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LATTICE APPROXIMATIONS OF REFLECTED STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS DRIVEN BY SPACE–TIME WHITE NOISE

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We introduce a discretization/approximation scheme for reflected stochastic partial differential equations driven by space–time white noise through systems of reflecting stochastic differential equations. To establish the convergence of the scheme, we study the existence and uniqueness of solutions of Skorohod-type deterministic systems on time-dependent domains. We also need to establish the convergence of an approximation scheme for deterministic parabolic obstacle problems. Both are of independent interest on their own.

1. Introduction. Consider the following stochastic partial differential equation (SPDE) with reflection:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} - \dfrac{\partial^2 u(t,x)}{\partial x^2} = f\big(t,x,u(t,x)\big) + \sigma\big(t,x,u(t,x)\big)\dot{W}(t,x) + \eta; \\ u(0,\cdot) = u_0, \qquad u(t,x) \geq 0; \\ u(t,0) = u(t,1) = 0, \end{cases} \tag{1.1}$$

where $\dot{W}$ denotes the space–time white noise defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, where $\mathcal{F}_t = \sigma(W(s,x) : x \in [0,1], 0 \leq s \leq t)$; u_0 is a nonnegative continuous function on $[0,1]$, which vanishes at 0 and 1; $\eta(t,x)$ is a random measure which is a part of the solution pair (u,η) and plays the role of a local time that prevents the solution u from being negative. The coefficients f and σ are measurable mappings from $R_+ \times [0,1] \times R$ into R. Let $C_0^2([0,1])$ denote the space of twice differentiable functions ϕ on $[0,1]$ satisfying $\phi(0) = \phi(1) = 0$. The following definition is taken from [5, 14].

DEFINITION 1.1. A pair (u,η) is said to be a solution of equation (1.1) if:

(i) u is a continuous random field on $R_+ \times [0,1]$; $u(t,x)$ is $\mathcal{F}_t$ measurable and $u(t,x) \geq 0$ a.s.

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(ii) η is a random measure on $R_+ \times (0,1)$ such that:

(a) $\eta(\{t\} \times (0,1)) = 0, \forall t \geq 0$.
(b) $\int_0^t \int_0^1 x(1-x)\eta(ds, dx) < \infty, t \geq 0$.
(c) η is adapted in the sense that for any measurable mapping ψ:

$$\int_0^t \int_0^1 \psi(s,x)\eta(ds,dx) \text{ is } \mathcal{F}_t\text{measurable.}$$

(iii) $\{u, \eta\}$ solves the parabolic SPDE in the following sense [$(\cdot,\cdot)$ denotes the scalar product in $L^2[0,1]$]: $\forall t \in R_+, \phi \in C_0^2([0,1])$ with $\phi(0) = \phi(1) = 0$,

$$\begin{aligned}
\big(u(t), \phi\big) - \int_0^t \big(u(s), \phi''\big)\, ds - \int_0^t \big(f\big(s, \cdot, u(s)\big), \phi\big)\, ds& \\
= (u_0, \phi) + \int_0^t \int_0^1 \phi(x)\sigma\big(s, x, u(s,x)\big) W(ds, dx)& \\
+ \int_0^t \int_0^1 \phi(x)\eta(ds, dx) \qquad \text{a.s.,}&
\end{aligned}$$

where $u(t) := u(t, \cdot)$.

(iv) $\int_Q u(t,x)\eta(dt,dx) = 0$, where $Q = R_+ \times (0,1)$.

The SPDEs with reflection driven by space–time white noise was first studied by Nualart and Pardoux in [14] when $\sigma(\cdot) = 1$, and by Donati-Martin and Pardoux in [5] for general diffusion coefficient σ. The uniqueness of the solution and large deviations were obtained by Xu and Zhang in [20]. SPDEs with reflection can also be used to model the evolution of random interfaces near a hard wall; see [8]. Various properties of the solution of equation (1.1) were studied since then. The hitting properties were investigated by Dalang, Mueller and Zambotti in [3]. Integration by parts formulae associated with SPDEs with reflection and the occupation densities were respectively established by Zambotti in [22, 23] and [21]. The strong Feller properties and the large deviations for invariant measures were studied by Zhang in [24, 25]. See also [4] and [6] for related works.

The purpose of this paper is to develop a numerical scheme (particle system approximations) for the reflected stochastic partial differential equations. This is a challenging problem which has been open for some time. Part of the difficulties is caused by the singularities of the space–time white noise. For example, Itô formula is not available for this type of equations. Part of the difficulties lie in the discretization of the random measure term η that appeared in equation (1.1). We introduce a discretization scheme through systems of reflecting stochastic differential equations. As the dimensions of the reflecting systems tend to infinity, the problem is to compare and control the systems with different dimensions. To this end, we study Skorohod-type deterministic problems on time-dependent domains and prove a useful a priori estimate for the solutions in terms of the time-dependent boundaries. To prove the convergence of the scheme, we also need to establish the convergence of a discretization scheme of deterministic parabolic obstacle problems. These preliminary results are of independent interest.

We note that Skorohod problems in time-dependent domains were studied in the nice papers [1, 2] and [15]. However, these results cannot be applied directly to our setting because the domains there are assumed to be bounded (or with some smooth boundaries). Moreover, for the purpose of this paper we need to establish dimension-free upper bounds of the solutions of the Skorohod-type problem in terms of the boundaries of the time-dependent domains. We like to mention that Funaki and Olla in [8] proved that the fluctuations of a $\nabla\phi$ interface model near a hard wall converge in law to the stationary solution of a SPDE with reflection. This is also an approximation by discrete systems under a different scaling. The discretization scheme for stochastic heat equations driven by space–time white noise was introduced by Funaki in [7] where convergence in law was established and by Gyöngy in [9, 10] where strong convergence was obtained. Approximation scheme for SPDEs of elliptic type was discussed by Martínez and Sanz-Solè in [13]. Discretizations for stochastic wave equations were investigated by Quer-Sardanyons and Sanz-Solè in [16]. Numerical schemes for stochastic evolution equations were obtained by Gyöngy and Millet in [11].

Let us now describe the content of the paper in more detail. In Section 2, we introduce the discretization scheme and state the main result. Section 3 is to study Skorohod-type problems on time-dependent domains in Euclidean spaces. We establish the existence and uniqueness of the solution of the Skorohod type problem on domains with boundaries being continuous functions of time. We provide a dimension-free bound of the solution in terms of the boundaries of the domains, which plays an important role in the rest of the paper. In Section 4, we introduce a discretization scheme for deterministic parabolic obstacle problems. We establish the convergence of the scheme first for smooth obstacles. In this case, we are able to show that the measure $\eta(dt, dx)$ appeared in the obstacle problem is absolutely continuous with respect to the Lebesgue measure $dt \times dx$ and the tightness of the approximating solutions. We prove the convergence of the scheme by identifying any limit of the approximating solutions as the unique solution of the parabolic obstacle problem. We then extend the scheme for continuous obstacles using the a priori estimate obtained in Section 3 for Skorohod-type problems. The Section 5 is devoted to the proof of the convergence of the discretization scheme for SPDEs with reflection. We first relate the SPDEs with reflection to a random parabolic obstacle problem. We obtain the convergence of the scheme by carefully comparing it with the discretization scheme introduced for obstacle problems in Section 4. Here, the results in Section 3 and Garsia lemma for random fields will play an important role.

REMARK 1.1. In this paper, we discretize only the space variable using systems of reflecting stochastic differential equations (SDEs). Now there is known procedure to further discretize the reflected SDEs (see [17]). Combining these two one could get the discretization for SPDEs with reflection both in time and space directions.

2. The discretization scheme and the main result. We first introduce the conditions on the coefficients. Let f, σ are two measurable mappings

$$f, \sigma : R_+ \times [0,1] \times R \to R$$

satisfying the following.

(H.1) For any $T > 0$, there exists a constant $c(T)$ such that for any $x, y \in [0,1], t \in [0,T], u, v \in R$,

$$\begin{aligned} &\bigl|f(t,x,u) - f(t,y,v)\bigr| + \bigl|\sigma(t,x,u) - \sigma(t,y,v)\bigr| \\ &\quad \le c(T)\bigl[|x-y| + |u-v|\bigr]. \end{aligned} \tag{2.1}$$

(H.2) For any $T > 0$, there exists a constant $c(T)$ such that for any $x \in [0,1], t \in [0,T], u \in R$,

$$\bigl|f(t,x,u)\bigr| \le c(T)\bigl(1+|u|\bigr). \tag{2.2}$$

For every integer $n \ge 1$ and $x = \frac{k}{n}, k = 1, 2, \dots, n-1$, define the processes $u^n(t, \frac{k}{n}), k = 1, \dots, n-1$ as the solution of the system of reflecting stochastic differential equations

$$\begin{aligned} du^n\left(t, \frac{k}{n}\right) &= n^2\left(u^n\left(t, \frac{k+1}{n}\right) - 2u^n\left(t, \frac{k}{n}\right) + u^n\left(t, \frac{k-1}{n}\right)\right)dt \\ &\quad + n\sigma\left(t, \frac{k}{n}, u^n\left(t, \frac{k}{n}\right)\right) d\left(W\left(t, \frac{k+1}{n}\right) - W\left(t, \frac{k}{n}\right)\right) \\ &\quad + f\left(t, \frac{k}{n}, u^n\left(t, \frac{k}{n}\right)\right)dt + d\eta_k^n(t), \\ u^n\left(t, \frac{k}{n}\right) &\ge 0, \qquad k = 1, 2, \dots, n-1, \\ u^n(t,0) &= u^n(t,1) = 0, \end{aligned} \tag{2.3}$$

with initial condition

$$u^n\left(0, \frac{k}{n}\right) = u_0\left(\frac{k}{n}\right), \qquad k = 1, \dots, n-1. \tag{2.4}$$

DEFINITION 2.1. We say that $\{u^n(t, \frac{k}{n}), \eta_k^n(t), k = 1, \dots, n-1\}$ is a solution to the reflecting system (2.3) if:

(i) for every $k \ge 1$, $u^n(t, \frac{k}{n}), t \ge 0$ is an adapted, nonnegative, continuous process,

(ii) for $k \geq 1$, $\eta_k^n(t), t \geq 0$ is an adapted, continuous, increasing process with $\eta_k^n(0) = 0$,

(iii) for every $t \geq 0$ and $k = 1, 2, \ldots, n-1$,

$$\begin{aligned} u^n\Big(t, \frac{k}{n}\Big) &= u_0\Big(\frac{k}{n}\Big) + n^2 \int_0^t \Big(u^n\Big(s, \frac{k+1}{n}\Big) - 2u^n\Big(s, \frac{k}{n}\Big) + u^n\Big(s, \frac{k-1}{n}\Big)\Big)\, ds \\ &\quad + \int_0^t f\Big(s, \frac{k}{n}, u^n\Big(s, \frac{k}{n}\Big)\Big)\, ds + \eta_k^n(t) \\ &\quad + \int_0^t n\sigma\Big(s, \frac{k}{n}, u^n\Big(s, \frac{k}{n}\Big)\Big)\, d\Big(W\Big(s, \frac{k+1}{n}\Big) - W\Big(s, \frac{k}{n}\Big)\Big) \end{aligned} \tag{2.5}$$

almost surely,

(iv) $\int_0^t u^n(s, \frac{k}{n}) \eta_k^n(ds) = 0$, for all $t \geq 0$, $k = 1, 2, \ldots, n-1$.

