ONE-DIMENSIONAL RANDOM WALKS WITH SELF-BLOCKING IMMIGRATION

BY MATTHIAS BIRKNER[1] AND RONGFENG SUN[2]

Johannes Gutenberg-Universität Mainz and National University of Singapore

We consider a system of independent one-dimensional random walkers where new particles are added at the origin at fixed rate whenever there is no older particle present at the origin. A Poisson ansatz leads to a semi-linear lattice heat equation and predicts that starting from the empty configuration the total number of particles grows as $c\sqrt{t}\log t$. We confirm this prediction and also describe the asymptotic macroscopic profile of the particle configuration.

1. Introduction: Model and results. Consider the following model of *random walks with self-blocking immigration* (RWSBI) at the origin. Let $\eta_x(t)$ be the number of particles at position $x \in \mathbb{Z}$ at time $t \geq 0$. Particles perform independent continuous-time random walks on $\mathbb{Z}$ with jump rate 1 and jump increments following a probability kernel $(a_x)_{x \in \mathbb{Z}}$ with

$$(1.1) \qquad \sum_x x a_x = 0 \quad \text{and} \quad \sigma^2 := \sum_x x^2 a_x \in (0, \infty).$$

In addition, at rate $\gamma > 0$ new particles attempt to "immigrate" at the origin 0 but are only successful if there is currently no other particle at 0. The system starts with no particles at time 0, that is, $\eta_x(0) \equiv 0$. See Remark 1.3 below for a discussion of the formal construction.

This system shows interesting self-organized behavior: It possesses an intrinsically defined "correct" growth rate and when particles are added to the system at a lower (resp., higher) rate than this correct rate, there is more (resp., less) vacant time at the origin, which results in more (resp., less) particles added, and the system is thus driven back toward the correct rate of addition of particles. The task is thus to identify this correct asymptotic rate at which particles are added to the system.

Obviously, more and more particles will be added to the system as time progresses and once created these perform independent random walks, which suggests hydrodynamic limit type arguments and results. While hydrodynamic limits

for interacting particle systems is a vigorous area of current research, it seems that our system is somewhat special in this framework, and that there is presently no readily applicable general theory to analyse its long-term behavior: It combines a "Kawasaki type" dynamics, namely the motion of particles which preserves total mass, and a very localized "Glauber-type" dynamics, namely the immigration mechanism which creates new mass, in a nontrivial and nonperturbative way. There is recent interest in extending hydrodynamic limits to models where nontrivial interactions among particles occur only in a very small part of the space, for example, Chen and Fan [4] study systems of walks in bounded domains where pairwise annihilation only happens at the boundary. Thus far, our analysis of RWSBI fits these efforts though our approach and the model details are quite different from [4]. Arguably, RWSBI is of a very special form, yet we believe that at this stage, with no general approach available, a detailed analysis of special cases is warranted.

Finally, we note that RWSBI first appeared in the literature as a caricature system for the analysis of a system of critically branching random walks with a density-dependent feedback; cf. Remark 1.4 below.

It is well known (see, e.g., [8], Chapter 1) that equilibrium states for systems of independent random walks are products of Poisson distributions. A Poisson ansatz leads to the heuristics that the particle density $\mathbb{E}[\eta_x(t)] \approx \rho_x(t)$, where $\rho_x(t)$ is the unique solution of the following ODE system, a semilinear discrete heat equation (the form of the nonlinearity in the first line of (1.2) arises by assuming $\eta_0(t)$ to be Poisson distributed with mean $\mathbb{E}[\eta_0(t)]$):

$$\partial_t \rho_x(t) = L_{\mathrm{rw}} \rho_x(t) + \gamma \delta_0(x) \exp(-\rho_0(t)), \qquad t \geq 0, x \in \mathbb{Z},$$

(1.2)

$$\rho_x(0) \equiv 0, \qquad x \in \mathbb{Z},$$

where L_{rw} is the adjoint of the generator of the random walk given in (1.1), with $(L_{\mathrm{rw}} f)_x := \sum_y a_{x-y}(f_y - f_x)$.

Denote the total mass of $\rho_.(t)$ by

$$(1.3) \qquad R(t) := \sum_{x \in \mathbb{Z}} \rho_x(t) = \int_0^t \gamma \exp(-\rho_0(s)) \, ds.$$

We have for $t \to \infty$ (see [2], Lemma 17, and also Lemma A.1 in Appendix A)

$$(1.4) \qquad \rho_0(t) = \frac{1}{2} \log t - \log \log t + \log(\sqrt{2\pi} \gamma / \sigma) + o(1),$$

$$(1.5) \qquad R(t) = \left[\sigma \left(\frac{2}{\pi} \right)^{1/2} \sqrt{t} \log t \right] (1 + o(1)).$$

Furthermore (cf. Lemma A.4 below),

$$\frac{1}{\log t} \rho_{[\sigma \sqrt{t} y]}(t) \underset{t \to \infty}{\longrightarrow} \tilde{\rho}(y) := \frac{1}{2\pi} \int_0^1 \frac{1}{\sqrt{s(1-s)}} e^{-y^2/(2s)} \, ds$$

(1.6)

$$= \frac{1}{\sqrt{2\pi}} \int_{|y|}^{\infty} e^{-z^2/2} \, dz, \qquad y \in \mathbb{R}.$$

Our main result is that the Poisson ansatz is indeed valid. The asymptotic behavior of the total number of particles in the system, as well as the particle distribution in space, agree with the behavior of $\rho_\cdot(t)$ under the Poisson ansatz.

THEOREM 1.1. *Let the model of random walks on $\mathbb{Z}$ with self-blocking immigration at the origin be defined as above, and recall $R(t)$ from* (1.3) *and* (1.5). *Almost surely, the total number of particles in the system satisfies*

$$(1.7) \qquad \lim_{t\to\infty} \frac{1}{R(t)} \sum_x \eta_x(t) = 1.$$

Using Theorem 1.1, we can further show that the "shape of the particle cloud," $(\eta_x(t))_{x\in\mathbb{Z}}$, follows the prediction from the Poisson ansatz.

THEOREM 1.2. *For any nonnegative bounded continuous function $f \in C_{b,+}(\mathbb{R})$, a.s. we have*

$$(1.8) \qquad \lim_{t\to\infty} \frac{1}{\sigma\sqrt{t}\log t} \sum_x \eta_x(t) f\left(\frac{x}{\sigma\sqrt{t}}\right) = \int_{\mathbb{R}} f(y)\tilde{\rho}(y)\,dy,$$

where $\tilde{\rho}(y) = \frac{1}{\sqrt{2\pi}}\int_{|y|}^{\infty} e^{-z^2/2}\,dz$, as in (1.6).

REMARK 1.3. 1. Starting from any finite initial condition, it is straightforward to construct the system η explicitly by using suitable Poisson processes, for example, as in Section 2 below; note that the total number of immigrated particles up to time t is dominated by a rate γ Poisson process, in particular the total number of particles is a.s. finite uniformly in any bounded time interval.

For a formal definition and suitable state space that allows infinite configurations, see [2], Section 3.1, and compare also the arguments in [2], Section 2.2, for the construction of the transition semigroup and a representation of η as a Poisson process-driven SDE system (a similar construction appears in [7]).

2. A much weaker version of (1.7) was previously shown in [2], Proposition 8, via the relative entropy method [10], namely that for any $\varepsilon > 0$,

$$(1.9) \qquad \sum_{x\in\mathbb{Z}} \eta_x(t) = o\big(t^{1/2+\varepsilon}\big) \qquad \text{in probability as } t\to\infty.$$

3. For the analogous system consisting of symmetric simple random walks on $\mathbb{Z}^2$, a Poisson ansatz predicts $\rho_0(t) = \log\log t - \log\log\log t - \log(2\pi) + o(1)$ and $R(t) \sim (2\pi t\log\log t)/\log t$; cf. [2], Remark 13. Using the techniques from Section 3, it is fairly straightforward to establish a corresponding upper bound for the total number of particles in the two-dimensional system in probability. It appears that in order to strengthen this bound to control the a.s. behavior and also to provide a matching lower bound using arguments parallel to those from Section 4, a very detailed study of the vacant time fluctuations of suitably tuned Poisson

systems of two-dimensional random walks with immigration will be required. We defer this question to future research.

REMARK 1.4 (Relation to self-catalytic branching random walks, [2], Chapter 2).   Let SCBRW(b) be a system of self-catalytic critical binary branching random walks on $\mathbb{Z}^d$ where each particle independently performs a random walk with kernel (1.1) and in addition while there are $k - 1$ other particles at its site, it splits in two or disappears with rate $b(k)$, where $b : \mathbb{N}_0 \to [0, \infty)$ is a branching rate function (when b is a linear function, this is a classical system of independent branching random walks). Starting from a homogeneous initial condition, say a Poisson field on $\mathbb{Z}^d$ with constant intensity, the long-term behavior of such systems exhibits a dichotomy between persistence (i.e., convergence to a nontrivial shift-invariant equilibrium) and clustering (i.e., local extinction combined with increasingly rare regions of diverging particle density), depending on the branching rate function b and the spatial dimension d. For general b and $d \leq 2$, it is believed ([2], Conjecture 1) but not rigorously known that clustering occurs. It is known (see, e.g., [2], Lemma 8) that in this case clustering is equivalent to the local divergence as time $t \to \infty$ of the configuration under the so-called Palm distribution (which re-weights configurations at time t proportional to the number of particles at the origin).

By a comparison result for the semigroups of SCBRW(b) with respect to convex order for different b's (cf. [2], Theorem 1 and Corollary 1), it suffices to consider the special case $b = b_{\text{sing}}$ with $b_{\text{sing}}(k) = \mathbf{1}_{\{k=1\}}$, that is, particles branch only if there is no other particle present at their site. The Palm distribution of SCBRW(b_{sing}) has a stochastic representation ([2], Proposition 5): It consists of the original SCBRW(b_{sing}) plus one special space–time path, which itself is drawn from the law of the time-reversed random walk, along which new particles immigrate at rate 1 but only when there is no other, older particle already present at this site; the special path and the immigrating particles have an interpretation as the family decomposition for a focal particle picked at the origin at time t. While this is conceptually appealing, it appears currently still too complex to allow a rigorous analysis of its long-time behavior.

Thus, we consider the following simplification or caricature, originally proposed by Anton Wakolbinger: Replace the random walk special path by a constant path and disallow branching away from the special path but keep the immigration mechanism along it unchanged. This yields RWSBI, our present object of study. In this sense, Theorems 1.1 and 1.2 corroborate Conjecture 1 from [2] in a quantitative way and in fact suggest that the typical number of particles under the Palm distribution of SCBRW(b_{sing}) should diverge like $\log t$ in $d = 1$. However, undoing the caricature steps to convert our findings into an actual proof of this conjecture will require new arguments, which is currently a work in progress [3].

The rest of the paper is organized as follows. Section 2 introduces and analyzes Poisson systems of random walks with immigration at the origin. The upper (resp., lower) bounds in Theorems 1.1 and 1.2 are proved in Section 3 (resp., Section 4) by suitable coupling and comparison with the Poisson system of random walks. In Appendix A, we derive the asymptotics (1.4)–(1.5), while in Appendix B, we derive an estimate for k-event "correlation functions" for Poisson processes.

2. Poisson systems of random walks. The key tool in our proof is an auxiliary Poisson system of random walks, $\widetilde{\eta} = (\widetilde{\eta}_x(t))_{x \in \mathbb{Z}, t \geq 0}$, where particles immigrate at $x = 0$ at time-dependent rate $\beta(t)$, for some suitable $\beta : [0, \infty) \to (0, \infty)$. Once arrived, they follow independent continuous-time random walks with jump rate 1. By coupling such a Poisson system with random walks with self-blocking immigration (RWSBI), in particular, by coupling the times when the origin is vacant in each process, we can obtain bounds on the number of particles added to the RWSBI in terms of the Poisson system. We will choose $\beta(t)$ to be perturbations of the rate $\gamma e^{-\rho_0(t)}$ dictated by the Poisson ansatz in (1.2).

We note that the system of random walks $\widetilde{\eta}$ can be characterized as a Poisson point process Π on the set $\mathcal{S}$ of all càdlàg paths $\bigcup_{t \geq 0} \{\zeta : [t, \infty) \to \mathbb{Z}\}$ (denote the starting time of ζ by τ_ζ), with intensity measure

$$(2.1) \qquad \nu(d\zeta) = \beta(\tau_\zeta) \, d\tau_\zeta \, \mathbb{P}(X \in d\zeta(\cdot - \tau_\zeta)),$$

where $X = (X_t)_{t \geq 0}$ is the rate 1 continuous time random walk as specified in (1.1), starting at $X_0 = 0$. Then

$$\widetilde{\eta}_x(t) = \Pi(\{\zeta : \zeta(t) = x\}), \qquad x \in \mathbb{Z}^d, t \geq 0,$$

in particular, $\widetilde{\eta}_x(t)$ is Poisson distributed with mean $\int_0^t \beta(u) p_x(t - u) \, du$, and

$$\mathbb{P}(\widetilde{\eta}_x(t) = 0) = \exp\left[-\int_0^t \beta(u) p_x(t - u) \, du\right],$$

where

$$(2.2) \qquad p_x(s) := \mathbb{P}(X_s = x).$$

Apart from the number of particles added to the system by time t, we will also be interested in the amount of time at which the origin is vacant, that is,

$$\widetilde{V}_{s,t} := \int_s^t \mathbf{1}_{\{\widetilde{\eta}_0(r)=0\}} \, dr, \qquad 0 \leq s \leq t.$$

We collect below results on the Poisson systems of random walks which we will need later. To prove the upper (resp., lower) bound in Theorems 1.1 and 1.2, it turns out that the appropriate choice of immigration rate $\beta(t)$ for the Poisson system $\widetilde{\eta}$ is

$$(2.3) \qquad \beta^{(+\varepsilon)}(t) := (1 + \varepsilon)\gamma e^{-\rho_0(t)} \qquad \text{resp. } \beta^{(-\varepsilon)}(t) := (1 - \varepsilon)\gamma e^{-\rho_0(t)},$$

where $\varepsilon > 0$ is chosen sufficiently close to 0, and $\rho_0(t)$ is as in (1.2). We will let $\widetilde{\eta}^{(\pm\varepsilon)}$ denote the respective Poisson system, and $\widetilde{V}_{s,t}^{(\pm\varepsilon)}$ its vacant time at the origin.

