Brazilian Journal of Probability and Statistics
2017, Vol. 31, No. 1, 87–110
DOI: 10.1214/15-BJPS304

Slash-elliptical nonlinear regression model

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Abstract. The aim of this paper is to develop nonlinear regression models with error distribution having the slash-elliptical family. A slash-elliptical random variable is defined as the quotient of two independent random variables, Z and $U^{1/q}$, where Z has an elliptical contoured distribution and U has a uniform distribution. A key advantage of the slash-elliptical distribution is the simplicity by which the well-known elliptical contoured distribution can be modified to support increase in kurtosis. The main properties of the slash-elliptical distribution is symmetry, heavy tails and convergence to the elliptical contoured distribution as the limiting case of the shape parameter. One of the advantages of this distribution is to allow larger kurtosis than the elliptical contoured distribution. In this paper, we propose estimation method, residual analysis and generalized leverage for the new class of regression models. We also develop diagnostic measures under local influence approach and present a real data analysis.

1 Introduction

In the context of data modelling with real support, the class of symmetrical models has been broadly studied, including the methods of estimation and diagnostic analysis for the linear models (Galea, Paula and Bolfarine, 1997, Galea, Paula and Uribe-Opazo, 2003, Cysneiros, Paula and Galea, 2007, Villegas et al., 2013) and nonlinear models (Galea, Paula and Cysneiros, 2005, Cysneiros and Vanegas, 2008, Vanegas and Cysneiros, 2010). The regression models in the univariate elliptical contoured family can be considered as an alternative to modelling with normal errors when the data has light (for example, logistic I distribution) or heavy tails (for instance, Student-t, power exponential, logistic II and slash distribution). A complete review on the symmetrical distributions can be found in Chmielewski (1981).

Among the elliptical distributions, the slash distribution allows for larger flexibility in the fit of real data and it has the normal distribution as a limit case. We say that a random variable Y has the slash distribution with location μ and scale ϕ parameters if it can be expressed as

$$Y = \mu + \sqrt{\phi}\frac{Z}{U^{1/q}}, \tag{1.1}$$

Key words and phrases. Slash-elliptical distribution, nonlinear model, residual, local influence.
Received October 2014; accepted November 2015.

where Z and $U^{1/q}$ are two independent random variables, Z having the standard normal distribution, U having the standard uniform distribution and $q > 0$ is a parameter that is related to the kurtosis of the slash distribution. When $\mu = 0$, $\phi = 1$ and $q = 1$, Y follows the standard slash distribution. Discussions about the slash distribution properties can be found in Rogers and Tukey (1972) and Mosteller and Tukey (1977). The maximum likelihood estimators (MLEs) for the location and scale parameters of the slash distribution were presented by Kafadar (1982). Lange and Sinsheimer (1993) developed a multivariate version where the variable Z in (1.1) is the multivariate normal distribution. In the same approach, Wang and Genton (2006) proposed the multivariate skew-slash distribution considering that the variable Z has a multivariate asymmetrical normal distribution in (1.1). A new class of distributions was introduced by Arslan (2008) by taking the variable Z having the generalized hyperbolic distribution. Arslan and Genç (2009) generalized the family of distributions proposed by Wang and Genton (2006) by constructing a family of multivariate distributions through of the scale mixture models of the multivariate Kotz type and uniform distributions.

The slash-elliptical distribution was proposed by Gómez, Quintana and Torres (2007) by replacing the distribution of Z by the family of univariate and multivariate elliptical distributions in (1.1). A key advantage of the slash-elliptical distribution is the simplicity by which the well-known symmetrical distribution can be modified to support increase in kurtosis. The main properties of the slash-elliptical distribution are symmetry, heavy tails and convergence to the elliptical distribution when $q \to \infty$.

The methods of estimation, hypothesis testing for the parameters, residuals and diagnostic analysis under the local influence approach and generalized leverage for linear regression models with univariate slash-elliptical errors were proposed by Alcantara and Cysneiros (2013). The aim of this paper is to extend their methodology for the class of nonlinear models with errors distribution in the family of univariate slash-elliptical distributions.

The paper is organized as follows. In Section 2, we present the slash-elliptical distribution. In Section 3, we define the slash-elliptical nonlinear regression model and develop the estimation procedure and inference methods. In addition, we propose a residual analysis and perform a simulation study for evaluating its behavior. In Section 4, we define some diagnostic measures for the class of slash-elliptical nonlinear models. In Section 5, we present an application to a real data set to illustrate the proposed methodology. Finally, in Section 6, we offer some conclusions.

2 Slash-elliptical distribution

We say that a random variable Y has the slash-elliptical distribution with location $\mu \in \mathbb{R}$ and scale $\phi > 0$ parameters if Y can be expressed as $Y = \mu + \sqrt{\phi} V U^{-1/q}$, where V and U are independent random variables, V has the standard elliptical contoured distribution, U has uniform distribution on $(0,1)$ and q is the shape parameter of slash-elliptical distribution (Gómez, Quintana and Torres, 2007). We denote a slash-elliptical random variable for $Y \sim \mathrm{SEL}(\mu, \phi, q, g)$ and its density function is defined by

$$f(y;\mu,\phi,q) = \frac{1}{\sqrt{\phi}} \begin{cases} \dfrac{qH(z^2)}{2|z|^{q+1}}, & z \neq 0, \\ \dfrac{qg(0)}{q+1}, & z = 0, \end{cases}$$

where $z = (y-\mu)/\sqrt{\phi}$ and $H(z^2) = \int_0^{z^2} t^{(q-1)/2} g(t)\,dt$, for some density generator function g with $g(t) > 0$ for $t > 0$ and $\int_0^\infty t^{-1/2} g(t)\,dt = 1$. The density generator function of some slash-elliptical distribution can be found in Table 1, where $\nu > 0$, $\sigma > 0$, $0 < \lambda < 1$, $g'(t)$ is the derivative of $g(t)$ with respect to t, $\gamma(\varsigma_1; \varsigma_2)$ is incomplete gamma function evaluated in ς_2 with ς_1 shape parameter and

$${}_2F_1(\iota_1, \iota_2; \iota_3; \iota_4) = \frac{\Gamma(\iota_3)}{\Gamma(\iota_2)\Gamma(\iota_3 - \iota_2)} \int_0^1 \frac{x^{\iota_2 - 1}(1-x)^{\iota_3 - \iota_2 - 1}}{(1 - \iota_4 x)^{\iota_1}}\,dx$$

is hypergeometric function as defined by Bailey (1935). In this paper, the slash-contaminated normal distribution is named Slash-CN. The plots of some slash-elliptical distributions, with $\mu = 0$, $\phi = 1$, $q = 1$ (dashed line), $q = 4$ (dotted line) and $q = 20$ (dash-dot line), compared with the standard normal distribution (solid line) are displayed in Figure 1.

3 Slash-elliptical nonlinear regression model

Let ε_i, $i = 1, \dots, n$, be independent random variables with slash-elliptical distributions, where $\varepsilon_i \sim \mathrm{SEL}(0, \phi, q, g)$, ϕ is the scale parameter and g is a density generator function that satisfies: $g(t) > 0$ for $t > 0$, and $\int_0^\infty t^{-1/2} g(t)\,dt = 1$. The nonlinear regression model is expressed as

$$Y_i = \mu_i(\boldsymbol{\beta}, \mathbf{x}_i) + \varepsilon_i, \qquad i = 1, 2, \dots, n,$$

where Y_i is the response variable, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p)^\top$ is the unknown parameter vector $(p \times 1)$, $\mathbf{x}_i = (1, x_{i2}, \dots, x_{ip})^\top$ is the regressor vector $(p \times 1)$, $\mu_i(\boldsymbol{\beta}, \mathbf{x}_i)$ is a nonlinear function of $\boldsymbol{\beta}$ that is twice continuously differentiable, such that the derivative matrix $\mathbf{D}_\beta = \partial \boldsymbol{\mu} / \partial \boldsymbol{\beta}$ has rank p $(p < n)$ for all $\boldsymbol{\beta}$ and fixed or known q.

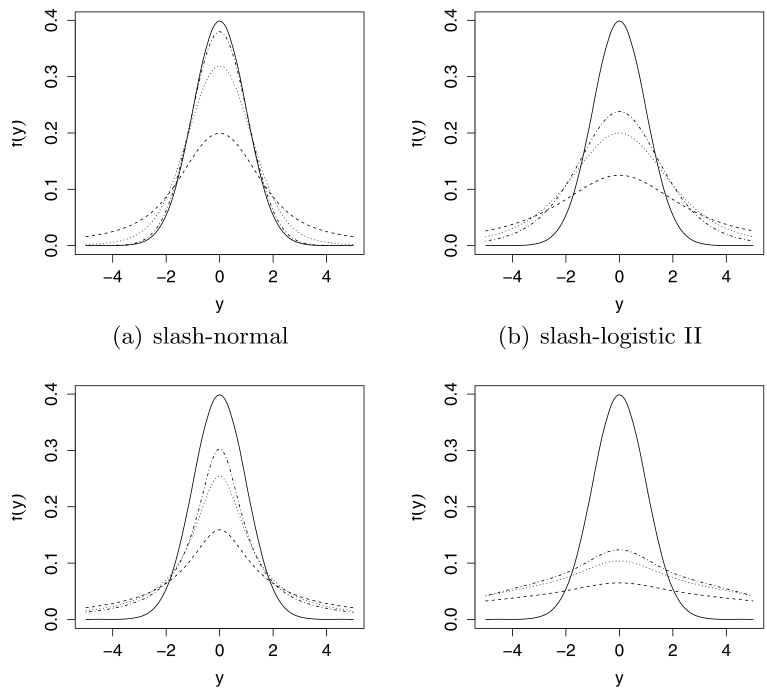
3.1 Parameter estimation

The MLE of $\boldsymbol{\theta} = (\boldsymbol{\beta}^\top, \phi)^\top$ for the slash-elliptical nonlinear model class is defined by $\hat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}} \ell(\boldsymbol{\theta})$, where $\ell(\boldsymbol{\theta})$ is the log-likelihood function given by

$$\ell(\boldsymbol{\theta}) = -\frac{n}{2}\log(\phi) + \sum_{i \in A} a(z_i) + K_q,$$