Set

$$u_k^n(t) := u^n\Big(t, \frac{k}{n}\Big),$$

$$W_k^n(t) = \sqrt{n}\Big(W\Big(t, \frac{k+1}{n}\Big) - W\Big(t, \frac{k+1}{n}\Big)\Big)$$

for $k = 1, \ldots, n-1$. Let $A^n = (A_{ki}^n)$ denote the $(n-1) \otimes (n-1)$ matrix with elements $A_{kk}^n = -2$, $A_{ki}^n = 1$ for $|k-i| = 1$, $A_{ki}^n = 0$ for $|k-i| > 1$. The system (2.3) is regarded as a $(n-1)$-dimensional reflected SDE on the domain $D_n = \{(z_1, \ldots, z_{n-1}); z_k \geq 0, k = 1, \ldots, n-1\}$ written as

$$\begin{aligned} du_k^n(t) &= n^2 \sum_{i=1}^{n-1} A_{ki}^n u_i^n(t)\, dt + f\Big(t, \frac{k}{n}, u_k^n(t)\Big)\, dt \\ &\quad + \sqrt{n}\sigma\Big(t, \frac{k}{n}, u_k^n(t)\Big)\, dW_k^n(t) + d\eta_k^n(t), \\ u_k^n(0) &= u_0\Big(\frac{k}{n}\Big), \qquad k = 1, 2, \ldots, n-1. \end{aligned} \tag{2.6}$$

As the domain D_n is convex, the existence and uniqueness of the solution of the system (2.6) is well known (see, e.g., [18] and [12]).

For every integer $n \geq 1$, define the random field

$$u^n(t, x) := u^n\Big(t, \frac{k}{n}\Big) + (nx - k)\Big(u^n\Big(t, \frac{k+1}{n}\Big) - u^n\Big(t, \frac{k}{n}\Big)\Big) \tag{2.7}$$

for $x \in [\frac{k}{n}, \frac{k+1}{n})$, $k = 0, \ldots, n-1$ with $u^n(t, 0) := 0$.

The main result of the paper reads as the following.

THEOREM 2.1. *Suppose* (H.1) *and* (H.2) *hold. Then for any* $p \geq 1$, *we have*

$$\lim_{n \to \infty} E\Big[\sup_{0 \leq t \leq T, 0 \leq x \leq 1} \big|u^n(t, x) - u(t, x)\big|^p\Big] = 0. \tag{2.8}$$

REMARK 2.1. Because of the appearance of the random measure (local time) term it seems hard to get the precise rate of the convergence. From the proof of the above theorem, we can see that the local time terms (i.e., the random measure terms) of the approximating systems also converge vaguely (as measures) to the corresponding term of the equation.

We end this section with a description of the group generated by the matrix A^n. For $j \geq 1$, define the vector

$$e_j := \left(\sqrt{\frac{2}{n}}\sin\left(j\frac{1}{n}\pi\right), \dots, \sqrt{\frac{2}{n}}\sin\left(j\frac{n-1}{n}\pi\right)\right).$$

One can easily check that $\{e_j, j = 1, \dots, n-1\}$ forms an orthonormal basis of R^{n-1}. Moreover, e_j, $j = 1, \dots, n-1$ are eigenvectors of $n^2 A^n$ with eigenvalues

$$\lambda_j^n := -j^2\pi^2 c_j^n,$$

where

$$\frac{4}{\pi^2} \leq c_j^n := \frac{\sin^2(\frac{j\pi}{2n})}{(\frac{j\pi}{2n})^2} \leq 1,$$

$j = 1, \dots, n-1$. Thus, the group $G^n(t) := \exp(n^2 A^n t)$ generated by $n^2 A^n$ on R^{n-1} admits the following representation:

$$G^n(t)e = \sum_{k=1}^{n-1} e^{\lambda_k^n t}\langle e, e_k\rangle e_k, \qquad e \in R^{n-1}. \tag{2.9}$$

3. Deterministic Skorohod-type systems. In this section, we study Skorohod-type problems on time dependent domains and obtain some a priori estimates.

Set $a^+ = a \vee 0$ and $a^- = (-a) \vee 0$ for $a \in R$. For a vector $b = (b_1, \dots, b_{n-1}) \in R^{n-1}$, we will use the following notation:

$$b^+ = (b_1^+, \dots, b_{n-1}^+), \qquad b^- = (b_1^-, \dots, b_{n-1}^-).$$

It is clear that $b = b^+ - b^-$.

Note that the matrix A^n introduced in Section 2 is negative definite. Furthermore, we also have the following.

LEMMA 3.1. *It holds that*

$$\langle b^+, A^n b\rangle \leq 0 \qquad \text{for all } b \in R^{n-1}. \tag{3.1}$$

PROOF. Write

$$\langle b^+, A^n b\rangle = \langle b^+, A^n b^+\rangle - \langle b^+, A^n b^-\rangle. \tag{3.2}$$

The first term on the right $\langle b^+, A^n b^+\rangle$ is nonpositive. Since $A^n_{ij} \geq 0$ for $i \neq j$, we have

$$\begin{aligned}\langle b^+, A^n b^-\rangle &= \sum_{i,j=1}^{n-1} b_i^+ A^n_{ij} b_j^- \\ &= \sum_{i=1}^{n-1} b_i^+ A^n_{ii} b_i^- + \sum_{i\neq j} b_i^+ A^n_{ij} b_j^- \\ &= \sum_{i\neq j} b_i^+ A^n_{ij} b_j^- \geq 0\end{aligned}$$

(3.1) follows. □

For $a = (a_1, \dots, a_{n-1}), b = (b_1, \dots, b_{n-1}) \in R^{n-1}$, we write $a \geq b$ if $a_i \geq b_i$ for all $i = 1, \dots, n-1$. Given $V = (V_1, \dots, V_{n-1}) \in C([0,\infty) \to R^{n-1})$ with $V(0) \geq 0$ (as a vector). Consider the following Skorohod-type problem with reflection in R^{n-1}:

$$\begin{cases} dZ(t) = n^2 A^n Z(t)\,dt + d\eta(t), & Z(0) = 0; \\ Z(t) \geq -V(t); & \\ \displaystyle\int_0^T \langle Z(t) + V(t), d\eta(t)\rangle = 0, & \text{for all } T > 0. \end{cases} \tag{3.3}$$

DEFINITION 3.1. A pair (Z, η) is called a solution to the problem (3.3) if it satisfies:

(1) $Z = (Z_1, \dots, Z_{n-1}) \in C([0,\infty) \to R^{n-1})$ and $Z(t) \geq -V(t)$,

(2) $\eta = (\eta_1, \dots, \eta_{n-1}) \in C([0,\infty) \to R^{n-1})$ and for each i, $\eta_i(t)$ is an increasing continuous function with $\eta_i(0) = 0$,

(3) for all $t \geq 0$,

$$Z(t) = \int_0^t n^2 A^n Z(s)\,ds + \eta(t),$$

(4) for $t \geq 0$, $\sum_{i=1}^{n-1} \int_0^t (Z_i(s) + V_i(s))\eta_i(ds) = 0$.

To prove the existence of the solution to equation (3.3), we need the following estimate which also plays an important role in the subsequent sections.

LEMMA 3.2. *If $(Z^i(t), \eta^i(t))$ is a solution to equation (3.3) with V replaced by V^i, $i = 1, 2$, then for $k \geq 1$, $T > 0$, we have*

$$\sup_{0\leq t\leq T} |Z_k^1(t) - Z_k^2(t)| \leq \sup_{0\leq t\leq T, 1\leq j\leq n-1} |V_j^1(t) - V_j^2(t)|. \tag{3.4}$$

PROOF. Set $m := \sup_{0\le t\le T, 1\le j\le n-1} |V_j^1(t) - V_j^2(t)|$ and $M = (m, m, \dots, m) \in R^{n-1}$. From the definition of the matrix A^n, it is easy to see that $A^n M = (-m, 0, \dots, 0, -m)$. We have

$$\begin{aligned} &d\big(Z^1(t) - Z^2(t) - M\big) \\ &\quad = n^2 A^n \big(Z^1(t) - Z^2(t) - M\big)\, dt + n^2 A^n M\, dt + d\eta^1(t) - d\eta^2(t). \end{aligned}$$

