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A note on combined inference on the common coefficient of variation using confidence distributions

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Abstract: This article considers inference for a common coefficient of variation (CV) shared by several normal populations. The confidence distributions (CD) are used to combine the information about each CV from different sources. A new procedure for constructing a confidence interval for the common CV is developed based on a combined confidence distribution for the inverse of the CV. The new derived CD interval has a theoretical exact frequentist property. Simulation results demonstrate that the new confidence intervals perform very well in terms of empirical coverage probability and average interval length. Finally, the proposed new procedure is illustrated on a real data example.

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1. Introduction

The coefficient of variation (CV) of a distribution is defined to be the ratio of the standard deviation σ to the mean μ, i.e., $\eta = \frac{\sigma}{\mu}$. This parameter is a useful measure of dispersion because it is not affected by the units of measurement, and it has many applications in different scientific fields. For example, in toxicology, the CV is often used as a measure of precision within and between laboratories, or among replicates for each treatment concentration. In climatology, Ananthakrishnan and Soman [3] used the CV in the analysis of rainfall data. In finance, Miller and Karson [20] used the CV as a measure of relative risks. Hamer et al. [11] also used the CV to assess homogeneity of bone test samples produced by a new method.

There has been a significant amount of work on the CV in the literature; e.g., McKay [19], Johnson and Welch [13], Koopmans et al. [14], Ahmed [1], Vangel [29], Feltz and Miller [7], Fung and Tsang [10], Nairy and Rao [21], Tian

[28], Verrill and Johnson [30], Forkman [9], Liu et al. [18], and Krishnamoorthy and Lee [15], etc. Among these works, Ahmed [1], Tian [28] and Forkman [9] considered the problem of estimating the coefficients of variance (CsV) when it is *apriori* suspected that several CsV are the same. As Tian [28] noted the need to infer the common population CsV from several samples, especially in meta-analysis, it is necessary to put these information together to give a more accurate pooling inference about the common CV after we accepted the equality of CsV from different populations.

In this paper, we are interested in the problem just mentioned above, that is, investigating how to pool the information about a common CV from different populations and give confidence intervals for it. For this purpose, we will use a confidence distribution (CD) as a main tool, take advantage of its good property for combining information and derive a new confidence interval based on a combined CD for the common inverse of the CV. It is worth noting that the combined CD for the common inverse of the CV is exact, so the derived confidence interval based on this CD naturally have exact frequentist property, and to our knowledge, no existing method has this feature for this problem so far.

The rest of this paper is organized as follows. Section 2 reviews the concept and some useful related conclusions about CD. As an illustration of how to construct CD, a CD for the inverse of the CV is given by using the idea of fiducial generalized pivotal quantity. In section 3, we construct the CD and the combined CD for a common inverse of the CV and give the confidence interval of the common CV on the basis of the combined CD. Section 4 provides the simulation results. In section 5, we illustrate the proposed new CD method with one real data set. The last section summarizes this paper.

2. A review of confidence distribution and related conclusions

2.1. Definition of confidence distribution

The concept of CD was formulated by Schweder and Hjort [23] and Singh et al. [24], which was pointed out as "Neymannian interpretation of Fisher's fiducial distribution". As a distribution estimator, a CD function contains a wealth of information for inference, such as point estimators, confidence intervals and p-values for frequentist statisticians. Schweder and Hjort [23] suggested that a CD is a "frequentist analogue of a Bayesian posterior."

Suppose $X_1, X_2, \ldots, X_n$ are n independent random draws from a population F and $\mathcal{X}$ is the sample space corresponding to the data set $\mathbf{X}_n = (X_1, X_2, \ldots, X_n)^T$. Let θ be a parameter of interest associated with F (F may contain other nuisance parameters), and let Θ be the parameter space. The following is the CD definition formally proposed by Singh et al. [24, 25].

Definition 2.1. A function $H_n(\mathbf{X}_n, \cdot) : \mathcal{X} \times \Theta \to [0, 1]$ is called a confidence distribution (CD) for a parameter θ if it satisfies the following two requirements: (i) $H_n(\cdot)$ is a continuous distribution function given $\mathbf{X}_n \in \mathcal{X}$; (ii) at the true parameter value $\theta = \theta_0$, $H_n(\theta_0) = H_n(\mathbf{X}_n, \theta_0)$, as a function of the sample $\mathbf{X}_n$, has the uniform distribution $U(0, 1)$.

The function $H_n(\cdot)$ is called an asymptotic confidence distribution (aCD) if requirement (ii) above is replaced by (ii)′, at $\theta = \theta_0, H_n(\theta_0) = H_n(\mathbf{X}_n, \theta_0) \overset{W}{\to} U(0,1)$ as $n \to +\infty$, and the continuity requirement on $H_n(\cdot)$ is dropped.

