Stochastic Systems
2016, Vol. 6, No. 2, 251–300
DOI: 10.1214/15-SSY198

ASYMPTOTIC BEHAVIOR OF A CRITICAL FLUID MODEL FOR A PROCESSOR SHARING QUEUE VIA RELATIVE ENTROPY

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In this paper, we develop a new approach to studying the asymptotic behavior of fluid model solutions for critically loaded processor sharing queues. For this, we introduce a notion of relative entropy associated with measure-valued fluid model solutions. In contrast to the approach used in [12], which does not readily generalize to networks of processor sharing queues, we expect the approach developed in this paper to be more robust. Indeed, we anticipate that similar notions involving relative entropy may be helpful for understanding the asymptotic behavior of critical fluid model solutions for stochastic networks operating under various resource sharing protocols naturally described by measure-valued processes.

1. Introduction. In the context of multiclass queueing networks operating under head-of-the-line (HL) service disciplines, Bramson [1] and Williams [15] have developed a modular approach for establishing heavy traffic diffusion approximations to such networks. In particular, they have given sufficient conditions under which asymptotic behavior of critical fluid model solutions can be used to prove state space collapse and thereby a heavy traffic limit theorem justifying a diffusion approximation. Although the HL assumption covers a wide variety of service disciplines, including first-in-first-out (FIFO) and static priorities, it requires that service for a given job class is concentrated on the job at the head-of-the-line. Consequently, it does not cover some disciplines that arise naturally in applications, such as the processor sharing discipline. While it is desirable to have a modular approach to proving diffusion approximations for stochastic networks with

Received July 2015.

‡Research supported in part by IPAM UCLA, an ROA supplement to NSF Grant DMS-1206772, and NSF Grant DMS-1510198.

§Research supported in part by NSF grant DMS-1206772.

MSC 2010 subject classifications: Primary 60K25, 60F17; secondary 60G57, 68M20, 90B22.

Keywords and phrases: Queueing, processor sharing, critical fluid model, fluid model asymptotics, relative entropy.

non-HL disciplines, the development of such an approach is only in its early stages. Following the general idea of such an approach, Gromoll et al. [5, 6] established a diffusion approximation for a measure-valued descriptor of a single server processor sharing queue. A key ingredient in that work was an analysis of the long time behavior of the measure-valued solutions of a critical fluid model. This analysis was performed in Puha and Williams [12] using coupling and renewal theory, which also provided a rate of convergence under suitable moment conditions. The results of [12] were then used by Gromoll [5] to prove state space collapse and the desired diffusion approximation for the single server processor sharing queue.

We are interested in developing the modular approach further, for processor sharing networks and for other stochastic networks having measure-valued state descriptors. While some aspects of the approach generalize readily, interactions between nodes may manifest complex behaviors not analyzable via the coupling and renewal methodology used in [12]. In this paper, we develop an alternative approach to that in [12] which employs a notion of relative entropy. We use this to derive the asymptotic behavior of the measure-valued solutions of a critical fluid model for a single server processor sharing queue. While we limit our analysis to this case for the moment, we believe that our notion of relative entropy is a natural one for studying the dynamic behavior of critical fluid models with measure-valued solutions. We expect it will be helpful for proving diffusion approximations for networks of processor sharing queues and some other stochastic networks having measure-valued state descriptors.

We now elaborate on the contents of this paper. We consider a critical fluid model for a single server processor sharing queue. This model was first introduced in [6] where it was shown that critical fluid model solutions arise as functional law of large numbers limits of measure-valued processes used to track the residual service times of jobs in heavily loaded processor sharing queues. The critical fluid model has one parameter, a Borel probability measure ν on $\mathbb{R}_+ = [0, \infty)$ that does not charge the origin and that has a finite positive mean $1/\alpha$, where $\alpha \in (0, \infty)$. The measure ν corresponds to the weak limit of the service time distributions for a sequence of heavily loaded processor sharing queues. The reciprocal α of its mean corresponds to the limiting average rate at which jobs arrive to the queue.

Let $\mathbf{M}$ denote the set of finite, nonnegative Borel measures on $\mathbb{R}_+$. A solution of the critical fluid model is a function $\mu : [0, \infty) \to \mathbf{M}$ that satisfies conditions (C.1)–(C.4) in Section 2 of this paper. For $t \in [0, \infty)$, we shall denote $\mu(t)$ by μ_t for convenience, and we refer to μ_t as the state at time t. We shall call μ_0 the initial state for μ, or equivalently the initial condition for μ. It was shown in [6] that if $\xi \in \mathbf{M}$ is continuous, i.e., it does not charge singletons, then there exists a unique fluid model solution μ such that $\mu_0 = \xi$. Here we shall focus on fluid model solutions with continuous initial conditions. In [12], it was shown that the critical fluid model has associated with it a collection $\mathbf{I}$ of invariant states given by

$$\mathbf{I} = \{\beta\nu_e : \beta \in \mathbb{R}_+\}.$$

Here ν_e is the so-called excess lifetime distribution associated with ν. It is the Borel probability measure on $\mathbb{R}_+$ that is absolutely continuous with respect to Lebesgue measure and that has density function

$$n_e(x) = \alpha\nu((x, \infty)), \quad x \in \mathbb{R}_+.$$

In [12], renewal theory arguments were used to show that, if $\xi \in \mathbf{M}$ is continuous and has a finite first moment, then the time t value μ_t of a critical fluid model solution μ converges weakly as $t \to \infty$ to an invariant state $\beta\nu_e$, for some $\beta \in \mathbb{R}_+$ [12, Theorem 1.2]. Under additional moment conditions on both ν and the initial state ξ, a rate of convergence was obtained using coupling techniques [12, Theorem 1.3]. In this paper, we develop an alternative approach to prove results of this nature that employs a relative entropy functional. Using this relative entropy approach, we are able to prove a uniform convergence result for initial states lying in certain relatively compact sets (see (9)) that holds under a finite second moment assumption on ν without appealing to renewal theoretic arguments. The latter condition is less restrictive than the strictly greater than two moments condition assumed in [12, Theorem 1.3]. We do this with an eye toward developing a technique that might apply to more general network models with interactions between network nodes.

A natural candidate for a relative entropy functional is the relative entropy of the time t value μ_t of a fluid model solution μ with respect to an appropriate invariant state $\beta\nu_e$. However, since invariant states are absolutely continuous with respect to Lebesgue measure and fluid model solutions do not necessarily satisfy this property, it is possible for such a function to be infinite for all time. We circumvent this issue by using an alternative relative entropy. Instead we use the relative entropy of the excess lifetime distribution $(\mu_t)_e$ associated with μ_t with respect to the excess lifetime distribution $(\nu_e)_e$ associated with ν_e. Indeed, we give sufficient conditions for this relative entropy to converge to zero as $t \to \infty$, uniformly for all initial states lying in certain relatively compact sets (see Theorem 3.2). Theorem 3.2 is one of two main results proved in this paper. Its proof is due to an absolute continuity in time property satisfied by our choice of relative entropy when applied to a fluid model solution. Indeed, we find an explicit representation for the almost everywhere defined time derivative that we work with to prove Theorem 3.2 (see Theorem 7.1). From Theorem 3.2, it is immediate that $(\mu_t)_e$ converges weakly to $(\nu_e)_e$ as $t \to \infty$, uniformly for all initial states lying in certain relatively compact sets. This uniform convergence of "shapes" of the excess lifetime distributions does not immediately imply the weak convergence of μ_t to an invariant state as $t \to \infty$. However, we are able to use this convergence of shapes, in conjunction with other properties of fluid model solutions, to establish the desired uniform weak convergence of μ_t in Theorem 3.1 and Corollary 3.1.

Theorem 3.1 is the other main result proved in this paper. This theorem provides sufficient conditions for μ_t to be uniformly close to $\mathbf{I}$ for all t sufficiently large and all initial states lying in suitable relatively compact sets. Theorem 3.1 almost follows from Theorem 3.2 by using a continuity property of the relative entropy functional and a compact containment property of fluid model solutions. The qualifier "almost" is due to the fact that the relative entropy functional takes the value zero for some measures that are not in $\mathbf{I}$, which we address separately to complete the proof of Theorem 3.1.

The paper is organized as follows. Section 2 contains the definition of a critical fluid model solution and a summary of its properties proved in [6]. Following that, in Section 3, the relative entropy functional is introduced and the main results of the paper, Theorems 3.1 and 3.2, and Corollary 3.1, are stated. In Section 4, we more fully develop the relative entropy functional as a measure of distance and specify its properties. In Section 5, we develop additional properties of fluid model solutions used in the relative entropy arguments given here. In Section 6, we prove that Theorem 3.1 follows from a combination of Theorem 3.2, properties of the relative entropy functional developed in Section 4, and properties of fluid model solutions developed in Section 5. In that section, we also develop some additional consequences of Theorem 3.2. The remainder of the paper is devoted to proving Theorem 3.2. In Section 7, we develop properties of the relative entropy functional as a function of time along fluid model solutions. These properties, along with properties of the relative entropy functional and of fluid model solutions, obtained in Sections 4 and 5, are used in Section 8 to prove Theorem 3.2.

1.1. *Notation.* The following notation will be used throughout the paper. Let $\mathbb{Z}$ denote the set of integers, $\mathbb{Z}_+$ denote the set of nonnegative integers and $\mathbb{N}$ denote the set of strictly positive integers. Let $\mathbb{R}$ denote the set of real numbers. For $x, y \in \mathbb{R}$, we write $x \vee y$ for the maximum of x and y and we write $x \wedge y$ for the minimum of x and y. Then, for $x \in \mathbb{R}$, we let x^+ denote the positive part of x and $|x|$ denote the absolute value of x, i.e., $x^+ = 0 \vee x$ and $|x| = x \vee (-x)$. The set of nonnegative real numbers $[0, \infty)$ will be denoted by $\mathbb{R}_+$. For a Borel set $B \subset \mathbb{R}_+$, we denote the indicator function of the set B by 1_B. We also define the real-valued function $\chi(x) = x$, for $x \in \mathbb{R}_+$.

Let $\mathbf{C}_b(\mathbb{R}_+)$ denote the set of bounded continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$ and $\mathbf{C}_b^1(\mathbb{R}_+)$ denote the set of functions in $\mathbf{C}_b(\mathbb{R}_+)$ that are once differentiable with derivative in $\mathbf{C}_b(\mathbb{R}_+)$. Given a nonnegative Borel measure ζ on $\mathbb{R}_+$, let $\mathbf{L}^1(\zeta)$ denote the set of Borel measurable functions from $\mathbb{R}_+$ to $\mathbb{R}$ that are integrable with respect to ζ. For $g \in \mathbf{L}^1(\zeta)$, we let $\langle g, \zeta \rangle = \int_{\mathbb{R}_+} g(x)\zeta(dx)$. We occasionally use this notation also if $g \geq 0$, in which case $\langle g, \zeta \rangle$ may take the value infinity. When ζ is Lebesgue measure, we simply write $\mathbf{L}^1$ for $\mathbf{L}^1(\zeta)$.

As mentioned in the introduction, we let $\mathbf{M}$ denote the set of finite, nonnegative Borel measures on $\mathbb{R}_+$. We denote the zero measure in $\mathbf{M}$ by $\mathbf{0}$ and the point mass at $x \in \mathbb{R}_+$ by δ_x. The set $\mathbf{M}$ is endowed with the topology of weak convergence. With this topology, $\mathbf{M}$ is a Polish space [13], and for $\zeta_n, \zeta \in \mathbf{M}$, $n \in \mathbb{N}$, we have $\zeta_n \stackrel{w}{\to} \zeta$ as $n \to \infty$ if and only if $\langle g, \zeta_n \rangle \to \langle g, \zeta \rangle$ as $n \to \infty$, for all $g \in \mathbf{C}_b(\mathbb{R}_+)$, where $\stackrel{w}{\to}$ denotes weak convergence. We note that $\mathbf{C}_b^1(\mathbb{R}_+)$ is convergence determining for this topology, i.e., if

$$\langle g, \zeta_n \rangle \to \langle g, \zeta \rangle \qquad \text{as } n \to \infty, \qquad \text{for all } g \in \mathbf{C}_b^1(\mathbb{R}_+), \tag{1}$$

then $\zeta_n \stackrel{w}{\to} \zeta$ as $n \to \infty$. To see this note that $g \equiv 1 \in \mathbf{C}_b^1(\mathbb{R}_+)$ so that (1) implies $\lim_{n\to\infty} \langle 1, \zeta_n \rangle = \langle 1, \zeta \rangle$. If (1) holds and $\langle 1, \zeta \rangle = 0$, then $\zeta_n \stackrel{w}{\to} \mathbf{0}$ as $n \to \infty$. If (1) holds and $\langle 1, \zeta \rangle > 0$, then without loss of generality we may assume that $\langle 1, \zeta_n \rangle > 0$ for all $n \in \mathbb{N}$. By applying [2, Exercise 10 on Page 151] to the sequence $\{\zeta_n / \langle 1, \zeta_n \rangle\}_{n \in \mathbb{N}}$ and $\zeta / \langle 1, \zeta \rangle$, it follows that $\zeta_n \stackrel{w}{\to} \zeta$ as $n \to \infty$.

A particular metric that induces the topology of weak convergence on $\mathbf{M}$ and under which $\mathbf{M}$ is a Polish space is the Prokhorov metric. For this, given a Borel set $B \subset \mathbb{R}_+$ and $\varepsilon > 0$, let

$$B^\varepsilon = \{y \in \mathbb{R}_+ : \inf_{x \in B} |x - y| < \varepsilon\}.$$

For $\zeta, \eta \in \mathbf{M}$, let

$$\begin{aligned} \mathbf{d}(\zeta, \eta) \;\; = \;\; & \inf\{\epsilon > 0 : \zeta(B) \leq \eta(B^\epsilon) + \epsilon \text{ and } \eta(B) \leq \zeta(B^\epsilon) + \epsilon, \\ & \qquad \text{for all closed sets } B \subset \mathbb{R}_+\}. \end{aligned}$$

Then $\mathbf{d}(\cdot, \cdot)$ is the Prokhorov metric. Given a subset $B \subset \mathbb{R}_+$ and $\zeta \in \mathbf{M}$ with slight abuse of notation, we further define

$$\mathbf{d}(\zeta, B) = \inf_{\eta \in B} \mathbf{d}(\zeta, \eta).$$

Let $\mathbf{K}$ denote the elements of $\mathbf{M}$ that are continuous, i.e., that do not charge singletons and let $\mathbf{A}$ denote the elements of $\mathbf{M}$ that are absolutely continuous with respect to Lebesgue measure. Additionally, let $\mathbf{P}$ denote the set of measures in $\mathbf{M}$ that are probability measures. We let $\mathbf{M}^+$ denote those elements of $\mathbf{M}$ with support intersecting $(0,\infty)$, i.e., $\mathbf{M}^+ = \mathbf{M} \setminus \{a\delta_0 : a \in \mathbb{R}_+\}$. We also let $\mathbf{K}^+ = \mathbf{M}^+ \cap \mathbf{K}$ and $\mathbf{A}^+ = \mathbf{M}^+ \cap \mathbf{A}$ and remark that $\mathbf{K}^+ = \mathbf{K} \setminus \{\mathbf{0}\}$ and $\mathbf{A}^+ = \mathbf{A} \setminus \{\mathbf{0}\}$ since a continuous measure is nonzero if and only if it has support intersecting $(0,\infty)$. For $k \in \mathbb{N}$, we say that a measure $\zeta \in \mathbf{M}$ has a finite kth moment if $\langle \chi^k, \zeta \rangle < \infty$ and we let $\mathbf{M}^k$ denote the set of all such measures. Let $\mathbf{M}^\dagger = \mathbf{M}^+ \cap \mathbf{M}^1$, $\mathbf{K}^\dagger = \mathbf{M}^\dagger \cap \mathbf{K}$, and $\mathbf{A}^\dagger = \mathbf{M}^\dagger \cap \mathbf{A}$. For $\zeta \in \mathbf{M}^\dagger$, there is an associated excess lifetime distribution $\zeta_e \in \mathbf{P} \cap \mathbf{A}$ that has density function p_ζ, where for each $x \in \mathbb{R}_+$,

$$p_\zeta(x) = \frac{\langle 1_{(x,\infty)}, \zeta \rangle}{\langle \chi, \zeta \rangle}. \tag{2}$$

Note that p_ζ is well defined since $\zeta \in \mathbf{M}^\dagger$ implies that $0 < \langle \chi, \zeta \rangle < \infty$. The mapping that takes $\zeta \in \mathbf{M}^\dagger$ to $\zeta_e \in \mathbf{P} \cap \mathbf{A}$ will play an important role in our analysis.

2. Critical fluid model. Here we recall the notion of a critical fluid model solution and some basic properties of such solutions that were developed in [6] and [12].

We begin by introducing the model parameters. Fix $\nu \in \mathbf{P}$ such that ν does not charge the origin and has finite mean $\langle \chi, \nu \rangle$, which is necessarily strictly positive. Let $\alpha = 1/\langle \chi, \nu \rangle$. The pair (α, ν) is referred to as critical fluid model data or simply *critical data*. The adjective "critical" refers to the fact that $\alpha \langle \chi, \nu \rangle = 1$, signifying that the rate at which work is arriving is equal to the rate at which it can be processed. Throughout this paper, we further assume that the critical data satisfies

$$\langle \chi^2, \nu \rangle < \infty, \tag{3}$$

i.e., $\nu \in \mathbf{M}^2$. For $x \in \mathbb{R}_+$, let

$$N(x) = \langle 1_{[0,x]}, \nu \rangle \qquad \text{and} \qquad \overline{N}(x) = 1 - N(x).$$

Then N and $\overline{N}$ denote the cumulative distribution function (cdf) and complementary cdf, respectively, associated with ν. Let

$$x_\nu = \inf\{x \in \mathbb{R}_+ : \overline{N}(x) = 0\}, \tag{4}$$

which is taken to be infinity if the set is empty. If x_ν is finite, then $\overline{N}(x_\nu) = 0$ by the right continuity of $\overline{N}$. Note that $x_\nu > 0$ since ν is nonzero and does not charge the origin. Since (α, ν) is critical data, i.e., since $\langle \chi, \nu \rangle = 1/\alpha$, the probability density function p_ν for the excess lifetime distribution ν_e is given by

$$p_\nu(x) = \alpha \overline{N}(x), \qquad \text{for } x \in \mathbb{R}_+.$$

For convenience, we let $n_e = p_\nu$. By (4), $n_e(x)$ is strictly positive for $x < x_\nu$ and zero for $x \geq x_\nu$. Furthermore, since $x_\nu > 0$, $\nu_e \in \mathbf{A}^+$. For $x \in \mathbb{R}_+$, set

$$N_e(x) = \left\langle 1_{[0,x]}, \nu_e \right\rangle \qquad \text{and} \qquad \overline{N}_e(x) = 1 - N_e(x).$$

A simple use of Fubini's theorem gives that $\langle \chi, \nu_e \rangle = \frac{\alpha}{2} \left\langle \chi^2, \nu \right\rangle < \infty$, and so $\nu_e \in \mathbf{A}^\dagger \subset \mathbf{M}^\dagger$. For convenience, we set

$$\alpha_e = \frac{1}{\langle \chi, \nu_e \rangle}. \tag{5}$$

Then the excess lifetime distribution $(\nu_e)_e$ for ν_e is well defined with density

$$p_{\nu_e}(x) = \alpha_e \overline{N}_e(x), \qquad \text{for } x \in \mathbb{R}_+.$$

Next we define what it means to be a fluid model solution. For this, let

$$\mathcal{C} = \{g \in \mathbf{C}_b^1(\mathbb{R}_+) : g(0) = 0, g'(0) = 0\}.$$

A *fluid model solution* for the critical data (α, ν) is a function $\mu : [0, \infty) \to \mathbf{M}$ that satisfies the following four conditions. As mentioned in the introduction, we write μ_t for the time t state $\mu(t)$.

(C.1) The function μ is continuous.
(C.2) For each $t \geq 0$, $\left\langle 1_{\{0\}}, \mu_t \right\rangle = 0$.
(C.3) For each $g \in \mathcal{C}$, μ satisfies

$$\langle g, \mu_t \rangle = \langle g, \mu_0 \rangle - \int_0^t \frac{\langle g', \mu_r \rangle}{\langle 1, \mu_r \rangle} dr + \alpha t \langle g, \nu \rangle\,, \tag{6}$$

for all $0 \leq t < t^* = \inf\{r \geq 0 : \langle 1, \mu_r \rangle = 0\}$.
(C.4) For all $t \geq t^*$, $\langle 1, \mu_t \rangle = 0$.

