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On the powers of polynomial logistic distributions

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Abstract. Let $P(x)$ be a polynomial monotone increasing on $(-\infty, +\infty)$. The probability distribution possessing the distribution function

$$F(x) = \frac{1}{1 + \exp\{-P(x)\}}$$

is called the *polynomial logistic distribution* associated with polynomial P and denoted by $\mathrm{PL}(P)$. It has recently been introduced, as a generalization of the logistic distribution, by V. M. Koutras, K. Drakos, and M. V. Koutras who have also demonstrated the importance of this distribution in modeling financial data. In the present paper, for a random variable $X \sim \mathrm{PL}(P)$, the analytical properties of its characteristic function are examined, the moment-(in)determinacy for the powers X^m, $m \in \mathbb{N}$ and $|X|^p$, $p \in (0, +\infty)$ depending on the values of m and p is investigated, and exemplary Stieltjes classes for the moment-indeterminate powers of X are constructed.

1 Introduction

This paper deals with a new class of probability distributions called 'polynomial logistic'. Polynomial logistic distribution has been introduced in Koutras, Drakos and Koutras (2014). It generalizes the classical logistic distribution, which has a wide spectrum of applications in social sciences, economy, biology, agriculture, and finance as well as other disciplines, see Balakrishnan (1992). Due to its high degree of importance, logistic distribution has a considerable number of various generalizations aiming to provide a better fitting to the available data sets. See, for example, Gupta and Kundu (2010), where skew logistic and proportional reversed hazard logistic distributions have been introduced and studied. The new generalization that is, polynomial logistic distribution, is found to be extremely helpful to describe real data arising in finance. This is because using polynomials of degree ≥ 3 in place of linear ones provides additional flexibility to fit data by employing a greater number of parameters. In the same paper by Koutras et al., a study of the properties of the polynomial logistic distribution has also been initiated and, afterwards, followed up in the present work. Having said so, let us introduce some notions and definitions to be used throughout the paper.

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Definition 1.1. Let $P(x)$ be a polynomial monotone increasing on $(-\infty, +\infty)$. The probability distribution with the distribution function

$$F(x) = \frac{1}{1 + \exp\{-P(x)\}}, \qquad -\infty < x < +\infty \tag{1.1}$$

is called the *polynomial logistic distribution* associated with polynomial P.

We denote this distribution by $\mathrm{PL}(P)$, and write $X \sim \mathrm{PL}(P)$ for a random variable following a polynomial logistic distribution. Here, P can be viewed as an underlying polynomial of the distribution. If $P(x) = \sum_{i=0}^{r} a_i x^i$, we can also write $\mathrm{PL}(a_0, a_1, \ldots, a_r)$ for $\mathrm{PL}(P)$ to indicate explicitly the degree and the coefficients of a polynomial P. Clearly, the degree r of an underlying polynomial P is odd and the leading coefficient a_r is positive. Meanwhile, P can serve as an underlying polynomial for $\mathrm{PL}(P)$ if and only if $P'(x) \geq 0$ for all $x \in \mathbb{R}$. There are various results related to the polynomials non-negative for all $x \in \mathbb{R}$. The most commonly known one states that any such polynomial is a sum of the squares of two polynomials (see, for example, Polya and Szegő (1998), Part 6, Section 6, problem 44). Nonnegative polynomials are also used in Bertsimas and Popescu (2002) for solving certain optimization problems of financial mathematics. Proposition 1(a) of that work provides a necessary and sufficient condition for a polynomial to be nonnegative on $\mathbb{R}$ in terms of positive semidefinite matrices.

In the particular case of $P(x) = x^r$, the corresponding polynomial logistic distribution is called *power logistic* and is designated by PL_r. In the case where $r = 1$, distribution PL_1 is the standard classical logistic distribution whose distribution function equals $F = 1/(1 + e^{-x})$, $x \in \mathbb{R}$.

It can be readily seen that if $X \sim \mathrm{PL}(P)$, then the density of X equals:

$$f(x) = \frac{P'(x)\exp\{-P(x)\}}{(1 + \exp\{-P(x)\})^2} = \frac{P'(x)}{4\cosh^2\{P(x)/2\}}, \tag{1.2}$$

while for $X \sim \mathrm{PL}_r$,

$$f(x) = \frac{r x^{r-1} \exp\{-x^r\}}{(1 + \exp\{-x^r\})^2}. \tag{1.3}$$

The aim of this paper is to study the moment determinacy and moment indeterminacy of the distributions of X^m, $m \in \mathbb{N}$ and $|X|^p$, $p \in (0, +\infty)$. It should be pointed out that the powers of random variables being involved in the Box–Cox transformation are widely used in statistical practices, see Box and Cox (1964).

