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Continuity results and estimates for the Lyapunov exponent of Brownian motion in stationary potential

Johannes Rueß

Eberhard Karls Universität Tübingen

Abstract. We collect some applications of the variational formula established by Schroeder [*J. Funct. Anal.* **77** (1988) 60–87] and Rueß [*ALEA Lat. Am. J. Probab. Math. Stat.* **11** (2014) 679–709] for the quenched Lyapunov exponent of Brownian motion in stationary and ergodic nonnegative potential. We show, for example, that the Lyapunov exponent for nondeterministic potential is strictly lower than the Lyapunov exponent for the averaged potential. The behaviour of the Lyapunov exponent under independent perturbations of the underlying potential is examined. And with the help of counterexamples, we are able to give a detailed picture of the continuity properties of the Lyapunov exponent.

1 Introduction

Schroeder (1988) and Rueß (2014) established a variational formula for the exponential decay rate of the Green function of Brownian motion evolving in a stationary and ergodic nonnegative potential. The purpose of this article is to collect some applications of this variational formula. A special focus is laid on continuity properties of the Lyapunov exponent. We give counterexamples in the last section in order to complete the picture.

We consider Brownian motion in $\mathbb{R}^d$, $d \in \mathbb{N}$. Let P_x be the law of standard Brownian motion with start in $x \in \mathbb{R}^d$ on the space $\Sigma := C([0,\infty), \mathbb{R}^d)$ equipped with the σ-algebra generated by the canonical projections, and let E_x be the associated expectation operator. With $(Z_t)_{t\geq 0}$ we denote the canonical process on Σ.

We assume that the Brownian motion is moving in a random potential: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and assume $(\mathbb{R}^d, +)$ is acting as a group on Ω via $\tau : \mathbb{R}^d \times \Omega \to \Omega$, $(x, \omega) \mapsto \tau_x\omega$. We always assume that $X := (\Omega, \mathcal{F}, \mathbb{P}, \tau)$ is a metric dynamical system, which means that τ is product measurable and $\mathbb{P}$ is invariant under τ_x for all $x \in \mathbb{R}^d$. Often $\mathbb{P}$ is required to be ergodic under $\{\tau_x : x \in \mathbb{R}^d\}$. Then X is called ergodic dynamical system. We denote the space of p-integrable functions on Ω by L^p, $p \geq 1$. Any nonnegative $V \in L^1$ is called *potential* throughout this article.

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Let V be a potential. We assume that Brownian motion Z is killed at rate V: Introduce the Green function as

$$g(x, y, \omega) := \int_0^\infty p^t(x, y) E^t_{x,y}\Big[\exp\Big\{-\int_0^t V(\tau_{Z_s}\omega)\, ds\Big\}\Big]\, dt,$$

where $x, y \in \mathbb{R}^d$, $\omega \in \Omega$, $E^t_{x,y}$ denotes the Brownian bridge measure, and $p^t(x, y)$ is the transition probability density of Brownian motion in $\mathbb{R}^d$. g can be interpreted as density for the expected occupation times measure for Brownian motion killed at rate V. Under natural assumptions, the Green function is the fundamental solution to

$$-\frac{1}{2}\Delta g(x, \cdot, \omega) + V_\omega g(x, \cdot, \omega) = \delta_x,$$

where δ_x denotes the Dirac measure at $x \in \mathbb{R}^d$, see, for example, (Pinsky, 1995, Theorem 4.3.8).

If X is an ergodic dynamical system and V satisfies certain boundedness and regularity assumptions, then it is shown in (Rueß, 2014, Theorem 1.2) that the Green function decays exponentially fast with a deterministic exponential decay rate called Lyapunov exponent, see also Theorem 1 below.

Deterministic exponential decay has been shown previously, for example, for periodic potentials by Schroeder (1988), and for Poissonian potentials by Sznitman (1994). Armstrong and Souganidis (2012) give analogous results in the context of Hamilton–Jacobi–Bellman equations. In discrete space, Zerner (1998) and Mourrat (2012) establish existence of Lyapunov exponents for random walks in random potentials.