Among conditions (C.1)–(C.4), condition (C.3) is perhaps the least intuitive. It arises by passing to the limit in a fluid scaled version of a certain prelimit equation describing the dynamics of a processor sharing queue (see [6, Equation (2.13) and Theorem 3.2]). The first and third terms on the righthand side of (6) respectively correspond to the initial distribution of fluid and the distribution of fluid upon arrival to the system. Then, the second term on the righthand side of (6) is purely due to processing, and the minus sign reflects this. To understand the form if this term intuitively, suppose that μ_t has a differentiable density, $h_t(\cdot)$, with respect to Lebesgue measure. Then, since processing by the server at time t corresponds to shifting "mass" to the left at rate $1/\langle 1, \mu_t\rangle$, the change in $h_t(\cdot)$ due to processing over a small time interval of length δ satisfies, for each $x \in \mathbb{R}_+$,

$$h_{t+\delta}(x) - h_t(x) \approx h_t(x + \delta/\langle 1, \mu_t\rangle) - h_t(x) \approx h_t'(x)\frac{\delta}{\langle 1, \mu_t\rangle}.$$

Then the change in $\langle g, \mu_t\rangle$ over this time interval due to processing is approximately

$$\int_{\mathbb{R}_+} g(x)h_t'(x)\frac{\delta}{\langle 1, \mu_t\rangle}dx.$$

For any compactly supported g in $\mathcal{C}$, integration by parts yields that the change in $\langle g, \mu_t\rangle$ due to processing is approximately

$$-\int_{\mathbb{R}_+} g'(x)h_t(x)\frac{\delta}{\langle 1, \mu_t\rangle}dx = -\frac{\delta\,\langle g', \mu_t\rangle}{\langle 1, \mu_t\rangle}.$$

Upon dividing by δ and letting $\delta \to 0$, one obtains the time derivative of the negative term in (6). Even when μ_t does not have a density, the "processing term" can be shown to be of this form. See [6, Section 3.1] for a more detailed interpretation of (C.1)–(C.4) in terms of the dynamics of a processor sharing queue.

Finally, we summarize the properties of fluid model solutions developed in [6] and [12] that are relevant to the work in this paper. For each $\xi \in \mathbf{K}$ there is a unique fluid model solution μ satisfying $\mu_0 = \xi$ (see [6, Theorem 3.1]). For clarity, we sometimes denote such a fluid model solution by μ^ξ. If $\xi = \mathbf{0}$, then $\mu_t^\xi = \mathbf{0}$ for all $t \in [0,\infty)$. If $\xi \in \mathbf{K}^+ = \mathbf{K} \setminus \{\mathbf{0}\}$, then $\mu_t^\xi \neq \mathbf{0}$ for any $t \in [0,\infty)$ and the associated t^* is infinity (see [6, Lemma 4.4]). In addition, if $\xi \in \mathbf{K}^+$, then μ_t^ξ has no atoms for all $t \in [0,\infty)$ (see [6, Proposition 4.6]). Hence, $\xi \in \mathbf{K}^+$ implies $\mu_t^\xi \in \mathbf{K}^+$ for all $t \in [0,\infty)$.

Next we define some related functionals of μ^ξ for $\xi \in \mathbf{K}^+$ and summarize their properties. Fix a fluid model solution μ such that $\mu_0 \in \mathbf{K}^+$. For each $t \in [0,\infty)$, let

$$q_t = \langle 1, \mu_t\rangle.$$

Since μ is a continuous function, so is q, and since $\mu_t \neq \mathbf{0}$ for all $t \in [0,\infty)$, $q_t \neq 0$ for all $t \in [0,\infty)$. For $t \in [0,\infty)$, let

$$s_t = \int_0^t \frac{1}{q_r}dr.$$

The function s is continuous, strictly increasing, and continuously differentiable with

$$\frac{d}{dt}s_t = \frac{1}{q_t}, \qquad \text{for all } t \in [0,\infty).$$

Furthermore, $\lim_{t\to\infty} s_t = \infty$ (see [6, Lemma 4.4]). Hence the inverse τ exists, and is continuous, strictly increasing, and continuously differentiable. In particular, for $x \in \mathbb{R}_+$,

$$\tau(x) = \inf\{t \in [0,\infty) : s_t \geq x\}, \tag{7}$$

$$\frac{d}{dx}\tau(x) = q_{\tau(x)}. \tag{8}$$

For each $t \in [0,\infty)$, let

$$w_t = \langle \chi, \mu_t \rangle .$$

Since $\mu_t \in \mathbf{K}^+$ for all $t \in [0,\infty)$, it follows that $w_t \neq 0$ for all $t \in [0,\infty)$. Also, since (α,ν) is critical data, $w_t = w_0$ for all $t \in [0,\infty)$ (see [6, Theorem 3.1]). Note that w_0 is the first moment of μ_0, which is finite if and only if $\mu_0 \in \mathbf{K}^\dagger$. Therefore, either $w_0 = \infty$ and so $w_t = \infty$ for all $t \in [0,\infty)$, or $w_0 < \infty$ and so $w_t < \infty$ for all $t \in [0,\infty)$. Hence, if $\mu_0 \notin \mathbf{K}^\dagger$, then $\mu_t \notin \mathbf{K}^\dagger$ for any $t \in [0,\infty)$. Otherwise, $\mu_0 \in \mathbf{K}^\dagger$ and $\mu_t \in \mathbf{K}^\dagger$ for all $t \in [0,\infty)$. For each $t \in [0,\infty)$ and $x \in \mathbb{R}_+$, let

$$M_t(x) = \left\langle 1_{[0,x]}, \mu_t \right\rangle \qquad \text{and} \qquad \overline{M}_t(x) = \left\langle 1_{(x,\infty)}, \mu_t \right\rangle .$$

If $\mu_0 \in \mathbf{K}^\dagger$, then the excess lifetime distribution $(\mu_t)_e$ associated with μ_t is well defined for each $t \in [0,\infty)$ with density function p_{μ_t} given by

$$p_{\mu_t}(x) = \frac{\overline{M}_t(x)}{w_0}, \qquad \text{for } x \in \mathbb{R}_+,$$

where we have used the fact that $w_t = w_0$ for all $t \in [0,\infty)$.

A measure $\xi \in \mathbf{K}$ is said to be an *invariant state* for the fluid model with critical data (α,ν) if the unique fluid model solution μ with initial state $\mu_0 = \xi$ satisfies $\mu_t = \xi$ for all $t \in [0,\infty)$. By [12, Theorem 1.1], the set of such invariant states $\mathbf{I}$ is given by

$$\mathbf{I} = \{\beta\nu_e : \beta \in \mathbb{R}_+\}.$$

Our interest will be in nonzero invariant states, and so we define

$$\mathbf{I}^+ = \{\beta\nu_e : \beta \in (0,\infty)\}.$$

3. Main results. Having introduced fluid model solutions and summarized some of their properties, we are now ready to state the main results of this paper. The first result is Theorem 3.1. This theorem focuses on fluid model solutions for which the initial condition lies in a member of a certain family of relatively compact sets. It states that the Prokhorov distance between the state at time t of any such fluid model solution and the set of invariant states tends to zero as time approaches infinity, uniformly with respect to all initial conditions lying in the relatively compact set containing the initial condition. To define the family of relatively compact sets, given $\zeta \in \mathbf{M}$, let $Z_\zeta(x) = \langle 1_{[0,x]}, \zeta \rangle$ and $\overline{Z}_\zeta(x) = \langle 1_{(x,\infty)}, \zeta \rangle$, for $x \in \mathbb{R}_+$. Note that $\langle \chi, \zeta \rangle = \int_0^\infty \overline{Z}_\zeta(x)dx$. Given $u > 0$, let

$$\mathbf{K}_u = \left\{ \xi \in \mathbf{K} : \overline{Z}_\xi(x) \le u\overline{N}_e(x) \text{ for all } x \in \mathbb{R}_+ \right\}.$$

Then, if $\xi \in \mathbf{K}_u$ for some $u > 0$, it follows that the tail of ξ is controlled from above by the constant u times the tail of ν_e. In particular, if $x_\nu < \infty$ and $\xi \in \mathbf{K}_u$ for some $u > 0$, then $\overline{Z}_\xi(x_\nu) = 0$ since $\overline{N}_e(x_\nu) = 0$. To illustrate how initial conditions in $\mathbf{K}_u$ might arise, consider initial conditions that are constant multiples of conditional (residual) lifetime distributions associated with ν. By conditional lifetime distributions associated with ν, we mean probability measures of the form ν_y, $y \in [0, x_\nu)$, such that $\nu_y(A) = \nu(A + y)/\nu(y,\infty)$ for all Borel sets $A \subset \mathbb{R}_+$. For certain choices of ν, there exists $u_\nu > 0$ such that each conditional lifetime distribution associated with ν is in $\mathbf{K}_{u_\nu}$. For example, if ν has a hyperexponential distribution, or if $\nu \in \mathbf{A}$ and ν has a bounded, increasing hazard rate, then this property holds. Hence, for such ν and for all $c > 0$, $c\nu_y \in \mathbf{K}_{cu_\nu}$ for all $y \in [0, x_\nu)$.

For each $u > 0$, $\mathbf{K}_u$ is relatively compact as a subset of $\mathbf{M}$ under the topology of weak convergence since

$$\sup_{\xi \in \mathbf{K}_u} \langle 1, \xi \rangle \le u \qquad \text{and} \qquad \sup_{\xi \in \mathbf{K}_u} \langle \chi, \xi \rangle \le u \langle \chi, \nu_e \rangle,$$

(cf. [8, Lemma 15.7.5]). Observe that for $u > 0$, the zero measure is in $\mathbf{K}_u$ so that $\mathbf{K}_u \not\subset \mathbf{K}^\dagger$. We wish to exclude the zero measure in order to use (2). Therefore, for $u, l > 0$, we define

$$\mathbf{K}_{u,l} = \mathbf{K}_u \cap \{\xi \in \mathbf{K} : \langle \chi, \xi \rangle \ge l\}, \tag{9}$$

which is a subset of the relatively compact set $\mathbf{K}_u$, and is therefore relatively compact. One can verify that the zero measure is not in the closure of $\mathbf{K}_{u,l}$ (see Lemma 4.6). Furthermore, note that if $\xi \in \mathbf{K}_u$ for some $u > 0$ and $\xi \ne \mathbf{0}$, then $\xi \in \mathbf{K}_{u,\langle \chi, \xi \rangle}$.

For the statement of Theorem 3.1 below, we recall that the unique fluid model solution with initial state $\xi \in \mathbf{K}$ is denoted by μ^ξ.

THEOREM 3.1. *Let $u, l > 0$. Then*

$$\lim_{t\to\infty} \sup_{\xi\in\mathbf{K}_{u,l}} \mathbf{d}(\mu_t^\xi, \mathbf{I}) = 0. \tag{10}$$

Furthermore, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sup_{\xi\in\mathbf{K}_{u,l}^\delta} \sup_{t\in[0,\infty)} \mathbf{d}(\mu_t^\xi, \mathbf{I}) < \varepsilon, \tag{11}$$

where $\mathbf{K}_{u,l}^\delta = \{\zeta \in \mathbf{K}_{u,l} : \mathbf{d}(\zeta, \mathbf{I}) < \delta\}$.

Theorem 3.1 is proved in Section 6, assuming Theorem 3.2 below holds. The result in Theorem 3.1 provides sufficient conditions for fluid model solutions with initial conditions lying in certain relatively compact sets to be uniformly close to the invariant manifold $\mathbf{I}$. The particular element of $\mathbf{I}$ to which the time t value of a given fluid model solution with initial condition in such a relatively compact set is close to is not identified in Theorem 3.1. However, this identification is not necessary in order to carry out the kinds of state space collapse arguments that contribute to proving diffusion limit results. Indeed, a careful examination of the arguments in [5] demonstrates that uniform proximity to $\mathbf{I}$ is all that is needed.

Even so, it is natural to ask about identifying the specific limit point. It turns out that we can use Theorem 3.1, together with properties of fluid model solutions, to obtain the following corollary, proved in Section 6 as well. For this recall that $\alpha_e = 1/\langle\chi, \nu_e\rangle$ (see (5)).

COROLLARY 3.1. *Let $u, l > 0$. Then*

$$\lim_{t\to\infty} \sup_{\xi\in\mathbf{K}_{u,l}} \mathbf{d}(\mu_t^\xi, \alpha_e w_0 \nu_e) = 0. \tag{12}$$

In particular, for each $\xi \in \mathbf{K}_u$, as $t \to \infty$,

$$\mu_t^\xi \xrightarrow{w} \alpha_e w_0 \nu_e. \tag{13}$$

Furthermore, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sup_{\xi\in\mathbf{K}_{u,l}^\delta} \sup_{t\in[0,\infty)} \mathbf{d}(\mu_t^\xi, \alpha_e w_0 \nu_e) < \varepsilon. \tag{14}$$

The results in Corollary 3.1 are similar to those in [12, Theorems 1.2 and 1.3]. The result in [12, Theorem 1.2] states that (13) holds under the condition $\langle\chi, \nu\rangle < \infty$ and $\langle\chi, \xi\rangle < \infty$, i.e., (3) does not necessarily need to hold (in which case $\alpha_e = 0$) and $\xi \in \mathbf{K}_u$ for some $u > 0$ is not required. However, [12, Theorem 1.2] does not imply the uniform convergence that is needed for the proofs of state space collapse given in [5]. The results in [12, Theorem 1.3] are slightly stronger than the uniform convergence in (12), since they provide rates of convergence and therefore suffice for the state space collapse arguments given in [5]. However, the results in [12, Theorem 1.3] also require the more restrictive condition of finite $2+\varepsilon$ moments on the service time distribution ν for some $\varepsilon > 0$, rather than just the finite second moment condition needed for (12) to hold. The proof in [12, Theorem 1.3] depends on renewal theory arguments, which aren't likely to generalize to the networks setting. We have included Corollary 3.1 to demonstrate how one obtains a result similar to the one in [12, Theorem 1.3] as a consequence of Theorem 3.1.

Theorem 3.1 is proved in Section 6 using a result (Theorem 3.2 below) about the convergence to zero of a certain relative entropy function. This relative entropy function has some similarities with cumulative residual entropy, which is introduced in [14] as an alternative to Shannon entropy, in that it is defined in terms of the tail mass $\overline{Z}_\zeta$ for $\zeta \in \mathbf{M}^\dagger$. For the description of our relative entropy function, note that for any $\zeta \in \mathbf{M}^\dagger$, one can compute the relative entropy of ζ_e with respect to $(\nu_e)_e$. This is so because $\zeta \in \mathbf{M}^\dagger$ implies that $0 < \langle \chi, \zeta \rangle < \infty$, which in turn implies that ζ_e is well-defined (see (2)). Then ζ_e and $(\nu_e)_e$ are both Borel probability measures that are absolutely continuous with respect to Lebesgue measure. Our idea is to first measure the proximity of a measure $\zeta \in \mathbf{M}^\dagger$ to the set $\mathbf{I}$ in terms of the relative entropy of ζ_e with respect to $(\nu_e)_e$. For this, let $h : \mathbb{R}_+ \to \mathbb{R}$ be given by

$$h(x) = x \ln x, \qquad \text{for } x \in \mathbb{R}_+,$$

where $h(0)$ is interpreted to be 0. Notice that h is continuous, strictly convex and bounded below on $\mathbb{R}_+$ with minimum value $-e^{-1}$ at e^{-1}. Furthermore, $\lim_{x\to\infty} h(x) = \infty$. Recall that for a measure $\zeta \in \mathbf{M}^\dagger$, p_ζ denotes the density function of ζ_e. In particular, p_{ν_e} denotes the density function of $(\nu_e)_e$. Let $H : \mathbf{M}^\dagger \to [0, \infty]$ be given by

$$H(\zeta) = \int_{\mathbb{R}_+} h\left(\frac{p_\zeta(x)}{p_{\nu_e}(x)}\right) p_{\nu_e}(x)dx, \qquad \text{for } \zeta \in \mathbf{M}^\dagger. \tag{15}$$

Here, the convention is that the integrand takes the value zero for all $x \in \mathbb{R}_+$ such that $p_\zeta(x) = 0$ and infinity for all $x \in \mathbb{R}_+$ such that $p_\zeta(x) > 0$ and $p_{\nu_e}(x) = 0$. In particular, $H(\zeta) = \infty$ whenever $\left\langle 1_{(x_\nu,\infty)}, \zeta \right\rangle > 0$. For each $\zeta \in \mathbf{M}^\dagger$, $H(\zeta)$ denotes the relative entropy of ζ_e with respect to $(\nu_e)_e$. Hence, it provides a kind of distance between these two probability measures.

In fact, one can regard $H(\zeta)$ as a measure of the distance between the "normalized shapes" of ζ and ν_e, as we will see. In Lemma 4.7, we show that $H(\zeta) < \infty$ if $\zeta \in \mathbf{K}_{u,l}$ for some $u, l > 0$.

In order to prove Theorem 3.1, we exploit the behavior of H along fluid model solutions. More specifically, given $\xi \in \mathbf{K}^{\dagger}$, we analyze the behavior of $H(\mu_t^{\xi})$ as a function of $t \in [0, \infty)$. To simplify the notation, for each $\xi \in \mathbf{K}^{\dagger}$ and $t \in [0, \infty)$, we let

$$\mathcal{H}_{\xi}(t) = H(\mu_t^{\xi}). \tag{16}$$

We establish the following theorem, which concerns the asymptotic behavior of $\mathcal{H}_{\xi}$ as time tends to infinity.

THEOREM 3.2. *Let $u, l > 0$. Then, for each $\xi \in \mathbf{K}_{u,l}$, $\mathcal{H}_{\xi}$ is monotone nonincreasing. Furthermore,*

$$\lim_{t \to \infty} \sup_{\xi \in \mathbf{K}_{u,l}} \mathcal{H}_{\xi}(t) = 0.$$

Theorem 3.2 is the other main result proved in this paper. It implies the uniform weak convergence of the excess lifetime distributions associated with the time t states of a particular collection of fluid model solutions to the unique excess lifetime distribution associated with the set of invariant states. It is a uniform convergence of "normalized shapes" of sorts and it provides a significant step toward proving Theorem 3.1. But, Theorem 3.1 does not follow as an immediate consequence of Theorem 3.2. Properties of the function H restricted to certain compact sets, as well as properties of fluid model solutions with initial states in $\mathbf{K}_{u,l}$ for some $u, l > 0$, play important roles in its proof as well, as we will see in Section 6.

Sections 7 and 8 are devoted to proving Theorem 3.2. An important step in the proof of Theorem 3.2 is to develop an absolute continuity property for $\mathcal{H}_{\xi}$ as a function of time for each $\xi \in \mathbf{K}_{u,l}$ and $u, l > 0$. This is done in Section 7 (see Theorem 7.1). In Theorem 7.1, an explicit formula is obtained for the density of $\mathcal{H}_{\xi}$ as a function of time for each $\xi \in \mathbf{K}_{u,l}$ and $u, l > 0$. This density is non-positive for all time, which immediately implies the nonincreasing property asserted in Theorem 3.2. In the proof of Theorem 7.1, the specifics of our notion of relative entropy are used to directly compute the density of $\mathcal{H}_{\xi}$ for each absolutely continuous $\xi \in \mathbf{K}_{u,l}$ and $u, l > 0$ (see Lemma 7.8 and its proof). The reader interested in such details will want to make note of the differential equation (65) satisfied by $\overline{M}_t(x) = \langle 1_{(x,\infty)}, \mu_t \rangle$, $t \geq 0$, for each fixed $x \in \mathbb{R}_+$ when the initial state μ_0 of the fluid model solution μ

is nonzero and is absolutely continuous with respect to Lebesgue measure, and the role that (65) plays in the proof of Lemma 7.8.

The remainder of the paper is organized as follows. Assuming Theorem 3.2 holds, Theorem 3.1 and Corollary 3.1 are proved in Section 6 after developing the necessary properties of the function H in Section 4 and the necessary properties of fluid model solutions with initial states in $\mathbf{K}_{u,l}$ for some $u, l > 0$ in Section 5. Following that, Theorem 7.1 is proved in Section 7. Finally Theorem 3.2 is proved in Section 8, as a consequence of Theorem 7.1 and additional properties of the density of $\mathcal{H}_\xi$ as a function of time established in Section 7.