Singh et al. [25] proposed several methods to construct a CD for a parameter of interest, such as using fiducial distributions, significant(p-value) functions, Bootstrap distributions and likelihood functions, etc. The following example of constructing a CD is relevant to our research in this paper.

Example 2.1 (CD for the inverse of the CV)**.** Suppose $X_1, X_2, \ldots, X_n$ are n independent random draws from a population $N(\mu, \sigma^2)$, and the parameter of interest is the inverse of the CV, i.e. $\rho = \frac{\mu}{\sigma}$. Now we try to construct the CD for ρ and use the idea of constructing a fiducial generalized pivotal quantity in Hannig et al. [12]. Denote $\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i$, $S_i^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \overline{X})^2$. According to Hannig et al. [12], we derive pivot equations,

$$\overline{X} \overset{d}{=} \mu + \frac{\sigma}{\sqrt{n}} E_1, \quad \sqrt{n} S \overset{d}{=} \sigma E_2, \tag{2.1}$$

where E_1, E_2 are independent, and $E_1 \sim N(0,1), E_2^2 \sim \chi^2(n-1)$. The symbol $\overset{d}{=}$ means equal in distribution. Solving μ, σ and substituting them into ρ, we obtain

$$\widehat{\rho}_F = \frac{\overline{X} - S \cdot E_1/E_2}{\sqrt{n} S/E_2}. \tag{2.2}$$

It can also be expressed as

$$\widehat{\rho}_F = \frac{\overline{X} - S(\overline{X}^* - \mu)/S^*}{\sigma S/S^*} \overset{\triangle}{=} \mathcal{R}_\rho(\mathbb{D}, \mathbb{D}^*; \xi) \tag{2.3}$$

by the form of a fiducial generalized pivotal quantity, where $\xi = (\mu, \sigma)$, and $\mathbb{D}^* = (\overline{X}^*, S^*)$ is the independent copy of $\mathbb{D} = (\overline{X}, S)$. Here $\widehat{\rho}_F$ can be viewed as a CD random variable for ρ, and the concept of a CD random variable is mentioned in Xie and Singh [32].

Denote $H_\rho(\mathbb{D}; \theta) = P(\mathcal{R}_\rho(\mathbb{D}, \mathbb{D}^*; \xi) \leq \theta)$; we will show that $H_\rho(\mathbb{D}; \theta)$ is a CD for the parameter ρ. Obviously, $H_\rho(\boldsymbol{d}; \theta)$ is a continuous cumulative distribution function given $\mathbb{D} = \boldsymbol{d} \overset{\triangle}{=} (\overline{x}, s)$, where $\boldsymbol{d}$ is the observed value of $\mathbb{D}$, the first condition (i) in Definition 2.1 is satisfied. When θ is taken to be the real parameter, that is, $\theta = \rho = \frac{\mu}{\sigma}$, $\mathcal{R}_\rho(\mathbb{D}, \mathbb{D}^*; \xi) = \frac{\overline{X} - S(\overline{X}^* - \mu)/S^*}{\sigma S/S^*} \leq \frac{\mu}{\sigma}$ can be expressed equally as $\frac{\overline{X}}{S} \leq \frac{\overline{X}^*}{S^*}$ after simplifying the expression. It is well known that $P(\frac{\overline{X}}{S} \leq \frac{\overline{X}^*}{S^*}) \sim U[0,1]$, so condition (ii) in Definition 2.1 is also satisfied. □

The CD derived above for the inverse of the CV is appealing because of the exact property. It is worth noting that we also can construct an asymptotic CD for the CV, i.e., $\eta = \frac{\sigma}{\mu}$ using a similar procedure to the one mentioned above. However, because there exists an exact CD for the inverse of the CV, we will take advantage of this property and focus on the CD of the inverse of the CV, and we will see later that it is convenient to do so.

2.2. The general framework of CD combination

Singh et al. [24] developed a general recipe for combining CD functions using a coordinate-wise monotone function from a k-dimensional cube $[0,1]^k$ to the real line $\mathbb{R} = (-\infty, +\infty)$. Assume $H_i(\theta) = H_i(\mathbf{X}_i, \theta), i = 1, \ldots, k$, are CD functions for the same parameter θ from k different samples $\mathbf{X}_i$ and the sample size of $\mathbf{X}_i$ is n_i. Let $g_c(u_1, \ldots, u_k)$ be a given continuous function on $[0,1]^k \to \mathbb{R}$ which is monotone. Singh et al. [24] suggested to combine the k CD functions as

