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EDGEWORTH EXPANSION FOR FUNCTIONALS OF CONTINUOUS DIFFUSION PROCESSES

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This paper presents new results on the Edgeworth expansion for high frequency functionals of continuous diffusion processes. We derive asymptotic expansions for weighted functionals of the Brownian motion and apply them to provide the Edgeworth expansion for power variation of diffusion processes. Our methodology relies on martingale embedding, Malliavin calculus and stable central limit theorems for semimartingales. Finally, we demonstrate the density expansion for Studentized statistics of power variations.

1. Introduction. Edgeworth expansions have been widely investigated by probabilists and statisticians in various settings. Nowadays, there exists a vast amount of literature on Edgeworth expansions in the case of independent random variables (cf. [4]), weakly dependent variables (cf. [7]) or in the framework of martingales [19, 22]. We refer to classical books [4, 8] and [17] for a comprehensive theory of asymptotic expansions and their applications. We remark that those authors mainly deal with Edgeworth expansions associated with a normal limit.

In the framework of high frequency data (or infill asymptotics), which refers to the sampling scheme in which the time step between two consecutive observations converges to zero while the time span remains fixed, a mixed normal limit appears as a typical asymptotic distribution. In the last years, a lot of research has been devoted to limit theorems for high frequency observations of diffusion processes or Itô semimartingales; see, for example, [2, 10, 11, 14] among many others. Such limit theorems find manifold applications in parametric and semiparametric inference for diffusion models, estimation of quadratic variation and related objects

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(see, e.g., [3, 18]), testing approaches for semimartingales (see, e.g., [1, 6]) or numerical analysis (see, e.g., [12]). While asymptotic mixed normality of high frequency functionals has been proved in various settings, the Edgeworth expansions associated with mixed normal limits have not been considered.

In this paper, we present the asymptotic expansion for high frequency statistics of continuous diffusion processes. More precisely, we study the Edgeworth expansion of weighted functionals of Brownian motion, where the weight arises from a continuous SDE, and apply the asymptotic results to power variations of continuous SDEs. Finally, we will obtain the density expansion for a Studentized version of the power variation.

Our approach is based on the recent work of Yoshida [25], who uses a martingale embedding method to obtain the asymptotic expansion of the characteristic function associated with a mixed normal limit. In a second step, the asymptotic density expansion is achieved via the Fourier inversion. Let us briefly sketch the main concepts of [25]. We are given a functional Z_n, which admits the decomposition

$$Z_n = M_n + r_n N_n,$$

where M_n is a leading term, r_n is a deterministic sequence with $r_n \to 0$ and N_n is some tight sequence of random variables. Here, M_n is a terminal value of a *continuous* martingale $(M_t^n)_{t\in[0,1]}$, which converges to a mixed normal limit in the functional sense. Under various technical conditions, including Malliavin differentiability of the involved objects, joint stable convergence of (M_n, N_n) and estimates of the tail behavior of the characteristic function, the paper [25] demonstrated the Edgeworth expansion for the density of Z_n [and, more generally, for the density of the pair (Z_n, F_n), where F_n is another functional usually used for studentization]. The asymptotic theory has been applied to quadratic functionals M_n in [24]. We would also like to refer to a related work of [22], where a martingale expansion in the case of normal limits has been presented. It was applied to the Edgeworth expansion for an ergodic diffusion process and an estimator of the volatility parameter (cf. [5]).

Although the paper [25] presents a general theory, its particular application to typical functionals of continuous diffusion processes is by far not straightforward. When dealing with commonly used high frequency statistics such as, for example, power variations, we are confronted with several levels of complications, which we list below:

(i) The computation of the second-order term N_n in the decomposition of Z_n appears to be rather involved (cf. Theorem 4.2). This stochastic second-order expansion requires a very precise treatment of the functional Z_n.

(ii) The joint asymptotic mixed normality of the vector (M^n, N_n, F_n, C^n), where C^n is the quadratic variation process associated with the martingale M^n and F_n is an external functional mentioned above, is required for the Edgeworth expansion (cf. Theorem 5.1). The proof of such results relies on stable limit theorems for semimartingales (cf. Theorems A.1, A.2 and 4.4).

(iii) Other ingredients of Edgeworth expansion are the adaptive random symbol $\underline{\sigma}$ and the anticipative random symbol $\overline{\sigma}$ (see [25] or Section 2 for the definition of random symbols). While the adaptive random symbol $\underline{\sigma}$ is given explicitly using the results of (ii), the anticipative random symbol $\overline{\sigma}$ is defined in an implicit way. We will show how this symbol can be determined in Sections 3.3 and 3.4. For this purpose, we will apply the Wiener chaos expansion and the duality between the kth Malliavin derivative D^k and its adjoint δ^k.

(iv) Checking the technical conditions presented in Section 2.3 is another difficult task. In particular, we need to show the existence of densities and to analyze the tail behavior of the characteristic function. This part involves many elements of Malliavin calculus (cf. Sections 3.5 and 3.6).

We see that the derivation of the Edgeworth expansion relies on a combination of various fields of stochastic calculus, such as limit theorems for semimartingales, Malliavin calculus and martingale methods. These steps require a completely new treatment in the power variation case, compared with those in simple quadratic functionals.

The paper is organized as follows. In Section 2, we review the main results of [25], which are crucial for this work. Section 3 is devoted to functionals of Brownian motion with random weights. We will deal with the treatment of the steps (i)–(iv), although the second-order term N_n remains absent. In Section 4, we show the asymptotic theory for the class of generalized power variations of continuous SDEs. In particular, we will determine the asymptotic behavior of the second-order term N_n. Section 5 combines the results of Sections 3 and 4, and we obtain an Edgeworth expansion for the power variation case. In Section 6, we deduce the formula for the asymptotic density associated with a Studentized version of power variation, which is probably most useful for applications. Section 7 is devoted to the derivation of the second-order term N_n. Finally, the Appendix collects the proofs of limit theorems for semimartingales, which are suitable for functionals considered in this paper.

2. Asymptotic expansion associated with mixed normal limit.

As we are applying various techniques from Malliavin calculus and stable central limit theorems for semimartingales, we start by introducing some notation.

(a) Throughout the paper, Δ_n denotes a sequence of positive real numbers with $\Delta_n \to 0$ and such that $1/\Delta_n$ is an integer. For the observation times $i\Delta_n$, $i \in \mathbb{N}$, we use a shorthand notation $t_i := i\Delta_n$. For any function $f : \mathbb{R} \to \mathbb{R}$, we denote by $f^{(k)}$ its kth derivative; for a function $f : \mathbb{R}^2 \to \mathbb{R}$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ the operator d^α is defined via $d^\alpha = d_{x_1}^{\alpha_1} d_{x_2}^{\alpha_2}$, where $d_{x_i}^k f$, $i = 1, 2$, denotes the kth partial derivative of f. The set $C_p^k(\mathbb{R})$ [resp., $C_b^k(\mathbb{R})$] denotes the space of k times differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that all derivatives up to order k have polynomial growth (resp., are bounded). Finally, $\mathtt{i} := \sqrt{-1}$.

(b) The set $\mathbb{L}^q$ denotes the space of random variables with finite qth moment; the corresponding $\mathbb{L}^q$-norms are denoted by $\|\cdot\|_{\mathbb{L}^q}$. The notation $Y_n \xrightarrow{d_{\mathrm{st}}} Y$ (resp., $Y_n \xrightarrow{\mathbb{P}} Y$, $Y_n \xrightarrow{d} Y$) stands for stable convergence (resp., convergence in probability, convergence in law).

(c) We now introduce some notions of Malliavin calculus (we refer to the books of Ikeda and Watanabe [9] and Nualart [20] for a detailed exposition of Malliavin calculus). Set $\mathbb{H} = \mathbb{L}^2([0,1], dx)$ and let $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ denote the usual scalar product on $\mathbb{H}$. We denote by D^k the kth Malliavin derivative operator and by δ^k its unbounded adjoint (also called Skrokhod integral of order k). The space $\mathbb{D}_{k,q}$ is the completion of the set of smooth random variables with respect to the norm

$$\|Y\|_{k,q} := \left(\mathbb{E}[|Y|^q] + \sum_{m=1}^{k} \mathbb{E}[\|D^m Y\|_{\mathbb{H}^{\otimes m}}^q] \right)^{1/q}.$$

For any smooth d-dimensional random variable Y, the Malliavin matrix is defined via $\sigma_Y := (\langle DY_i, DY_j \rangle_{\mathbb{H}})_{1 \le i,j \le d}$. We sometimes write $\Delta_Y := \det \sigma_Y$ for the determinant of the Malliavin matrix. Finally, we set $\mathbb{D}_{k,\infty} = \bigcap_{q \ge 2} \mathbb{D}_{k,q}$.

We start this section by reviewing the theoretical results from [25], which concern the Edgeworth expansion associated with a mixed normal limit. On a filtered Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,1]}, \mathbb{P})$, we consider a one-dimensional functional Z_n, which admits the decomposition

$$Z_n = M_n + r_n N_n, \tag{2.1}$$

where r_n is a deterministic sequence with $r_n \to 0$ and N_n is some tight sequence of random variables (in this paper we will have $r_n = \Delta_n^{1/2}$). We assume that the leading term M_n is a terminal value of some continuous $(\mathcal{F}_t)$-martingale $(M_t^n)_{t \in [0,1]}$, that is, $M_n = M_1^n$. In this paper, we are interested in cases where M_n (and so Z_n) converges stably in law to a mixed normal variable M (stable convergence has been originally introduced in [21]). This means

$$M_n \xrightarrow{d_{\mathrm{st}}} M, \tag{2.2}$$

where the random variable M is defined on an extension $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ of the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and, conditionally on $\mathcal{F}$, M has a normal law with mean 0 and conditional variance C. In this case, we use the notation

$$M \sim MN(0, C).$$

We recall that a sequence of random variables $(Y_n)_{n \in \mathbb{N}}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a metric space E is said to converge stably with limit Y, written

$Y_n \xrightarrow{d_{\mathrm{st}}} Y$, where Y is defined on an extension $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ of the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$, iff for any bounded, continuous function g and any bounded $\mathcal{F}$-measurable random variable X it holds that

$$\mathbb{E}\big[g(Y_n)X\big] \to \overline{\mathbb{E}}\big[g(Y)X\big], \qquad n \to \infty. \tag{2.3}$$

For statistical applications, it is not sufficient to consider the Edgeworth expansion of the law of Z_n. It is much more adequate to study the asymptotic expansion for the pair (Z_n, F_n), where F_n is another functional which converges in probability:

$$F_n \xrightarrow{\mathbb{P}} F.$$

When F_n is a consistent estimator of the conditional variance C (i.e., $F = C$), which is the most important application, we would obtain by the properties of stable convergence:

$$\frac{Z_n}{\sqrt{F_n}} \xrightarrow{d} \mathcal{N}(0, 1).$$

In this case, the asymptotic expansion of the law of (Z_n, F_n) would imply the Edgeworth expansion for the Studentized statistic $Z_n/\sqrt{F_n}$.

We consider the stochastic processes $(M_t)_{t\in[0,1]}$ and $(C^n_t)_{t\in[0,1]}$ with

$$M = M_1, \qquad C_t = \langle M \rangle_t, \qquad C^n_t = \langle M^n \rangle_t, \qquad C_n = \langle M^n \rangle_1. \tag{2.4}$$

Here, the process $(M_t)_{t\in[0,1]}$, defined on $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$, represents the stable limit of the continuous $(\mathcal{F}_t)$-martingale $(M^n_t)_{t\in[0,1]}$, while C^n denotes the quadratic variation process associated with M^n. Now, let us set

$$\widehat{C}_n = r_n^{-1}(C_n - C), \qquad \widehat{F}_n = r_n^{-1}(F_n - F). \tag{2.5}$$

Apart from various technical conditions, presented in the Section 2.3, our main assumption will be the following:

(A1) (i) $(M^n_\cdot, N_n, \widehat{C}_n, \widehat{F}_n) \xrightarrow{d_{\mathrm{st}}} (M_\cdot, N, \widehat{C}, \widehat{F})$.
(ii) $M_t \sim MN(0, C_t)$.

In order to present an Edgeworth expansion for the pair (Z_n, F_n), we need to define two *random symbols* $\underline{\sigma}$ and $\overline{\sigma}$, which play a crucial role in what follows. We call $\underline{\sigma}$ the adaptive (or classical) random symbol and $\overline{\sigma}$ the anticipative random symbol.

2.1. *The classical random symbol* $\underline{\sigma}$. Let $\widetilde{\mathcal{F}} = \mathcal{F} \vee \sigma(M)$. We take a random function $\widetilde{C}(z)$ such that

$$\widetilde{C}(M) = \mathbb{E}[\widehat{C}|\widetilde{\mathcal{F}}]. \tag{2.6}$$

In the same way we define the variables $\widetilde{F}(z)$ and $\widetilde{N}(z)$ such that

$$\widetilde{F}(M) = \mathbb{E}[\widehat{F}|\widetilde{\mathcal{F}}], \qquad \widetilde{N}(M) = \mathbb{E}[N|\widetilde{\mathcal{F}}].$$

REMARK 2.1. Due to Assumption (A1)(i), we have the pointwise stable convergence $(M_n, N_n, \widehat{C}_n, \widehat{F}_n) \xrightarrow{d_{\text{st}}} (M, N, \widehat{C}, \widehat{F})$. Usually, the limit $(M, N, \widehat{C}, \widehat{F})$ is jointly mixed normal with expectation $\mu \in \mathbb{R}^4$ (and $\mu_1 = 0$) and conditional covariance matrix $\Sigma \in \mathbb{R}^{4\times 4}$. We deduce, for instance, that

$$\widetilde{N}(M) = \mu_2 + \frac{\Sigma_{12}}{\Sigma_{11}} M.$$

Consequently, we have $\widetilde{N}(z) = \mu_2 + \frac{\Sigma_{12}}{\Sigma_{11}} z$. The quantities $\widetilde{C}(z)$ and $\widetilde{F}(z)$ are computed similarly.

Now, the adaptive random symbol $\underline{\sigma}$ is defined by

$$\underline{\sigma}(z, \mathrm{i}u, \mathrm{i}v) = \frac{(\mathrm{i}u)^2}{2} \widetilde{C}(z) + \mathrm{i}u\widetilde{N}(z) + \mathrm{i}v\widetilde{F}(z). \tag{2.7}$$

Notice that $\underline{\sigma}$ is a second-order polynomial in $(\mathrm{i}u, \mathrm{i}v)$. The random symbol $\underline{\sigma}(z, \mathrm{i}u, \mathrm{i}v)$ is called classical, because it appears already in the martingale expansion in the central limit theorem [22, 23], that is, in the case where C is a deterministic constant. In contrast, the anticipative random symbol $\overline{\sigma}$, which will be defined in the next subsection, is due to the mixed normality of the limit. In fact, it disappears if C is nonrandom.

2.2. *The anticipative random symbol* $\overline{\sigma}$. The second random symbol $\overline{\sigma}$ is given in an implicit way. Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ with $|\alpha| = \alpha_1 + \alpha_2$. Set

$$\partial^\alpha = \mathrm{i}^{-|\alpha|} d^\alpha.$$

We define the quantity Φ_n by

$$\Phi_n(u, v) = \mathbb{E}\left[\exp\left(-\frac{u^2}{2}C + \mathrm{i}vF\right)\left(\mathcal{E}(\mathrm{i}uM^n)_1 - 1\right)\psi_n\right],$$

where $\mathcal{E}(H)_t$ denotes the exponential martingale associated with a continuous martingale H, that is,

$$\mathcal{E}(H)_t = \exp\left(H_t - \frac{1}{2}\langle H\rangle_t\right) = 1 + \int_0^t \mathcal{E}(H)_s \, dH_s,$$

and the random variable ψ_n plays a role of a threshold that ensures the integrability of the above expression, whose precise definition is given in Section 2.3 below. In particular, ψ_n converges to 1 in probability.

REMARK 2.2. Recalling the definition of the exponential martingale $\mathcal{E}(\mathrm{i}uM^n)$, we observe that $\Phi_n(u, v)$ is closely related to the joint characteristic function of (M_n, F). Condition (A5) of Section 2.3 specifies the tail behavior of $\Phi_n(u, v)$. When $C = F$ is deterministic, that is, we are in the framework of a standard central limit theorem, the truncation ψ_n can be dropped and we obtain that $\Phi_n(u, v) = 0$, since $(\mathcal{E}(\mathrm{i}uM^n)_t - 1)_{t\in[0,1]}$ is a martingale with mean 0.

Now, we assume that the limit $\Phi^\alpha(u,v) := \lim_{n\to\infty} r_n^{-1}\partial^\alpha \Phi_n(u,v)$ (if it exists) admits the representation

$$\Phi^\alpha(u,v) = \partial^\alpha \mathbb{E}\left[\exp\left(-\frac{u^2}{2}C + \mathtt{i}vF\right)\overline{\sigma}(\mathtt{i}u,\mathtt{i}v)\right], \quad (u,v)\in\mathbb{R}^2, \alpha\in\mathbb{Z}_+^2, \tag{2.8}$$

where the random symbol $\overline{\sigma}(\mathtt{i}u,\mathtt{i}v)$ has the form

$$\overline{\sigma}(\mathtt{i}u,\mathtt{i}v) = \sum_j \overline{c}_j(\mathtt{i}u)^{m_j}(\mathtt{i}v)^{n_j} \qquad \text{(finite sum)} \tag{2.9}$$

with $\overline{c}_j \in \mathbb{D}_{l,\infty}$ for a certain $l\in\mathbb{N}$ [cf. assumption (A4) in Section 2.3]. We remark that $\overline{\sigma}(\mathtt{i}u,\mathtt{i}v)$ is a polynomial with random coefficients.