Measurable functions f on Ω give rise to functions f_ω on $\mathbb{R}^d$, called realisations of f, defined by $f_\omega(x) = f(\tau_x\omega)$ for $x \in \mathbb{R}^d$ and $\omega \in \Omega$. If f_ω is differentiable for all $\omega \in \Omega$, we call f (classically) differentiable and we denote the derivative by $(Df)(\omega) := D(f_\omega)(0)$. Let $y \in \mathbb{R}^d$. We recall the variational expression as introduced in (Rueß, 2014, (1.4)),

$$\Gamma_V(y) := 2 \inf_{f\in\mathbb{F}}\Big[\Big(\int \frac{|\nabla f|^2}{8f} + Vf\, d\mathbb{P}\Big)\Big(\inf_{\phi\in\Phi_y} \int \frac{|\phi|^2}{2f}\, d\mathbb{P}\Big)\Big]^{1/2}.$$

Here the space $\mathbb{F}$ is the space of probability densities $f \in L^1$ with the following properties:

- $\mathbb{E}f = 1$ and there exists $c_f > 0$ such that $f \geq c_f$,
- f_ω is differentiable of any order for all ω, and $\sup_\Omega |D^n f| < \infty$ for $n \in \mathbb{N}_0$.

The space Φ_y is the space of divergence-free vector fields $\phi \in (L^1)^d$ such that:

- ϕ_ω is differentiable of any order for all ω, and $\sup_\Omega |D^n\phi| < \infty$ for all $n \in \mathbb{N}_0$,
- $\mathbb{E}\phi = y$, and $\nabla \cdot \phi = 0$ for all ω.

The Lyapunov exponent can be expressed by Γ_V under the following conditions on the potential V, see Theorem 1 below:

(i) $\sup_\Omega V < \infty$.
(ii) $\inf_\Omega V > 0$.
(iii) $g(0,\cdot,\omega) \in C^2(\mathbb{R}^d \setminus \{0\})$ for $\omega \in \Omega$, $L_\omega g(0,\cdot,\omega) = 0$ on $\mathbb{R}^d \setminus \{0\}$, where $L_\omega := (1/2)\Delta - V_\omega$, for $\omega \in \Omega$.
(iv) $V_\omega(x)$ is uniformly continuous in x and ω, that is

$$\lim_{\delta\to 0} \sup_{|x|\le\delta} \sup_{\omega\in\Omega} \left|V_\omega(x) - V_\omega(0)\right| = 0.$$

Slightly more general but more complicated versions of assumptions (ii) and (iv) are given by Rueß (2014). That (iv) implies (Rueß, 2014, (E1)) follows from homogenisation as outlined in (Rueß, 2014, Section 4.1).

Like in the article of Rueß (2014) we call a potential satisfying this set of assumptions shortly a *regular potential*. Note that the convolution of a potential satisfying the boundedness conditions (i) and (ii) with a compactly supported, nonnegative, smooth kernel leads to *regular potentials*, use, for example, (Pinsky, 1995, Theorem 4.2.5) and (Rueß, 2014, (4.13)). One has the following representation of the Lyapunov exponent.

Theorem 1 ((Schroeder, 1988, (1.1)), (Rueß, 2014, Theorem 1.2)). *If X is an ergodic dynamical system and V a regular potential, then for all $y \in \mathbb{R}^d \setminus \{0\}$ $\mathbb{P}$-a.s. the limit in the following exists and is given as*

$$\alpha_V(y) := \lim_{r\to\infty} -\frac{1}{r} \ln g(0, ry, \omega) = \Gamma_V(y). \tag{1.1}$$

In Sections 2–4, we derive properties of the Lyapunov exponent α_V from its variational representation Γ_V. Among others, we derive a strict inequality $\alpha_V < \alpha_{\mathbb{E}V}$ and we establish continuity properties of the Lyapunov exponent. We state the results for the variational expression Γ_V having in mind that as soon as the underlying dynamical system is ergodic and the considered potentials are *regular* these results do hold by Theorem 1 for α_V as well.