LEMMA 2.1.   *Let $\widetilde{\eta}^{(\pm\varepsilon)}$ be the Poisson system of random walks with immigration rate $\beta^{(\pm\varepsilon)}$ for some $\varepsilon \in (0, 1)$. Then*

$$(2.4) \qquad \frac{\sum_x \widetilde{\eta}_x^{(\pm\varepsilon)}(t)}{R(t)} \to 1 \pm \varepsilon \qquad \text{a.s. as } t \to \infty,$$

where $R_t = \sum_x \rho_x(t) = [\sigma(\frac{2}{\pi})^{1/2}\sqrt{t}\log t](1 + o(1))$ as defined in (1.3), and

$$(2.5) \qquad \frac{\widetilde{V}_{0,t}^{(+\varepsilon)}}{R(t)} \to 0 \qquad \text{a.s. as } t \to \infty.$$

LEMMA 2.2.   *Let $\widetilde{\eta}^{(-\varepsilon)}$ be the Poisson system of random walks with immigration rate $\beta^{(-\varepsilon)}$ for some $\varepsilon \in (0, 1)$. Then there exists $t_0 > 0, c > 0$ such that for all $t/2 \le s < t$ with $t \ge t_0$, we have*

$$(2.6) \qquad \mathbb{E}[\widetilde{V}_{s,t}^{(-\varepsilon)}] \ge c(t - s)t^{-\frac{1-\varepsilon}{2}}.$$

If $\xi \in (\frac{1-\varepsilon}{2}, 1)$, then there exists $b \in (0, \infty)$ such that for every $k \in \mathbb{N}$, there exist $t_0, C \in (0, \infty)$ so that for all s, t with $t_0 \le t/2 \le s \le t - t^\xi$,

$$(2.7) \qquad \mathbb{E}[(\widetilde{V}_{s,t}^{(-\varepsilon)} - \mathbb{E}[\widetilde{V}_{s,t}^{(-\varepsilon)}])^k] \le Ct^{-bk}\mathbb{E}[\widetilde{V}_{s,t}^{(-\varepsilon)}]^k$$

(we can choose $b = (\xi - \frac{1-\varepsilon}{2})/48$).

This shows that the vacant time $\widetilde{V}_{s,t}^{(-\varepsilon)}$ is concentrated around its mean with high probability.

We now give the proofs of Lemmas 2.1 and 2.2. The proof of Lemma 2.2 is of independent interest, but is quite involved and, therefore, can be read after the proof of Theorems 1.1 and 1.2 in Sections 3 and 4.

PROOF OF LEMMA 2.1.   Recalling (2.3), we have

$$\widetilde{\rho}_x^{(\pm\varepsilon)}(t) := \mathbb{E}[\widetilde{\eta}_x^{(\pm\varepsilon)}(t)] = \int_0^t \beta^{(\pm\varepsilon)}(s) p_x(t - s)\, ds$$

$$= (1 \pm \varepsilon) \int_0^t \gamma e^{-\rho_0(t)} p_x(t - s)\, ds = (1 \pm \varepsilon)\rho_x(t).$$

In particular, $\mathbb{E}[\sum_x \widetilde{\eta}_x^{(\pm\varepsilon)}(t)] = (1 \pm \varepsilon)\sum_x \rho_x(t) = (1 \pm \varepsilon)R(t)$. Since $\sum_x \widetilde{\eta}_x^{(\pm\varepsilon)}(t)$ is nothing but a time-changed Poisson process with mean $(1 \pm \varepsilon)R(t)$, (2.4) follows immediately.

To prove (2.5), note that $\beta^{(\pm\varepsilon)}(t) \sim \frac{1\pm\varepsilon}{\sqrt{2\pi}}\sigma t^{-1/2}\log t$ by (1.4), and hence

$$\mathbb{E}[\widetilde{V}_{0,t}^{(+\varepsilon)}] = \int_0^t e^{-\widetilde{\rho}_0^{(+\varepsilon)}(u)}\, du = \int_0^t e^{-(1+\varepsilon)\rho_0(u)}\, du$$

$$\le 1 + C\int_1^t \frac{(\log u)^{1+\varepsilon}}{u^{(1+\varepsilon)/2}}\, du \le 2Ct^{(1-\varepsilon)/2}(\log t)^{1+\varepsilon}.$$

For any $\delta > 0$, by Markov inequality and the asymptotics of $R(t)$ in (1.5),

$$\mathbb{P}\big(\widetilde{V}_{0,t}^{(+\varepsilon)} > \delta R(t)\big) \leq \frac{\mathbb{E}[\widetilde{V}_{0,t}^{(+\varepsilon)}]}{\delta R(t)} \leq \frac{C'}{t^{\varepsilon/4}}$$

for all t sufficiently large. By Borel–Cantelli, along the sequence of times $t_n = c^n$ for any $c > 1$, we then have $\limsup_{n\to\infty} \widetilde{V}_{0,c^n}^{(+\varepsilon)}/R(c^n) \leq \delta$ almost surely. Since $\widetilde{V}_{0,s}^{(+\varepsilon)} \leq \widetilde{V}_{0,t}^{(+\varepsilon)}$ for $s \leq t$, together with the asymptotics of $R(t)$ given in (1.5), we obtain

$$\limsup_{t\to\infty} \frac{\widetilde{V}_{0,t}^{(+\varepsilon)}}{R(t)} \leq \limsup_{n\to\infty} \frac{\widetilde{V}_{0,c^n}^{(+\varepsilon)}}{R(c^{n-1})} \leq \delta\sqrt{c}.$$

Since $\delta > 0$ can be chosen arbitrarily, (2.5) then follows. $\square$

PROOF OF LEMMA 2.2. Using the asymptotics of $\rho_0(\cdot)$ given in (1.4), (2.6) holds because

$$\begin{aligned}
\mathbb{E}\big[\widetilde{V}_{s,t}^{(-\varepsilon)}\big] &= \int_s^t e^{-\mathbb{E}[\widetilde{\eta}_0^{(-\varepsilon)}(u)]}\, du = \int_s^t e^{-(1-\varepsilon)\rho_0(u)}\, du \\
&\geq \int_s^t u^{-\frac{1-\varepsilon}{2}}\, du = \frac{2}{1+\varepsilon}\big(t^{\frac{1+\varepsilon}{2}} - s^{\frac{1+\varepsilon}{2}}\big) \\
&\geq c(t-s)t^{-\frac{1-\varepsilon}{2}}.
\end{aligned}$$

Next, we prove the centered moment bound (2.7). To lighten notation, we will drop the dependence on ε in the remainder of the proof and write $\widetilde{V}_{s,t} = \widetilde{V}_{s,t}^{(-\varepsilon)}$, $\widetilde{\eta}_0 = \widetilde{\eta}_0^{(-\varepsilon)}$, etc. Note

$$\begin{aligned}
(2.8) \quad &\mathbb{E}\big[\big(\widetilde{V}_{s,t} - \mathbb{E}[\widetilde{V}_{s,t}]\big)^k\big] \\
&= k! \int \cdots \int_{s \leq u_1 < \cdots < u_k \leq t} \mathbb{E}\Bigg[\prod_{i=1}^k \big(\mathbf{1}(\widetilde{\eta}_0(u_i) = 0) - \mathbb{P}(\widetilde{\eta}_0(u_i) = 0)\big)\Bigg]\, du_k \cdots du_1.
\end{aligned}$$

The idea to estimate (2.8) is the following. When the u_i's are close, the contribution to the integral is small due to the restricted range of integration; when the u_i's are far apart, we can use the decorrelation of the Poisson system as quantified by Lemma B.1. We thus group u_i's into blocks as follows, where each block contains consecutive u_i's that are close to each other, and different groups are far apart.

We group the time points $u_1, \ldots, u_k$ into blocks that are separated from each other by at least t^δ, with $\delta = \frac{2}{3}(\xi - \frac{1-\varepsilon}{2})(> 0)$. A block structure is determined by $\ell \in \{1, 2, \ldots, \lfloor k/2 \rfloor\}$, and ℓ pairs of indices g_i, h_i with

$$1 \leq g_1 < h_1 < g_2 < h_2 < \cdots < g_\ell < h_\ell \leq k.$$

Let $B(g, h)$ denote the set of all $\vec{u} := (u_1, \ldots, u_k)$ with $s \le u_1 < \cdots < u_k \le t$, such that for each $1 \le i \le \ell$, $J_i := [g_i, h_i] \cap \mathbb{N}$ is a block of indices with

$$u_{r+1} - u_r \le t^\delta \qquad \text{for all } r \in [g_i, h_i - 1] \cap \mathbb{N} \quad \text{and}$$

$$\min\{u_{g_i} - u_{g_i-1}, u_{h_i+1} - u_{h_i}\} > t^\delta,$$

where $u_0 := -\infty$, $u_{k+1} := +\infty$. Indices in the set $J_0 := \{1, \ldots, k\} \setminus (J_1 \cup \cdots \cup J_\ell)$ are the blocks of singletons, that is, for each $i \in J_0$, u_i is separated from all the other u_j's by at least t^δ.

Now consider a fixed block structure as determined by ℓ and $g_1, h_1, \ldots, g_\ell, h_\ell$, and let $(u_1, \ldots, u_k) \in B(g, h)$. Write

(2.9)
$$\prod_{i=1}^{k} \big(\mathbf{1}(\widetilde{\eta}_0(u_i) = 0) - \mathbb{P}(\widetilde{\eta}_0(u_i) = 0)\big)$$

$$= \prod_{m=0}^{\ell} \prod_{i \in J_m} \big(\mathbf{1}(\widetilde{\eta}_0(u_i) = 0) - \mathbb{P}(\widetilde{\eta}_0(u_i) = 0)\big).$$

To apply Lemma B.1, for each block J_m with $1 \le m \le \ell$, we need to rewrite the product of the centered indicators as linear combinations of centered indicators. More precisely, for each $1 \le m \le \ell$, we write

$$\prod_{i \in J_m} \big(\mathbf{1}(\widetilde{\eta}_0(u_i) = 0) - \mathbb{P}(\widetilde{\eta}_0(u_i) = 0)\big)$$

$$= \sum_{J_m' \subset J_m} (-1)^{|J_m \setminus J_m'|} \mathbf{1}(\widetilde{\eta}_0(u_i) = 0, i \in J_m') \prod_{i \in J_m \setminus J_m'} \mathbb{P}(\widetilde{\eta}_0(u_i) = 0)$$

(2.10)
$$= \sum_{J_m' \subset J_m} (-1)^{|J_m \setminus J_m'|} \big(\mathbf{1}(\widetilde{\eta}_0(u_i) = 0, i \in J_m')$$

$$- \mathbb{P}(\widetilde{\eta}_0(u_i) = 0, i \in J_m')\big) \prod_{i \in J_m \setminus J_m'} \mathbb{P}(\widetilde{\eta}_0(u_i) = 0)$$

$$+ \sum_{J_m'' \subset J_m} (-1)^{|J_m \setminus J_m''|} \mathbb{P}(\widetilde{\eta}_0(u_i) = 0, i \in J_m'') \prod_{i \in J_m \setminus J_m''} \mathbb{P}(\widetilde{\eta}_0(u_i) = 0),$$

where we centered the indicator function $\mathbf{1}(\widetilde{\eta}_0(u_i) = 0, i \in J_m')$, and $\mathbb{P}(\widetilde{\eta}_0(u_i) = 0, i \in J_m')$ is interpreted to be 1 if $J_m' = \varnothing$. Note that for blocks of singletons, that is, $i \in J_0$, the indicator function is already centered and there is no constant term as in (2.10), which is why the singleton blocks are separated from the other blocks $J_1, \ldots, J_\ell$.

Applying (2.10) for indices in blocks $J_1, \ldots, J_\ell$ in (2.9), and expanding and grouping terms, we can then rewrite (2.9) as a sum of

$$\pm A(\widehat{J}, \vec{J}', \vec{J}'', \vec{u}),$$

where $\widehat{J} \subset \{1, \ldots, \ell\}$ determine the blocks for which we choose a centered indicator function (instead of a constant) from the expansion in (2.10), and $\vec{J}' = (J_1', \ldots, J_\ell')$, $\vec{J}'' = (J_1'', \ldots, J_\ell'')$, with $J_m', J_m'' \subset J_m$ as in (2.10) for each block J_m. More precisely,

$$A(\widehat{J}, \vec{J}', \vec{J}'', \vec{u})$$

$$= \prod_{i \in J_0} \left(\mathbf{1}(\tilde{\eta}_0(u_i) = 0) - \mathbb{P}(\tilde{\eta}_0(u_i) = 0) \right)$$

$$\times \prod_{m \in \widehat{J}} \left(\left(\mathbf{1}(\tilde{\eta}_0(u_i) = 0, i \in J_m') \right. \right.$$

$$\left. - \mathbb{P}(\tilde{\eta}_0(u_i) = 0, i \in J_m') \right) \prod_{i \in J_m \setminus J_m'} \mathbb{P}(\tilde{\eta}_0(u_i) = 0) \right)$$

(2.11)
$$\times \prod_{m \in \{1, \ldots, \ell\} \setminus \widehat{J}} \left(\mathbb{P}(\tilde{\eta}_0(u_i) = 0, i \in J_m'') \prod_{i \in J_m \setminus J_m''} \mathbb{P}(\tilde{\eta}_0(u_i) = 0) \right)$$

$$= \prod_{i \in J_0} \left(\mathbf{1}(\tilde{\eta}_0(u_i) = 0) - \mathbb{P}(\tilde{\eta}_0(u_i) = 0) \right)$$

$$\times \prod_{m \in \widehat{J}} \left(\mathbf{1}(\tilde{\eta}_0(u_i) = 0, i \in J_m') - \mathbb{P}(\tilde{\eta}_0(u_i) = 0, i \in J_m') \right)$$

$$\times \prod_{m \in \{1, \ldots, \ell\} \setminus \widehat{J}} \mathbb{P}(\tilde{\eta}_0(u_i) = 0, i \in J_m'') \times \prod_{i \in \breve{J}} \mathbb{P}(\tilde{\eta}_0(u_i) = 0),$$

where in the last line, $\breve{J} := \bigcup_{m \in \widehat{J}} (J_m \setminus J_m') \cup \bigcup_{m \in \{1, \ldots, \ell\} \setminus \widehat{J}} (J_m \setminus J_m'')$.