Let D be a probability distribution having finite moments of all orders. If the moment problem for D has a unique solution, then D is said to be *moment determinate* or *M-determinate*; otherwise, it is *moment indeterminate* or *M-indeterminate*. A number of criteria is available both for M-determinacy and M-indeterminacy, among which the mostly known ones are the Cramér, Carleman, and Krein conditions (cf. Stoyanov (2000) and Stoyanov (2013), Section 11). However, in the case of M-indeterminacy, those criteria do not indicate the way to exhibit explicitly the distributions which are different from D but have the same moments. For an absolutely continuous distribution, this can be achieved by finding a family of probability densities called the *Stieltjes class*. The name "Stieltjes class" has been implemented by J. Stoyanov and it reflects the contributions made by Stieltjes who produced the first M-indeterminate distributions by the same method of perturbation functions as used today to create Stieltjes classes. See Letter 325 in Vol. 2 of Baillaud and Bourget (1905) and Stieltjes (1894). Before we begin, let us recall the necessary notations and definitions introduced in Stoyanov (2004).

Definition 1.2. Let f be a probability density possessing finite moments of all orders, and h be an integrable function on $(-\infty, \infty)$, such that $\operatorname{vraisup}|h(x)| = 1$. If, for all $n \in \mathbb{N}_0$,

$$\int_{\mathbb{R}} x^n h(x) f(x)\, dx = 0,$$

then h is called a *perturbation* of f. Equivalently, one can also say that the product hf has its *all moments vanishing*.

Definition 1.3. Given a probability density f and its perturbation h, the set

$$\mathbf{S} = \mathbf{S}(f, h) := \{\omega_\varepsilon(x) : \omega_\varepsilon(x) = f(x)[1 + \varepsilon h(x)], x \in \mathbb{R}, \varepsilon \in [-1, 1]\}$$

is said to be a *Stieltjes class* for density f based on perturbation h.

Evidently, $\mathbf{S}$ is an infinite family of probability densities all having the same moments as f. Notice that, for a given probability density f, there exist different Stieltjes classes based on different perturbation functions h. Currently, the investigation of Stieltjes classes has been drawing the attention of many researchers and new studies on the subject are constantly coming out. See, for example, Kleiber (2013, 2014), Pakes (2007).

The present paper is organized as follows. In Section 2, the results on the analytical properties of the characteristic functions of X^m and $|X|^p$ are presented. These results imply that both X^m and $|X|^p$ are moment-determinate for $m \le r$ and $p \le r$. Further, in Section 3, it is shown that moment determinacy for $|X|^p$ is valid for all $p \le 2r$, while for $p > 2r$, the distribution of $|X|^p$ becomes moment-indeterminate. The situation concerning X^m is more subtle and somewhat similar to the phenomenon described in Berg (1988). Namely, if m is *even*, then X^m is M-determinate for $m \le 2r$ and M-indeterminate otherwise. Meanwhile, if m is *odd*, then X^m is M-determinate if and only if $m \le r$. For example, when $r = 3$, the distributions of X^3, X^4, and X^6 are M-determinate, whereas the distribution of X^5 is M-indeterminate. Finally, in Section 4 examples of Stieltjes classes are constructed pertinent to the M-indeterminate powers of X and $|X|$ in the case when X has a power logistic distribution.

2 The characteristic functions of polynomial logistic distributions

Given a random variable X possessing the distribution function F, its characteristic function is denoted by $\varphi(t; F)$; that is,

$$\varphi(t; F) = \mathbf{E}\big[\exp\{itX\}\big], \qquad t \in \mathbb{R}.$$

For any real-valued random variable X, its characteristic function exists for all $t \in \mathbb{R}$. What is more, it is uniformly continuous on $(-\infty, +\infty)$. If, in addition,

$$\exists a > 0: \qquad \varphi(t; F) = \sum_{k=0}^{\infty} c_k t^k, \qquad t \in (-a, a), \tag{2.1}$$

then $\varphi(t; F)$ is said to be *analytic* on $(-a, a)$, and with the help of the series in (2.1) it admits an analytic continuation into the disc $\{z : |z| < a\} \subset \mathbb{C}$. If the power series expansion (2.1) holds for all $t \in (-\infty, +\infty)$, then $\varphi(t; f)$ is an *entire* characteristic function. In this case, the power series $\sum_{k=0}^{\infty} c_k z^k$ converges for all $z \in \mathbb{C}$ and defines an entire function of the complex variable.

The important characteristics of an entire function are its *order*, ρ, and *type*, σ, which describe its rate of growth. To be specific, if $g(z)$ is an entire function and $M(r; g) = \max_{|z| \le r} |g(z)|$, then the order and type of g are given by:

$$\rho = \rho(g) = \limsup_{r \to \infty} \frac{\ln \ln M(r; g)}{\ln r} \tag{2.2}$$

and

$$\sigma = \sigma(g) = \limsup_{r \to \infty} \frac{\ln M(r; g)}{r^{\rho}}, \tag{2.3}$$

provided $0 < \rho < \infty$. Alternatively, the order and type of an entire function can be expressed directly from the coefficient of its power series development. For detailed information see, for example, Levin (1996), Lecture I, Sections 1.2 and 1.3.