4. Properties of the relative entropy function *H*. Given $\zeta \in \mathbf{M}^\dagger$, $H(\zeta)$, as defined in (15), denotes the relative entropy of ζ_e with respect to $(\nu_e)_e$. Relative entropy, regarded as an extended real-valued function on the space of pairs of Borel probability measures on $\mathbb{R}_+$ that are absolutely continuous with respect to Lebesgue measure, has proved to be a useful non-metric distance measure. It is non-metric in the sense that it is not symmetric, and doesn't satisfy the triangle inequality. Even so, it provides a measure of distance in the sense that it is nonnegative and takes the value zero if and only if the two probability measures are the same. In this section, we develop properties of the function H. The treatment here is self-contained. However, we refer the interested reader to [3, Chapter 15.1] for a more general development of relative entropy and its properties.

Let $\psi : \mathbb{R}_+ \to \mathbb{R}$ be given by

$$\psi(x) = 1 - x + h(x), \qquad x \in \mathbb{R}_+.$$

Then ψ is nonnegative and strictly convex with a minimum value of zero when its argument equals one. In addition, for all $\zeta \in \mathbf{M}^\dagger$ such that $\langle 1_{(x_\nu,\infty)}, \zeta\rangle = 0$, we have

$$H(\zeta) = \int_0^{x_\nu} h\left(\frac{p_\zeta(x)}{p_{\nu_e}(x)}\right) p_{\nu_e}(x)dx = \int_0^{x_\nu} \psi\left(\frac{p_\zeta(x)}{p_{\nu_e}(x)}\right) p_{\nu_e}(x)dx. \tag{17}$$

This has two immediate consequences. For this, let

$$\mathbf{J} = \{\zeta \in \mathbf{M} : \zeta = a\delta_0 + \beta\nu_e \text{ for some } a, \beta \in \mathbb{R}_+\}, \tag{18}$$

$$\mathbf{J}^+ = \{\zeta \in \mathbf{J} : \zeta = a\delta_0 + \beta\nu_e \text{ for some } a \in \mathbb{R}_+ \text{ and } \beta > 0\}. \tag{19}$$

LEMMA 4.1. *For each* $\zeta \in \mathbf{M}^\dagger$,

(i) $H(\zeta) \in [0, \infty]$;

(ii) $H(\zeta)=0$ *if and only if* $\zeta \in \mathbf{J}^+$.

PROOF. To verify (i), note that if $\zeta \in \mathbf{M}^\dagger$ satisfies $H(\zeta)=\infty$, then the result holds trivially. Otherwise, $H(\zeta)<\infty$ and $\left\langle 1_{(x_\nu,\infty)}, \zeta \right\rangle = 0$. Then by (17) and nonnegativity of ψ and p_{ν_e} it follows that $H(\zeta)\in[0,\infty]$. For the proof of (ii), fix $\zeta \in \mathbf{M}^\dagger$. If $\zeta \in \mathbf{J}^+$, then $p_\zeta = p_{\nu_e}$ and so $H(\zeta)=0$. Conversely, if $H(\zeta)=0$, then $\left\langle 1_{(x_\nu,\infty)}, \zeta \right\rangle = 0$. Hence, by (17), nonnegativity of ψ and positivity and monotonicity of p_{ν_e} on $[0,x_\nu)$, it follows that $\psi\left(p_\zeta(x)/p_{\nu_e}(x)\right)=0$ for Lebesgue almost every $x\in[0,x_\nu)$. Thus, for Lebesgue almost every $x\in[0,x_\nu)$, $p_\zeta(x)=p_{\nu_e}(x)$. But then, for Lebesgue almost every $x\in\mathbb{R}_+$,

$$\left\langle 1_{(x,\infty)}, \zeta \right\rangle = \frac{\langle \chi, \zeta\rangle\, \overline{N}_e(x)}{\langle \chi, \nu_e\rangle}.$$

Indeed, since both sides are right continuous in x, the above equality holds for all $x\in\mathbb{R}_+$. This together with the fact that $\zeta\in\mathbf{M}^\dagger$ implies that $\zeta \in \mathbf{J}^+$. ☐

For each $\zeta \in \mathbf{M}^\dagger$ as noted in [4], we have the following relationship between $H(\zeta)$ and the Prokhorov distance between ζ_e and $(\nu_e)_e$:

$$\mathbf{d}(\zeta_e,(\nu_e)_e) \le \sqrt{\frac{H(\zeta)}{2}}. \tag{20}$$

This inequality is actually a combination of two results. Firstly, the Prokhorov distance is bounded above by the total variation distance (see [7, page 34] for example). Secondly, the total variation distance is bounded above by the square root of one half of the relative entropy distance (see [9]). Hence, if $\{\zeta_n\}_{n\in\mathbb{N}} \subset \mathbf{M}^\dagger$ is such that $\lim_{n\to\infty} H(\zeta_n)=0$, then $(\zeta_n)_e \xrightarrow{w} (\nu_e)_e$ as $n\to\infty$.

This raises the question as to whether or not $\{\zeta_n\}_{n\in\mathbb{N}} \subset \mathbf{M}^\dagger$ and $\lim_{n\to\infty} H(\zeta_n)=0$ implies that for some $\zeta\in\mathbf{J}^+$,

$$\zeta_n \xrightarrow{w} \zeta \qquad \text{as } n\to\infty. \tag{21}$$

We provide some sufficient conditions for this in Lemma 4.3, but first we record Lemma 4.2, which more generally relates weak convergence of excess lifetime distributions to weak convergence of the original measures.

LEMMA 4.2. *Suppose that* $\{\zeta_n\}_{n\in\mathbb{N}} \subset \mathbf{M}^\dagger$, $\zeta\in\mathbf{M}^\dagger$, *and* $(\zeta_n)_e \xrightarrow{w} \zeta_e$ *as* $n\to\infty$. *If*

$$\lim_{n\to\infty} \langle 1, \zeta_n\rangle = \langle 1,\zeta\rangle \qquad \textit{and} \qquad \lim_{n\to\infty} \langle \chi, \zeta_n\rangle = \langle \chi,\zeta\rangle,$$

then, as $n \to \infty$,

$$\zeta_n \stackrel{w}{\to} \zeta.$$

PROOF. Since $(\zeta_n)_e \stackrel{w}{\to} \zeta_e$ as $n \to \infty$, it follows that for all $g \in \mathbf{C}_b^1(\mathbb{R}_+)$,

$$\lim_{n\to\infty} \left\langle g', (\zeta_n)_e \right\rangle = \left\langle g', \zeta_e \right\rangle.$$

Using integration by parts, we obtain that for each $g \in \mathbf{C}_b^1(\mathbb{R}_+)$ and $n \in \mathbb{N}$,

$$\begin{aligned} \left\langle g', (\zeta_n)_e \right\rangle &= \frac{-g(0)\langle 1, \zeta_n\rangle}{\langle \chi, \zeta_n\rangle} + \frac{\langle g, \zeta_n\rangle}{\langle \chi, \zeta_n\rangle}, \\ \left\langle g', \zeta_e \right\rangle &= \frac{-g(0)\langle 1, \zeta\rangle}{\langle \chi, \zeta\rangle} + \frac{\langle g, \zeta\rangle}{\langle \chi, \zeta\rangle}. \end{aligned}$$

Hence,

$$\begin{aligned} \lim_{n\to\infty} \langle g, \zeta_n\rangle &= \lim_{n\to\infty} \left[\langle \chi, \zeta_n\rangle \left\langle g', (\zeta_n)_e \right\rangle + g(0)\langle 1, \zeta_n\rangle\right] \\ &= \langle \chi, \zeta\rangle \left\langle g', \zeta_e \right\rangle + g(0)\langle 1, \zeta\rangle \\ &= \langle g, \zeta\rangle. \end{aligned}$$

Then, since $\mathbf{C}_b^1(\mathbb{R}_+)$ is convergence determining for measures in $\mathbf{M}^\dagger$, the desired result holds. □

LEMMA 4.3. *Suppose that* $\{\zeta_n\}_{n\in\mathbb{N}} \subset \mathbf{M}^\dagger$ *and* $\lim_{n\to\infty} H(\zeta_n) = 0$. *Further suppose that there exist* $a \in \mathbb{R}_+$ *and* $\beta > 0$ *such that*

$$\lim_{n\to\infty} \langle 1, \zeta_n\rangle = a + \beta \qquad \text{and} \qquad \lim_{n\to\infty} \langle \chi, \zeta_n\rangle = \beta \langle \chi, \nu_e\rangle.$$

Then, as $n \to \infty$,

$$\zeta_n \stackrel{w}{\to} \zeta = a\delta_0 + \beta\nu_e \in \mathbf{J}^+.$$

PROOF. By (20), we have that $(\zeta_n)_e \stackrel{w}{\to} (\nu_e)_e$ as $n \to \infty$. Set $\zeta = a\delta_0 + \beta\nu_e \in \mathbf{J}^+ \subset \mathbf{M}^\dagger$. Then $\langle 1, \zeta\rangle = a + \beta$, $\langle \chi, \zeta\rangle = \beta\langle \chi, \nu_e\rangle$, and $\zeta_e = (\nu_e)_e$. So the result follows from Lemma 4.2. □

Note that in Lemma 4.3 the limit is in $\mathbf{J}^+$ rather than $\mathbf{I}^+$. The following result bounds the Prokhorov distance to $\mathbf{I}$ in terms of the Prokhorov distance to $\mathbf{J}$ and the size of the atom at the origin.

LEMMA 4.4. *Suppose that* $\zeta \in \mathbf{M}$, $\eta \in \mathbf{J}$, $\varepsilon > 0$, *and* $\mathbf{d}(\zeta, \eta) \le \varepsilon$. *Then* $\mathbf{d}(\zeta, \beta\nu_e) \le \varepsilon + a$, *where* $\eta = a\delta_0 + \beta\nu_e$ *for some* $a \in \mathbb{R}_+$ *and* $\beta > 0$. *Furthermore,* $a \le \zeta([0, \varepsilon)) + \varepsilon$ *so that*

$$\mathbf{d}(\zeta, \mathbf{I}) \le \zeta([0, \varepsilon)) + 2\varepsilon.$$

PROOF. For any closed set $A \subset \mathbb{R}_+$, we have that

$$\begin{aligned} \zeta(A) &\leq \eta(A^\varepsilon) + \varepsilon \leq a + \beta\nu_e(A^\varepsilon) + \varepsilon \leq \beta\nu_e(A^{\varepsilon+a}) + \varepsilon + a, \\ \beta\nu_e(A) \leq \eta(A) &\leq \zeta(A^\varepsilon) + \varepsilon \leq \zeta(A^{\varepsilon+a}) + \varepsilon + a. \end{aligned}$$

Hence, $\mathbf{d}(\zeta, \beta\nu_e) \leq \varepsilon + a$. By considering $A = \{0\}$, we obtain $A^\varepsilon = [0, \varepsilon)$ and so

$$a = \eta(\{0\}) \leq \zeta([0, \varepsilon)) + \varepsilon.$$

Since $\beta\nu_e \in \mathbf{I}$, the result follows. □

Finally, one might wonder if $\{\zeta_n\}_{n\in\mathbb{N}} \subset \mathbf{M}^\dagger$, $\zeta \in \mathbf{J}^+$, and $\zeta_n \xrightarrow{w} \zeta$ as $n \to \infty$ implies $\lim_{n\to\infty} H(\zeta_n) = 0$. Note that the inequality in (20) is not helpful in this regard. While we have the following sufficient condition for lower semicontinuity of H, it is really continuity of H that we require. We provide a sufficient condition for this further below.

LEMMA 4.5. *Suppose that* $\{\zeta_n\}_{n\in\mathbb{N}} \subset \mathbf{M}^\dagger$, $\zeta \in \mathbf{M}^\dagger$, *and* $\zeta_n \xrightarrow{w} \zeta$ *and* $\langle\chi, \zeta_n\rangle \to \langle\chi, \zeta\rangle$ *as* $n \to \infty$. *Then*

$$\liminf_{n\to\infty} H(\zeta_n) \geq H(\zeta). \tag{22}$$

PROOF. The assumptions imply that, for almost all $x \in \mathbb{R}_+$,

$$\lim_{n\to\infty} p_{\zeta_n}(x) = p_\zeta(x). \tag{23}$$

If there exists $x > x_\nu$ such that $p_\zeta(x) > 0$, then $H(\zeta) = \infty$. We may suppose that x is such that (23) holds at x. Then, for all n sufficiently large, $p_{\zeta_n}(x) > 0$ and so $H(\zeta_n) = \infty$. Hence, (22) holds. Otherwise, $p_\zeta(x) = 0$ for all $x > x_\nu$ and then by right continuity of p_ζ, $p_\zeta(x_\nu) = 0$. Since h is continuous, for almost all $x \in [0, x_\nu)$,

$$\lim_{n\to\infty} h\left(\frac{p_{\zeta_n}(x)}{p_{\nu_e}(x)}\right) = h\left(\frac{p_\zeta(x)}{p_{\nu_e}(x)}\right).$$

This together with (17) and Fatou's lemma implies (22). □

If we restrict the domain of H, we can verify that H is continuous on this restricted domain. To describe this, given $\zeta \in \mathbf{M}$, recall that for $x \in \mathbb{R}_+$, $Z_\zeta(x) = \langle 1_{[0,x]}, \zeta\rangle$ and $\overline{Z}_\zeta(x) = \langle 1_{(x,\infty)}, \zeta\rangle$. Then, given $u, l > 0$, define

$$\mathbf{M}_u = \{\zeta \in \mathbf{M} : \langle 1, \zeta\rangle \leq u \text{ and } \overline{Z}_\zeta(x) \leq u\overline{N}_e(x) \text{ for all } x \in \mathbb{R}_+\}, \tag{24}$$

$$\mathbf{M}_{u,l} = \{\zeta \in \mathbf{M}_u : l \leq \langle\chi, \zeta\rangle\}. \tag{25}$$

Note that for all $u,l > 0$, $\mathbf{K}_{u,l} \subset \mathbf{M}_{u,l} \subset \mathbf{M}^{\dagger}$. In addition, note that if $u,l > 0$ are such that $\mathbf{M}_{u,l} \neq \emptyset$, then

$$l \leq u \langle \chi, \nu_e \rangle . \tag{26}$$

Sets of the form $\mathbf{M}_{u,l}$ for $u,l > 0$ are natural in our context for several reasons. For one, they are compact (see Lemma 4.6 below). In addition, we will show that given $u,l > 0$, there exist $u^*, l^* > 0$ such that for all $\xi \in \mathbf{K}_{u,l}$, $\mu_t^{\xi} \in \mathbf{K}_{u^*,l^*}$ for all $t \in [0,\infty)$ (see Corollary 5.1). In particular fluid model solutions starting in a compact set of this form, remain in a (possibly enlarged) compact set of this form for all time. Finally, on sets of this form, one may invoke bounded convergence in order to demonstrate that H is continuous on this restricted domain (see Lemma 4.8 and its proof). The combination of these properties will be important for the relative entropy arguments given here.

We begin by verifying compactness.

LEMMA 4.6. *Given $u,l > 0$, $\mathbf{M}_u$ and $\mathbf{M}_{u,l}$ are compact.*

PROOF. Fix $u,l > 0$. For all $\zeta \in \mathbf{M}_u$, we have that $\langle 1, \zeta \rangle \leq u$ and $\langle \chi, \zeta \rangle \leq u \langle \chi, \nu_e \rangle$. Hence, $\mathbf{M}_u$ and $\mathbf{M}_{u,l}$ are relatively compact (cf. [8, Lemma 15.7.5]). Therefore, it suffices to show that $\mathbf{M}_u$ and $\mathbf{M}_{u,l}$ are closed. First suppose that $\{\zeta_n\}_{n\in\mathbb{N}} \subset \mathbf{M}_u$ and $\zeta_n \stackrel{w}{\to} \zeta \in \mathbf{M}$ as $n \to \infty$. We must show that $\zeta \in \mathbf{M}_u$. We have that $\lim_{n\to\infty} \langle 1, \zeta_n \rangle = \langle 1, \zeta \rangle$, and so $\langle 1, \zeta \rangle \leq u$. Furthermore, for all ζ-continuity points $x \in \mathbb{R}_+$,

$$\overline{Z}_{\zeta}(x) = \lim_{n\to\infty} \overline{Z}_{\zeta_n}(x) \leq u\overline{N}_e(x).$$

Since the continuity points are dense and using right continuity of both sides above, we have for all $x \in \mathbb{R}_+$,

$$\overline{Z}_{\zeta}(x) \leq u\overline{N}_e(x).$$

Then $\zeta \in \mathbf{M}_u$ and so $\mathbf{M}_u$ is compact. Next suppose that $\{\zeta_n\}_{n\in\mathbb{N}} \subset \mathbf{M}_{u,l}$ and $\zeta_n \stackrel{w}{\to} \zeta \in \mathbf{M}$ as $n \to \infty$. Since $\mathbf{M}_{u,l} \subset \mathbf{M}_u$ and $\mathbf{M}_u$ is compact, $\zeta \in \mathbf{M}_u$. We must show that $\zeta \in \mathbf{M}_{u,l}$. For this, note that for any $\xi \in \mathbf{M}$,

$$\left\langle \chi 1_{(x,\infty)}, \xi \right\rangle = x\overline{Z}_{\xi}(x) + \int_x^{\infty} \overline{Z}_{\xi}(y) dy. \tag{27}$$

This together with $\zeta_n \in \mathbf{M}_{u,l}$ for all $n \in \mathbb{N}$ implies that, for all $x \in \mathbb{R}_+$,

$$\sup_{n\in\mathbb{N}} \left\langle \chi 1_{(x,\infty)}, \zeta_n \right\rangle \leq u \left\langle \chi 1_{(x,\infty)}, \nu_e \right\rangle .$$

Combining this with the facts that $\zeta_n \stackrel{w}{\to} \zeta$ as $n \to \infty$ and $\langle \chi, \nu_e \rangle < \infty$, we can conclude that

$$\lim_{n\to\infty} \langle \chi, \zeta_n \rangle = \langle \chi, \zeta \rangle .$$

Therefore, $l \leq \langle \chi, \zeta \rangle \leq u \langle \chi, \nu_e \rangle < \infty$. Thus, $\zeta \in \mathbf{M}_{u,l}$. □

The bounds stated in the next proposition are used to verify continuity of H on $\mathbf{M}_{u,l}$ for $u, l > 0$.

LEMMA 4.7. *For $u, l > 0$, $\zeta \in \mathbf{M}_{u,l}$, and $x \in [0, x_\nu)$,*

$$\left| h\left(\frac{p_\zeta(x)}{p_{\nu_e}(x)}\right)\right| \leq \max\left(e^{-1}, h\left(\frac{u\langle \chi, \nu_e\rangle}{l}\right)\right), \tag{28}$$

and

$$H(\zeta) \leq h\left(\frac{u\langle \chi, \nu_e\rangle}{l}\right). \tag{29}$$

In particular, $H(\zeta) < \infty$ for all $\zeta \in \mathbf{M}_{u,l}$.