The sign corresponding to a given choice of $\widehat{J}, \vec{J}', \vec{J}''$ is

$$(-1)^{\sum_{m \in \widehat{J}} |J_m \setminus J_m'| + \sum_{m \in \{1, \ldots, \ell\} \setminus \widehat{J}} |J_m \setminus J_m''|}.$$

Using Lemma B.1, we can bound the expectation of the product of centered indicator functions in (2.11) by

$$\mathbb{E}\left[\prod_{i \in J_0} \left(\mathbf{1}(\xi(E_i) = 0) - \mathbb{P}(\xi(E_i) = 0) \right) \right.$$

(2.12) $$\times \prod_{m \in \widehat{J}} \left(\mathbf{1}(\xi(F_m) = 0) - \mathbb{P}(\xi(F_m) = 0) \right) \right]$$

$$\leq C t^{-\delta(|J_0| + |\widehat{J}|)/8} \times \prod_{i \in J_0} \mathbb{P}(\tilde{\eta}_0(u_i) = 0) \times \prod_{m \in \widehat{J}} \mathbb{P}(\tilde{\eta}_0(u_i) = 0, i \in J_m'),$$

where ξ is a Poisson point process on the random walk paths space with intensity measure ν given by (2.1), with $\beta(t) = \beta^{(-\varepsilon)}(t)$; and

$$E_i := \{\text{random walk paths } \zeta \colon \text{with } \zeta(u_r) = 0\}, \qquad r \in J_0,$$

$$F_m := \{\text{random walk paths } \zeta \colon \text{with } \zeta(u_r) = 0 \text{ for some } r \in J'_m\}, \qquad m \in \widehat{J}.$$

Let us reorder and relabel the sets $(E_i)_{i \in J_0}$ and $(F_m)_{m \in \widehat{J}}$ by $(\widetilde{E}_i)_{1 \le i \le |J_0| + |\widehat{J}|}$, where each $\widetilde{E}_i$ is of the form $\{\zeta : \zeta(u_r) = 0 \text{ for some } r \in \widetilde{J}_i\}$ for some distinct index set $\widetilde{J}_i \subset \{1, \dots, k\}$, and elements of $\widetilde{J}_1$ being smaller than those of $\widetilde{J}_2$, etc.

To see how does (2.12) follow from Lemma B.1, note that for any $I \subset \{1, 2, \dots, |J_0| + |\widehat{J}|\}$,

$$\nu\left(\bigcap_{i \in I} \widetilde{E}_i\right) = \nu(\zeta : \text{ for each } i \in I, \zeta(u_r) = 0 \text{ for some } r \in \widetilde{J}_i)$$

$$= \int_0^t \beta^{(-\varepsilon)}(v) \mathbb{P}(X_{u_r} = 0 \text{ for some } r \in \widetilde{J}_i \text{ for each } i \in I | X_v = 0) \, dv$$

$$\le \left(\frac{C'}{t^{\delta/2}}\right)^{|I|-1} \int_0^t \beta^{(-\varepsilon)}(v) \mathbb{P}(X_{u_r} = 0 \text{ for some } r \in \widetilde{J}_1 | X_v = 0) \, dv$$

$$\le \left(\frac{C'}{t^{\delta/2}}\right)^{|I|-1} \int_0^t \beta^{(-\varepsilon)}(v) \sum_{r \in \widetilde{J}_1} \mathbb{P}(X_{u_r} = 0 | X_v = 0) \, dv$$

$$= \left(\frac{C'}{t^{\delta/2}}\right)^{|I|-1} \sum_{r \in \widetilde{J}_1} \mathbb{E}[\widetilde{\eta}_0(u_r)] = (1 - \varepsilon)\left(\frac{C'}{t^{\delta/2}}\right)^{|I|-1} \sum_{r \in \widetilde{J}_1} \rho_0(u_r)$$

$$\le \frac{C}{(t^{\delta/4})^{|I|-1}},$$

where we applied the local central limit theorem in the first inequality, noting that the random walk returns to the origin at least $|I| - 1$ times over intervals of length at least t^δ, and we applied (1.4) to bound $\rho_0(u_r)$ in the last inequality. For any $I_1, \dots, I_n \subset \{1, \dots, |J_0| + |\widehat{J}|\}$ with $|I_1|, \dots, |I_n| \ge 2$ and $I_1 \cup \dots \cup I_n = \{1, 2, \dots, |J_0| + |\widehat{J}|\}$, we then have

$$\prod_{j=1}^n \nu\left(\bigcap_{i \in I_j} \widetilde{E}_i\right) \le C^n \left(t^{-\delta/4}\right)^{\sum_{i=1}^n |I_i| - n}$$

$$\le \min\{C^n \left(t^{-\delta/4}\right)^{|J_0| + |\widehat{J}| - n}, (Ct^{-\delta/4})^n\}.$$

Substituting these bounds into (B.1), where the first bound is used for $1 \le n \le \frac{|J_0| + |\widehat{J}|}{2}$, and the second bound is used for $n > \frac{|J_0| + |\widehat{J}|}{2}$, it is then easily seen that (2.12) follows [note that we only need to consider (2.12) for the case $|J_0| + |\widehat{J}| \ge 2$, since otherwise the inequality is trivial].

Having verified (2.12), we can then apply (2.11) to bound

$$\mathbb{E}[A(\widehat{J}, \vec{J}', \vec{J}'', \vec{u})]$$

(2.13) $$\le Ct^{-\delta(|J_0| + |\widehat{J}|)/8} \prod_{i \in J_0 \cup \check{J}} \mathbb{P}(\widetilde{\eta}_0(u_i) = 0) \prod_{m \in \widehat{J}} \mathbb{P}(\widetilde{\eta}_0(u_i) = 0, i \in J'_m)$$

$$\times \prod_{m \in \{1,\ldots,\ell\} \setminus \widehat{J}} \mathbb{P}(\widetilde{\eta}_0(u_i) = 0, i \in J_m'')$$

$$\leq C t^{-\delta |J_0|/8} \prod_{i \in J_0 \cup J_0'} \mathbb{P}(\widetilde{\eta}_0(u_i) = 0),$$

where J_0' contains the smallest index from each block J_m, $1 \leq m \leq \ell$.

Therefore, following the discussion after (2.10), we have

$$\int_{B(g,h)} \cdots \int \mathbb{E}\left[\prod_{i=1}^{k} (\mathbf{1}(\widetilde{\eta}_0(u_i) = 0) - \mathbb{P}(\widetilde{\eta}_0(u_i) = 0)) \right] du_k \cdots du_1$$

$$\leq C t^{-\delta |J_0|/8} \sum_{\widehat{J}, \vec{J}', \vec{J}''} \int_{B(g,h)} \cdots \int \prod_{i \in J_0 \cup J_0'} \mathbb{P}(\widetilde{\eta}_0(u_i) = 0) \, du_k \cdots du_1$$

(2.14)

$$\leq C' t^{-\delta |J_0|/8} t^{\delta (|J_1| + \cdots + |J_\ell| - \ell)} \mathbb{E}[\widetilde{V}_{s,t}]^{|J_0| + \ell}$$

$$= C' t^{-\delta |J_0|/8} \left(\frac{t^\delta}{\mathbb{E}[\widetilde{V}_{s,t}]} \right)^{k - |J_0| - \ell} \mathbb{E}[\widetilde{V}_{s,t}]^k,$$

where C' contains combinatorial factors that depend only on k, but not on s and t (and we used $|J_1| + \cdots + |J_\ell| = k - |J_0|$ in the last line).

Since $\mathbb{E}[\widetilde{V}_{s,t}] \geq c(t-s)t^{-\frac{1-\varepsilon}{2}} \geq ct^{\xi - \frac{1-\varepsilon}{2}}$ by (2.6) and the assumption on s and t, the term in (2.14) is bounded by

$$(2.15) \qquad C'' t^{-\delta |J_0|/8} t^{(\delta - \xi + \frac{1-\varepsilon}{2})(k - |J_0| - \ell)} \mathbb{E}[\widetilde{V}_{s,t}]^k.$$

Note that $\delta - \xi + \frac{1-\varepsilon}{2} < 0$, $k - |J_0| - \ell \geq 0$, and $\ell \leq (k - |J_0|)/2$ (since each block J_m, $1 \leq m \leq \ell$, contains at least two indices). Thus, when $|J_0| \geq k/4$, we can bound (2.15) by

$$(2.16) \qquad C'' t^{-k\delta/32} \mathbb{E}[\widetilde{V}_{s,t}]^k,$$

whereas when $|J_0| < k/4$ (and hence $k - |J_0| - \ell \geq k/4$), we can bound (2.15) by

$$(2.17) \qquad C'' t^{-k(\xi - \frac{1-\varepsilon}{2} - \delta)/4} \mathbb{E}[\widetilde{V}_{s,t}]^k.$$

Either way, we find that the bound in (2.14) can be bounded by $C'' t^{-bk} \mathbb{E}[\widetilde{V}_{s,t}]^k$ for some C'' depending only k, and $b > 0$ depending only on ξ and ε.

Since for given k there are only finitely many choices for ℓ and $g_1, h_1; \ldots; g_\ell, h_\ell$, summing over all possible $B(g,h)$ then yields the claimed bound (2.7) with $b = \delta/32 = (\xi - \frac{1-\varepsilon}{2})/48$. $\quad\square$

3. Upper bounds in Theorems 1.1 and 1.2. Here is the basic idea for the upper bound on the system of random walks with self-blocking immigration (RWSBI), $\eta = (\eta_x(t))_{x \in \mathbb{Z}, t \geq 0}$. Let $\widetilde{\eta}^{(+\varepsilon)}$ be the Poisson system of random walks introduced in Section 2. We then attempt to add extra particles (labeled as $\widehat{\eta}$ particles) to the Poisson system $\widetilde{\eta}^{(+\varepsilon)}$ at the origin with rate γ provided that the origin is vacant under $\widetilde{\eta}^{(+\varepsilon)}$, and these attempted additions are coupled with those in the η system. In particular, a particle added in the η system can be coupled either to an $\widehat{\eta}$ particle added at the same time if the origin is vacant under $\widetilde{\eta}^{(+\varepsilon)}$, or to a particle in the $\widetilde{\eta}^{(+\varepsilon)}$ system if the origin is occupied under $\widetilde{\eta}^{(+\varepsilon)}$. This coupling constructs the η particles as a subset of the $\widetilde{\eta}^{(+\varepsilon)}$ and $\widehat{\eta}$ particles, for which explicit calculations can be carried out.

3.1. *Coupling with the Poisson system.* We now formulate precisely the coupling between the Poisson system $\widetilde{\eta}^{(+\varepsilon)}$, the system of particles $\widehat{\eta}^{(+\varepsilon)}$ added during the times when $\widetilde{\eta}^{(+\varepsilon)}$ is vacant at the origin, and the true RWSBI system η.

Suppose that the Poisson system $\widetilde{\eta}^{(+\varepsilon)}$ has been constructed. Let $0 < T_1 < T_2 < \cdots$ be the times of an independent rate γ Poisson point process on $[0, \infty)$. At each time T_i, we add a particle at the origin to the $\widehat{\eta}^{(+\varepsilon)}$ system if the origin is vacant under $\widetilde{\eta}^{(+\varepsilon)}$. The successfully added particles then perform independent random walks. We now construct the η system from $\widetilde{\eta}^{(+\varepsilon)}$ and $\widehat{\eta}^{(+\varepsilon)}$ as follows:

- At time T_1, the origin is either occupied by a particle in the Poisson system $\widetilde{\eta}^{(+\varepsilon)}$, or a particle is added at the origin to the $\widehat{\eta}^{(+\varepsilon)}$ system. In either case, we add a particle to η at the origin, which follows the same random walk as the particle (pick one if there is more than one) at the origin in the union of $\widetilde{\eta}^{(+\varepsilon)}$ and $\widehat{\eta}^{(+\varepsilon)}$.
- Assume that by time T_k for some $k \geq 1$, particles have been added to η in such a way that each particle in η is coupled to a distinct particle in the union of $\widetilde{\eta}^{(+\varepsilon)}$ and $\widehat{\eta}^{(+\varepsilon)}$. We now attempt to add a particle at time T_{k+1} to η that preserves this coupling condition.
  - If the origin is occupied at time T_{k+1} under η, then no particle is added to η.
  - If the origin is vacant at time T_{k+1} under η, we note that it is either occupied under the Poisson system $\widetilde{\eta}^{(+\varepsilon)}$, or a particle is added at the origin to the $\widehat{\eta}^{(+\varepsilon)}$ system. In either case, the origin is occupied by particles in the union of $\widetilde{\eta}^{(+\varepsilon)}$ and $\widehat{\eta}^{(+\varepsilon)}$, and none of these particles could have been coupled with any particle in η. We then add a particle at the origin to η, which follows the same random walk as a corresponding particle in the union of $\widetilde{\eta}^{(+\varepsilon)}$ and $\widehat{\eta}^{(+\varepsilon)}$ at the origin.

From the above inductive construction of η, it is clear that each particle in η is coupled to a distinct particle in the union of $\widetilde{\eta}^{(+\varepsilon)}$ and $\widehat{\eta}^{(+\varepsilon)}$, and hence almost surely,

$$(3.1) \qquad \eta_x(t) \leq \widetilde{\eta}_x^{(+\varepsilon)}(t) + \widehat{\eta}_x^{(+\varepsilon)}(t) \qquad \text{for all } x \in \mathbb{Z}, t \geq 0$$

and in particular

$$(3.2) \qquad \sum_x \eta_x(t) \leq \sum_x \widetilde{\eta}_x^{(+\varepsilon)}(t) + \sum_x \widehat{\eta}_x^{(+\varepsilon)}(t) \qquad \text{for all } t \geq 0.$$

3.2. *Proof of Theorem* 1.1 (*upper bound*). By (3.2), for any $\varepsilon > 0$, we have

$$\limsup_{t \to \infty} \frac{1}{R(t)} \sum_x \eta_x(t) \leq \limsup_{t \to \infty} \frac{1}{R(t)} \sum_x \widetilde{\eta}_x^{(+\varepsilon)}(t)$$

$$+ \limsup_{t \to \infty} \frac{1}{R(t)} \sum_x \widehat{\eta}_x^{(+\varepsilon)}(t),$$

where the first term equals $1 + \varepsilon$ by (2.4). The second term equals 0 because by construction, conditioned on $\widetilde{\eta}^{(+\varepsilon)}$, $\sum_x \widehat{\eta}_x^{(+\varepsilon)}(t)$ is a time-changed Poisson process with mean $\gamma \widetilde{V}_{0,t}^{(+\varepsilon)}$, and $\widetilde{V}_{0,t}^{(+\varepsilon)}/R(t) \to 0$ a.s. as $t \to \infty$ by (2.5). Therefore,

$$\limsup_{t \to \infty} \frac{1}{R(t)} \sum_x \eta_x(t) \leq 1 + \varepsilon,$$

which gives the desired upper bound if we let $\varepsilon \downarrow 0$.