Although, for every random variable X—or, equivalently, for every distribution function F—its characteristic function exists for all $t \in \mathbb{R}$, in many problems it is more convenient to deal with the *moment-generating* function of X defined by

$$M_X(z) := \mathbf{E}\big[\exp\{zX\}\big], \qquad z \in \mathbb{C}. \tag{2.4}$$

Evidently, $M_X(it) = \varphi(t)$ for $t \in \mathbb{R}$. Notice that, in general, $M_X(z)$ may not exist in any (arbitrarily small) neighborhood of 0. The existence of $M_X(z)$ is closely related to the analyticity of the characteristic function $\varphi(t; F_X)$. To be specific, $M_X(z)$ exists for $|z| < a, a > 0$ if and only if $\varphi(t; F_X)$ is analytic on $(-a, a)$; and $M_X(z)$ exists for all $z \in \mathbb{C}$ if and only if $\varphi(t; F_X)$ is entire. We refer to Stoyanov (2013), Chapter 8, pp. 68–70.

The analytical properties of characteristic and moment-generating functions are used widely not only within Probability Theory itself, but also in a variety of applications. See, for example, Del Baño Rollin, Ferreiro-Castilla and Utzet (2010), where the characteristic functions provide a main tool in investigating the densities of the random variables appearing in volatility models of mathematical finance.

It is known that the characteristic function of the standard logistic distribution equals:

$$\varphi(t) = \frac{\pi t}{\sinh(\pi t)}.$$

This function has an infinite number of poles at $\pm ik, k \in \mathbb{N}$, and, as such, is not entire. Clearly, $\varphi(t)$ is analytic on $(-1, 1)$ and admits an analytic continuation in horizontal strip $\{z : -1 < \operatorname{Im} z < 1\}$. The first theorem of this paper demonstrates that, when $\deg P \geq 3$, the characteristic function of the polynomial logistic distribution is entire, unlike that in the classical case.

Theorem 2.1. *Let $P(x) = \sum_{i=0}^{r} a_i x^i$ be a polynomial of degree $r \geq 3$ monotone increasing on $(-\infty, +\infty)$. If a random variable $X \sim \mathrm{PL}(a_0, a_1, \ldots, a_r)$, then the characteristic function $\varphi(t; F)$ is entire of order $\rho = r/(r-1)$ and type $\sigma = (r-1)r^{\rho} a_r^{1/(r-1)}$.*

Proof. Given a distribution function F, consider the "tail" function

$$W_F(x) := 1 - F(x) + F(-x), \qquad x > 0. \tag{2.5}$$

Obviously, $W_F(x)$ is a nonincreasing function which tends to 0 as $x \to +\infty$. It is known (see Linnik and Ostrovskii (1977), Theorem 2.2.2) that the characteristic function $\varphi(t; F)$ is entire if and only if

$$\forall a > 0, \qquad W_F(x) = O\bigl(\exp\{-ax\}\bigr) \qquad \text{as } x \to +\infty. \tag{2.6}$$

If $F \sim \mathrm{PL}(P)$, then

$$W_F(x) = \frac{\exp\{-P(x)\}}{1 + \exp\{-P(x)\}} + \frac{\exp\{P(-x)\}}{1 + \exp\{P(-x)\}}. \tag{2.7}$$

Clearly, $W_F(x) \leq \exp\{-P(x)\} + \exp\{P(-x)\}$ and hence, for $\deg P = r \geq 3$, condition (2.6) is satisfied, implying that $\varphi(t; F)$ is an entire function. To determine its order and type, we refer to Theorem 2.4.4 of the same book by Linnik and Ostrovskii. First, evaluate

$$\kappa := \lim_{x \to +\infty} \frac{\ln^+ \ln^+ (1/W_F(x))}{\ln x}. \tag{2.8}$$

To calculate the above limit, it can be noticed from (2.7) that

$$1/\bigl[1 + \exp\bigl\{P(x)\bigr\}\bigr] \leq W_F(x) \leq \exp\bigl\{-P(x)\bigr\} + \exp\bigl\{P(-x)\bigr\},$$

whence

$$-\ln\bigl[\exp\bigl\{-P(x)\bigr\} + \exp\bigl\{P(-x)\bigr\}\bigr] \leq \ln\bigl(1/W_F(x)\bigr) \leq \ln\bigl[1 + \exp\bigl\{P(x)\bigr\}\bigr].$$

Writing $P(x) := a_r x^r + q(x)$ and $Q(x) := q(x) + q(-x)$, one has:

$$P(x) - \ln[1 + \exp\{Q(x)\}] \le \ln(1/W_F(x)) \le P(x) + \ln[1 + \exp\{-P(x)\}]$$

yielding

$$1 + o(1) \le \frac{\ln(1/W_F(x))}{P(x)} \le 1 + o(1), \qquad x \to +\infty.$$

As a result,

$$\ln(1/W_F(x)) \sim P(x) \sim a_r x^r, \qquad x \to +\infty. \tag{2.9}$$

Thence

$$\ln\ln(1/W_F(x)) = \ln a_r + r \ln x + o(1), \qquad x \to +\infty$$

and one obtains $\kappa = r$. By using $\kappa^{-1} + \rho^{-1} = 1$, see formula (2.4.3) in Linnik and Ostrovskii (1977), we conclude that the order of the characteristic function of F equals $\rho = r/(r-1)$, as stated.