Table 1 *The $g(t)$, $g'(t)$ and $H(z^2)$ functions of some slash-elliptical distributions*

Distribution	$g(t)$	$g'(t)$	$H(z^2)$
Slash-normal	$\frac{e^{-t/2}}{\sqrt{2\pi}}$	$\frac{-e^{-t/2}}{2\sqrt{2\pi}}$	$\frac{2^{q/2}}{\sqrt{\pi}}\gamma(\frac{q+1}{2};\ \frac{z^2}{2})$
Slash-logistic II	$\frac{e^{-\sqrt{t}}}{(1+e^{-\sqrt{t}})^2}$	$\frac{e^{-\sqrt{t}}(e^{-\sqrt{t}}-1)}{2\sqrt{t}(1+e^{-\sqrt{t}})^3}$	$\int_0^{z^2} \frac{t^{(q-1)/2}e^{-\sqrt{t}}}{(1+e^{-\sqrt{t}})^2}\,dt$
Slash-Student-t	$\frac{(\nu+t)^{-(\nu+1)/2}}{\nu^{-\nu/2}B(1/2,\nu/2)}$	$-\frac{(\nu+1)(\nu+t)^{-(\nu+3)/2}}{2\nu^{-\nu/2}B(1/2,\nu/2)}$	$\frac{2\,{}_2F_1((\nu+1)/2,(q+1)/2;(q+3)/2;-z^2/\nu)}{(q+1)z^{-(q+1)}\sqrt{\nu}B(1/2,\nu/2)}$
Slash-CN	$\frac{(1-\lambda)}{\sqrt{2\pi}e^{t/2}}+\frac{\lambda e^{-t/2\sigma^2}}{\sigma\sqrt{2\pi}}$	$-\frac{(1-\lambda)}{2\sqrt{2\pi}e^{t/2}}-\frac{\lambda e^{-t/2\sigma^2}}{2\sigma^3\sqrt{2\pi}}$	$\frac{(1-\lambda)\gamma((q+1)/2;z^2/2)+\lambda\sigma^q\gamma((q+1)/2;z^2/(2\sigma^2))}{2^{-q/2}\sqrt{\pi}}$



(a) slash-normal (b) slash-logistic II

(c) slash-Student-t($\nu = 1$) (d) slash-CN($\lambda = 0.9, \sigma = 4$)

Figure 1 *Plots of the standard normal density* (*solid line*) *and slash-elliptical densities with* $\mu = 0$, $\phi = 1$, $q = 1$ (*dashed line*), $q = 4$ (*dotted line*) *and* $q = 20$ (*dash-dot line*) *for*: (a) *slash-normal*, (b) *slash-logistic II*, (c) *slash-Student-t* ($\nu = 1$) *and* (d) *slash-CN* ($\lambda = 0.9$, $\sigma = 4$).

where $A = \{i : z_i \neq 0\}$, $z_i = (y_i - \mu_i)/\sqrt{\phi}$, $\mu_i = \mu_i(\boldsymbol{\beta}, \mathbf{x}_i)$, $a(z_i) = \log(H(z_i^2)) - (q+1)\log|z_i|$, $H(z_i^2) = \int_0^{z_i^2} t^{(q-1)/2} g(t)\,dt$, $K_q = n_A \log(q/2) + (n - n_A) \times \log(qg(0)/(q+1))$ and n_A is the number of elements in the set A. The score function of $\boldsymbol{\theta}$ is given by $\mathbf{U}_\theta = (\mathbf{U}_\beta^\top, \mathbf{U}_\phi)^\top$, $\mathbf{U}_\beta = -(1/\sqrt{\phi})\mathbf{D}_\beta^\top \mathbf{D}(a')$ and $\mathbf{U}_\phi = -(1/(2\phi))\mathbf{z}^\top \mathbf{v}(l)$, where $\mathbf{D}_\beta = (\mathbf{d}_1^\top, \ldots, \mathbf{d}_n^\top)^\top$, $\mathbf{d}_i = \mu_i'(\boldsymbol{\beta}, \mathbf{x}_i)$, $\mu_i'(\boldsymbol{\beta}, \mathbf{x}_i)$ is the derivative of the function $\mu_i(\boldsymbol{\beta}, \mathbf{x}_i)$ with respect to $\boldsymbol{\beta}$, $\mathbf{z} = (z_1, \ldots, z_n)^\top$, $\mathbf{D}(a') = \operatorname{diag}(a'(z_1), \ldots, a'(z_n))$, $a'(z_i) = 2|z_i|^q g(z_i^2)/H(z_i^2) - (q+1)/z_i$, $\mathbf{v}(l) = (l(z_1), \ldots, l(z_n))^\top$ and $l(z_i) = n/(n_A z_i) + a'(z_i)$. The observed information matrix of the slash-elliptical nonlinear model is given by

$$-\ddot{\mathbf{L}}_{\hat{\theta}\hat{\theta}} = -\left(\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}\, \partial \boldsymbol{\theta}^\top}\right)\bigg|_{\theta=\hat{\theta}} = -\begin{bmatrix} \ddot{\mathbf{L}}_{\beta\beta} & \ddot{\mathbf{L}}_{\beta\phi} \\ \ddot{\mathbf{L}}_{\phi\beta} & \ddot{\mathbf{L}}_{\phi\phi} \end{bmatrix}\bigg|_{\theta=\hat{\theta}},$$

$$\ddot{\mathbf{L}}_{\beta\beta} = -\frac{1}{\phi}(\mathbf{D}_\beta^\top \mathbf{B} \mathbf{D}_\beta),$$

$$\ddot{\mathbf{L}}_{\beta\phi} = \ddot{\mathbf{L}}_{\phi\beta}^{\top} = \frac{1}{2\phi^{3/2}}\mathbf{D}_{\beta}^{\top}\mathbf{v}(m) \qquad \text{and}$$

$$\ddot{\mathbf{L}}_{\phi\phi} = -\frac{1}{4\phi^2}\left[\sum_{i\in A} z_i\big(z_i c(z_i) - a'(z_i)\big) - 2n\right] = -d,$$

where $\mathbf{B} = \mathbf{C}[\sqrt{\phi}\sum_{i\in A}\mathbf{D}_{\beta\beta}(i)a'(z_i)]\mathbf{C}^{\top} + \mathbf{D}(c)$, $\mathbf{C} = \mathbf{D}(c)\mathbf{D}_{\beta}(\mathbf{D}_{\beta}^{\top}\mathbf{D}(c)\mathbf{D}_{\beta})^{-1}$, $c(z_i) = 4z_i^{2q}g^2(z_i^2)H^{-2}(z_i^2) - 2|z_i|^{q-1}[qg(z_i^2) - 2z_i^2 g'(z_i^2)]H^{-1}(z_i^2) - (q+1)z_i^{-2}$, $\mathbf{D}(c) = \mathrm{diag}(c(z_1),\ldots,c(z_n))$, $\mathbf{D}_{\beta\beta}(i) = \partial^2\mu_i(\beta,\mathbf{x}_i)/\partial\beta\,\partial\beta^{\top}$, $m(z_i) = a'(z_i) - z_i c(z_i)$ and $\mathbf{v}(m) = (m(z_1),\ldots,m(z_n))^{\top}$. The K_q, $a(z)$, $a'(z)$, $c(z)$, $m(z)$ and $o(z)$ functions of some slash-elliptical distributions can be seen in Tables 2 and 3.

The system of equations $\mathbf{U}_{\theta} = \mathbf{0}$ does not have closed form solutions. In this study, we use the quasi-newton optimization method to obtain estimates of the parameter vector $\boldsymbol{\theta}$. The initial values for the estimates of the iterative process can be obtained from the fitted normal nonlinear regression.

Table 2 *The K_q, $a(z)$ and $a'(z)$ functions for some slash-elliptical distributions*

Distribution	K_q
Slash-normal	$\log(\frac{q^n(q+1)^{-(n-n_A)}}{2^{(n+n_A)/2}\pi^{(n-n_A)/2}})$
Slash-logistic II	$\log(\frac{q^n 2^{-(2n-n_A)}}{(q+1)^{n-n_A}})$
Slash-Student-t	$\log(\frac{q^n \nu^{-(n-n_A)/2}(q+1)^{-(n-n_A)}}{2^{n_A}(B(1/2,\nu/2))^{n-n_A}})$
Slash-CN	$\log(\frac{q^n(\sigma(1-\lambda)+\lambda)^{n-n_A}}{2^{(n+n_A)/2}\pi^{(n-n_A)/2}(q+1)^{(n-n_A)}})$
	$a(z)$
Slash-normal	$\log(\frac{2^{q/2}\gamma((q+1)/2;z^2/2)}{\lvert z\rvert^{q+1}\sqrt{\pi}})$
Slash-logistic II	$\log(\frac{H(z^2)}{\lvert z\rvert^{q+1}})$
Slash-Student-t	$\log(\frac{2\,{}_2F_1((\nu+1)/2,(q+1)/2;(q+3)/2;-z^2/\nu)}{(q+1)\sqrt{\nu}B(1/2,\nu/2)})$
Slash-CN	$\log(\frac{(1-\lambda)\gamma((q+1)/2;z^2/2)+\lambda\sigma^q\gamma((q+1)/2;z^2/(2\sigma^2))}{\lvert z\rvert^{q+1}2^{-q/2}\sqrt{\pi}})$
	$a'(z)$
Slash-normal	$\frac{z^q e^{-z^2/2}}{2^{(q-1)/2}\gamma((q+1)/2;z^2/2)} - \frac{q+1}{z}$
Slash-logistic II	$\frac{2z^q e^{-\lvert z\rvert}}{(1+e^{-\lvert z\rvert})^2 H(z^2)} - \frac{q+1}{z}$
Slash-Student-t	$\frac{(q+1)\nu^{(\nu+1)/2}(\nu+z^2)^{-(\nu+1)/2}}{z\,{}_2F_1((\nu+1)/2,(q+1)/2;(q+3)/2;-z^2/\nu)} - \frac{q+1}{z}$
Slash-CN	$\frac{z^q}{2^{(q-1)/2}}[\frac{(1-\lambda)e^{-z^2/2}+\lambda e^{-z^2/2\sigma^2}}{(1-\lambda)\gamma((q+1)/2;z^2/2)+\lambda\sigma^q\gamma((q+1)/2;z^2/(2\sigma^2))}] - \frac{q+1}{z}$