$$H^c(\theta) = G_c\{g_c(H_1(\theta), \ldots, H_k(\theta))\} \tag{2.4}$$

Here, the function G_c is totally determined by the monotone g_c function: $G_c(t) = P(g_c(U_1, \ldots, U_k) \le t)$, where $U_1, \ldots, U_k$ are independent $U[0,1]$ random variables. When the underlying true parameter values of the k individual CD functions $H_i(\theta)$s are the same, it is easy to verify that $H_c(\theta)$ is also a CD function for θ, and $H_c(\theta)$ contains information from all k samples, and it is referred to as a *combined CD function*. Usually we can take

$$g_c(u_1, \ldots, u_k) = F_0^{-1}(u_1) + \cdots + F_0^{-1}(u_k), \tag{2.5}$$

where $F_0(\cdot)$ is a given cumulative distribution function. Singh et al. [24] discussed the performance of combined CD functions in terms of Bahadur slope when using different F_0, such as the standard normal distribution $F_0(t) = \Phi(t)$, exponential distribution $F_0(t) = (1 - e^{-t})\mathbf{1}_{(t>0)}$ or $F_0(t) = e^t \mathbf{1}_{(t \le 0)}$.

For further details about CD and combined CD we refer readers to Singh et al. [24, 25], Schweder and Hjort [23], Efron [6], Xie et al. [31], Xie and Singh [32], Claggett et al. [4], Liu et al. [16, 17] and Yang et al. [33], etc.

3. A new confidence interval for the common CV based on the combined CD

3.1. A combined CD for the common inverse of the CV

We start by introducing the necessary notation. Let $X_{i1}, X_{i2}, \ldots, X_{in_i}$ be the ith random sample from a normal population with mean μ_i and variance $\sigma_i^2 = \mu_i^2 \eta$ for $i = 1, 2, \ldots, k$, so that η is the common population CV, the samples are mutually independent. Denote $\overline{X}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij}$, $S_i^2 = \frac{1}{n_i} \sum_{j=1}^{n_i} (X_{ij} - \overline{X}_i)^2$, $i = 1, 2, \ldots, k$ and $\overline{x}_i$, s_i^2, $i = 1, 2, \ldots, k$ are their observed values, respectively.

It is easy to obtain k independent CDs for the common inverse of the CV $\rho = \frac{1}{\eta}$ from each sample according to Example 2.1 in section 2. Denote the CD for the common inverse of the CV based on the ith sample is

$$H_\rho^i(\mathbb{D}_i; \theta) = P(\mathcal{R}_i(\mathbb{D}_i, \mathbb{D}_i^*; \xi_i) \le \theta), \tag{3.1}$$

where $\mathcal{R}_i(\mathbb{D}_i, \mathbb{D}_i^*; \xi_i) = \frac{\overline{X}_i - S_i(\overline{X}_i^* - \mu_i)/S_i^*}{\sigma_i S_i / S_i^*}$, $\mathbb{D}_i^* = (\overline{X}_i^*, S_i^*)$ is the independent copy of $\mathbb{D}_i = (\overline{X}_i, S_i)$, $\xi_i = (\mu_i, \sigma_i)$ for $i = 1, 2, \ldots, k$ and denote $\mathbb{D} = (\mathbb{D}_1, \ldots, \mathbb{D}_k)$, $\mathbb{D}^* = (\mathbb{D}_1^*, \ldots, \mathbb{D}_k^*)$.

Theorem 3.1. *Suppose any continuous function $g_c(u_1, u_2, \ldots, u_d) : [0,1]^k \to \mathbb{R}$ is monotone in each coordinate, $U_1, U_2, \ldots, U_k$ are independent $U[0,1]$ distributed random variables, $G_c(\cdot)$ is the continuous cumulative distribution function of $g_c(U_1, U_2, \ldots, U_k)$; denote $H_c(u_1, u_2, \ldots, u_k) = G_c(g_c(u_1, u_2, \ldots, u_k))$, then*

$$H_\rho^C(\theta) = H_c(H_\rho^1(\mathbb{D}_1; \theta), \ldots, H_\rho^k(\mathbb{D}_k; \theta)) \tag{3.2}$$

is a CD for the common inverse of the CV, it is a combined CD, where $H_\rho^i(\mathbb{D}_i; \theta)$, $i = 1, \ldots, k$ are defined as (3.1).

The proof is straightforward and is omitted.

3.2. Constructing confidence interval of the common CV

As Cox [5] mentioned, a confidence distribution can be viewed as a "sample-dependent distribution function that can represent confidence intervals of all levels" for a parameter of interest. It is evident from requirement (ii) in Definition 2.1 that the intervals $(-\infty, H_n^{-1}(1-\alpha)]$ and $[H_n^{-1}(\alpha), +\infty)$ provide level $100(1-\alpha)\%$ one-sided confidence intervals for the parameter of interest θ, for any $\alpha \in (0,1)$. Also, $[H_n^{-1}(\alpha_1), H_n^{-1}(1-\alpha_2)]$ is a level $100(1-\alpha_1-\alpha_2)\%$ confidence interval for the parameter θ, for any $\alpha_1 > 0, \alpha_2 > 0$, and $\alpha_1 + \alpha_2 < 1$.