To determine type $\sigma(\varphi)$, one has to calculate the following:

$$\lambda := \lim_{x \to +\infty} \frac{\ln^+(1/W_F(x))}{x^\kappa}. \tag{2.10}$$

Using (2.9) and the fact that $\kappa = r$, it is immediate that $\lambda = a_r$. Then, by applying relation (2.4.4) from Linnik and Ostrovskii (1977), which states that $(\kappa\lambda)^{\rho-1} \times \sigma\rho = 1$, one obtains:

$$\sigma = \frac{r-1}{r^{r/(r-1)} \cdot a_r^{1/(r-1)}}.$$

□

The corollaries below once again emphasize the difference between the properties of the logistic distribution and the polynomial logistic associated with a polynomial of degree at least 3.

Corollary 2.1. *If $X \sim \mathrm{PL}(P)$, $\deg P \ge 3$, then $M_X(z)$ exists for all $z \in \mathbb{C}$.*

Corollary 2.2. *Polynomial logistic distribution associated with a polynomial of degree ≥ 3 is not infinitely divisible.*

Proof. It is known that a characteristic function of an infinitely divisible distribution does not vanish—see Stoyanov (2013), Section 9, p. 78. Meanwhile, the characteristic function of a polynomial logistic distribution $\mathrm{PL}(P)$ with $\deg P \ge 3$ is entire of a noninteger order and, consequently, it has infinitely many zeroes. Thus, the corresponding distribution is not infinitely divisible. □

Following the same line of reasoning, one can reach results of similar nature for the powers polynomial logistic distributions.

Theorem 2.2. *Let* $X \sim \mathrm{PL}(P) = \mathrm{PL}(a_0, a_1, \ldots, a_r)$. *Denote by* $F_m, m \in \mathbb{N}$, *and* $G_p, p \in (-\infty, +\infty)$, *the distribution functions of* X^m *and* $|X|^p$, *respectively. Then, for* $1 \leq m \leq r-1$, *and* $0 < p < r$, *the characteristic functions* $\varphi(t; F_m)$ *and* $\varphi(t; G_p)$ *are entire of orders* $r_m = r/(r-m)$ *and* $r_p = r/(r-p)$, *whose types are* $\sigma_m = (r-m)r^{-r/(r-m)}(a_r/m)^{-m/(r-m)}$ *and* $\sigma_p = (r-p)r^{-r/(r-p)}(a_r/p)^{-p/(r-p)}$, *respectively.*

Proof. Straightforward calculations reveal that $F_m(x) = F(x^{1/m})$ if m is odd and $F_m(x) = F(x^{1/m}) - F(-x^{1/m}), x > 0$ if m is even, while $G_p(x) = F(x^{1/p}) - F(-x^{1/p}), x > 0$. Therefore, in both cases, the tail function (2.5) can be expressed in the form $W_p(x) = W_F(x^{1/p}), x > 0$, where for F_m only $p \in \mathbb{N}$ is allowed. Due to the fact that

$$W_p(x) \leq \exp\{-P(x^{1/p})\} + \exp\{P(-x^{1/p})\} = O(\exp\{-a_r x^{r/p}\}) \quad \text{as } x \to +\infty,$$

the condition (2.6) is satisfied whenever $p < r$. This implies that, for these powers, the characteristic functions $\varphi(z; F_m)$ and $\varphi(z; G_p)$ are entire. Since the order and type of a characteristic function depend solely on the behavior of its tail, one may use (2.9) to derive:

$$\ln(1/W_p(x)) = \ln(1/W_F(x^{1/p})) \sim P(x^{1/p}) \sim a_r x^{r/p}, \qquad x \to +\infty. \tag{2.11}$$

Therefore, by formulae (2.8) and (2.10), one may calculate $\kappa = r/p$ and $\lambda = a_r$ and derive the required values by applying Theorem 2.4.4 of Linnik and Ostrovskii (1977). □

It is clear that condition (2.6) is not satisfied if $p \geq r$. Therefore, for these values of p (or m), the respective characteristic functions are not entire. However, in the case $m = r$ or $p = r$, the characteristic functions will be analytic in some neighborhood of 0, as the next theorem shows.

Theorem 2.3. *Let* $X \sim \mathrm{PL}(P) = \mathrm{PL}(a_0, a_1, \ldots, a_r)$, F_m *and* G_p *be defined as in Theorem* 2.2. *Then, the characteristic functions* $\varphi(t; F_r)$ *and* $\varphi(t; G_r)$ *are analytic for* $|t| < a_r$.