3.3. *Proof of Theorem* 1.2 (*upper bound*). By (3.1), for any $\varepsilon > 0$ and any bounded nonnegative continuous function $f \in C_{b,+}(\mathbb{R})$, we have

$$(3.3) \qquad \frac{1}{\sigma \sqrt{t} \log t} \sum_{x \in \mathbb{Z}} \eta_x(t) f\left(\frac{x}{\sigma \sqrt{t}}\right) \leq \frac{1}{\sigma \sqrt{t} \log t} \sum_{x \in \mathbb{Z}} \widetilde{\eta}_x^{(+\varepsilon)}(t) f\left(\frac{x}{\sigma \sqrt{t}}\right)$$

$$+ \|f\|_\infty \frac{\sum_x \widehat{\eta}_x^{(+\varepsilon)}(t)}{\sigma \sqrt{t} \log t}.$$

Since $R(t) \sim \sigma(\frac{2}{\pi})^{\frac{1}{2}} \sqrt{t} \log t$ by (1.5), the second term tends to 0 as $t \to \infty$ as shown above in the proof of Theorem 1.1, and hence almost surely,

$$(3.4) \qquad \limsup_{t \to \infty} \frac{1}{\sigma \sqrt{t} \log t} \sum_{x \in \mathbb{Z}} \eta_x(t) f\left(\frac{x}{\sigma \sqrt{t}}\right)$$

$$\leq \limsup_{t \to \infty} \frac{1}{\sigma \sqrt{t} \log t} \sum_{x \in \mathbb{Z}} \widetilde{\eta}_x^{(+\varepsilon)}(t) f\left(\frac{x}{\sigma \sqrt{t}}\right).$$

Denote $\Xi_t := \sum_{x \in \mathbb{Z}} \widetilde{\eta}_x^{(+\varepsilon)}(t) f(\frac{x}{\sigma \sqrt{t}})$. First, we note that

$$\frac{\mathbb{E}[\Xi_t]}{\sigma \sqrt{t} \log t} = (1 + \varepsilon) \sum_{x \in \mathbb{Z}} \frac{\rho_x(t)}{\sigma \sqrt{t} \log t} f\left(\frac{x}{\sigma \sqrt{t}}\right)$$

$$\xrightarrow[t \to \infty]{} (1 + \varepsilon) \int_{\mathbb{R}} f(y) \widetilde{\rho}(y) \, dy =: M,$$

where the convergence follows from Lemma A.4 and a Riemann sum approximation of the integral. To show that $\Xi_t/(\sigma\sqrt{t}\log t)$ converges a.s. to the same limit M, we note that Ξ_t is a weighted sum of independent Poisson random variables with mean $m_t := \mathbb{E}[\Xi_t] = (M + o(1))\sigma\sqrt{t}\log t$, and each individual weight is uniformly bounded by $\|f\|_\infty$. By elementary large deviation estimates for Poisson random processes, for any $\delta > 0$, we have

$$\mathbb{P}(|\Xi_t - m_t| \geq \delta m_t) \leq C_1 e^{-C_2 m_t} \leq C_1 e^{-C_3\sqrt{t}\log t},$$

and hence by Borel–Cantelli, $\Xi_t/m_t \to 1$ a.s. along the time sequence $t_n = (\log n)^2$. To extend it to all $t \uparrow \infty$, by Borel–Cantelli, it suffices to show that for each $\delta > 0$,

$$(3.5) \qquad \sum_{n=1}^{\infty} \mathbb{P}\left(\sup_{t \in [t_n, t_{n+1}]} |\Xi_t - \Xi_{t_n}| \geq \delta m_{t_n} \right) < \infty.$$

Note that $t_{n+1} - t_n \sim 2\log n/n$, and $\sup_{t \in [t_n, t_{n+1}]}|\Xi_t - \Xi_{t_n}|$ can be bounded in terms of the number of particles added to the $\widetilde{\eta}^{(+\varepsilon)}$ system during the time interval $[t_n, t_{n+1}]$ (which is Poisson distributed), plus the number of particles in $\widetilde{\eta}_{t_n}^{(+\varepsilon)}$ which have unusually large displacements (of order $\sqrt{t_n}$) during $[t_n, t_{n+1}]$ (note that these displacements are independent). Elementary large deviation estimates then give (3.5).

In conclusion, the RHS of (3.4) converges a.s. to $(1 + \varepsilon)\int_{\mathbb{R}} f(y)\tilde{\rho}(y)\,dy$. Since $\varepsilon > 0$ can be arbitrary, this implies the desired upper bound in Theorem 1.2.

4. Lower bounds in Theorems 1.1 and 1.2.

Here is the basic strategy for the lower bound on the system of random walks with self-blocking immigration (RWSBI), $\eta = (\eta_x(t))_{x \in \mathbb{Z}, t \geq 0}$. Let $\widetilde{\eta} := \widetilde{\eta}^{(-\varepsilon)}$ be a Poisson system of random walks with immigration rate $\beta^{(-\varepsilon)}$ as introduced in Section 2. To get a lower bound on the η system, we will construct an auxiliary system of $\widehat{\eta}$ particles, where particles are added at rate at most γ and only when the origin is vacant under $\widehat{\eta}$, and $\widehat{\eta}$ particles may be killed from time to time. Such an $\widehat{\eta}$ system will be embedded as a subset of the η system. To have explicit control on the rate at which particles are added in the $\widehat{\eta}$ system, which will lead to a lower bound on η, we couple $\widehat{\eta}$ with the Poisson system $\widetilde{\eta}$ in such a way that each particle added to $\widehat{\eta}$ is coupled with a particle in $\widetilde{\eta}$ (albeit starting at a different time), so that when we attempt to add a new particle to $\widehat{\eta}$, the origin being vacant under $\widetilde{\eta}$ ensures that it is also vacant under $\widehat{\eta}$. We can then bound from below the rate at which $\widehat{\eta}$ particles are added in terms of the vacant time (at the origin) of the Poisson system $\widetilde{\eta}$, which can be estimated explicitly. This strategy will be made more precise in the following subsections.

4.1. *Coupling of one-dimensional random walks.* We will need the following result, which shows that for two random walks X and Y starting respective at $x \in \mathbb{Z}$ and 0 at time 0 with $|x| \gg 1$, there is a coupling between X and Y such that with high probability, the coupling is successful in the sense that X and Y coalesce and become a single walk before time $\tau_0 := \inf\{t \geq 0 : X(t) = 0\}$. Furthermore, whether the coupling is successful or not is independent of $(X(\tau_0 + t))_{t \geq 0}$.

LEMMA 4.1. *For $n \in \mathbb{N}$, let X_n and Y_n be two rate 1 continuous time random walks on $\mathbb{Z}$ with increment distribution $(a_x)_{x \in \mathbb{Z}}$ as specified in (1.1), starting respectively at x_n and 0 at time 0. Then there exists a coupling between X_n and Y_n with a coupling time T_n, such that:*

(i) *Either $T_n \leq \tau_0^{X_n} := \inf\{t \geq 0 : X_n(t) = 0\}$ and $X_n(t) = Y_n(t)$ for all $t \geq T_n$, in which case we call the coupling successful; or $T_n = \infty$ and we call the coupling unsuccessful;*

(ii) *The event $F_n := \{T_n \leq \tau_0^{X_n}\}$ is measurable w.r.t. Y_n and $(X_n(t))_{0 \leq t \leq \tau_0^{X_n}}$, and on its complement $\{T_n = \infty\}$, $(X_n(\tau_0^{X_n} + t))_{t \geq 0}$ is independent of Y_n and $(X_n(t))_{0 \leq t \leq \tau_0^{X_n}}$.*

(iii) *If $|x_n| \to \infty$ as $n \to \infty$, then $\mathbb{P}(F_n) \to 1$.*

Furthermore, when p is symmetric, the coupling can be chosen such that the joint dynamics of $(X_n(t), Y_n(t))_{t \geq 0}$ is Markovian.

REMARK 4.1. When X_n and Y_n are simple symmetric random walks on $\mathbb{Z}$, there is a simple Markovian coupling such that the coupling is successful with probability 1 for all $n \in \mathbb{N}$.

If $X_n(0)$ is even, then we let X_n and Y_n jump simultaneously but in opposite directions until the first time that the two walks meet, and from this time on they perform identical jumps. This ensures that X_n and Y_n coalesce before X_n hits 0. If $X_n(0)$ is odd, then we wait for the first jump by either X_n or Y_n, when the difference becomes even, and then couple as before.

PROOF OF LEMMA 4.1. Without loss of generality, we may assume that $x_n \to \infty$.

When p is symmetric, we can couple X_n and Y_n such that they take opposite steps [simply putting $Y_n(t) := x_n - X_n(t)$] until they get close (i.e., either they meet or exchange order), and then run them as independent random walks until either they meet or X_n hits 0, whichever happens first. In the first case, we set the meeting time to be T_n and let the two walks move together afterward; in the second case, we just set $T_n = \infty$.

In the general case, we can still couple X_n and Y_n such that they take "essentially" opposite steps until they get close by a suitable coupling to Brownian

motion, and then proceed as above. To implement this strategy, let

$$\tau_{1/2}^{Y_n} := \inf\{t \geq 0 : Y_n(t) \geq x_n/2\} \quad \text{and}$$

$$\tau_{1/2}^{X_n} := \inf\{t \geq 0 : X_n(t) \leq x_n/2\}.$$

By Donsker's invariance principle, as $x_n \to \infty$,

(4.1)
$$((x_n^{-1} Y_n(x_n^2 t))_{t \geq 0}, x_n^{-2} \tau_{1/2}^{Y_n}) \underset{n \to \infty}{\Longrightarrow} ((B_t)_{t \geq 0}, \tau_{1/2}),$$

$$((x_n^{-1} X_n(x_n^2 t))_{t \geq 0}, x_n^{-2} \tau_{1/2}^{X_n}) \underset{n \to \infty}{\Longrightarrow} ((1 - B_t)_{t \geq 0}, \tau_{1/2}),$$

where $(B_t)_{t \geq 0}$ is a Brownian motion with $\mathbb{E}[B_t^2] = \sigma^2 t$ and $\tau_{1/2} := \inf\{t \geq 0 : B_t \geq 1/2\}$. By Skorohod's representation theorem, we can couple $(X_n)_{n \geq 1}$ and B, and also $(Y_n)_{n \geq 1}$ and B, first possibly on different probability spaces, such that in both lines of (4.1) the convergence holds almost surely. Then, using regular versions of the conditional distribution given B on both probability spaces together with the same Brownian motion, we can construct copies of $(X_n)_{n \geq 1}$, $(Y_n)_{n \geq 1}$ and B on the same probability space such that the convergence in both lines of (4.1) holds simultaneously almost surely. We will use this coupling, which forces X_n and Y_n to take essentially opposite steps.

Since $\tau_{1/2}^{X_n}$ and $\tau_{1/2}^{Y_n}$ are stopping times, we may resample $(X_n(t))_{t \geq \tau_{1/2}^{X_n}}$ and $(Y_n(t))_{t \geq \tau_{1/2}^{Y_n}}$ [conditional on $X_n(\tau_{1/2}^{X_n})$, respectively on $Y_n(\tau_{1/2}^{Y_n})$] independently of their past and of each other without changing the law of X_n, resp. Y_n. Assume this resampling from now on, and let

$$\tau_{X_n, Y_n} := \inf\{t \geq \tau_{1/2}^{X_n} \vee \tau_{1/2}^{Y_n} : X_n(t) = Y_n(t)\}.$$

On the event $\tau_{X_n, Y_n} \leq \tau_0^{X_n} = \inf\{t \geq 0 : X_n(t) = 0\}$, we set the coupling time $T_n = \tau_{X_n, Y_n}$ and resample Y_n to be equal to X_n from time T_n onward. The coupling is then successful.

On the event $\tau_0^{X_n} < \tau_{X_n, Y_n}$, we set $T_n = \infty$ and the coupling is unsuccessful, and we do not make any further modification of X_n and Y_n.

With the above coupling, properties (i) and (ii) in Lemma 4.1 are clearly satisfied. To verify (iii), we need to show that under our coupling,

(4.2) $$\mathbb{P}(\tau_{X_n, Y_n} \leq \tau_0^{X_n}) \to 1 \qquad \text{as } |x_n| \to \infty.$$

Note that the above probability does not change if we assume $(X_n(t))_{t \geq \tau_{1/2}^{X_n}}$ and $(Y_n(t))_{t \geq \tau_{1/2}^{Y_n}}$ are coalescing random walks starting respectively at the space–time points $(X_n(\tau_{1/2}^{X_n}), \tau_{1/2}^{X_n})$ and $(Y_n(\tau_{1/2}^{Y_n}), \tau_{1/2}^{Y_n})$, where under our coupling, $(x_n^{-1} X_n(\tau_{1/2}^{X_n}), x_n^{-2} \tau_{1/2}^{X_n})$ and $(x_n^{-1} Y_n(\tau_{1/2}^{Y_n}), x_n^{-2} \tau_{1/2}^{Y_n})$ converge almost surely to the same space–time point $(1/2, \tau_{1/2})$. Therefore, by the weak convergence of coalescing random walks to coalescing Brownian motions (proved in [9], Section 5,

for discrete time random walks and is easily seen to hold also in continuous time), $(x_n^{-1} X_n(x_n^2 t))_{t \geq x_n^{-2} \tau_{1/2}^{X_n}}$ and $(x_n^{-1} Y_n(x_n^2 t))_{t \geq x_n^{-2} \tau_{1/2}^{Y_n}}$ converge to the same Brownian motion W starting at $1/2$ at time $\tau_{1/2}$, and the rescaled time of coalescence, $x_n^{-2} \tau_{X_n, Y_n}$, converges to $\tau_{1/2}$. In particular, the probability that $(X_n(t))_{t \geq \tau_{1/2}^{X_n}}$ hits 0 before meeting $(Y_n(t))_{t \geq \tau_{1/2}^{Y_n}}$ tends to 0 as n tends to infinity. This implies (4.2) and the claim in (iii). $\square$

4.2. *Coupling with the Poisson system.*

We now formulate precisely the coupling between the true system η, the Poisson system $\widetilde{\eta} := \widetilde{\eta}^{(-\varepsilon)}$ with immigration rate $\beta^{(-\varepsilon)}(t) = (1 - \varepsilon)\gamma e^{-\rho_0(t)}$, and the auxiliary system $\widehat{\eta}$ as outlined at the start of this section. To simplify notation, in the remainder of the subsection, we will drop $(-\varepsilon)$ from the superscript and simply write $\widetilde{\eta}$ instead of $\widetilde{\eta}^{(-\varepsilon)}$.