PROOF. Fix $u, l > 0$. Given $\zeta \in \mathbf{M}_{u,l}$, we have $\overline{Z}_\zeta(x_\nu) = 0$ if $x_\nu < \infty$. Therefore, given $\zeta \in \mathbf{M}_{u,l}$,

$$H(\zeta) = \int_0^{x_\nu} h\left(\frac{p_\zeta(x)}{p_{\nu_e}(x)}\right) p_{\nu_e}(x)dx.$$

Then, given $\zeta \in \mathbf{M}_{u,l}$, for all $x \in [0, x_\nu)$,

$$0 \leq \frac{p_\zeta(x)}{p_{\nu_e}(x)} = \frac{\overline{Z}_\zeta(x)\langle \chi, \nu_e\rangle}{\langle \chi, \zeta\rangle \overline{N}_e(x)} \leq \frac{u\langle \chi, \nu_e\rangle}{\langle \chi, \zeta\rangle} \leq \frac{u\langle \chi, \nu_e\rangle}{l}.$$

Since h is nonpositive and bounded below by $-e^{-1}$ on $[0,1]$ and is nonnegative and increasing on $[1, \infty)$, it follows that (28) holds. In addition, (29) follows by combining the inequality above with the fact that $u\langle \chi, \nu_e\rangle / l \geq 1$ by (26). □

LEMMA 4.8. *Given $u, l > 0$, H is continuous on $\mathbf{M}_{u,l}$.*

PROOF. Suppose that $\{\zeta_n\}_{n\in\mathbb{N}} \subset \mathbf{M}_{u,l}$ and $\zeta_n \stackrel{w}{\to} \zeta \in \mathbf{M}$ as $n \to \infty$. Then, by Lemma 4.6, $\zeta \in \mathbf{M}_{u,l}$. As demonstrated in the proof of Lemma 4.6, for all ζ-continuity points $x \in \mathbb{R}_+$,

$$\lim_{n\to\infty} \overline{Z}_{\zeta_n}(x) = \overline{Z}_\zeta(x) \qquad \text{and} \qquad \lim_{n\to\infty} \langle \chi, \zeta_n\rangle = \langle \chi, \zeta\rangle .$$

Since $0 < l \leq \langle \chi, \zeta \rangle$, it follows that for all but countably many $x \in \mathbb{R}_+$,

$$\lim_{n\to\infty} p_{\zeta_n}(x) = p_\zeta(x).$$

Therefore, since h is continuous, for all but countably many $x \in [0, x_\nu)$,

$$\lim_{n\to\infty} h\left(\frac{p_{\zeta_n}(x)}{p_{\nu_e}(x)}\right) = h\left(\frac{p_\zeta(x)}{p_{\nu_e}(x)}\right).$$

This together with (28) and the bounded convergence theorem implies that $\lim_{n\to\infty} H(\zeta_n) = H(\zeta)$. □

5. Bounds for fluid model solutions.

Here we develop several bounds satisfied by fluid model solutions. In particular, we prove two compact containment properties of fluid model solutions that will be used in the proofs of Theorems 3.1 and 3.2 (see Corollaries 5.1 and 5.4). We will also obtain an upper bound on the mass near the origin that will be used in the proof of Theorem 3.1 (see Lemma 5.2 and Corollary 5.2).

Let μ be a fluid model solution with $\mu_0 \in \mathbf{K}^\dagger$. Recall the notation associated with such a fluid model solution, which is introduced in Section 2. In particular, n_e is the density for ν_e. As a consequence of [6, Lemma 4.3] and the fact that $\mu_t \neq \mathbf{0}$ for all $t \in [0, \infty)$, for all $t \in [0, \infty)$ and $x \in \mathbb{R}_+$,

$$\overline{M}_t(x) = \overline{M}_0(x + s_t) + \int_0^t n_e(x + s_t - s_v)\,dv. \tag{30}$$

Given $0 \leq r \leq t < \infty$ and $x \in \mathbb{R}_+$, consider (30) with r in place of t and $x + s_t - s_r$ in place of x. Subtract this from (30) as stated. Then, it follows that for all $0 \leq r \leq t < \infty$ and $x \in \mathbb{R}_+$,

$$\overline{M}_t(x) = \overline{M}_r(x + s_t - s_r) + \int_r^t n_e(x + s_t - s_v)\,dv. \tag{31}$$

Setting $x = 0$ in (31), using property (C.2) of fluid model solutions and using the fact that the integrand is bounded above by α yields that for all $0 \leq r \leq t < \infty$,

$$q_t \leq \overline{M}_r(s_t - s_r) + \alpha(t - r). \tag{32}$$

LEMMA 5.1. *Let μ be a fluid model solution with $\mu_0 \in \mathbf{K}^\dagger$. Set*

$$u_0 = \frac{3\max(q_0, 6\alpha w_0)}{2}. \tag{33}$$

Then $q_t \leq u_0$ for all $t \in [0, \infty)$.

PROOF. Fix a fluid model solution μ with $\mu_0 \in \mathbf{K}^\dagger$. Let $f(x) = x^2 - 6w_0\alpha x + w_0^2\alpha^2$ for $x \in \mathbb{R}$. This quadratic has two distinct positive roots. Let r denote the larger of those two roots and note that $f(x) \geq 0$ for all $x \geq r$. Also note that $3\alpha w_0 < r < 6\alpha w_0$. Set

$$a = \max(q_0, r), \qquad \ell = \frac{a - \alpha w_0}{2\alpha}, \qquad \text{and} \qquad b = a + \alpha\ell = \frac{3a}{2} - \frac{\alpha w_0}{2}.$$

Then $a \geq r$, which implies that $\ell > 0$ and $f(a) \geq 0$. Since $b \leq 3a/2 \leq u_0$, to prove the lemma, it suffices to show that $q_t \leq b$ for all $t \in [0, \infty)$. To prove this, we will prove by induction that for each $n = 0, 1, 2, \ldots$

$$q_{n\ell} \leq a \qquad \text{and} \qquad q_t \leq b \quad \text{for all } t \in [n\ell, (n+1)\ell]. \tag{34}$$

We begin with the base case, $n = 0$. By definition of a, $q_0 \leq a$. By (32) for $t \in [0, \ell]$, we have

$$q_t \leq q_0 + \alpha\ell \leq a + \alpha\ell = b.$$

Hence, (34) holds for $n = 0$.

For the induction step, fix $m \in \mathbb{N}$ and assume that (34) holds for $n = 0, \ldots, m-1$. We wish to show that (34) holds for $n = m$. By (32), the generalization of Markov's inequality to finite measures, and the fact that $w_t = w_0$ for all $t \in [0, \infty)$, we have

$$q_{m\ell} \leq \overline{M}_{(m-1)\ell}\left(s_{m\ell} - s_{(m-1)\ell}\right) + \alpha\ell \leq \frac{w_0}{s_{m\ell} - s_{(m-1)\ell}} + \alpha\ell.$$

By the definition of s and the induction hypothesis,

$$s_{m\ell} - s_{(m-1)\ell} = \int_{(m-1)\ell}^{m\ell} \frac{1}{q_v} dv \geq \frac{\ell}{b}.$$

Hence,

$$q_{m\ell} \leq \frac{w_0 b}{\ell} + \alpha\ell = \frac{w_0 a 2\alpha}{a - w_0\alpha} + \frac{a}{2} + \frac{w_0\alpha}{2}.$$

In order to show that $q_{m\ell} \leq a$, it suffices to show that

$$\frac{2w_0\alpha a}{a - w_0\alpha} + \frac{a}{2} + \frac{w_0\alpha}{2} \leq a.$$

This holds if and only if

$$0 \leq (a - w_0\alpha)^2 - 4w_0\alpha a.$$

The righthand side is equal to $f(a)$, and it is true that $0 \leq f(a)$. Hence, $q_{m\ell} \leq a$. This together with (32) implies that for $t \in [m\ell, (n+1)\ell]$,

$$q_t \leq q_{m\ell} + \alpha\ell \leq a + \alpha\ell = b.$$

Hence, (34) holds for $n = m$. Therefore, by the principle of mathematical induction, (34) holds for each $n = 0, 1, 2, \ldots$. □

Given $u > 0$, let

$$\mathbf{K}_u^+ = \mathbf{K}_u \setminus \{\mathbf{0}\}.$$

Lemma 5.1 implies that the total mass of any fluid model solution in $\mathbf{K}_u^+$ is uniformly bounded above for all time. From this, we obtain the following corollary.

COROLLARY 5.1. *Let $u, l > 0$. Set*

$$u^* = \frac{3u \max(1, 6\alpha \langle \chi, \nu_e \rangle)}{2} \qquad \textit{and} \qquad l^* = l. \tag{35}$$

Then, for all fluid model solutions μ with $\mu_0 \in \mathbf{K}_u^+$, $\mu_t \in \mathbf{K}_{u^*}^+$ for all $t \in [0, \infty)$. Moreover, for all fluid model solutions μ with $\mu_0 \in \mathbf{K}_{u,l}$, $\mu_t \in \mathbf{K}_{u^*,l^*}$ for all $t \in [0, \infty)$.

PROOF. Fix $u > 0$ and a fluid model solution μ with $\mu_0 \in \mathbf{K}_u^+$. Let u^* be given by (35). We must show that $\overline{M}_t(x) \leq u^* \overline{N}_e(x)$ for all $t \in [0, \infty)$ and $x \in \mathbb{R}_+$. Since $\mu_0 \in \mathbf{K}_u^+$, $q_0 \leq u$ and $w_0 \leq u \langle \chi, \nu_e \rangle$. Hence, by Lemma 5.1, $q_t \leq u^*$ for all $t \in [0, \infty)$. Then, for each $t \in [0, \infty)$ and $x \in \mathbb{R}_+$,

$$\int_0^t n_e(x + s_t - s_v) dv = \int_0^t n_e(x + s_t - s_v) \frac{q_v}{q_v} dv \leq u^* \int_0^t n_e(x + s_t - s_v) \frac{1}{q_v} dv.$$

Therefore, for each $t \in [0, \infty)$ and $x \in \mathbb{R}_+$, using the change of variables $y = x + s_t - s_v$ gives,

$$\int_0^t n_e(x + s_t - s_v) dv \leq u^* \int_x^{x+s_t} n_e(y) dy = u^* (\overline{N}_e(x) - \overline{N}_e(x + s_t)). \tag{36}$$

This together with (30), the definition of $\mathbf{K}_u^+$, and the fact that $u \leq u^*$ implies that for all $t \in [0, \infty)$ and $x \in \mathbb{R}_+$

$$\overline{M}_t(x) \leq u \overline{N}_e(x + s_t) + u^* \left(\overline{N}_e(x) - \overline{N}_e(x + s_t) \right) \leq u^* \overline{N}_e(x).$$

Hence, $\mu_t \in \mathbf{K}_{u^*}^+$ for all $t \in [0, \infty)$.

Lastly, fix $u, l > 0$ and a fluid model solution μ with $\mu_0 \in \mathbf{K}_{u,l}$. Let u^*, l^* be given by (35). Since $\mu_0 \in \mathbf{K}_{u,l}$, $\mu_0 \in \mathbf{K}_u^+$. Then, by what was proved above, $\mu_t \in \mathbf{K}_{u^*}^+$ for all $t \in [0,\infty)$. This, together with the fact that the fluid analog of the workload process is constant for critical data, implies that $w_t = w_0 \geq l = l^*$ for all $t \in [0,\infty)$. Then $\mu_t \in \mathbf{K}_{u^*,l^*}$ for all $t \in [0,\infty)$. □

Next, we obtain upper bounds on the mass that fluid model solutions have in neighborhoods of the origin.

LEMMA 5.2. *Let $u > 0$ and let u^* be given by (35). For all fluid model solutions μ with $\mu_0 \in \mathbf{K}_u^+$, $t \in [0,\infty)$, and $x \in \mathbb{R}_+$,*

$$M_t(x) \leq \overline{M}_0(s_t) - \overline{M}_0(s_t + x) + \alpha u^* x.$$

PROOF. Fix $u > 0$ and a fluid model solution μ with $\mu_0 \in \mathbf{K}_u^+$, $t \in [0,\infty)$, and $x \in \mathbb{R}_+$. Let u^* be given by (35). By property (C.2) of fluid model solutions and (30), we have that

$$\begin{aligned} M_t(x) &= \overline{M}_t(0) - \overline{M}_t(x) \\ &= \overline{M}_0(s_t) - \overline{M}_0(x+s_t) + \int_0^t (n_e(s_t - s_v) - n_e(x + s_t - s_v))\, dv. \end{aligned}$$

With the change of variables, $r = s_v$, by (7) we have $\tau(r) = v$. Then, from (8), it follows that,

$$M_t(x) = \overline{M}_0(s_t) - \overline{M}_0(x+s_t) + \int_0^{s_t} (n_e(s_t - r) - n_e(x + s_t - r))\, q_{\tau(r)} dr.$$

Hence,

$$M_t(x) = \overline{M}_0(s_t) - \overline{M}_0(x+s_t) + \alpha \int_0^{s_t} \left\langle 1_{(s_t - r, x + s_t - r]}, \nu \right\rangle q_{\tau(r)} dr.$$

Therefore, by Corollary 5.1,

$$M_t(x) \leq \overline{M}_0(s_t) - \overline{M}_0(x+s_t) + u^* \alpha \int_0^{s_t} \left\langle 1_{(s_t - r, x + s_t - r]}, \nu \right\rangle dr.$$

Then, using the change of variables $y = s_t - r$, we obtain

$$M_t(x) \leq \overline{M}_0(s_t) - \overline{M}_0(x+s_t) + u^* \alpha \int_0^{s_t} \left\langle 1_{(y, x+y]}, \nu \right\rangle dy. \tag{37}$$

If $s_t \leq x$, then the result follows since $\left\langle 1_{(y,x+y]}, \nu \right\rangle \leq 1$ for all $y \in \mathbb{R}_+$. Otherwise, $s_t > x$. Then, by splitting the integrand on the righthand side of (37) into a difference, distributing the integral sign and using a change of variables in the first integral, canceling the common portion of the two integrals that result, and dropping the remaining portion of the second integral since it is negative, and finally using the fact that the integrand is bounded above by one, we obtain

$$\begin{aligned}
M_t(x) &\le \overline{M}_0(s_t) - \overline{M}_0(x+s_t) + u^*\alpha \int_0^{s_t} \left(\langle 1_{[0,x+y]}, \nu\rangle - \langle 1_{[0,y]}, \nu\rangle\right) dy \\
&= \overline{M}_0(s_t) - \overline{M}_0(x+s_t) + u^*\alpha \left(\int_x^{x+s_t} \langle 1_{[0,y]}, \nu\rangle\, dy - \int_0^{s_t} \langle 1_{[0,y]}, \nu\rangle\, dy\right) \\
&= \overline{M}_0(s_t) - \overline{M}_0(x+s_t) + u^*\alpha \left(\int_{s_t}^{x+s_t} \langle 1_{[0,y]}, \nu\rangle\, dy - \int_0^{x} \langle 1_{[0,y]}, \nu\rangle\, dy\right) \\
&\le \overline{M}_0(s_t) - \overline{M}_0(x+s_t) + u^*\alpha \int_{s_t}^{x+s_t} \langle 1_{[0,y]}, \nu\rangle\, dy \\
&\le \overline{M}_0(s_t) - \overline{M}_0(x+s_t) + u^*\alpha x.
\end{aligned}$$

So the result holds. □

An immediate consequence of Lemma 5.2 is the following corollary, which yields an upper bound on the mass near the origin that is uniform over initial conditions in $\mathbf{K}_u^+$, for $u > 0$.

COROLLARY 5.2. *Let $u > 0$ and let u^* be given by (35). For all fluid model solutions μ with $\mu_0 \in \mathbf{K}_u^+$, $t \in [0, \infty)$, and $x \in \mathbb{R}_+$,*

$$M_t(x) \le u^* \left(\overline{N}_e(t/u^*) + \alpha x\right). \tag{38}$$

Furthermore, given $\varepsilon > 0$, there exists $\delta, x^* > 0$ such that if μ is a fluid model solution satisfying $\mu_0 \in \mathbf{K}_u^+$ and $\mathbf{d}(\mu_0, \mathbf{I}) < \delta$, then

$$\sup_{0 \le x \le x^*} \sup_{t \in [0,\infty)} M_t(x) < \varepsilon. \tag{39}$$

PROOF. Fix $u > 0$ and a fluid model solution μ with $\mu_0 \in \mathbf{K}_u^+$, $t \in [0, \infty)$, and $x \in \mathbb{R}_+$. Let u^* be given by (35).

First we verify (38). By Corollary 5.1, $\mu_t \in \mathbf{K}_{u^*}^+$ for $t \in [0, \infty)$. Hence, for $t \in [0, \infty)$, $0 < q_t \le u^*$ and so

$$s_t = \int_0^t \frac{1}{q_v} dv \ge \frac{t}{u^*}.$$

This yields that for $t \in [0, \infty)$, $\overline{M}_0(s_t) \le \overline{M}_0(t/u^*) \le u^* \overline{N}_e(t/u^*)$. Then (38) follows from Lemma 5.2.

Next we verify (39). For this, fix $\varepsilon > 0$. Since ν_e is continuous, $\overline{N}_e$ is uniformly continuous and so there exists $h > 0$ such that

$$\sup_{y \in \mathbb{R}_+} \left(\overline{N}_e(y) - \overline{N}_e(y+h) \right) < \frac{\varepsilon}{6u}. \tag{40}$$

Fix $\delta \in (0, h/2 \wedge \varepsilon/6)$ and further assume that $\mathbf{d}(\mu_0, \mathbf{I}) < \delta$. Then there exists $\beta < u + \delta$ such that $\mathbf{d}(\mu_0, \beta\nu_e) < \delta$. Hence, using the definition of the Prokhorov distance $\mathbf{d}(\cdot,\cdot)$, the continuity of μ_0 and ν_e, and (40), we have, for all $y \le h - 2\delta$,

$$\begin{aligned}
\overline{M}_0(s_t) - \overline{M}_0(s_t + y) &= \left\langle 1_{(s_t, s_t+y]}, \mu_0 \right\rangle = \left\langle 1_{[s_t, s_t+y]}, \mu_0 \right\rangle \\
&\le \beta \left\langle 1_{((s_t-\delta)^+, s_t+y+\delta)}, \nu_e \right\rangle + \delta \\
&= \beta \left\langle 1_{((s_t-\delta)^+, s_t+y+\delta]}, \nu_e \right\rangle + \delta \\
&= \beta \left(\overline{N}_e((s_t-\delta)^+) - \overline{N}_e\left(s_t+y+\delta\right) \right) + \delta \\
&\le (u+\delta) \left(\overline{N}_e((s_t-\delta)^+) - \overline{N}_e(s_t+y+\delta) \right) + \delta \\
&\le u \left(\overline{N}_e((s_t-\delta)^+) - \overline{N}_e(s_t+y+\delta) \right) + 2\delta \\
&\le u \left(\overline{N}_e((s_t-\delta)^+) - \overline{N}_e(s_t+h-\delta) \right) + 2\delta \\
&< \frac{\varepsilon}{6} + \frac{2\varepsilon}{6} = \frac{\varepsilon}{2}.
\end{aligned}$$

This together with Lemma 5.2 implies that for $y \le x^* = \min(h - 2\delta, \varepsilon/(3u^*\alpha))$, we have that $M_t(y) < 5\varepsilon/6$. Hence, (39) holds. $\square$

Next, we turn our attention to lower bounds. One can obtain a lower bound on the total mass that holds for all time, but it is not uniform over initial conditions in $\mathbf{K}_u^+$ for any $u > 0$. One should not expect a uniform lower bound since the zero measure is a limit point of $\mathbf{K}_u^+$ for any $u > 0$. However, if one considers initial conditions in sets of the form $\mathbf{K}_{u,l}$ for $u, l > 0$, the total mass is uniformly bounded away from zero due to relative compactness. Indeed, given $u, l > 0$, since $\mathbf{K}_{u,l} \subset \mathbf{M}_{u,l}$, where the latter is compact and $\mathbf{0} \notin \mathbf{M}_{u,l}$,

$$\lambda = \inf\{\langle 1, \zeta \rangle : \zeta \in \mathbf{M}_{u,l}\}, \tag{41}$$

is strictly positive. Similarly, for u^*, l^* given by (35),

$$\lambda^* = \inf\{\langle 1, \zeta \rangle : \zeta \in \mathbf{M}_{u^*,l^*}\}, \tag{42}$$

is strictly positive. This together with Corollary 5.1 implies the following corollary.

COROLLARY 5.3. *Let $u, l > 0$. Let $u^*, l^*, \lambda^* > 0$ be given by (35) and (42). If μ is a fluid model solution such that $\mu_0 \in \mathbf{K}_{u,l}$, then $\lambda^* \le q_t$ for all $t \in [0, \infty)$.*

Lastly, we would like to know if, given $u, l > 0$, there exists $l_* > 0$ such that $l_* \overline{N}_e(x) \le \overline{M}_t(x)$ for all $x \in \mathbb{R}_+$, $t \in [0, \infty)$, and fluid model solutions μ with initial state in $\mathbf{K}_{u,l}$. Such a condition is not true at time zero unless it is imposed, and we wish to avoid such restrictions if possible. Instead, in what follows, we obtain a lower bound that is asymptotically of the desired form (see (43) and recall that $\lim_{t\to\infty} s_t = \infty$).