Section 5 does not rely on the variational representation for the Lyapunov exponent. It consists of counterexamples which show, for example, that the Lyapunov exponent is not continuous with respect to L^p convergence of the potential, $1 \le p < \infty$, even for *regular potentials*.

Notation

At some point, we use notation of Rueß (2014) and in order to keep this note compact we introduce several objects in a short way and refer the reader to the first two sections of the article of Rueß (2014) for more detailed descriptions.

We denote by S^{d-1} the set of unit vectors in $\mathbb{R}^d$. The Lebesgue measure on $\mathbb{R}^d$ is denoted by $\mathscr{L}$. We write $|\cdot|$ for the Euclidean norm on $\mathbb{R}^d$, $d \in \mathbb{N}$.

We need the concept of weak differentiability on X: A measurable function $f : \Omega \to \mathbb{R}^d$ is called weakly differentiable in direction i if $\mathbb{P}$-a.e. realisation of f is weakly differentiable in direction i, and if there exists a measurable function g on Ω such that $\mathbb{P}$-a.s. $\mathscr{L}$-a.e. $g_\omega = \partial_i(f_\omega)$. Then g is called the weak derivative $\partial_i f$ of f in direction i. The weak derivative is uniquely determined $\mathbb{P}$-a.s., and coincides with the classical derivative if the realisations of f are classically differentiable. We have the differential operator $\nabla f = (\partial_i f)_i$, if the weak derivatives in any direction exist. We introduce

$$\mathcal{D}(\partial_i) := \{f \in L^2 : f \text{ weakly differentiable in direction } i, \partial_i f \in L^2\}.$$

On $\bigcap_i \mathcal{D}(\partial_i)$ we have the norm $\|f\|_\nabla := \|f\|_2 + \sum_i \|\partial_i f\|_2$. In addition to $\mathbb{F}$ and Φ_y, we need the following function spaces: Let $y \in \mathbb{R}^d$, define $\mathbb{D}_w := \bigcap_{i=1}^d \mathcal{D}(\partial_i)$, and

$$\mathbb{D} := \Big\{f \in L^1 : f_\omega \in C^\infty(\mathbb{R}^d)\ \forall \omega \in \Omega, \sup_\Omega |D^n f| < \infty\ \forall n \in \mathbb{N}_0\Big\},$$

$$\mathfrak{D} := \{D \subset \mathbb{D}_w : D \text{ dense in } \mathbb{D}_w \text{ w.r.t. } \|\cdot\|_\nabla\},$$

$$\mathbb{F}_w := \{f \in \mathbb{D}_w : \mathbb{E}f = 1, \exists c_f > 0 \text{ s.t. } f > c_f\ \mathbb{P}\text{-a.s.}, \|f\|_\infty, \|\nabla f\|_\infty < \infty\},$$

$$\mathfrak{F} := \Big\{F \subset \mathbb{F}_w : \forall f \in \mathbb{F}_w\ \exists (f_n)_n \subset F \text{ and } c > 0 \text{ s.t.}$$
$$f_n \to f \text{ w.r.t. } \|\cdot\|_\nabla \text{ and } \inf_n f_n > c\ \mathbb{P}\text{-a.s.}\Big\},$$

$$\Phi_y^w := \{\phi \in (L^2)^d : \mathbb{E}[\phi \cdot \nabla w] = 0\ \forall w \in \mathbb{D}, \mathbb{E}\phi = y\},$$

$$\mathfrak{P}_y := \{\phi_y \subset \Phi_y^w : \phi_y \text{ dense in } \Phi_y^w \text{ w.r.t. } \|\cdot\|_2\}.$$

Examples

We give two main examples for dynamical systems $X = (\Omega, \mathcal{F}, \mathbb{P}, \tau)$ which fit into our framework. As a special example in Section 5, we encounter the Poisson line process.