Let $t_0 = 0 < t_1 < t_2 < t_3 < \cdots$, and consider the time intervals $I_n = (t_{3n-3}, t_{3n-2}]$, $I_n' = (t_{3n-2}, t_{3n-1}]$, $I_n'' = (t_{3n-1}, t_{3n}]$. The precise values of the t_n's will be determined later in (4.4), with $|I_n| = |I_n'| \ll |I_n''|$. We will attempt to add exactly one $\widehat{\eta}$ particle in each time interval I_n, which will be coupled with the first $\widetilde{\eta}$ particle added during the time interval I_n'', with the coupling prescribed in Lemma 4.1.

More precisely, let $(\widetilde{N}_t)_{t \geq 0}$ be a Poisson process with rate $\beta^{(-\varepsilon)}$, which determine the times when particles are added to $\widetilde{\eta}$, and let $(N_t)_{t \geq 0}$ be an independent Poisson process with rate γ, which determines the times when we might attempt to add particles to $\widehat{\eta}$ and η. Start with $\widetilde{\eta}.(0) = \eta.(0) = \widehat{\eta}.(0) \equiv 0$, and as an inductive hypothesis, assume that particles have been added to $\widetilde{\eta}$, η and $\widehat{\eta}$ up to time $t_{3(n-1)}$ for some $n \geq 1$, such that the following properties hold:

(a) The paths of all added $\widetilde{\eta}$ particles have been sampled to time ∞, while the path of each added η particle has been sampled till its first return to the origin after time $t_{3(n-1)}$, and the same for each $\widehat{\eta}$ particle unless it has been killed earlier;

(b) Each $\widehat{\eta}$ particle is coupled to a distinct η particle so that they follow the same path till the time of death of the $\widehat{\eta}$ particle. In particular, there are always more η particles at the origin than $\widehat{\eta}$ particles;

(c) Each $\widehat{\eta}$ particle added during the time interval I_k, for any $k \leq n - 1$, is either killed at its first return to the origin after time t_{3k}, or it lives forever and is successfully coupled as in Lemma 4.1 to an $\widetilde{\eta}$ particle added during the subsequent time interval I_k'';

(d) As a consequence of (c), at any time $t \geq t_{3(n-1)}$, if one of the $\widehat{\eta}$ particles added before time $t_{3(n-1)}$ is at the origin, then so is one of the $\widetilde{\eta}$ particles added before time $t_{3(n-1)}$.

We now add particles to $\widetilde{\eta}$, η and $\widehat{\eta}$ in the time intervals $(t_{3(n-1)}, t_{3n}]$ as follows:

- Add particles to $\widetilde{\eta}$ during the time interval $I_n \cup I_n'$ according to the Poisson process $\widetilde{N}$, with particle trajectories sampled to time ∞ according to independent

random walks. Evolve existing η and $\widehat{\eta}$ particles further till their first return to the origin after time t_{3n}.

-   Let

$$\widehat{T}_n := \inf\{t \in I_n : \widetilde{\eta}_0(t) = 0, \Delta N_t = 1\},$$

$$\widetilde{T}_n := \inf\{t \in I_n'' : \Delta \widetilde{N}_t = 1\},$$

where $\inf \varnothing := \infty$. If $\widehat{T}_n = \infty$, then no $\widehat{\eta}$ particles are added during $(t_{3(n-1)}, t_{3n}]$, and η and $\widetilde{\eta}$ particles are added independently according to their own rules until time t_{3n}, and their paths are sampled such that property (a) above continues to hold by t_{3n};

-   If $\widehat{T}_n < \infty$, then we add an $\widehat{\eta}$ particle at the origin at time $\widehat{T}_n$. If the origin is occupied in η at time $\widehat{T}_n$, then we let the added $\widehat{\eta}$ particle follow the same path X_n as one of the η particles at the origin chosen at random. If the origin is vacant in η at time $\widehat{T}_n$, then we also add an η particle at time $\widehat{T}_n$ and let both particles follow the same random walk X_n, sampled independently of everything else till its first return to the origin after time t_{3n}. (Should the $\widehat{\eta}$ particle be killed later in the construction, we understand that the η particle will be unaffected.)

-   Continue to add η and $\widetilde{\eta}$ particles independently according to their own rules until time $t_{3n} \wedge \widetilde{T}_n$, and sample their paths so that property (a) continues to hold by t_{3n}. If $\widetilde{T}_n = \infty$, then kill the added $\widehat{\eta}$ particle at time $\tau_n := \inf\{t \geq t_{3n} : X_n(t) = 0\}$.

-   If $\widetilde{T}_n < \infty$, then add an $\widetilde{\eta}$ particle at the origin at time $\widetilde{T}_n$ and sample its path Y_n according to the conditional law of Y_n, conditioned on $(X_n(t))_{\widetilde{T}_n \leq t \leq \tau_n}$, so that (X_n, Y_n) follows the law of the coupled random walks (X_n, Y_n) in Lemma 4.1. Denote

$$E_n := \{\widehat{T}_n < \infty, \widetilde{T}_n < \infty \text{ and } X_n \text{ and}$$

$$Y_n \text{ are coupled successfully as in Lemma 4.1}\}.$$

    If the coupling is successful, then let the added $\widehat{\eta}$ particle live forever, otherwise kill it at time $\tau_n = \inf\{t \geq t_{3n} : X_n(t) = 0\}$.

-   Continue to add η and $\widetilde{\eta}$ particles independently according to their own rules till time t_{3n}, with their trajectories sampled so that property (a) continues to hold by t_{3n}.

We note a subtle point in the above coupling, namely that we need to show that the $\widetilde{\eta}$ particles added at times $(\widetilde{T}_n)_{n \in \mathbb{N}}$ are indeed distributed as independent random walks. This is true because by construction, conditioned on $X_n(\widetilde{T}_n)$ for $n \in \mathbb{N}$ with $\widehat{T}_n, \widetilde{T}_n < \infty$, the $\widehat{\eta}$ particle trajectories $(X_n(t))_{\widetilde{T}_n \leq t \leq \tau_n}$ are jointly independent, while the path Y_n of each $\widetilde{\eta}$ particle coupled to X_n depends only on $(X_n(t))_{\widetilde{T}_n \leq t \leq \tau_n}$ by Lemma 4.1.

It is clear that properties (a)–(d) above continue to hold after adding all particles up to time t_{3n}, and hence by induction, they hold for all time. In particular, by the

coupling between η and $\widehat{\eta}$, for all $n \in \mathbb{N}$, we have

$$(4.3) \qquad \sum_{x \in \mathbb{Z}} \eta_x(t_{3n}) \geq \sum_{x \in \mathbb{Z}} \widehat{\eta}_x(t_{3n}) \geq \sum_{j=1}^n \mathbf{1}_{E_j} = \sum_{j=1}^n \mathbf{1}_{\{\widehat{T}_j < \infty, \widetilde{T}_j < \infty\}} \mathbf{1}_{E_j}.$$

To prove the lower bounds in Theorems 1.1 and 1.2, we will use the following choice of $(t_i)_{i \in \mathbb{N}}$:

$$t_{3n} = \varepsilon^2 \frac{n^2}{(\log(n \vee 3))^2} \quad \text{and}$$

$$(4.4)$$

$$t_{3n+2} - t_{3n+1} = t_{3n+1} - t_{3n} = \varepsilon^2 (n+1)^{1-\varepsilon/2}, \qquad n \geq 0.$$

The choice of t_i is motivated by the fact that from (1.5), the time until the nth particle appears in the true system should be of order $n^2/(\log n)^2$. Note that (4.4) implies

$$(4.5) \qquad \sum_{n: t_{3n} \leq t} 1 \underset{t \to \infty}{\sim} (\sqrt{t} \log t)/2\varepsilon.$$

REMARK 4.2. When the random walk jump kernel $p(\cdot)$ is symmetric, we can use the Markovian coupling of random walks guaranteed by Lemma 4.1 to construct the coupled $\widetilde{\eta}$, $\widehat{\eta}$ and η particle systems jointly as a Markov process, with the use of labels to distinguish whether a particle is an $\widetilde{\eta}$, $\widehat{\eta}$ or η particle, or it has multiple labels due to the coupling.

Spelling out the generator of such a system is straightforward, though lengthy, so we do not make it explicit here. Briefly, at a time $\widehat{T}_n < \infty$, if the origin is empty in the η system, we add a particle X_n with label "$\eta \& \widehat{\eta}$"; while if the origin is occupied in η but empty in $\widehat{\eta}$, we change the label of one of the η particles to "$\eta \& \widehat{\eta}$" (and call this the X_n particle). The X_n particle evolves as a free random walk until time $\widehat{T}_n \wedge t_{3n}$. If $\widehat{T}_n < t_{3n}$, then we add a particle Y_n with label "$\widetilde{\eta}$," and X_n and Y_n then evolve jointly as a Markov process according to the Markovian coupling from Lemma 4.1 until either they meet (at which time the particles merge and henceforth evolve as a free random walk with label "$\eta \& \widehat{\eta} \& \widetilde{\eta}$"), or the X_n particle visits the origin before meeting Y_n (from this time the X_n particle evolves as a free random walk with label "η" and the Y_n particle evolves independently as a free random walk with label "$\widetilde{\eta}$"). If $\widehat{T}_n \geq t_{3n}$, then we change the label of the X_n particle to "η" at time t_{3n}. In between, all other particles (with their labels) evolve independently, and additions of η, respectively, $\widetilde{\eta}$ particles are executed according to their respective rules.

4.3. *Proof of Theorem* 1.1 (*lower bound*). First, we note that the number of particles added to the Poisson system $\widetilde{\eta}$ during the time interval I_n'', which we

denote by $\widetilde{M}_n$, is a Poisson random variable with mean

$$
\begin{aligned}
\mathbb{E}[\widetilde{M}_n] &= \int_{t_{3n-1}}^{t_{3n}} \beta^{(-\varepsilon)}(s)\, ds = \int_{t_{3n-1}}^{t_{3n}} \frac{\sigma(1-\varepsilon)}{\sqrt{2\pi}} \frac{\log s}{\sqrt{s}}(1+o(1))\, ds \\
&\underset{n\to\infty}{\sim} \frac{2\sigma(1-\varepsilon)}{\sqrt{2\pi}}(\sqrt{t_{3n}} - \sqrt{t_{3n-1}})\log t_{3n} \\
&\sim \ 4\sigma \frac{\varepsilon(1-\varepsilon)}{\sqrt{2\pi}},
\end{aligned}
$$

where we used the form of $\beta^{(-\varepsilon)}$ given in (2.3), the asymptotics for $\rho_0(t)$ given in (1.4), and the choice of $(t_i)_{i\in\mathbb{N}}$ given in (4.4). Therefore,

$$
\mathbb{P}(\widetilde{T}_n < \infty) = \mathbb{P}(\widetilde{M}_n > 0) = 1 - e^{-\mathbb{E}[\widetilde{M}_n]} \underset{n\to\infty}{\longrightarrow} 1 - e^{-4\sigma\frac{\varepsilon(1-\varepsilon)}{\sqrt{2\pi}}}.
$$

Since $(\widetilde{M}_n)_{n\in\mathbb{N}}$ are independent, almost surely, we have

$$
(4.6) \qquad \sum_{j=1}^{n} \mathbf{1}_{\{\widetilde{T}_j < \infty\}} \underset{n\to\infty}{\sim} n\big(1 - e^{-4\sigma\frac{\varepsilon(1-\varepsilon)}{\sqrt{2\pi}}}\big).
$$

Next, we observe that on each time interval I_n, conditioned on the Poisson system $\widetilde{\eta}$,

$$
\mathbb{P}(\widehat{T}_n = \infty \mid \widetilde{\eta}) = e^{-\gamma \widetilde{V}_{t_{3n-3}, t_{3n-2}}},
$$

where by (2.6),

$$
\begin{aligned}
\mathbb{E}[\widetilde{V}_{t_{3n-3}, t_{3n-2}}] &\geq c(t_{3n-2} - t_{3n-3}) t_{3n-2}^{-(1-\varepsilon)/2} \geq c\varepsilon^2 n^{1-\varepsilon/2} t_{3n}^{-(1-\varepsilon)/2} \\
&= c\varepsilon^{1+\varepsilon} n^{\varepsilon/2} (\log n)^{1-\varepsilon}
\end{aligned}
$$

for some $c > 0$. By the moment bound in Lemma 2.2 for $(\widetilde{V}_{t_{3n-3}, t_{3n-2}} - \mathbb{E}[\widetilde{V}_{t_{3n-3}, t_{3n-2}}])^k$ for a sufficiently large even k (note that the conditions are fulfilled), we can apply Markov's inequality and Borel–Cantelli to conclude that a.s., $\widetilde{V}_{t_{3n-3}, t_{3n-2}}/\mathbb{E}[\widetilde{V}_{t_{3n-3}, t_{3n-2}}] \to 1$, and hence $\{\widetilde{V}_{t_{3n-3}, t_{3n-2}} > n^{\varepsilon/2}/2\}$ occurs for all large enough n. Therefore, a.s., $\sum_n \mathbb{P}(\widehat{T}_n = \infty \mid \widetilde{\eta}) < \infty$, and hence almost surely,

$$
(4.7) \qquad \{\widehat{T}_j < \infty\} \qquad \text{occurs for all } j \text{ sufficiently large.}
$$