LEMMA 5.3. *Let $u, l > 0$ and let $u^*, l^*, \lambda^* > 0$ be given by (35) and (42). Given a fluid model solution μ such that $\mu_0 \in \mathbf{K}_{u,l}$, for $t \in [0, \infty)$, and $x \in \mathbb{R}_+$,*

$$\overline{M}_t(x) \ge \lambda^* \left(\overline{N}_e(x) - \overline{N}_e(x + s_t)\right). \tag{43}$$

Proof. Fix $u, l > 0$ and let $u^*, l^*, \lambda^* > 0$ be given by (35) and (42). Fix a fluid model solution μ such that $\mu_0 \in \mathbf{K}_{u,l}$ and $t \in [0, \infty)$. By Corollary 5.1, $\mu_t \in \mathbf{K}_{u^*,l^*}$ for all $t \in [0, \infty)$. Then, by (30) and (42), given $t \in [0, \infty)$ and $x \in \mathbb{R}_+$,

$$\begin{aligned} \overline{M}_t(x) &\ge \int_0^t n_e(x + s_t - s_v)dv \ge \lambda^* \int_0^t n_e(x + s_t - s_v)\frac{1}{q_v}dv \\ &= \lambda^* \int_x^{x+s_t} n_e(y)dy = \lambda^* \left(\overline{N}_e(x) - \overline{N}_e(x + s_t)\right). \quad \square \end{aligned}$$

The result in Lemma 5.3 motivates the following definition. For $u, l, \theta > 0$, let λ be given by (41) and set

$$\mathbf{M}_{u,l,\theta} = \left\{\zeta \in \mathbf{M}_{u,l} : \overline{Z}_\zeta(x) \ge \lambda\left(\overline{N}_e(x) - \overline{N}_e(x + \theta)\right)\right\}, \tag{44}$$

$$\mathbf{K}_{u,l,\theta} = \mathbf{M}_{u,l,\theta} \cap \mathbf{K}. \tag{45}$$

LEMMA 5.4. *Given $u, l, \theta > 0$, $\mathbf{M}_{u,l,\theta}$ is compact.*

Proof. Fix $u, l, \theta > 0$. Since $\mathbf{M}_{u,l,\theta} \subset \mathbf{M}_u$, $\mathbf{M}_{u,l,\theta}$ is relatively compact. Hence, it suffices to show that $\mathbf{M}_{u,l,\theta}$ is closed. Suppose that $\{\zeta_n\}_{n\in\mathbb{N}} \subset \mathbf{M}_{u,l,\theta}$ and $\zeta_n \xrightarrow{w} \zeta \in \mathbf{M}$ as $n \to \infty$. We must show that $\zeta \in \mathbf{M}_{u,l,\theta}$. Since $\mathbf{M}_{u,l,\theta} \subset \mathbf{M}_{u,l}$, Lemma 4.6 implies $\zeta \in \mathbf{M}_{u,l}$. Therefore, it suffices to show that for all $x \in \mathbb{R}_+$,

$$\overline{Z}_\zeta(x) \ge \lambda\left(\overline{N}_e(x) - \overline{N}_e(x + \theta)\right). \tag{46}$$

For all ζ-continuity points $x \in \mathbb{R}_+$,

$$\lim_{n\to\infty} \overline{Z}_{\zeta_n}(x) = \overline{Z}_\zeta(x).$$

Therefore, (46) holds for all ζ-continuity points $x \in \mathbb{R}_+$. The set of ζ-discontinuity points is countable, $\overline{Z}_\zeta$ is right continuous, and $\overline{N}_e$ is continuous. Therefore, (46) holds for all $x \in \mathbb{R}_+$. □

The following corollary implies that, given $u, l, \theta > 0$, any fluid model solution with initial value in $\mathbf{K}_{u,l}$ necessarily enters $\mathbf{K}_{u^*,l^*,\theta^*}$ by a uniform time T^* and stays in $\mathbf{K}_{u^*,l^*,\theta^*}$ thereafter, where u^*, l^*, λ^* are given by (35) and (42) and $\theta^* = \theta$.

COROLLARY 5.4. *Let $u, l, \theta > 0$. Let $u^*, l^*, \lambda^* > 0$ be given by (35) and (42) and set $\theta^* = \theta$ and $T^* = \theta^* u^*$. If μ is a fluid model solution such that $\mu_0 \in \mathbf{K}_{u,l}$, then $\mu_t \in \mathbf{K}_{u^*,l^*,\theta^*}$ for all $t \geq T^*$.*

PROOF. Fix $u, l, \theta > 0$. Let $u^*, l^*, \lambda^* > 0$ be the constants given by (35) and (42) and set $\theta^* = \theta$ and $T^* = \theta^* u^*$. Fix a fluid model solution μ such that $\mu_0 \in \mathbf{K}_{u,l}$. Then, for any $t > 0$, $s_t > 0$ and by Corollary 5.1 and Lemma 5.3, $\mu_t \in \mathbf{K}_{u^*,l^*,s_t}$ for all $t > 0$. For $t \geq T^*$, $s_t \geq s_{T^*} \geq \theta^*$ and so $\mathbf{K}_{u^*,l^*,s_t} \subset \mathbf{K}_{u^*,l^*,\theta^*}$. □

6. Proof of Theorem 3.1 via relative entropy. In this section, we assume that Theorem 3.2 holds and we obtain some consequences of it, including using it to prove Theorem 3.1. First we use it to prove Corollary 6.1, stated below. The statement of Corollary 6.1 is similar to that of Theorem 3.1. The distinction is that the set $\mathbf{J}$ appears in Corollary 6.1 rather than the set $\mathbf{I}$. To obtain the full result in Theorem 3.1, we use Corollary 6.1 in conjunction with Lemma 4.4 and Corollary 5.2. This is done following the proof of Corollary 6.1. Corollary 3.1 is proved as a consequence of Theorem 3.1 at the end of this section.

Recall that given $u, l, \delta > 0$, $\mathbf{K}^\delta_{u,l} = \{\zeta \in \mathbf{K}_{u,l} \ : \mathbf{d}(\zeta, \mathbf{I}) < \delta\}$ and $\mathbf{J}$ is given by (18).

COROLLARY 6.1. *Let $u, l > 0$. Then,*

$$\lim_{t\to\infty} \sup_{\xi \in \mathbf{K}_{u,l}} \mathbf{d}(\mu^\xi_t, \mathbf{J}) = 0. \tag{47}$$

Furthermore, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sup_{\xi \in \mathbf{K}^\delta_{u,l}} \sup_{t \in [0,\infty)} \mathbf{d}(\mu^\xi_t, \mathbf{J}) < \varepsilon. \tag{48}$$

PROOF. Fix $u,l>0$. Let $u^*,l^*>0$ be given by (35). Then, by Corollary 5.1, $\mu_t^\xi \in \mathbf{K}_{u^*,l^*}$ for all $\xi \in \mathbf{K}_{u,l}$ and $t\in[0,\infty)$. Given $x\in\mathbb{R}_+$, let

$$\mathbf{D}_x = \{\zeta \in \mathbf{M}_{u^*,l^*} : \mathbf{d}(\zeta,\mathbf{J}) \ge x\} \qquad \text{and} \qquad \mathbf{H}_x = \{\zeta \in \mathbf{M}_{u^*,l^*} : H(\zeta) \ge x\}.$$

For any given $x\in\mathbb{R}_+$, $\mathbf{D}_x$ is a closed subset of a compact set and is therefore compact. Furthermore, for $x>0$, $\mathbf{D}_x \cap \mathbf{J} = \emptyset$. Then, by Lemma 4.1, $H(\zeta)>0$ for all $\zeta\in\mathbf{D}_x$ and $x>0$. Since H is continuous on $\mathbf{D}_x \subset \mathbf{M}_{u^*,l^*}$ (see Lemma 4.8) and a continuous function on a compact set achieves its minimum value, it follows that for any given $x>0$, there exists $y>0$ such that $H(\zeta)\ge y$ for all $\zeta\in\mathbf{D}_x$. Fix $\varepsilon>0$ and let $\gamma>0$ be such that

$$\mathbf{D}_\varepsilon \subset \mathbf{H}_\gamma.$$

We begin by verifying (47). By Theorem 3.2, there exists $T>0$ such that $\mathcal{H}_\xi(t)<\gamma$ for all $t\ge T$ and $\xi\in\mathbf{K}_{u,l}$. Hence, $\mu_t^\xi \notin \mathbf{H}_\gamma$ for all $t\ge T$ and $\xi\in\mathbf{K}_{u,l}$. Therefore, $\mu_t^\xi\notin\mathbf{D}_\varepsilon$ for all $t\ge T$ and $\xi\in\mathbf{K}_{u,l}$. But $\mu_t^\xi\in\mathbf{M}_{u^*,l^*}$ for all $t\in[0,\infty)$ and $\xi\in\mathbf{K}_{u,l}$. Therefore, it follows that $\mathbf{d}(\mu_t^\xi,\mathbf{J})<\varepsilon$ for all $t\ge T$ and $\xi\in\mathbf{K}_{u,l}$. Since $\varepsilon>0$ was arbitrary, (47) holds.

Next we verify (48). By Lemmas 4.1, 4.6, and 4.8, H is uniformly continuous on $\mathbf{M}_{u,l}$ and, for $\zeta\in\mathbf{M}_{u,l}$, $H(\zeta)=0$ if and only if $\zeta\in\mathbf{J}^+$. Therefore, there exists $\delta>0$ such that $H(\zeta)<\gamma$ for all $\zeta\in\mathbf{M}_{u,l}$ satisfying $\mathbf{d}(\zeta,\mathbf{J}^+)<\delta$. Observe that $\mathbf{K}_{u,l}^\delta\subset\mathbf{M}_{u,l}$, $\mathbf{I}^+\subset\mathbf{J}^+$ and $\mathbf{d}(\zeta,\mathbf{I})<\delta$ implies $\mathbf{d}(\zeta,\mathbf{I}^+)<\delta$. Therefore, $\mathbf{d}(\xi,\mathbf{J}^+)<\delta$ for all $\xi\in\mathbf{K}_{u,l}^\delta$. Hence, $\xi\in\mathbf{K}_{u,l}^\delta$ implies that $\mathcal{H}_\xi(0)=H(\xi)<\gamma$. By the monotonicity asserted in Theorem 3.2, it follows that for all $\xi\in\mathbf{K}_{u,l}^\delta$ and $t\in[0,\infty)$,

$$H(\mu_t^\xi)=\mathcal{H}_\xi(t)\le\mathcal{H}_\xi(0)=H(\xi)<\gamma.$$

Hence, $\mu_t^\xi\notin\mathbf{H}_\gamma$ for all $\xi\in\mathbf{K}_{u,l}^\delta$ and $t\in[0,\infty)$. Therefore, $\mu_t^\xi\notin\mathbf{D}_\varepsilon$ for all $\xi\in\mathbf{K}_{u,l}^\delta$ and $t\in[0,\infty)$. But, $\mu_t^\xi\in\mathbf{K}_{u^*,l^*}$ for all $\xi\in\mathbf{K}_{u,l}^\delta$ and $t\in[0,\infty)$. Consequently, $\mathbf{d}(\mu_t^\xi,\mathbf{J})<\varepsilon$ for all $\xi\in\mathbf{K}_{u,l}^\delta$ and $t\in[0,\infty)$. □

PROOF OF THEOREM 3.1. Fix $u,l,\varepsilon>0$ and let u^*,l^* be given by (35). We will show that there exists $T,\delta>0$ such that if either $\xi\in\mathbf{K}_{u,l}$ and $t\ge T$ or $\xi\in\mathbf{K}_{u,l}^\delta$ and $t\in[0,\infty)$, then

$$\mathbf{d}(\mu_t^\xi,\mathbf{I})<\varepsilon.$$

Suppose that $\kappa,T,\delta>0$ are such that if either $\xi\in\mathbf{K}_{u,l}$ and $t\ge T$ or $\xi\in\mathbf{K}_{u,l}^\delta$ and $t\in[0,\infty)$, then

$$\mathbf{d}(\mu_t^\xi,\mathbf{J})<\kappa.$$

Then, by Lemma 4.4, if either $\xi \in \mathbf{K}_{u,l}$ and $t \geq T$ or $\xi \in \mathbf{K}^{\delta}_{u,l}$ and $t \in [0,\infty)$, then

$$\mathbf{d}(\mu^{\xi}_t, \mathbf{I}) < \left\langle 1_{[0,\kappa)}, \mu^{\xi}_t \right\rangle + 2\kappa.$$

Hence it suffices to find $0 < \kappa \leq \varepsilon/3$ and $T, \delta > 0$ such that if either $\xi \in \mathbf{K}_{u,l}$ and $t \geq T$ or $\xi \in \mathbf{K}^{\delta}_{u,l}$ and $t \in [0,\infty)$, then

$$\mathbf{d}(\mu^{\xi}_t, \mathbf{J}) < \kappa \qquad \text{and} \qquad \left\langle 1_{[0,\kappa)}, \mu^{\xi}_t \right\rangle \leq \frac{\varepsilon}{3}. \tag{49}$$

We will find such a $\kappa, T, \delta > 0$.

To begin, let $\theta > 0$ be such that $u^* \overline{N}_e(\theta) < \varepsilon/6$ and set $T' = u^*\theta$. Then, for all $t \geq T'$, $u^* \overline{N}_e(t/u^*) < \varepsilon/6$. Next, let $x' > 0$ be such that $u^* \alpha x' \leq \varepsilon/6$. Then, by Corollary 5.2, there exists $\delta' > 0$ and $0 < \kappa < \min(x', \varepsilon/3)$ such that if either $\xi \in \mathbf{K}_{u,l}$ and $t \geq T'$ or $\xi \in \mathbf{K}^{\delta}_{u,l}$ for any $0 < \delta \leq \delta'$ and $t \in [0,\infty)$, then

$$\left\langle 1_{[0,\kappa)}, \mu^{\xi}_t \right\rangle \leq \frac{\varepsilon}{3}.$$

By Corollary 6.1, there exists $T \geq T'$ and $0 < \delta \leq \delta'$ such that if either $\xi \in \mathbf{K}_{u,l}$ and $t \geq T$ or $\xi \in \mathbf{K}^{\delta}_{u,l}$ and $t \in [0,\infty)$, then

$$\mathbf{d}(\mu^{\xi}_t, \mathbf{J}) < \kappa.$$

Hence (49) holds. □

Proof of Corollary 3.1. Fix $u, l, \varepsilon > 0$ and let u^*, l^* be given by (35). We begin by verifying (12). In particular, we will show that there exists $T > 0$ such that if $\xi \in \mathbf{K}_{u,l}$ and $t \geq T$, then

$$\mathbf{d}(\mu^{\xi}_t, \alpha_e w_0 \nu_e) < \varepsilon. \tag{50}$$

Fix $x^* > 0$ such that

$$\int_{x^*}^{\infty} \overline{N}_e(x)dx < \frac{\varepsilon}{3\alpha_e\,(2u^* + 1)}. \tag{51}$$

Also fix $0 < \kappa < \frac{\varepsilon}{12\alpha_e x^*} \wedge \frac{\varepsilon}{3} \wedge 1$ such that

$$\sup_{x \in \mathbb{R}_+} \nu_e(x, x + 3\kappa) < \frac{\varepsilon}{12\alpha_e(u^* + 1)x^*}. \tag{52}$$

By Theorem 3.1, there exists $T > 0$ such that for each $\xi \in \mathbf{K}_{u,l}$ and each $T \geq t$, there exists a nonnegative constant $c(\xi, t)$ such that

$$\mathbf{d}(\mu^{\xi}_t, c(\xi, t)\nu_e) < \kappa. \tag{53}$$

Fix such a $T > 0$ and a collection $\{c(\xi,t) : \xi \in \mathbf{K}_{u,l},\ t \geq T\}$ of nonnegative constants satisfying (53). Then for each $\xi \in \mathbf{K}_{u,l}$ and $t \geq T$,

$$\mathbf{d}(\mu_t^\xi, \alpha_e w_0 \nu_e) \leq \mathbf{d}(\mu_t^\xi, c(\xi,t)\nu_e) + \mathbf{d}(c(\xi,t)\nu_e, \alpha_e w_0 \nu_e). \tag{54}$$

For each $\xi \in \mathbf{K}_{u,l}$ and $t \geq T$, $w_0 = \left\langle \chi, \mu_t^\xi \right\rangle$ and, since ν_e is a probability measure,

$$\begin{aligned}
\mathbf{d}(c(\xi,t)\nu_e, \alpha_e w_0 \nu_e) &\leq |c(\xi,t) - \alpha_e w_0| = \alpha_e \left| \langle \chi, c(\xi,t)\nu_e \rangle - \left\langle \chi, \mu_t^\xi \right\rangle \right| \\
&= \alpha_e \left| \int_0^\infty c(\xi,t)\overline{N}_e(x)dx - \int_0^\infty \overline{M}_t^\xi(x)dx \right| \\
&\leq \alpha_e \left| \int_0^{x^*} c(\xi,t)\overline{N}_e(x)dx - \int_0^{x^*} \overline{M}_t^\xi(x)dx \right| \\
&\quad + \alpha_e \int_{x^*}^\infty c(\xi,t)\overline{N}_e(x)dx + \alpha_e \int_{x^*}^\infty \overline{M}_t^\xi(x)dx.
\end{aligned}$$

Since $\mu_t^\xi \in \mathbf{K}_{u^*,l^*}$ for all $\xi \in \mathbf{K}_{u,l}$ and $t \geq 0$, it follows that for each $\xi \in \mathbf{K}_{u,l}$ and $t \geq T$,

$$\begin{aligned}
\mathbf{d}(c(\xi,t)\nu_e, \alpha_e w_0 \nu_e) &\leq \alpha_e \left| \int_0^{x^*} c(\xi,t)\overline{N}_e(x)dx - \int_0^{x^*} \overline{M}_t^\xi(x)dx \right| \\
&\quad + \alpha_e (c(\xi,t) + u^*) \int_{x^*}^\infty \overline{N}_e(x)dx.
\end{aligned}$$

By (53) and the choice of κ, $c(\xi,t) \leq u^* + \kappa < u^* + 1$ for each $\xi \in \mathbf{K}_{u,l}$ and $t \geq T$. This together with (51) yields that, for each $\xi \in \mathbf{K}_{u,l}$ and $t \geq T$,

$$\mathbf{d}(c(\xi,t)\nu_e, \alpha_e w_0 \nu_e) < \alpha_e \int_0^{x^*} \left| c(\xi,t)\overline{N}_e(x) - \overline{M}_t^\xi(x) \right| dx + \frac{\varepsilon}{3}.$$

Combining this with (54), (53) and the choice of κ yields that for each $\xi \in \mathbf{K}_{u,l}$ and $t \geq T$,

$$\mathbf{d}(\mu_t^\xi, \alpha_e w_0 \nu_e) < \frac{2\varepsilon}{3} + \alpha_e \int_0^{x^*} \left| c(\xi,t)\overline{N}_e(x) - \overline{M}_t^\xi(x) \right| dx.$$

By applying (53) twice, recalling that $c(\xi,t) \leq u^* + 1$ for each $\xi \in \mathbf{K}_{u,l}$ and $t \geq T$, applying (52), and appealing to the choice of κ, it follows that for each $\xi \in \mathbf{K}_{u,l}$, $t \geq T$, and $x \in [0, x^*]$,

$$\left| c(\xi,t)\overline{N}_e(x) - \overline{M}_t^\xi(x) \right| \leq c(\xi,t) \left\langle 1_{((x-\kappa)^+,x)}, \nu_e \right\rangle + \left\langle 1_{((x-\kappa)^+,x)}, \mu_t^\xi \right\rangle + \kappa$$

$$
\begin{aligned}
&\leq c(\xi,t)\left\langle 1_{((x-\kappa)^+,x)},\nu_e\right\rangle \\
&\quad +c(\xi,t)\left\langle 1_{((x-2\kappa)^+,x+\kappa)},\nu_e\right\rangle+2\kappa \\
&\leq 2(u^*+1)\left\langle 1_{((x-2\kappa)^+,x+\kappa)},\nu_e\right\rangle+2\kappa \\
&< \frac{\varepsilon}{3\alpha_e x^*}.
\end{aligned}
$$

Combining the previous two displays yields (50). Hence, (12) holds. The verification of (14) follows a similar line of reasoning. To see that (13) holds, fix $u>0$ and $\xi\in\mathbf{K}_u$. If $\xi=\mathbf{0}$, then $\mu^\xi\equiv\mathbf{0}$ and the result holds trivially. If $\xi\neq\mathbf{0}$, then $\xi\in\mathbf{K}_{u,\langle\chi,\xi\rangle}$ and the result follows from (12). □

7. Relative entropy along fluid paths. In this section, we develop properties of the relative entropy functional H as a function of time along fluid model solutions. In particular, given $\xi\in\mathbf{K}^\dagger$, we analyze the behavior of $\mathcal{H}_\xi$. A main result developed in this section is Theorem 7.1, which is an absolute continuity property satisfied by $\mathcal{H}_\xi$ for $\xi\in\mathbf{K}_{u,l}$ for $u,l>0$. This absolute continuity property implies a monotonicity property for $\mathcal{H}_\xi$. The proof of Theorem 3.2, given in Section 8, exploits this absolutely continuity property of $\mathcal{H}_\xi$, as well as the resulting monotonicity property.