Table 3 *The $c(z)$, $m(z)$ and $o(z)$ functions of some slash-elliptical distributions*

Distributions	$c(z)$
Slash-normal	$\frac{z^{2q}e^{-z^2}}{2^{q-1}\gamma^2((q+1)/2;z^2/2)} - \frac{z^{q-1}(q+z^2)e^{-z^2/2}}{2^{(q-1)/2}\gamma((q+1)/2;z^2/2)} - \frac{q+1}{z^2}$
Slash-logistic II	$\frac{4z^{2q}e^{-2\|z\|}}{H^2(z^2)(1+e^{-\|z\|})^4} - \frac{2z^{q-1}(q-z)e^{-\|z\|}}{H(z^2)(1+e^{-\|z\|})^2} - \frac{q+1}{z^2}$
Slash-Student-t	$\frac{(q+1)^2(1+z^2/\nu)^{-(\nu+1)}}{z^2\,{}_2F_1^2((\nu+1)/2,(q+1)/2;(q+3)/2;-z^2/\nu)} - \frac{(q+1)(z^2(q+\nu+1)+q\nu)(1+z^2/\nu)^{-(\nu+3)/2}}{z^2\nu\,{}_2F_1((\nu+1)/2,(q+1)/2;(q+3)/2;-z^2/\nu)} - \frac{q+1}{z^2}$
Slash-CN	$\frac{4z^{2q}[(1-\lambda)e^{-z^2/2}+\lambda e^{-z^2/2\sigma^2}]^2}{[(1-\lambda)\gamma((q+1)/2;z^2/2)+\lambda\sigma^q\gamma((q+1)/2;z^2/(2\sigma^2))]^2} - \frac{z^{q-1}[(1-\lambda)(q+z^2)e^{-z^2/2}+(\lambda/\sigma)(q+z^2/\sigma^2)e^{-z^2/2\sigma^2}]}{2^{(q-1)/2}[(1-\lambda)\gamma((q+1)/2;z^2/2)+\lambda\sigma^q\gamma((q+1)/2;z^2/(2\sigma^2))]} - \frac{q-1}{z^2}$
	$m(z)$
Slash-normal	$\frac{z^q(q+1+z^2)e^{-z^2/2}}{2^{(q-1)/2}\gamma((q+1)/2;z^2/2)} - \frac{z^{2q+1}e^{-z^2}}{2^{q-1}\gamma^2((q+1)/2;z^2/2)}$
Slash-logistic II	$\frac{2z^q(q+1-z)e^{-\|z\|}}{H(z^2)(1+e^{-\|z\|})^2} - \frac{4z^{2q+1}e^{-2\|z\|}}{H^2(z^2)(1+e^{-\|z\|})^4}$
Slash-Student-t	$\frac{(q+1)(z^2(q+\nu+2)+\nu(q+1))(1+z^2/\nu)^{-(\nu+3)/2}}{z\nu\,{}_2F_1((\nu+1)/2,(q+1)/2;(q+3)/2;-z^2/\nu)} - \frac{(q+1)^2(1+z^2/\nu)^{-(\nu+1)}}{z\,{}_2F_1^2((\nu+1)/2,(q+1)/2;(q+3)/2;-z^2/\nu)}$
Slash-CN	$\frac{z^q[(1-\lambda)(q+1+z^2)e^{-z^2/2}+(\lambda/\sigma)(q+1+z^2/\sigma^2)e^{-z^2/2\sigma^2}]}{2^{(q-1)/2}[(1-\lambda)\gamma((q+1)/2;z^2/2)+\lambda\sigma^q\gamma((q+1)/2;z^2/(2\sigma^2))]} - \frac{4z^{2q+1}[(1-\lambda)e^{-z^2/2}+\lambda e^{-z^2/2\sigma^2}]^2}{[(1-\lambda)\gamma((q+1)/2;z^2/2)+\lambda\sigma^q\gamma((q+1)/2;z^2/(2\sigma^2))]^2}$
	$o(z)$
Slash-normal	$\frac{z^{q+1}e^{-z^2/2}}{2^{(q-1)/2}\gamma((q+1)/2;z^2/2)} - q$
Slash-logistic II	$\frac{2z^{q+1}e^{-\|z\|}}{(1+e^{-\|z\|})^2H(z^2)} - q$
Slash-Student-t	$\frac{(q+1)\nu^{(\nu+1)/2}(\nu+z^2)^{-(\nu+1)/2}}{{}_2F_1((\nu+1)/2,(q+1)/2;(q+3)/2;-z^2/\nu)} - q$
Slash-CN	$\frac{z^{q+1}}{2^{(q-1)/2}}[\frac{(1-\lambda)e^{-z^2/2}+\lambda e^{-z^2/2\sigma^2}}{(1-\lambda)\gamma((q+1)/2;z^2/2)+\lambda\sigma^q\gamma((q+1)/2;z^2/(2\sigma^2))}] - q$

3.2 Simulation study 1

In order to evaluate the empirical behavior of the estimator $\hat{\boldsymbol{\theta}}$, we conduct a simulation study in the model

$$Y_i = \beta_0 + \exp(\beta_1 + \beta_2 x_i) + \varepsilon_i, \qquad i = 1, 2, \dots, n, \tag{3.1}$$

where $\beta_0 = 1$, $\beta_1 = 1$, $\beta_2 = 1$ and $\phi = 2$ and $x_i \sim N(0, 1)$ are kept fixed among 10,000 replications. Six study scenarios for the error distribution ε_i are considered: slash-normal, slash-Student-t ($\nu = 5$) and slash-CN ($\lambda = 0.3$, $\sigma = 2$) distributions, all three with $q = 5$ and $q = 25$. We use the programming language Ox (Doornik, 2007) to implement all the functions developed in this paper. Codes in Ox can be obtained from the authors upon request. We generate 10,000 Monte Carlo replicates of the slash-elliptical nonlinear model in (3.1) and evaluated the MLEs $\boldsymbol{\theta} = (\beta_0, \beta_1, \beta_2, \phi)^\top$ for each scenario, with the sample size $n = 50$ and $n = 100$. Empirical statistical measures for parameter estimates on the simulated models are obtained and presented in Table 4. In all scenarios, we observe that the MLEs of the parameters, β_0, β_1 and τ present good statistical properties as n increases, as expected, for the true values taken for these parameters, in particular the biases are close to zero.

3.3 Goodness of fit and hypothesis testing

Galea, Paula and Cysneiros (2005), Vanegas and Cysneiros (2010) and Villegas et al. (2013) showed that the MLEs from symmetrical nonlinear regression models with heavy tails are less sensitive to extreme observations than the estimates from the normal models The similiar behaviour is also noted for the slash nonlinear regression model Lucas (1997) developed an interesting study on the robust aspects of the Student-t model. He showed that the protection against outliers is preserved only if the degrees of freedom extra parameter ν is fixed. Otherwise, if the degrees of freedom parameter is also estimated by maximum likelihood, the influence functions for ϕ and ν and the change-of-variance function of the location parameter are not bounded. In this direction, we keep all extra parameters, $\boldsymbol{\delta}$, fixed or known. For instance, q in the slash-normal and slash-logistic II, $(q, \nu)^\top$ in the slash-Student-t and $(q, \lambda, \sigma)^\top$ in the slash-CN keep fixed.

We adopt a profile likelihood function defined by $\ell_{\boldsymbol{\delta}}(\boldsymbol{\theta}) = \ell(\boldsymbol{\theta}; \boldsymbol{\delta})$, where $\boldsymbol{\delta}$ is a parameters vector known or fixed. Thus, the MLE $\boldsymbol{\theta}$ for $\boldsymbol{\delta}$ fixed or known is $\hat{\boldsymbol{\theta}}_{\boldsymbol{\delta}} = \arg\max_{\boldsymbol{\theta}} \ell_{\boldsymbol{\delta}}(\boldsymbol{\theta})$. We use a selection procedure based on the AIC (Akaike, 1974) and BIC (Schwarz, 1978) information criteria, for which the $\hat{\boldsymbol{\delta}}_{\text{AIC}}$ and $\hat{\boldsymbol{\delta}}_{\text{BIC}}$ estimators are the $\boldsymbol{\delta}$ values that minimizes the quantities $\text{AIC} = -2\ell_{\boldsymbol{\delta}}(\hat{\boldsymbol{\theta}}) + 2(p + 1)$ and $\text{BIC} = -2\ell_{\boldsymbol{\delta}}(\hat{\boldsymbol{\theta}}) + (p + 1)\log(n)$, respectively. The other method is plug in a consistent estimator for extra parameter, for example, moment estimator.

