.3)], it is enough to prove the lemma for a constant initial profile. Assume, therefore, that $f_0(j) = a$ for all $j \in \Lambda_N$ and for some $a > 0$. A simple computation shows that

$$\frac{d}{dt} \frac{1}{N} \sum_{j=0}^{N-1} f_t(j) m_N(j) = \frac{1}{N} \sum_{j=0}^{N-1} (\Omega_n f_t)(j) m_N(j) \ge -\alpha R_n f_t(0) m_N(0).$$

By Lemma 10.3, $f_t(0) \le N^{-1}\sum_{0\le j\le N-1} f_t(j)$. On the other hand, $m_N(0) = 1 \le m_N(j)e^{-\gamma}$ for all $j \in \Lambda_N$. Hence,

$$\frac{d}{dt}\frac{1}{N}\sum_{j=0}^{N-1} f_t(j)m_N(j) \ge -\alpha R_n e^{-\gamma}\frac{1}{N}\sum_{j=0}^{N-1} f_t(j)m_N(j).$$

Therefore, by Gronwall's inequality and since R_n is bounded above by a finite constant independent of N,

$$\frac{1}{N}\sum_{j=0}^{N-1} f_t(j)m_N(j) \ge e^{-At}\frac{1}{N}\sum_{j=0}^{N-1} f_0(j)m_N(j) \ge ae^{\gamma}e^{-At}.$$

A constant profile satisfies both conditions of Lemma 10.3. We may therefore apply this lemma to bound above $N^{-1}\sum_{j\in\Lambda_N} f_t(j)$ by $C_0 \min_{k\in\Lambda_N} f_t(k)$, which completes the proof since $m_N(j) \le 1$. $\square$

The next result provides a bound for the fundamental solution of (10.2). The proof is based on the classical arguments of hypercontractivity [4, 8]. We need, however, to estimate additional terms which appear because Ω_n is not a generator.

For $\varepsilon > 0$, let $\delta = \varepsilon/(1+\varepsilon)$, and let $\varphi_\varepsilon : [0,1] \to [\delta, 1-2\varepsilon]$ be given by

$$\varphi_\varepsilon(t) = \begin{cases} \sqrt{\delta^2 + t}, & \text{for } 0 \le t \le 1/8, \\ 1 - \sqrt{4\varepsilon^2 + 1 - t}, & \text{for } 7/8 \le t \le 1. \end{cases}$$

We complete the definition of φ_ε in the interval $[1/8, 7/8]$ in a way to obtain an increasing C^1 function whose derivative in the interval $[1/8, 7/8]$ is bounded by 2. Note that this bound is compatible with $\varphi'_\varepsilon(1/8)$ and $\varphi'_\varepsilon(7/8)$, which are both bounded by $\sqrt{2}$.

Actually, the exact form of φ_ε is irrelevant for the proof of Lemma 10.6. The only important properties needed are that

$$\int_0^1 \frac{1}{\varphi_\varepsilon(t)[1-\varphi_\varepsilon(t)]}\,dt < \infty \quad \text{and} \quad \int_0^1 \dot\varphi_\varepsilon(t)\log\frac{\dot\varphi_\varepsilon(t)}{\varphi_\varepsilon(t)[1-\varphi_\varepsilon(t)]}\,dt < \infty,$$

where $\dot\varphi_\varepsilon(t)$ represents the derivative of φ_ε.

LEMMA 10.6. *Fix $n \ge 1$ and recall that we denote by $q_t(j,\cdot)$ the solution of the linear equation (10.2) with initial condition $q_0(j,k) = \delta_{j,k}$. Assume that $N \ge n+1$ and let $A_1 = -\gamma n\beta$. There exists a finite constants C_0, depending only on E, β and n, such that*

$$\max_{0\le j,k\le N-1} q_T(j,k) \le \frac{C_0 e^{C_0 T}}{\sqrt{N^2 T}}$$

for all T *such that*

$$\begin{aligned}\log(TN^2) \ge 16, \qquad \log(TN^2) \le \sqrt{\frac{TN^2}{8A_0}},\\ \log(TN^2) \le N\Big(1 \wedge \frac{1}{8e^E A_1}\Big),\end{aligned} \tag{10.7}$$

where A_0 *is given in* (10.1).

PROOF. Here, we follow [14, 15]. In this proof, C_0 represents a finite constant depending only on β, E and n, which may change from line to line.