By the chain rule,

$$\begin{aligned} &d\left[\sum_{k=1}^{n-1} \big((Z_k^1(t) - Z_k^2(t) - m)^+\big)^2\right] \\ &\quad = 2\sum_{k=1}^{n-1} \big(Z_k^1(t) - Z_k^2(t) - m\big)^+ d\big(Z_k^1(t) - Z_k^2(t) - m\big) \\ &\quad = 2n^2 \left[\sum_{k=1}^{n-1} \big(Z_k^1(t) - Z_k^2(t) - m\big)^+ \sum_{i=1}^{n-1} A_{ki}^n \big(Z_i^1(t) - Z_i^2(t) - m\big)\right] dt \\ &\qquad + 2n^2 \big\langle \big(Z^1(t) - Z^2(t) - M\big)^+, A^n M\big\rangle\, dt \\ &\qquad + 2\sum_{k=1}^{n-1} \big(Z_k^1(t) - Z_k^2(t) - m\big)^+ d\eta_k^1(t) \\ &\qquad - 2\sum_{k=1}^{n-1} \big(Z_k^1(t) - Z_k^2(t) - m\big)^+ d\eta_k^2(t) \\ &\quad := I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.5}$$

By Lemma 3.1,

$$I_1 = 2n^2 \big\langle \big(Z^1(t) - Z^2(t) - M\big)^+, A^n \big(Z^1(t) - Z^2(t) - M\big)\big\rangle \le 0. \tag{3.6}$$

In view of the expression of $A^n M$, we have

$$I_2 = 2n^2\big[-m\big(Z_1^1(t) - Z_1^2(t) - m\big)^+ - m\big(Z_{n-1}^1(t) - Z_{n-1}^2(t) - m\big)^+\big] \le 0. \tag{3.7}$$

Observe that

$$\begin{aligned} &\{t; Z_k^1(t) - Z_k^2(t) > m\} \\ &\quad \subset \{t; Z_k^1(t) > Z_k^2(t) + m\} \\ &\quad \subset \{t; Z_k^1(t) > -V_k^2(t) + m\} \\ &\quad \subset \{t; Z_k^1(t) > -V_k^1(t)\}. \end{aligned}$$

Therefore,

$$I_3 \le 2\sum_{k=1}^{n-1}\left(Z_k^1(t) - Z_k^2(t) - m\right)^+ \chi_{\{t;Z_k^1(t)>-V_k^1(t)\}}\,d\eta_k^1(t)$$
$$= 0. \tag{3.8}$$

Clearly, $I_4 \le 0$ because of the negative sign. It follows from (3.5)–(3.8) that

$$d\left[\sum_{k=1}^{n-1}\left(\left(Z_k^1(t) - Z_k^2(t) - m\right)^+\right)^2\right] \le 0.$$

Hence,

$$\sum_{k=1}^{n-1}\left(\left(Z_k^1(t) - Z_k^2(t) - m\right)^+\right)^2 \le \sum_{k=1}^{n-1}\left((-m)^+\right)^2 = 0$$

proving the lemma. □

THEOREM 3.1. *There exists a unique solution (Z, η) to the system (3.3).*

PROOF. We first prove the existence. Assume for the moment $V \in C^1([0, \infty) \to R^{n-1})$. Consider the following system with reflecting boundary on the convex domain $D_n = \{(z_1, \dots, z_{n-1}); z_k \ge 0, k = 1, \dots, n-1\}$:

$$\begin{cases} du(t) = n^2A^n u(t)\,dt - n^2A^n V(t)\,dt + V'(t)\,dt + d\eta(t); \\ u(0) = V(0), \qquad u_i(t) \ge 0, \qquad i = 1, \dots, n-1; \\ \displaystyle\int_0^t u_i(s)\,d\eta_i(t) = 0, \qquad i = 1, \dots, n-1. \end{cases} \tag{3.9}$$

It is well known that the above system admits a unique solution (u, η); see [18] and [12]. Let $Z(t) := u(t) - V(t)$. It is easy to verify that (Z, η) is the unique solution to the system (3.3). Now consider the general case $V \in C([0, \infty) \to R^{n-1})$. Take a sequence $V^m \in C^1([0, \infty) \to R^{n-1})$ with $V^m(0) \ge 0$, $m \ge 1$ that converges to V uniformly on any finite interval. Let (Z^m, η^m) denote the unique solution to the system:

$$\begin{cases} dZ^m(t) = n^2A^n Z^m(t)\,dt + d\eta^m(t), \qquad Z^m(0) = 0; \\ Z^m(t) \ge -V^m(t); \\ \displaystyle\int_0^T \langle Z^m(t) + V^m(t), d\eta^m(t)\rangle = 0. \end{cases} \tag{3.10}$$

By Lemma 3.2, it follows that for $T > 0$,

$$\lim_{m,l\to\infty}\sup_{0\le t\le T}\left|Z^m(t) - Z^l(t)\right|$$
$$\le \lim_{m,l\to\infty}\sup_{0\le t\le T}\left|V^m(t) - V^l(t)\right| = 0.$$

Thus, there exists $Z \in C([0,\infty) \to R^{n-1})$ such that $Z^m \to Z$ uniformly on finite intervals. From equation (3.10), we see that η^m also converges uniformly on finite intervals to some $\eta \in C([0,\infty) \to R^{n-1})$. Furthermore, letting $m \to \infty$ in (3.10), we see that (Z,η) is a solution to the system (3.3). We show now the uniqueness. Let (Z,η), $(\hat{Z},\hat{\eta})$ be two solutions to the system (3.3). By the chain rule,

$$
\begin{aligned}
&|Z(t) - \hat{Z}(t)|^2 \\
&\quad = 2n^2 \int_0^t \langle Z(s) - \hat{Z}(s), A^n(Z(s) - \hat{Z}(s))\rangle\, ds \\
&\qquad + 2\int_0^t \langle Z(s) - \hat{Z}(s), d\eta(s) - d\hat{\eta}(s)\rangle \\
&\quad \le 2\int_0^t \langle Z(s) + V(s) - V(s) - \hat{Z}(s), d\eta(s) - d\hat{\eta}(s)\rangle \\
&\quad = -2\int_0^t \langle V(s) + \hat{Z}(s), d\eta(s)\rangle - 2\int_0^t \langle Z(s) + V(s), d\hat{\eta}(s)\rangle \\
&\quad \le 0,
\end{aligned}
\tag{3.11}
$$

where we have used the fact that $V(s) + \hat{Z}(s) \ge 0$, $V(s) + Z(s) \ge 0$ (as vectors). Hence, $Z = \hat{Z}$ which further implies $\eta = \hat{\eta}$ from equation (3.3). $\square$

4. A discretization scheme for deterministic obstacle problems. In this section, we will introduce a discretization scheme for parabolic obstacle problems and establish the convergence of the scheme. Consider the following parabolic obstacle problem:

$$
\begin{cases}
\dfrac{\partial Z(t,x)}{\partial t} - \dfrac{\partial^2 Z(t,x)}{\partial x^2} = \dot{\eta}(t,x), & x \in [0,1]; \\
Z(t,x) \ge -V(t,x); \\
\displaystyle\int_0^t \int_0^1 (Z(s,x) + V(s,x))\eta(ds,dx) = 0,
\end{cases}
\tag{4.1}
$$

where $V \in C(R_+ \times [0,1])$ with $V(0,x) = u_0(x) \ge 0$.

DEFINITION 4.1. If a pair (Z,η) satisfies:

(1) Z is a continuous function on $R_+ \times [0,1]$ and

$$Z(0,x) = 0,\ Z(t,0) = Z(t,1) = 0, \qquad Z(t,x) \ge -V(t,x),$$

(2) η is a measure on $(0,1) \times R_+$ such that for all $\varepsilon > 0$, $T > 0$

$$\eta([0,T] \times (\varepsilon,(1-\varepsilon))) < \infty,$$

(3) for all $t \ge 0$, $\phi \in C_0^2(0,1)$,

$$(Z(t),\phi) - \int_0^t (Z(s),\phi'')\,ds = \int_0^t \int_0^1 \phi(x)\eta(ds,dx),$$

(4) $\int_0^t\int_0^1 (Z(s,x)+V(s,x))\eta(ds,dx)=0$, $t\geq 0$ then (Z,η) is called a solution to problem (4.1).

The following result was proved in [14], Theorem 1.4.

PROPOSTION 4.1 ([NP]). *If $V(0,x)=u_0(x)$, $V(t,0)=V(t,1)=0$ for all $t\geq 0$, equation (4.1) admits a unique solution. Moreover, if Z^1, Z^2 are solutions of the obstacle problem (4.1) with V replaced respectively with V^1 and V^2, then $|Z^1-Z^2|_\infty^T\leq |V^1-V^2|_\infty^T$, for $T>0$. Where $|Z^1-Z^2|_\infty^T=\sup_{0\leq t\leq T, 0\leq x\leq 1}|Z^1(t,x)-Z^2(t,x)|$ and $|V^1-V^2|_\infty^T$ is defined accordingly.*

We now introduce the discretization scheme for the deterministic obstacle problem (4.1). For very positive integer $n\geq 1$, define

$$V^n(t)=\left(V\left(t,\frac{1}{n}\right),\dots,V\left(t,\frac{n-1}{n}\right)\right),$$

where $V(t,x)$ is the function appeared in equation (4.1).

Consider the following Skorohod-type reflecting system in R^{n-1}:

$$\begin{cases} dZ^n(t)=n^2A^nZ^n(t)\,dt+d\eta^n(t);\\ Z^n(t)\geq -V^n(t);\\ \int_0^T\langle Z^n(t)+V^n(t),d\eta^n(t)\rangle=0. \end{cases}\tag{4.2}$$

The existence and uniqueness of the solution of the above system was proved in Section 3. For $n\geq 1$, define the continuous functions Z^n by

$$Z^n(t,x):=Z_k^n(t)+(nx-k)\big(Z_{k+1}^n(t)-Z_k^n(t)\big)\tag{4.3}$$

for $x\in[\frac{k}{n},\frac{k+1}{n})$, $k=0,\dots,n-1$, where $Z_0^n(t)$, $Z_n^n(t)$ are set to be zero. We have

THEOREM 4.1. *Let Z be the solution to equation (4.1). Then for $T>0$,*

$$\lim_{n\to\infty}\sup_{0\leq t\leq T,0\leq x\leq 1}\big|Z^n(t,x)-Z(t,x)\big|=0.\tag{4.4}$$

PROOF. We divide the proof into two steps.