Table 4 *Means, standard errors (SE), biases and mean square errors (MSEs) for the parameter estimates on the simulated models*

			Slash-normal				Slash-Student-t ($\nu = 5$)				Slash-CN ($\lambda = 0.3$, $\sigma = 2$)			
n	q		β_0	β_1	β_2	τ	β_0	β_1	β_2	τ	β_0	β_1	β_2	τ
50	5	Mean	0.96	1.00	1.00	1.89	0.95	1.00	1.00	1.90	0.94	0.99	1.00	1.89
		SE	0.55	0.20	0.09	0.44	0.66	0.19	0.07	0.52	0.71	0.26	0.12	0.51
		Bias	−0.04	0.00	0.00	−0.11	−0.05	0.00	0.00	−0.10	−0.06	−0.01	0.00	−0.11
		MSE	0.56	0.20	0.10	0.45	0.44	0.04	0.01	0.28	0.71	0.26	0.10	0.52
	25	Mean	0.97	1.00	1.00	1.88	0.97	1.00	1.00	1.90	0.96	1.00	1.00	1.88
		EP	0.46	0.17	0.08	0.39	0.56	0.16	0.06	0.50	0.60	0.22	0.10	0.47
		Bias	−0.03	0.00	0.00	−0.12	−0.03	0.00	0.00	−0.10	−0.04	−0.01	0.00	−0.12
		MSE	0.46	0.17	0.10	0.41	0.31	0.03	0.00	0.26	0.60	0.22	0.10	0.49
100	5	Mean	0.98	1.00	1.00	1.95	0.98	1.00	1.00	1.95	0.97	1.00	1.00	1.95
		SE	0.44	0.15	0.07	0.31	0.45	0.15	0.06	0.37	0.55	0.19	0.09	0.37
		Bias	−0.02	0.00	0.00	−0.05	−0.02	0.00	0.00	−0.05	−0.03	0.00	0.00	−0.05
		MSE	0.45	0.14	0.10	0.32	0.20	0.02	0.00	0.14	0.55	0.20	0.10	0.37
	25	Mean	0.99	1.00	1.00	1.94	0.99	1.00	1.00	1.95	0.98	1.00	1.00	1.94
		EP	0.37	0.13	0.06	0.28	0.38	0.12	0.05	0.35	0.46	0.16	0.07	0.34
		Bias	−0.01	0.00	0.00	−0.06	−0.01	0.00	0.00	−0.05	−0.02	0.00	0.00	−0.06
		MSE	0.36	0.14	0.00	0.28	0.15	0.02	0.00	0.13	0.46	0.17	0.10	0.35

The inclusion or not of the explanatory variables in the model can be tested using likelihood ratio, Wald and score tests as presented in Li (2001). The likelihood ratio (ξ_{RV}), Wald (ξ_W) and score (ξ_{RV}) statistics for the hypotheses tests, $H_0 : \boldsymbol{\beta} = \boldsymbol{\beta}^0$ versus $H_1 : \boldsymbol{\beta} \neq \boldsymbol{\beta}^0$, for a particular known vector $\boldsymbol{\beta}^0$ of dimension $(s \times 1)$, respectively, are given by

$$\xi_{\text{RV}} = 2\{\ell(\hat{\boldsymbol{\beta}}, \hat{\phi}) - \ell(\boldsymbol{\beta}^0, \hat{\phi}^0)\},$$

$$\xi_W = [\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}^0]^\top \widehat{\text{Var}}(\hat{\boldsymbol{\beta}})^{-1}[\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}^0] \qquad \text{and}$$

$$\xi_{\text{SR}} = \mathbf{U}_{\hat{\boldsymbol{\beta}}^0}^\top \widehat{\text{Var}}_0(\hat{\boldsymbol{\beta}}^0)\mathbf{U}_{\hat{\boldsymbol{\beta}}^0},$$

where $\hat{\boldsymbol{\beta}}$ and $\hat{\phi}$ are the unrestricted MLEs of $\boldsymbol{\beta}$ and ϕ, respectively, $\hat{\phi}^0$ is the restricted MLE of ϕ, $\widehat{\text{Var}}(\hat{\boldsymbol{\beta}}) = \hat{\phi}(\mathbf{D}_\beta^\top \widehat{\mathbf{W}}\mathbf{D}_\beta)^{-1}$ and $\widehat{\text{Var}}_0(\hat{\boldsymbol{\beta}}^0) = \hat{\phi}^0(\mathbf{D}_\beta^\top \widehat{\mathbf{W}}^0\mathbf{D}_\beta)^{-1}$ are respectively, the variance and covariance matrices of $\boldsymbol{\beta}$ under the unrestricted and restricted models where $\widehat{\mathbf{W}} = \mathbf{B} - \frac{\mathbf{v}(m)\mathbf{v}^\top(m)}{4\hat{\phi}^2 d}$. Under standard regularity conditions, Cox and Hinkley (1974, Cap. 9), demonstrated that the statistics ξ_{RV}, ξ_W, and ξ_{SR}, under H_0, follow an asymptotic chi-square distribution with s degrees of freedom.

3.4 Residuals

The residual analysis is part of the modelling that seeks to measure the discrepancy between the observed values y_i and the fitted values $\hat{y}_i$ is statistically significant. In this study, we define a residual for the class of slash-elliptical nonlinear models based on the standardized residual proposed by Pregibon (1981). This residual is based on the statistic

$$\text{LR}_i = 2[\ell(\widetilde{\mu}_i) - \ell(\widehat{\mu}_i)], \tag{3.2}$$

where $\ell_i(\mu_i)$ is the logarithm of the slash-elliptical density evaluated at $\mu_i = \mu_i(\beta, \mathbf{x}_i)$ and $\widetilde{\mu}_i$ is the MLE of μ_i on the saturated model, that is, $\widetilde{\mu}_i = y_i$. Therefore, replacing $\widetilde{\mu}_i = y_i$ in (3.2), we obtain

$$\text{LR}_i = 2I_{(y_i \neq \hat{\mu}_i)}\left\{\log\left(\frac{q}{q+1}g(0)\right) - \log\left(\frac{q}{2}\right) - a(\hat{z}_i)\right\},$$

where $\hat{z}_i = (y_i - \hat{\mu}_i)/\hat{\phi}$ and $I(\cdot)$ is indicator function and the deviance component residual defined by $r_{d_i} = \text{sign}(\hat{z}_i)\sqrt{\text{LR}_i}$.

3.5 Simulation study 2

We conduct a simulation study to evaluate the empirical behavior of the proposed deviance component residual of the slash-elliptical nonlinear model. We use the same setup in Section 3.2, modifying only the scenarios for the error distribution ε_i, and considered to be: slash-normal, slash-logistic II, slash-Student-t ($\nu = 5$)

and slash-CN ($\lambda = 0.3, \sigma = 2$), all four with $q = 5$ and $q = 25$. We generate 10,000 Monte Carlo replicates of the slash-elliptical nonlinear model in (3.1) and evaluate the deviance component residual for each scenario, with the sample size at $n = 20$. Empirical statistical measures, mean ($\bar{r}$), standard deviation (S), skewness (b_1) and kurtosis (b_2) of the simulated residual are obtained and presented in Tables 5 and 6.

Overall, this simulation study proves empirically that the deviance component residual has mean close to zero, standard deviation less than 1.1, slightly asymmetrical and positive kurtosis. We observe that the mean simulated residuals are close to zero within the range of -0.029 to 0.033. Furthermore, the mean and asymmetry coefficient tend to zero when q increases. The slash-normal model has the lowest standard deviation in the range of 0.8 to 0.9, while for the other current models, the standard deviations less than 1.1 and positive kurtosis are slightly asymmetrical.

4 Diagnostics analysis

The purpose of the diagnostic analysis is to detect observations that exert a disproportionate influence on the modelling. The analysis of local influence is a diagnostic method that aims to evaluate the effect of small perturbations in the data or model based on deviation measures, for example, likelihood displacement. In this study, we consider the likelihood displacement proposed by Cook (1986). Let $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_n)^\top$ a perturbation vector ($n \times 1$) restricted to some open set Ω. The local influence measures aimed at comparing $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\theta}}_\omega$ based on an influence measure when $\boldsymbol{\omega}$ varies on Ω. We also consider a generalized leverage measure proposed by Wei, Hu and Fung (1998), which identifies observations with high influence on their predicted values.

4.1 Local influence

Consider the likelihood displacement $\mathrm{LD}(\boldsymbol{\omega}) = 2\{\ell(\hat{\boldsymbol{\theta}}) - \ell(\hat{\boldsymbol{\theta}}_\omega)\}$, where $\ell(\boldsymbol{\theta})$ and $\ell(\boldsymbol{\theta}_\omega)$ are log-likelihood functions of the postulate and perturbed models, respectively. If $\boldsymbol{\omega}_0$ is the nonperturbation vector, then $\mathrm{LD}(\boldsymbol{\omega}_0) = 0$ and $\mathrm{LD}(\boldsymbol{\omega}) \geq 0$ for $\boldsymbol{\omega} \neq \boldsymbol{\omega}_0$, then $\boldsymbol{\omega}_0$ is a local minimum point of $\mathrm{LD}(\boldsymbol{\omega})$.

The normal curvature of the surface $(\omega^\top, \mathrm{LD}(\omega))^\top$ in the direction of a vector $\mathbf{d}$ is equal to $C_{\mathbf{d}}(\boldsymbol{\theta}) = 2|\mathbf{d}^\top \ddot{\mathbf{F}} \mathbf{d}|$ (Cook, 1986), where

$$\ddot{\mathbf{F}} = \frac{\partial^2 \mathrm{LD}(\boldsymbol{\omega})}{\partial \boldsymbol{\omega}\, \partial \boldsymbol{\omega}^\top} = \boldsymbol{\Delta}^\top (-\ddot{\mathbf{L}}_{\theta\theta})^{-1} \boldsymbol{\Delta}$$

is the local influence matrix and $\Delta = \partial^2 \ell(\boldsymbol{\theta}_\omega)/\partial \boldsymbol{\theta}\, \partial \boldsymbol{\omega}^\top$ is evaluated at $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$ and $\boldsymbol{\omega} = \boldsymbol{\omega}_0$. One can also obtain the normal curvature of $\mathrm{LD}(\boldsymbol{\omega})$ for the parameters $\boldsymbol{\beta}$ and ϕ separately through the following expressions: $C_{\mathbf{d}}(\boldsymbol{\beta}) = 2|\mathbf{d}^\top \boldsymbol{\Delta}^\top (\ddot{\mathbf{L}}_{\theta\theta}^{-1} -$

Table 5 *Empirical statistical measures of the proposed residuals for some slash-eliptical models with $q = 5$*