Fix $0 \le k \le N-1$ and T in the range (10.7). Let $\varepsilon^{-1} = \log(TN^2)$, $p : [0, T] \to [1+\varepsilon, 2\varepsilon^{-1}]$ be given by $p(t) = [1 - \varphi_\varepsilon(t/T)]^{-1}$. Set $f_t(\cdot) = q_t(x, \cdot)$, $u_t^2 = f_t^{p(t)}$, $v_t^2 = u_t^2/\|u_t\|_2^2$. An elementary computation, identical to the one presented at the beginning of the proof of Theorem 2.1 in [14], gives that

$$\begin{aligned}\frac{d}{dt}\log\|f_t\|_{p(t)} \le \frac{\dot p(t)}{p(t)^2}\int v_t^2 \log v_t^2\, dm_N \\ - \frac{2[p(t)-1]}{p(t)^2} D_N(v_t) + A_1 N v_t(N-1)^2.\end{aligned} \tag{10.8}$$

Set

$$\ell(t)^2 = N^2\Big\{\frac{p(t)-1}{4A_0\dot p(t)} \wedge 1\Big\} = \frac{TN^2}{A_0}\Big\{\frac{\varphi_\varepsilon(t/T)[1-\varphi_\varepsilon(t/T)]}{4\dot\varphi_\varepsilon(t/T)} \wedge \frac{A_0}{T}\Big\}.$$

By the second condition in (10.7), $\ell(t) \ge 1$. Divide the interval Λ_N in subintervals of length $\ell(t)$. The last interval has length between $\ell(t)$ and $2\ell(t)-1$. By the logarithmic Sobolev inequality (10.1) and by the proof of Lemma 4.3 of [14], since $m_N(k) \ge e^\gamma$, the first term on the right-hand side of (10.8) is less than or equal to

$$\frac{\dot p(t)}{p(t)^2}\Big\{A_0 \frac{4\ell(t)^2}{N^2} D_N(v_t) - \log\big[e^\gamma \ell(t)\big]\Big\}.$$

By definition of $\ell(t)$, the right-hand side of (10.8) is bounded by

$$-\frac{\dot p(t)}{2p(t)^2}\log\big[e^{2\gamma}\ell(t)^2\big] - \frac{[p(t)-1]}{p(t)^2} D_N(v_t) + A_1 N v_t(N-1)^2. \tag{10.9}$$

Let

$$m(t) = N\frac{p(t)-1}{p(t)^2}\Big\{\frac{1}{2e^E A_1} \wedge 4\Big\} = N\varphi_\varepsilon(t/T)\big[1-\varphi_\varepsilon(t/T)\big]\Big\{\frac{1}{2e^E A_1} \wedge 4\Big\}.$$

Notice that $m(t) \le N$, because $0 \le p(t)^{-1} \le 1$. On the other hand, as $p(t)^{-1}[1 - p(t)^{-1}] \ge \{4\log(TN^2)\}^{-1}$ and $N \ge \log(TN^2)\{8e^E A_1 \vee 1\}$, we have that

$m(t) \ge 1$. Adding and subtracting the average of $v_t(j)$ over the interval $\{N - m(t), \dots, N-1\}$, and repeating the same argument as in the proof of Lemma 10.2, since $-\gamma \le E$, we obtain that

$$v_t(N-1)^2 \le 2m(t) \sum_{j=N-m(t)}^{N-2} \{v_t(j+1) - v_t(j)\}^2 + \frac{2}{m(t)} \sum_{j=N-m(t)}^{N-1} v_t(j)^2$$
$$\le \frac{2e^E m(t)}{N^2} D_N(v_t) + \frac{2e^E}{m(t)}$$

because $\|v(t)\|_2 = 1$. By the definition of $m(t)$, the first term of this expression multiplied by $A_1 N$ may be absorbed by the Dirichlet form in (10.9). Hence, (10.9) is less than or equal to

$$-\frac{\dot p(t)}{2p(t)^2} \log[e^{2\gamma} \ell(t)^2] + C_0 \frac{p(t)^2}{p(t)-1}\cdot$$

Up to this point, we proved that

$$\log\left(\frac{\|f_T\|_{p(T)}}{\|f_0\|_{p(0)}}\right) \le -\int_0^T \frac{\dot p(t)}{2p(t)^2} \log[e^{2\gamma}\ell(t)^2]\, dt + C_0 \int_0^T \frac{p(t)^2}{p(t)-1}\, dt\cdot \tag{10.10}$$