Step 1. Suppose $V\in C^{1,2}([0,\infty)\times[0,1])$. In this case, we first show that the function $\eta^n(t)$ in (4.2) is absolutely continuous and

$$\int_0^T|\dot\eta^n|^2(t)\,dt=\int_0^T\sum_{k=1}^{n-1}(\dot\eta_k^n(t))^2\,dt\leq C\left(\int_0^T|\dot V^n(t)|^2\,dt+n\right),\tag{4.5}$$

for some constant C independent of n, where $\dot{V}^n(t)$ stands for the derivative of V^n. Indeed, let $U^n(t) := Z^n(t) + V^n(t)$. Then (U^n, η^n) is the solution of the reflecting system:

$$\begin{cases} dU^n(t) = n^2 A^n U^n(t)\,dt + \dot{V}^n(t)\,dt - n^2 A^n V^n(t)\,dt + d\eta^n(t); \\ U^n(t) \geq 0; \\ \int_0^T \langle U^n(t), d\eta^n(t)\rangle = 0. \end{cases} \tag{4.6}$$

Define $\phi(z) := \sum_{k=1}^{n-1}(z_k^-)^2$ for $z \in R^{n-1}$, where z_k^- stands for the negative part of z_k. Consider the following penalized equation:

$$\begin{aligned} dU^{n,\varepsilon}(t) &= n^2 A^n U^{n,\varepsilon}(t)\,dt + \dot{V}^n(t)\,dt - n^2 A^n V^n(t)\,dt \\ &\quad - \frac{1}{\varepsilon}\nabla\phi\bigl(U^{n,\varepsilon}(t)\bigr)\,dt. \end{aligned} \tag{4.7}$$

According to [12], it holds that

$$\lim_{\varepsilon\to 0}\sup_{0\leq t\leq T}\bigl|U^{n,\varepsilon}(t) - U^n(t)\bigr| = 0, \tag{4.8}$$

$$\eta^n(t) = -\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t \nabla\phi\bigl(U^{n,\varepsilon}(s)\bigr)\,ds, \qquad \text{for } t > 0. \tag{4.9}$$

On the other hand, using the chain rule we obtain

$$\begin{aligned} \phi\bigl(U^{n,\varepsilon}(t)\bigr) &= n^2\int_0^t \langle\nabla\phi\bigl(U^{n,\varepsilon}(s)\bigr), A^n U^{n,\varepsilon}(s)\rangle\,ds \\ &\quad + \int_0^t \langle\nabla\phi\bigl(U^{n,\varepsilon}(s)\bigr), \dot{V}^n(s)\rangle\,ds \\ &\quad - n^2\int_0^t \langle\nabla\phi\bigl(U^{n,\varepsilon}(s)\bigr), A^n V^n(s)\rangle\,ds - \frac{1}{\varepsilon}\int_0^t |\nabla\phi|^2\bigl(U^{n,\varepsilon}(s)\bigr)\,ds. \end{aligned} \tag{4.10}$$

As in the proof of Lemma 3.2, we can show that $\langle b^-, A^n b\rangle \geq 0$ for all $b \in R^{n-1}$. Thus,

$$\begin{aligned} & n^2\int_0^t \langle\nabla\phi\bigl(U^{n,\varepsilon}(s)\bigr), A^n U^{n,\varepsilon}(s)\rangle\,ds \\ &\qquad = -2n^2\int_0^t \langle\bigl(U^{n,\varepsilon}(s)\bigr)^-, A^n U^{n,\varepsilon}(s)\rangle\,ds \leq 0. \end{aligned} \tag{4.11}$$

As $\phi \geq 0$, it follows from (4.10) and (4.11) that

$$\begin{aligned} & \frac{1}{\varepsilon}\int_0^t |\nabla\phi|^2\bigl(U^{n,\varepsilon}(s)\bigr)\,ds \\ &\quad \leq \int_0^t \langle\nabla\phi\bigl(U^{n,\varepsilon}(s)\bigr), \dot{V}^n(s)\rangle\,ds - n^2\int_0^t \langle\nabla\phi\bigl(U^{n,\varepsilon}(s)\bigr), A^n V^n(s)\rangle\,ds \end{aligned} \tag{4.12}$$

$$\le \left(\int_0^t |\nabla\phi|^2\big(U^{n,\varepsilon}(s)\big)\,ds\right)^{\frac{1}{2}} \times \left\{\left(\int_0^t \big|\dot V^n(s)\big|^2\,ds\right)^{\frac{1}{2}} + \left(\int_0^t \big|n^2A^nV^n(s)\big|^2\,ds\right)^{\frac{1}{2}}\right\},$$

which yields that

$$\int_0^t \left|\frac{1}{\varepsilon}\nabla\phi\right|^2\big(U^{n,\varepsilon}(s)\big)\,ds \le C\left\{\int_0^t \big|\dot V^n(s)\big|^2\,ds + \int_0^t \big|n^2A^nV^n(s)\big|^2\,ds\right\} \qquad \text{for all } t>0. \tag{4.13}$$

By selecting a subsequence if necessary, we conclude that $\frac{1}{\varepsilon}\nabla\phi(U^{n,\varepsilon}(\cdot))$ converges weakly in $L^2([0,T]\to R^{n-1})$ as $\varepsilon\to 0$. Combing with (4.9) we deduce that $\eta^n(t)$ is absolutely continuous and

$$\begin{aligned}\int_0^T |\dot\eta^n|^2(t)\,dt &\le \liminf_{\varepsilon\to 0}\int_0^T \left|\frac{1}{\varepsilon}\nabla\phi\right|^2\big(U^{n,\varepsilon}(s)\big)\,ds \\ &\le C\left\{\int_0^T \big|\dot V^n(s)\big|^2\,ds + \int_0^T \big|n^2A^nV^n(s)\big|^2\,ds\right\} \qquad \text{for all } t>0.\end{aligned} \tag{4.14}$$

From the definition of A^n, it is seen that

$$A^nV^n(t) = \begin{pmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_{n-1}\end{pmatrix}, \quad \text{where } a_k = V\left(t,\frac{k+1}{n}\right) - 2V\left(t,\frac{k}{n}\right) + V\left(t,\frac{k-1}{n}\right). \tag{4.15}$$

Observe that

$$\begin{aligned}\big|n^2a_k\big| &= n^2\left|\int_{\frac{k}{n}}^{\frac{k+1}{n}} dy\int_{\frac{k}{n}}^{y}\frac{\partial^2V(t,z)}{\partial z^2}\,dz + \int_{\frac{k-1}{n}}^{\frac{k}{n}} dy\int_{y}^{\frac{k}{n}}\frac{\partial^2V(t,z)}{\partial z^2}\,dz\right| \\ &\le 2\sup_{0\le t\le T,0\le z\le 1}\left|\frac{\partial^2V(t,z)}{\partial z^2}\right|.\end{aligned} \tag{4.16}$$

Substitute (4.16) back to (4.14) to complete the proof of (4.5). Next, we show that the family $\{Z^n(t,x), n\ge 1\}$ defined in (4.3) is relatively compact in the space

$C([0,T]\times[0,1])$. Recall that $G^n(t)=e^{n^2A^nt}$ was defined in Section 2. By the variation of constant formula, we have

$$Z^n(t)=\int_0^t G^n(t-s)\dot{\eta}^n(s)\,ds. \tag{4.17}$$

For $n\geq 1$, define

$$\dot{\eta}^n(t,x)=\dot{\eta}_k^n(t)+(nx-k)\big(\dot{\eta}_{k+1}^n(t)-\dot{\eta}_k^n(t)\big) \quad \text{for } x\in\left[\frac{k}{n},\frac{k+1}{n}\right), k=0,\dots,n-1, \tag{4.18}$$

with $\dot{\eta}_0^n(t):=0$, $\dot{\eta}_n^n(t):=0$. Set $\varphi_j(x):=\sqrt{2}\sin(jx\pi)$. As in [9], introduce the kernel $G^n(t,x,y)$ by

$$G^n(t,x,y)=\sum_{j=1}^{n-1}\exp(\lambda_j^n t)\varphi_j^n(x)\varphi_j\big(k_n(y)\big), \tag{4.19}$$

where $k_n(y)=\frac{[ny]}{n}$ and for $x\in[\frac{k}{n},\frac{k+1}{n}]$, define

$$\varphi_j^n(x)=\varphi_j\left(\frac{k}{n}\right)+(nx-k)\left(\varphi_j\left(\frac{k+1}{n}\right)-\varphi_j\left(\frac{k}{n}\right)\right). \tag{4.20}$$

The following statements were proved in [9] (see the proof of Lemma 3.6 there):

$$\int_0^s\int_0^1\left|G^n(t-r,x,y)-G^n(s-r,x,y)\right|^2dr\,dy\leq C_1\sqrt{t-s}, \tag{4.21}$$

for $x\in[0,1]$ and $s\leq t\leq T$.

$$\int_s^t\int_0^1\left|G^n(t-r,x,y)\right|^2dr\,dy\leq C_2\sqrt{t-s}, \tag{4.22}$$

for $x\in[0,1]$ and $s\leq t\leq T$.

$$\int_0^t\int_0^1\left|G^n(t-r,x,z)-G^n(t-r,y,z)\right|^2dr\,dz\leq C_3|x-y|, \tag{4.23}$$

for $x,y\in[0,1]$ and $0\leq t\leq T$.

The constants C_1, C_2, C_3 in the above estimates are independent of n.