	Slash-normal				Slash-logistic II				Slash-Student-t ($\nu = 5$)				Slash-CN ($\lambda = 0.3, \sigma = 2$)			
Obs.	$\bar{r}$	S	b_1	b_2	$\bar{r}$	S	b_1	b_2	$\bar{r}$	S	b_1	b_2	$\bar{r}$	S	b_1	b_2
1	−0.007	0.9	−0.004	0.5	−0.014	1.0	0.058	0.2	0.004	1.1	0.005	0.2	−0.012	1.1	0.001	0.3
2	0.007	0.9	0.023	0.6	0.008	1.0	0.007	0.3	−0.007	1.1	0.049	0.4	−0.029	1.0	−0.006	0.4
3	0.001	0.9	0.086	0.7	−0.023	1.0	0.002	0.2	−0.019	1.1	0.017	0.3	−0.004	1.0	−0.023	0.4
4	0.008	0.9	−0.012	0.5	−0.017	1.0	−0.052	0.2	0.018	1.1	0.021	0.5	−0.013	1.0	0.041	0.1
5	−0.002	0.9	−0.017	0.7	0.022	1.0	−0.015	0.4	0.012	1.0	−0.032	0.6	0.033	1.0	−0.016	0.6
6	0.010	0.9	−0.006	0.8	−0.003	1.0	−0.027	0.4	0.019	1.0	0.017	0.6	0.015	1.0	−0.019	0.6
7	0.009	0.8	−0.040	0.8	0.018	1.0	−0.028	0.8	0.007	1.0	0.007	0.7	−0.001	1.0	−0.063	0.7
8	−0.006	0.9	0.017	0.7	−0.010	1.0	0.054	0.2	−0.028	1.1	0.035	0.4	−0.013	1.0	0.018	0.4
9	−0.010	0.9	0.091	0.6	−0.024	1.0	0.044	0.2	−0.012	1.0	−0.013	0.6	−0.015	1.0	−0.021	0.5
10	0.009	0.9	0.052	1.0	0.031	1.0	−0.077	0.5	0.009	1.0	−0.026	0.6	0.031	1.0	−0.031	0.7
11	0.004	0.8	−0.048	0.6	0.031	1.0	0.073	0.4	0.015	1.0	−0.054	0.7	0.026	1.0	−0.020	0.7
12	−0.006	0.9	0.114	0.6	−0.011	1.0	0.033	0.4	−0.010	1.1	0.007	0.3	−0.003	1.0	−0.001	0.3
13	0.018	0.9	0.099	0.7	0.021	1.0	0.028	0.3	0.000	1.0	0.005	0.5	0.010	1.0	0.013	0.5
14	0.004	0.9	0.030	0.7	−0.014	1.0	−0.012	0.5	0.013	1.0	−0.054	0.4	0.001	1.0	−0.005	0.5
15	−0.002	0.9	0.002	0.6	−0.014	1.0	−0.045	0.3	0.000	1.1	−0.032	0.4	−0.014	1.1	−0.044	0.5
16	−0.010	0.9	−0.013	0.6	0.001	1.0	0.032	0.2	−0.002	1.1	0.009	0.3	−0.028	1.0	0.039	0.4
17	0.007	0.9	−0.094	0.9	0.001	1.0	−0.006	0.4	−0.004	1.0	0.001	0.5	0.006	1.0	0.024	0.6
18	−0.009	0.9	−0.017	0.5	0.005	1.0	−0.042	0.3	−0.010	1.1	0.029	0.5	−0.005	1.0	−0.036	0.5
19	0.003	0.9	0.008	0.8	0.016	1.0	0.014	0.4	0.007	1.1	0.018	0.3	0.022	1.0	0.009	0.4
20	−0.015	0.9	0.027	0.6	−0.024	1.0	−0.033	0.3	0.002	1.1	0.047	0.3	−0.005	1.0	0.067	0.3

Table 6 *Empirical statistical measures of the proposed residuals for some slash-eliptical models with $q = 25$*

	Slash-normal				Slash-logistic II				Slash-Student-t ($\nu = 5$)				Slash-CN ($\lambda = 0.3, \sigma = 2$)			
Obs.	$\bar{r}$	S	b_1	b_2	$\bar{r}$	S	b_1	b_2	$\bar{r}$	S	b_1	b_2	$\bar{r}$	S	b_1	b_2
1	−0.006	0.9	−0.036	0.1	−0.009	1.0	0.069	0.1	0.007	1.0	0.009	0.2	−0.009	1.0	0.004	0.2
2	0.004	0.9	−0.014	0.1	0.010	1.0	0.021	0.1	−0.005	1.0	0.057	0.5	−0.029	1.0	−0.015	0.4
3	0.001	0.9	0.045	0.0	−0.017	1.0	0.004	0.2	−0.016	1.0	0.025	0.3	0.000	1.0	−0.003	0.3
4	0.012	0.9	0.001	0.0	−0.016	1.0	−0.051	0.2	0.017	1.0	0.038	0.6	−0.009	1.0	0.045	0.2
5	−0.005	0.8	−0.014	0.1	0.018	0.9	0.002	0.3	0.010	1.0	−0.032	0.6	0.028	1.0	−0.039	0.6
6	0.005	0.8	−0.022	0.1	−0.005	0.9	−0.016	0.3	0.016	1.0	0.032	0.5	0.010	1.0	−0.039	0.5
7	0.005	0.7	−0.027	0.2	0.014	0.9	−0.039	0.5	0.006	1.0	0.009	0.7	−0.003	0.9	−0.064	0.7
8	−0.006	0.9	0.021	0.2	−0.010	1.0	0.054	0.2	−0.025	1.0	0.028	0.5	−0.010	1.0	0.016	0.3
9	−0.006	0.9	0.049	0.1	−0.020	1.0	−0.007	0.2	−0.010	1.0	−0.038	0.7	−0.012	1.0	−0.011	0.6
10	0.005	0.8	0.061	0.1	0.023	0.9	−0.072	0.3	0.006	1.0	−0.049	0.6	0.027	0.9	−0.004	0.6
11	0.003	0.8	−0.006	0.0	0.025	0.9	0.058	0.3	0.016	1.0	−0.008	0.7	0.022	1.0	−0.010	0.6
12	−0.006	0.8	0.066	0.0	−0.010	1.0	0.013	0.2	−0.007	1.0	0.003	0.3	−0.002	1.0	−0.002	0.3
13	0.012	0.8	0.040	0.0	0.021	1.0	0.049	0.2	0.000	1.0	0.023	0.5	0.007	1.0	−0.004	0.5
14	0.004	0.8	0.037	0.1	−0.014	0.9	−0.004	0.3	0.010	1.0	−0.050	0.4	−0.001	1.0	−0.020	0.4
15	−0.003	0.8	−0.015	0.2	−0.011	1.0	−0.032	0.2	−0.001	1.0	−0.032	0.5	−0.015	1.0	−0.048	0.3
16	−0.007	0.9	0.005	0.1	0.003	1.0	0.024	0.2	0.001	1.0	0.002	0.3	−0.024	1.0	0.019	0.3
17	0.007	0.8	−0.061	0.1	0.001	1.0	−0.002	0.3	−0.005	1.0	−0.006	0.5	0.006	1.0	0.026	0.5
18	−0.007	0.8	−0.005	0.1	0.005	1.0	−0.044	0.2	−0.009	1.0	0.044	0.5	−0.004	1.0	−0.028	0.4
19	0.002	0.8	0.036	0.1	0.016	0.9	0.018	0.3	0.005	1.0	0.006	0.4	0.018	1.0	0.008	0.3
20	−0.012	0.9	0.045	0.1	−0.019	1.0	−0.027	0.2	0.002	1.0	0.046	0.3	−0.004	1.0	0.065	0.2

$\mathbf{L}_1)\boldsymbol{\Delta}\mathbf{d}|$ and $C_{\mathbf{d}}(\phi) = 2|\mathbf{d}^\top\boldsymbol{\Delta}^\top(\ddot{\mathbf{L}}_{\theta\theta}^{-1} - \mathbf{L}_2)\boldsymbol{\Delta}\mathbf{d}|$, respectively, where $\mathbf{L}_1 = \begin{bmatrix} 0 & 0 \\ 0 & \ddot{\mathbf{L}}_{\phi\phi}^{-1} \end{bmatrix}$ and $\mathbf{L}_2 = \begin{bmatrix} \ddot{\mathbf{L}}_{\beta\beta}^{-1} & 0 \\ 0 & 0 \end{bmatrix}$. The index plot of the vector $\mathbf{d}_{\max}$, which is defined as the eigenvector corresponding to higher absolute eigenvalue of $\ddot{\mathbf{F}}$, can reveal the most influential observations in $\hat{\boldsymbol{\theta}}$. Another approach for diagnostic analysis was proposed by Lesaffre and Verbeke (1998), which is based on index plot of $C_i = C_{d_i}$, where d_i is a vector formed by zeros with one in the ith position for each $i = 1, 2, \ldots, n$, taking as atypical observations those which are superior to $2\overline{C}$ with $\overline{C} = \sum_{i=1}^n C_i/n$.

4.2 Curvature calculation

The matrix $\boldsymbol{\Delta} = (\boldsymbol{\Delta}_\beta^\top, \Delta_\phi)^\top$, where $\boldsymbol{\Delta}_\beta = \partial^2\ell(\boldsymbol{\theta}_\omega)/\partial\boldsymbol{\beta}\,\partial\boldsymbol{\omega}^\top$ and $\Delta_\phi = \partial^2\ell(\boldsymbol{\theta}_\omega)/\partial\phi\,\partial\boldsymbol{\omega}^\top$, defines the curvature $C_{\mathbf{d}}(\boldsymbol{\theta}) = 2|\mathbf{d}^\top\ddot{\mathbf{F}}\mathbf{d}|$ of the local influence measure on likelihood displacement. Hereafter, we define the matrix $\boldsymbol{\Delta}$ for the slash-elliptical nonlinear model according to the perturbation schemes: cases-weight, scale and response.

Cases-weight perturbation. The log-likelihood function under perturbed model is given by ω_i in the ith case, that is,

$$\ell(\boldsymbol{\theta}_\omega) = \sum_{i=1}^n \omega_i \log[f_{y_i}(y_i)],$$

for $0 \le \omega_i \le 1$. According to this perturbation scheme, the matrix $\boldsymbol{\Delta}$ is estimated by

$$\hat{\boldsymbol{\Delta}} = \begin{pmatrix} -\dfrac{1}{\sqrt{\hat{\phi}}}\mathbf{D}_{\hat{\beta}}^\top\mathbf{D}(\hat{a}') \\ -\dfrac{1}{2\hat{\phi}}\mathbf{v}^\top(\hat{o}) \end{pmatrix},$$

where $\mathbf{D}(\hat{a}') = \mathrm{diag}(a'(\hat{z}_1), \ldots, a'(\hat{z}_n))$, $o(\hat{z}_i) = 1 + \hat{z}_i a'(\hat{z}_i)$ and $\mathbf{v}^\top(\hat{o}) = (o(\hat{z}_1), \ldots, o(\hat{z}_n))$.