Proof. It is known that a characteristic function $\varphi(t; F)$ is analytic for $|t| < R$ if and only if

$$\forall a < R, \qquad W_F(x) = O(\exp\{-ax\}) \qquad \text{as } x \to +\infty.$$

See, for example, Linnik and Ostrovskii (1977), formula (2.2.3). Meanwhile, since $W_r(x) = W_F(x^{1/r})$, it follows that, $\forall \varepsilon > 0$,

$$W_r(x) \leq \exp\{-P(x^{1/r})\} + \exp\{P(-x^{1/r})\} = O(\exp\{-(a_r - \varepsilon)x\}, \quad x \to +\infty.$$

Therefore, functions $\varphi(t; F_r)$ and $\varphi(t; G_r)$ are analytic for $|t| < a_r$. In addition, they are not analytic on any interval $(-R, R)$, where $R > a_r$. □

Corollary 2.3. *The characteristic functions $\varphi(t; F_r)$ and $\varphi(t; G_r)$ admit an analytic continuation in the strip $\{z : -a_r < \operatorname{Im} z < a_r\}$.*

Corollary 2.4. *For $1 \le m \le r$ and $0 < p \le r$, the moment-generating functions of X^m and $|X|^p$ exist whenever $|z| < a_r$. This means that the Cramér condition—see, for example, Stoyanov (2013), Section 11, p. 100, is satisfied and, consequently the distributions of these random variables are M-determinate. This fact can also be established by the unicity theorem for analytic functions or, alternatively, with the help the Carleman condition as in Theorem 3.1.*

3 M-determinate and M-indeterminate powers of polynomial logistic distributions

As it has been stated by Corollary 2.4, random variables X^m, $m \in \mathbb{N}$ and $|X|^p$, $p \in \mathbb{R}_+$, are M-determinate whenever $m \le r$ and $p \le r$ as a consequence of the corresponding characteristic functions being analytic in some neighborhood of 0. In this section, the M-(in)determinacy for the distributions of $|X|^p$ and X^m in the case when $m, p > r$ is investigated. As it turns out, despite the fact that the characteristic functions are no longer analytic, the distributions $|X|^p$ are M-determinate also for all $p \le 2r$. This bound proves to be sharp, that is, for $p > 2r$, the related distributions are M-indeterminate. For the distributions of X^m, $m > r$ the situation appears to be more complicated, as Theorem 3.2 implies.

From here on, letter C (with or without indices) denotes positive constants whose values do not need to be specified.

Theorem 3.1. *Let $X \sim \mathrm{PL}(P)$, $\deg P = r$. Then:*

(i) *For $p \le 2r$, the distribution of $|X|^p$ is M-determinate;*
(ii) *For $p > 2r$, the distribution of $|X|^p$ is M-indeterminate.*

Proof.

(i) An explicit formula for the density of $|X|^p$ can be written as:

$$\begin{aligned} g_p(x) = \frac{1}{4p} x^{1/p-1} \biggl\{ & \frac{P'(x^{1/p})}{\cosh^2(P(x^{1/p})/2)} \\ & + \frac{P'(-x^{1/p})}{\cosh^2(P(-x^{1/p})/2)} \biggr\}, \qquad x > 0. \end{aligned} \tag{3.1}$$

Consequently, the density enjoys the following estimate:

$$g_p(x) \le C_1 x^{r/p-1} \exp\{-C_2 x^{r/p}\}, \qquad x \ge x_0. \tag{3.2}$$

Hence, the moments $m_n = \mathbf{E}[(|X|^p)^n]$ satisfy the inequalities below:

$$m_n \le x_0^n + C_1 \int_0^{+\infty} x^{n+r/p-1} \exp\{-C_2 x^{r/p}\}\,dx = x_0^n + C_1 \cdot C_2^{-np/r} \Gamma(np/r+1)$$
$$\le C_3^n \Gamma(2n+1) = C_3^n (2n)! \qquad \text{because } p/r \le 2.$$

Therefore, $\sum_{n=0}^{\infty} (m_n)^{-1/(2n)} = \infty$ and, by the Carleman criterion, the distribution of $|X|^p$ is moment-determinate in the Stieltjes sense. Since the distribution of $|X|^p$ is not discrete, by virtue of the Chihara theorem (cf. Chihara (1968)) it is also moment-determinate in the sense of Hamburger, that is, M-determinate.