By (4.17) and a simple calculation, we find that

$$Z^n(t,x)=\int_0^t\int_0^1 G^n(t-s,x,y)\dot{\eta}^n\big(s,k_n(y)\big)\,ds\,dy. \tag{4.24}$$

The estimate (4.4) yields that

$$\begin{aligned}\int_0^T\int_0^1|\dot{\eta}^n|^2\big(s,k_n(y)\big)\,ds\,dy&=\sum_{k=0}^{n-1}\frac{1}{n}\int_0^T|\dot{\eta}_k^n|^2(s)\,ds\\&\leq C\left(\int_0^T\int_0^1\left|\frac{\partial V(t,k_n(y))}{\partial t}\right|^2dt\,dy+1\right)\\&\leq C_T,\end{aligned} \tag{4.25}$$

where we have used the smoothness assumption on V, the definition of V^n and (4.2). Using the above estimate and Hölder's inequality, it follows from (4.24), (4.23), (4.22) and (4.21) that there exists a constant C, independent of n, such that

$$\begin{aligned}\big|Z^n(t,x)-Z^n(s,y)\big|^2 &\le C\big(\sqrt{|t-s|}+|x-y|\big),\\ &\qquad s,t\in[0,T], x,y\in[0,1].\end{aligned} \tag{4.26}$$

By the Arzela–Ascoli theorem, $\{Z^n(t,x), n\ge 1\}$ is relatively compact. On the other hand, (4.25) implies that $\{\dot\eta^n(\cdot,k_n(\cdot)), n\ge 1\}$ is relatively compact in $L^2([0,T]\times[0,1])$ with respect to the weak topology. Selecting a subsequence if necessary, we can assume that $Z^n(\cdot,\cdot)$ converges uniformly to some function $Z(\cdot,\cdot)\in C([0,T]\times[0,1])$ and $\dot\eta^n(\cdot,k_n(\cdot))$ converges weakly to some $\dot\eta(\cdot,\cdot)\in L^2([0,T]\times[0,1])$. We complete the proof of step 1 by showing that $(Z,\eta(dt,dy):=\dot\eta(t,y)\,dt\,dy)$ is the solution to the system (4.1). For $\phi\in C_0^2((0,1))$, set $\phi^n:=(\phi(\frac{1}{n}),\dots,\phi(\frac{n-1}{n}))$. By the symmetry of the matrix A^n, it follows from (4.4) that

$$\langle Z^n(t),\phi^n\rangle=\int_0^t\langle n^2A^n\phi^n,Z^n(s)\rangle\,ds+\int_0^t\langle\phi^n,\dot\eta^n(s)\rangle\,ds. \tag{4.27}$$

Multiply the above equation by $\frac{1}{n}$ to get

$$\begin{aligned}\int_0^1 Z^n\big(t,k_n(y)\big)\phi\big(k_n(y)\big)\,dy &=\int_0^t ds\int_0^1\Delta_n\phi\big(k_n(y)\big)Z^n\big(s,k_n(y)\big)\,dy\\ &\quad+\int_0^t ds\int_0^1\phi\big(k_n(y)\big)\dot\eta^n\big(s,k_n(y)\big)\,dy,\end{aligned} \tag{4.28}$$

where $\Delta_n\phi(x):=n^2(\phi(x+\frac{1}{n})-2\phi(x)+\phi(x-\frac{1}{n}))$ is the discrete Laplacian operator. Letting $n\to\infty$ in (4.28) we obtain

$$\begin{aligned}&\int_0^1 Z(t,y)\phi(y)\,dy\\ &\qquad=\int_0^t ds\int_0^1\phi''(y)Z(s,y)\,dy+\int_0^t ds\int_0^1\phi(y)\dot\eta(s,y)\,dy,\end{aligned} \tag{4.29}$$

where we have used the fact that $\phi(k_n(y))\to\phi(y)$ (strongly) in $L^2([0,T]\times[0,1])$. On the other hand, it follows from the definition that

$$\int_0^t\int_0^1\big(Z^n\big(s,k_n(y)\big)+V\big(s,k_n(y)\big)\big)\dot\eta^n\big(s,k_n(y)\big)\,ds\,dy=0. \tag{4.30}$$

Invoking (4.26) and the dominated convergence theorem, we get

$$\begin{aligned}&\int_0^T\int_0^1\big(Z^n\big(s,k_n(y)\big)+V\big(s,k_n(y)\big)-Z(s,y)-V(s,y)\big)^2\,ds\,dy\\ &\qquad\le C\int_0^T\int_0^1\big(Z^n\big(s,k_n(y)\big)-Z^n(s,y)\big)^2\,ds\,dy\end{aligned} \tag{4.31}$$

$$
\begin{aligned}
&\quad + C\int_0^T\int_0^1 \big(Z^n(s,y)-Z(s,y)\big)^2\,ds\,dy\\
&\quad + \int_0^T\int_0^1 \big(V\big(s,k_n(y)\big)-V(s,y)\big)^2\,ds\,dy\\
&\le C\left(\frac{1}{n}\right)^2 + C\int_0^T\int_0^1 \big(Z^n(s,y)-Z(s,y)\big)^2\,ds\,dy\\
&\quad + \int_0^T\int_0^1 \big(V\big(s,k_n(y)\big)-V(s,y)\big)^2\,ds\,dy\\
&\to 0 \qquad \text{as } n\to\infty.
\end{aligned}
$$

Letting $n\to\infty$ in (4.30), the weak convergence of $\dot{\eta}^n$ and (4.31) yield

$$
\int_0^t\int_0^1 \big(Z(s,y)+V(s,y)\big)\dot{\eta}(s,y)\,ds\,dy = 0.
$$

We have shown that conditions (3), (4) in the Definition 4.1 are satisfied by (Z,η). It is straightforward to check that (Z,η) also satisfies (1), (2) in the Definition 4.1. Thus, (Z,η) is the solution to equation (4.1).

Step 2. The general case $V\in C(R_+\times[0,1])$. Take a sequence $V_m\in C^{1,2}(R_+\times[0,1]), m\ge 1$ such that $\sup_{0\le t\le T,0\le x\le 1}|V_m(t,x)-V(t,x)|\to 0$ as $m\to\infty$ for any $T>0$. For every integer $n\ge 1$, define

$$
V^{m,n}(t) = \left(V_m\left(t,\frac{1}{n}\right),\dots,V_m\left(t,\frac{n-1}{n}\right)\right).
$$

Let $(Z^{m,n},\eta^{m,n})$ be the solution to the following Skorohod-type problem in R^{n-1}:

$$
\begin{cases}
dZ^{m,n}(t) = n^2A^nZ^{m,n}(t)\,dt + d\eta^{m,n}(t);\\
Z^{m,n}(t)\ge -V^{m,n}(t);\\
\displaystyle\int_0^T \langle Z^{m,n}(t)+V^{m,n}(t), d\eta^{m,n}(t)\rangle = 0.
\end{cases}
\tag{4.32}
$$

Set $Z_0^{m,n}(t)=0$ and $Z_n^{m,n}(t)=0$. Introduce the continuous functions $Z^{m,n}$ by

$$
Z^{m,n}(t,x) := Z_k^{m,n}(t) + (nx-k)\big(Z_{k+1}^{m,n}(t)-Z_k^{m,n}(t)\big)
\tag{4.33}
$$

for $x\in[\frac{k}{n},\frac{k+1}{n})$, $k=0,\dots,n-1$. According to the result proved in step 1, for every $m\ge 1$ we have

$$
\lim_{n\to\infty}\sup_{0\le t\le T,0\le x\le 1}\big|Z^{m,n}(t,x)-Z^{(m)}(t,x)\big| = 0,
\tag{4.34}
$$

where $Z^{(m)}(t,x)$ is the solution of the following parabolic obstacle problem:

$$
\begin{cases}
\dfrac{\partial Z^{(m)}(t,x)}{\partial t} - \dfrac{\partial^2 Z^{(m)}(t,x)}{\partial x^2} = \dot{\eta}^{(m)}(t,x);\\
Z^{(m)}(t,x)\ge -V_m(t,x);\\
\displaystyle\int_0^t\int_0^1 \big(Z^{(m)}(t,x)+V_m(s,x)\big)\eta^{(m)}(ds,dx) = 0.
\end{cases}
\tag{4.35}
$$

On the other hand, applying Lemma 3.2 we have

$$\begin{aligned}
&\sup_{0\le t\le T,0\le x\le 1}\big|Z^{m,n}(t,x)-Z^n(t,x)\big| \\
&\qquad = \sup_{0\le t\le T,1\le k\le n-1}\big|Z_k^{m,n}(t)-Z_k^n(t)\big| \\
&\qquad \le \sup_{0\le t\le T,1\le k\le n-1}\big|V_k^{m,n}(t)-V_k^n(t)\big| \\
&\qquad = \sup_{0\le t\le T,0\le x\le 1}\big|V_m\big(t,k_n(x)\big)-V\big(t,k_n(x)\big)\big| \\
&\qquad \le \sup_{0\le t\le T,0\le x\le 1}\big|V_m(t,x)-V(t,x)\big|.
\end{aligned} \tag{4.36}$$

Now we are in the position to complete the proof of the theorem. For every $m \ge 1$, by (4.36) and Proposition 4.1 we have

$$\begin{aligned}
&\sup_{0\le t\le T,0\le x\le 1}\big|Z^n(t,x)-Z(t,x)\big| \\
&\qquad \le \sup_{0\le t\le T,0\le x\le 1}\big|Z^n(t,x)-Z^{(m)}(t,x)\big| \\
&\qquad\quad + \sup_{0\le t\le T,0\le x\le 1}\big|Z^{(m)}(t,x)-Z(t,x)\big| \\
&\qquad \le \sup_{0\le t\le T,0\le x\le 1}\big|Z^n(t,x)-Z^{m,n}(t,x)\big| \\
&\qquad\quad + \sup_{0\le t\le T,0\le x\le 1}\big|Z^{m,n}(t,x)-Z^{(m)}(t,x)\big| \\
&\qquad\quad + \sup_{0\le t\le T,0\le x\le 1}\big|V_m(t,x)-V(t,x)\big| \\
&\qquad \le 2\sup_{0\le t\le T,0\le x\le 1}\big|V_m(t,x)-V(t,x)\big| \\
&\qquad\quad + \sup_{0\le t\le T,0\le x\le 1}\big|Z^{m,n}(t,x)-Z^{(m)}(t,x)\big|.
\end{aligned} \tag{4.37}$$

Given a positive constant $\varepsilon > 0$. First, choose m sufficiently large such that

$$2\sup_{0\le t\le T,0\le x\le 1}\big|V_m(t,x)-V(t,x)\big| \le \frac{\varepsilon}{2}. \tag{4.38}$$

For such a fixed m, by (4.34) there exists an integer N such that for $n \ge N$,

$$\sup_{0\le t\le T,0\le x\le 1}\big|Z^{m,n}(t,x)-Z^{(m)}(t,x)\big| \le \frac{\varepsilon}{2}. \tag{4.39}$$