Proposition 3.1. *Assume $h_{\frac{\alpha}{2}}(\boldsymbol{d}), h_{1-\frac{\alpha}{2}}(\boldsymbol{d})$ are $\frac{\alpha}{2}, 1-\frac{\alpha}{2}$ quantiles of $H_\rho^C(\theta)$ respectively given sample values $\mathbb{D} = \boldsymbol{d}$, then $[h_{\frac{\alpha}{2}}(\boldsymbol{d}), h_{1-\frac{\alpha}{2}}(\boldsymbol{d})]$ is a confidence interval of ρ with $100(1-\alpha)\%$ confidence level. For the common CV η,*

$$\left[\min\{h_{1-\frac{\alpha}{2}}^{-1}(\boldsymbol{d}), h_{\frac{\alpha}{2}}^{-1}(\boldsymbol{d})\}, \max\{h_{1-\frac{\alpha}{2}}^{-1}(\boldsymbol{d}), h_{\frac{\alpha}{2}}^{-1}(\boldsymbol{d})\}\right] \tag{3.3}$$

is the confidence interval with $100(1-\alpha)\%$ confidence level.

According to the definition of a CD, this kind of confidence intervals have exact frequentist property. In order to derive the confidence interval for CV, we choose g_c just as function (2.5). Regarding the choice of F_0, at first, we consider an exponential distribution, i.e., $F_0(t) = e^t \mathbf{1}_{(t \le 0)}$. There are two reasons for this choice. On one hand, in theory, theorem 3.2 in Singh et al. [24] showed that an exponential distribution based combining method is optimal in terms of achieving the largest possible value of Bahadur slopes. On the other hand, we also conducted numerical simulation for three kinds of exponential distributions mentioned in Singh et al. [24], and $F_0(t) = e^t \mathbf{1}_{(t \le 0)}$ was the best choice according to the simulation results. In addition, the standard normal distribution is often used in combining methods, so we tried it too. Therefore, according to Singh et al. [24], we have two kinds of combined CD as the following,

$$H_E(y) \quad = \quad P\left(\chi_{2k}^2 \ge -2\sum_{i=1}^k \log H_\rho^i(\mathbb{D}_i; y)\right), \tag{3.4}$$

$$H_N(y) \quad = \quad \Phi\left(\frac{1}{\sqrt{k}}\sum_{i=1}^k \Phi^{-1}\left(H_\rho^i(\mathbb{D}_i; y)\right)\right), \tag{3.5}$$

where $H_\rho^i(\mathbb{D}_i; y)$ is CD that is defined in (3.1) based on the ith sample for $i = 1, 2, \ldots, k$.

3.3. Computing algorithm

The new proposed confidence intervals can be calculated through Monte Carlo simulation. For a given data set $X_{ij}, i = 1, 2, \ldots, k, j = 1, 2, \ldots, n_i$, the algorithm follows

Step 1. Compute $\overline{x}_i, s_i^2$ for $i = 1, 2, \ldots, k$.
Step 2. For the calculated values $(\overline{x}_i, s_i^2)$, simulate the empirical distribution according to formula (3.1) for $i = 1, 2, \ldots, k$.
Step 3. Simulate the empirical distribution following (3.4) and (3.5).
Step 4. Obtain the empirical $\frac{\alpha}{2}$ and $1 - \frac{\alpha}{2}$ quantiles of (3.4) and (3.5) according to Step 3.
Step 5. Compute the new intervals according to Proposition 3.1.

4. Simulation study

In this section, we describe the simulation studies that we conducted to compare the new proposed confidence intervals (denote the methods as CDEM for exponential distribution based combining method and CDNM for standard normal distribution based combining method, respectively) for the common CV η with two previously existing methods proposed by Tian [28](denote the method as TM) and Forkman [9] (denote the method as FM). It is worth noting that both TM and FM are asymptotic inference methods, in fact, to our knowledge, there is no other exact inference method for this problem besides the CD method we proposed in this paper. As Fung and Tsang [10] stated, the range from 0.05 to 0.5 is chosen since the CV rarely exceeds 0.5 in medical and biological studies, so we set the common population coefficient of variation to $\frac{1}{20}$, $\frac{1}{10}$, $\frac{1}{5}$, $\frac{1}{3}$ and $\frac{1}{2}$. We consider cases with 3, 5 and 8 samples with sample size from 3 to 20. For each parameter setting, 5000 random samples are generated.