Scale perturbation. In scale perturbation scheme where the scale parameter $\phi_{\omega i} = \phi/\omega_i$ for $\omega_i > 0$ and $i = 1, 2, \ldots, n$, the log-likelihood function under perturbed model is given by

$$\ell(\boldsymbol{\theta}_\omega) = -\frac{n}{2}\log\phi - \frac{1}{2}\sum_{i=1}^n \log\omega_i + \sum_{i\in A} a(z_{i\omega}) + K_q,$$

where $z_{i\omega} = \sqrt{\omega_i}(y_i - \mu_i)/\sqrt{\phi} = \sqrt{\omega_i}z_i$. Note that, when $\omega_i = 1$, $\phi_i = \phi$ and the perturbed model reduces to the postulate one. For $0 < \omega_i < 1$, inflating occurs in ϕ, and when $\omega_i > 1$, there is a reduction of ϕ. In this case, the matrix $\boldsymbol{\Delta}$ to scale perturbation is estimated by

$$\hat{\Delta} = \begin{pmatrix} -\dfrac{1}{2\sqrt{\hat{\phi}}}\mathbf{D}_{\hat{\beta}}^\top \mathbf{D}(\hat{m}) \\ -\dfrac{1}{4\hat{\phi}}\hat{\mathbf{z}}^\top \mathbf{D}(\hat{m}) \end{pmatrix},$$

where the $\mathbf{D}(\hat{m}) = \mathrm{diag}(m(\hat{z}_1), \dots, m(\hat{z}_n))$.

Response perturbation. Finally, let us consider an additive perturbation in response variable expressed as $y_{i\omega} = y_i + \omega_i$, so as that when $\omega_i = 0$, the perturbed model is equal to postulate model. The log-likelihood function for the perturbed model is given by

$$\ell(\boldsymbol{\theta}, \omega) = -\frac{n}{2}\log(\phi) + \sum_{i \in A} a(z_{i\omega}) + K_q,$$

where $z_{i\omega} = (y_i + \omega_i - \mathbf{x}_i^\top \boldsymbol{\beta})/\sqrt{\phi} = z_i + \omega_i/\sqrt{\phi}$. The matrix $\boldsymbol{\Delta}$ for the response perturbation is estimated by

$$\hat{\boldsymbol{\Delta}} = \begin{pmatrix} \dfrac{1}{\hat{\phi}}\mathbf{D}_{\hat{\beta}}^\top \mathbf{D}(\hat{c}) \\ -\dfrac{1}{2\hat{\phi}^{3/2}}\mathbf{v}^\top(\hat{m}) \end{pmatrix}.$$

4.3 Generalized leverage

The leverage is a measure that allows to verify the influence of the response variable y_i on its own predicted value $\hat{y}_i$. In linear regression model, the leverage is defined by $\partial \hat{y}_i/\partial y_i$, that is, the leverage reflects the instantaneous rate of change of the predicted value $\hat{y}_i$ when the response variable y_i is increased by an infinitesimal value. Furthermore, the ith leverage coincides with the ith element of the diagonal of the projection matrix $\mathbf{H} = \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top$.

A general expression for $\partial \hat{y}_i/\partial y_i$ has been proposed by Wei, Hu and Fung (1998) for the case where the log-likelihood function $\ell(\boldsymbol{\theta}, \mathbf{y})$ has continuous second-order derivatives with respect to $\boldsymbol{\theta}$ and $\mathbf{y}$, and the MLE of $\boldsymbol{\theta}$ exists and is unique. The generalized leverage matrix is defined by $\mathbf{GL}(\theta) = \mathbf{D}_\theta(-\ddot{\mathbf{L}}_{\theta\theta})^{-1}\ddot{\mathbf{L}}_{\theta y}$, where $\mathbf{D}_\theta = \frac{\partial \hat{\boldsymbol{\mu}}}{\partial \boldsymbol{\theta}^\top}|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}(\mathbf{y})}$ is a matrix $(n \times (p+1))$ and $\ddot{\mathbf{L}}_{\theta y} = \frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}\, \partial \mathbf{y}^\top}|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}(\mathbf{y})}$ is a matrix $((p+1) \times n)$. For the class of slash-elliptic nonlinear models, we have $\mathbf{D}_\theta = [\mathbf{D}_\beta, 0_{n\times 1}]$,

$$(-\ddot{\mathbf{L}}_{\theta\theta})^{-1} = \begin{bmatrix} \phi(\mathbf{D}_\beta^\top \mathbf{W}\mathbf{D}_\beta)^{-1} & \dfrac{(\mathbf{D}_\beta^\top \mathbf{W}\mathbf{D}_\beta)^{-1}\mathbf{D}_\beta^\top \mathbf{v}(m)}{2\sqrt{\phi}d} \\ \dfrac{\mathbf{v}(m)^\top \mathbf{D}_\beta(\mathbf{D}_\beta^\top \mathbf{W}\mathbf{D}_\beta)^{-1}}{2\sqrt{\phi}d} & \dfrac{1}{d} + \dfrac{\mathbf{v}(m)^\top \mathbf{D}_\beta(\mathbf{D}_\beta^\top \mathbf{W}\mathbf{D}_\beta)^{-1}\mathbf{D}_\beta^\top \mathbf{v}(m)}{4\phi^2 d^2} \end{bmatrix}$$

and $\ddot{\mathbf{L}}_{\theta y} = \begin{bmatrix} \phi^{-1}\mathbf{D}_\beta^\top \mathbf{D}(c) \\ -\frac{1}{2\phi^{3/2}}\mathbf{v}(m)^\top \end{bmatrix}$. Therefore, the generalized leverage matrix of the slash-elliptical nonlinear model is defined by

$$\mathbf{GL}(\hat{\boldsymbol{\theta}}) = \mathbf{D}_{\hat{\beta}}(\mathbf{D}_{\hat{\beta}}^\top \hat{\mathbf{W}}\mathbf{D}_{\hat{\beta}})^{-1}\mathbf{D}_{\hat{\beta}}^\top \hat{\mathbf{W}}^*,$$

where $\hat{\mathbf{W}}^* = (\mathbf{D}(c) - \mathbf{v}(m)\mathbf{v}^\top(m)/4\phi^2 d)$ is evaluated at $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$. The matrix $\mathbf{GL}(\hat{\boldsymbol{\theta}})$ is not symmetrical and neither idempotent. Suppose ϕ is given, we can write a relationship between the generalized leverage matrix with the local influence matrix on likelihood displacement $\ddot{\mathbf{F}} = 2\boldsymbol{\Delta}^\top(-\ddot{\mathbf{L}}_{\beta\beta})^{-1}\boldsymbol{\Delta}$, for response additive perturbation scheme $y_{i\omega} = y_i + \omega_i$. We obtain

$$\begin{aligned}
\ddot{\mathbf{F}}|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}} &= 2\hat{\boldsymbol{\Delta}}^\top(-\ddot{\mathbf{L}}_{\hat{\beta}\hat{\beta}})^{-1}\hat{\boldsymbol{\Delta}} \\
&= 2\hat{\boldsymbol{\Delta}}^\top(\mathbf{D}_{\hat{\beta}}^\top\mathbf{D}_{\hat{\beta}})^{-1}\mathbf{D}_{\hat{\beta}}^\top[\mathbf{D}_{\hat{\beta}}(-\ddot{\mathbf{L}}_{\hat{\beta}\hat{\beta}})^{-1}\hat{\boldsymbol{\Delta}}] \\
&= \frac{2}{\phi}\mathbf{D}(\hat{c})\mathbf{D}_{\hat{\beta}}(\mathbf{D}_{\hat{\beta}}^\top\mathbf{D}_{\hat{\beta}})^{-1}\mathbf{D}_{\hat{\beta}}^\top\mathbf{D}_{\hat{\beta}}(\mathbf{D}_{\hat{\beta}}^\top\hat{\mathbf{W}}\mathbf{D}_{\hat{\beta}})^{-1}\mathbf{D}_{\hat{\beta}}^\top\mathbf{W}^* \\
&= \frac{2}{\phi}\mathbf{D}(c)\mathbf{D}_{\hat{\beta}}(\mathbf{D}_{\hat{\beta}}^\top\mathbf{D}_{\hat{\beta}})^{-1}\mathbf{D}_{\hat{\beta}}^\top\,\mathrm{GL}(\hat{\boldsymbol{\theta}}).
\end{aligned}$$

Consequently, the total local influence C_i reduces to the simple form $2\phi^{-1}c_i\tilde{h}_{ii} \times \mathbf{GL}_{ii}(\hat{\boldsymbol{\theta}})$, if $c_i\tilde{h}_{ii}/\phi$ and (or) $\mathbf{GL}_{ii}(\hat{\boldsymbol{\theta}})$ where $\tilde{h}_{ii}$ is ith element of the diagonal $\tilde{\mathbf{H}} = \mathbf{D}_\beta(\mathbf{D}_\beta^\top\mathbf{D}_\beta)^{-1}\mathbf{D}_\beta^\top$ is an orthogonal projection matrix on subspace spanned by the columns of $\mathbf{D}_{\hat{\beta}}$.