2.3. *Assumptions and truncation functionals*. In this subsection, we state the conditions $(A2)_\ell$, (A3), $(A4)_{\ell,\mathfrak{n}}$, (A5) and $(A6)_\ell$ required in Theorem 2.3 below. Localization techniques will be essential to carry out the computations rigorously. For this purpose, we need two auxiliary functionals s_n and $\tilde{\xi}_n$, which will be introduced in details later.

$(A2)_\ell$ (i) $F\in\mathbb{D}_{\ell+1,\infty}$ and $C\in\mathbb{D}_{\ell,\infty}$.

(ii) $M_n\in\mathbb{D}_{\ell+1,\infty}$, $F_n\in\mathbb{D}_{\ell+1,\infty}$, $C_n\in\mathbb{D}_{\ell,\infty}$, $N_n\in\mathbb{D}_{\ell+1,\infty}$ and $s_n\in\mathbb{D}_{\ell,\infty}$. Moreover,

$$\sup\{\|M_n\|_{\ell+1,p} + \|\widehat{C}_n\|_{\ell,p} + \|\widehat{F}_n\|_{\ell+1,p} + \|N_n\|_{\ell+1,p} + \|s_n\|_{\ell,p}\} < \infty$$

for every $p\ge 2$.

(A3) (i) $\mathbb{P}[\Delta_{(M_n,F)} < s_n] = O(r_n^{1+\kappa})$ for some positive constant κ. Recall that $\Delta_{(M_n,F)}$ denotes the determinant of the Malliavin matrix of (M_n,F).

(ii) For every $p\ge 2$,

$$\limsup_{n\to\infty} \mathbb{E}[s_n^{-p}] < \infty,$$

and moreover $C^{-1}\in\mathbb{L}^\infty$.

$(A4)_{\ell,\mathfrak{n}}$ (i) $\widetilde{C}(z)$, $\widetilde{N}(z)$ and $\widetilde{F}(z)$ are random polynomials with coefficients in $\mathbb{D}_{4,\infty}$.

(ii) The random symbol $\overline{\sigma}$, which satisfies (2.8), admits a representation

$$\overline{\sigma}(\mathtt{i}u,\mathtt{i}v) = \sum_j \overline{c}_j(\mathtt{i}u)^{m_j}(\mathtt{i}v)^{n_j} \qquad \text{(finite sum)},$$

where the numbers $n_j\in\mathbb{N}$ satisfy $n_j\le\mathfrak{n}$ and $\bar{c}_j\in\mathbb{D}_{\ell,\infty}$.

Let $\Phi_n^\alpha = \partial^\alpha\Phi_n$. We remark that the functional Φ_n^α depends on a truncation functional ψ_n we will specify later.

(A5) For some $q \in (1/3, 1/2)$,

$$\sup_n \sup_{(u,v)\in\Lambda_n^0(2,q)} |(u,v)|^3 r_n^{-1} |\Phi_n^{\alpha}(u,v)| < \infty$$

for every $\alpha \in \mathbb{Z}_+^2$, where $\Lambda_n^0(2,q) = \{(u,v) \in \mathbb{R}^2; |(u,v)| \le r_n^{-q}\}$.

(A6)$_\ell$ $\tilde{\xi}_n \in \mathbb{D}_{\ell,\infty}$, $\sup_n \|\tilde{\xi}_n\|_{\ell,p} < \infty$ for every $p > 1$, and $P[|\tilde{\xi}_n| > 1/2] = O(r_n^{1+\kappa})$ as $n \to \infty$ for some positive constant κ.

Truncation techniques will play an essential role in derivation of the asymptotic expansion. We shall construct a truncation functional ψ_n below, which has been introduced in the definition of $\Phi_n(u,v)$. Let $\psi \in C^\infty([0,1])$ be a real-valued function with $\psi(x) = 1$ for $|x| \le 1/2$ and $\psi(x) = 0$ for $|x| \ge 1$. Recalling that $C_1 = C$, we define a random variable ξ_n by

$$\xi_n = 10^{-1} r_n^{-2c} (C_1^n - C)^2 + 2[1 + 4\Delta_{(M_1^n, C)} s_n^{-1}]^{-1} + r_n^{2c_1} C^2, \tag{2.10}$$

where $c_1 > 0$, c satisfies $2q < c < 1$ and the constant q is given in (A5). Define the 2×2 random matrix R_n' by

$$R_n' = \sigma_{Q_n}^{-1} \big(r_n \langle DQ_n, DR_n \rangle_{\mathbb{H}} + r_n \langle DR_n, DQ_n \rangle_{\mathbb{H}} + r_n^2 \langle DR_n, DR_n \rangle_{\mathbb{H}} \big),$$

where $Q_n = (M_n, F)$ and $R_n = (N_n, \widehat{F}_n)$. Obviously,

$$\sigma_{(Z_n, F_n)} = \sigma_{Q_n} \big(I_2 + R_n' \big), \tag{2.11}$$

where I_2 is the 2×2 identity matrix. Let $\xi_n' = r_n^{-1} |R_n'|^2$. We define ψ_n by

$$\psi_n = \psi(\xi_n) \psi(\xi_n') \psi(\tilde{\xi}_n). \tag{2.12}$$

We remark that the random variables appearing in the definition of ξ_n and ξ_n' are bounded under truncation. Since we later deal with exponentials of these variables, we need to exclude their large values to obtain finite expectations. This is exactly the intuition behind the definition of the truncation functional ψ_n. A priori the meaning of the random variables s_n and $\tilde{\xi}_n$, which enter the formulas (2.10) and (2.12), respectively, is not clear at this stage. The variable $\tilde{\xi}_n$ will play again a role of truncation, which is proof specific. The term s_n will be set up in Section 3.2.

2.4. *The asymptotic expansion of the density of* (Z_n, F_n). We set

$$\sigma = \underline{\sigma} + \overline{\sigma}. \tag{2.13}$$

We remark that due to the definition of $\underline{\sigma}$ and $\overline{\sigma}$ the random symbol σ admits the representation

$$\sigma(z, \mathrm{i}u, \mathrm{i}v) = \sum_j c_j(z) (\mathrm{i}u)^{m_j} (\mathrm{i}v)^{n_j} \qquad \text{(finite sum)} \tag{2.14}$$

for some $c_j(z) \in \bigcap_{p>1} \mathbb{L}^p$. The approximative density of (Z_n, F_n) is defined as

$$(2.15)\quad \begin{aligned} p_n(z,x) &= \mathbb{E}[\phi(z;0,C)|F=x]p^F(x) \\ &\quad + r_n \sum_j (-d_z)^{m_j}(-d_x)^{n_j}\big(\mathbb{E}[c_j(z)\phi(z;0,C)|F=x]p^F(x)\big), \end{aligned}$$

where p^F denotes the density of F and $\phi(\cdot;a,b^2)$ is the density of $\mathcal{N}(a,b^2)$-distribution. Obviously, we will require certain regularity conditions in terms of Malliavin calculus in order to validate the existence of the density p^F and the derivatives in (2.15) as well as to validate the estimate of the approximation error.

For any integrable function $h : \mathbb{R}^2 \to \mathbb{R}$, we set

$$(2.16)\quad \Delta_n(h) = \left| \mathbb{E}[h(Z_n, F_n)] - \int h(z,x) p_n(z,x)\, dz\, dx \right|.$$

The following theorem is Theorem 2 of [25] (see also [24]).

THEOREM 2.3. *Let $\ell = 5 \vee 2[(\mathfrak{n}+3)/2]$ with $\mathfrak{n} = \max_j n_j$, where the integers n_j are defined at* (2.14). *Define the set $\mathcal{E}(K,\gamma) = \{h : \mathbb{R}^2 \to \mathbb{R} | h$ is measurable and $|h(z,x)| \le K(|z| + |x|)^\gamma\}$ for $K, \gamma > 0$. Then under the assumptions* (A1), (A2)$_\ell$, (A3), (A4)$_{\ell,\mathfrak{n}}$, (A5) *and* (A6)$_\ell$,

$$(2.17)\quad \sup_{h \in \mathcal{E}(K,\gamma)} \Delta_n(h) = o(r_n).$$

3. Functionals of Brownian motion with random weights. In this section, we consider general weighted functionals of a Brownian motion with weights depending on a given stochastic differential equation, and we shall derive an expansion formula. Here, the stochastic second-order term N_n is still absent. In later sections, we will meet an expansion with nonvanishing N_n when considering the power variations of diffusion processes. However, we will solve two essential problems in this general but concrete situation, that is, identification of the anticipative random symbol in this model, and proof of the nondegeneracy of the functionals.

On a given Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, \mathbb{P})$ we consider a 1-dimensional stochastic differential equation of the form

$$(3.1)\quad dX_t = b^{[1]}(X_t)\, dW_t + b^{[2]}(X_t)\, dt,$$

where X_0 is a bounded random variable, $b^{[1]}, b^{[2]} : \mathbb{R} \to \mathbb{R}$ are two deterministic functions and W is a standard Brownian motion. Sometimes we will use the notation

$$b_t^{[1]} = b^{[1]}(X_t), \qquad b_t^{[2]} = b^{[2]}(X_t).$$

The somewhat unusual notation $b^{[1]}$, $b^{[2]}$ refers to the fact that the diffusion term $b^{[1]}$ dominates the drift term $b^{[2]}$ in all asymptotic expansions (so $b^{[1]}$ is the first-order term and $b^{[2]}$ is the second-order term). Under standard smoothness conditions, the processes $b_t^{[k]}$, $k = 1, 2$, also satisfy a SDE of the type (3.1) by Itô's formula; in this case we denote by $b_t^{[k.1]}$ (resp., $b_t^{[k.2]}$) the diffusion term (resp., the drift term) of $b_t^{[k]}$. In the same manner, we introduce the processes $b_t^{[k_1\cdots k_d]}$, $k_1,\dots,k_d = 1, 2$, recursively. We will assume that $b^{[1]}$ and $b^{[2]}$ are in $C^\infty_{b,1}(\mathbb{R})$ (the set of smooth functions such that each derivative of positive order is bounded).

In this section, we consider weighted functionals of the Brownian motion of the type

$$M_n = \Delta_n^{1/2} \sum_{i=1}^{1/\Delta_n} a(X_{t_{i-1}}) f\left(\frac{\Delta_i^n W}{\sqrt{\Delta_n}}\right), \qquad \Delta_i^n W = W_{i\Delta_n} - W_{(i-1)\Delta_n}, \tag{3.2}$$

where $a \in C^\infty_p(\mathbb{R})$ and $f \in C^{11}_p(\mathbb{R})$. Since f has polynomial growth, it holds that $\mathbb{E}[f^2(Z)] < \infty$ with $Z \sim \mathcal{N}(0,1)$. Consequently, the function f exhibits a Hermite expansion. We assume that the function f has the form

$$f(x) = \sum_{k=2}^\infty \lambda_k H_k(x) \qquad \text{in } \mathbb{L}^2\big(\mathbb{R}; \phi(x;0,1)\,dx\big) \tag{3.3}$$

with $\lambda_k = \mathbb{E}[f(Z)H_k(Z)]/k!$ and $Z \sim \mathcal{N}(0,1)$, where H_k is the kth Hermite polynomial, that is, $H_0(x) = 1$ and

$$H_k(x) = (-1)^k e^{\frac{x^2}{2}} \frac{d^k}{dx^k}\big(e^{-\frac{x^2}{2}}\big), \qquad k \ge 1.$$

In particular, the Hermite rank of the function f is at least 2 and $\mathbb{E}[f(Z)] = 0$ for $Z \sim \mathcal{N}(0,1)$. We will see later that the Hermite rank 1 would not lead to the asymptotic mixed normal distribution with conditional mean 0. In this section, we will consider

$$F_n = \Delta_n \operatorname{Var}\big[f(Z)\big] \sum_{i=1}^{1/\Delta_n} a^2(X_{t_{i-1}}), \tag{3.4}$$

which is a Riemann sum approximation of $C = \langle M \rangle_1$, as the reference variable. The second convergence of the following proposition is a straightforward consequence of [2], Section 8.

PROPOSITION 3.1. *It holds that*

$$F_n \xrightarrow{\mathbb{P}} C = \operatorname{Var}\big[f(Z)\big] \int_0^1 a^2(X_s)\,ds \quad \textit{and} \quad \Delta_n^{-1/2}(F_n - C) \xrightarrow{\mathbb{P}} 0.$$

3.1. *A limit theorem for $(M_n, \widehat{C}_n)$ and the adaptive random symbol.* First, we note that for $H = f(\frac{\Delta_i^n W}{\sqrt{\Delta_n}})$ it holds

$$H = \int_0^1 \mathbb{E}[D_s H | \mathcal{F}_s]\,dW_s,$$

which is the Clark–Ocone formula. Consequently, we deduce the identity

$$f\left(\frac{\Delta_i^n W}{\sqrt{\Delta_n}}\right) = \Delta_n^{-1/2} \int_{t_{i-1}}^{t_i} \mathbb{E}\left[f'\left(\frac{\Delta_i^n W}{\sqrt{\Delta_n}}\right)\Big|\mathcal{F}_s\right] dW_s.$$

Thus, we naturally have a continuous square-integrable $(\mathcal{F}_t)$-martingale $M^n = (M_t^n)_{t\in[0,1]}$ given by

$$(3.5)\quad \begin{aligned} M_t^n &= \int_0^t b_s^n \, dW_s, \\ b_s^n &= a(X_{\Delta_n[s/\Delta_n]})\mathbb{E}\left[f'\left(\frac{W_{\Delta_n[s/\Delta_n]+\Delta_n} - W_{\Delta_n[s/\Delta_n]}}{\sqrt{\Delta_n}}\right)\Big|\mathcal{F}_s\right] \end{aligned}$$

and we deduce that

$$(3.6)\quad \begin{aligned} C_t^n &= \langle M^n \rangle_t \\ &= \int_0^t a^2(X_{\Delta_n[s/\Delta_n]})\mathbb{E}^2\left[f'\left(\frac{W_{\Delta_n[s/\Delta_n]+\Delta_n} - W_{\Delta_n[s/\Delta_n]}}{\sqrt{\Delta_n}}\right)\Big|\mathcal{F}_s\right] ds. \end{aligned}$$

From this identity, we obtain the convergence

$$C_t^n \xrightarrow{\mathbb{P}} C_t = \mathrm{Var}[f(Z)] \int_0^t a^2(X_s)\, ds.$$

The latter follows from Theorem A.1 in the Appendix applied to the function $g : \mathbb{R} \times C([0,1]) \to \mathbb{R}_+$ defined via

$$g(z,w) := z^2 \int_0^1 \mathbb{E}^2[f'(U_s + w(s))]\, ds \qquad \text{with } U_s \sim \mathcal{N}(0, 1-s).$$

By Theorem A.2 of the Appendix, we deduce the following result.