Table 1 presents the empirical coverage probabilities and average interval lengths of 95% two-sided confidence intervals for the situations of 3 samples. Figures 1 and 2 summarize the simulation results in Table 1 through boxplots. Figure 1 shows that all of the four methods have good average empirical confidence level that is very close to the nominal level 95%, but TM has larger range than CDEM, CDNM and FM. Figure 2 shows the differences of the average confidence interval lengths, relative to the CDEM interval, for the other three procedures. These relative lengths are denoted by RL, which is defined as $\frac{AL_M - AL_{CDEM}}{AL_{CDEM}}$, where AL_M denotes the average length of a competing interval and AL_{CDEM} denotes the average length of CDEM interval. From Figure 2, we see that CDEM has smaller average lengths than CDNM, TM and FM. In some settings, the average lengths of TM intervals are 1.4 times or more of CDEM's intervals. We also notice that the interval length performance of

TABLE 1
Empirical coverage probabilities and average interval length of 95% two-sided confidence intervals for 3 samples

CV	$Method$	α_{CV}	AVL	α_{CV}	AVL	α_{CV}	AVL
		$(5,5,5)^*$		$(10,10,10)$		$(20,20,20)$	
	$CDEM$	.952	.0541	.947	.0277	.952	.0190
$\frac{1}{20}$	$CDNM$	.954	.0599	.948	.0287	.955	.0191
	TM	.955	.1068	.955	.0330	.951	.0203
	FM	.951	.0557	.945	.0283	.947	.0190
	$CDEM$	.949	.1090	.951	.0557	.950	.0379
$\frac{1}{10}$	$CDNM$	.949	.1215	.950	.0575	.952	.0383
	TM	.954	.2172	.953	.0660	.950	.0405
	FM	.947	.1135	.946	.0571	.949	.0382
	$CDEM$	.951	.2258	.949	.1153	.950	.0786
$\frac{1}{5}$	$CDNM$	.952	.2506	.951	.1200	.952	.0789
	TM	.952	.4629	.947	.1340	.945	.0815
	FM	.954	.2463	.952	.1202	.948	.0793
	$CDEM$	.959	.3984	.951	.2055	.945	.1390
$\frac{1}{3}$	$CDNM$	.951	.4326	.949	.2135	.946	.1404
	TM	.926	.9306	.934	.2290	.926	.1368
	FM	.939	.5161	.951	.2230	.945	.1435
	$CDEM$	.955	.5125	.953	.3360	.949	.2342
$\frac{1}{2}$	$CDNM$	.955	.5124	.954	.3515	.949	.2362
	TM	.929	2.006	.921	.3612	.895	.2092
	FM	.978†	.9936	.959	.4128	.956	.2517

$*$*Sample Size*

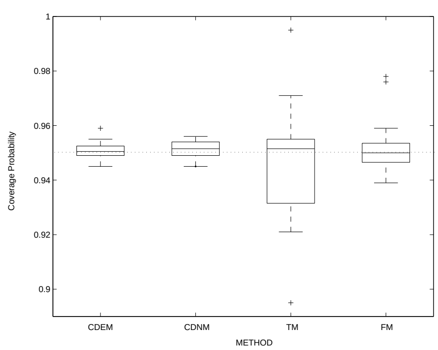

FIG 1. *Comparison of empirical coverage probabilities for settings with 3 samples.*

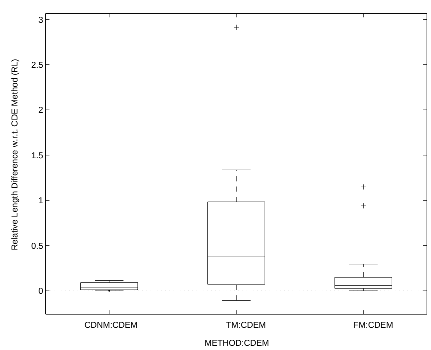

FIG 2. *Relative difference of average confidence interval length for settings with 3 samples.*

CDNM is worse than CDEM in most settings, and even worse than FM, especially when CV is relatively small, such as $\frac{1}{20}$, $\frac{1}{10}$ and $\frac{1}{5}$. With the sample size increasing, the performance of CDNM is approaching to CDEM, as shown in situations of $(20, 20, 20)$.

In addition, we notice that FM did not work sometimes in several settings of small sample sizes, such as $(5, 5, 5)$, when we conducted Monte Carlo simulation.