In order to state Theorem 7.1, we introduce some notation. Let $u,l>0$. In Theorem 7.1 we establish that for each fluid model solution μ^ξ with initial state $\xi\in\mathbf{K}_{u,l}$, $\mathcal{H}_\xi$ is absolutely continuous with respect to Lebesgue measure on $[0,\infty)$ and that the density function is nonpositive. In fact, we obtain an explicit representation of the density function. For this, for $x\in(0,\infty)$, let

$$k(x)=x-1-\ln x.$$

We further define $k(0)=\infty$ so that $k:\mathbb{R}_+\to[0,\infty]$ is continuous. Given a fluid model solution μ^ξ with initial state $\xi\in\mathbf{K}^\dagger$ and $t\in[0,\infty)$, let

$$\mathcal{K}_\xi(t)=\int_0^{x_\nu} k\left(\frac{\overline{M}_t(x)}{q_t\overline{N}_e(x)}\right)n_e(x)dx. \tag{55}$$

In this section, we show that $\mathcal{H}_\xi$ is absolutely continuous with respect to Lebesgue measure with density function equal to a strictly negative constant multiple of $\mathcal{K}_\xi$ for $\xi\in\mathbf{K}_{u,l}$. In particular, we prove the following theorem.

THEOREM 7.1. *Let $u,l>0$ and $\xi\in\mathbf{K}_{u,l}$. The function $\mathcal{H}_\xi$ is absolutely continuous on $[0,\infty)$, with respect to Lebesgue meaure, with density function κ_ξ given by*

$$\kappa_\xi(t)=\frac{-1}{w_0}\mathcal{K}_\xi(t),\qquad \text{for all } t\in(0,\infty). \tag{56}$$

In particular, $\mathcal{H}_\xi$ is nonincreasing on $[0,\infty)$.

In order to prove Theorem 7.1, we first establish an absolute continuity result for initial states that are absolutely continuous (see Lemma 7.8). For this, recall (24), (25), and (44) and given $u, l, \theta > 0$ let

$$\mathbf{A}_u = \mathbf{A} \cap \mathbf{M}_u, \qquad \mathbf{A}_{u,l} = \mathbf{A} \cap \mathbf{M}_{u,l}, \qquad \text{and} \qquad \mathbf{A}_{u,l,\theta} = \mathbf{A} \cap \mathbf{M}_{u,l,\theta}.$$

Then, in Section 7.4, we use an approximation argument to extend this result to include certain continuous initial states, and thereby complete the proof of Theorem 7.1.

7.1. *The function K.* Recall that given $\xi \in \mathbf{K}^\dagger$, $\mathcal{H}_\xi(t) = H(\mu_t^\xi)$ for all $t \in [0,\infty)$, where H is given by (15). Similarly for $K : \mathbf{M} \setminus \mathbf{0} \to [0,\infty]$ defined by

$$K(\zeta) = \int_0^{x_\nu} k\left(\frac{\overline{Z}_\zeta(x)}{\langle 1, \zeta\rangle\, \overline{N}_e(x)}\right) n_e(x)dx, \tag{57}$$

we have that $\mathcal{K}_\xi(t) = K(\mu_t^\xi)$ for all $t \in [0,\infty)$ and $\xi \in \mathbf{K}^\dagger$. We now develop some properties of K.

LEMMA 7.1. *For $\zeta \in \mathbf{M} \setminus \mathbf{0}$, $K(\zeta) = 0$ if and only if $\zeta \in \mathbf{I}^+$.*

PROOF. Fix $\zeta \in \mathbf{M} \setminus \mathbf{0}$. If $\zeta \in \mathbf{I}^+$, then $\zeta = \beta\nu_e$ for some $\beta \in (0,\infty)$. Then, since $k(1) = 0$, it is immediate that $K(\zeta) = 0$. On the other hand, given $\zeta \in \mathbf{M}\setminus\mathbf{0}$ such that $K(\zeta) = 0$, since $k \geq 0$, it follows that for Lebesgue almost every $x \in [0, x_\nu)$,

$$k\left(\frac{\overline{Z}_\zeta(x)}{\langle 1, \zeta\rangle\, \overline{N}_e(x)}\right) = 0.$$

But k is finite and positive on $(0,1) \cup (1,\infty)$ and infinity at 0. Hence, for Lebesgue almost every $x \in [0, x_\nu)$,

$$\frac{\overline{Z}_\zeta(x)}{\langle 1, \zeta\rangle\, \bar{N}_e(x)} = 1.$$

Since both the numerator and denominator are right continuous in x, the above equality holds for all $x \in [0, x_\nu)$. This together with $\zeta \in \mathbf{M}\setminus\mathbf{0}$ implies that $\zeta \in \mathbf{J}^+$. By taking $x = 0$, it also implies that $\overline{Z}_\zeta(0) = \langle 1, \zeta\rangle$ so that $\zeta(\{0\}) = 0$. Hence, $\zeta \in \mathbf{I}^+$. $\square$

Lemma 7.2. *Suppose that* $\{\zeta_n\}_{n\in\mathbb{N}} \subset \mathbf{M}\setminus\mathbf{0}$, $\zeta\in\mathbf{M}\setminus\mathbf{0}$, *and* $\zeta_n \overset{w}{\to} \zeta$ *as* $n\to\infty$. *Then*

$$\liminf_{n\to\infty} K(\zeta_n) \geq K(\zeta). \tag{58}$$

In particular, K *is lower semicontinuous on* $\mathbf{M}\setminus\mathbf{0}$.

Proof. The assumptions imply that $\lim_{n\to\infty}\langle 1,\zeta_n\rangle = \langle 1,\zeta\rangle > 0$ and that for almost all $x\in\mathbb{R}_+$,

$$\lim_{n\to\infty} \overline{Z}_{\zeta_n}(x) = \overline{Z}_{\zeta}(x).$$

Then, since k is continuous, for almost all $x\in[0,x_\nu)$,

$$\lim_{n\to\infty} k\left(\frac{\overline{Z}_{\zeta_n}(x)}{\langle 1,\zeta_n\rangle\,\overline{N}_e(x)}\right) = k\left(\frac{\overline{Z}_{\zeta}(x)}{\langle 1,\zeta\rangle\,\overline{N}_e(x)}\right).$$

Further, since the range of k is contained in $[0,\infty]$, Fatou's lemma implies that $\liminf_{n\to\infty} K(\zeta_n)\geq K(\zeta)$. □

Next we identify some conditions under which K is finite.

Lemma 7.3. *For each* $u,l,\theta>0$ *and* $\zeta\in\mathbf{M}_{u,l,\theta}$, $K(\zeta)<\infty$.

The proof of this lemma uses the results stated in the next two lemmas.

Lemma 7.4. *Given a probability density function* $g:\mathbb{R}_+\to\mathbb{R}_+$ *with associated cumulative distribution function* $G:\mathbb{R}_+\to[0,1]$, *let* $\overline{G}(x) = 1-G(x)$ *for all* $x\in\mathbb{R}_+$ *and set*

$$y^* = \inf\{x\in\mathbb{R}_+ : \overline{G}(x)=0\}.$$

Then

$$-\int_0^{y^*} \ln\left(\overline{G}(x)\right) g(x)dx = 1.$$

Proof. Consider the change of variables $y=-\ln\left(\overline{G}(x)\right)$. For $x=0$, $y=0$ and for $x=y^*$, $y=\infty$. Also $\exp(-y)=\overline{G}(x)$ and so $\exp(-y)dy = g(x)dx$. Therefore,

$$-\int_0^{y^*} \ln\left(\overline{G}(x)\right) g(x)dx = \int_0^{\infty} y\exp(-y)dy = 1. \quad \square$$

LEMMA 7.5. *For each $\theta > 0$, let*

$$j_\theta(x) = -\ln\left(\overline{N}_e(x) - \overline{N}_e(x+\theta)\right), \qquad x \in [0, x_\nu).$$

Then, for each $\theta > 0$, $j_\theta \geq 0$ and $j_\theta \in \mathbf{L}^1(\nu_e)$.

PROOF. Fix $\theta > 0$. Observe that for $x \in [0, x_\nu)$, $0 \leq j_\theta(x) < \infty$ since $0 < \overline{N}_e(x) - \overline{N}_e(x+\theta) \leq 1$. Also note that for such x,

$$\frac{dj_\theta(x)}{dx} = \frac{n_e(x) - n_e(x+\theta)}{\overline{N}_e(x) - \overline{N}_e(x+\theta)} > 0.$$

So j_θ is nonnegative, finite and strictly increasing on $[0, x_\nu)$.

First consider the case $x_\nu < \infty$. Then,

$$\int_0^{x_\nu} j_\theta(x) n_e(x) dx = \int_0^{(x_\nu-\theta)^+} j_\theta(x) n_e(x) dx + \int_{(x_\nu-\theta)^+}^{x_\nu} \ln\left(\overline{N}_e(x)\right) n_e(x) dx. \tag{59}$$

The second integral on the righthand side of (59) converges by Lemma 7.4. If $x_\nu \leq \theta$, the first integral on the righthand side of (59) is zero and the result follows. Otherwise, $\theta < x_\nu < \infty$. Then using the fact that j_θ is strictly increasing, we see that

$$\int_0^{x_\nu-\theta} j_\theta(x) n_e(x) dx \leq \alpha \int_0^{x_\nu-\theta} j_\theta(x) dx \leq \alpha(x_\nu - \theta) j_\theta(x_\nu - \theta) < \infty.$$

So the result holds if $x_\nu < \infty$.

Next consider the case where $x_\nu = \infty$. Then, we have that

$$\int_0^\infty j_\theta(x) n_e(x) dx = \int_0^\infty j_\theta(x) \left(n_e(x) - n_e(x+\theta)\right) dx + \int_0^\infty j_\theta(x) n_e(x+\theta) dx.$$

In a manner similar to the proof of Lemma 7.4, we can use the change of variables $u = j_\theta(x)$ to demonstrate that

$$\int_0^\infty j_\theta(x) \left(n_e(x) - n_e(x+\theta)\right) dx \leq 1.$$

Hence, it suffices to show that

$$\int_0^\infty j_\theta(x) n_e(x+\theta) dx < \infty. \tag{60}$$

Note that by the monotonicity of n_e for all $x \in \mathbb{R}_+$,

$$\overline{N}_e(x) - \overline{N}_e(x+\theta) \geq \theta n_e(x+\theta) = \alpha\theta\overline{N}(x+\theta).$$

Therefore,

$$\begin{aligned}\int_0^\infty j_\theta(x) n_e(x+\theta)dx &\leq -\int_0^\infty \ln\left(\alpha\theta\overline{N}(x+\theta)\right) n_e(x+\theta)dx \\ &= -\ln(\alpha\theta)\,\overline{N}_e(\theta) - \alpha\int_0^\infty \ln\left(\overline{N}(x+\theta)\right)\overline{N}(x+\theta)dx \\ &= -\ln(\alpha\theta)\,\overline{N}_e(\theta) - \alpha\int_\theta^\infty \ln\left(\overline{N}(x)\right)\overline{N}(x)dx \\ &\leq -\ln(\alpha\theta)\,\overline{N}_e(\theta) - \alpha\int_0^\infty \ln\left(\overline{N}(x)\right)\overline{N}(x)dx.\end{aligned}$$

Therefore, in order to verify (60), it suffices to show that

$$-\int_0^\infty \ln\left(\overline{N}(x)\right)\overline{N}(x)dx < \infty. \tag{61}$$

To verify (61) note that, for all $\gamma \in [1/e, 1)$, elementary calculus can be used to demonstrate that $-\ln(x) \leq 1/x^\gamma$ for all $x \in (0, 1]$. Hence, for all $\gamma \in [1/e, 1)$ and $x \in [0, x_\nu)$,

$$-\ln\left(\overline{N}(x)\right) \leq \left(\frac{1}{\overline{N}(x)}\right)^\gamma.$$

Further, for all $\gamma \in [1/e, 1/2)$, we have that $2(1-\gamma) > 1$. Then, since $x_\nu = \infty$, it follows that for all $\gamma \in [1/e, 1/2)$,

$$\begin{aligned}-\int_0^\infty \ln\left(\overline{N}(x)\right)\overline{N}(x)dx &\leq \int_0^\infty \left(\frac{1}{\overline{N}(x)}\right)^\gamma \overline{N}(x)dx \\ &= \int_0^1 \overline{N}(x)^{1-\gamma}dx + \int_1^\infty \overline{N}(x)^{1-\gamma}dx \\ &\leq 1 + \int_1^\infty \left(\frac{\langle\chi^2, \nu\rangle}{x^2}\right)^{1-\gamma} dx \\ &< \infty.\end{aligned}$$

Hence, the result holds if $x_\nu = \infty$. ☐

Proof of Lemma 7.3. Fix $u, l, \theta > 0$ and $\zeta \in \mathbf{M}_{u,l,\theta}$. Let λ be given by (41). For $x \in [0, x_\nu)$,

$$0 < \lambda\left(\overline{N}_e(x) - \overline{N}_e(x+\theta)\right) \leq \overline{Z}_\zeta(x).$$

Therefore, using the above for the lower bound and the fact that $\zeta \in \mathbf{M}_{u,l,\theta}$ and the definition of λ for the upper bound, we have for $x \in [0, x_\nu)$,

$$0 < \frac{\overline{Z}_\zeta(x)}{\langle 1, \zeta \rangle \, \overline{N}_e(x)} \le \frac{u}{\lambda}.$$

Hence, one only needs to be concerned with the integral of the natural logarithm term, which could blow up if the ratio were to approach zero too fast relative to the rate at which n_e tends to zero as x increases to x_ν. By Lemma 7.4,

$$\int_0^{x_\nu} \ln\left(\frac{\overline{Z}_\zeta(x)}{\langle 1, \zeta \rangle \, \overline{N}_e(x)}\right) n_e(x)dx = \int_0^{x_\nu} \ln\left(\frac{\overline{Z}_\zeta(x)}{\langle 1, \zeta \rangle}\right) n_e(x)dx + 1.$$

Since $\zeta \in \mathbf{M}_{u,l,\theta}$,

$$-\int_0^{x_\nu} \ln\left(\frac{\overline{Z}_\zeta(x)}{\langle 1, \zeta \rangle}\right) n_e(x)dx \le -\int_0^{x_\nu} \ln\left[\frac{\lambda}{u}\left(\overline{N}_e(x) - \overline{N}_e(x+\theta)\right)\right] n_e(x)dx.$$

By Lemma 7.5, the righthand side is finite. Hence, $K(\zeta) < \infty$. $\square$

Having determined the zero set, verified lower semicontinuity, and given sufficient conditions for finiteness of K, next we demonstrate that K is continuous on certain compact sets.

LEMMA 7.6. *For each $u, l, \theta > 0$, K is continuous on $\mathbf{M}_{u,l,\theta}$.*

PROOF. Fix $u, l, \theta > 0$. As previously noted, $\mathbf{M}_{u,l,\theta}$ is compact (see Lemma 5.4). Fix $\{\zeta_n\}_{n\in\mathbb{N}} \subset \mathbf{M}_{u,l,\theta}$ such that $\zeta_n \overset{w}{\to} \zeta$ as $n \to \infty$. Then $\zeta \in \mathbf{M}_{u,l,\theta}$, $\lim_{n\to\infty} \langle 1, \zeta_n \rangle = \langle 1, \zeta \rangle$, and for all ζ-continuity points $x \in \mathbb{R}_+$,

$$\lim_{n\to\infty} \overline{Z}_{\zeta_n}(x) = \overline{Z}_\zeta(x).$$

Hence, for almost all $x \in [0, x_\nu)$,

$$\lim_{n\to\infty} \frac{\overline{Z}_{\zeta_n}(x)}{\langle 1, \zeta_n \rangle \, \overline{N}_e(x)} = \frac{\overline{Z}_\zeta(x)}{\langle 1, \zeta \rangle \, \overline{N}_e(x)}.$$

Since k is continuous, it follows that for almost all $x \in [0, x_\nu)$,

$$\lim_{n\to\infty} k\left(\frac{\overline{Z}_{\zeta_n}(x)}{\langle 1, \zeta_n \rangle \, \overline{N}_e(x)}\right) = k\left(\frac{\overline{Z}_\zeta(x)}{\langle 1, \zeta \rangle \, \overline{N}_e(x)}\right).$$

Note that by the definition of $\mathbf{M}_{u,l,\theta}$ for all $x \in [0, x_\nu)$ and $n \in \mathbb{N}$,

$$\frac{\lambda(\overline{N}_e(x) - \overline{N}_e(x+\theta))}{u\overline{N}_e(x)} \leq \frac{\overline{Z}_{\zeta_n}(x)}{\langle 1, \zeta_n \rangle \overline{N}_e(x)} \leq \frac{u}{\lambda}.$$

Here the lower bound is positive and bounded above by one, and the upper bound is greater than or equal to one. Furthermore, $0 \leq k(x) \leq \max(x - 1, -\ln x)$ for all $x \in \mathbb{R}_+$. Therefore, for all $t \in [0, \infty)$ and $x \in [0, x_\nu)$,

$$0 \leq k\left(\frac{\overline{Z}_{\zeta_n}(x)}{\langle 1, \zeta_n \rangle \overline{N}_e(x)}\right) \leq \frac{u}{\lambda} - 1 - \ln\left(\frac{\lambda(\overline{N}_e(x) - \overline{N}_e(x+\theta))}{u\overline{N}_e(x)}\right).$$

For $x \in [0, x_\nu)$, let

$$g(x) = \frac{u}{\lambda} - 1 + \ln\left(\frac{u}{\lambda}\right) + \ln\left(\overline{N}_e(x)\right) - \ln\left(\overline{N}_e(x) - \overline{N}_e(x+\theta)\right).$$

By Lemmas 7.4 and 7.5, $g \in \mathbf{L}^1(\nu_e)$. Hence, by the dominated convergence theorem, $\lim_{n\to\infty} K(\zeta_n) = K(\zeta)$. □

7.2. *Finiteness and continuity of $\mathcal{H}_\xi$ and $\mathcal{K}_\xi$.* In preparation for proving the absolute continuity results, we state and prove the following lemma.

LEMMA 7.7. *Let $u, l > 0$ and $\xi \in \mathbf{K}_{u,l}$. The function $\mathcal{H}_\xi$ is finite and continuous on $[0, \infty)$ and the function $\mathcal{K}_\xi$ is finite and continuous on $(0, \infty)$.*

PROOF. Fix $u, l > 0$ and $\xi \in \mathbf{K}_{u,l}$. Let $u^*, l^* > 0$ be the constants given by (35). By Corollary 5.1, $\mu_t^\xi \in \mathbf{K}_{u^*,l^*} \subset \mathbf{M}_{u^*,l^*}$, for all $t \in [0, \infty)$. For $t \in [0, \infty)$, we have that $\mathcal{H}_\xi(t) = H(\mu_t^\xi)$. By Lemmas 4.7 and 4.8, H is finite and continuous on $\mathbf{M}_{u^*,l^*}$. Thus, $\mathcal{H}_\xi$ is finite. Further, by property (C.1) of fluid model solutions, fluid model solutions are continuous functions of time. Then $\mathcal{H}_\xi$ is continuous since it is a composition of continuous functions.

Next we turn our attention to $\mathcal{K}_\xi$. It suffices to verify finiteness and continuity on (t, ∞) for each $t > 0$. For this fix $t > 0$. Set $\theta = t/u^*$. Let θ^* and T^* be as in the statement of Corollary 5.4. Then $T^* = t$. Hence, by Corollary 5.4, $\mu_r^\xi \in \mathbf{K}_{u^*,l^*,\theta^*} \subset \mathbf{M}_{u^*,l^*,\theta^*}$ for all $r \geq t$. For each $r \in (t, \infty)$, $\mathcal{K}_\xi(r) = K(\mu_r^\xi)$. The result follows from property (C.1) of fluid model solutions and Lemmas 7.3 and 7.6. □

7.3. *Absolutely continuous initial states.* In this section, we prove the following lemma, which is a version of Theorem 7.1 for fluid model solutions with absolutely continuous initial measures.

LEMMA 7.8. *Let $u, l > 0$ and $\xi \in \mathbf{A}_{u,l}$. The function $\mathcal{H}_\xi$ is absolutely continuous with respect to Lebesgue measure on $[0, \infty)$ with density function κ_ξ given by*

$$\kappa_\xi(t) = \frac{-1}{w_0} \mathcal{K}_\xi(t), \qquad \text{for all } t \in (0, \infty) \tag{62}$$

In particular, $\mathcal{H}_\xi$ is nonincreasing on $[0, \infty)$.