Putting (4.37), (4.38) and (4.39) together we obtain that

$$\sup_{0\le t\le T,0\le x\le 1}\big|Z^n(t,x)-Z(t,x)\big| \le \varepsilon$$

for $n \ge N$. As ε is arbitrary, the proof is complete. $\square$

5. The convergence of the scheme. After all the preparations in the previous sections, this part is devoted to the proof of the main result. For $y \in R^{n-1}$, set

$$F_n(t,y) = \begin{pmatrix} f\Big(t, \frac{1}{n}, y_1\Big) \\ f\Big(t, \frac{2}{n}, y_2\Big) \\ \cdot \\ \cdot \\ \cdot \\ f\Big(t, \frac{n-1}{n}, y_{n-1}\Big) \end{pmatrix},$$

$$\Sigma_n(t,y) = \begin{pmatrix} \sigma\Big(t, \frac{1}{n}, y_1\Big) & 0 & \cdot\cdot & 0 \\ 0 & \sigma\Big(t, \frac{2}{n}, y_2\Big) & \cdot\cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot\cdot & \sigma\Big(t, \frac{n-1}{n}, y_{n-1}\Big) \end{pmatrix}.$$

The system (2.6) can be written as

$$\begin{aligned} du^n(t) &= n^2 A^n u^n(t)\,dt + F_n\big(t, u^n(t)\big)\,dt \\ &\quad + \sqrt{n}\Sigma_n\big(t, u^n(t)\big)\,dW^n(t) + d\eta^n(t), \end{aligned} \tag{5.1}$$

where $W^n = (W_1^n(t), \dots, W_{n-1}^n(t))$.

By the variation of constant formula, it follows that

$$\begin{aligned} u^n(t) &= G^n(t)u^n(0) + \int_0^t G^n(t-s)F_n\big(s, u^n(s)\big)\,ds \\ &\quad + \sqrt{n}\int_0^t G^n(t-s)\Sigma_n\big(s, u^n(s)\big)\,dW^n(s) + \int_0^t G^n(t-s)\,d\eta^n(s), \end{aligned} \tag{5.2}$$

where as before $G^n(t) = \exp(n^2 A^n t)$. Denote

$$\begin{aligned} v^n(t) &= G^n(t)u^n(0) + \int_0^t G^n(t-s)F_n\big(s, u^n(s)\big)\,ds \\ &\quad + \sqrt{n}\int_0^t G^n(t-s)\Sigma_n\big(s, u^n(s)\big)\,dW^n(s). \end{aligned} \tag{5.3}$$

Then v^n is the solution of the SDE

$$\begin{aligned} dv^n(t) &= n^2 A^n v^n(t)\,dt + F_n\big(t, u^n(t)\big)\,dt \\ &\quad + \sqrt{n}\Sigma_n\big(t, u^n(t)\big)\,dW^n(t), \end{aligned} \tag{5.4}$$

and $(Z^n(t) := u^n(t) - v^n(t), \eta^n(t))$ is the solution of the system

$$
\begin{cases}
dZ^n(t) = n^2 A^n Z^n(t)\,dt + d\eta^n(t); \\
Z^n(t) \geq -v^n(t); \\
\int_0^T \langle Z^n(t) + v^n(t), d\eta^n(t)\rangle = 0, \qquad \text{for all } T > 0.
\end{cases}
\tag{5.5}
$$

Recall the random field $u^n(t,x)$ defined in (2.7) in Section 2 and introduce the random fields

$$
\eta^n(t,x) = \eta_k^n(t) + (nx-k)\bigl(\eta_{k+1}^n(t) - \eta_k^n(t)\bigr) \qquad \text{for } x \in \left[\frac{k}{n}, \frac{k+1}{n}\right), k = 0, \dots, n-1,
\tag{5.6}
$$

$$
v^n(t,x) = v_k^n(t) + (nx-k)\bigl(v_{k+1}^n(t) - v_k^n(t)\bigr) \qquad \text{for } x \in \left[\frac{k}{n}, \frac{k+1}{n}\right), k = 0, \dots, n-1,
\tag{5.7}
$$

where $\eta_0^n(t) := 0, \eta_n^n(t) := 0$, $v_0^n(t) := 0$, $v_n^n(t) := 0$. Let the kernel $G^n(t,x,y)$ be defined as in (4.19) in Section 4. It is easy to verify that u^n and v^n satisfies the equations

$$
\begin{aligned}
u^n(t,x) = {} & \int_0^1 G^n(t,x,y) u^n\bigl(0, k_n(y)\bigr)\,dy \\
& + \int_0^t \int_0^1 G^n(t-s,x,y) f\bigl(s, k_n(y), u^n\bigl(s, k_n(y)\bigr)\bigr)\,dy\,ds \\
& + \int_0^t \int_0^1 G^n(t-s,x,y) \sigma\bigl(s, k_n(y), u^n\bigl(s, k_n(y)\bigr)\bigr) W(ds,dy) \\
& + \int_0^t \int_0^1 G^n(t-s,x,y) \eta^n\bigl(ds, k_n(y)\bigr)\,dy,
\end{aligned}
\tag{5.8}
$$

$$
\begin{aligned}
v^n(t,x) = {} & \int_0^1 G^n(t,x,y) u^n\bigl(0, k_n(y)\bigr)\,dy \\
& + \int_0^t \int_0^1 G^n(t-s,x,y) f\bigl(s, k_n(y), u^n\bigl(s, k_n(y)\bigr)\bigr)\,dy\,ds \\
& + \int_0^t \int_0^1 G^n(t-s,x,y) \sigma\bigl(s, k_n(y), u^n\bigl(s, k_n(y)\bigr)\bigr) W(ds,dy),
\end{aligned}
\tag{5.9}
$$

where $k_n(y) = \frac{[ny]}{n}$ as in Section 4. Let $G(t,x,y)$ denote the heat kernel of the Laplacian $\frac{\partial^2}{\partial x^2}$ on the interval [0, 1] with the Dirichlet boundary condition, that is,

$$
G(t,x,y) = \sum_{k=1}^{\infty} \exp\bigl(-k^2\pi^2 t\bigr)\varphi_k(x)\varphi_k(y).
$$

The following lemma was proved in [9].

LEMMA 5.1. *The following statements hold*:

(i) *There exists a constant c such that*

$$\int_0^\infty \int_0^1 \left|G(t,x,y) - G^n(t,x,y)\right|^2 dy\, dt \le \frac{c}{n}, \tag{5.10}$$

for $x \in [0,1]$ and $n \ge 1$.

(ii) *For every $t > 0$, $\gamma \in (0,1)$, $\beta > \frac{\gamma}{2} + \frac{1}{2}$ there is a constant C such that*

$$\int_0^1 \left|G(t,x,y) - G^n(t,x,y)\right|^2 dy \le C n^{-\gamma} t^{-\beta}, \tag{5.11}$$

for $x \in [0,1]$ and $n \ge 1$.

We have the following estimate for u^n.

LEMMA 5.2. *For any $T > 0$, we have*

$$\sup_{0\le t\le T, 0\le x\le 1} \left|u^n(t,x)\right| \le 2 \sup_{0\le t\le T, 0\le x\le 1} \left|v^n(t,x)\right|. \tag{5.12}$$

PROOF. Keeping in mind that $u^n(t,x)$, $v^n(t,x)$ are piecewise linear in x and applying Lemma 3.2 to the system (5.5), we have

$$\begin{aligned} &\sup_{0\le t\le T, 0\le x\le 1} \left|u^n(t,x)\right| \\ &\quad = \sup_{0\le t\le T, 1\le k\le n-1} \left|u^n\left(t, \frac{k}{n}\right)\right| = \sup_{0\le t\le T, 1\le k\le n-1} \left|Z_k^n(t) + v^n\left(t, \frac{k}{n}\right)\right| \\ &\quad \le 2 \sup_{0\le t\le T, 1\le k\le n-1} \left|v^n\left(t, \frac{k}{n}\right)\right| = 2 \sup_{0\le t\le T, 0\le x\le 1} \left|v^n(t,x)\right| \end{aligned}$$

proving the lemma. □

PROPOSTION 5.1. *Assume the linear growth condition* (H.2) *in Section* 2. *Then for $p \ge 1$ and $T > 0$, there exists a constant C_p such that*

$$\sup_n E\Big[\sup_{0\le t\le T, 0\le x\le 1} \left|u^n(t,x)\right|^p\Big] \le C_p. \tag{5.13}$$

PROOF. We will use the notation $|u|_\infty^t := \sup_{0\le s\le t, 0\le x\le 1} |u(s,x)|$. We can assume $p > 20$. By Lemma 5.2, we have

$$\begin{aligned} &\left(\left|u^n\right|_\infty^T\right)^p \le 2^p \left(\left|v^n\right|_\infty^T\right)^p \\ &\qquad \le c(p)|u_0|_\infty^p + c(p)\left(\sup_{x\in[0,1], t\in[0,T]} \left|\int_0^t \int_0^1 G^n(t-s,x,y)\right.\right. \end{aligned} \tag{5.14}$$

$$\times f\big(s, k_n(y), u^n\big(s, k_n(y)\big)\big)\, dy\, ds\bigg|\bigg)^p$$

$$+ c(p)\bigg(\sup_{x\in[0,1],t\in[0,T]}\bigg|\int_0^t\int_0^1 G^n(t-s,x,y)$$

$$\times \sigma\big(s, k_n(y), u^n\big(s, k_n(y)\big)\big) W(ds, dy)\bigg|\bigg)^p.$$

Set

$$I_1(t,x) := \int_0^t\int_0^1 G^n(t-s,x,y) f\big(s, k_n(y), u^n\big(s, k_n(y)\big)\big)\, dy\, ds,$$

$$I_2(t,x) := \int_0^t\int_0^1 G^n(t-s,x,y)\sigma\big(s, k_n(y), u^n\big(s, k_n(y)\big)\big) W(ds, dy).$$