(ii) To prove M-indeterminacy for $p > 2r$, the Krein criterion will be applied which provides a sufficient condition for a distribution to be M-indeterminate. To satisfy this criterion, it has to be shown see, for example, Stoyanov (2013), Section 11 that

$$\int_0^{\infty} \frac{-\ln g_p(x^2)}{1+x^2}\,dx < \infty. \tag{3.3}$$

Formula (3.1) implies that $\ln g_p(x^2)$ is integrable on any bounded interval. Hence, it suffices to check that $\int_a^{\infty} \frac{-\ln g_p(x^2)}{1+x^2}\,dx < \infty$ for some $a > 0$. Select $a > 0$ in such a way that $g_p(x^2) \ne 0$ for $x \in [a, \infty)$. This is possible because $g_p(x)$ has at most a finite number of zeroes. Since

$$g_p(x^2) \sim C_5 x^{2r/p-2} \exp\{-P(x^{2/p})\}, \qquad x \to +\infty,$$

it follows that

$$-\ln g_p(x^2) \sim P(x^{2r/p}) \sim a_r x^{2r/p}, \qquad x \to +\infty.$$

As $2r/p < 1$, the logarithmic integral (3.3) converges. □

Corollary 3.1. *Let $m \in \mathbb{N}$ be even. Then, the distribution of X^m is M-determinate if and only if $m \le 2r$.*

Remark 3.1. Similar to the classical case when $r = 1$, the distribution of $Y = |X|, x \sim \mathrm{PL}_r$, can be called *half-power logistic*. The results of Theorem 3.1 in the case $r = 1$ coincide with the previously established ones for the half-logistic distribution. See, for example, Lin and Stoyanov (2015).

Remark 3.2. At this stage, it is worth pointing out that part (i) of Theorem 3.1 can also be established by using Hardy's condition, stating that if X is a non-negative random variable such that

$$\exists C > 0: \qquad \mathbf{E}\big[\exp\{C\sqrt{X}\}\big] < \infty,$$

then the distribution of X is M-determinate. Here, we refer to Theorem 1 and Corollary 3(i) of Stoyanov and Lin (2013). One can deduce that estimate (3.2) implies Hardy's condition for $|X|^p$ to be satisfied for $C \in (0, C_2)$ whenever $p \le 2r$.

Now, what about the *odd* powers of X? By Corollary 2.5, if $m \le r$, then the distribution of X^m is M-determinate. The next theorem shows that this bound is sharp.

Theorem 3.2. *Let $m \in \mathbb{N}$ be odd. If $m > r$, then the distribution of X^m is M-indeterminate.*

Proof. The statement can be derived from the application of the Krein condition for distributions with support $(-\infty, +\infty)$. See, for example, Stoyanov (2013), Section 11. In the case when m is odd, the density of X^m equals:

$$f_m(x) = \frac{1}{m} x^{1/m-1} \cdot \frac{P'(x^{1/m}) \exp\{-P(x^{1/m})\}}{(1+\exp\{-P(x^{1/m})\})^2}, \qquad -\infty < x < \infty.$$

The Krein condition, sufficient for the M-indeterminacy, is satisfied if the integral

$$\int_{-\infty}^{+\infty} \frac{-\ln f_m(x)}{1+x^2}\, dx < \infty. \tag{3.4}$$

The integrand of (3.4) is integrable on any bounded interval and, thence, it is sufficient to show that $\int_{|x|\ge a} \frac{-\ln f_m(x)}{1+x^2}\, dx < \infty$ for some $a > 0$. Obviously, $f_m(x) = 0 \Leftrightarrow P'(x^{1/m}) = 0$, and, as such, $f_m(x)$ has at most $(r-1)/2$ distinct real zeroes, say, $a_1, a_2, \dots, a_s$. Set $a := 1 + \max_{1\le j\le s} |a_j|$. Then $\frac{-\ln f_m(x)}{1+x^2}$ is continuous whenever $|x| \ge a$. In addition,

$$-\ln f_m(x) \sim P\bigl(|x|^{1/m}\bigr) \sim a_r |x|^{r/m} \qquad \text{as } x \to \pm\infty.$$

Therefore, for $r/m < 1$ integral (3.4) converges, yielding that the distribution of X^m is M-indeterminate when $m > r$. □

Remark 3.3. If $r = 1$, one can reveal the previously known results on the classical logistic distribution, see Lin and Huang (1997), Stoyanov and Tolmatz (2005), Section 6, and Stoyanov, Lin and Dasgupta (2013), Section 3.2. Meanwhile, tails (2.7) of polynomial logistic distributions with $r \ge 3$ are lighter than those of the standard logistic one. Thereby justifying the fact that larger powers of X stay M-determinate compared with powers of the classical (half)-logistic.

4 Stieltjes classes for powers of PL_r distributions

In this part, the Stieltjes classes for M-indeterminate powers of PL_r—that is, power logistic—distribution will be constructed. This is done to avoid cumbersome calculations, although a similar approach can be exploited in more general cases. First, a Stieltjes class for $|X|^p$, $p > 2r$, will be presented.

Theorem 4.1. *Let $X \sim \mathrm{PL}_r$ and $p > 2r$. If*

$$\tilde{h}_p(x) = \begin{cases} (1 + e^{-x^{r/p}})^2 \sin\big(x^{r/p} \tan(\pi r/p) - \pi r/p\big), & x > 0, \\ 0, & x < 0, \end{cases} \tag{4.1}$$

and $M_p := \max_{x \geq 0} |\tilde{h}_p(x)|$, then

$$h_p(x) := \frac{\tilde{h}_p(x)}{M_p} \tag{4.2}$$

is a perturbation for the density $g_p(x)$ of $|X|^p$.