$X^{\mathbb{T},d}$—*the d-dimensional torus* $\mathbb{T}^d$: Choose $\Omega := \mathbb{T}^d$, let $\mathcal{F} := \mathcal{B}(\mathbb{T}^d)$ be the Borel σ-algebra on $\mathbb{T}^d$, and set $\tau_x\omega := \omega + x \pmod 1$ for $x \in \mathbb{R}^d$, $\omega \in \mathbb{T}^d$. With $\mathbb{P}$ being the Lebesgue measure $\mathscr{L}$, the dynamical system X becomes stationary and ergodic.

Stationary ergodic random measures: Let $\Omega := \mathcal{M}(\mathbb{R}^d)$ be the set of locally finite measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ equipped with the topology of vague convergence. Let $\mathcal{F}$ be the Borel σ-algebra on Ω, and set $\tau_x\omega[A] := \omega[A + x]$ for $x \in \mathbb{R}^d$, $A \in \mathcal{B}(\mathbb{R}^d)$ and $\omega \in \Omega$. Then for any distribution $\mathbb{P}$ of a stationary ergodic random measure on $(\mathcal{M}(\mathbb{R}^d), \mathcal{F})$ the dynamical system X becomes an ergodic dynamical system, use (Daley and Vere-Jones, 2008, Exercise 12.1.1(a)).

Let $W : \mathbb{R}^d \to [0, \infty)$ be a Borel measurable function and set

$$V(\omega) := \int_{\mathbb{R}^d} W(x)\omega(dx).$$

Under suitable conditions on W and the random measure, V is a potential. If $\mathbb{P}$ is a Poisson point process with constant intensity this leads to the so called Poissonian potentials, see (Sznitman, 1998). We denote such a dynamical system where $\mathbb{P}$ is a Poisson point process with constant intensity $\nu > 0$ by $X^{\mathrm{poi},\nu}$.

2 Elementary properties

We deduce elementary properties of Γ_V:

Proposition 2. *Assume V is a potential. For $c \geq 0$, for $y \in \mathbb{R}^d$,*

$$\Gamma_V(cy) = c\Gamma_V(y). \tag{2.1}$$

Let $c \geq 1$, then

$$\Gamma_{cV}^2 \leq c\Gamma_V^2. \tag{2.2}$$

Analogously if $0 \leq c \leq 1$, one has $\Gamma_{cV}^2 \geq c\Gamma_V^2$. In constant potential $c \geq 0$, for $y \in \mathbb{R}^d$,

$$\Gamma_c(y) = \sqrt{2c}|y|. \tag{2.3}$$

Γ is concave in the following sense: Let λ_i, $1 \leq i \leq k$, be positive real numbers s.t. $\sum_{i=1}^k \lambda_i = 1$. Let $V_1, \dots, V_k$ be potentials on Ω, then $\Gamma^2_{\sum_{i=1}^k \lambda_i V_i} \geq \sum_{i=1}^k \lambda_i \Gamma^2_{V_i}$, in particular,

$$\Gamma_{\sum_{i=1}^k \lambda_i V_i} \geq \sum_{i=1}^k \lambda_i \Gamma_{V_i}. \tag{2.4}$$

For sums of a constant potential and some other potential V

$$\Gamma^2_{c+V} \geq \Gamma^2_V + \Gamma^2_c. \tag{2.5}$$

If $\sigma_s(0) := \inf_{f\in\mathbb{F}} \mathbb{E}[|\nabla f|^2/(8f) + Vf] > 0$, we have for $x, y \in \mathbb{R}^d$,

$$\Gamma_V(x+y) \leq \Gamma_V(x) + \Gamma_V(y). \tag{2.6}$$

Γ_V is monotone in V: Assume $V_1 \leq V_2$ are potentials, then

$$\Gamma_{V_1} \leq \Gamma_{V_2}. \tag{2.7}$$

For $y \in \mathbb{R}^d \setminus \{0\}$,

$$\Gamma_V^2(y/|y|) \geq 2\sigma_s(0). \tag{2.8}$$

For completeness, we restated (2.1) which is shown in (Rueß, 2014, Lemma 3.1). Many of these properties are already established for the Lyapunov exponent of Brownian motion in Poissonian potential, see, for example, (Sznitman, 1998, Chapter 5). In the discrete space setting of random walk in random potential, such results are obtained in (Zerner, 1998, Proposition 4). Formula (2.3) for the Lyapunov exponent of constant potential is well known, a calculation can be found in (Rueß, 2012, (2.9)).