The proof of Lemma 7.8 relies on absolute continuity properties in both space and time satisfied by fluid model solutions having absolutely continuous initial conditions. These properties are developed in Section 7.3.1. Following that, Lemma 7.8 is proved in Section 7.3.2.

7.3.1. *Partial derivatives for fluid model solutions.* In this section, we develop a differential equation (see (65) below), which plays a key role in proving Lemma 7.8. Throughout this section, we restrict attention to fluid model solutions μ such that $\mu_0 \in \mathbf{A}$. Here we show that all such fluid model solutions remain in $\mathbf{A}$ for all time by developing a formula for the density with respect to Lebesgue measure of μ_t for each fixed $t \in [0, \infty)$ (see Lemma 7.9 below). This is used to show that for each fixed $x \in \mathbb{R}_+$, $\overline{M}_t(x) = \langle 1_{(x,\infty)}, \mu_t \rangle$ as a function of time is absolutely continuous with respect to Lebesgue measure. In fact, we develop a formula for the associated density in time, which gives rise to the differential equation (65) (see Lemma 7.10 below). The differential equation (65) suffices for the proof of Lemma 7.8 given in Section 7.3.2.

Moreover, when $\mu_0 \in \mathbf{A}$ has a continuous density and ν does not charge points, the results in Lemmas 7.9 and 7.10 can be combined to show $\overline{M}_t(x) = \langle 1_{(x,\infty)}, \mu_t \rangle$ satisfies a certain partial differential equation (PDE) (see (66)). This type of PDE is not new to the literature. Indeed, a version of this PDE was used in [10] to study stability properties of subcritical bandwidth sharing models. However, the authors of [10] assumed that their fluid model solutions are absolutely continuous with respect to Lebesgue measure for all time and that fluid model solutions are sufficiently smooth for their PDE to be satisfied. Here, in Corollary 7.1, we provide a rigorous connection between our PDE and fluid model solutions. Specifically, we provide sufficient conditions for $\overline{M}_t(x) = \langle 1_{(x,\infty)}, \mu_t \rangle$ to satisfy (66).

Henceforth, we adopt the convention that when $\xi = \mathbf{0}$ so that $\mu_t^\xi = \mathbf{0}$ for all $t \in [0, \infty)$, $s_t \equiv 0$ for all $t \in [0, \infty)$. This is needed for the statement of Lemma 7.9.

LEMMA 7.9. *Let μ be a fluid model solution with $\mu_0 \in \mathbf{A}$. For each $t \in [0,\infty)$, $\mu_t \in \mathbf{A}$. In particular, for each $t \in [0,\infty)$, the following is a density (with respect to Lebesgue measure) for μ_t:*

$$m_t(x) = m_0(x+s_t) + \alpha \left\langle 1_{(x,x+s_t]}(\cdot) q_{\tau(x+s_t-\cdot)}, \nu \right\rangle, \qquad \text{for all } x \in \mathbb{R}_+, \tag{63}$$

where m_0 denotes a density for μ_0. In particular, for each $t \in [0,\infty)$,

$$\frac{\partial \overline{M}_t(x)}{\partial x} = -m_t(x), \qquad \text{for all almost all } x \in \mathbb{R}_+. \tag{64}$$

Furthermore, if ν does not charge atoms and m_0 is continuous on $\mathbb{R}_+$, then $m_t(x)$ is continuous as a function of $x \in \mathbb{R}_+$ for each $t \in [0,\infty)$ and continuous as a function of $t \in [0,\infty)$ for each $x \in \mathbb{R}_+$, in which case (64) holds for all $t \in [0,\infty)$ and $x \in \mathbb{R}_+$, where partial derivatives at $t = 0$ and $x = 0$ are from the right.

PROOF. Fix a fluid model solution μ such that $\mu_0 \in \mathbf{A}$. The result is trivial if $\mu_0 = \mathbf{0}$ since $\mathbf{0}$ is an invariant state. Henceforth assume that $\mu_0 \in \mathbf{A}^+$. For $t \in [0,\infty)$ and $x \in \mathbb{R}_+$, let $m_t(x)$ be defined by (63). We will show that for each $t \in [0,\infty)$, m_t is integrable with respect to Lebesgue measure on $\mathbb{R}_+$ and moreover that for each $x \in \mathbb{R}_+$, $\overline{M}_t(x) = \left\langle 1_{(x,\infty)}, \mu_t \right\rangle = \int_x^\infty m_t(y)dy$. Fix $t \in [0,\infty)$ and $x \in \mathbb{R}_+$. We have

$$\begin{aligned} \int_x^\infty m_t(y)dy &= \int_{(x,\infty)} m_0(y+s_t)dy + \alpha \int_{(x,\infty)} \left\langle 1_{(y,y+s_t]}(\cdot) q_{\tau(y+s_t-\cdot)}, \nu \right\rangle dy \\ &= \overline{M}_0(x+s_t) + \alpha \int_{(x,\infty)} \int_{(y,y+s_t]} q_{\tau(y+s_t-v)} \nu(dv)dy. \end{aligned}$$

Interchanging the order of integration yields that

$$\begin{aligned} \int_x^\infty m_t(y)dy &= \overline{M}_0(x+s_t) + \alpha \int_{(x,x+s_t]} \int_{(x,v)} q_{\tau(y+s_t-v)} dy \nu(dv) \\ &\quad + \alpha \int_{(x+s_t,\infty)} \int_{[v-s_t,v)} q_{\tau(y+s_t-v)} dy \nu(dv). \end{aligned}$$

Using the change of variables $r = y + s_t - v$ in the interior integrals, we obtain

$$\begin{aligned} \int_x^\infty m_t(y)dy &= \overline{M}_0(x+s_t) + \alpha \int_{(x,x+s_t]} \int_{(x+s_t-v,s_t)} q_{\tau(r)} dr \nu(dv) \\ &\quad + \alpha \int_{(x+s_t,\infty)} \int_{[0,s_t)} q_{\tau(r)} dr \nu(dv). \end{aligned}$$

Again interchanging the order of integration in both integrals gives

$$\begin{aligned}\int_x^\infty m_t(y)dy &= \overline{M}_0(x+s_t)+\alpha\int_{(0,s_t)}\int_{(x+s_t-r,x+s_t]}\nu(dv)q_{\tau(r)}dr\\ &\quad +\alpha\int_{[0,s_t)}\int_{(x+s_t,\infty)}\nu(dv)q_{\tau(r)}dr\\ &= \overline{M}_0(x+s_t)+\alpha\int_{[0,s_t)}\int_{(x+s_t-r,\infty)}\nu(dv)q_{\tau(r)}dr\\ &= \overline{M}_0(x+s_t)+\int_{[0,s_t)} n_e(x+s_t-r)q_{\tau(r)}dr.\end{aligned}$$

Let $v=\tau(r)$. Then, by (8), for the last integral $dv = q_{\tau(r)}dr$ and $0\le v<t$. Hence,

$$\int_x^\infty m_t(y)dy = \overline{M}_0(x+s_t)+\int_0^t n_e(x+s_t-s_v)dv.$$

This together with (30) gives

$$\overline{M}_t(x)=\int_x^\infty m_t(y)dy.$$

Since $t\in[0,\infty)$ and $x\in\mathbb{R}_+$ were arbitrary, the above holds for all such t and x. Thus, (64) holds.

Finally assume that ν does not charge atoms and m_0 is continuous on $\mathbb{R}_+$. Fix $t\in[0,\infty)$ and $x\in\mathbb{R}_+$. Let $\{x_n\}_{n\in\mathbb{N}}\subset\mathbb{R}_+$ be such that $\lim_{n\to\infty}x_n=x$. By Property (C.1) of fluid model solutions and the definition of τ (see (7)), $q_{\tau(\cdot)}$ is a composition of functions that are continuous on $\mathbb{R}_+$ and is therefore continuous on $\mathbb{R}_+$. Using this, it follows that for all $y\in\mathbb{R}_+$ such that $y\neq x, x+s_t$,

$$\lim_{n\to\infty} 1_{(x_n,x_n+s_t]}(y)q_{\tau(x_n+s_t-y)} = 1_{(x,x+s_t]}(y)q_{\tau(x+s_t-y)}.$$

Since $q_{\tau(\cdot)}$ is continuous on $\mathbb{R}_+$, $q_{\tau(\cdot)}$ is bounded on compact intervals. By (63), the bounded convergence theorem, the fact that ν does not charge atoms and that m_0 is continuous, it follows that $\lim_{n\to\infty}m_t(x_n)=m_t(x)$. A similar argument demonstrates that $\lim_{n\to\infty}m_{t_n}(x)=m_t(x)$ for $\{t_n\}_{n\in\mathbb{N}}\subset[0,\infty)$ such that $\lim_{n\to\infty}t_n=t$. □

LEMMA 7.10. *Let μ be a fluid model solution with $\mu_0\in\mathbf{A}^+$. For each fixed $x\in\mathbb{R}_+$, the function $\overline{M}_t(x)$ is absolutely continuous in time with respect to Lebesgue measure and has density function*

$$n_e(x)-\frac{m_t(x)}{q_t},\qquad t\in[0,\infty).$$

Consequently, for each $x \in \mathbb{R}_+$, for Lebesgue almost every $t \in [0,\infty)$,

$$\frac{\partial \overline{M}_t(x)}{\partial t} = n_e(x) - \frac{m_t(x)}{q_t}. \tag{65}$$

Furthermore, if ν does not charge atoms and μ_0 has a continuous density m_0, then (65) holds for all $x \in \mathbb{R}_+$ and $t \in [0,\infty)$.

Together Lemmas 7.9 and 7.10 imply that $\overline{M}_t(x)$ satisfies a partial differential equation (see (66) below) for $(t,x) \in (0,\infty)\times(0,\infty)$ in the weak sense. In the next corollary, we specify conditions under which $\overline{M}_t(x)$ satisfies (66) (in the strong sense). While we do not use either property to prove Lemma 7.8, this may be of independent interest.

COROLLARY 7.1. *Suppose that ν does not charge atoms. Let μ be a fluid model solution with $\mu_0 \in \mathbf{A}^+$ such that μ_0 has a continuous density m_0. For every $t \in [0,\infty)$ and $x \in \mathbb{R}_+$,*

$$\frac{\partial \overline{M}_t(x)}{\partial t} = n_e(x) + \frac{1}{q_t}\frac{\partial \overline{M}_t(x)}{\partial x}. \tag{66}$$

PROOF OF LEMMA 7.10. Fix a fluid model solution μ such that $\mu_0 \in \mathbf{A}^+$. Fix $t \in [0,\infty)$ and $x \in \mathbb{R}_+$. We have

$$\int_0^t \left(n_e(x) - \frac{m_v(x)}{q_v} \right) dv = t n_e(x) - \int_0^t \frac{m_v(x)}{q_v} dv. \tag{67}$$

By (63),

$$\int_0^t \frac{m_v(x)}{q_v} dv = \int_0^t \frac{m_0(x+s_v)}{q_v} dv + \int_0^t \frac{\alpha \left\langle 1_{(x,x+s_v]}(\cdot) q_{\tau(x+s_v-\cdot)}, \nu \right\rangle}{q_v} dv. \tag{68}$$

Using the change of variables $y = x + s_v$, we obtain

$$\int_0^t \frac{m_0(x+s_v)}{q_v} dv = \int_x^{x+s_t} m_0(y)dy = \overline{M}_0(x) - \overline{M}_0(x+s_t). \tag{69}$$

Also, after interchanging the order of integration,

$$\begin{aligned} \int_0^t \frac{\alpha \left\langle 1_{(x,x+s_v]}(\cdot) q_{\tau(x+s_v-\cdot)}, \nu \right\rangle}{q_v} dv &= \alpha \int_0^t \int_{(x,x+s_v]} \frac{q_{\tau(x+s_v-y)}}{q_v} \nu(dy) dv \\ &= \alpha \int_{(x,x+s_t]} \int_{\tau(y-x)}^t \frac{q_{\tau(x+s_v-y)}}{q_v} dv \nu(dy). \end{aligned}$$

So then, using the change of variables $r = \tau(x + s_v - y)$ in the interior integral and (8) gives $dr = \frac{q_{\tau(x+s_v-y)}}{q_v} dv$. Hence,

$$\int_0^t \frac{\alpha \left\langle 1_{(x,x+s_v]}(\cdot) q_{\tau(x+s_v-\cdot)}, \nu \right\rangle}{q_v} dv = \alpha \int_{(x,x+s_t]} \int_0^{\tau(x+s_t-y)} dr \nu(dy).$$

Then after interchanging the order of integration yet again

$$\begin{aligned} \int_0^t \frac{\alpha \left\langle 1_{(x,x+s_v]}(\cdot) q_{\tau(x+s_v-\cdot)}, \nu \right\rangle}{q_v} dv &= \alpha \int_0^t \int_{(x,x+s_t-s_r]} \nu(dy) dr \\ &= \alpha \int_0^t \left[N(x + s_t - s_r) - N(x) \right] dr \\ &= \int_0^t \left[n_e(x) - n_e(x + s_t - s_r) \right] dr \\ &= t n_e(x) - \int_0^t n_e(x + s_t - s_r) dr. \end{aligned} \tag{70}$$

Combining (67)–(70), and then using (30) gives

$$\begin{aligned} \int_0^t \left(n_e(x) - \frac{m_v(x)}{q_v} \right) dv &= \overline{M}_0(x + s_t) - \overline{M}_0(x) + \int_0^t n_e(x + s_t - s_v) dv \\ &= \overline{M}_t(x) - \overline{M}_0(x), \end{aligned}$$

from which (65) follows. The last part of the lemma holds since for each $x \in \mathbb{R}_+$ the righthand side of (65) is continuous under the given conditions. □

7.3.2. *Proof of Lemma 7.8.* Lemma 7.8 is proved in this section. For this, the following fact is needed.

LEMMA 7.11. *Let $u, l > 0$ and μ be a fluid model solution with $\mu_0 \in \mathbf{K}_{u,l}$. For all $t \geq 0$,*

$$\lim_{x \nearrow x_\nu} \overline{M}_t(x) \ln \left(\frac{\overline{M}_t(x)}{\overline{N}_e(x)} \right) = 0. \tag{71}$$

PROOF. Fix $u, l > 0$ and μ a fluid model solution with $\mu_0 \in \mathbf{K}_{u,l}$. Note that $|h|$ is bounded above on any closed interval of the form $[0, y]$ for each $y \in \mathbb{R}_+$. Hence, for each $t \in [0, \infty)$, Corollary 5.1 implies that

$$\limsup_{x \nearrow x_\nu} \left| h \left(\frac{\overline{M}_t(x)}{\overline{N}_e(x)} \right) \right| < \infty. \tag{72}$$

For each $t \in [0, \infty)$ and $x \in [0, x_\nu)$,

$$\overline{M}_t(x) \ln \left(\frac{\overline{M}_t(x)}{\overline{N}_e(x)} \right) = \overline{N}_e(x) h \left(\frac{\overline{M}_t(x)}{\overline{N}_e(x)} \right). \tag{73}$$

Since $\lim_{x \to x_\nu} \overline{N}_e(x) = 0$, (72) and (73) imply (71). □

PROOF OF LEMMA 7.8. Fix $u, l > 0$ and $\xi \in \mathbf{A}_{u,l}$. Let $u^*, l^*, \lambda^* > 0$ be the constants given by (35) and (42). The proof of (62) proceeds in two main steps. For this, for each $t \in [0, \infty)$, let

$$V_t(x) = \frac{M_t(x)}{q_t} - N_e(x), \qquad \text{for } x \in \mathbb{R}_+.$$

Then for each fixed $t \in [0, \infty)$, V_t is continuous and of bounded variation. Hence, for each $t \in [0, \infty)$, dV_t corresponds to a finite signed Borel measure on $\mathbb{R}_+$. The first step is to show that for each $t \in (0, \infty)$,

$$\mathcal{K}_\xi(t) = \int_0^{x_\nu} h' \left(\frac{p_{\mu_t}(x)}{p_{\nu_e}(x)} \right) dV_t(x). \tag{74}$$

The second step is to integrate this expression in order to show that for each $0 < r \leq t < \infty$

$$\int_r^t \mathcal{K}_\xi(v) dv = -w_0 \left(\mathcal{H}_\xi(t) - \mathcal{H}_\xi(r) \right). \tag{75}$$

Then $\mathcal{H}_\xi$ is absolutely continuous on $(0, \infty)$. By Lemma 7.7, $\mathcal{H}_\xi$ is continuous on $[0, \infty)$. Hence, letting $r \searrow 0$ in (75) completes the proof of (62).

First we show that (75) holds when (74) holds. For this, fix $0 < r \leq t < \infty$. By integrating (74) and using (63) and Fubini's theorem and (65), we obtain that

$$\begin{aligned} \int_r^t \mathcal{K}_\xi(v) dv &= \int_r^t \int_0^{x_\nu} h' \left(\frac{p_{\mu_v}(x)}{p_{\nu_e}(x)} \right) \left(\frac{m_v(x)}{q_v} - n_e(x) \right) dx dv \\ &= \int_0^{x_\nu} \int_r^t h' \left(\frac{p_{\mu_v}(x)}{p_{\nu_e}(x)} \right) \left(\frac{m_v(x)}{q_v} - n_e(x) \right) dv dx \\ &= - \int_0^{x_\nu} \int_r^t h' \left(\frac{p_{\mu_v}(x)}{p_{\nu_e}(x)} \right) \frac{\partial}{\partial v} \overline{M}_v(x) dv dx. \end{aligned}$$

Note that for each $x \in [0, x_\nu)$ and $v \in [0, \infty)$, $\overline{M}_v(x) = w_0 p_{\mu_v}(x)$ and w_0 is a constant. Hence,

$$\int_r^t \mathcal{K}_\xi(v) dv = -w_0 \int_0^{x_\nu} \int_r^t h' \left(\frac{p_{\mu_v}(x)}{p_{\nu_e}(x)} \right) \left(\frac{\frac{\partial}{\partial v} p_{\mu_v}(x)}{p_{\nu_e}(x)} \right) dv \; p_{\nu_e}(x) dx$$

$$= -w_0 \int_0^{x_\nu} \left[h\left(\frac{p_{\mu_t}(x)}{p_{\nu_e}(x)}\right) - h\left(\frac{p_{\mu_r}(x)}{p_{\nu_e}(x)}\right)\right] p_{\nu_e}(x)dx$$
$$= -w_0 \left(\mathcal{H}_\xi(t) - \mathcal{H}_\xi(r)\right).$$

So (75) holds if (74) holds.