PROPOSITION 3.2. *It holds that*

$$(M_n, \widehat{C}_n) \xrightarrow{d_{\mathrm{st}}} (M, \widehat{C}) \sim MN(0, \Sigma) \qquad \textit{with } \Sigma = \int_0^1 \Sigma_s \, ds,$$

where the matrix Σ_s *is defined by*

$$\Sigma_s^{11} = \mathrm{Var}[f(Z)]a^2(X_s), \qquad \Sigma_s^{22} = \Gamma_1 a^4(X_s), \qquad \Sigma_s^{12} = \Sigma_s^{21} = \Gamma_2 a^3(X_s),$$

with

$$\begin{aligned} \Gamma_1 &= \mathrm{Var}\left[\int_0^1 \mathbb{E}^2[f'(W_1)|\mathcal{F}_s]\, ds\right], \\ \Gamma_2 &= \mathrm{Cov}\left[f(W_1), \int_0^1 \mathbb{E}^2[f'(W_1)|\mathcal{F}_s]\, ds\right]. \end{aligned}$$

Notice that the stable convergence in the above proposition does not hold if f has Hermite rank 1, since in this case the process $(v_s)_{s\geq 0}$ defined in Theorem A.2 is not identically 0. As in the previous section, we immediately obtain the adaptive random symbol

$$\underline{\sigma}(z, \mathtt{i}u, \mathtt{i}v) = \frac{4z(\mathtt{i}u)^2 \int_0^1 a^3(X_s)\,ds}{3\,\mathrm{Var}[f(Z)]\int_0^1 a^2(X_s)\,ds} =: z(\mathtt{i}u)^2 \mathcal{C}_1. \tag{3.7}$$

3.2. *Setting s_n.* We need to define the functionals s_n (and consequently ξ_n) to go further. We set $r_n = \Delta_n^{1/2}$, $\beta(x) = \mathrm{Var}[(f(Z))]a(x)^2$ with $Z \sim \mathcal{N}(0,1)$ and $a_t = a(X_t)$, $\beta_t = \beta(X_t)$. Let

$$\sigma_{22}(t) = \int_0^t \left[\int_r^1 \beta_s' D_r X_s\,ds\right]^2 dr. \tag{3.8}$$

Define a matrix $\tilde{\sigma}(n,t)$ by

$$\tilde{\sigma}(n,t) = \begin{bmatrix} \tilde{\sigma}_{11}(n,t) & \tilde{\sigma}_{12}(n,t) \\ \tilde{\sigma}_{12}(n,t) & \sigma_{22}(t) \end{bmatrix}$$

with

$$\begin{aligned}\tilde{\sigma}_{11}(n,t) = \Delta_n &\sum_{i:t_i\leq t} \left[a_{t_{i-1}} f'(\Delta_n^{-1/2}\Delta_i^n W)\right]^2 \\ &+ \sum_{i:t_i\leq t}\int_{t_{i-1}}^{t_i}\left[\Delta_n^{1/2}\sum_{k=i+1}^n a'_{t_{k-1}} f(\Delta_n^{-1/2}\Delta_k^n W)1_{\{t_k\leq t\}} D_r X_{t_{k-1}}\right]^2 dr\end{aligned}$$

and

$$\begin{aligned}\tilde{\sigma}_{12}(n,t) = \sum_{i:t_i\leq t}\int_{t_{i-1}}^{t_i}&\left(\left[\Delta_n^{1/2}\sum_{k=i+1}^n a'_{t_{k-1}} f(\Delta_n^{-1/2}\Delta_k^n W)1_{\{t_k\leq t\}} D_r X_{t_{k-1}}\right]\right. \\ &\left.\times \int_r^1 \beta_s' D_r X_s\,ds\right) dr\end{aligned}$$

for $t \in \Pi^n$. Define s_n by

$$s_n = \frac{1}{2}\det\left[\tilde{\sigma}\left(n, \frac{1}{2}\right) + \psi\left(\frac{m_n}{2\mathsf{c}_1}\right) I_2\right], \tag{3.9}$$

where I_2 is the 2×2 unit matrix, $\psi : \mathbb{R} \to [0,1]$ is a smooth function such that $\psi(x) = 1$ if $|x| \leq 1/2$ and $\psi(x) = 0$ if $|x| \geq 1$, c_1 is a positive number, and

$$m_n = \Delta_n \sum_{i=1}^{[1/2\Delta_n]} \left[f'(\Delta_n^{-1/2}\Delta_i^n W)\right]^2.$$

Let

$$\tilde{\xi}_n = L^* \int_{[0,1]^2} \left(\frac{r_n^{-2q} |C_t^n - C_t - C_s^n + C_s|}{|t-s|^{3/8}} \right)^8 dt\, ds,$$

where L^* is a sufficiently large constant. We will later show that the random variable s_n satisfies assumption (A3). We define ξ_n using s_n as in (2.10) of Section 2.3.

3.3. *Decompositions of the torsion.* In this subsection, we present some preparatory decompositions for the computation of $\overline{\sigma}$. Recall that $f \in C_p^{11}(\mathbb{R})$ and it admits the Hermite expansion $f(x) = \sum_{k=2}^{\infty} \lambda_k H_k(x)$. Consequently, it holds that $\sum_{k=2}^{\infty} k! k^{11} \lambda_k^2 < \infty$.

The martingale $M^n = (M_t)_{t \in [0,1]}$ admits the local chaos expansion

$$\begin{aligned} M_t^n = {} & \Delta_n^{1/2} \sum_{i=1}^{1/\Delta_n} a_{t_{i-1}} \sum_{k=2}^{\infty} k! \lambda_k \Delta_n^{-k/2} \\ & \times \int_{t_{i-1} \wedge t}^{t_i \wedge t} \int_{t_{i-1}}^{s_1} \cdots \int_{t_{i-1}}^{s_{k-1}} dW_{s_k} \cdots dW_{s_2}\, dW_{s_1}. \end{aligned} \tag{3.10}$$

Obviously, each infinite sum in (3.10) is well defined as an $\mathbb{L}^2$-limit when $k \to \infty$. Since

$$H_k\big(\Delta_n^{-1/2} \Delta_i^n W\big) = k! \Delta_n^{-k/2} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{s_1} \cdots \int_{t_{i-1}}^{s_{k-1}} dW_{s_k} \cdots dW_{s_2}\, dW_{s_1},$$

we find (3.2) again. Now, we set

$$e_t^n(u) = \mathcal{E}\big(\mathtt{i} u M^n\big)_t, \qquad \Psi(u,v) = \exp\left(\left(-\frac{u^2}{2} + \mathtt{i} v\right) C\right). \tag{3.11}$$

Now, we recall the integration by parts (or duality) formula (see, e.g., [20]): For any $w \in \operatorname{Dom} \delta$ and any smooth random variable $Y \in \mathbb{D}_{1,2}$, it holds that

$$\mathbb{E}\big[\delta(w) Y\big] = \mathbb{E}\big[\langle w, DY \rangle_{\mathbb{H}}\big]. \tag{3.12}$$

For each $n \in \mathbb{N}$, there exists a positive constant a_n such that $\mathsf{a}_n \max\{C, C_1^n\} < 1/2$ whenever $\psi_n > 0$ [cf. (2.12)]. Thus, on the event $\{\psi_n > 0\}$, $e_t^n(u) = \bar{e}_t^n(u)$ ($t \in [0,1], u \in \mathbb{R}$) and $\Psi(u,v) = \bar{\Psi}_n(u,v)$ ($u, v \in \mathbb{R}$) [$\Psi(u,v)$ is defined before (2.8)], where

$$\bar{e}_t^n(u) = e_t^n(u) \psi\big(\mathsf{a}_n C_1^n\big) \quad \text{and} \quad \bar{\Psi}_n(u,v) = \Psi(u,v) \psi(\mathsf{a}_n C).$$

By definition,

$$\bar{e}_\cdot^n(u) \int_{t_{i-1}}^{\cdot} \cdots \int_{t_{i-1}}^{s_{k-1}} dW_{s_k} \cdots dW_{s_2} \in \operatorname{Dom} \delta$$

and $\bar{\Psi}_n(u,v) \in \mathbb{D}_{1,p}$ for $p > 1$. Therefore, for

$$\mathbb{Y} = \mathbb{E}\Big[\int_{t_{i-1}}^{t_i} e^n_{s_1}(u)\Big(\int_{t_{i-1}}^{s_1} \cdots \int_{t_{i-1}}^{s_{k-1}} dW_{s_k} \cdots dW_{s_2}\Big) dW_{s_1} \Psi(u,v)\psi_n a_{t_{i-1}}\Big],$$

we obtain

$$\mathbb{Y} = \mathbb{E}\Big[\int_0^1 \bar{e}^n_{s_1}(u)\Big(\int_{t_{i-1}}^{s_1} \cdots \int_{t_{i-1}}^{s_{k-1}} dW_{s_k} \cdots dW_{s_2}\Big) \times 1_{I_i^n}(s_1) D_{s_1}\big(\bar{\Psi}_n(u,v)\psi_n a_{t_{i-1}}\big)\, ds_1\Big]$$

by (3.12). Moreover, since $\bar{e}^n_t(u) \in \mathbb{D}_{1,p}$ for $p > 1$ and $D_{s_1}(\bar{\Psi}_n(u,v)\psi_n a_{t_{i-1}}) \in \mathbb{D}_{1,p}$ for $p > 1$ as well as

$$\int_{t_{i-1}}^{\cdot} \cdots \int_{t_{i-1}}^{s_{k-1}} dW_{s_k} \cdots dW_{s_3} \in \operatorname{Dom} \delta,$$

we also have

$$\mathbb{Y} = \int_{t_{i-1}}^{t_i} \mathbb{E}\Big[\int_{t_{i-1}}^{s_1} \int_{t_{i-1}}^{s_2} \Big(\int_{t_{i-1}}^{s_3} \cdots \int_{t_{i-1}}^{s_{k-1}} dW_{s_k} \cdots dW_{s_4}\Big) dW_{s_3} \times D_{s_2}\big(\bar{e}^n_{s_1}(u) D_{s_1}\big(\bar{\Psi}_n(u,v)\psi_n a_{t_{i-1}}\big)\big)\, ds_2\Big] ds_1.$$

In what follows, we will identify $e^n_t(u)$ with $\bar{e}^n_t(u)$ and $\Psi(u,v)$ with $\bar{\Psi}_n(u,v)$, respectively, and apply such procedures for taming exponential type functionals, without explicitly mentioned.

Since the infinite sums in k of (3.10) are also limits of $\mathbb{L}^2$-martingales, we can validate the exchange of the limit and the sum, and then use the duality between the Skorokhod integral and the derivative operator D at (3.12) to carry out

$$\begin{aligned}
&\sum_{i=1}^{1/\Delta_n} \mathbb{E}\Big[\int_{t_{i-1}}^{t_i} e^n_t(u)\, dM^n_t \Psi(u,v)\psi_n\Big] \\
&\quad = \Delta_n^{1/2} \sum_{i=1}^{1/\Delta_n} \sum_{k=2}^{\infty} k!\lambda_k \Delta_n^{-k/2} \\
&\qquad \times \mathbb{E}\Big[\int_{t_{i-1}}^{t_i} e^n_{s_1}(u)\Big(\int_{t_{i-1}}^{s_1} \cdots \int_{t_{i-1}}^{s_{k-1}} dW_{s_k} \cdots dW_{s_2}\Big) dW_{s_1} \Psi(u,v)\psi_n a_{t_{i-1}}\Big] \\
&\quad = \Delta_n^{1/2} \sum_{i=1}^{1/\Delta_n} \sum_{k=2}^{\infty} k!\lambda_k \Delta_n^{-k/2} \int_{t_{i-1}}^{t_i} ds_1 \int_{t_{i-1}}^{s_1} ds_2 \\
&\qquad \times \mathbb{E}\Big[\int_{t_{i-1}}^{s_2} \Big(\int_{t_{i-1}}^{s_3} \cdots \int_{t_{i-1}}^{s_{k-1}} dW_{s_k} \cdots dW_{s_4}\Big) \\
&\qquad \times dW_{s_3} D_{s_2}\big(e^n_{s_1}(u) D_{s_1}\big(\Psi(u,v)\psi_n a_{t_{i-1}}\big)\big)\Big].
\end{aligned} \tag{3.13}$$

Applying the duality once again, we obtain the decomposition

$$\begin{aligned}
\mathfrak{A}_n(u,v) &:= \Delta_n^{-1/2}\sum_{i=1}^{1/\Delta_n}\mathbb{E}\Big[\int_{t_{i-1}}^{t_i} e_t^n(u)\,dM_t^n\Psi(u,v)\psi_n\Big] \\
&= 2\sum_{i=1}^{1/\Delta_n}\lambda_2\Delta_n^{-1}\int_{t_{i-1}}^{t_i} ds_1\int_{t_{i-1}}^{s_1} ds_2\mathbb{E}\big[D_{s_2}\big(e_{s_1}^n(u)D_{s_1}\big(\Psi(u,v)\psi_n a_{t_{i-1}}\big)\big)\big] \\
&\quad + \sum_{i=1}^{1/\Delta_n}\sum_{k=3}^{\infty} k!\lambda_k\Delta_n^{-k/2}\int_{t_{i-1}}^{t_i} ds_1\int_{t_{i-1}}^{s_1} ds_2\int_{t_{i-1}}^{s_2} ds_3 \\
&\quad \times\mathbb{E}\Big[\int_{t_{i-1}}^{s_3}\cdots\int_{t_{i-1}}^{s_{k-1}} dW_{s_k}\cdots dW_{s_4} \\
&\quad \times D_{s_3}\big\{D_{s_2}\big(e_{s_1}^n(u)D_{s_1}\big(\Psi(u,v)\psi_n a_{t_{i-1}}\big)\big)\big\}\Big] \\
&= \ddot{\mathfrak{A}}_n(u,v)+\dddot{\mathfrak{A}}_n(u,v),
\end{aligned}$$

where

$$\ddot{\mathfrak{A}}_n(u,v) = 2\sum_{i=1}^{1/\Delta_n}\lambda_2\Delta_n^{-1}\int_{t_{i-1}}^{t_i} ds_1\int_{t_{i-1}}^{s_1} ds_2\mathbb{E}\big[D_{s_2}\big(e_{s_1}^n(u)D_{s_1}\big(\Psi(u,v)\psi_n a_{t_{i-1}}\big)\big)\big],$$

$$\begin{aligned}
\dddot{\mathfrak{A}}_n(u,v) &= \sum_{i=1}^{1/\Delta_n}\Delta_n^{-3/2}\int_{t_{i-1}}^{t_i} ds_1\int_{t_{i-1}}^{s_1} ds_2\int_{t_{i-1}}^{s_2} ds_3 \\
&\quad\times\mathbb{E}\Big[\Big(\sum_{k=3}^{\infty}k!\lambda_k\Delta_n^{-(k-3)/2}\int_{t_{i-1}}^{s_3}\cdots\int_{t_{i-1}}^{s_{k-1}} dW_{s_k}\cdots dW_{s_4}\Big) \\
&\quad\times D_{s_3}\big\{D_{s_2}\big(e_{s_1}^n(u)D_{s_1}\big(\Psi(u,v)\psi_n a_{t_{i-1}}\big)\big)\big\}\Big].
\end{aligned}$$

Here, we used three times Malliavin differentiability of the objects. We remark that the first term $\ddot{\mathfrak{A}}_n(u,v)$, which is associated with the second-order Wiener chaos, is a dominating quantity, while $\dddot{\mathfrak{A}}_n(u,v)$ will turn out to be negligible.

3.4. *Identification of the anticipative random symbol.* We shall specify the limit of $\mathfrak{A}_n(u,v)$. First,

$$\begin{aligned}
\big|\dddot{\mathfrak{A}}_n(u,v)\big| &\le \sum_{i=1}^{1/\Delta_n}\Delta_n^{-3/2}\int_{t_{i-1}}^{t_i} ds_1\int_{t_{i-1}}^{s_1} ds_2\int_{t_{i-1}}^{s_2} ds_3 \\
&\quad\times\Big\|\sum_{k=3}^{\infty}k!\lambda_k\Delta_n^{-(k-3)/2}\int_{t_{i-1}}^{s_3}\cdots\int_{t_{i-1}}^{s_{k-1}} dW_{s_k}\cdots dW_{s_4}\Big\|_{\mathbb{L}^2}
\end{aligned}$$

$$\times \big\| D_{s_3}\big\{D_{s_2}\big(e^n_{s_1}(u) D_{s_1}\big(\Psi(u,v)\psi_n a_{t_{i-1}}\big)\big)\big\}\big\|_{\mathbb{L}^2}$$
$$\le \frac{\Delta_n^{1/2}}{6}\sqrt{\sum_{k=3}^{\infty} k!k^3\lambda_k^2}$$
$$\times \sup_{\substack{n\in\mathbb{N}, s_1,s_2,s_3\in[0,1]\\ t_{i-1}<s_3<s_2<s_1\le t_i}} \big\| D_{s_3}\big\{D_{s_2}\big(e^n_{s_1}(u) D_{s_1}\big(\Psi(u,v)\psi_n a_{t_{i-1}}\big)\big)\big\}\big\|_{\mathbb{L}^2} \to 0$$

as $n \to \infty$ for every (u, v), since the above supremum is bounded due to assumption $(A2)_8$, and product and chain rule for the Malliavin derivative.