Proof. According to (3.1), the probability density of $|X|^p$ equals:

$$g_p(x) = \frac{r x^{r/p-1}}{2p \cosh^2(x^{r/p}/2)}, \qquad x > 0.$$

Then,

$$\forall \alpha \in (r/p, 1/2), \exists C > 0: \qquad g_p(x) \geq C \exp\{-x^\alpha\}, \qquad x > 0. \tag{4.3}$$

A method to construct Stieltjes classes for densities on $(0, +\infty)$ having a lower bound of the form (4.3) with $\alpha \in (0, 1/2)$ is presented in Ostrovska (2014). For the convenience of the readers, the procedure is outlined briefly below. Let

$$\varphi(z) = z^{r/p-1} \exp\left\{-\frac{z^{r/p}}{\cos(\pi r/p)}\right\}, \qquad z \in \mathbb{C}. \tag{4.4}$$

Clearly, $\varphi(z)$ is analytic in $\{z : \operatorname{Im} z \geq 0\} \setminus \{0\}$, where it enjoys the estimate:

$$|\varphi(z)| \leq |z|^{r/p-1} \exp\{-|z|^{r/p}\}. \tag{4.5}$$

Select real numbers $0 < \rho < R$ and consider in the upper half-plane a closed contour $L := l_1 \cup l_2 \cup l_3 \cup l_4$, consisting of two segments: $l_1 = [\rho, R]$ and $l_3 = [-R, -\rho]$, and two arcs: $l_2 = \{z : |z| = R, 0 < \arg z < \pi\}$, and $l_4 = \{z : |z| = \rho, 0 < \arg z < \pi\}$. By virtue of the Cauchy theorem:

$$\oint_L z^n \varphi(z)\, dz = 0, \qquad \forall n \in \mathbb{N}_0,$$

where the positive (i.e., counterclockwise) direction of the path has been taken. Obviously,

$$\oint_L z^n \varphi(z)\, dz = I_1 + I_2 + I_3 + I_4 \qquad \text{where } I_j := \int_{l_j} z^n \varphi(z)\, dz, \qquad j = 1, 2, 3, 4.$$

As under condition (4.5) the integrals along the arcs tend to 0 as $R \to \infty$ and $\rho \to 0$ for each $n \in \mathbb{N}_0$, passing to limit as $R \to \infty$ and $\rho \to 0$ leads to:

$$\int_0^\infty x^n \varphi(x)\, dx + (-1)^n \int_0^\infty x^n \varphi(-x)\, dx = 0, \qquad n \in \mathbb{N}_0.$$

Taking the imaginary part of the latter equality yields:

$$\int_0^\infty x^n\big[\operatorname{Im}\varphi(-x)\big]\,dx = 0.$$

It can be readily seen that

$$\operatorname{Im}\varphi(-x) = x^{r/p-1}e^{-x^{r/p}}\sin\left(x^{r/p}\tan\frac{\pi r}{p} - \frac{\pi r}{p}\right)$$

and, therefore, (4.1) equals

$$\frac{2r}{p}\cdot\frac{\operatorname{Im}\varphi(-x)}{g_p(x)},$$

implying that $g_p\tilde{h}_p$ has all vanishing moments. Thus, (4.2) is a perturbation function for g_p. □

Corollary 4.1. *Given $p > 2r$, the set*

$$\mathbf{S}_p = \{\omega_\varepsilon : \omega_\varepsilon(x) = g_p(x)\big[1 + \varepsilon h_p(x)\big], \varepsilon \in [-1, 1]\}$$

is a Stieltjes class for g_p.

Remark 4.1. Function $\varphi(z)$ in (4.4) can be chosen in different ways under the stipulation that the necessary estimates are satisfied. Selecting

$$\varphi(z) = \exp\left\{-\frac{z^\beta}{\cos\pi\beta}\right\}, \qquad \beta \in (r/p, 1/2),$$

one obtains a perturbation function of the form

$$h_p(x) = Cx^{1-r/p}\big(1 + e^{-x^{r/p}}\big)^2\sin\big(x^\beta\tan\pi\beta\big)\exp\{x^{r/p} - x^\beta\}, \qquad x > 0.$$

It should be mentioned that, for the case $r = 1$, the latter coincides with the perturbation function (18), obtained through other methods in Stoyanov and Tolmatz (2005), for the M-indeterminate powers of the classical logistic distribution.

Now, let $m \in \mathbb{N}$. Since $X^m = |X|^m$ for m being even, all that remains is to construct a Stieltjes class only in the case when m is odd. Let us remind that the situation is somewhat similar to the one described in Berg (1988) and Stoyanov (2013), Section 11.1. Namely, if $m > r$ is *odd*, then the distribution of X^m is M-indeterminate, though for m being *even*, the distribution remains M-indeterminate unless $m > 2r$. The proof of the next assertion is a slight modification of the preceding one, and it follows the ideas presented in Proposition 1 of Berg (1988), where odd powers of the standard normal distribution have been studied. For this reason, it will be presented without going into details.