TABLE 2
Empirical coverage probabilities and average interval length of 95% two-sided confidence intervals for 5 samples

CV	$Method$	α_{CV}	AVL	α_{CV}	AVL	α_{CV}	AVL
		$(5,5,5,5,5)^*$		$(10,10,10,10,10)$		$(20,20,20,20,20)$	
$\frac{1}{20}$	$CDEM$	.953	.0333	.955	.0217	.954	.0150
	$CDNM$	.951	.0350	.954	.0218	.953	.0147
	TM	.935	.0540	.949	.0258	.945	.0157
	FM	.942	.0337	.955	.0215	.957	.0147
$\frac{1}{10}$	$CDEM$	.952	.0670	.952	.0435	.957	.0299
	$CDNM$	.951	.0708	.954	.0438	.958	.0293
	TM	.937	.1092	.948	.0517	.949	.0313
	FM	.948	.0683	.955	.0435	.956	.0296
$\frac{1}{5}$	$CDEM$	.949	.1382	.949	.0897	.953	.0613
	$CDNM$	.947	.1462	.945	.0903	.949	.0599
	TM	.933	.2275	.927	.1049	.942	.0630
	FM	.942	.1432	.945	.0905	.948	.0607
$\frac{1}{3}$	$CDNM$	.962	.2522	.951	.1593	.950	.1092
	$CDNM$	.955	.2634	.953	.1606	.950	.1068
	TM	.933	.4294	.926	.1791	.933	.1063
	FM	.941	.2727	.949	.1645	.951	.1088
$\frac{1}{2}$	$CDEM$	.955	.4055	.957	.2696	.950	.1835
	$CDNM$	.954	.4057	.955	.2706	.948	.1788
	TM	.883	.8839	.923	.2837	.932	.1632
	FM	.937†	.5754	.957	.2941	.952	.1892

$*$*Sample Size*

The reason is that the $\chi^2_{1-\frac{\alpha}{2}} - \sum_{i=1}^{k}(n_i-1)u_i$ and $\chi^2_{\frac{\alpha}{2}} - \sum_{i=1}^{k}(n_i-1)u_i$ in formula (12) of Forkman [9] may be negative. We discarded these negative outcomes and computed the empirical coverage probabilities only using those effective results. The simulated probabilities of these situations are denoted with symbol † following the numbers in Tables 1–3. We also see that TM and FM have bad performances of both empirical coverage probabilities and interval lengths, especially in the setting of small samples and simultaneously relative large CV value such as $\frac{1}{2}$. On the basis of the above analysis, we recommend the CD intervals for the common CV.

Tables 2 and 3 are analogous to Table 1 for 5 and 8 samples, respectively, and Figures 3–6 are analogous to Figures 1 and 2 for 5 and 8 samples, respectively. Tables 2 and 3 show that TM are liberal for small sample sizes and large CVs. Regarding interval lengths, in Tables 2 and 3, we notice that the interval lengths of CDNM are a little bit shorter than the counterpart of CDEM in the last situation of sample size design. It shows that average interval length performance of CDNM intervals becomes closer to or even better than CDEM as sample size increases to some extent. The anonymous referee points out that the CD combining method allows use of non-trivial weights to improve the efficiency, so we use a weighted CDNM (CDWNM) to study the CD combining method further. The weighted combined CD formula based on the standard normal

TABLE 3
Empirical coverage probabilities and average interval length of 95% two-sided confidence intervals for 8 samples

CV	$Method$	α_{CV}	AVL	α_{CV}	AVL	α_{CV}	AVL
		$(5,5,5,5,5,5,5,5)^*$		$(10,10,10,10,10,10,10,10)$		$(20,20,20,20,20,20,20,20)$	
$\frac{1}{20}$	$CDEM$	.953	.0258	.956	.0171	.955	.0116
	$CDNM$	.956	.0270	.953	.0171	.956	.0116
	TM	.901	.0436	.930	.0205	.940	.0125
	FM	.945	.0259	.949	.0168	.948	.0114
$\frac{1}{10}$	$CDEM$	.951	.0519	.948	.0344	.956	.0238
	$CDNM$	.952	.0541	.947	.0341	.959	.0229
	TM	.917	.0881	.923	.0412	.946	.0248
	FM	.947	.0523	.944	.0339	.956	.0230
$\frac{1}{5}$	$CDEM$	.956	.1079	.952	.0702	.951	.0487
	$CDNM$	.955	.1122	.952	.0700	.951	.0471
	TM	.890	.1858	.923	.0829	.930	.0499
	FM	.955	.1095	.945	.0699	.948	.0483
$\frac{1}{3}$	$CDEM$	.957	.1945	.953	.1248	.952	.0863
	$CDNM$	.955	.2003	.952	.1243	.950	.0837
	TM	.870	.3549	.898	.1423	.925	.0843
	FM	.945	.2037	.947	.1271	.954	.0853
$\frac{1}{2}$	$CDEM$	.953	.3234	.951	.2112	.949	.1440
	$CDNM$	.952	.3296	.951	.2088	.951	.1384
	TM	.859	.7518	.845	.2270	.873	.1289
	FM	.938	.3813	.949	.2241	.952	.1462