Inequality (2.2) is stronger than inequality $\alpha_{cV} \le c\alpha_V$, $c \ge 1$, which one obtains applying Jensen inequality to the representation of the Lyapunov exponent given in (Rueß, 2014, (1.9)). This here allows to deduce the correct asymptotics given in (4.9). In the same way (2.4) could be deduced with Hölder inequality from (Rueß, 2014, (1.9)). The "squared" inequality however is a stronger result.

Inequality (2.8) can be interpreted as a relation between the Lyapunov exponent and the quenched free energy

$$\Lambda_\omega(0) := \limsup_{t\to\infty} \frac{1}{t} \ln E_0\left[\exp\left\{-\int_0^t V_\omega(Z_s)\,ds\right\}\right].$$

For example, in (Rueß, 2014, Corollary 1.4) under suitable assumptions, we could relate the quenched free energy of Brownian motion with drift $\lambda \in \mathbb{R}^d$ in potential V to the variational expression σ_s with an additional "drift term".

Proof of Proposition 2. For (2.2), note that

$$\int \frac{|\nabla f|^2}{8f} + cVf\,d\mathbb{P} \le c\int \frac{|\nabla f|^2}{8f} + Vf\,d\mathbb{P}.$$

For (2.3) recall the "inverse" Hölder inequality: If g, h are measurable, $h \neq 0$ $\mathbb{P}$-a.s., then for $r \in (1,\infty)$,

$$\mathbb{E}\big[|g|^{1/r}\big]^r \mathbb{E}\big[|h|^{-1/(r-1)}\big]^{-(r-1)} \le \mathbb{E}\big[|gh|\big]. \tag{2.9}$$

This follows by an application of Hölder's inequality $\|f_1 f_2\|_1 \le \|f_1\|_p \|f_2\|_q$, $1 \le p,q \le \infty$, $p^{-1}+q^{-1}=1$ to $f_1 := |gh|^{1/r}$, $f_2 := |h|^{-1/r}$, $p=r$, $q=r/(r-1)$. (2.9) applied to $r=2$, $g:=|\phi|^2$, $h:=f^{-1}$, and Jensen inequality give

$$\inf_{\phi\in\Phi_y} \int \frac{|\phi|^2}{2f}\,d\mathbb{P} \ge |y|^2/2. \tag{2.10}$$

Estimate with (2.10)

$$\begin{aligned}\Gamma_c^2(y) &= 4\inf_{f\in\mathbb{F}}\left(\int \frac{|\nabla f|^2}{8f} + cf\,d\mathbb{P}\right)\left(\inf_{\phi\in\Phi_y}\int \frac{|\phi|^2}{2f}\,d\mathbb{P}\right)\\ &\ge 4c \inf_{f\in\mathbb{F}}\inf_{\phi\in\Phi_y}\int \frac{|\phi|^2}{2f}\,d\mathbb{P} \ge 2c|y|^2.\end{aligned}$$

On the other hand, choosing $f \equiv 1$ and $\phi \equiv y$ in the variational expression for $\Gamma_c^2(y)$, one has $\Gamma_c^2(y) \le 2c|y|^2$.

For the inequality preceding (2.4), observe that:

$$\begin{aligned}\Gamma^2_{\sum_i \lambda_i V_i}(y) &= 4 \inf_{f\in\mathbb{F}} \inf_{\phi\in\Phi_y} \left(\int \sum_i \lambda_i \left(\frac{|\nabla f|^2}{8f} + V_i f\right) d\mathbb{P}\right)\left(\int \frac{|\phi|^2}{2f}\, d\mathbb{P}\right)\\ &\ge \sum_i \lambda_i 4 \inf_{f\in\mathbb{F}} \inf_{\phi\in\Phi_y}\left(\int \frac{|\nabla f|^2}{8f} + V_i f\, d\mathbb{P}\right)\left(\int \frac{|\phi|^2}{2f}\, d\mathbb{P}\right)\\ &= \sum_i \lambda_i \Gamma^2_{V_i}(y).\end{aligned}$$

Since the square root is concave and monotone the “non-squared” inequality (2.4) is valid.