By the linear growth of f and the Hölder inequality,

$$\begin{aligned}
E\big(|I_1|_\infty^T\big)^p &\le C_p(T)\bigg(\sup_{x\in[0,1],t\in[0,T]}\int_0^t\int_0^1 \big(G^n(s,x,y)\big)^2\, dy\, ds\bigg)^{\frac{p}{2}} \\
&\quad\times E\int_0^T \big(\big|f\big(\cdot, k_n(\cdot), u^n\big(\cdot, k_n(\cdot)\big)\big)\big|_\infty^t\big)^p\, dt \\
&\le C_p(T)\bigg(1 + E\int_0^T \big(\big|u^n\big(\cdot, k_n(\cdot)\big)\big|_\infty^t\big)^p\, dt\bigg) \\
&\le C_p(T)\bigg(1 + E\int_0^T \big(|u^n|_\infty^t\big)^p\, dt\bigg),
\end{aligned} \tag{5.15}$$

where $\frac{1}{q} + \frac{1}{p} = 1$, $C_p(T)$ denotes a generic constant depending on T, p. The fact that

$$\sup_n \sup_{x\in[0,1],t\in[0,T]}\int_0^t\int_0^1 \big(G^n(s,x,y)\big)^2\, dy\, ds < \infty \tag{5.16}$$

has also been used in the derivation of (5.15). (5.16) follows from (5.10) and the well-known fact (see, e.g., [19])

$$\sup_{x\in[0,1],t\in[0,T]}\int_0^t\int_0^1 \big(G(s,x,y)\big)^2\, dy\, ds < \infty.$$

In view of (4.21), (4.22), (4.23), following a similar calculation as in the proof of Corollary 3.4 in [19] and Lemma 3.6 in [9], we obtain that

$$\begin{aligned}
&E\big|I_2(t,x) - I_2(s,y)\big|^p \\
&\quad\le c\bigg(E\int_0^{(t\vee s)} \big(\big|u^n\big(\cdot, k_n(\cdot)\big)\big|_\infty^r\big)^p\, dr\bigg) \times \big|(t,x) - (s,y)\big|^{\frac{p}{4}-3} \\
&\quad\le c\bigg(E\int_0^{(t\vee s)} \big(|u^n|_\infty^r\big)^p\, dr\bigg) \times \big|(t,x) - (s,y)\big|^{\frac{p}{4}-3}.
\end{aligned} \tag{5.17}$$

Applying Garsia–Rodemich–Rumsey's lemma (see, e.g., Theorem 1.1 and Corollary 1.2 in [19]), we get from (5.17) that

$$\begin{aligned}&\left|I_2(t,x)-I_2(s,y)\right|^p\\ &\quad\le N(\omega)^p\left|(t,x)-(s,y)\right|^{\frac{p}{4}-5}\left(\log\left(\frac{\gamma}{|(t,x)-(s,y)|}\right)\right)^2,\end{aligned}\tag{5.18}$$

where $N(\omega)$ is a random variable satisfying

$$E\left[N^p\right]\le ac\left(E\int_0^{(t\vee s)}\left(|u^n|_\infty^r\right)^p\,dr\right),\tag{5.19}$$

where a,γ are constants depending only on p and c is the constant appeared in (5.17). Choosing $s=0$ in (5.18), we see that there exists a constant c_T such that

$$E\Big(\sup_{x\in[0,1],t\in[0,T]}\left|I_2(t,x)\right|^p\Big)\le c_T E\int_0^T\left(|u^n|_\infty^t\right)^p\,dt.\tag{5.20}$$

Putting (5.14), (5.15), (5.20) together, we get that

$$E\left(|u^n|_\infty^T\right)^p\le c(p,K,T)\left(E\int_0^T\left(|u^n|_\infty^t\right)^p\,dt+1\right),\tag{5.21}$$

where $c(p,K,T)$ is a constant depending on p,K,T. Applying Grownwall's lemma, we prove the proposition. $\square$

PROOF OF THE MAIN RESULT (THEOREM 2.1). Recall

$$\begin{aligned}u(t,x)&=\int_0^1 G(t,x,y)u_0(y)\,dy\\ &\quad+\int_0^t\int_0^1 G(t-s,x,y)f\big(s,y,u(s,y)\big)\,dy\,ds\\ &\quad+\int_0^t\int_0^1 G(t-s,x,y)\sigma\big(s,y,u(s,y)\big)W(ds,dy)\\ &\quad+\int_0^t\int_0^1 G(t-s,x,y)\eta(ds,dy).\end{aligned}\tag{5.22}$$

Set

$$\begin{aligned}\bar{v}(t,x)&=\int_0^1 G(t,x,y)u_0(y)\,dy\\ &\quad+\int_0^t\int_0^1 G(t-s,x,y)f\big(s,y,u(s,y)\big)\,dy\,ds\\ &\quad+\int_0^t\int_0^1 G(t-s,x,y)\sigma\big(s,y,u(s,y)\big)W(ds,dy).\end{aligned}\tag{5.23}$$

Then $(\bar{z}(t,x) := u(t,x) - \bar{v}(t,x), \eta(dt,dx))$ solves the following random parabolic obstacle problem:

$$\begin{cases} \dfrac{\partial \bar{z}(t,x)}{\partial t} - \dfrac{\partial^2 \bar{z}(t,x)}{\partial x^2} = \dot{\eta}(t,x), & x \in [0,1]; \\ \bar{z}(t,x) \ge -\bar{v}(t,x); \\ \displaystyle\int_0^t \int_0^1 \big(\bar{z}(s,x) + \bar{v}(s,x)\big)\eta(ds,dx) = 0, & t \ge 0. \end{cases} \tag{5.24}$$

For very positive integer $n \ge 1$, define

$$\bar{v}^n(t) = \left(\bar{v}\left(t, \frac{1}{n}\right), \dots, \bar{v}\left(t, \frac{n-1}{n}\right)\right).$$

Let $(\bar{z}^n, \bar{\eta}^n)$ be the solution of the following random Skorohod-type problem in R^{n-1}:

$$\begin{cases} d\bar{z}^n(t) = n^2 A^n \bar{z}^n(t)\,dt + d\bar{\eta}^n(t); \\ \bar{z}^n(t) \ge -\bar{v}^n(t); \\ \displaystyle\int_0^T \langle \bar{z}^n(t) + \bar{v}^n(t), d\bar{\eta}^n(t)\rangle = 0. \end{cases} \tag{5.25}$$

Introduce the continuous random field $\bar{z}^n$:

$$\bar{z}^n(t,x) := \bar{z}^n_k(t) + (nx-k)\big(\bar{z}^n_{k+1}(t) - \bar{z}^n_k(t)\big) \tag{5.26}$$

for $x \in [\frac{k}{n}, \frac{k+1}{n})$, $k = 0, \dots, n-1$, with $\bar{z}^n_0(t) := 0$, $\bar{z}^n_n(t) := 0$. By Theorem 4.1, we conclude that

$$\lim_{n\to\infty} \sup_{0\le t\le T, 0\le x\le 1} \left|\bar{z}^n(t,x) - \bar{z}(t,x)\right| = 0 \tag{5.27}$$

almost surely. Let $\bar{v}^n$ denote the random field:

$$\bar{v}^n(t,x) := \bar{v}\left(t, \frac{k}{n}\right) + (nx-k)\left(\bar{v}\left(t, \frac{k+1}{n}\right) - \bar{v}\left(t, \frac{k}{n}\right)\right) \tag{5.28}$$

for $x \in [\frac{k}{n}, \frac{k+1}{n})$, $k = 1, \dots, n-1$. Since $\bar{v}(t,x)$ is a continuous random field with bounded moments of any order, it is clear that for any $p \ge 1$,

$$\lim_{n\to\infty} E\Big[\sup_{0\le t\le T, 0\le x\le 1} \left|\bar{v}^n(t,x) - \bar{v}(t,x)\right|^p\Big] = 0. \tag{5.29}$$

Set $\bar{u}^n(t,x) := \bar{v}^n(t,x) + \bar{z}^n(t,x)$. Since $u(t,x) := \bar{v}(t,x) + \bar{z}(t,x)$, it follows from (5.27) and (5.29) that

$$\lim_{n\to\infty} E\Big[\sup_{0\le t\le T, 0\le x\le 1} \left|\bar{u}^n(t,x) - u(t,x)\right|^p\Big] = 0. \tag{5.30}$$

Recall the definition of the random fields $u^n(t,x)$ defined in (5.8) or (2.7). To prove the theorem, that is,

$$\lim_{n\to\infty} E\Big[\sup_{0\le t\le T, 0\le x\le 1} \left|u^n(t,x) - u(t,x)\right|^p\Big] = 0,$$

in view of (5.30) it is sufficient to show that

$$\lim_{n\to\infty} E\Big[\sup_{0\le t\le T, 0\le x\le 1} \big|\bar{u}^n(t,x) - u^n(t,x)\big|^p\Big] = 0. \tag{5.31}$$

Applying Lemma 3.2 to the systems (5.5) and (5.25), it follows that

$$\begin{aligned}
&\sup_{0\le t\le T, 0\le x\le 1} \big|\bar{u}^n(t,x) - u^n(t,x)\big| \\
&\qquad = \sup_{0\le t\le T, 0\le k\le n-1} \left|\bar{u}^n\left(t, \frac{k}{n}\right) - u^n\left(t, \frac{k}{n}\right)\right| \\
&\qquad = \sup_{0\le t\le T, 0\le k\le n-1} \left|\bar{v}^n\left(t, \frac{k}{n}\right) - v^n\left(t, \frac{k}{n}\right) + \left(\bar{z}_k^n(t) - Z_k^n(t)\right)\right| \\
&\qquad \le 2 \sup_{0\le t\le T, 0\le k\le n-1} \left|\bar{v}^n\left(t, \frac{k}{n}\right) - v^n\left(t, \frac{k}{n}\right)\right| \\
&\qquad \le 2 \sup_{0\le t\le T, 0\le x\le 1} \big|\bar{v}^n(t,x) - v^n(t,x)\big|.
\end{aligned} \tag{5.32}$$