5 Application

We apply the proposed methods to the data set due to Dudzinski and Mykytowycz (1961), where the dry weight of the eye lens w (in mg) was measured for 71 free-living wild rabbits with age x (in days). The ecological motivation of the study of Dudzinski and Mykytowycz (1961) was to propose a method to determine the age of European rabbits (*Oryctolagus cuniculus*) caught in the wild from the weight of the eye lens. Eye lens weight tends to vary less with environmental conditions than does total body weight, and therefore may be a more accurate indicator of the age. The nonlinear model fitted by Dudzinski and Mykytowycz (1961) to these data was

$$y_i = \log(w_i) = \alpha - \frac{\kappa}{x_i + \psi} + \varepsilon_i, \qquad i = 1, 2, \ldots, 71, \tag{5.1}$$

where ε_i is the random error and $\boldsymbol{\beta} = (\alpha, \kappa, \psi)^\top$ is the unknown parameter vector. Based on the least square estimates, they concluded that the weight of the eye lens is a reliable indicator of the age of the rabbit for the first 150 days of the

Table 7 *MLEs (approximated SE) for the parameters of the fitted slash-elliptical models*

Model	α	κ	ψ	ϕ	BIC	AIC
Normal	5.64	130.58	37.60	0.004	−148.43	−157.48
	(0.02)	(5.60)	(2.27)	(0.0006)		
Logistic II	5.63	127.26	35.86	0.001	−180.99	−190.05
	(0.02)	(4.99)	(2.01)	(0.0002)		
Student-t ($\nu = 4$)	5.63	126.28	35.29	0.002	−182.20	−191.25
	(0.02)	(4.65)	(1.87)	(0.0004)		
CN ($\lambda = 0.2, \sigma = 2$)	5.63	125.86	35.20	0.002	−182.44	−191.49
	(0.02)	(4.61)	(1.86)	(0.0004)		
Slash-normal ($q = 3$)	5.63	125.77	35.16	0.0014	−181.56	−190.61
	(0.02)	(4.65)	(1.89)	(0.0003)		
Slash-logistic II ($q = 3$)	5.63	126.29	35.26	0.0005	−182.14	−191.19
	(0.02)	(4.48)	(1.77)	(0.0001)		
Slash-Student-t ($q = 7, \nu = 4$)	5.63	126.31	35.28	0.0015	−182.33	−191.38
	(0.02)	(4.49)	(1.78)	(0.0003)		
Slash-CN ($q = 5, \lambda = 0.2, \sigma = 2$)	5.63	125.92	35.19	0.0014	−182.54	−191.59
	(0.02)	(4.58)	(1.84)	(0.0003)		

animal. This data set has been used to illustrate various types of nonlinear models, such as the nonlinear regression model (Ratkowsky, 1983), the exponential family nonlinear model (Wei, 1998) and the symmetrical nonlinear model (Galea, Paula and Cysneiros, 2005, Vanegas and Cysneiros, 2010, Cao, Lin and Zhu, 2010). The statistical motivation of these data is the suspicion that the observations 4, 5, 16 and 17 are the influential points in the nonlinear model with symmetrical errors (Galea, Paula and Cysneiros, 2005).

In this study, we propose for the error of the model defined in (5.1) some distributions belonging to the slash-elliptical class. Based on the BIC criterion to select the extra parameters as discussed in Section 3.3, we choose the models: slash-normal ($q = 3$), slash-logistic II ($q = 3$), slash-Student-t ($q = 7$, $\nu = 4$) and slash-CN ($q = 5$, $\lambda = 0.2$, $\sigma = 2$). Statistical measures for the selected models are given in Table 7.

For all cases, we note that the MLEs in the slash-elliptical and symmetrical models are similar. However, the asymptotic standard error estimates of the slash-elliptical models are always smaller. Furthermore, the estimates of the slash-Student-t ($q = 7$, $\nu = 4$) model have the lowest estimated asymptotic standard errors. All parameters are significant for the proposed models with p-values close to zero in all cases.

Figure 2 displays the fitted curve plot to the data set of the age and eye lens weight of European rabbits in Australia according to the slash-CN model. The influential points are highlighted in this figure.

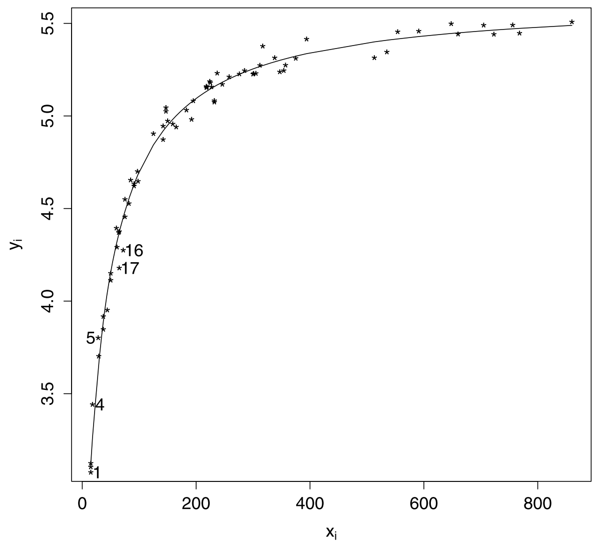


Figure 2 *Fitted curve plot of the slash-CN* ($q = 5$, $\lambda = 0.2$, $\sigma = 2$).

5.1 The deviance component residual

We conduct a residual and diagnostic analysis of this data set in order to illustrate the proposed methods for the class of slash-elliptical nonlinear models. The index plot of the deviance component residual is given in Figure 3, where we can verify that the residual had a random behaviour and between -3.1 and 3. The points ♯4, ♯5, ♯16 and ♯17 are the most distant from zero in the slash-elliptical models, as they occur in symmetrical models (see Figure 1 in Galea, Paula and Cysneiros (2005)). For investigating possible violations of the homoscedastic assumptions, the plots of the fitted values versus the deviance component residuals are displayed in Figure 3. For all plots, we note that there is no systematic behaviour of the residuals compared to the fitted values, indicating that the assumption of constant variance is reasonable.

5.2 The generalized leverage and the local influence under response perturbation

In Section 3.3, we show analytically that the local influence measure with response perturbation is proportional to the generalized leverage measure. This result is showed for the European rabbits data set, when comparing the index plots: generalized leverage (Figure 3) and local influence under response perturbation (Figure 4). The observations highlighted, ♯1, ♯2 and ♯3, are considered as high leverage for the slash-normal and slash-CN models. For the slash-logistic II model, we highlight the observations ♯2 and ♯3. For the slash-Student-t model, the other observations are highlighted. However, as noted in Table 8, the slash-Student-t model is not as sensitive to the observations ♯4, ♯5, ♯16 and ♯17.

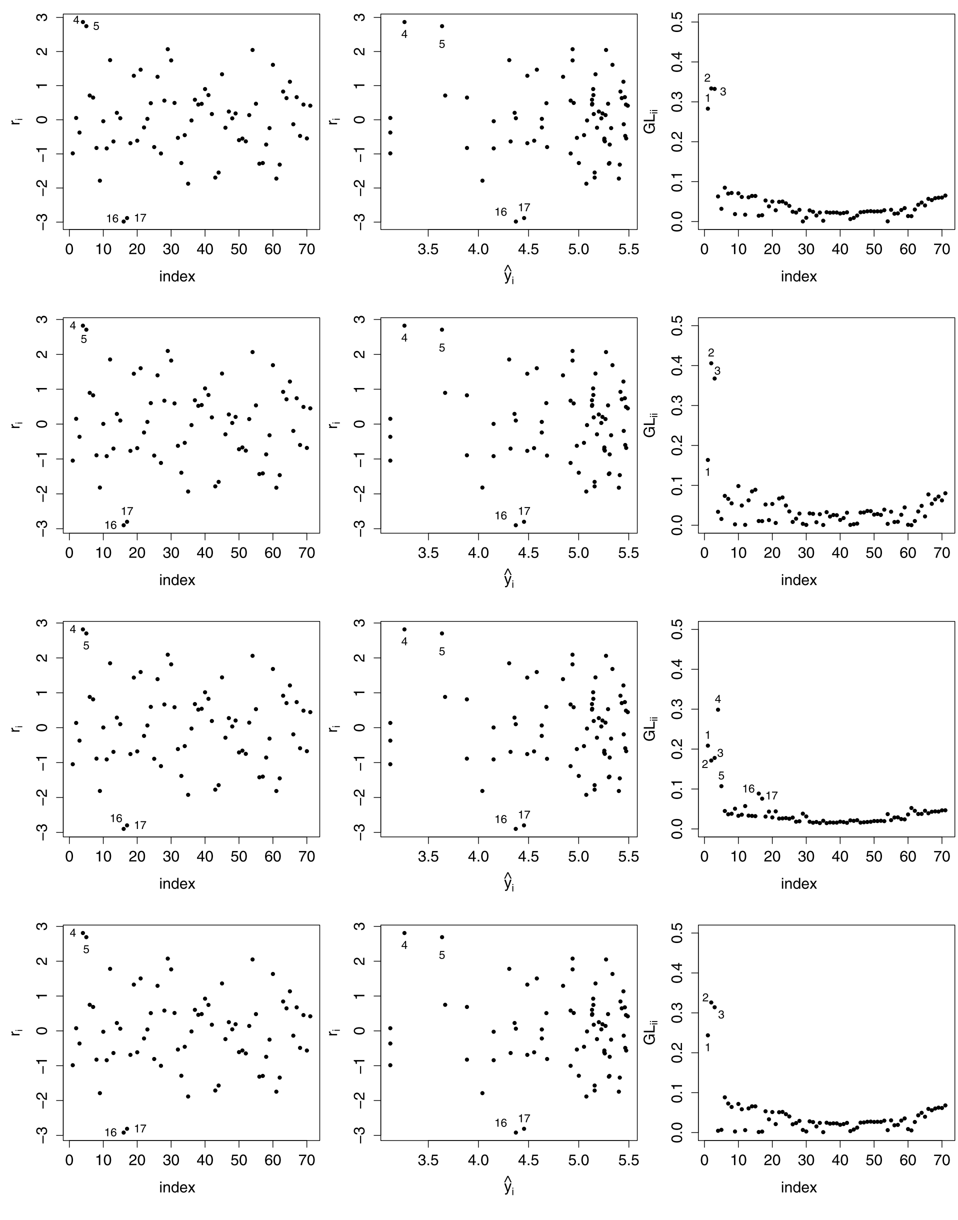


Figure 3 *Index plot deviance component residual (first column), deviance component residual versus fitted values (second column) and generalized leverage (third column) of the distributions: slash-normal* ($q = 3$) *(first row), slash-logistic II* ($q = 3$) *(second row), slash-Student-t* ($q = 7$, $\nu = 4$) *(third row) and slash-CN* ($q = 5$, $\lambda = 0.2$, $\sigma = 2$) *(fourth row).*