For (2.5) use (2.10), (2.3) and

$$\begin{aligned}\Gamma^2_{c+V}(y) \ge{}& 4 \inf_{f\in\mathbb{F}} \left\{\left(\int \frac{|\nabla f|^2}{8f} + V f\, d\mathbb{P}\right) \inf_{\phi\in\Phi_y}\left(\int \frac{|\phi|^2}{2f}\, d\mathbb{P}\right)\right\}\\ &+ 4c \inf_{f\in\mathbb{F}} \inf_{\phi\in\Phi_y} \int \frac{|\phi|^2}{2f}\, d\mathbb{P}\\ \ge{}& \Gamma_V(y) + 2c|y|^2.\end{aligned}$$

For (2.6), we argue with the representation of Γ_V given in (Rueß, 2014, Propositions 3.13, 3.15): For $f \in \mathbb{F}$, $\eta \in S^{d-1}$ let $K(f) := \mathbb{E}[|\nabla f|^2/(8f) + V f]$ and $H(\eta, f) := \inf_{w\in\mathbb{D}} \mathbb{E}[|\nabla w - \eta|^2 f]$. Then

$$\begin{aligned}\Gamma_V(x+y) &= \sup_{\eta\in S^{d-1}} \big|\langle x+y,\eta\rangle\big| \inf_{f\in\mathbb{F}}\left[\frac{2K(f)}{H(\eta,f)}\right]^{1/2}\\ &\le \sup_{\eta\in S^{d-1}} \big(\big|\langle x,\eta\rangle\big| + \big|\langle y,\eta\rangle\big|\big) \inf_{f\in\mathbb{F}}\left[\frac{2K(f)}{H(\eta,f)}\right]^{1/2} \le \Gamma_V(x) + \Gamma_V(y).\end{aligned}$$

For (2.7) note that $\mathbb{E}[|\nabla f|^2/(8f) + V f]\mathbb{E}[|\phi|^2/(2f)]$ is monotone in V for all $f \in \mathbb{F}$.

For (2.8) use (2.10) and (2.1). □

3 Inequalities

3.1 Effect of randomness

In (Rueß, 2014, Corollary 1.3) as a direct consequence of Theorem 1, we have seen that

$$\Gamma_V \le \Gamma_{\mathbb{E}V}. \tag{3.1}$$

The following theorem is a refinement of (3.1).

Theorem 3. *Let $X := (\Omega, \mathcal{F}, \mathbb{P}, \tau)$ and $Y := (\Omega, \mathcal{G}, \mathbb{P}|_{\mathcal{G}}, \tau)$ be metric dynamical systems with $\mathcal{G} \subset \mathcal{F}$. Let V be a potential on X. Then with obvious notation,*

$$\Gamma_V^X \leq \Gamma_{\mathbb{E}[V|\mathcal{G}]}^Y.$$

Proof. Let $\mathbb{F}_s^X$, Φ_y^X and $\mathbb{F}_s^Y$, Φ_y^Y denote the spaces $\mathbb{F}$, Φ_y for the dynamics of X and Y, respectively. One has $\mathbb{F}_s^Y \subset \mathbb{F}_s^X$ and $\Phi_y^Y \subset \Phi_y^X$. Hence,

$$\begin{aligned}
&\inf_{f \in \mathbb{F}_s^X} \mathbb{E}\left[\frac{|\nabla f|^2}{8f} + Vf\right] \inf_{\phi \in \Phi_y^X} \mathbb{E}\left[\frac{|\phi|^2}{2f}\right] \\
&\leq \inf_{f \in \mathbb{F}_s^Y} \mathbb{E}\left[\frac{|\nabla f|^2}{8f} + \mathbb{E}[V|\mathcal{G}]f\right] \inf_{\phi \in \Phi_y^Y} \mathbb{E}\left[\frac{|\phi|^2}{2f}\right]
\end{aligned}$$

which shows the statement. □

3.2 Strict inequality

It is natural to ask whether the randomness of the potential has significant effect on the Lyapunov exponent. The following theorem gives a positive answer to this question.