Theorem 4.2. *Let $X \sim \mathrm{PL}_r$ and let $m > r$ be an odd positive integer. If*

$$\tilde{h}_m(x) := \frac{m}{r}\sin\left[\frac{\pi r}{2m} - x^{r/m}\tan\frac{\pi r}{2m}\right](1+e^{-x^{r/m}})^2, \qquad x > 0,$$

and $K_m = \max_{-\infty<x<+\infty}|\tilde{h}_m(|x|)|$, then

$$h_m(x) = \frac{\tilde{h}_m(|x|)}{K_m} \tag{4.6}$$

is a perturbation for the density f_m of X^m.

Proof. Given odd $m \in \mathbb{N}$, the density of X^m is expressed by:

$$f_m(x) = \frac{r x^{r/m-1} e^{-x^{r/m}}}{m(1+e^{-x^{r/m}})^2}. \tag{4.7}$$

Obviously, for odd values of m, the density f_m is symmetric and, therefore, if $\tilde{h}_m(x)$ is a bounded function on $[0, +\infty)$ such that

$$\int_0^{+\infty} x^{2n} f_m(x)\tilde{h}_m(x)\,dx = 0 \qquad \text{for all } n \in \mathbb{N}_0, \tag{4.8}$$

then (4.6) is a perturbation for f_m. To find $\tilde{h}_m(x)$, a contour integration will be used. However, since f_m does not satisfy estimate (4.3), the direct application of the result from Ostrovska (2014) is no longer possible and using the same approach one has to consider a contour different from the one in Theorem 4.1. Denote $\alpha = r/m \in (0, 1)$ and put

$$\varphi(z) = z^{\alpha-1}\exp\left\{-\frac{z^\alpha}{\cos(\pi\alpha/2)}\right\}.$$

Clearly, $\varphi(z)$ is analytic in $\mathbb{C} \setminus (-\infty, 0]$. In addition, the following estimate is valid:

$$\left|\varphi(z)\right| \le |z|^{\alpha-1}\exp\{-|z|^\alpha\} \qquad \text{whenever } 0 \le \arg z \le \frac{\pi}{2}. \tag{4.9}$$

For real numbers $0 < \rho < R$, consider in the first quadrant the closed contour $L := l_1 \cup l_2 \cup l_3 \cup l_4$, consisting of two segments: $l_1 = [\rho, R]$ and $l_3 = [i\rho, iR]$, and two arcs: $l_2 = \{z : |z| = R, 0 < \arg z < \frac{\pi}{2}\}$, and $l_4 = \{z : |z| = \rho, 0 < \arg z < \frac{\pi}{2}\}$. It can be readily seen that, under condition (4.9), the integrals of $z^{2n}\varphi(z)$ along the arcs tend to 0 as $R \to \infty$ and $\rho \to 0$ for each $n \in \mathbb{N}_0$. Therefore, after applying the Cauchy theorem to $\int_L z^{2n}\varphi(z)\,dz$ and passing to limit as $R \to \infty$ and $\rho \to 0$, one obtains:

$$\int_0^{+\infty} x^{2n}\varphi(x)\,dx - (-1)^n i\int_0^{+\infty} x^{2n}\varphi(ix)\,dx = 0, \qquad n \in \mathbb{N}_0.$$

Taking the imaginary part of the latter equality leads to:

$$\int_0^{+\infty} x^{2n}\,\mathrm{Re}\big[\varphi(ix)\big]\,dx = 0, \qquad \forall n \in \mathbb{N}_0.$$

By the straightforward calculations, $\mathrm{Re}\,\varphi(ix) = x^{\alpha-1}\exp\{-x^{\alpha}\}\sin[\frac{\pi\alpha}{2} - x^{\alpha} \times \tan\frac{\pi\alpha}{2}]$, where $\alpha = r/m$. Now, for $x > 0$, set:

$$\tilde{h}_m(x) := \frac{\mathrm{Re}[\varphi(ix)]}{f_m(x)} = \frac{m}{r}\sin\left[\frac{\pi r}{2m} - x^{r/m}\tan\frac{\pi r}{2m}\right]\big(1 + e^{-x^{r/m}}\big)^2.$$

Obviously, $\tilde{h}_m(x)$ is bounded on $[0, +\infty)$ and satisfy (4.8). Thus, (4.6) is a perturbation of density f_m. □

Corollary 4.2. *Given an odd positive integer* $m \geq 2r + 1$, *the set*

$$\mathbf{S}_m = \big\{\omega_\varepsilon : \omega_\varepsilon(x) = f_m(x)\big[1 + \varepsilon h_m(x)\big], \varepsilon \in [-1, 1]\big\}$$

is a Stieltjes class for f_m.

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