Next, we will treat $\ddot{\mathfrak{A}}_n(u, v)$. We deform it as $\ddot{\mathfrak{A}}_n(u, v) = \check{\mathfrak{A}}_n(u, v) + \hat{\mathfrak{A}}_n(u, v)$ with

$$\check{\mathfrak{A}}_n(u,v) = 2\sum_{i=1}^{1/\Delta_n} \lambda_2 \Delta_n^{-1}\int_{t_{i-1}}^{t_i} ds_1 \int_{t_{i-1}}^{s_1} ds_2 \mathbb{E}\big[a_{t_{i-1}} e^n_{t_{i-1}}(u) D_{s_2}\big(D_{s_1}\big(\Psi(u,v)\psi_n\big)\big)\big],$$

thanks to $D_s a_{t_{i-1}} = 0$ and $D_s e^n_{t_{i-1}}(u) = 0$ for $s > t_{i-1}$, and

$$\hat{\mathfrak{A}}_n(u,v) = 2\sum_{i=1}^{1/\Delta_n} \lambda_2 \Delta_n^{-1}\int_{t_{i-1}}^{t_i} ds_1 \int_{t_{i-1}}^{s_1} ds_2 \mathbb{E}\big[D_{s_2}\big(e^n_{s_1}(u) - e^n_{t_{i-1}}(u)\big)$$
$$\times D_{s_1}\big(\Psi(u,v)\psi_n a_{t_{i-1}}\big)\big].$$

Then by continuity of $e^n_\cdot(u)$ in $\mathbb{D}_{1,p}$ [see again (A2)], we conclude $\hat{\mathfrak{A}}_n(u, v) \to 0$ as $n \to \infty$ for every (u, v). Since $e^n_{s_1}(u)\Psi(u, v)$ is bounded under truncation by ψ_n or even by its derivative, the $\mathbb{L}^p$-continuity of the objects yields

$$\check{\mathfrak{A}}_n(u,v) \to \lambda_2 \mathbb{E}\bigg[\int_0^1 a_t \exp\bigg(\mathtt{i}uM_t + \frac{1}{2}u^2 C_t\bigg) D_t D_t \Psi(u,v)\, dt\bigg], \tag{3.14}$$

where $D_t D_t \Psi(u, v) = \lim_{s\uparrow t} D_s D_t \Psi(u, v)$. It should be noted that the integrability and this limiting procedure are valid because $D_t D_t \Psi(u, v) = \Psi(u, v)A_t$ with a sum A_t of regular variables, and

$$\operatorname*{ess\,sup}_{\omega} \sup_{t\in[0,1]} \big(C^n_t - C_1\big)1_{\{|\xi_n|<1\}} \le \Delta_n^{c/2} \le 1 < \infty$$

for all n, due to $C^n_t \le C^n_1$ and the construction of the quantity ξ_n in (2.10) of Section 2.3. Furthermore,

$$\lambda_2 \mathbb{E}\bigg[\int_0^1 a_t \exp\bigg(\mathtt{i}uM_t + \frac{1}{2}u^2 C_t\bigg) D_t D_t \Psi(u,v)\, dt\bigg]$$
$$= \lambda_2 \mathbb{E}\bigg[\int_0^1 a_t \mathbb{E}\big[\exp(\mathtt{i}uM_t)|\mathcal{F}\big]\bigg\{\exp\bigg(\frac{1}{2}u^2 C_t\bigg) \times \Psi(u,v)\bigg\} A_t\, dt\bigg]$$
$$= \lambda_2 \mathbb{E}\bigg[\int_0^1 a_t D_t D_t \Psi(u,v)\, dt\bigg].$$

Consequently, for

$$\Phi_n^\alpha(u,v) = \mathtt{i}^{-|\alpha|} d_{(u,v)}^\alpha \mathbb{E}\big[L_1^n(u)\Psi(u,v)\psi_n\big]$$
$$= \mathtt{i}^{-|\alpha|} d_{(u,v)}^\alpha \mathbb{E}\Big[\mathtt{i}u \int_0^1 e_t^n(u)\, dM_t^n \Psi(u,v)\psi_n\Big],$$

where $L_t^n(u) = e_t^n(u) - 1$, we obtain

$$\begin{aligned}\tilde{\Phi}^\alpha(u,v) &= \lim_{n\to\infty} \Delta_n^{-1/2}\Phi_n^\alpha(u,v)\\ &= \lim_{n\to\infty} \mathtt{i}^{-|\alpha|} d_{(u,v)}^\alpha\big(\mathtt{i}u\mathfrak{A}_n(u,v)\big)\\ &= \lambda_2 \mathtt{i}^{-|\alpha|} d_{(u,v)}^\alpha \mathbb{E}\Big[\int_0^1 \mathtt{i}u a_t D_t D_t \Psi(u,v)\, dt\Big]\\ &= \lambda_2 \mathtt{i}^{-|\alpha|} d_{(u,v)}^\alpha \mathbb{E}\Big[\Psi(u,v)\cdot \int_0^1 \mathtt{i}u a_t\Big(\Big(-\frac{u^2}{2}+\mathtt{i}u\Big)^2 (D_tC)^2\\ &\qquad + \Big(-\frac{u^2}{2}+\mathtt{i}u\Big) D_t D_t C\Big)\, dt\Big].\end{aligned}$$

Therefore,

$$\begin{aligned}\overline{\sigma}(\mathtt{i}u,\mathtt{i}v) = \lambda_2 \int_0^1 \mathtt{i}u a_t\Big(\Big(-\frac{u^2}{2}+\mathtt{i}u\Big)^2 (D_tC)^2\\ + \Big(-\frac{u^2}{2}+\mathtt{i}u\Big) D_t D_t C\Big)\, dt.\end{aligned} \tag{3.15}$$

We recall that the process DX_t is given as the solution of the SDE

$$D_s X_t = b^{[1]}(X_s) + \int_s^t \big(b^{[2]}\big)'(X_u) D_s X_u\, du + \int_s^t \big(b^{[1]}\big)'(X_u) D_s X_u\, dW_u$$

for $s \le t$ (and 0 when $s > t$), and

$$\begin{aligned}D_r D_s X_t = \big(b^{[1]}\big)'(X_s) D_r X_s &+ \int_s^t \big(b^{[2]}\big)''(X_u) D_r X_u D_s X_u\, du\\ &+ \int_s^t \big(b^{[2]}\big)'(X_u) D_r D_s X_u\, du\\ &+ \int_s^t \big(b^{[1]}\big)''(X_u) D_r X_u D_s X_u\, dW_u + \int_s^t \big(b^{[1]}\big)'(X_u) D_r D_s X_u\, dW_u\end{aligned}$$

for $r < s \le t$. Then (3.15) implies the identity

$$\begin{aligned}\overline{\sigma}(\mathtt{i}u,\mathtt{i}v) = \mathtt{i}u\lambda_2\Big(\Big(-\frac{u^2}{2}+\mathtt{i}v\Big)^2 \mathrm{Var}^2\big[f(Z)\big]\mathcal{C}_2\\ + \Big(-\frac{u^2}{2}+\mathtt{i}v\Big)\mathrm{Var}\big[f(Z)\big](\mathcal{C}_3+\mathcal{C}_4)\Big)\end{aligned} \tag{3.16}$$

with

$$\mathcal{C}_2 = \int_0^1 a(X_s)\Big(\int_s^1 (a^2)'(X_u) D_s X_u \, du\Big)^2 ds,$$
$$\mathcal{C}_3 = \int_0^1 a(X_s)\Big(\int_s^1 (a^2)''(X_u)(D_s X_u)^2 \, du\Big) ds,$$
$$\mathcal{C}_4 = \int_0^1 a(X_s)\Big(\int_s^1 (a^2)'(X_u) D_s D_s X_u \, du\Big) ds.$$

Now, having obtained the full random symbol $\sigma = \underline{\sigma} + \overline{\sigma}$ and hence the density $p_n(z, x)$ for σ, we can formulate the following statement, which generalizes the results of [24], Theorem 1, on the quadratic form to the weighted power variation of Brownian motion.

THEOREM 3.3. *Let $b^{[1]}, b^{[2]} \in C^\infty_{b,1}(\mathbb{R})$, $a \in C^\infty_p(\mathbb{R})$ and $f \in C^{11}_p(\mathbb{R})$. Let the functional F_n be given by (3.4). Recall the definition $\beta(x) = \mathrm{Var}[f(Z)]a(x)^2$ and $\beta_t = \beta(X_t)$ for a standard normal random variable Z. Assume that the following conditions are satisfied*:

(C1) $\inf_x |b^{[1]}(x)| > 0$ *and* $\inf_x |a(x)| > 0$.
(C2) *For each $x_0 \in \operatorname{supp} \mathcal{L}\{X_0\}$, there exists $k \geq 1$ such that $\beta^{(k)}(x_0) \neq 0$.*

Then for any positive numbers K and γ, it holds that

$$\sup_{h \in \mathcal{E}(K,\gamma)} \left| \mathbb{E}\big[h(M_n, F_n)\big] - \int h(z, x) p_n(z, x) \, dz \, dx \right| = o(\sqrt{\Delta_n})$$

as $n \to \infty$, where the set $\mathcal{E}(K, \gamma)$ was defined in Theorem 2.3.

In the rest of this section, we will prove Theorem 3.3. In this situation, we will verify conditions (A1), $(\mathrm{A2})_\ell$, (A3), $(\mathrm{A4})_{\ell,\mathfrak{n}}$ (A5) and (A6) of Theorem 2.3 for $\ell = 10$. The conditions of Theorem 3.3 trivially imply (A1) and $(\mathrm{A2})_\ell$. We already have $(\mathrm{A4})_{\ell,\mathfrak{n}}$. Condition (A6) is also easy to check. In the following subsections, we concentrate on proving (A3) and (A5).

3.5. *Estimate of the characteristic functions.* We shall now show condition (A5) of Section 2.3 under the assumptions of Theorem 3.3, namely

$$\sup_n \sup_{(u,v) \in \Lambda_n^0(2,q)} \big|(u, v)\big|^3 \Delta_n^{-1/2} \big|\Phi_n^\alpha(u, v)\big| < \infty \tag{3.17}$$

for

$$\Phi_n^\alpha(u, v) = \mathtt{i}^{-|\alpha|} d^\alpha_{(u,v)} \mathbb{E}\big[L_1^n(u)\Psi(u, v)\psi_n\big], \qquad L_t^n(u) = e_t^n(u) - 1.$$

We apply the duality formula twice and use nondegeneracy of the Malliavin matrix of (M_t^n, F) together with that of $C - C_t$, in the expression

$$\Phi_n^{\alpha}(u,v) = \mathtt{i}^{-|\alpha|} d_{(u,v)}^{\alpha} \mathbb{E}\left[\int_0^1 e_t^n(u) d\big(\mathtt{i} u M_t^n\big) \Psi(u,v)\psi_n\right].$$

For this purpose, the representation (3.13) is useful. By the $\mathbb{L}^2$-convergence, we see that

$$\begin{aligned}
&\sum_{i=1}^{1/\Delta_n} \mathbb{E}\left[\int_{t_{i-1}}^{t_i} e_t^n(u)\, dM_t^n \Psi(u,v)\psi_n\right] \\
&\quad = \Delta_n^{-1/2} \sum_{i=1}^{1/\Delta_n} \int_{t_{i-1}}^{t_i} ds_1 \int_{t_{i-1}}^{s_1} ds_2 \\
&\qquad \times \mathbb{E}\Bigg[\sum_{k=2}^{\infty} k!\lambda_k \Delta_n^{-(k-2)/2} \int_{t_{i-1}}^{s_2} \int_{t_{i-1}}^{s_3} \cdots \\
&\qquad \times \int_{t_{i-1}}^{s_{k-1}} dW_{s_k} \cdots dW_{s_4}\, dW_{s_3} a_{t_{i-1}} D_{s_2}\big(e_{s_1}^n(u) D_{s_1}\big(\Psi(u,v)\psi_n\big)\big)\Bigg] \\
&\quad = \Delta_n^{-1/2} \sum_{i=1}^{1/\Delta_n} \int_{t_{i-1}}^{t_i} ds_1 \int_{t_{i-1}}^{s_1} ds_2 \mathbb{E}\big[f_{n,i,s_2}^{\dagger} a_{t_{i-1}} D_{s_2}\big(e_{s_1}^n(u) D_{s_1}\big(\Psi(u,v)\psi_n\big)\big)\big],
\end{aligned}$$

where

$$f_{n,i,s_2}^{\dagger} = \sum_{k=2}^{\infty} k!\lambda_k \Delta_n^{-(k-2)/2} \int_{t_{i-1}}^{s_2} \int_{t_{i-1}}^{s_3} \cdots \int_{t_{i-1}}^{s_{k-1}} dW_{s_k} \cdots dW_{s_4}\, dW_{s_3},$$

and consequently reach the representation

$$\begin{aligned}
&\mathtt{i} u \sum_{i=1}^{1/\Delta_n} \mathbb{E}\left[\int_{t_{i-1}}^{t_i} e_t^n(u)\, dM_t^n \Psi(u,v)\psi_n\right] \\
&\qquad = \Delta_n^{-1/2} \sum_{i=1}^{1/\Delta_n} \int_{t_{i-1}}^{t_i} ds_1 \int_{t_{i-1}}^{s_1} ds_2 E_i^n(u,v)_{s_1,s_2},
\end{aligned}$$

where

$$E_i^n(u,v)_{s_1,s_2} = iu\mathbb{E}\big[f_{n,i,s_2}^{\dagger} a_{t_{i-1}} D_{s_2}\big(e_{s_1}^n(u) D_{s_1}\big(\Psi(u,v)\psi_n\big)\big)\big]. \tag{3.18}$$

Let

$$\mathbb{E}_s^n(u,v) = e_s^n(u)\Psi(u,v).$$

Then $\mathbb{E}_s(u,v)$ has the FGH-decomposition (cf. [25], page 911):

$$\mathbb{E}_s^n(u,v) = \mathbb{F}_s^n(u,v)\mathbb{G}_s(u)\mathbb{H}_s^n(u)$$

with

$$\mathbb{F}_s^n(u,v) = \exp\big(\mathrm{i}uM_s^n + \mathrm{i}vC\big), \qquad \mathbb{G}_s(u) = \exp\left(-\frac{1}{2}u^2(C - C_s)\right),$$
$$\mathbb{H}_s^n(u) = \exp\left(\frac{1}{2}u^2\big(C_s^n - C_s\big)\right).$$

From (3.18) and the FGH-decomposition,

$$E_i^n(u,v)_{s_1,s_2} = \mathbb{E}\big[\mathbb{F}_{s_1}^n(u,v)\mathbb{G}_{s_1}(u)\mathbb{H}_{s_1}^n(u)\psi_{s_1,s_2}^n(u,v) f_{n,i,s_2}^{\dagger} a_{t_{i-1}}\big], \tag{3.19}$$

where

$$\begin{aligned}\psi_{s_1,s_2}^n(u,v) &= \mathrm{i}u\big(e_{s_1}^n(u)\Psi(u,v)\big)^{-1} D_{s_2}\big(e_{s_1}^n(u) D_{s_1}\big(\Psi(u,v)\psi_n\big)\big)\\ &= \left\{\psi_n\left(-\frac{u^2}{2} + \mathrm{i}v\right) D_{s_1}C_1 + D_{s_1}\psi_n\right\}\mathrm{i}u\left(\mathrm{i}uD_{s_2}M_{s_1}^n + \frac{u^2}{2}D_{s_2}C_{s_1}^n\right)\\ &\quad + \psi_n \mathrm{i}u\left(-\frac{u^2}{2} + \mathrm{i}v\right)^2 (D_{s_2}C_1)(D_{s_1}C_1)\\ &\quad + 2(D_{s_2}\psi_n)\mathrm{i}u\left(-\frac{u^2}{2} + \mathrm{i}v\right) D_{s_1}C_1 + D_{s_2}D_{s_1}\psi_n \mathrm{i}u.\end{aligned}$$

Suppose that the following condition, which we will prove in the next subsection, is satisfied for $\ell = 10$:

($C2^\flat$) The variables s_n ($n \in \mathbb{N}$) satisfy the following conditions:

- (i) $\sup_{t\geq\frac{1}{2}} \mathbb{P}[\det\sigma_{(M_t^n, C_1)} < s_n] = O(\Delta_n^{4/3+\varepsilon})$ as $n \to \infty$ for some $\varepsilon > 0$.
- (ii) $\limsup_{n\to\infty} \mathbb{E}[s_n^{-p}] < \infty$ for every $p > 1$.
- (iii) $\limsup_{n\to\infty} \|s_n\|_{\ell,p} < \infty$ for every $p \geq 2$.

Now following the (a)–(h) procedure of [25], pages 911–912, and the argument of the proof of Theorem 4 therein, we can obtain

$$\sup_n \sup_{i=1,\dots,n} \sup_{s_1,s_2: t_{i-1}<s_1<s_2\leq t_i} \sup_{(u,v)\in\Lambda_n^0(2,q)} \big|(u,v)\big|^3 \big|E_i^n(u,v)_{s_1,s_2}\big| < \infty \tag{3.20}$$

by applying the integration-by-parts formula at most 8 times. More precisely, we introduce a new truncation

$$\psi_{n,s_1} = \psi\big(2\big[1 + 4\Delta_{(M_{s_1}^n, C)} s_n^{-1}\big]^{-1}\big),$$

which will be used when the integration-by-parts formula for $(M_{s_1}^n, C)$ is applied for $s_1 \geq 1/2$. We have the decomposition of $E_i^n(u,v)_{s_1,s_2}$ expressed by (3.19):

$$\begin{aligned}E_i^n(u,v)_{s_1,s_2} &= \mathbb{E}\big[\mathbb{F}_{s_1}^n(u,v)\mathbb{G}_{s_1}(u)\mathbb{H}_{s_1}^n(u)\psi_{s_1,s_2}^n(u,v)\psi_{s_1}^n f_{n,i,s_2}^{\dagger} a_{t_{i-1}}\big]\\ &\quad + R_{n,s_1,s_2}(u,v)\end{aligned}$$

with

$$\left|R_{n,s_1,s_2}(u,v)\right| \le K\Delta_n^{-5q/2}\sup_{s'}\|1-\psi_{n,s'}\|_{\mathbb{L}^p}$$

for all n, s_1 and restricted (u, v), where K denotes a generic positive constant. The right-hand side can be shown to be of order $o(\Delta_n^{3q/2})$ for sufficiently small numbers $q > 1/3$ [cf. Assumption (A5)] and $p > 1$. Then, as already noticed, we can follow the (a)–(h) procedure of [25], by using the FGH-decomposition, but with $\psi(\xi_n)\psi_{n,s_1}$ for truncation, to obtain (3.20).

Finally, we obtain (3.17) for $\alpha = 0$ from (3.20). When $\alpha \neq 0$, the argument of the proof is essentially the same as above. As a conclusion, (3.17) [and consequently (A5)] holds for every α under the assumptions (C1) and (C2$^\flat$).