Theorem 4. *Assume V is nondeterministic and weakly differentiable with $\|\nabla V\|_\infty < \infty$. Assume $0 < v_{\min} \leq V \leq v_{\max} < \infty$. Then for $y \neq 0$,*

$$\Gamma_V(y) < \Gamma_{\mathbb{E}V}(y).$$

Proof. Without restriction, we consider the set of functions $\mathbb{F}_w$ instead of $\mathbb{F}$ in the definition of Γ_V. This is possible by Rueß (2014, Proposition 2.2). Let $0 < p < \infty$ and choose $f_p := \beta V^{-p}$ with $\beta := \mathbb{E}[V^{-p}]^{-1}$. Then $f_p \in \mathbb{F}_w$. One has

$$\nabla f_p = -\beta p V^{-p-1} \nabla V,$$

using the chain rule for weak derivatives, see (Gilbarg and Trudinger, 1983, Lemma 7.5). Choosing $\phi \equiv y$, we get

$$\begin{aligned}
\Gamma_V^2(y) &\leq 2|y|^2 \mathbb{E}\left[\frac{|\nabla f_p|^2}{8f_p} + Vf_p\right] \mathbb{E}\left[\frac{1}{f_p}\right] \\
&= 2|y|^2 \mathbb{E}\left[\beta p^2 \frac{|\nabla V|^2}{8V^{p+2}} + \beta V^{1-p}\right] \mathbb{E}\left[\frac{V^p}{\beta}\right] \qquad (3.2) \\
&= 2|y|^2 \mathbb{E}\left[p^2 \frac{|\nabla V|^2}{8V^{p+2}} + V^{1-p}\right] \mathbb{E}[V^p].
\end{aligned}$$

We start considering $\psi(p) := \mathbb{E}[V^{1-p}]\mathbb{E}[V^p]$. ψ is differentiable on $\mathbb{R}$ with derivative

$$\psi'(p) = -\mathbb{E}[V^{1-p} \ln V]\mathbb{E}[V^p] + \mathbb{E}[V^{1-p}]\mathbb{E}[V^p \ln V],$$

where we used the theorem on differentiation under the integral sign, see, for example, (Bauer, 2001, Lemma 16.2). At $p = 0$ one has

$$\psi'(0) = -\mathbb{E}[V \ln V] + \mathbb{E}[V]\mathbb{E}[\ln V] = -\operatorname{Cov}(V, \ln V).$$

One has $\operatorname{Cov}(V, \ln V) > 0$ by FKG-inequality, see, for example, (Rinott and Saks, 1992, Theorem 1.2). Therefore, choosing a constant $\operatorname{Cov}(V, \ln V) > C_1 > 0$ one has for $p > 0$ small enough,

$$\psi(p) = \mathbb{E}\big[V^{1-p}\big]\mathbb{E}\big[V^p\big] < \mathbb{E}[V] - C_1 p. \tag{3.3}$$

On the other hand, there is a constant $C_2 > 0$ such that for $p > 0$,

$$\mathbb{E}\bigg[p^2 \frac{|\nabla V|^2}{8V^{p+2}}\bigg]\mathbb{E}\big[V^p\big] \le 8^{-1} p^2 v_{\min}{}^{-2} (v_{\max}/v_{\min})^p \|\nabla V\|_2^2 \le C_2 p^2. \tag{3.4}$$

Estimates (3.2), (3.3) and (3.4) give for $p > 0$ small

$$\Gamma_V^2(y) \le 2|y|^2\big(\mathbb{E}[V] - C_1 p + C_2 p^2\big),$$

which for $p > 0$ small enough is strictly lower than $\Gamma^2_{\mathbb{E}V}(y) = 2\mathbb{E}[V]|y|^2$, see (2.3). □

3.3 Perturbation and extension

The potential V may be perturbed by an external input or extended into "new" dimensions. Extensions are of interest if one considers for example, random chessboard potentials, see, for example, Dal Maso and Modica (1986). In the following, we elaborate a framework for external input and extensions and give estimates on Lyapunov exponents for perturbed or extended potentials.