Introduce

$$\begin{aligned}
\hat{v}^n(t,x) &= \int_0^1 G^n(t,x,y) u\big(0, k_n(y)\big)\, dy \\
&\quad + \int_0^t \int_0^1 G^n(t-s,x,y) f\big(s, k_n(y), \bar{u}^n\big(s, k_n(y)\big)\big)\, dy\, ds \\
&\quad + \int_0^t \int_0^1 G^n(t-s,x,y) \sigma\big(s, k_n(y), \bar{u}^n\big(s, k_n(y)\big)\big) W(ds,dy).
\end{aligned} \tag{5.33}$$

Recalling the expression of v^n in (5.9) we have

$$\begin{aligned}
&\hat{v}^n(t,x) - v^n(t,x) \\
&\qquad = \int_0^t \int_0^1 G^n(t-s,x,y) \big[f\big(s, k_n(y), \bar{u}^n\big(s, k_n(y)\big)\big) \\
&\qquad\quad - f\big(s, k_n(y), u^n\big(s, k_n(y)\big)\big)\big]\, dy\, ds \\
&\qquad\quad + \int_0^t \int_0^1 G^n(t-s,x,y) \big[\sigma\big(s, k_n(y), \bar{u}^n\big(s, k_n(y)\big)\big) \\
&\qquad\quad - \sigma\big(s, k_n(y), u^n\big(s, k_n(y)\big)\big)\big] W(ds,dy).
\end{aligned} \tag{5.34}$$

Using the above representation, the Lipschitz continuity of the coefficients and the similar arguments leading to the proof of (5.21) we can show that

$$E\big(\big|\hat{v}^n - v^n\big|_\infty^T\big)^p \le c(p,K,T) E \int_0^T \big(\big|\bar{u}^n - u^n\big|_\infty^t\big)^p\, dt. \tag{5.35}$$

Combining (5.32) and (5.35), we obtain that

$$
\begin{aligned}
E\big[\big(|\bar u^n - u^n|_{\infty}^{T}\big)^p\big]
&\le cE\Big[\sup_{0\le t\le T, 0\le x\le 1}\big|\bar v^n(t,x)-\hat v^n(t,x)\big|^p\Big]\\
&\quad + cE\Big[\sup_{0\le t\le T, 0\le x\le 1}\big|\hat v^n(t,x)-v^n(t,x)\big|^p\Big]\\
&\le CE\Big[\sup_{0\le t\le T, 0\le x\le 1}\big|\bar v^n(t,x)-\hat v^n(t,x)\big|\Big]\\
&\quad + CE\int_0^T\big(|\bar u^n - u^n|_{\infty}^{t}\big)^p\,dt.
\end{aligned}
\tag{5.36}
$$

By the Grownwall's inequality, we derive that

$$
E\big[\big(|\bar u^n - u^n|_{\infty}^{T}\big)^p\big] \le C(T)E\Big[\sup_{0\le t\le T, 0\le x\le 1}\big|\bar v^n(t,x)-\hat v^n(t,x)\big|^p\Big].
\tag{5.37}
$$

It remains to show

$$
\lim_{n\to\infty} E\Big[\sup_{0\le t\le T, 0\le x\le 1}\big|\bar v^n(t,x)-\hat v^n(t,x)\big|^p\Big] = 0.
\tag{5.38}
$$

From (5.28), we deduce that

$$
\begin{aligned}
\bar v^n(t,x) &= \int_0^1 \bar G^n(t,x,y)u(0,y)\,dy\\
&\quad + \int_0^t\int_0^1 \bar G^n(t-s,x,y) f\big(s,y,u(s,y)\big)\,dy\,ds\\
&\quad + \int_0^t\int_0^1 \bar G^n(t-s,x,y)\sigma\big(s,y,u(s,y)\big)W(ds,dy),
\end{aligned}
\tag{5.39}
$$

where $\bar G^n$ is defined as follows:

$$
\bar G^n(t,x,y) := G\Big(t,\frac{k}{n},y\Big) + (nx-k)\Big(G\Big(t,\frac{k+1}{n},y\Big) - G\Big(t,\frac{k}{n},y\Big)\Big)
\tag{5.40}
$$

for $x \in [\frac{k}{n}, \frac{k+1}{n})$, $k = 1,\dots,n-1$. Recall the definition of $\varphi_k^n(x)$ in (4.20). It is easy to check that

$$
\bar G^n(t,x,y) = \sum_{k=1}^{\infty} \exp\big(-k^2\pi t\big)\varphi_k^n(x)\varphi_k(y),
$$

and moreover, for $T > 0$,

$$
\sup_{0\le x\le 1}\int_0^T\int_0^1 \big(G(s,x,y) - \bar G^n(s,x,y)\big)^2\,ds\,dy \to 0
\tag{5.41}
$$

as $n \to \infty$.

Now,

$$
\begin{aligned}
&\hat{v}^n(t,x) - \bar{v}^n(t,x) \\
&\quad = \int_0^t \int_0^1 \left[G^n(t-s,x,y) - \bar{G}^n(t-s,x,y)\right] \\
&\qquad \times f\big(s, k_n(y), \bar{u}^n\big(s, k_n(y)\big)\big)\, ds\, dy \\
&\qquad + \int_0^t \int_0^1 \left[G^n(t-s,x,y) - \bar{G}^n(t-s,x,y)\right] \\
&\qquad \times \sigma\big(s, k_n(y), \bar{u}^n\big(s, k_n(y)\big)\big) W(ds,dy) \\
&\qquad + \int_0^t \int_0^1 \bar{G}^n(t-s,x,y)\big[f\big(s, k_n(y), \bar{u}^n\big(s, k_n(y)\big)\big) \\
&\qquad - f\big(s, y, u(s,y)\big)\big]\, dy\, ds \\
&\qquad + \int_0^t \int_0^1 \bar{G}^n(t-s,x,y)\big[\sigma\big(s, k_n(y), \bar{u}^n\big(s, k_n(y)\big)\big) \\
&\qquad - \sigma\big(s, y, u(s,y)\big)\big] W(ds,dy) \\
&\quad := B_1^n(t,x) + B_2^n(t,x) + B_3^n(t,x) + B_4^n(t,x).
\end{aligned}
\tag{5.42}
$$

We will show that each of the four terms tends to zero. In view of (5.10) and (5.41), by the linear growth of f, we have

$$
\begin{aligned}
&E\Big[\sup_{0\le t\le T, 0\le x\le 1} \big|B_1^n(t,x)\big|^2\Big] \\
&\qquad \le C\Big(\sup_{0\le t\le T, 0\le x\le 1} \int_0^t \int_0^1 \big(G^n\big(t-s, k_n(x), y\big) \\
&\qquad\quad - \bar{G}^n\big(t-s, k_n(x), y\big)\big)^2\, ds\, dy\Big) \\
&\qquad\quad \times \int_0^T \int_0^1 \big(1 + E\big[\big|\bar{u}^n\big(s, k_n(y)\big)\big|^2\big]\big)\, ds\, dy \\
&\qquad \le C_T \sup_{0\le t\le T, 0\le x\le 1} \int_0^t \int_0^1 \big(G^n\big(t-s, k_n(x), y\big) \\
&\qquad\quad - \bar{G}^n\big(t-s, k_n(x), y\big)\big)^2\, ds\, dy \to 0.
\end{aligned}
\tag{5.43}
$$

By the similar arguments as in the proof of Corollary 3.4 in [19] and in the proof of Lemma 3.6 in [9], we can show that there exists a constant K_p depending on $\sup_n \sup_{0\le t\le T, 0\le x\le 1} E[|u^n(t,x)|^{2p}]$ and $\sup_{0\le t\le T, 0\le x\le 1} E[|u(t,x)|^{2p}]$ such that

$$
E\big[\big|B_i^n(t,x) - B_i^n(s,y)\big|^{2p}\big] \le K_p\big(|t-s|^{\frac{1}{2}} + |x-y|\big)^p, \tag{5.44}
$$

for all $s, t \in [0, T]$, $x, y \in [0, 1]$, where $i = 2, 3, 4$. On the other hand, for fixed $(t, x) \in [0, T] \times [0, 1]$, we have

$$\lim_{n\to\infty} E\big[\big|B_i^n(t,x)\big|^{2p}\big] = 0, \qquad i = 2, 3, 4. \tag{5.45}$$

Let us prove (5.45) for B_4^n. Other cases are similar. By Burkholder's inequality and the Lipschitz continuity of σ,

$$\begin{aligned}
&E\big[\big|B_4^n(t,x)\big|^{2p}\big] \\
&\quad \le C_p E\biggl[\biggl(\int_0^t \int_0^1 \bar{G}^n(t-s,x,y)^2 \bigl(\sigma\bigl(s, k_n(y), \bar{u}^n\bigl(s, k_n(y)\bigr)\bigr) \\
&\qquad\quad - \sigma\bigl(s, y, u(s,y)\bigr)\bigr)^2 \, ds\, dy\biggr)^p\biggr] \\
&\quad \le C_p \biggl(\int_0^t \int_0^1 \bar{G}^n(t-s,x,y)^2 \, ds\, dy\biggr)^p \\
&\qquad \times \biggl\{\frac{1}{n} + E\Bigl[\sup_{0\le t\le T, 0\le x\le 1} \bigl|\bar{u}^n\bigl(t, k_n(x)\bigr) - u\bigl(t, k_n(x)\bigr)\bigr|^{2p}\Bigr] \\
&\qquad + E\Bigl[\sup_{0\le t\le T, 0\le x\le 1} \bigl|u\bigl(t, k_n(x)\bigr) - u(t,x)\bigr|^{2p}\Bigr]\biggr\} \\
&\quad \longrightarrow 0, \qquad \text{as } n \to \infty,
\end{aligned} \tag{5.46}$$

where (5.30) has been used. By virtue of (5.44), (5.45) and a standard procedure (see, e.g., [26]) we can deduce that

$$\lim_{n\to\infty} E\Bigl[\sup_{0\le t\le T, 0\le x\le 1} \bigl|B_i^n(t,x)\bigr|^{2p}\Bigr] = 0, \qquad i = 2, 3, 4. \tag{5.47}$$

Putting (5.42), (5.43) and (5.47) together we complete the proof of (5.38), and hence the theorem. □

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