Obviously, condition (A3) is valid under (C1) and (C2$^\flat$). In particular, the nondegeneracy of C simply follows from $\inf_x |a(x)| > 0$. Thus, we are left to proving condition (C2$^\flat$).

3.6. *Proof of* (C2$^\flat$). We shall now prove that condition (C2$^\flat$) holds under the assumptions of Theorem 3.3. Recall that

$$M_t^n = \Delta_n^{1/2}\sum_{i:t_i\le t} a(X_{t_{i-1}})f\left(\Delta_n^{-1/2}\Delta_i^n W\right)$$

for $t \in \Pi^n = \{t_i\}$. We deduce that

$$\begin{aligned}
D_r M_t^n &= \sum_{i:t_i\le t} a_{t_{i-1}} f'\left(\Delta_n^{-1/2}\Delta_i^n W\right)1_{(t_{i-1},t_i]}(r) \\
&\quad + \Delta_n^{1/2}\sum_{i:t_i\le t} a'_{t_{i-1}} D_r X_{t_{i-1}} f\left(\Delta_n^{-1/2}\Delta_i^n W\right)1_{\{r\le t_{i-1}\}} \\
&= \sum_{i:t_i\le t}\Bigg[a_{t_{i-1}} f'\left(\Delta_n^{-1/2}\Delta_i^n W\right) \\
&\quad + \Delta_n^{1/2}\sum_{k=i+1}^{n} a'_{t_{k-1}} f\left(\Delta_n^{-1/2}\Delta_k^n W\right)1_{\{t_k\le t\}} D_r X_{t_{k-1}}\Bigg]1_{(t_{i-1},t_i]}(r)
\end{aligned}$$

for $t \in \Pi^n$, where $\sum_{k=n+1}^n \cdots = 0$. Hence,

$$\begin{aligned}
\sigma_{11}(n,t) := \sigma_{M_t^n} = \sum_{i:t_i\le t}\int_{t_{i-1}}^{t_i}\Bigg[a_{t_{i-1}} f'\left(\Delta_n^{-1/2}\Delta_i^n W\right) \\
+ \Delta_n^{1/2}\sum_{k=i+1}^{n} a'_{t_{k-1}} f\left(\Delta_n^{-1/2}\Delta_k^n W\right)1_{\{t_k\le t\}} D_r X_{t_{k-1}}\Bigg]^2 dr
\end{aligned}$$

for $t \in \Pi^n$. We have $C_t = \int_0^t \beta(X_s)\,ds$. Since $D_r C_t = \int_r^t \beta'_s D_r X_s\,ds$ for $t \in [0,1]$, we obtain

$$\sigma_{12}(n,t) := \langle DM^n, DC\rangle_{\mathbb{H}} = \sum_{i:t_i \le t} \int_{t_{i-1}}^{t_i} \Biggl(\Biggl[a_{t_{i-1}} f'(\Delta_n^{-1/2}\Delta_i^n W) + \Delta_n^{1/2} \sum_{k=i+1}^{n} a'_{t_{k-1}} f(\Delta_n^{-1/2}\Delta_k^n W) 1_{\{t_k \le t\}} D_r X_{t_{k-1}}\Biggr] \times \int_r^1 \beta'_s D_r X_s\,ds\Biggr)\,dr$$

for $t \in \Pi^n$. The Malliavin matrix of (M_t^n, C) is

$$\sigma_{(M_t^n, C)} = \begin{bmatrix} \sigma_{11}(n,t) & \sigma_{12}(n,t) \\ \sigma_{12}(n,t) & \sigma_{22}(1) \end{bmatrix}$$

for $t \in \Pi^n$. Let

$$\sigma(n,t) = \begin{bmatrix} \sigma_{11}(n,t) & \sigma_{12}(n,t) \\ \sigma_{12}(n,t) & \sigma_{22}(t) \end{bmatrix}.$$

By the Clark–Ocone representation formula, we have $f'(\Delta_n^{-1/2}\Delta_i^n W) = \Delta_n^{-1/2} \int_{t_{i-1}}^{t_i} a_{n,i}(s)\,dW_s$ with

$$a_{n,i}(s) = \Delta_n^{1/2} \mathbb{E}\bigl[D_s\bigl(f'(\Delta_n^{-1/2}\Delta_i^n W)\bigr) | \mathcal{F}_s\bigr],$$

and moreover,

$$\begin{aligned} a_{n,i}(s) &= \mathbb{E}\bigl[f''(\Delta_n^{-1/2}\Delta_i^n W) | \mathcal{F}_s\bigr] 1_{(t_{i-1}, t_i]}(s) \\ &= g_s\bigl(\Delta_n^{-1/2}(W_s - W_{t_{i-1}})\bigr) 1_{(t_{i-1}, t_i]}(s), \\ g_r(z) &= \int_{\mathbb{R}} f''\biggl(z + \sqrt{\frac{t_i - r}{\Delta_n}}x\biggr) \phi(x; 0, 1)\,dx \end{aligned}$$

for $r \in (t_{i-1}, t_i]$. Then obviously $\sup_{s \in (t_{i-1}, t_i], i=1,\dots,n, n \in \mathbb{N}} \|a_{n,i}(s)\|_{9,p} < \infty$ for every $p > 1$. In the same way, we see that

$$f(\Delta_n^{-1/2}\Delta_i^n W) = \Delta_n^{-1/2} \int_{t_{i-1}}^{t_i} \alpha_{n,i}(s)\,dW_s$$

with some predictable processes $\alpha_{n,i}(s)$ satisfying

$$\sup_{s \in (t_{i-1}, t_i], i=1,\dots,n, n \in \mathbb{N}} \|a_{n,i}(s)\|_{10,p} < \infty$$

for every $p > 1$. By Lemma 5 of [25],

$$\begin{aligned}
&\Bigg\| \sum_{i:t_i \le t} \int_{t_{i-1}}^{t_i} a_{t_{i-1}} f'\big(\Delta_n^{-1/2} \Delta_i^n W\big) \Delta_n^{1/2} \\
&\quad \times \sum_{k=i+1}^{n} a'_{t_{k-1}} f\big(\Delta_n^{-1/2} \Delta_k^n W\big) 1_{\{t_k \le t\}} D_r X_{t_{k-1}} \, dr \Bigg\|_{\mathbb{L}^9} \\
&\quad = \Bigg\| \Delta_n \sum_{i:t_i \le t} \Bigg[a_{t_{i-1}} \bigg(\Delta_n^{-1/2} \int_{t_{i-1}}^{t_i} a_{n,i}(s_1) \, dW_{s_1} \bigg) \\
&\qquad \times \Bigg(\Delta_n^{1/2} \sum_{k=i+1}^{n} \bigg\{ \int_{t_{i-1}}^{t_i} a'_{t_{k-1}} 1_{\{t_k \le t\}} \Delta_n^{-1} D_r X_{t_{k-1}} \, dr \bigg\} \Delta_n^{-1/2} \\
&\qquad \times \int_{t_{k-1}}^{t_k} \alpha_{n,i}(s) \, dW_s \Bigg) \Bigg] \Bigg\|_{\mathbb{L}^9} \\
&\quad = O\big(\Delta_n^{1/2}\big)
\end{aligned}$$

for $t \in \Pi^n$. Hence,

$$\sup_{n \in \mathbb{N}} \sup_{t \in \Pi^n} \big\| \sigma_{11}(n,t) - \tilde{\sigma}_{11}(n,t) \big\|_{\mathbb{L}^9} = O\big(\Delta_n^{1/2}\big)$$

as $n \to \infty$, where the term $\tilde{\sigma}_{11}(n,t)$ is defined in Section 3.2. Furthermore, by the same lemma, we have

$$\begin{aligned}
&\sup_{t \in \Pi^n} \Bigg\| \sum_{i:t_i \le t} \int_{t_{i-1}}^{t_i} \bigg(a_{t_{i-1}} f'\big(\Delta_n^{-1/2} \Delta_i^n W\big) \int_r^1 \beta'_s D_r X_s \, ds \bigg) dr \Bigg\|_{\mathbb{L}^{10}} \\
&\quad = \sup_{t \in \Pi^n} \Bigg\| \Delta_n \sum_{i:t_i \le t} a_{t_{i-1}} \bigg(\Delta_n^{-1/2} \int_{t_{i-1}}^{t_i} a_{n,i}(s_1) \, dW_{s_1} \bigg) \\
&\qquad \times \bigg(\Delta_n^{-1} \int_{t_{i-1}}^{t_i} \bigg(\int_r^1 \beta'_s D_r X_s \, ds \bigg) dr \bigg) \Bigg\|_{\mathbb{L}^9} \\
&\quad = O\big(\Delta_n^{1/2}\big).
\end{aligned}$$

Therefore,

$$\sup_{t \in \Pi^n} \big\| \sigma_{12}(n,t) - \tilde{\sigma}_{12}(n,t) \big\|_{\mathbb{L}^9} = O\big(\Delta_n^{1/2}\big).$$

From these estimates,

$$\sup_{t \in \Pi^n} \big\| \sigma(n,t) - \tilde{\sigma}(n,t) \big\|_{\mathbb{L}^9} = O\big(\Delta_n^{1/2}\big).$$

One has

$$\begin{aligned}\det\tilde\sigma(n,t) &= \tilde\sigma_{11}(n,t)\sigma_{22}(t)-\tilde\sigma_{12}(n,t)^2\\ &\geq \Delta_n\sum_{i:t_i\leq t}\bigl[a_{t_{i-1}}f'\bigl(\Delta_n^{-1/2}\Delta_i^n W\bigr)\bigr]^2\sigma_{22}(t)\geq \inf_x\bigl|a(x)\bigr|^2 m_n\sigma_{22}(t)\end{aligned}\tag{3.21}$$

for $t\in\Pi^n$, where the random variable m_n is defined in Section 3.2.

Now, we shall verify (C2$^\flat$). Checking (C2$^\flat$)(iii) is not difficult if one estimates the $\mathbb{H}^{\otimes m}$-norms of $D_{r_1,\dots,r_m}$-derivative of the objects, in part with the aid of the Burkhölder inequality. For (C2$^\flat$)(ii), it suffices to show

$$\limsup_{n\to\infty}\mathbb{E}\bigl[1_{\{m_n\geq \mathsf{c}_1\}}\bigl(\det\tilde\sigma(n,1/2)\bigr)^{-p}\bigr]<\infty\tag{3.22}$$

for every $p>1$ since $s_n\geq 1/2$ when $m_n<\mathsf{c}_1$. Consider the two-dimensional stochastic process $\bar X_t=(X_t^{(1)},X_t^{(2)})$ defined by the stochastic integral equations with smooth coefficients

$$\bar X_t=\bar X_0+\int_0^t V_1(\bar X_s)\circ dW_s+\int_0^t V_0(\bar X_s)\,ds,\tag{3.23}$$

for $t\in[0,1]$, where the first integral is given in the Stratonovich sense and

$$V_1(x)=\begin{bmatrix} b^{[1]}(x^1)\\ 0\end{bmatrix},\qquad V_0(x)=\begin{bmatrix}\tilde b^{[2]}(x^1)\\ \beta(x^1)\end{bmatrix}$$

for $x=(x^1,x^2)$, $\tilde b^{[2]}=b^{[2]}-2^{-1}b^{[1]}(b^{[1]})'$. Under (C2), the system (3.23) satisfies the Hörmander condition

$$\mathrm{Lie}[V_0;V_1]\bigl(x^1,0\bigr)=\mathbb{R}^2\qquad\bigl(\forall x^1\in\operatorname{supp}\mathcal{L}\{X_0\}\bigr),\tag{3.24}$$

where $\mathrm{Lie}[V_0;V_1]$ denotes the Lie algebra generated by V_1 and V_0. That is, $\mathrm{Lie}[V_0;V_1]=\operatorname{span}(\bigcup_{j=0}^\infty\Sigma_j)$, where $\Sigma_0=\{V_1\}$ and $\Sigma_j=\{[V,V_i];V\in\Sigma_{j-1},i=0,1\}$ $(j\geq 1)$ with the Lie bracket $[\cdot,\cdot]$. $\mathrm{Lie}[V_0;V_1](x)$ is $\mathrm{Lie}[V_0;V_1]$ evaluated at x.

As a result, for any $t\in(0,1]$ and $p>1$, there exists a constant K_p such that

$$\sup_{\mathbf{v}\in\mathbb{R}^2:|\mathbf{v}|=1}\mathbb{P}\biggl[\mathbf{v}^\star\int_0^t \bar Y_s^{-1}V_1(\bar X_s)V_1(\bar X_s)^\star\bigl(\bar Y_s^{-1}\bigr)^\star\,ds\,\mathbf{v}\leq\varepsilon\biggr]\leq K_p\varepsilon^p\tag{3.25}$$

for all $\varepsilon\in(0,1)$. Here, $\bar Y_t$ denotes a unique solution of the variational equation corresponding to (3.23). See [20], Theorems 2.3.2, 2.3.3 and Lemma 2.3.1, or Kusuoka and Stroock [15, 16], Ikeda and Watanabe [9] for the implication of (3.25) through (3.24). Therefore, we obtain

$$\mathbb{E}\biggl[\biggl(\det\int_0^t\bar Y_s^{-1}V_1(\bar X_s)V_1(\bar X_s)^\star\bigl(\bar Y_s^{-1}\bigr)^\star\,ds\biggr)^{-p}\biggr]<\infty$$

for every $p > 1$. Since $\bar{Y}_1^{-1}$ is bounded in $\bigcap_{p>1} \mathbb{L}^p$, we have

$$\mathbb{E}\left[\left(\det \int_0^t \bar{Y}_1 \bar{Y}_s^{-1} V_1(\bar{X}_s) V_1(\bar{X}_s)^\star (\bar{Y}_s^{-1})^\star \bar{Y}_1^\star \, ds\right)^{-p}\right] < \infty$$

for every $p > 1$. Recalling the definition at (3.8), this inequality gives $\sigma_{22}(t)^{-1} \in \bigcap_{p>1} \mathbb{L}^p$ for every $t \in (0, 1]$, and consequently, in view of (3.21), we obtained (3.22), and hence $(\mathrm{C}2^\flat)$(ii) for arbitrary $\mathsf{c}_1 > 0$. Finally,

$$\begin{aligned}
&\sup_{t\geq \frac{1}{2}} \mathbb{P}[\det \sigma_{(M_t^n, C_1)} < s_n] \\
&\quad \leq \sup_{t\geq \frac{1}{2}} \mathbb{P}\big[\det \sigma(n,t) < s_n\big] \\
&\quad \leq \sup_{t\in \Pi^n : t\geq \frac{1}{2}} \mathbb{P}\big[\det \sigma(n,t) < 1.5 s_n\big] \\
&\qquad + \sup_{s,t:|t-s|\leq \Delta_n} \mathbb{P}\big[\big|\det \sigma(n,t) - \det \sigma(n,s)\big| > 0.5 s_n\big] \\
&\quad \leq \mathbb{P}\big[\det \sigma(n, 1/2) < 1.75 s_n\big] + O\big(\Delta_n^{1.35}\big) \\
&\quad \leq \mathbb{P}\big[\det \tilde{\sigma}(n, 1/2) < 2 s_n\big] + \mathbb{P}\big[\big|\det \sigma(n, 1/2) - \det \tilde{\sigma}(n, 1/2)\big| > 0.25 s_n\big] \\
&\qquad + O\big(\Delta_n^{1.35}\big) \\
&\quad \leq \mathbb{P}\big[m_n > 2\mathsf{c}_1, \det \tilde{\sigma}(n, 1/2) < 2 s_n\big] + \mathbb{P}[m_n \leq 2\mathsf{c}_1] \\
&\qquad + \Delta_n^{-3/19} \mathbb{E}\big[\big|\det \sigma(n, 1/2) - \det \tilde{\sigma}(n, 1/2)\big|^3\big] + 2^{5\times 19/3} \Delta_n^{5/3} \mathbb{E}\big[s_n^{-5\times 19/3}\big] \\
&\qquad + O\big(\Delta_n^{1.35}\big) \\
&\quad = O\big(\Delta_n^{51/38}\big)
\end{aligned}$$

as $n \to \infty$ if we take $\mathsf{c}_1 < \mathbb{E}[f'(Z)^2]/2$. Thus, we have verified $(\mathrm{C}2^\flat)$(i), which completes the proof.

4. Stochastic expansion of generalized power variation of diffusions. Hereafter, we will concentrate on the stochastic expansion of the type (2.1) for the class of generalized power variation. The results of this section are necessary for the derivation of the Edgeworth expansion for power variation, which is presented in Section 5, but they might be also useful for other expansion problems in high frequency framework. We again consider a one-dimensional diffusion process $X = (X_t)_{t\in[0,1]}$ satisfying the stochastic differential equation

$$dX_t = b^{[1]}(X_t)\, dW_t + b^{[2]}(X_t)\, dt.$$

Our aim is to study the stochastic expansion of generalized power variations of the form

$$V_n(f) = \Delta_n \sum_{i=1}^{1/\Delta_n} f\left(\frac{\Delta_i^n X}{\sqrt{\Delta_n}}\right), \qquad \Delta_i^n X = X_{t_i} - X_{t_{i-1}}, \tag{4.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given *even* function, that is, $f(x) = f(-x)$ for all $x \in \mathbb{R}$. These types of functionals play a very important role in mathematical finance, where they are used for various estimation and testing procedures; see, for example, [2, 3, 6] and [11] among many others. The most classical subclass of statistics (4.1) are power variations, which correspond to functions of the form $f(x) = |x|^p$; we will concentrate on Edgeworth expansion of power variations in the next section. We introduce the notation

$$\rho_x(f) = \mathbb{E}\big[f(xZ)\big], \qquad x \in \mathbb{R}, Z \sim \mathcal{N}(0,1) \tag{4.2}$$

whenever the latter is finite. Now, let us recall the law of large numbers and the central limit theorem for the functional $V_n(f)$ derived in [2].