Next we verify (74). Again fix $t \in (0,\infty)$. We have

$$\mathcal{K}_\xi(t) = \int_0^{x_\nu} k\left(\frac{\overline{M}_t(x)}{q_t\overline{N}_e(x)}\right) n_e(x)dx$$
$$= \int_0^{x_\nu} \left(\frac{\overline{M}_t(x)}{q_t\overline{N}_e(x)} - \ln\left(\frac{\overline{M}_t(x)}{q_t\overline{N}_e(x)}\right) - 1\right) n_e(x)dx$$
$$= \int_0^{x_\nu} \left(\frac{\overline{M}_t(x)}{q_t\overline{N}_e(x)} - \ln\left(\frac{\overline{M}_t(x)}{q_t\overline{N}_e(x)}\right)\right) n_e(x)dx - 1. \tag{76}$$

As a consequence of integration by parts and Lemma 7.11, we have

$$\int_0^{x_\nu} \frac{\overline{M}_t(x)}{q_t\overline{N}_e(x)} n_e(x)dx = -\int_0^{x_\nu} \frac{-n_e(x)}{\overline{N}_e(x)}\frac{\overline{M}_t(x)}{q_t}dx$$
$$= \lim_{y\nearrow x_\nu}\left(-\ln\left(\overline{N}_e(y)\right)\frac{\overline{M}_t(y)}{q_t} + \int_0^y \ln\left(\overline{N}_e(x)\right)\frac{d\overline{M}_t(x)}{q_t}\right)$$
$$= \int_0^{x_\nu} \ln\left(\overline{N}_e(x)\right)\frac{d\overline{M}_t(x)}{q_t}$$
$$= -\int_0^{x_\nu} \ln\left(\overline{N}_e(x)\right)\frac{dM_t(x)}{q_t}.$$

By Lemma 7.4 with $g(x) = m_t(x)/q_t$ for $x \in \mathbb{R}_+$,

$$-1 = \int_0^{x_\nu} \ln\left(\frac{\overline{M}_t(x)}{q_t}\right)\frac{dM_t(x)}{q_t}.$$

So we obtain that

$$\int_0^{x_\nu} \frac{\overline{M}_t(x)}{q_t\overline{N}_e(x)} n_e(x)dx - 1 = \int_0^{x_\nu} \ln\left(\frac{\overline{M}_t(x)}{q_t\overline{N}_e(x)}\right)\frac{dM_t(x)}{q_t}. \tag{77}$$

Hence, by (76) and (77),

$$\mathcal{K}_\xi(t) = \int_0^{x_\nu} \ln\left(\frac{\overline{M}_t(x)}{q_t\overline{N}_e(x)}\right) dV_t(x).$$

Note that

$$\int_0^{x_\nu} dV_t(x) = \int_0^{x_\nu} \frac{dM_t(x)}{q_t} - \int_0^{x_\nu} dN_e(x) = 1 - 1 = 0.$$

Therefore,

$$\begin{aligned}\mathcal{K}_\xi(t) &= \int_0^{x_\nu}\left[\ln\left(\frac{\overline{M}_t(x)}{q_t\overline{N}_e(x)}\right)+\ln\left(\frac{q_t\langle\chi,\nu_e\rangle}{w_0}\right)+1\right]dV_t(x)\\ &= \int_0^{x_\nu}\left[\ln\left(\frac{p_{\mu_t}(x)}{p_{\nu_e}(x)}\right)+1\right]dV_t(x),\end{aligned}$$

which implies (74). Hence, (62) holds. □

7.4. *Extension to continuous initial states.* In this section, we use an approximation argument to prove Theorem 7.1. We prepare for this by stating and proving three lemmas related to this approximation. For this, given $n\in\mathbb{N}$, let $\varphi_n\in\mathbf{C}_c(\mathbb{R})$ be such that $\varphi_n\geq 0$, $\varphi_n(x)=0$ for all $x\in(-\infty,-1/n]\cup[1/n,\infty)$ and $\int_{\mathbb{R}}\varphi_n(x)dx=1$. Given $\xi\in\mathbf{K}$ and $n\in\mathbb{N}$, set $\xi_n=\varphi_n*\xi$. Here $*$ denotes the convolution operator. In particular, for each $n\in\mathbb{N}$, ξ_n is the nonnegative Borel measure on $\mathbb{R}_+$ that is absolutely continuous with respect to Lebesgue measure and has density $d_n(x)=\int_{\mathbb{R}_+}\varphi_n(y-x)\xi(dy)$ for $x\in\mathbb{R}_+$. Then, given $n\in\mathbb{N}$ and $f\in\mathbf{C}_b(\mathbb{R}_+)$, by Fubini's theorem,

$$\langle f,\xi_n\rangle=\int_{\mathbb{R}_+}f(x)\int_{\mathbb{R}_+}\varphi_n(y-x)\xi(dy)dx=\langle f*\varphi_n,\xi\rangle\,.$$

Given $\xi\in\mathbf{K}$, we refer to $\{\xi_n\}_{n\in\mathbb{N}}$ as the approximating sequence.

LEMMA 7.12. *Let $\xi\in\mathbf{K}^\dagger$. For each $n\in\mathbb{N}$ and $x\in\mathbb{R}_+$,*

$$\left\langle 1_{(x+1/n,\infty)},\xi\right\rangle\leq\left\langle 1_{(x,\infty)},\xi_n\right\rangle\leq\left\langle 1_{((x-1/n)^+,\infty)},\xi\right\rangle, \tag{78}$$

$$\langle\chi,\xi\rangle-\frac{\langle 1,\xi\rangle}{n}\leq\langle\chi,\xi_n\rangle\leq\langle\chi,\xi\rangle+\frac{\langle 1,\xi\rangle}{n}. \tag{79}$$

In particular, $\xi_n\in\mathbf{A}^\dagger$ for each $n\in\mathbb{N}$ and as $n\to\infty$,

$$\xi_n\overset{w}{\to}\xi\qquad\textit{and}\qquad\langle\chi,\xi_n\rangle\to\langle\chi,\xi\rangle\,. \tag{80}$$

PROOF. Fix $\xi\in\mathbf{K}^\dagger$ and $x\in\mathbb{R}_+$. By Fubini's theorem, for $n\in\mathbb{N}$,

$$\begin{aligned}\left\langle 1_{(x,\infty)},\xi_n\right\rangle &= \int_x^\infty\int_{\mathbb{R}_+}\varphi_n(v-y)\xi(dv)dy\\ &= \int_{\mathbb{R}_+}\int_x^\infty\varphi_n(v-y)dy\xi(dv)\\ &= \int_{\mathbb{R}_+}\int_{(v-1/n)\vee x}^{v+1/n}\varphi_n(v-y)dy\xi(dv)\end{aligned}$$

$$= \int_{(x-1/n)^+}^{\infty} \int_{(v-1/n)\vee x}^{v+1/n} \varphi_n(v-y)dy\xi(dv).$$

Then, (78) follows since $\int_{(v-1/n)\vee x}^{v+1/n} \varphi_n(v-y)dy \le 1$ for all $v > (x-1/n)^+$, and $\int_{(v-1/n)\vee x}^{v+1/n} \varphi_n(v-y)dy = 1$ for all $v > x+1/n$. Similarly, for each $n \in \mathbb{N}$,

$$\langle \chi, \xi_n \rangle = \int_{\mathbb{R}_+} \int_{(v-1/n)^+}^{v+1/n} y\varphi_n(v-y)dy\xi(dv).$$

So then replacing the factor of y in the integrand of the interior integral above with $v+1/n$ or $v-1/n$, (79) follows. □

Given $\xi \in \mathbf{K}$ and $n \in \mathbb{N}$, let $\{\mu_t^n\}_{t\ge 0}$ denote the unique fluid model solution such that $\mu_0^n = \xi_n$ and let μ denote the unique fluid model solution such that $\mu_0 = \xi$.

LEMMA 7.13. *Let $\xi \in \mathbf{K}^\dagger$. For each $t \in [0,\infty)$, as $n \to \infty$,*

$$\mu_t^n \xrightarrow{w} \mu_t. \tag{81}$$

PROOF. Fix $\xi \in \mathbf{K}^\dagger$. By [6, Lemma 4.9], it suffices to show that $\xi_n \xrightarrow{w} \xi$ as $n \to \infty$, which follows from Lemma 7.12. □

LEMMA 7.14. *Let $u, l > 0$. Given $T > 0$, there exist positive constants $\tilde{u}$, $\tilde{l}$, and $\tilde{\theta}$, and $N \in \mathbb{N}$ such that $\mu_t^n \in \mathbf{A}_{\tilde{u},\tilde{l},\tilde{\theta}}$ for each $\xi \in \mathbf{K}_{u,l}$, $n \ge N$ and $t \ge T$, where $\tilde{\lambda} = \inf\{\langle 1, \zeta\rangle : \zeta \in \mathbf{M}_{\tilde{u},\tilde{l}}\}$ is used to define $\mathbf{A}_{\tilde{u},\tilde{l},\tilde{\theta}}$.*

PROOF. Fix $u, l, T > 0$ and $\xi \in \mathbf{K}_{u,l}$. By Lemma 7.12, for all $n \in \mathbb{N}$, $\langle 1, \xi_n \rangle \le \langle 1, \xi \rangle \le u$ and

$$\langle \chi, \xi_n \rangle \le \langle \chi, \xi \rangle + \frac{\langle 1, \xi \rangle}{n} \le u \langle \chi, \nu_e \rangle + \frac{u}{n} \le u \left(\langle \chi, \nu_e \rangle + 1 \right).$$

Let

$$u_0 = \frac{3 \max(u, 6\alpha u \left(\langle \chi, \nu_e \rangle + 1 \right))}{2}.$$

By Lemma 5.1, it follows that $q_t^n \le u_0$ for all $t \in [0,\infty)$ and $n \in \mathbb{N}$. From this, it follows that $s_t^n \ge t/u_0$ for all $n \in \mathbb{N}$. Fix N' such that $(T/2u_0)-(1/N') > 0$. Then, for $n \ge N'$,

$$s_{T/2}^n - \frac{1}{n} \ge \frac{T}{2u_0} - \frac{1}{n} \ge \frac{T}{2u_0} - \frac{1}{N'} > 0.$$

In addition, by Lemma 7.12, for all $n \in \mathbb{N}$ and $x \in \mathbb{R}_+$,

$$\overline{M}_0^n(x) = \left\langle 1_{(x,\infty)}, \xi_n \right\rangle \quad \leq \quad \left\langle 1_{((x-1/n)^+,\infty)}, \xi \right\rangle = \overline{M}_0((x-1/n)^+).$$

Therefore, for all $x \in \mathbb{R}_+$ and $n \geq N'$,

$$\overline{M}_0^n(x + s_{T/2}^n) \leq \overline{M}_0(x + s_{T/2}^n - 1/n) \leq \overline{M}_0(x) \leq u\overline{N}_e(x) \leq u_0\overline{N}_e(x).$$

This together with (30) and (36) (with u_0 in place of u^*) gives that for all $x \in \mathbb{R}_+$ and $n \geq N'$,

$$\begin{aligned} \overline{M}_{T/2}^n(x) &= \overline{M}_0^n(x + s_{T/2}^n) + \int_0^{T/2} n_e\left(x + s_{T/2}^n - s_v^n\right) dv \\ &\leq u_0\overline{N}_e(x) + u_0\left(\overline{N}_e(x) - \overline{N}_e(x + s_{T/2}^n)\right) \leq 2u_0\overline{N}_e(x). \end{aligned}$$

Hence, $\mu_{T/2}^n \in \mathbf{A}_{2u_0}$ for all $n \geq N'$.

For $n \geq N'$ and $t \in [0,\infty)$, set

$$\tilde{\mu}_t^n = \mu_{T/2+t}^n.$$

Then, for all $n \geq N'$, $\tilde{\mu}^n$ is a fluid model solution such that $\tilde{\mu}_0^n \in \mathbf{A}_{2u_0}$. Let

$$\tilde{u} = \frac{3\max(2u_0, 12\alpha u_0 \langle \chi, \nu_e \rangle)}{2} = 3\max(u_0, 6\alpha u_0 \langle \chi, \nu_e \rangle).$$

Then, by Corollary 5.1, for all $n \geq N'$, $\tilde{\mu}_t^n \in \mathbf{A}_{\tilde{u}}$ for all $t \in [0,\infty)$. Furthermore, for $n \geq N'$, since the first moment of any fluid model solution is constant, Lemma 7.12 implies that

$$\langle \chi, \tilde{\mu}_0^n \rangle \quad = \quad \left\langle \chi, \mu_{T/2}^n \right\rangle = \langle \chi, \xi_n \rangle \geq \langle \chi, \xi \rangle - \frac{\langle 1, \xi \rangle}{n} \geq l - \frac{u}{n}.$$

Let N be such that $N \geq N'$ and $l - u/n > 0$ for all $n \geq N$ and set $\tilde{l} = l - u/N$. Since $\langle \chi, \tilde{\mu}_t^n \rangle = \langle \chi, \tilde{\mu}_0^n \rangle$ for all $t \in [0,\infty)$ and $n \geq N'$, it follows that $\tilde{\mu}_t^n \in \mathbf{A}_{\tilde{u},\tilde{l}}$ for all $t \in [0,\infty)$ and $n \geq N$. Set $\tilde{\theta} = T/2\tilde{u}$. Then, by Corollary 5.4, it follows that for all $n \geq N$, $\tilde{\mu}_t^n \in \mathbf{A}_{\tilde{u},\tilde{l},\tilde{\theta}}$ for all $t \geq T/2$. But, given $n \geq N$ and $t \geq T$, we have $t - T/2 \geq T/2$ and $\mu_t^n = \tilde{\mu}_{t-T/2}^n \in \mathbf{A}_{\tilde{u},\tilde{l},\tilde{\theta}}$. □

Proof of Theorem 7.1. Fix $u, l > 0$ and $\xi \in \mathbf{K}_{u,l}$. It suffices to show that for all $0 < r < t < \infty$,

$$\mathcal{H}_\xi(t) - \mathcal{H}_\xi(r) = \frac{-1}{w_0}\int_r^t \mathcal{K}_\xi(v)dv. \tag{82}$$

Then $\mathcal{H}_\xi$ is absolutely continuous on $(0,\infty)$. By Lemma 7.7, $\mathcal{H}_\xi$ is continuous on $[0,\infty)$. Hence, once (82) is established, absolute continuity on $[0,\infty)$ follows by letting $r \searrow 0$ in (82).

In order to verify (82), fix $0 < r < t < \infty$. By Lemma 7.14, there exist positive constants $\tilde{u}$, $\tilde{l}$, $\tilde{\theta}$, and N such that $\mu_v^n \in \mathbf{A}_{\tilde{u},\tilde{l},\tilde{\theta}}$ for all $v \geq r$ and $n \geq N$. For $n \geq N$ and $v \in [0,\infty)$, let

$$\tilde{\mu}_v^n = \mu_{v+r}^n.$$

Then $\tilde{\mu}^n$ is the unique fluid model solution such that $\tilde{\mu}_0^n = \mu_r^n \in \mathbf{A}_{\tilde{u},\tilde{l},\tilde{\theta}}$ for all $n \geq N$. Therefore, by Lemma 7.8, for all $n \geq N$ and $t \geq r$,

$$\mathcal{H}_{\tilde{\mu}_0^n}(t-r) - \mathcal{H}_{\tilde{\mu}_0^n}(0) = \frac{-1}{\langle \chi, \tilde{\mu}_0^n \rangle} \int_0^{t-r} \mathcal{K}_{\tilde{\mu}_0^n}(v)dv.$$

Then, by the definition of $\tilde{\mu}^n$, $\mathcal{H}_{\xi_n}$, $\mathcal{K}_{\xi_n}$ and the fact that $\langle \chi, \tilde{\mu}_0^n \rangle = \langle \chi, \mu_r^n \rangle = \langle \chi, \mu_0^n \rangle$, for $n \geq N$ and $t \geq r$, we obtain

$$\mathcal{H}_{\xi_n}(t) - \mathcal{H}_{\xi_n}(r) = \frac{-1}{\langle \chi, \mu_0^n \rangle} \int_r^t \mathcal{K}_{\xi_n}(v)dv. \tag{83}$$

Since $\mu_v^n \in \mathbf{A}_{\tilde{u},\tilde{l},\tilde{\theta}} \subset \mathbf{M}_{\tilde{u},\tilde{l},\tilde{\theta}}$ for all $v \geq r$ and $n \geq N$, Lemmas 4.8, 7.6, and 7.13 imply that for each $v \geq r$,

$$\lim_{n\to\infty} \mathcal{H}_{\xi_n}(v) = \mathcal{H}_\xi(v) \qquad \text{and} \qquad \lim_{n\to\infty} \mathcal{K}_{\xi_n}(v) = \mathcal{K}_\xi(v). \tag{84}$$

Since $\mu_v^n \in \mathbf{M}_{\tilde{u},\tilde{l},\tilde{\theta}}$ for all $v \geq r$ and $n \geq N$, compactness of $\mathbf{M}_{\tilde{u},\tilde{l},\tilde{\theta}}$ and Lemma 7.6 imply that

$$\sup_{n\geq N, v\geq r} \mathcal{K}_{\xi_n}(v) < \infty.$$

This together with the bounded convergence theorem, (83), and (84) imply (82). □

8. Uniform convergence of relative entropy to zero. Theorem 7.1 is now used in conjunction with compactness of $\mathbf{M}_{u,l}$ and $\mathbf{M}_{u,l,\theta}$ for $u,l,\theta > 0$ and continuity properties $\mathcal{H}_\xi$ and $\mathcal{K}_\xi$ for $\xi \in \mathbf{M}_{u,l}$ for $u,l > 0$ to prove Theorem 3.2.

Proof of Theorem 3.2. Fix $u,l > 0$ such that $\mathbf{K}_{u,l} \neq \emptyset$. The asserted monotonicity is an immediate consequence of Theorem 7.1. Therefore, it suffices to show that for each $\varepsilon > 0$, there exists $T > 0$ such that for all $\xi \in \mathbf{K}_{u,l}$ and $t \geq T$, $\mathcal{H}_\xi(t) < \varepsilon$.

Let $u^*, l^*, \lambda^* > 0$ be the constants given by (35) and (42). For $\theta > 0$, set $\theta^* = \theta$ and $T^* = \theta^* u^*$. By Corollary 5.4, $\mu_t^\xi \in \mathbf{K}_{u^*,l^*,\theta^*}$ for all $t \geq T^*$ and $\xi \in \mathbf{K}_{u,l}$. In particular, $\mathbf{K}_{u^*,l^*,\theta^*} \neq \emptyset$. Let

$$\mathbf{H}'_\varepsilon = \{\zeta \in \mathbf{M}_{u^*,l^*,\theta^*} : H(\zeta) \geq \varepsilon\}.$$

We wish to show that there exists $T > T^*$ such that for all $\xi \in \mathbf{K}_{u,l}$ and $t \geq T$, $\mu_t^\xi \notin \mathbf{H}'_\varepsilon$. Then it follows that $\mathcal{H}_\xi(t) = H(\mu_t^\xi) < \varepsilon$ for all $\xi \in \mathbf{K}_{u,l}$ and $t \geq T$. Since $\mathcal{H}_\xi$ is monotone nonincreasing, it suffices to show that there exists $T > T^*$ such that for each $\xi \in \mathbf{K}_{u,l}$, there exists $t \in [0, T]$ such that $\mu_t^\xi \notin \mathbf{H}'_\varepsilon$.

Note that $\mathbf{H}'_\varepsilon$ is relatively compact since it is contained in $\mathbf{M}_{u^*,l^*,\theta^*}$. Hence, its closure $\overline{\mathbf{H}}'_\varepsilon$ is compact and contained in $\mathbf{M}_{u^*,l^*,\theta^*}$. Since H is continuous on $\mathbf{M}_{u^*,l^*}$, it follows that $\zeta \in \overline{\mathbf{H}}'_\varepsilon$ satisfies $H(\zeta) \geq \varepsilon$. Therefore, by Lemma 4.1, any $\zeta \in \overline{\mathbf{H}}'_\varepsilon$ satisfies $\zeta \notin \mathbf{J}$. Then, by Lemma 7.1, $K(\zeta) > 0$ for all $\zeta \in \overline{\mathbf{H}}'_\varepsilon$. But K is lower semicontinuous, and any lower semicontinuous function on a compact set achieves its minimum. Hence, there exists $\delta > 0$ such that $K(\zeta) \geq \delta$ for all $\zeta \in \overline{\mathbf{H}}'_\varepsilon$. Let

$$T = T^* + \frac{u^* \langle \chi, \nu_e \rangle}{\delta} h\left(\frac{u^* \langle \chi, \nu_e \rangle}{l^*}\right).$$

To complete the proof, we show that for each $\xi \in \mathbf{K}_{u,l}$, there exists $t \in [0, T]$ such that $\mu_t^\xi \notin \mathbf{H}'_\epsilon$.

Suppose that $\xi \in \mathbf{K}_{u,l}$ and $t \geq T^*$ are such that $\mu_r^\xi \in \mathbf{H}'_\varepsilon$ for all $r \in [T^*, t]$. It suffices to show that $t < T$. Since $\mu_r^\xi \in \mathbf{H}'_\varepsilon$ for each $r \in [T^*, t]$, $K(\mu_r^\xi) \geq \delta$ for each $r \in [T^*, t]$. Then, for each $r \in [T^*, t]$,

$$\kappa_\xi(r) = \frac{-1}{w_0} \mathcal{K}_\xi(r) = \frac{-1}{w_0} K(\mu_r^\xi) \leq \frac{-\delta}{u^* \langle \chi, \nu_e \rangle}. \tag{85}$$

This together with Theorem 7.1 and (29) implies that

$$\begin{aligned}
\varepsilon &\leq \mathcal{H}_\xi(t) \\
&\leq \frac{-\delta}{u^* \langle \chi, \nu_e \rangle}(t - T^*) + \mathcal{H}_\xi(T^*) \\
&\leq \frac{-\delta}{u^* \langle \chi, \nu_e \rangle}(t - T^*) + h\left(\frac{u^* \langle \chi, \nu_e \rangle}{l^*}\right) \\
&= \frac{\delta}{u^* \langle \chi, \nu_e \rangle}(T - t).
\end{aligned}$$

Hence, $0 < T - t$, and so $t < T$. □

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