Lastly, we consider the events E_j in (4.3). In our coupling construction of $\widetilde{\eta}$, η and $\widehat{\eta}$, let $\mathcal{F}_n$ denote the σ-algebra generated by: the Poisson point process $\widetilde{N}$ up to time $\widetilde{T}_n \wedge t_{3n}$ and the trajectories of the $\widetilde{\eta}$ particles added before that time, as well as the Poisson point process N up to time $\widehat{T}_n \wedge t_{3n-2}$ and the trajectories of the $\widehat{\eta}$ particles added before that time. Then $(\mathcal{F}_n)_{n\in\mathbb{N}}$ defines a filtration, with $\{\widetilde{T}_n < \infty, \widehat{T}_n < \infty\} \in \mathcal{F}_n$, and $E_n \in \mathcal{F}_{n+1}$. Furthermore, since $|I'_n| \to \infty$, on the

event $\{\widehat{T}_n < \infty, \widetilde{T}_n < \infty\}$, the path X_n of the $\widehat{\eta}$ particle added at time $\widehat{T}_n$ satisfies $|X_n(\widetilde{T}_n)| \to \infty$ in probability as $n \to \infty$. Therefore, by Lemma 4.1,

$$
\begin{aligned}
& \big|\mathbb{P}(E_n|\mathcal{F}_n) - \mathbf{1}_{\{\widehat{T}_n < \infty, \widetilde{T}_n < \infty\}}\big| \\
(4.8) \quad & \leq \mathbf{1}_{\{\widehat{T}_n < \infty, \widetilde{T}_n < \infty\}}\big|\mathbb{P}(X_n \text{ and } Y_n \text{ are successfully coupled}) - 1\big| \\
& \xrightarrow[n \to \infty]{} 0.
\end{aligned}
$$

Note that (4.6) and (4.7) imply

$$
\sum_{j=1}^{n} \mathbf{1}_{\{\widehat{T}_j < \infty, \widetilde{T}_j < \infty\}} \underset{n \to \infty}{\sim} \sum_{j=1}^{n} \mathbf{1}_{\{\widetilde{T}_j < \infty\}} \underset{n \to \infty}{\sim} n\big(1 - e^{-4\sigma \frac{\varepsilon(1-\varepsilon)}{\sqrt{2\pi}}}\big) \longrightarrow \infty \qquad \text{a.s.,}
$$

which together with (4.8) gives

$$
(4.9) \quad \sum_{j=1}^{n} \mathbb{P}(E_j|\mathcal{F}_j) \underset{n \to \infty}{\sim} \sum_{j=1}^{n} \mathbf{1}_{\{\widehat{T}_j < \infty, \widetilde{T}_j < \infty\}} \underset{n \to \infty}{\sim} \sum_{j=1}^{n} \mathbf{1}_{\{\widetilde{T}_j < \infty\}} \longrightarrow \infty \qquad \text{a.s.}
$$

On the other hand, by the second Borel–Cantelli lemma ([5], (4.11)),

$$
\frac{\sum_{j=1}^{n} \mathbf{1}_{E_j}}{\sum_{j=1}^{n} \mathbb{P}(E_j|\mathcal{F}_j)} \xrightarrow[n \to \infty]{} 1 \qquad \text{a.s. on } \left\{\sum_{j=1}^{\infty} \mathbb{P}(E_j|\mathcal{F}_j) = \infty\right\},
$$

which event is seen to have probability 1 by (4.9). Therefore, we also have

$$
\begin{aligned}
(4.10) \quad & \sum_{j=1}^{n} \mathbf{1}_{E_j} \underset{n \to \infty}{\sim} \sum_{j=1}^{n} \mathbb{P}(E_j|\mathcal{F}_j) \underset{n \to \infty}{\sim} \sum_{j=1}^{n} \mathbf{1}_{\{\widetilde{T}_j < \infty\}} \\
& \underset{n \to \infty}{\sim} n\big(1 - e^{-4\sigma \frac{\varepsilon(1-\varepsilon)}{\sqrt{2\pi}}}\big) \qquad \text{a.s.}
\end{aligned}
$$

Since $t_{3j} = \varepsilon^2 \frac{j^2}{(\log j)^2}$, by (4.3) and (4.5), this implies

$$
\begin{aligned}
(4.11) \quad & \sum_{x} \eta_x(t) \geq \sum_{j: t_{3j} \leq t} \mathbf{1}_{E_j} \underset{t \to \infty}{\sim} \sum_{j=1}^{\sqrt{t} \log t / 2\varepsilon} \mathbf{1}_{E_j} \\
& \underset{t \to \infty}{\sim} \frac{\sqrt{t} \log t}{2\varepsilon}\big(1 - e^{-4\sigma \frac{\varepsilon(1-\varepsilon)}{\sqrt{2\pi}}}\big) \qquad \text{a.s.}
\end{aligned}
$$

Letting $\varepsilon \downarrow 0$ then gives the desired lower bound on $\sum_x \eta_x(t)$ in Theorem 1.1.

4.4. *Proof of Theorem* 1.2 (*lower bound*). The lower bound on the rate at which η particles arrive readily leads to a lower bound on the spatial distribution of these particles at time t, since once an η particle arrives, it evolves independently from all other particles.

First, note that it suffices to verify Theorem 1.2 for $f \in C_{b,+}(\mathbb{R})$ with a uniformly bounded derivative f', since Theorem 1.1 on the convergence of the total mass of the measure $\frac{\sum_x \eta_x(t)\delta_x}{\sigma\sqrt{t}\log t}$ implies that it suffices to verify Theorem 1.2 for $f \in C_{b,+}(\mathbb{R})$ with compact support, and any such f can then be approximated in supremum norm by functions in $C_{b,+}(\mathbb{R})$ with bounded derivatives.

Let us recall our construction of $\hat{\eta}$ in Section 4.2. For each $n \in \mathbb{N}$, an $\hat{\eta}$ particle is added at time $\hat{T}_n$ and then follows a random walk X_n and lives forever, provided that $\hat{T}_n < \infty$, $\tilde{T}_n < \infty$, and X_n can be successfully coupled to the random walk Y_n that governs the motion of the $\tilde{\eta}$ particle added at time $\tilde{T}_n$. Then analogous to (4.11), for any $f \in C_{b,+}(\mathbb{R})$, almost surely

$$\liminf_{t\to\infty} \frac{1}{\sigma\sqrt{t}\log t} \sum_x \eta_x(t) f\left(\frac{x}{\sigma\sqrt{t}}\right)$$

$$\geq \liminf_{t\to\infty} \frac{1}{\sigma\sqrt{t}\log t} \sum_{n:t_{3n}\leq t} \mathbf{1}_{E_n} f\left(\frac{X_n(t)}{\sigma\sqrt{t}}\right).$$

We can replace $\mathbf{1}_{E_n}$ above by $\mathbf{1}_{\{\tilde{T}_n<\infty\}}$, since by (4.10) and (4.5), we have

$$\frac{1}{\sigma\sqrt{t}\log t}\left|\sum_{n:t_{3n}\leq t} \mathbf{1}_{E_n} f\left(\frac{X_n(t)}{\sigma\sqrt{t}}\right) - \sum_{n:t_{3n}\leq t} \mathbf{1}_{\{\tilde{T}_n<\infty\}} f\left(\frac{X_n(t)}{\sigma\sqrt{t}}\right)\right|$$

$$= \frac{1}{\sigma\sqrt{t}\log t} \sum_{n:t_{3n}\leq t} (\mathbf{1}_{\{\tilde{T}_n<\infty\}} - \mathbf{1}_{E_n}) f\left(\frac{X_n(t)}{\sigma\sqrt{t}}\right)$$

$$\leq \frac{\|f\|_\infty}{\sigma\sqrt{t}\log t}\left(\sum_{n:t_{3n}\leq t} \mathbf{1}_{\{\tilde{T}_n<\infty\}} - \sum_{n:t_{3n}\leq t} \mathbf{1}_{E_n}\right) \xrightarrow[t\to\infty]{} 0.$$

For any $\hat{\eta}$ particle that gets killed, let us extend its path X_n beyond its death by an independent random walk, and for n with $\hat{T}_n = \infty$, we let X_n be an independent random walk starting from t_{3n-2}. We then have

(4.12)
$$\liminf_{t\to\infty} \frac{1}{\sigma\sqrt{t}\log t} \sum_x \eta_x(t) f\left(\frac{x}{\sigma\sqrt{t}}\right)$$

$$\geq \liminf_{t\to\infty} \frac{1}{\sigma\sqrt{t}\log t} \sum_{n:t_{3n}\leq t} \mathbf{1}_{\{\tilde{T}_n<\infty\}} f\left(\frac{X_n(t)}{\sigma\sqrt{t}}\right).$$

Note that the space–time shifted random walks $(W_n(s))_{s\geq 0} := (X_n(\tilde{T}_n + s) - X_n(\tilde{T}_n))_{s\geq 0}$, $n \in \mathbb{N}$, are i.i.d. and independent of the Poisson process $\tilde{N}$ that determines the times when a particle is added to $\tilde{\eta}$. We can then use $(W_n)_{n\in\mathbb{N}}$ and $\tilde{N}$ to construct another Poisson system of random walks $\tilde{\xi}$ with the same distribution as $\tilde{\eta}$. More precisely, for each $n \in \mathbb{N}$ with $\tilde{T}_n < \infty$, we add a $\tilde{\xi}$ particle at the origin at time $\tilde{T}_n$ which follows the trajectory $(W_n(s - \tilde{T}_n))_{s\geq\tilde{T}_n} = (X_n(s) - X_n(\tilde{T}_n))_{s\geq\tilde{T}_n}$. For all other times t with $\Delta\tilde{N}_t = 1$, we add a $\tilde{\xi}$ particle at the origin at time t, which follows an independent random walk trajectory.

We claim that a.s., the RHS of (4.12) does not change if we replace the trajectories of the $\widehat{\eta}$ particles therein by those of the $\widetilde{\xi}$ particles added at times $(\widetilde{T}_n)_{n\in\mathbb{N}}$. Indeed, the absolute difference arising from such a replacement (before taking $\liminf_{t\to\infty}$) is

$$\left| \frac{1}{\sigma\sqrt{t}\log t} \sum_{n:t_{3n}\leq t} \mathbf{1}_{\{\widetilde{T}_n<\infty\}} \left(f\left(\frac{X_n(t)}{\sigma\sqrt{t}}\right) - f\left(\frac{X_n(t)-X_n(\widetilde{T}_n)}{\sigma\sqrt{t}}\right) \right) \right|$$

$$(4.13) \qquad \leq \frac{\|f'\|_\infty}{\sigma\sqrt{t}\log t} \sum_{n:t_{3n}\leq t} \mathbf{1}_{\{\widetilde{T}_n<\infty\}} \min\left\{ \|f\|_\infty, \frac{|X_n(\widetilde{T}_n)|}{\sigma\sqrt{t}} \right\}$$

$$\leq \frac{\|f'\|_\infty}{\sigma\sqrt{t}\log t} \sum_{n:t_{3n}\leq t} \min\left\{ \|f\|_\infty, \frac{\sup_{s\in[0,t_{3n}-t_{3n-3}]} |\widehat{X}_n(s)|}{\sigma\sqrt{t}} \right\},$$

where $\widehat{X}_n$ is the random walk obtained from X_n by shifting its starting time to 0, which are i.i.d. and independent of $\widetilde{N}$. If we denote the minima in (4.13) by $U_{n,t}$, then by Doob's L^2 maximal inequality for $\widehat{X}_n$,

$$\mathbb{E}[U_{n,t}] \leq \frac{2}{\sigma\sqrt{t}}\mathbb{E}[\widehat{X}_{t_{3n}-t_{3n-3}}^2]^{\frac{1}{2}} = 2\sqrt{t_{3n}-t_{3n-3}} \leq \frac{C\sqrt{n}}{\sqrt{t}\log n}.$$

Therefore, using (4.5),

$$(4.14) \qquad \frac{\|f'\|_\infty}{\sigma\sqrt{t}\log t} \sum_{n:t_{3n}\leq t} \mathbb{E}[U_{n,t}] \leq \frac{C'\|f'\|_\infty}{t\log t}(\sqrt{t}\log t)^{\frac{3}{2}} \xrightarrow[t\to\infty]{} 0.$$

Since $(U_{n,t})_{n\in\mathbb{N}}$ are independent random variables uniformly bounded by $\|f\|_\infty$, a standard fourth moment calculation applied to $\frac{\|f'\|_\infty}{\sigma\sqrt{t}\log t}\sum_{n:t_{3n}\leq t}(U_{n,t} - \mathbb{E}[U_{n,t}])$, together with Markov inequality and Borel–Cantelli, show that this sequence converges a.s. to 0 along the times $(t_{3N})_{N\in\mathbb{N}}$ (and hence also along $t\uparrow\infty$). Together with (4.14), this implies that the bound in (4.13) converges a.s. to 0 as $t\to\infty$, and hence we can replace the $\widehat{\eta}$ particle trajectories in the RHS of (4.12) by those of the $\widetilde{\xi}$ particles added at times $(\widetilde{T}_n)_{n\in\mathbb{N}}$.

We now make one more reduction, namely that including all particles in the $\widetilde{\xi}$ system (not just those added at times $\widetilde{T}_n$) only introduces a small ε-dependent error. More precisely,

$$\frac{1}{\sigma\sqrt{t}\log t}\left| \sum_{n:t_{3n}\leq t} \mathbf{1}_{\{\widetilde{T}_n<\infty\}} f\left(\frac{X_n(t)-X_n(\widetilde{T}_n)}{\sigma\sqrt{t}}\right) - \sum_x \widetilde{\xi}_x(t)f\left(\frac{x}{\sigma\sqrt{t}}\right) \right|$$

$$\leq \frac{\|f\|_\infty}{\sigma\sqrt{t}\log t}\left| \sum_x \widetilde{\xi}_x(t) - \sum_{n:t_{3n}\leq t} \mathbf{1}_{\{\widetilde{T}_n<\infty\}} \right|$$

$$\xrightarrow[t\to\infty]{} \|f\|_\infty\left| \left(\frac{2}{\pi}\right)^{1/2}(1-\varepsilon) - \frac{1}{2\varepsilon\sigma}\left(1 - e^{-4\sigma\frac{\varepsilon(1-\varepsilon)}{\sqrt{2\pi}}}\right) \right| =: A_\varepsilon$$

by the a.s. asymptotics for $\sum_x \widetilde{\widetilde{\xi}}_x(t)$ in Lemma 2.1 and the a.s. asymptotics for $\sum_{n:t_{3n}\leq t} 1_{\{\widetilde{T}_n<\infty\}}$ given in (4.10) and (4.11). Note that the limit A_ε above tends to 0 as $\varepsilon \downarrow 0$.