THEOREM 4.1. (i) *Assume that* $b^{[1]}, b^{[2]} \in C(\mathbb{R})$ *and* $f \in C_p(\mathbb{R})$. *Then it holds that*

$$V_n(f) \xrightarrow{\mathbb{P}} V(f) = \int_0^1 \rho_{b_s^{[1]}}(f)\, ds. \tag{4.3}$$

(ii) *If moreover* $b^{[1]} \in C^2(\mathbb{R})$ *and* $f \in C_p^1(\mathbb{R})$, *we obtain the stable convergence*

$$\Delta_n^{-1/2}\big(V_n(f) - V(f)\big) \xrightarrow{d_{\text{st}}} M \sim MN\left(0, \int_0^1 \rho_{b_s^{[1]}}(f^2) - \rho^2_{b_s^{[1]}}(f)\, ds\right). \tag{4.4}$$

Now, we derive the second-order stochastic expansion associated with the central limit theorem (4.4). Let us introduce the notation

$$\alpha_i^n = \Delta_n^{-1/2} b_{t_{i-1}}^{[1]} \Delta_i^n W, \tag{4.5}$$

which serves as an approximation of the increment $\Delta_i^n X / \sqrt{\Delta_n}$. One of the main results of this section is the following theorem. We remark that this result might be of independent interest for other expansion problems in probability and statistics.

THEOREM 4.2. *Assume that* $b^{[2]} \in C^2(\mathbb{R})$, $b^{[1]} \in C^4(\mathbb{R})$ *and* $f \in C_p^2(\mathbb{R})$. *Then we obtain the stochastic expansion*

$$\widetilde{V}_n(f) := \Delta_n^{-1/2}\big(V_n(f) - V(f)\big) = M_n + \Delta_n^{1/2} N_n + o_{\mathbb{P}}\big(\Delta_n^{1/2}\big) \tag{4.6}$$

with

$$M_n = \Delta_n^{1/2} \sum_{i=1}^{1/\Delta_n} \big(f(\alpha_i^n) - \rho_{b_{t_{i-1}}^{[1]}}\big), \tag{4.7}$$

and $N_n = \sum_{k=1}^5 N_{n,k}$

$$N_{n,1} = \Delta_n^{1/2} \sum_{i=1}^{1/\Delta_n} f'(\alpha_i^n)\Big(b_{t_{i-1}}^{[2]} + \frac{1}{2} b_{t_{i-1}}^{[1.1]} H_2(\Delta_i^n W/\sqrt{\Delta_n})\Big),$$

$$N_{n,2} = \Delta_n^{-1/2} \sum_{i=1}^{1/\Delta_n} f'(\alpha_i^n)\Big(b_{t_{i-1}}^{[2.1]} \int_{t_{i-1}}^{t_i} (W_s - W_{t_{i-1}})\,ds$$

$$+ b_{t_{i-1}}^{[1.2]} \int_{t_{i-1}}^{t_i} \{s - t_{i-1}\}\,dW_s + \frac{\Delta_n^{3/2} b_{t_{i-1}}^{[1.1.1]}}{6} H_3(\Delta_i^n W/\sqrt{\Delta_n})\Big),$$

$$N_{n,3} = \frac{\Delta_n}{2} \sum_{i=1}^{1/\Delta_n} f''(\alpha_i^n)\Big(b_{t_{i-1}}^{[2]} + \frac{1}{2} b_{t_{i-1}}^{[1.1]} H_2(\Delta_i^n W/\sqrt{\Delta_n})\Big)^2, \tag{4.8}$$

$$N_{n,4} = \frac{1}{2\Delta_n} \sum_{i=1}^{1/\Delta_n} \Big(-\rho''_{b_{t_{i-1}}^{[1]}}(f)|b_{t_{i-1}}^{[1.1]}|^2 \int_{t_{i-1}}^{t_i} (W_s - W_{t_{i-1}})^2\,ds$$

$$- \Delta_n^2 \rho'_{b_{t_{i-1}}^{[1]}}(f) b_{t_{i-1}}^{[1.2]}\Big),$$

$$N_{n,5} = -\Delta_n^{-1} \sum_{i=1}^{1/\Delta_n} \rho'_{b_{t_{i-1}}^{[1]}}(f) b_{t_{i-1}}^{[1.1]} \int_{t_{i-1}}^{t_i} (W_s - W_{t_{i-1}})\,ds,$$

where $(H_k)_{k\ge 0}$ *denote the Hermite polynomials and the processes* $b_t^{[k_1\cdots k_d]}$ *were defined in Section* 3.

PROOF. See Section 7. □

To describe the limits of the quantities $N_{n,k}$, $1 \le k \le 5$, we need to introduce some further notation.

NOTATION. We introduce the functions $g_k : \mathbb{R}^6 \to \mathbb{R}$, $1 \le k \le 5$, as follows:

$$g_1(x_1, \dots, x_6) = \mathbb{E}\Big[U f'(x_2 U)\Big(x_1 + \frac{1}{2} x_5 H_2(U)\Big) - \rho'_{x_2}(f) x_5 U V\Big],$$

$$g_2(x_1, \dots, x_6) = \mathbb{E}\Big[f'(x_2 U)\Big((x_3 + x_4)V + \frac{1}{6} x_6 H_3(U)\Big)\Big],$$

$$g_3(x_1, \dots, x_6) = \frac{1}{2}\mathbb{E}\Big[f''(x_2 U)\Big(x_1 + \frac{1}{2} x_5 H_2(U)\Big)^2\Big],$$

$$g_4(x_1, \dots, x_6) = -\frac{1}{4}\rho''_{x_2}(f) x_5^2 - \frac{1}{2}\rho'_{x_2}(f) x_4,$$

$$g_5(x_1, \dots, x_6) = \mathbb{E}\Big[\Big\{f'(x_2 U)\Big(x_1 + \frac{1}{2} x_5 H_2(U)\Big) - \rho'_{x_2}(f) x_5 V\Big\}^2\Big]$$

with $(U, V) \sim \mathcal{N}_2(0, \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{pmatrix})$.

Theorem A.1 implies the convergence in probability

$$N_{n,k} \xrightarrow{\mathbb{P}} N_k = \int_0^1 g_k\big(b_s^{[2]}, b_s^{[1]}, b_s^{[2.1]}, b_s^{[1.2]}, b_s^{[1.1]}, b_s^{[1.1.1]}\big)\,ds, \qquad k = 2, 3, 4 \tag{4.9}$$

under the assumptions of Theorem 4.2. The terms $N_{n,1}$ and $N_{n,5}$ converge stably in law due to Theorem A.2; their asymptotic distributions will be specified later.

REMARK 4.3. The fact that we consider the drift and volatility processes of the type $b_s^{[k]} = b^{[k]}(X_s)$ is not essential for developing the stochastic expansion of Theorem 4.2. In general, the processes $b_s^{[k_1 \cdots k_l]}$ that appear in Theorem 4.2 may depend on different Brownian motions, which are not perfectly correlated with W that drives the process X. In this case, a similar stochastic expansion can be deduced; however, it will contain additional terms, which are due to new Brownian motions.

In the next section, we will require a consistent estimator of the asymptotic variance of M_n, that is,

$$C = \int_0^1 \{\rho_{b_s^{[1]}}(f^2) - \rho_{b_s^{[1]}}^2(f)\}\,ds.$$

A rather natural one is given by

$$F_n = \Delta_n \sum_{i=1}^{1/\Delta_n} f^2\left(\frac{\Delta_i^n X}{\sqrt{\Delta_n}}\right) - f\left(\frac{\Delta_i^n X}{\sqrt{\Delta_n}}\right) f\left(\frac{\Delta_{i+1}^n X}{\sqrt{\Delta_n}}\right). \tag{4.10}$$

The next theorem, which follows from the combination of central limit theorems presented in [2] and Theorem A.2, describes the joint asymptotic distribution of (M_n, F_n, N_n). This result is crucial for the derivation of the Edgeworth expansion.

THEOREM 4.4. *Assume that conditions of Theorem 4.2 are satisfied. Then we obtain the stable convergence*

$$\big(M_n, \Delta_n^{-1/2}(F_n - C), N_n\big) \xrightarrow{d_{\mathrm{st}}} (M, \widehat{F}, N) \sim MN\left(\mu, \int_0^1 \Xi_s\,ds\right),$$

where the matrix Ξ_s is given as

$$\Xi_s^{11} = \rho_{b_s^{[1]}}(f^2) - \rho_{b_s^{[1]}}^2(f),$$

$$\Xi_s^{12} = \Xi_s^{21} = \rho_{b_s^{[1]}}(f^3) - 3\rho_{b_s^{[1]}}(f^2)\rho_{b_s^{[1]}}(f) + 2\rho_{b_s^{[1]}}^3(f),$$

$$\Xi_s^{22} = \rho_{b_s^{[1]}}(f^4) - 4\rho_{b_s^{[1]}}(f^3)\rho_{b_s^{[1]}}(f) + 6\rho_{b_s^{[1]}}(f^2)\rho_{b_s^{[1]}}^2(f) - 3\rho_{b_s^{[1]}}^4(f),$$

$$\Xi_s^{33} = (g_5 - g_1^2)\big(b_s^{[2]}, b_s^{[1]}, b_s^{[2.1]}, b_s^{[1.2]}, b_s^{[1.1]}, b_s^{[1.1.1]}\big),$$

and $\Xi_s^{13} = \Xi_s^{23} = 0$, *and* $\mu_1 = \mu_2 = 0$,

$$\mu_3 = \int_0^1 g_1\big(b_s^{[2]}, b_s^{[1]}, b_s^{[2.1]}, b_s^{[1.2]}, b_s^{[1.1]}, b_s^{[1.1.1]}\big)\, dW_s + \sum_{k=2}^{4} N_k.$$

5. Asymptotic expansion for the power variation. Now we have all instruments at hand to obtain the Edgeworth expansion for the case of power variation $V_n(f_p)$ with

$$f_p(x) = |x|^p,$$

which is our leading example. As we mentioned in Section 4, this would be the most important class of functionals in mathematical finance. In order to obtain the Edgeworth expansion for power variation, we will combine the results of Sections 3 and 4. Applying Theorem 4.2 to the function f_p, we see that the martingale part M_n is given as

$$M_n = \Delta_n^{1/2} \sum_{i=1}^{1/\Delta_n} |b^{[1]}(X_{t_{i-1}})|^p \left(\left| \frac{\Delta_i^n W}{\sqrt{\Delta_n}} \right|^p - m_p \right)$$

with $m_p = \mathbb{E}[|\mathcal{N}(0,1)|^p]$. In particular, M_n is a weighted power variation studied in Section 3. Consequently, we can apply the results of Section 3 with

$$a(x) = |b^{[1]}(x)|^p, \qquad f(x) = f_p(x) - m_p \quad \text{and} \quad p \in 2\mathbb{N} \cup (11, \infty).$$

Now, we will compute all quantities from previous sections required for the Edgeworth expansion. First, we obtain the Hermite expansion

$$f(x) = \sum_{k=2}^{\infty} \lambda_k H_k(x)$$

with $\lambda_k = 0$ if k is odd (because f is an even function), and $\lambda_2 = (m_{p+2} - m_p)/2$. We start with the computation of the random symbol $\underline{\sigma}$. Here, we mainly need to determine the functions $g_1, \dots, g_5$ defined in Section 4. We observe that, for any $k \geq 0$ with $k < p$,

$$f_p^{(k)}(x) = \operatorname{sgn}(x)^k p(p-1)\cdots(p-k+1)|x|^{p-k}, \qquad \rho_x(f_p) = m_p|x|^p.$$

Now, a straightforward calculation gives the identities

$$g_1(x_1, \dots, x_6) = p \operatorname{sgn}(x_2)|x_2|^{p-1}\left(x_1 m_p + \frac{1}{2} x_5 (m_{p+2} - 2m_p) \right),$$

$$g_2(x_1, \dots, x_6) = p \operatorname{sgn}(x_2)|x_2|^{p-1}\left(\frac{1}{2}(x_3 + x_4) m_p + \frac{1}{6} x_6 (m_{p+2} - m_p) \right),$$

$$g_3(x_1, \dots, x_6) = \frac{p(p-1)}{2}|x_2|^{p-2}\Big(x_1^2 m_{p-2} + x_1 x_5 (m_p - m_{p-2})$$

$$+\frac{x_5^2}{4}(m_{p+2}-2m_p+m_{p-2})\bigg),$$

$$g_4(x_1,\dots,x_6)=\frac{p}{4}m_p\big(-(p-1)|x_2|^{p-2}x_5^2-2x_4\operatorname{sgn}(x_2)|x_2|^{p-1}\big),$$

$$g_5(x_1,\dots,x_6)=p^2|x_2|^{2p-2}\bigg(x_1^2m_{2p-2}+x_1x_5(m_{2p}-m_{2p-2})$$
$$+\frac{x_5^2}{4}(m_{2p+2}-2m_{2p}+m_{2p-2})$$
$$+\frac{x_5^2}{3}m_p^2-x_5m_p\bigg(x_1m_p+\frac{x_5}{2}[m_{p+2}-m_p]\bigg)\bigg).$$

As in the previous section, we consider the quantity

$$F_n=\Delta_n\sum_{i=1}^{1/\Delta_n}f_{2p}\bigg(\frac{\Delta_i^nX}{\sqrt{\Delta_n}}\bigg)-f_p\bigg(\frac{\Delta_i^nX}{\sqrt{\Delta_n}}\bigg)f_p\bigg(\frac{\Delta_{i+1}^nX}{\sqrt{\Delta_n}}\bigg)$$

as a consistent estimator of C. We obtain the following result, which again follows from Theorem A.2.

THEOREM 5.1. *Assume that conditions of Theorem 4.2 are satisfied. Then we obtain the stable convergence*

$$\big(M_n,\Delta_n^{-1/2}(F_n-C),N_n,\Delta_n^{-1/2}(C_n-C)\big)$$
$$\xrightarrow{d_{\mathrm{st}}}(M,\widehat{F},N,\widehat{C})\sim MN\bigg(\mu,\int_0^1\Xi_s\,ds\bigg),$$

where the entries Ξ_s^{ij}, $1\le i,j\le 3$, *of the matrix* $\Xi_s\in\mathbb{R}^{4\times 4}$ *and* μ_j, $1\le j\le 3$ *of the vector* $\mu\in\mathbb{R}^4$ *are given in Theorem 4.4, and* $\mu_4=\Xi_s^{34}=0$,

$$\Xi_s^{14}=\Xi_s^{41}=\Gamma_2\big|b^{[1]}(X_s)\big|^{3p},\qquad \Xi_s^{24}=\Xi_s^{42}=\overline{\Gamma}\big|b^{[1]}(X_s)\big|^{4p},$$
$$\Xi_s^{44}=\Gamma_1\big|b^{[1]}(X_s)\big|^{4p},$$

where the constants Γ_1,Γ_2 *are given in Proposition 3.2 and* $\overline{\Gamma}$ *is defined as*

$$\overline{\Gamma}=\operatorname{Cov}\bigg[f_{2p}(W_1),\int_0^1\mathbb{E}^2[f_p'(W_1)|\mathcal{F}_s]\,ds\bigg]$$
$$-2\operatorname{Cov}\bigg[f_p(W_1)f_p(W_2-W_1),\int_0^1\mathbb{E}^2[f_p'(W_1)|\mathcal{F}_s]\,ds\bigg].$$

As a consequence of Theorem 5.1 and Remark 2.1, we conclude that

(5.1) $$\underline{\sigma}(z,\mathrm{i}u,\mathrm{i}v)=(\mathrm{i}u)^2\mathcal{H}_1(z)+\mathrm{i}u\mathcal{H}_2+\mathrm{i}v\mathcal{H}_3(z)$$

with

$$\mathcal{H}_1(z) = z\frac{\int_0^1 \Xi_s^{14}\,ds}{2\int_0^1 \Xi_s^{11}\,ds}, \qquad \mathcal{H}_2 = \mu_3, \qquad \mathcal{H}_3(z) = z\frac{\int_0^1 \Xi_s^{12}\,ds}{\int_0^1 \Xi_s^{11}\,ds}.$$

It should be noted that $\underline{\sigma}$ of (5.1) is essentially the same but different from $\underline{\sigma}$ of (3.7) since the reference functional F_n is now defined by (4.10) not by (3.4) while the limits of both coincide with each other and the ways of derivation of two adaptive random symbols are the same except for $\widehat{F}$. Using the results of Section 3, we immediately obtain the anticipative random symbol

$$\overline{\sigma}(\mathtt{i}u, \mathtt{i}v) = \mathtt{i}u\left(\mathtt{i}v - \frac{u^2}{2}\right)^2\mathcal{H}_4 + \mathtt{i}u\left(\mathtt{i}v - \frac{u^2}{2}\right)\mathcal{H}_5 \tag{5.2}$$

with $\mathcal{H}_4 = \lambda_2(m_{2p} - m_p^2)^2\mathcal{C}_2$, $\mathcal{H}_5 = \lambda_2(m_{2p} - m_p^2)(\mathcal{C}_3 + \mathcal{C}_4)$, where

$$\mathcal{C}_2 = \int_0^1 |b^{[1]}(X_s)|^p\left(\int_s^1 (|b^{[1]}|^{2p})'(X_u)D_sX_u\,du\right)^2 ds,$$

$$\mathcal{C}_3 = \int_0^1 |b^{[1]}(X_s)|^p\left(\int_s^1 (|b^{[1]}|^{2p})''(X_u)(D_sX_u)^2\,du\right) ds,$$

$$\mathcal{C}_4 = \int_0^1 |b^{[1]}(X_s)|^p\left(\int_s^1 (|b^{[1]}|^{2p})'(X_u)D_sD_sX_u\,du\right) ds.$$

In the power variation case, $a(x) = |b^{[1]}(x)|^p$ and we assumed in (C1) that $a(x)$ is bounded away from zero. So, in our situation, $a(x)$ is smooth in a neighborhood of X_0. By a certain large deviation argument, we may assume that $a(x)$ is smooth and even having bounded derivatives, from the beginning, at least in the proof of asymptotic nondegeneracy.