$*$*Sample Size*

distribution is given by

$$H_{WN}(y) \quad = \quad \Phi\left(\frac{1}{\sqrt{\sum_{j=1}^{k} w_j^2}} \sum_{i=1}^{k} w_i \Phi^{-1}\left(H_\rho^i(\mathbb{D}_i; y)\right)\right), \tag{4.1}$$

where $H_\rho^i(\mathbb{D}_i; y)$ is the CD defined as before and w_i is the weight for the ith component, here we try to use the sample size, i.e., n_i, as the weight. Obviously, based on this weighted plan, CDNM and CDWNM are equivalent under the situations of equal sample sizes. Hence, we only consider the settings of unequal sample sizes. Table 4 presents the design and corresponding simulation results. In theory, the CDWNM still has the exact frequentist property as we mentioned before, so the focus is on the average interval length. From the interval length results in Table 4, it is obvious that the average lengths of CDWNM are indeed shorter than the counterparts of CDNM, and also shorter than CDEM in many situations. The above weighted procedure shows an appealing property of the combining CD method.

According to Tables 2–4, TM is also much longer than the other methods. FM has similar performances as CDEM and CDWNM in some settings of 5 and 8 samples. However, considering the exact property of CD based method and

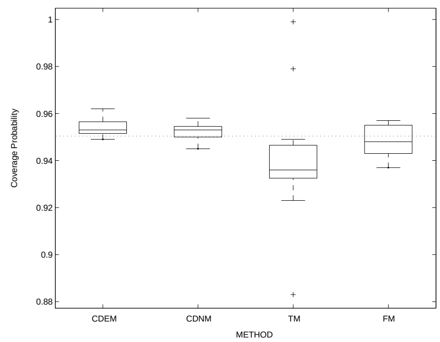

FIG 3. *Comparison of empirical coverage probabilities for settings with 5 samples.*

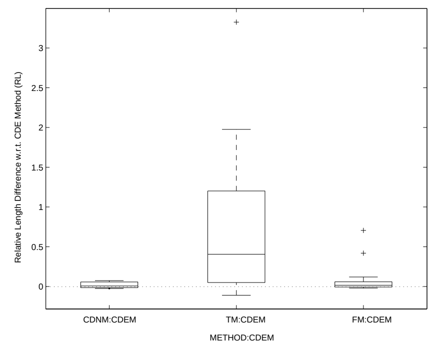

FIG 4. *Relative difference of average confidence interval length for settings with 5 samples.*

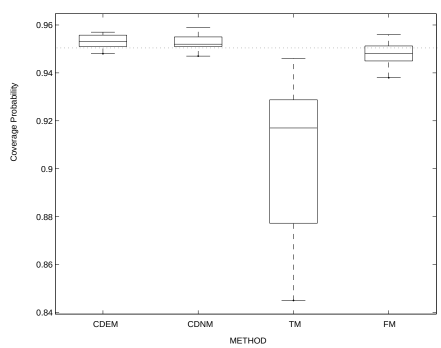

FIG 5. *Comparison of empirical coverage probabilities for settings with 8 samples.*

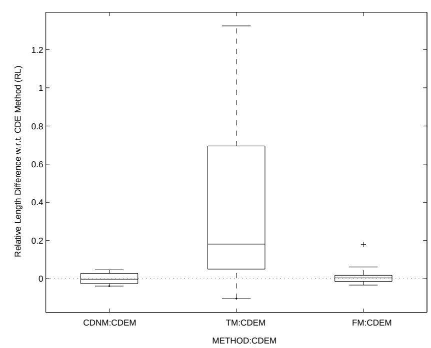

FIG 6. *Relative difference of average confidence interval length for settings with 8 samples.*

above mentioned possibly negative values during the process of calculating FM intervals, the CD based intervals are better choices for practice use.

5. An application to a real data example

For illustration purposes, we consider a real data set derived from a field experiment in two experiment stations of the China Agricultural University. In agriculture, the CV is a very important criterion to measure experimental variability. In general, lower magnitude of CV is the reflection of reliability (precision) of the experimental results (Patel et al. [22], Taylor et al. [26]). Therefore, agriculturalists want to know the CV information in field experiments before further study. They are also concerned with the estimation of the common CV if the equality of CsV in different experimental fields can be reasonably assumed, because many agricultural experiments are conducted in different fields of different locations, the common CV shows the common uniformity of these different fields.