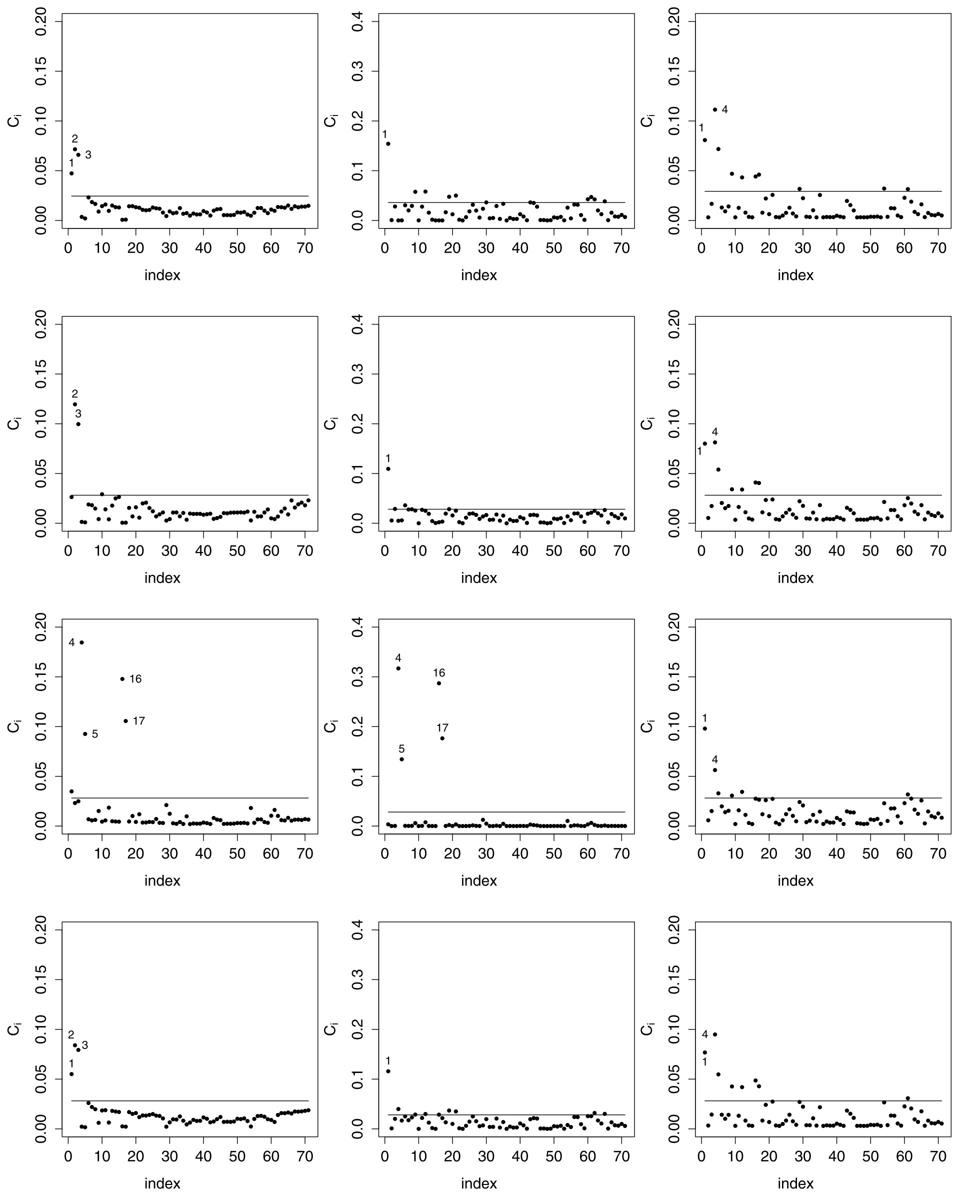


Figure 4 *Index plot C_i under response perturbation (first column), under scale perturbation (second column) and under case-weight perturbation (third column) of the distributions: slash-normal ($q = 3$) (first row), slash-logistic II ($q = 3$) (second row), slash-Student-t ($q = 7$, $\nu = 4$) (third row) and slash-CN ($q = 5$, $\lambda = 0.2$, $\sigma = 2$) (fourth row).*

Table 8 *Sensitivity analysis when dropout the influential points*

Models	Removed points	α	κ	ψ	ϕ	SE($\hat{\alpha}$)	SE($\hat{\kappa}$)	SE($\hat{\psi}$)	SE($\hat{\phi}$)
Slash-normal	4, 5, 16, 17	−0.07	−1.55	−3.09	−24.29	−11.68	−14.22	−16.26	−28.53
Slash-logistic II	4, 5, 16, 17	−0.04	−1.16	−2.34	−25.82	−9.88	−11.84	−13.82	−27.60
Slash-Student-t	4, 5, 16, 17	−0.05	−1.20	−2.43	−26.48	−13.45	−10.46	−8.40	−1.20
Slash-CN	4, 5, 16, 17	−0.06	−1.49	−2.93	−26.32	−11.16	−12.99	−14.70	−22.75
Slash-normal	1	0.06	1.28	2.88	1.92	3.49	7.98	15.11	3.30
Slash-logistic II	1	0.05	1.02	2.30	0.83	2.09	5.59	11.43	1.56
Slash-Student-t	1	0.05	1.04	2.33	0.70	2.65	5.78	9.05	4.30
Slash-CN	1	0.05	1.14	2.54	0.48	2.40	6.13	11.95	2.34
Slash-normal	1, 4	0.02	0.26	0.37	−6.66	−1.70	−0.12	2.75	−7.79
Slash-logistic II	1, 4	0.02	0.41	0.70	−7.08	−1.81	−0.71	1.19	−6.80
Slash-Student-t	1, 4	0.02	0.40	0.67	−7.19	−0.96	3.88	12.46	−6.79
Slash-CN	1, 4	0.02	0.34	0.52	−6.49	−1.32	0.61	4.14	−4.97

5.3 The local influence under scale perturbation

In Figure 4, we display the index plot C_i under scale perturbation according to the current models. We observe that only the observation $\sharp 1$ appears as a possible influential point for slash-normal, slash-logistic II and slash-CN models and observations $\sharp 4$, $\sharp 5$, $\sharp 16$ and $\sharp 17$ for slash-Student-t model.

5.4 The local influence under cases-weight perturbation

The index plot C_i under cases-weight perturbation are given in Figure 4, in which we note that the observations $\sharp 1$ and $\sharp 4$ are highlighted in the four models used for European rabbits data set.

5.5 Sensitivity analysis

In order to verify if the observations $\sharp 4$, $\sharp 5$, $\sharp 16$ and $\sharp 17$ are outliers for slash-elliptical models, we conduct a sensitivity analysis of these observations. We remove these observations and refit the models, and then we evaluate the rates of change of the estimates of the parameters and their respective asymptotic standard errors estimated by

$$\mathrm{RC}(\tau) = \left(\frac{\tau^{(j)} - \tau}{\tau}\right) \times 100,$$

where τ and $\tau^{(j)}$ are the estimates of the parameters of the models with all observations and with excluded points, respectively, and $\tau = \alpha, \kappa, \psi$, or ϕ. The rates of change in the estimates after removing the points $\sharp 4$, $\sharp 5$, $\sharp 16$ and $\sharp 17$ are listed in Table 8 (values in %).

We observe that there is a small reduction in the estimates of the parameters α, κ, ψ and ϕ, as well as their estimated asymptotic standard errors. This analysis was also done by Galea, Paula and Cysneiros (2005) for the symmetrical models. In comparative terms, the rates of change of the estimates in slash-elliptical models are smaller than those of the corresponding symmetrical models. Thus, we conclude that the points $\sharp 4$, $\sharp 5$, $\sharp 16$ and $\sharp 17$ are not considered as influence points under the fitted slash-elliptical model.

We also evaluate the rate of changes on the estimates when the observation $\sharp 1$ is removed. The results are displayed in Table 8 (values in %). We observe a small increase in the parameter and asymptotic standard errors estimates after removing the observation $\sharp 1$, indicating that this observation is not an influential point on location and the scale parameters.

In order to verify if the observations $\sharp 1$ and $\sharp 4$ are influencing the model with respect of cases-weight perturbations, we remove these observations and estimated the models again, obtaining the results given in Table 8 (values in %). Note that removing these two observations, we obtain small variation in these rates, less than 12%, thus indicating that the observations $\sharp 1$ and $\sharp 4$ are not influential as cases-weight perturbation.

6 Concluding remarks

In this paper, we propose a methodology for estimation, inference, and diagnostic analysis for the class of nonlinear models with slash-elliptical error distribution. We develop an inferential methodology based on asymptotic standard error estimators obtained from the observed information matrix. We suggest the AIC and BIC criteria for choosing the extra parameters, when these are not known. We present the asymptotic tests (likelihood ratio, Wald and score) that can be used to test for inclusion and exclusion of explanatory variables in the slash-elliptical nonlinear model.

We developed deviance component residual for the proposed regression model and conduct a simulation study to evaluate the behaviour of the residuals. We conclude that the deviance component residual has good statistical properties. In particular, the mean and standard deviation are close to zero and one, respectively, and a small asymmetry.

We also develop generalized leverage and local influence measures for slash-elliptical nonlinear models, considering perturbation scheme in the response, scale and cases-weight. We show that there is a relationship between the local influence matrix under the response perturbation and the generalized leverage matrix. The data analysis on the age and eye lens weight of European rabbits in Australia reveals that for the slash-elliptical models the standard error are smaller than those for the estimates of the symmetrical models. The observations $\sharp 4$, $\sharp 5$, $\sharp 16$ and $\sharp 17$ are considered as influential observations in the symmetrical models Galea, Paula and Cysneiros (2005) but in slash-elliptical models, these observations are not confirmed as an influential observations.

Acknowledgments

This study was partially supported by CNPq and Capes, Brazil. The authors thank the Editor, an anonymous Associate Editor and two anonymous referees for their constructive comments on an earlier version of this manuscript, which provides an improved version.

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