Since $\dot p(t)/p(t)^2 = T^{-1}\dot\varphi_\varepsilon(t/T)$, in view of (10.7), the first term on the right-hand side is less than or equal to

$$-\frac{1}{2}\log(TN^2) + C_0 + \frac{1}{2}\int_0^1 \dot\varphi_\varepsilon(t) \log\left\{\frac{\dot\varphi_\varepsilon(t)}{\varphi_\varepsilon(t)[1-\varphi_\varepsilon(t)]} \vee \frac{T}{4A_0}\right\} dt.$$

Since $\log(a \vee b) \le \log_+ a + \log_+ b$, where $\log_+ a = \log a \vee 0$, the previous integral can be estimated by the sum of two terms. The first one is $\log_+(T/4A_0) \le C_0 T$, while the second one is

$$\frac{1}{2}\int_0^1 \dot\varphi_\varepsilon(t) \log_+\left\{\frac{\dot\varphi_\varepsilon(t)}{\varphi_\varepsilon(t)[1-\varphi_\varepsilon(t)]}\right\} dt.$$

On the interval $[1/8, 7/8]$, $\dot\varphi_\varepsilon(t)$ is bounded by 2 and $\varphi_\varepsilon(t)[1-\varphi_\varepsilon(t)]$ is bounded below by a positive constant independent of the parameters. On the other hand, on the interval $[0, 1/8]$, in view of (10.7), $\dot\varphi_\varepsilon(t)/\{\varphi_\varepsilon(t)[1-\varphi_\varepsilon(t)]\} \ge [\delta^2 + t]^{-1} \ge 1$. Hence, in this interval, the previous integral is bounded by

$$\frac{1}{4}\int_0^{1/8} \frac{1}{\sqrt{\delta^2+t}} \log\frac{1}{\delta^2+t}\, dt \le C_0.$$

A similar analysis can be carried out in the interval $[7/8, 1]$.

The second term on the right-hand side of (10.10) is equal to

$$C_0 T \int_0^1 \frac{1}{\varphi_\varepsilon(t)[1-\varphi_\varepsilon(t)]}\, dt \le C_0' T.$$

Therefore,

$$\log\left(\frac{\|f_T\|_{p(T)}}{\|f_0\|_{p(0)}}\right) \le -(1/2)\log\{N^2T\} + C_0 + C_0T.$$

To complete the proof of the lemma, it remains to observe that $\|f_T\|_M \le e^{E/2} \|f_T\|_{p(T)}$, $\|f_0\|_{p(0)} \le 1$. $\square$

COROLLARY 10.7. *Fix $n \ge 1$, $T_0 > 0$, and denote by $q_t(j,\cdot)$ the solution of the linear equation* (10.2) *with initial condition $q_0(j,k) = \delta_{j,k}$. There exist a finite constant C_0 and $N_0 \ge 1$, depending only on E, β and n, such that*

$$q_t(j,k) \le \frac{C_0 e^{C_0 t}}{\sqrt{N^2 t}}$$

for all $0 \le t \le T_0$, $N \ge N_0$ and $0 \le j,k \le N-1$.

PROOF. Fix $n \ge 1$, $T_0 > 0$, and $0 \le j \le N-1$. There exists $N_0 \ge n+1$ for which the last condition in (10.7) is satisfied for all $0 \le t \le T_0$, $N \ge N_0$.

There exists $a > 0$ such that $\sup_{x \ge a} \log x/\sqrt{x} \le (8A_0)^{-1/2}$. Let $b = \max\{a, e^{16}\}$. Fix $0 \le t \le T_0$. If $tN^2 \le b$, by Lemma 10.4,

$$\max_{0\le k\le N-1} q_t(j,k) \le C_0 e^{C_0 t} \le \frac{\sqrt{b} C_0 e^{C_0 t}}{\sqrt{N^2 t}}.$$

On the other hand, if $tN^2 \ge b$, t fulfills all the assumptions of the previous lemma. This completes the proof. $\square$

We conclude this section with a remark used several times in the previous sections. Let $f_t(k) = \sum_{j\in\Lambda_N} q_t(j,k)$. Thus, f is the solution of (10.2) with initial condition $f(k) = 1$ for all $k \in \Lambda_N$. By Lemma 10.4, for all $T > 0$, there exists a finite constant C_0, depending only on E, β and T such that

$$\sup_{0\le t\le T} \max_{k\in\Lambda_N} \sum_{j\in\Lambda_N} q_t(j,k) = \sup_{0\le t\le T} \max_{k\in\Lambda_N} f_t(k) \le C_0. \tag{10.11}$$

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