From the above argument, we obtain an asymptotic expansion for the power variation. Recall $\widetilde{V}_n(f) = \Delta_n^{-1/2}(V_n(f) - V(f))$.

THEOREM 5.2. *Let $b^{[1]}, b^{[2]} \in C_{b,1}^{\infty}(\mathbb{R})$ and $f_p(x) = |x|^p$ with $p \in 2\mathbb{N} \cup (13,\infty)$. Assume that $\inf_x |b^{[1]}(x)| > 0$, $\sum_{k=1}^{\infty} |(b^{[1]})^{(k)}(X_0)| > 0$ and let the functional F_n be given by (3.4). Then for the density $p_n(z,x)$ corresponding to the random symbol σ determined by (5.1) and (5.2), it holds that*

$$\sup_{h\in\mathcal{E}(K,\gamma)}\left|\mathbb{E}\big[h\big(\widetilde{V}_n(f_p), F_n\big)\big] - \int h(z,x)p_n(z,x)\,dz\,dx\right| = o(\sqrt{\Delta_n})$$

as $n \to \infty$, for any positive numbers K and γ.

Theorem 5.2 is proved by applying Theorems 3.3 and 5.1. In the present situation, N_n involves f'' and that is the reason why the number 13 appears. However, it would be possible to reduce it to 11 if the estimations related with N_n-part is refined, though we do not pursue this point in this article.

Theorem 5.2 and the corresponding Edgeworth expansion for the Studentized statistics at (6.1) are the main results of this paper. In particular, these asymptotic expansions can be applied to distribution analysis of various statistics in financial mathematics as power variation-type estimators are frequently used in this field. Another potential area of application is Euler approximation of continuous SDEs of the form (3.1). As is well known from [12], the Euler approximation scheme is asymptotically mixed normal and its limit depends on the asymptotic theory for quadratic variation. Thus, our Edgeworth expansion results can be potentially applied to numerical analysis of SDEs to obtain a more precise formula for the error distribution.

6. Studentization. As we mentioned in the beginning, we are mainly interested in the Edgeworth expansion connected with standard central limit theorem

$$\frac{Z_n}{\sqrt{F_n}} \xrightarrow{d} \mathcal{N}(0,1),$$

where F_n is a consistent estimator of C defined in (4.10). In the following, we present such an Edgeworth expansion for the case of power variation discussed in the previous section. First of all, we remark that the random symbol $\sigma(z, \mathtt{i}u, \mathtt{i}v)$ is given as

$$\sigma(z, \mathtt{i}u, \mathtt{i}v) = \sum_{j=1}^{8} c_j(z)(\mathtt{i}u)^{m_j}(\mathtt{i}v)^{n_j},$$

where $m = (m_j)_{1\le j\le 8}$, $n = (n_j)_{1\le j\le 8}$, $c(z) = (c_j(z))_{1\le j\le 8}$ are given by

$$m = (1,0,2,1,3,1,3,5), \qquad n = (0,1,0,1,0,2,1,0),$$
$$c(z) = \left(\mathcal{H}_2, \mathcal{H}_3(z), \mathcal{H}_1(z), \mathcal{H}_5, \frac{1}{2}\mathcal{H}_5, \mathcal{H}_4, \mathcal{H}_4, \frac{1}{4}\mathcal{H}_4\right).$$

As a consequence, we obtain the following decomposition for the density $p_n(z,x)$ of (Z_n, F_n):

$$p_n(z,x) = \phi(z;0,x)p^C(x) + \Delta_n^{1/2}\sum_{j=1}^{8} p_j(z,x)$$

with

$$p_j(z,x) = (-d_z)^{m_j}(-d_x)^{n_j}\big(\phi(z;0,x)p^C(x)\mathbb{E}[c_j(z)|C=x]\big), \qquad 1\le j\le 8.$$

We start with the following observation. Let Π be a finite measure on $\mathbb{R}$ with density π, such that all moments of Π are finite. Then it trivially holds that

$$\lim_{x\to\infty}|x|^k\pi(x) = 0, \qquad \lim_{x\to-\infty}|x|^k\pi(x) = 0, \qquad k\ge 0.$$

Given that the density π is a C^k function and g is a polynomial, we also have

$$\int_{\mathbb{R}} g^{(k)}(x)\pi(x)\,dx = (-1)^k \int_{\mathbb{R}} g(x)\pi^{(k)}(x)\,dx$$

by induction. Let g be an arbitrary polynomial and $\kappa(x) = \mathbb{E}[H|C = x]p^C(x)$ for an integrable random variable H, and note that

$$\int_{\mathbb{R}} m(x)\kappa(x)\,dx = \mathbb{E}\big[m(C)H\big],$$

whenever the integral makes sense. We define the polynomials $q_{\beta,v}(z, x)$ via

$$d_x^{\beta} g(z/\sqrt{x}) = \sum_{v\le\beta} q_{\beta,v}(z/\sqrt{x}, 1/\sqrt{x})g^{(v)}(z/\sqrt{x}),$$

where $g^{(v)}$ denotes the vth derivative of g. Let $(\alpha, \beta) \in \mathbb{N}_0^2$. Then it holds that

$$\begin{aligned}
&\int_{\mathbb{R}^2} g\left(\frac{z}{\sqrt{x}}\right) d_z^{\alpha} d_x^{\beta}\big[\phi(z; 0, x)\kappa(x)\big]\,dz\,dx \\
&\quad= (-1)^{\beta} \int_{\mathbb{R}^2} d_x^{\beta} g\left(\frac{z}{\sqrt{x}}\right) d_z^{\alpha}\phi(z; 0, x)\kappa(x)\,dz\,dx \\
&\quad= (-1)^{\beta} \int_{\mathbb{R}^2} \sum_{v\le\beta} q_{\beta,v}\left(\frac{z}{\sqrt{x}}, \frac{1}{\sqrt{x}}\right) g^{(v)}\left(\frac{z}{\sqrt{x}}\right) d_z^{\alpha}\phi(z; 0, x)\kappa(x)\,dz\,dx \\
&\quad= (-1)^{\beta} \int_{\mathbb{R}^2} \sum_{v\le\beta} q_{\beta,v}\left(y, \frac{1}{\sqrt{x}}\right) g^{(v)}(y)x^{-\alpha/2} d_y^{\alpha}\phi(y; 0, 1)\kappa(x)\,dy\,dx \\
&\quad= (-1)^{\beta} \int_{\mathbb{R}} g(y) \sum_{v\le\beta} (-1)^v d_y^v \bigg\{ d_y^{\alpha}\phi(y; 0, 1) \\
&\qquad\quad \times \int_{\mathbb{R}} q_{\beta,v}\left(y, \frac{1}{\sqrt{x}}\right) x^{-\alpha/2}\kappa(x)\,dx \bigg\}\,dy \\
&\quad= \int_{\mathbb{R}} g(y) \sum_{v\le\beta} (-1)^{\beta+v} d_y^v \big\{ d_y^{\alpha}\phi(y; 0, 1)\mathbb{E}\big[HC^{-\alpha/2} q_{\beta,v}\big(y, C^{-1/2}\big)\big]\big\}\,dy.
\end{aligned}$$

Clearly, the above identity will enable us to compute the Edgeworth expansion for the Studentized statistic $Z_n/\sqrt{F_n}$. We need to determine the polynomials $q_{\beta,v}$ for $\beta = 0, 1, 2$:

$$\begin{aligned}
&q_{0,0}(a, b) = 1, \qquad q_{1,0}(a, b) = 0, \qquad q_{1,1}(a, b) = -\frac{1}{2}ab^2, \\
&q_{2,0}(a, b) = 0, \qquad q_{2,1}(a, b) = \frac{3}{4}ab^4, \qquad q_{2,2}(a, b) = \frac{1}{4}a^2b^4.
\end{aligned}$$

Recall the identity $d_y^{\alpha}\phi(y; 0, 1) = (-1)^{\alpha} H_{\alpha}(y)\phi(y; 0, 1)$ and

$$H_1(x) = x, \qquad H_3(x) = x^3 - 3x, \qquad H_5(x) = x^5 - 10x^3 + 15x.$$

A straightforward computation shows that

$$\int_{\mathbb{R}^2} g\left(\frac{z}{\sqrt{x}}\right) p_1(z,x)\,dz\,dx = \mathbb{E}[\mathcal{H}_2 C^{-1/2}] \int_{\mathbb{R}} g(y) y \phi(y;0,1)\,dy,$$

$$\int_{\mathbb{R}^2} g\left(\frac{z}{\sqrt{x}}\right) \sum_{j=4}^{5} p_j(z,x)\,dz\,dx = -\frac{1}{2}\mathbb{E}[\mathcal{H}_5 C^{-3/2}] \int_{\mathbb{R}} g(y) y \phi(y;0,1)\,dy,$$

$$\int_{\mathbb{R}^2} g\left(\frac{z}{\sqrt{x}}\right) \sum_{j=6}^{8} p_j(z,x)\,dz\,dx = \frac{3}{4}\mathbb{E}[\mathcal{H}_4 C^{-5/2}] \int_{\mathbb{R}} g(y) y \phi(y;0,1)\,dy.$$

The corresponding computation for the terms $p_2(z,x)$ and $p_3(z,x)$ has to be performed separately, since the random variables c_2 and c_3 depend on z. Recall that the quantities $\mathcal{H}_1(z)$ and $\mathcal{H}_3(z)$ are linear in z, that is, $\mathcal{H}_1(z) = z\widetilde{\mathcal{H}}_1$, $\mathcal{H}_3(z) = z\widetilde{\mathcal{H}}_3$. We deduce as above (here $\kappa(x) = \mathbb{E}[\widetilde{\mathcal{H}}_3 | C = x] p^C(x)$)

$$\begin{aligned}
&\int_{\mathbb{R}^2} g\left(\frac{z}{\sqrt{x}}\right) p_2(z,x)\,dz\,dx \\
&\quad = -\int_{\mathbb{R}^2} z g\left(\frac{z}{\sqrt{x}}\right) d_x[\phi(z;0,x)\kappa(x)]\,dz\,dx \\
&\quad = \int_{\mathbb{R}} g(y) d_y\{y\phi(y;0,1)\mathbb{E}[\widetilde{\mathcal{H}}_3 q_{1,1}(y, C^{-1/2}) C^{1/2}]\}\,dy \\
&\quad = \frac{1}{2}\mathbb{E}[\widetilde{\mathcal{H}}_3 C^{-1/2}] \int_{\mathbb{R}} g(y)\phi(y;0,1)(2y - y^3)\,dy.
\end{aligned}$$

Finally, we obtain that (here $\kappa(x) = \mathbb{E}[\widetilde{\mathcal{H}}_1 | C = x] p^C(x)$)

$$\begin{aligned}
&\int_{\mathbb{R}^2} g\left(\frac{z}{\sqrt{x}}\right) p_3(z,x)\,dz\,dx \\
&\quad = \int_{\mathbb{R}^2} g\left(\frac{z}{\sqrt{x}}\right) d_z^2[z\phi(z;0,x)\kappa(x)]\,dz\,dx \\
&\quad = \int_{\mathbb{R}^2} x^{-1} g''(y) y\phi(y;0,1)\kappa(x)\,dy\,dx \\
&\quad = \mathbb{E}[\widetilde{\mathcal{H}}_1 C^{-1/2}] \int_{\mathbb{R}} g(y) d_y^2[y\phi(y;0,1)]\,dy \\
&\quad = \mathbb{E}[\widetilde{\mathcal{H}}_1 C^{-1/2}] \int_{\mathbb{R}} g(y) H_3(y)\phi(y;0,1)\,dy.
\end{aligned}$$

Combining the above results, we deduce the Edgeworth expansion for the density of $Z_n/\sqrt{F_n}$, which is one of the main statements of the paper.

COROLLARY 6.1. *Assume that the conditions of Theorem 5.2 hold. Then we obtain the expansion*

$$p^{Z_n/\sqrt{F_n}}(y) = \phi(y;0,1) + \Delta_n^{1/2}\phi(y;0,1)\Big(y\Big\{\mathbb{E}[\mathcal{H}_2 C^{-1/2}] - \frac{1}{2}\mathbb{E}[\mathcal{H}_5 C^{-3/2}] + \frac{3}{4}\mathbb{E}[\mathcal{H}_4 C^{-5/2}] + \mathbb{E}[\widetilde{\mathcal{H}}_3 C^{-1/2}] - 3\mathbb{E}[\widetilde{\mathcal{H}}_1 C^{-1/2}]\Big\} + y^3\Big\{\mathbb{E}[\widetilde{\mathcal{H}}_1 C^{-1/2}] - \frac{1}{2}\mathbb{E}[\widetilde{\mathcal{H}}_3 C^{-1/2}]\Big\}\Big). \tag{6.1}$$

If we consider the quantity M_n from (3.2) with $a \equiv 1$ and $\Delta_n = n^{-1}$, that is,

$$M_n = n^{-1/2}\sum_{i=1}^{n} f\big(\sqrt{n}\Delta_i^n W\big) \qquad \text{with } \mathbb{E}\big[f(W_1)\big] = 0,$$

and $F_n = C = \mathbb{E}[f^2(W_1)]$, a straightforward application of the formula (6.1) implies the identity

$$p^{M_n/C} = \phi(y;0,1) + \frac{\mathbb{E}[f^3(W_1)]}{6\sqrt{nC^3}}\phi(y;0,1)H_3(y),$$

where H_3 denotes the third Hermite polynomial. This identity corresponds to the classical Edgeworth expansion for sums of i.i.d. random variables.

REMARK 6.2. In practice, the application of the asymptotic expansion at (6.1) requires the knowledge of the coefficients of the type $b^{[k_1\cdots k_d]}$ [cf. (4.9)]. While the volatility related processes $b^{[1]}$, $b^{[1.1]}$, $b^{[1.1.1]}$ can be estimated from high frequency data X_{t_i}, the drift related processes $b^{[2]}$, $b^{[2.1]}$, $b^{[1.2]}$ cannot be consistently estimated on a fixed time span. Thus, the applicability of the Edgeworth expansion at (6.1) relies on the knowledge of the drift related coefficients or their estimation on an infinite time span.

7. Proofs.

7.1. *A stochastic expansion.* Below, we denote by K a generic positive constant, which may change from line to line. We also write K_p if the constant depends on an external parameter p.