By the successive reductions we have made, we have thus shown that a.s.,

(4.15)

$$\liminf_{t\to\infty} \frac{1}{\sigma\sqrt{t}\log t} \sum_x \eta_x(t) f\left(\frac{x}{\sigma\sqrt{t}}\right)$$

$$\geq \liminf_{t\to\infty} \frac{1}{\sigma\sqrt{t}\log t} \sum_x \widetilde{\widetilde{\xi}}_x(t) f\left(\frac{x}{\sigma\sqrt{t}}\right) - A_\varepsilon,$$

where the above limit for the $\widetilde{\widetilde{\xi}}$ system equals $(1-\varepsilon)\int_{\mathbb{R}} f(y)\tilde{\rho}(y)\,dy$ by the same argument as that in Section 3.3 for the $\tilde{\eta}^{(+\varepsilon)}$ system. Letting $\varepsilon \downarrow 0$ then completes the proof of the a.s. lower bound in Theorem 1.2.

APPENDIX A: ASYMPTOTICS OF A SEMILINEAR LATTICE HEAT EQUATION

This section is adapted from [2], Section 3.6.1, for ease of reference and the reader's convenience.

We consider the long-time behavior of the solution of the following inhomogeneous heat equation on $\mathbb{Z}$ (which reduces to (1.2) upon choosing $\alpha = 1$):

(A.1)

$$\partial_t \rho_x(t) = L_{\mathrm{rw}}\rho_x(t) + \gamma\delta_0(x)\exp(-\alpha\rho_0(t)), \qquad t \geq 0, x \in \mathbb{Z},$$

$$\rho_x(0) \equiv 0,$$

where $\gamma, \alpha > 0$ are parameters, and L_{rw} is the generator of a rate 1 continuous time random walk X on $\mathbb{Z}$, whose jump increments follow the probability kernel $(a_{-x})_{x\in\mathbb{Z}}$ with mean 0 and variance σ^2, as specified in (1.1).

REMARK A.1.   1. In integral form (sometimes called "Duhamel's principle"), (A.1) reads

(A.2) $$\rho_x(t) = \gamma \int_0^t p_x(t-s)\exp(-\alpha\rho_0(s))\,ds, \qquad x \in \mathbb{Z}, t \geq 0,$$

where $p_x(t) = \mathbb{P}_0(X(t) = x)$ is the transition probability of a continuous-time random walk with generator L_{rw}.

2. Let ρ be the solution of (A.1). Then $\vartheta_x(t) := \alpha\rho_x(t)$ solves $\partial_t\vartheta_x(t) = L_{\mathrm{rw}}\vartheta_x(t) + \gamma'\delta_0(x)\exp(-\vartheta_0(t))$ with $\gamma' := \gamma\alpha$, hence it suffices to consider the case $\alpha = 1$.

3. (A.1) [and hence also (1.2)] has a unique solution: Let $\rho^{(1)}, \rho^{(2)}$ be solutions, then

$$\frac{\partial}{\partial t} \sum_x (\rho_x^{(2)}(t) - \rho_x^{(1)}(t))^2 = 2\sum_x (\rho_x^{(2)}(t) - \rho_x^{(1)}(t)) L_{\mathrm{rw}}(\rho_\cdot^{(2)}(t) - \rho_\cdot^{(1)}(t))_x$$

$$+ 2\gamma(\rho_0^{(2)}(t) - \rho_0^{(1)}(t))(e^{-\alpha\rho_0^{(2)}(t)} - e^{-\alpha\rho_0^{(1)}(t)}) \leq 0,$$

noting that $\sum_x f_x (L_{\mathrm{rw}} f)_x \leq 0$ for any $f \in \ell^2(\mathbb{Z})$, and $(a - b)(e^{-\alpha a} - e^{-\alpha b}) \leq 0$ for any $a, b \in \mathbb{R}$. Hence, $\rho^{(1)} \equiv \rho^{(2)}$.

LEMMA A.1. *Let ρ be the solution of* (A.1). *Then $\rho_0(t)$ is increasing in t, and as $t \to \infty$,*

$$(A.3) \qquad \rho_0(t) = \frac{1}{\alpha} \left\{ \frac{1}{2} \log t - \log \log t + \log(\sqrt{2\pi} \gamma \alpha / \sigma) \right\} + o(1),$$

$$(A.4) \qquad \sum_x \rho_x(t) = \gamma \int_0^t e^{-\alpha \rho_0(s)} \, ds \sim \frac{\sigma}{\alpha} \left(\frac{2}{\pi} \right)^{1/2} \sqrt{t} \log t.$$

PROOF. Assume w.l.o.g. $\alpha = 1$; cf. Remark A.1. We see from (A.2) for $x = 0$ that $\rho_0(t)$ is the solution of the functional equation

$$(A.5) \qquad f(t) = \int_0^t \gamma p_0(t - s) \exp(-f(s)) \, ds, \qquad t \geq 0.$$

Let us call a function $\bar\varphi : \mathbb{Z}^d \times \mathbb{R}_+ \to \mathbb{R}_+$ with $\bar\varphi_\cdot(0) \equiv 0$ a *strict supersolution to* (A.1) if it solves

$$(A.6) \qquad \begin{aligned} &\partial_t \bar\varphi_x(t) = L_{\mathrm{rw}} \bar\varphi_x(t) + \gamma r^{\bar\varphi}(t) \delta_0(x), \qquad t \geq 0, x \in \mathbb{Z} \\ &\text{with an } r^{\bar\varphi}(t) > \exp(-\bar\varphi_0(t)). \end{aligned}$$

Then we see that $\bar\varphi_0(t) \geq \rho_0(t)$ for all $t \geq 0$: Indeed, $\psi_x(t) := \bar\varphi_x(t) - \rho_x(t)$ solves

$$\partial_t \psi_x(t) = L_{\mathrm{rw}} \psi_x(t) + \gamma (r^{\bar\varphi}(t) - e^{-\rho_0(t)}) \delta_0(x)$$

and $\psi_0(t) > 0$ for small t. Assume that $t_0 := \inf\{t : \psi_0(t) < 0\} < \infty$. Then we would have $\psi_0(t_0) = 0$ by continuity, but also $\psi_x(t_0) \geq 0$ for all x. To see this, observe that $\psi_x(t)$, $x \neq 0$ has a representation (ψ solves the heat equation away from 0, consider $\psi_0(t)$ as exogenous input)

$$\psi_x(t) = \int_0^t \psi_0(t - s) \mathbb{P}_x(T_0 \in ds) + \mathbb{E}_x[\psi_{X(t)}(0); T_0 > t]$$

$$\left(= \mathbb{E}_x[\psi_{X(t \wedge T_0)}(t - (t \wedge T_0))] \right),$$

where $T_0 := \inf\{s : X_s = 0\}$ (see Lemma A.3). Hence, $\psi_x(t_0) \geq 0$ for all x because $\psi_\cdot(0) \equiv 0$ and $\psi_0(s) \geq 0$ for $0 \leq s \leq t_0$ by definition. Consequently, $L_{\mathrm{rw}} \psi_0(t_0) \geq 0$ and we conclude that $\gamma^{-1} \partial_t \psi_0(t_0) \geq r^{\bar\varphi}(t_0) - e^{-\rho_0(t_0)} > \exp(-\bar\varphi_0(t_0)) - \exp(-\rho_0(t_0)) = 0$ in contradiction to the definition of t_0.

We can construct a supersolution to (A.1) from a strict subsolution to (A.5): Assume $\underline{f} : [0, \infty) \to \mathbb{R}$ satisfies

$$(A.7) \qquad \underline{f}(t) < \int_0^t \gamma p_0(t - s) \exp(-\underline{f}(s)) \, ds \qquad \text{for } t \geq 0.$$

Then

(A.8)
$$\bar{\varphi}_x(t) := \int_0^t \gamma p_x(t-s) \exp(-\underline{f}(s)) \, ds$$

solves

$$\partial_t \bar{\varphi}_x(t) = L_{\mathrm{rw}} \bar{\varphi}_x(t) + \gamma \exp(-\underline{f}(t)) \delta_0(x)$$

and in particular $\bar{\varphi}_0(t) > \underline{f}(t)$, hence $\exp(-\underline{f}(t)) > \exp(-\bar{\varphi}_0(t))$.

Similarly, if $\underline{\varphi}$ is a strict subsolution we have $\underline{\varphi}_0(t) \le \rho_0(t)$ for all $t \ge 0$ and such a $\underline{\varphi}$ can be constructed analogously from a supersolution $\bar{f}$ to (A.5).

Observe that the solution ρ of (A.1) has the property that $\rho_0(t)$ is an increasing function: Obviously, $\partial_t \rho_0(t) > 0$ for t small. Assume that $t_0 := \inf\{t : \partial_t \rho_0(t) < 0\} < \infty$. Then by continuity $\partial_t \rho_0(t_0) = 0$. We have for $x \in \mathbb{Z} \setminus \{0\}$ by the representation given in Lemma A.3

$$\partial_t \rho_x(t_0) = \lim_h \frac{1}{h} \left[\int_0^{t_0} \rho_0(t_0 - s) \mathbb{P}_x(T_0 \in ds) - \int_0^{t_0 - h} \rho_0(t_0 - h - s) \mathbb{P}_x(T_0 \in ds) \right]$$

$$= \lim_h \int_0^{t_0 - h} \frac{1}{h} (\rho_0(t_0 - s) - \rho_0(t_0 - h - s)) \mathbb{P}_x(T_0 \in ds)$$

$$+ \lim_h \frac{1}{h} \int_{t_0 - h}^{t_0} \rho_0(t_0 - s) \mathbb{P}_x(T_0 \in ds)$$

$$\ge \int_0^{t_0} \partial_t \rho_0(t - s) \mathbb{P}_x(T_0 \in ds) + \underbrace{\rho_0(0)}_{=0} \left. \frac{\mathbb{P}_x(T_0 \in dt)}{dt} \right|_{t=t_0} > 0,$$

because $\partial_t \rho_0(t) > 0$ in $[0, t_0)$ and $\mathrm{supp}(\mathcal{L}_x(T_0)) = \mathbb{R}_+$, and we applied Fatou's lemma in the first inequality. Thus,

$$\partial_t^2 \rho_0(t_0) = \sum_x a_x (\partial_t \rho_x(t_0) - \partial_t \rho_0(t_0)) - \partial_t \rho_0(t_0) \gamma \exp(-\rho_0(t_0)) > 0,$$

contradicting the definition of t_0.

LEMMA A.2. *Assume $\alpha = 1$.*

(i) *For $C < \log(\sqrt{2\pi}\gamma/\sigma)$ there exists a $K > 0$ such that*

$$\underline{f}(t) := \begin{cases} \dfrac{1}{2}\log t - \log\log t + C, & \textit{if } t \ge K, \\ -1, & \textit{if } 0 \le t < K, \end{cases}$$

is a strict subsolution for (A.5).

(ii) *For $C > \log(\sqrt{2\pi}\gamma/\sigma)$, there exist $K, K' > 0$ such that*

$$\bar{f}(t) := \begin{cases} \dfrac{1}{2}\log t - \log\log t + C, & \textit{if } t \ge K, \\ K', & \textit{if } 0 \le t < K, \end{cases}$$

is a strict supersolution for (A.5).

PROOF. This is a straightforward computation using the local central limit theorem, $p_0(t) \sim (2\pi\sigma^2 t)^{-1/2}$. Here are some details:

(i) Let $e^{-C} = (1 + 3\varepsilon)/(\sqrt{2\pi}\gamma/\sigma)$ with $\varepsilon > 0$ small. We have $p_0(t) \geq (1 - \varepsilon)/\sqrt{2\pi\sigma^2 t}$ for $t \geq t_0(\varepsilon)$. For $\underline{f}$ as in (i) and any $t \geq K \vee t_0(\varepsilon)$, we estimate

$$\int_0^t p_0(t-s)\gamma e^{-\underline{f}(s)}\, ds$$

$$\geq \int_K^{t-K} p_0(t-s)e^{-C}\gamma \frac{\log s}{\sqrt{s}}\, ds$$

$$\geq \frac{1+\varepsilon}{2\pi} \int_K^{t-K} \frac{\log s}{\sqrt{s(t-s)}}\, ds = \frac{1+\varepsilon}{2\pi} \int_{K/t}^{1-K/t} \frac{\log t + \log u}{\sqrt{u(1-u)}}\, du$$

$$\geq \frac{1+\varepsilon}{2\pi} \left\{ \log t \left[\int_0^1 \frac{1}{\sqrt{u(1-u)}}\, du - 4\sqrt{2K/t} \right] + \int_0^1 \frac{\log u}{\sqrt{u(1-u)}}\, du \right\}.$$

Observing that $\int_0^1 (u(1-u))^{-1/2}\, du = \pi$ and $\int_0^1 (u(1-u))^{-1/2} \log u\, du \in (-\infty, 0)$ we see that there exists $n(= n(\varepsilon)) \geq 1$ such that for all $K \geq 1$

$$\int_0^t \gamma p_0(t-s)e^{-\underline{f}(s)}\, ds \geq \frac{1+\varepsilon/2}{2} \log t > \underline{f}(t) \qquad \text{whenever } t \geq nK.$$

On the other hand for $t_0(\varepsilon) \leq K < t < nK$, we have

$$\int_0^t \gamma p_0(s) \exp(-\underline{f}(t-s))\, ds$$

$$\geq \frac{\gamma(1-\varepsilon)}{\sqrt{2\pi\sigma^2}} \int_{t-K}^t \frac{1}{\sqrt{u}}\, du \geq \frac{\gamma(1-\varepsilon)}{\sqrt{2\pi\sigma^2}} \int_{(n-1)K}^{nK} u^{-1/2}\, du$$

$$= \frac{\gamma(1-\varepsilon)\sqrt{2}}{\sigma\sqrt{\pi}} \left(\sqrt{nK} - \sqrt{(n-1)K} \right) \geq \frac{\gamma}{2\sigma} \sqrt{\frac{K}{2\pi n}}$$

and $\underline{f}(t) \leq \log(nK)$. So we just have to chose $K \geq 1$ so big that $\frac{\gamma}{2\sigma}\sqrt{\frac{K}{2\pi n}} > \log(nK)$.