TABLE 4
Empirical coverage probabilities and average interval length of 95% two-sided confidence intervals for comparison of CDNM and CDWNM

CV	$Method$	α_{CV}	AVL	α_{CV}	AVL	α_{CV}	AVL
		$(3,4,5)$		$(5,8,15)$		$(10,20,30)$	
$\frac{1}{20}$	$CDEM$	.949	.0541	.948	.0314	.955	.0201
	$CDNM$	.951	.0597	.955	.0314	.956	.0196
	$CDWNM$	.951	.0589	.951	.0306	.955	.0193
	TM	.948	.1067	.952	.0369	.949	.0204
	FM	.946	.0556	.948	.0297	.951	.0190
$\frac{1}{5}$	$CDEM$	.950	.2260	.948	.1300	.951	.0829
	$CDNM$	.953	.2508	.947	.1300	.955	.0811
	$CDWNM$	.953	.2473	.948	.1272	.954	.0799
	TM	.964	.4657	.945	.1500	.953	.0821
	FM	.946	.2455	.944	.1271	.955	.0792
$\frac{1}{2}$	$CDEM$	.946	.5076	.958	.3610	.953	.2474
	$CDNM$	.951	.5003	.960	.3690	.953	.2428
	$CDWNM$	.949	.4940	.956	.3633	.955	.2400
	TM	.995	2.075	.899	.4451	.901	.2125
	FM	.976†	1.206	.952	.4444	.960	.2513
		$(3,3,4,$ $5,5)$		$(5,8,10,$ $12,15)$		$(10,15,20,$ $25,30)$	
$\frac{1}{20}$	$CDEM$	.962	.0404	.953	.0225	.952	.0155
	$CDNM$	.958	.0433	.955	.0223	.952	.0149
	$CDWNM$	.958	.0426	.953	.0219	.953	.0148
	TM	.935	.0900	.945	.0266	.956	.0158
	FM	.947	.0405	.952	.0214	.948	.0145
$\frac{1}{5}$	$CDEM$	.951	.1702	.955	.0928	.948	.0636
	$CDNM$	.956	.1819	.953	.0922	.949	.0614
	$CDWNM$	.954	.1792	.953	.0909	.951	.0609
	TM	.940	.4111	.942	.1087	.936	.0635
	FM	.946	.1738	.952	.0899	.943	.0610
$\frac{1}{2}$	$CDEM$	.960	.4596	.959	.2788	.952	.1898
	$CDNM$	.957	.4571	.957	.2768	.955	.1823
	$CDWNM$	.951	.4496	.950	.2736	.953	.1811
	TM	.998	1.998	.875	.3183	.890	.1650
	FM	.955†	.8100	.947	.2963	.957	.1890

In general, a field, or a block is divided into many plots with equal size in agricultural experiments, then different types of seeds are randomly planted to these plots for selecting high yield type of seed or other experimental purposes. Among these planted seeds, one type of seed such as wild type with stable traits must be planted as control group, and the yield data of plots with this kind of seed will be used to compute CV. The data set we used here describes three groups of crop yields of plots with wild type seeds from three different experimental blocks, respectively; one block lies in Beijing experiment station of China Agriculture University, the other two in Hebei experiment station of China Agriculture University. Some descriptive statistics of the original data are given in Table 5. The unit of the original data is kilogram per hectare (Kg/Ha).

Table 5
Some descriptive statistics for crop yields from three different experimental blocks

	Sample Size	$\overline{X}_i$(Kg/Ha)	S_i
Block 1	32	4881.3	1599.1
Block 2	32	6628.2	2032.1
Block 3	21	6512.0	1612.9

Table 6
Nominally 90% and 95% Confidence Intervals (CI) on common CV for the data set of crop yields

Method	90%CI	90%CI Length	95%CI	95%CI Length
CDEM	(0.2610, 0.3372)	0.0762	(0.2568, 0.3510)	0.0942
CDNM	(0.2634, 0.3486)	0.0852	(0.2574, 0.3594)	0.1020
CDWNM	(0.2622, 0.3480)	0.0858	(0.2556, 0.3588)	0.1032
TM	(0.2709, 0.3527)	0.0818	(0.2633, 0.3621)	0.0987
FM	(0.2652, 0.3518)	0.0866	(0.2589, 0.3628)	0.1039

Shapiro-Wilk method is used to test the normality of every sample and there is no sufficient evidence to reject the normal assumption. Then we use the methods mentioned in Nairy and Rao [21] to test the homogeneity of CsV, since there is also no sufficient evidence to reject the equality of CsV, it may be reasonable to assume that the CsV of the three different blocks data are common. Therefore, on the basis of above analysis and assumptions, we are interested in making inference about the common CV based on these three samples.

The 90% and 95% confidence intervals for the common CV based on the above methods are given in Table 6. It is easy to see that the new CDEM confidence intervals have the shortest interval lengths.

6. Concluding remarks

In this article, we propose a new interval estimation procedure for a common CV shared by several normal populations using a CD and combined CD approach. We report simulation studies conducted to compare confidence intervals for common CV η with two other confidence intervals from the existing literature. The results of simulation studies show that the proposed CD intervals for η are satisfactory in terms of both coverage probability and average interval length. In theory, the new CD intervals have exact frequentist property because of the exact combined CD for the inverse of the CV ρ. We also present a real data set to illustrate the use of the proposed procedure. All these results confirm that the CD intervals can be recommended for practical use instead of the methods previously discussed in the literature.

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