PROOF OF THEOREM 4.2. First, we remark that all processes of the type $(b_s^{[k_1\cdots k_m]})_{s\geq 0}$ ($k_j \in \{1,2\}$), which we consider below, are continuous and so locally bounded. Applying the localization technique described in Section 3 of [2] we can assume w.l.o.g. that these processes are bounded in (ω, s), which we do from now on. We decompose

$$\Delta_n^{-1/2}\big(V_n(f) - V(f)\big) = M_n + R_n^{(1)} + R_n^{(2)}$$

with

$$R_n^{(1)} = \Delta_n^{-1/2}\Bigg(V_n(f) - \Delta_n \sum_{i=1}^{1/\Delta_n} f\big(\alpha_i^n\big)\Bigg),$$

$$R_n^{(2)} = \Delta_n^{-1/2}\Bigg(\Delta_n \sum_{i=1}^{1/\Delta_n} \rho_{b^{[1]}_{t_{i-1}}} - V(f)\Bigg).$$

We start with the asymptotic expansion of the quantity $R_n^{(2)}$. Due to Burkhölder inequality any process Y of the form (3.1) with bounded coefficients $b^{[1]}$, $b^{[2]}$ satisfies the inequality

$$\mathbb{E}\big[|Y_t - Y_s|^p\big] \le C_p |t-s|^{p/2} \tag{7.1}$$

for any $p \ge 0$. In particular, this inequality holds for the processes $b^{[1]}$, $b^{[2]}$, $b^{[2.2]}$, $b^{[2.1]}$, $b^{[1.2]}$, $b^{[1.1]}$ as they are diffusion processes (due to Itô formula). Applying (7.1) and the Taylor expansion, we deduce that

$$\begin{aligned}
R_n^{(2)} &= \Delta_n^{-1/2} \sum_{i=1}^{1/\Delta_n} \int_{t_{i-1}}^{t_i} \{\rho_{b^{[1]}_{t_{i-1}}} - \rho_{b^{[1]}_s}\}\, ds \\
&= \Delta_n^{-1/2} \sum_{i=1}^{1/\Delta_n} \int_{t_{i-1}}^{t_i} \Big\{\rho'_{b^{[1]}_{t_{i-1}}}\big(b^{[1]}_{t_{i-1}} - b^{[1]}_s\big) - \frac{1}{2}\rho''_{b^{[1]}_{t_{i-1}}}\big(b^{[1]}_{t_{i-1}} - b^{[1]}_s\big)^2\Big\}\, ds \\
&\quad + o_{\mathbb{P}}\big(\Delta_n^{1/2}\big) \\
&=: R_n^{(2.1)} + R_n^{(2.2)} + o_{\mathbb{P}}\big(\Delta_n^{1/2}\big).
\end{aligned}$$

Recall that $db^{[1]}_t = b^{[1.2]}_t\, dt + b^{[1.1]}_t\, dW_t$. We conclude the identity

$$\begin{aligned}
R_n^{(2.2)} &= -\frac{\Delta_n^{-1/2}}{2} \sum_{i=1}^{1/\Delta_n} \rho''_{b^{[1]}_{t_{i-1}}} \int_{t_{i-1}}^{t_i} \big(b^{[1]}_{t_{i-1}} - b^{[1]}_s\big)^2\, ds \\
&= -\frac{\Delta_n^{-1/2}}{2} \sum_{i=1}^{1/\Delta_n} \rho''_{b^{[1]}_{t_{i-1}}} \big|b^{[1.1]}_{t_{i-1}}\big|^2 \int_{t_{i-1}}^{t_i} (W_{t_{i-1}} - W_s)^2\, ds + o_{\mathbb{P}}\big(\Delta_n^{1/2}\big) \\
&=: \Delta_n^{1/2}\big(N_{n,4}^{(1)} + o_{\mathbb{P}}(1)\big).
\end{aligned}$$

For the term $R_n^{(2.1)}$, we obtain the decomposition

$$\begin{aligned}
R_n^{(2.1)} &= -\Delta_n^{-1/2} \sum_{i=1}^{1/\Delta_n} \rho'_{b^{[1]}_{t_{i-1}}} \int_{t_{i-1}}^{t_i} \Big(\int_{t_{i-1}}^{s} b^{[1.2]}_u\, du + \int_{t_{i-1}}^{s} b^{[1.1]}_u\, dW_u\Big)\, ds \\
&= -\Delta_n^{-1/2} \sum_{i=1}^{1/\Delta_n} \rho'_{b^{[1]}_{t_{i-1}}} \Big(\frac{\Delta_n^2}{2} b^{[1.2]}_{t_{i-1}} + b^{[1.1]}_{t_{i-1}} \int_{t_{i-1}}^{t_i} (W_s - W_{t_{i-1}})\, ds\Big) + o_{\mathbb{P}}\big(\Delta_n^{1/2}\big) \\
&=: \Delta_n^{1/2}\big(N_{n,4}^{(2)} + N_{n,5} + o_{\mathbb{P}}(1)\big).
\end{aligned}$$

We remark that $N_{n,4} = N_{n,4}^{(1)} + N_{n,4}^{(2)}$. The treatment of the quantity $R_n^{(1)}$ is a bit more involved. We apply again (7.1) and Taylor expansion:

$$\begin{aligned} R_n^{(1)} &= \Delta_n^{1/2} \sum_{i=1}^{1/\Delta_n} \left(f'(\alpha_i^n) \left\{ \frac{\Delta_i^n X}{\sqrt{\Delta_n}} - \alpha_i^n \right\} + \frac{1}{2} f''(\alpha_i^n) \left\{ \frac{\Delta_i^n X}{\sqrt{\Delta_n}} - \alpha_i^n \right\}^2 \right) + o_{\mathbb{P}}(\Delta_n^{1/2}) \\ &=: R_n^{(1.1)} + R_n^{(1.2)} + o_{\mathbb{P}}(\Delta_n^{1/2}). \end{aligned}$$

For the term $R_n^{(1.2)}$, we obtain the decomposition

$$\begin{aligned} R_n^{(1.2)} &= \frac{\Delta_n^{-1/2}}{2} \sum_{i=1}^{1/\Delta_n} f''(\alpha_i^n) \left(\int_{t_{i-1}}^{t_i} b_s^{[2]} \, ds + \int_{t_{i-1}}^{t_i} b_s^{[1]} - b_{t_{i-1}}^{[1]} \, dW_s \right)^2 \\ &= \frac{\Delta_n^{-1/2}}{2} \sum_{i=1}^{1/\Delta_n} f''(\alpha_i^n) \left(\Delta_n b_{t_{i-1}}^{[2]} + b_{t_{i-1}}^{[1.1]} \int_{t_{i-1}}^{t_i} (W_s - W_{t_{i-1}}) \, dW_s \right)^2 \\ &\quad + o_{\mathbb{P}}(\Delta_n^{1/2}) \\ &= \frac{\Delta_n^{3/2}}{2} \sum_{i=1}^{1/\Delta_n} f''(\alpha_i^n) \left(b_{t_{i-1}}^{[2]} + \frac{1}{2} b_{t_{i-1}}^{[1.1]} H_2 \left(\frac{\Delta_i^n W}{\sqrt{\Delta_n}} \right) \right)^2 + o_{\mathbb{P}}(\Delta_n^{1/2}) \\ &= \Delta_n^{1/2} \big(N_{n,3} + o_{\mathbb{P}}(1) \big). \end{aligned}$$

The quantity $R_n^{(1.1)}$ is decomposed as

$$R_n^{(1.1)} = \sum_{i=1}^{1/\Delta_n} f'(\alpha_i^n) \left(\int_{t_{i-1}}^{t_i} b_s^{[2]} \, ds + \int_{t_{i-1}}^{t_i} \{ b_s^{[1]} - b_{t_{i-1}}^{[1]} \} \, dW_s \right) =: R_n^{(1.1.1)} + R_n^{(1.1.2)}$$

with

$$\begin{aligned} R_n^{(1.1.1)} &= \Delta_n \sum_{i=1}^{1/\Delta_n} f'(\alpha_i^n) \left(b_{t_{i-1}}^{[2]} \, ds + \frac{1}{2} b_{t_{i-1}}^{[1.1]} H_2 \left(\frac{\Delta_i^n W}{\sqrt{\Delta_n}} \right) \right), \\ R_n^{(1.1.2)} &= \sum_{i=1}^{1/\Delta_n} f'(\alpha_i^n) \left(\int_{t_{i-1}}^{t_i} \{ b_s^{[2]} - b_{t_{i-1}}^{[2]} \} \, ds \right. \\ &\quad \left. + \int_{t_{i-1}}^{t_i} \left(\int_{t_{i-1}}^{s} b_u^{[1.2]} \, du + \int_{t_{i-1}}^{s} \{ b_u^{[1.1]} - b_{t_{i-1}}^{[1.1]} \} \, dW_u \right) dW_s \right). \end{aligned}$$

We remark that $R_n^{(1.1.1)} = \Delta_n^{1/2} N_{n,1}$. Since f' is an odd function (because f is even), we deduce that

$$\begin{aligned} R_n^{(1.1.2)} &= \sum_{i=1}^{1/\Delta_n} f'(\alpha_i^n) \left(b_{t_{i-1}}^{[2.1]} \int_{t_{i-1}}^{t_i} (W_s - W_{t_{i-1}}) \, ds + \frac{\Delta_n^{3/2} b_{t_{i-1}}^{[1.1.1]}}{6} H_3 \left(\frac{\Delta_i^n W}{\sqrt{\Delta_n}} \right) \right. \\ &\quad \left. + b_{t_{i-1}}^{[1.2]} \int_{t_{i-1}}^{t_i} (s - t_{i-1}) \, dW_s \right) + o_{\mathbb{P}}(\Delta_n^{1/2}). \end{aligned}$$

As $R_n^{(1.1.2)} = \Delta_n^{1/2}(N_{n,2} + o_{\mathbb{P}}(1))$, we are done. $\square$

APPENDIX

In this subsection, we present a law of large numbers and a multivariate functional stable convergence theorem, which is frequently used in this paper. For any $k = 1, \dots, d$, let $g^k : C([0,1]) \to \mathbb{R}$ be a measurable function with polynomial growth, that is,

$$|g^k(x)| \le K(1 + \|x\|_\infty^p),$$

for some $K > 0$, $p > 0$ and $\|x\|_\infty = \sup_{z\in[0,1]} |x(z)|$. In most cases, g^k will be a function of $x(1)$; the path-dependent version is only required to account for the asymptotic behavior of the functional C_n. Let $(a_s)_{s\ge 0}$ be an $\mathbb{R}^d$-valued, $(\mathcal{F}_s)$-adapted, continuous and bounded stochastic process. Our first result is the following theorem.

THEOREM A.1. *Let $g : \mathbb{R}^d \times C([0,1]) \to \mathbb{R}$ be a measurable function with polynomial growth in the last variable and $a = (a_1, \dots, a_d)$. Then it holds that*

$$\Delta_n \sum_{i=1}^{1/\Delta_n} g\big(a_{t_{i-1}}, \Delta_n^{-1/2}\{W_{t_{i-1}+s\Delta_n} - W_{t_{i-1}}\}_{0\le s\le 1}\big) \xrightarrow{\mathbb{P}} \int_0^1 \rho(a_s, g)\,ds$$

with $\rho(z,g) := \mathbb{E}[g(z, \{W_s\}_{0\le s\le 1})]$, $z \in \mathbb{R}^d$.

PROOF. Since $\Delta_n^{-1/2}\{W_{t_{i-1}+s\Delta_n} - W_{t_{i-1}}\}_{0\le s\le 1} \overset{d}{=} \{W_s\}_{0\le s\le 1}$, we obtain that

$$\begin{aligned}
&\Delta_n \sum_{i=1}^{1/\Delta_n} \mathbb{E}\big[g\big(a_{t_{i-1}}, \Delta_n^{-1/2}\{W_{t_{i-1}+s\Delta_n} - W_{t_{i-1}}\}_{0\le s\le 1}\big)|\mathcal{F}_{t_{i-1}}\big] \\
&\quad = \Delta_n \sum_{i=1}^{1/\Delta_n} \rho(a_{t_{i-1}}, g) \xrightarrow{\mathbb{P}} \int_0^1 \rho(a_s, g)\,ds.
\end{aligned}$$

On the other hand, we deduce that

$$\begin{aligned}
&\Delta_n \sum_{i=1}^{1/\Delta_n} g\big(a_{t_{i-1}}, \Delta_n^{-1/2}\{W_{t_{i-1}+s\Delta_n} - W_{t_{i-1}}\}_{0\le s\le 1}\big) \\
&\quad - \Delta_n \sum_{i=1}^{1/\Delta_n} \mathbb{E}\big[g\big(a_{t_{i-1}}, \Delta_n^{-1/2}\{W_{t_{i-1}+s\Delta_n} - W_{t_{i-1}}\}_{0\le s\le 1}\big)|\mathcal{F}_{t_{i-1}}\big] \xrightarrow{\mathbb{P}} 0,
\end{aligned}$$

because

$$\Delta_n^2 \sum_{i=1}^{1/\Delta_n} \mathbb{E}\big[g^2\big(a_{t_{i-1}}, \Delta_n^{-1/2}\{W_{t_{i-1}+s\Delta_n} - W_{t_{i-1}}\}_{0\le s\le 1}\big)|\mathcal{F}_{t_{i-1}}\big] \xrightarrow{\mathbb{P}} 0.$$

This completes the proof. □

Next, we consider a sequence of d-dimensional processes $Y_t^n = (Y_{1,t}^n, \dots, Y_{d,t}^n)$ defined via

$$Y_{k,t}^n = \Delta_n^{1/2} \sum_{i=1}^{[t/\Delta_n]} a_{t_{i-1}}^k \big[g^k\big(\Delta_n^{-1/2}\{W_{t_{i-1}+s\Delta_n} - W_{t_{i-1}}\}_{0\le s\le 1}\big) - \mathbb{E}g^k\big(\Delta_n^{-1/2}\{W_{t_{i-1}+s\Delta_n} - W_{t_{i-1}}\}_{0\le s\le 1}\big)\big], \qquad k = 1, \dots, d,$$

where the functions g^k satisfy the assumption of Theorem A.1. The stable convergence of Y^n is as follows.

THEOREM A.2. *It holds that*

$$Y_t^n \xrightarrow{d_{\text{st}}} Y_t = \int_0^t v_s \, dW_s + \int_0^t \big(w_s - v_s v_s^\star\big)^{1/2} \, dW_s',$$

where the functional convergence is stable in law, W' is a d-dimensional Brownian motion independent of $\mathcal{F}$, and the processes $(v_s)_{s\ge 0}$ in $\mathbb{R}^d$ and $(w_s)_{s\ge 0}$ in $\mathbb{R}^{d\times d}$ are defined as

$$v_s^k = a_s^k \mathbb{E}\big[g^k\big(\{W_s\}_{0\le s\le 1}\big)W_1\big],$$
$$w_s^{kl} = a_s^k a_s^l \operatorname{Cov}\big[g^k\big(\{W_s\}_{0\le s\le 1}\big), g_l\big(\{W_s\}_{0\le s\le 1}\big)\big],$$

with $1 \le k, l \le d$. In particular, it holds that $\int_0^t w_s^{1/2} \, dW_s' \sim MN(0, \int_0^t w_s \, ds)$.

PROOF. We write $Y_t^n = \sum_{i=1}^{[t/\Delta_n]} \chi_i^n$ with

$$\chi_{i,k}^n = \Delta_n^{1/2} a_{t_{i-1}}^k \big[g^k\big(\Delta_n^{-1/2}\{W_{t_{i-1}+s\Delta_n} - W_{t_{i-1}}\}_{0\le s\le 1}\big) - \mathbb{E}g^k\big(\Delta_n^{-1/2}\{W_{t_{i-1}+s\Delta_n} - W_{t_{i-1}}\}_{0\le s\le 1}\big)\big], \qquad k = 1, \dots, d.$$

According to Theorem IX.7.28 of [13], we need to show that

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}\big[\chi_{i,k}^n \chi_{i,l}^n | \mathcal{F}_{t_{i-1}}\big] \xrightarrow{\mathbb{P}} \int_0^t w_s^{kl} \, ds, \tag{A.1}$$

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}\big[\chi_{i,k}^n \Delta_i^n W | \mathcal{F}_{t_{i-1}}\big] \xrightarrow{\mathbb{P}} \int_0^t v_s^k \, ds, \tag{A.2}$$

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}\big[|\chi_{i,k}^n|^2 1_{\{|\chi_{i,k}^n| > \epsilon\}} | \mathcal{F}_{t_{i-1}}\big] \xrightarrow{\mathbb{P}} 0 \qquad \forall \epsilon > 0, \tag{A.3}$$

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}\big[\chi_{i,k}^n \Delta_i^n Q | \mathcal{F}_{t_{i-1}}\big] \xrightarrow{\mathbb{P}} 0\,, \tag{A.4}$$

where $1 \le k, l \le d$ and the last condition must hold for all bounded continuous martingales Q with $[W, Q] = 0$. Conditions (A.1) and (A.2) are obvious since

$$\Delta_n^{-1/2}\{W_{t_{i-1}+s\Delta_n} - W_{t_{i-1}}\}_{0\le s\le 1} \stackrel{d}{=} \{W_s\}_{0\le s\le 1}.$$

Condition (A.3) follows from

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}[|\chi_{i,k}^n|^2 1_{\{|\chi_{i,k}^n|>\epsilon\}}|\mathcal{F}_{t_{i-1}}] \le \epsilon^{-2} \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}[|\chi_{i,k}^n|^4|\mathcal{F}_{t_{i-1}}] \le K\Delta_n \to 0,$$

which holds since the process a is bounded and g^k is of polynomial growth. In order to prove the last condition, we use the Itô–Clark representation theorem

$$g^k(\Delta_n^{-1/2}\{W_{t_{i-1}+s\Delta_n} - W_{t_{i-1}}\}_{0\le s\le 1}) - \mathbb{E}g^k(\Delta_n^{-1/2}\{W_{t_{i-1}+s\Delta_n} - W_{t_{i-1}}\}_{0\le s\le 1})$$
$$= \int_{t_{i-1}}^{t_i} \eta_{k,s}^n \, dW_s$$

for some predictable process η_k^n. Itô isometry implies the identity

$$\mathbb{E}[\chi_{i,k}^n \Delta_i^n Q|\mathcal{F}_{t_{i-1}}] = \Delta_n^{1/2} a_{t_{i-1}}^k \mathbb{E}\left[\int_{t_{i-1}}^{t_i} \eta_{k,s}^n d[W, Q]_s|\mathcal{F}_{t_{i-1}}\right] = 0.$$

This completes the proof of Theorem A.2. □

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