(ii) Can be treated similarly. $\square$

PROOF OF LEMMA A.1, CONTINUED. Constructing $\bar{\varphi}$ and $\underline{\varphi}$ as in (A.8) from the functions $\underline{f}$ and $\bar{f}$ given in Lemma A.2, with $\underline{\varphi} \leq \rho \leq \bar{\varphi}$, we see easily that

$$(A.9) \qquad \rho_0(t) \sim \frac{1}{2} \log t \qquad \text{as } t \to \infty.$$

But we need a finer result, namely $\rho_0(t) = \frac{1}{2}\log t - \log\log t + \log(\sqrt{2\pi}\gamma/\sigma) + o(1)$. We use Laplace transforms to strengthen the asymptotics (A.9):

Denoting $\xi(t) := \gamma \exp(-\rho_0(t))$ we can write (A.5) as $\rho_0 = p_0 * \xi$, after taking Laplace transforms this reads

$$(A.10) \qquad \widehat{\rho_0}(\lambda) = \widehat{p_0}(\lambda)\widehat{\xi}(\lambda), \qquad \lambda > 0.$$

We have $\widehat{p_0}(\lambda) \sim (2\sigma^2\lambda)^{-1/2}$ as $\lambda \downarrow 0$. From (A.9) and a Tauberian theorem (see, e.g., [6], Chapter XIII.5, Theorem 4), we conclude that $\widehat{\rho_0}(\lambda) \sim \frac{1}{2\lambda}\log(1/\lambda)$, hence $\widehat{\xi}(\lambda) \sim \sigma(2\lambda)^{-1/2}\log(1/\lambda)$ for $\lambda \downarrow 0$. Invoking the Tauberian theorem in the other direction, we get

$$\gamma\exp(-\rho_0(t)) = (\sigma(2\pi)^{-1/2}t^{-1/2}\log t)(1 + o(1)).$$

Equation (A.3) follows by taking logarithms. Observe that the use of the Tauberian theorem is justified because ρ_0, and hence also ξ, are monotone functions. Finally, observe that $\int(\log s)/\sqrt{s}\,ds = 2\sqrt{s}\log s - 4\sqrt{s}$ to obtain (A.4). $\quad\square$

LEMMA A.3. *Let $\psi.(0) : \mathbb{Z} \to \mathbb{R}$ and $\psi_0(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ be given real-valued continuous functions and define ψ on $\mathbb{Z} \times \mathbb{R}_+$ as the solution of the heat equation corresponding to L_{rw} away from 0 with given boundary behavior, that is, ψ solves*

$$\partial_t\psi_x(t) = L_{\mathrm{rw}}\psi_x(t), \qquad x \in \mathbb{Z} \setminus \{0\}, t \geq 0.$$

Then ψ has the stochastic representation

$$\psi_x(t) = \mathbb{E}_x[\psi_{X(t \wedge T_0)}(t - (t \wedge T_0))],$$

where $(X(t))_{t \geq 0}$ is a continuous-time random walk on $\mathbb{Z}$ with generator L_{rw} and $T_0 := \inf\{s > 0 : X(s) = 0\}$ the hitting time of the origin.

LEMMA A.4. *Let ρ be the solution of (A.1). Then uniformly in $x \in \mathbb{R}$, we have the following convergence*:

$$\frac{1}{\log t}\rho_{[\sigma\sqrt{t}x]}(t) \xrightarrow[t \to \infty]{} \frac{1}{2\pi}\int_0^1 \frac{1}{\sqrt{s(1-s)}}e^{-x^2/(2s)}\,ds$$

(A.11)
$$=: \tilde{\rho}(x) = 1 - \Phi(|x|),$$

where $\Phi(a) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^a e^{-z^2/2}\,dz$.

PROOF.   By (A.3), we have $\exp(-\alpha\rho_0(t)) = (\sigma(\log t)/\gamma\sqrt{2\pi t})(1+o(1))$, and by the local CLT, $p_{[\sigma\sqrt{t}x]}(t) = (2\pi\sigma^2 t)^{-1/2}(e^{-x^2/2} + o(1))$ uniformly in $x \in \mathbb{R}$. Thus, by (A.2),

$$\frac{1}{\log t}\rho_{[\sigma\sqrt{t}x]}(t)$$

$$= \frac{\gamma}{\log t}\int_0^t \frac{\log s}{\gamma\sqrt{2\pi s}}\frac{1}{\sqrt{2\pi(t-s)}}e^{-tx^2/2(t-s)}\,ds + o(1)$$

$$= \frac{1}{2\pi}\int_0^t \frac{\log t + \log(s/t)}{\log t}((s/t)(1-s/t))^{-1/2}e^{-x^2/2(1-s/t)}\frac{ds}{t} + o(1)$$

$$= \frac{1}{2\pi}\int_0^1 \frac{1}{\sqrt{(1-u)u}}e^{-x^2/2u}\,du + o(1),$$

where we substituted $u = 1 - s/t$ in the last line.

To prove the identity in (A.11), substituting $1/s = z$ and then $z - 1 = y$, we find

$$\frac{1}{2\pi} \int_0^1 \frac{1}{\sqrt{s(1-s)}} e^{\frac{-x^2}{2s}} \, ds$$

$$= \frac{1}{2\pi} \int_1^\infty (z^{-1}(1 - z^{-1}))^{-1/2} \exp\left(\frac{-x^2}{2} z\right) \frac{dz}{z^2}$$

$$= \frac{1}{2\pi} \int_1^\infty \frac{1}{z\sqrt{z-1}} \exp\left(\frac{-x^2}{2} z\right) dz$$

$$= \frac{1}{2\pi} e^{-x^2/2} \int_0^\infty \exp\left(\frac{-x^2}{2} y\right) \frac{1}{\sqrt{y}(y+1)} \, dy$$

$$= \frac{1}{2\pi} e^{-x^2/2} \hat{f}(x^2/2),$$

where $\hat{f}$ is the Laplace transform of $f(t) = 1/((y + 1)\sqrt{y})$. A table of Laplace transforms (e.g., [1], Formula 29.3.114) shows that $\hat{f}(z) = 2\pi e^z (1 - \Phi(\sqrt{2z}))$. $\qquad\square$

APPENDIX B: CORRELATION FUNCTIONS FOR POISSON VACANT EVENTS

In this section, we compute the correlation function for the events that a Poisson point process is vacant on each of k given sets. This is used to prove Lemma 2.2 on the centered moments of the origin's vacant time for a Poisson system of random walks.

LEMMA B.1. *Let $(S, \mathcal{B})$ be a measurable space, ξ a Poisson point process on S with intensity measure ν. Then for $k \in \mathbb{N}$, $E_1, E_2, \ldots, E_k \in \mathcal{S}$ with $\nu(E_1), \ldots, \nu(E_k) < \infty$, and $M \in \mathbb{N} \cup \{0\}$,*

$$\mathbb{E}\left[\prod_{i=1}^k \left(\mathbf{1}(\xi(E_i) = 0) - \mathbb{P}(\xi(E_i) = 0)\right)\right]$$

(B.1)
$$= e^{-\sum_{i=1}^k \nu(E_i)} \sum_{n=1}^\infty \frac{1}{n!} \sum_{\substack{I_1,\ldots,I_n \subset \{1,\ldots,k\} \\ |I_1|,\ldots,|I_n| \geq 2 \\ I_1 \cup \cdots \cup I_n = \{1,\ldots,k\}}} (-1)^{\sum_{j=1}^n |I_j|} \prod_{j=1}^n \nu\left(\bigcap_{\ell \in I_j} E_\ell\right)$$

$$= e^{-\sum_{i=1}^k \nu(E_i)} \sum_{n=1}^M \frac{1}{n!} \sum_{\substack{I_1,\ldots,I_n \subset \{1,\ldots,k\} \\ |I_1|,\ldots,|I_n| \geq 2 \\ I_1 \cup \cdots \cup I_n = \{1,\ldots,k\}}} (-1)^{\sum_{j=1}^n |I_j|} \prod_{j=1}^n \nu\left(\bigcap_{\ell \in I_j} E_\ell\right)$$

$$(B.2) \qquad + e^{-\sum_{i=1}^{k} \nu(E_i)} R_{M+1}\Big(2^k \max_{1 \le i < j \le k} \nu(E_i \cap E_j)\Big),$$

where $|R_{M+1}(x)| \le 2^k \frac{|x|^{M+1}}{(M+1)!} e^{|x|}$.

REMARK B.1. Lemma B.1 allows us to control the k-point correlation function quantitatively in terms of $\nu(E_i \cap E_j)$, $1 \le i < j \le k$. This result should be well known, but we sketch the proof below for completeness and lack of a precise reference.

PROOF OF LEMMA B.1. Since $\mathbb{P}(\xi(B) = 0) = e^{-\nu(B)}$ for any set $B \in \mathcal{B}$, we have

$$\mathbb{E}\left[\prod_{i=1}^{k}\big(\mathbf{1}(\xi(E_i) = 0) - \mathbb{P}(\xi(E_i) = 0)\big)\right]$$

$$= \sum_{I' \subset \{1,\ldots,k\}} (-1)^{k-|I'|} \mathbb{P}\Big(\xi\Big(\bigcup_{\ell \in I'} E_\ell\Big) = 0\Big) \prod_{j \notin I'} \mathbb{P}(\xi(E_j) = 0)$$

$$(B.3) \qquad = \sum_{I' \subset \{1,\ldots,k\}} (-1)^{k-|I'|} \exp\left[-\nu\Big(\bigcup_{\ell \in I'} E_\ell\Big) - \sum_{j \notin I'} \nu(E_j)\right]$$

$$= e^{-\sum_{i=1}^{k} \nu(E_i)} \sum_{I' \subset \{1,\ldots,k\}} (-1)^{k-|I'|} \exp\left[-\nu\Big(\bigcup_{\ell \in I'} E_\ell\Big) + \sum_{j \in I'} \nu(E_j)\right]$$

$$= e^{-\sum_{i=1}^{k} \nu(E_i)} \sum_{I' \subset \{1,\ldots,k\}} (-1)^{k-|I'|} \exp\left[\sum_{I \subset I', |I| \ge 2} (-1)^{|I|} \nu\Big(\bigcap_{\ell \in I} E_\ell\Big)\right],$$

where we used the inclusion-exclusion principle in the last line.

Note that when we Taylor expand the rightmost exponential in (B.3), the zeroth order term is $\sum_{I' \subset \{1,\ldots,k\}} (-1)^{k-|I'|} = 0$. For a fixed $I' \subset \{1, \ldots, k\}$ and $n \in \mathbb{N}$, the nth order term of the Taylor expansion for the exponential is

$$\sum_{I' \subset \{1,\ldots,k\}} (-1)^{k-|I'|} \frac{1}{n!} \sum_{\substack{I_1,\ldots,I_n \subset I' \\ |I_1|,\ldots,|I_n| \ge 2}} (-1)^{|I_1|+\cdots+|I_n|} \prod_{j=1}^{n} \nu\Big(\bigcap_{\ell \in I_j} E_\ell\Big)$$

$$= \frac{1}{n!} \sum_{\substack{I_1,\ldots,I_n \subset \{1,\ldots,k\} \\ |I_1|,\ldots,|I_n| \ge 2}} (-1)^{\sum_{j=1}^{k} |I_j|} \prod_{j=1}^{n} \nu\Big(\bigcap_{\ell \in I_j} E_\ell\Big) \sum_{\substack{I' \subset \{1,\ldots,k\} \\ I' \supset I_1 \cup \cdots \cup I_n}} (-1)^{k-|I'|}$$

$$= \frac{1}{n!} \sum_{\substack{I_1,\ldots,I_n \subset \{1,\ldots,k\} \\ |I_1|,\ldots,|I_n| \ge 2 \\ I_1 \cup \cdots \cup I_n = \{1,\ldots,k\}}} (-1)^{\sum_{j=1}^{n} |I_j|} \prod_{j=1}^{n} \nu\Big(\bigcap_{\ell \in I_j} E_\ell\Big)$$

since whenever $I_1 \cup \cdots \cup I_n \neq \{1, \ldots, k\}$, the summation over I' gives 0. This proves (B.1).

To check (B.2), let $\phi(I') := \sum_{I \subset I', |I| \geq 2} (-1)^{|I|} \nu(\bigcap_{\ell \in I} E_\ell)$. Note that

$$\left|\phi(I')\right| \leq 2^k \max_{1 \leq i < j \leq k} \nu(E_i \cap E_j).$$

Applying the bound $|e^x - \sum_{n=0}^{M} \frac{x^n}{n!}| \leq \frac{|x|^{M+1}}{(M+1)!} e^{|x|}$ to $e^{\phi(I')}$ in (B.3) then gives (B.2). $\quad\square$

REFERENCES

[1] ABRAMOWITZ, M. and STEGUN, I. A. (1964). *Handbook of Mathematical Functions with Formulas*, *Graphs*, *and Mathematical Tables*. *National Bureau of Standards Applied Mathematics Series* **55**. U.S. Government Printing Office, Washington, DC. MR0167642

[2] BIRKNER, M. (2003). Particle systems with locally dependent branching: Long-time behaviour, genealogy and critical parameters. Dissertation, Johann Wolfgang Goethe-Universität Frankfurt am Main.

[3] BIRKNER, M. and SUN, R. (2016). Low-dimensional lonely branching random walks die out. Manuscript in progress.

[4] CHEN, Z.-Q. and FAN, W.-T. (2016). Hydrodynamic limits and propagation of chaos for interacting random walks in domains. *Ann. Appl. Probab.* To appear. Available at arXiv:1311.2325.

[5] DURRETT, R. (1996). *Probability*: *Theory and Examples*, 2nd ed. Duxbury Press, Belmont, CA. MR1609153

[6] FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications*. *Vol. II*, 2nd ed. Wiley, New York. MR0270403

[7] GARCIA, N. L. and KURTZ, T. G. (2006). Spatial birth and death processes as solutions of stochastic equations. *ALEA Lat. Am. J. Probab. Math. Stat.* **1** 281–303. MR2249658

[8] KIPNIS, C. and LANDIM, C. (1999). *Scaling Limits of Interacting Particle Systems*. *Grundlehren der Mathematischen Wissenschaften* **320**. Springer, Berlin. MR1707314

[9] NEWMAN, C. M., RAVISHANKAR, K. and SUN, R. (2005). Convergence of coalescing non-simple random walks to the Brownian web. *Electron. J. Probab.* **10** 21–60. MR2120239

[10] YAU, H.-T. (1991). Relative entropy and hydrodynamics of Ginzburg–Landau models. *Lett. Math. Phys.* **22** 63–80. MR1121850

INSTITUT FÜR MATHEMATIK
JOHANNES GUTENBERG-UNIVERSITÄT MAINZ
STAUDINGERWEG 9
55099 MAINZ
GERMANY
E-MAIL: birkner@mathematik.uni-mainz.de

DEPARTMENT OF MATHEMATICS
NATIONAL UNIVERSITY OF SINGAPORE
10 LOWER KENT RIDGE ROAD
119076
SINGAPORE
E-MAIL: matsr@nus.edu.sg