Let $X := (\Omega_1, \mathcal{F}_1, \mathbb{P}_1, \tau^{(1)})$ and $Y := (\Omega_2, \mathcal{F}_2, \mathbb{P}_2, \tau^{(2)})$ be metric dynamical systems of dimensions d_1 and d_2, respectively. Consider some measure $\mathbb{P}$ on $(\Omega, \mathcal{F}) := (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$ with marginal distributions $\mathbb{P}_1$ and $\mathbb{P}_2$. We introduce two possible actions on Ω:

Extension of X: Define for $x = (x_1, x_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and $\omega \in \Omega$ the action

$$\tau_x^e \omega := \big(\tau_{x_1}^{(1)} \omega_1, \tau_{x_2}^{(2)} \omega_2\big).$$

Perturbation of X: Assume $d_1 = d_2$, define for $x \in \mathbb{R}^{d_1}$ and $\omega \in \Omega$

$$\tau_x^p \omega := \big(\tau_x^{(1)} \omega_1, \tau_x^{(2)} \omega_2\big).$$

Note that τ^e as well as τ^p are indeed product measurable actions on the respective spaces.

If $\mathbb{P}$ is invariant under τ^p or τ^e, then $\mathbb{P}$ is called a joining of $\mathbb{P}_1$ and $\mathbb{P}_2$. We denote the set of joinings with respect to τ^p by $J_p(X, Y)$ and the set of joinings with respect to τ^e by $J_e(X, Y)$. The product measure $\mathbb{P}_1 \otimes \mathbb{P}_2$ is always a joining with respect to τ^p and τ^e.

Note that joinings of ergodic dynamical systems are not necessarily ergodic any more. For example consider the torus $X^{\mathbb{T},d}$, see page 438. On $(\mathbb{T}^d \times \mathbb{T}^d, \mathcal{B}(\mathbb{T}^d) \otimes \mathcal{B}(\mathbb{T}^d), \mathscr{L} \otimes \mathscr{L})$ the shift τ^p is not ergodic.

Joinings are discussed in literature in great extent. Existence and ergodicity of joinings in general, and the question when the product measure leads to an ergodic joining are addressed, for example, in (Furstenberg, 1981, Chapter 5,6), (Cornfeld, Fomin and Sinaĭ, 1982, Chapter 10), (Rudolph, 1990, Chapter 6), Ryzhikov (1991). However, we want to mention that often in literature actions of only one transformation or actions of $\mathbb{Z}$ on Ω are considered, instead of studying the action of more general groups.

We recall a result in this direction: The definition of weak mixing for one shift can be found in (Rudolph, 1990, Definition 4.1).

Lemma 5 (See, e.g., (Rudolph, 1990, Proposition 4.19)). *Let ϕ be a measurable transformation of a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Then ϕ is weakly mixing under $\tilde{\mathbb{P}}$, if and only if the product $\phi \times \psi$ of ϕ with any other ergodic transformation ψ of some probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ is an ergodic transformation on $(\tilde{\Omega} \times \hat{\Omega}, \tilde{\mathcal{F}} \otimes \hat{\mathcal{F}}, \tilde{\mathbb{P}} \otimes \hat{\mathbb{P}})$.*

Here the product of ϕ and ψ is defined by $\phi \times \psi : \tilde{\Omega} \times \hat{\Omega} \to \tilde{\Omega} \times \hat{\Omega}$, $(\omega_1, \omega_2) \mapsto (\phi(\omega_1), \psi(\omega_2))$. On page 438 we have introduced the ergodic dynamical system $X^{\mathrm{poi},\nu}$, where $\mathbb{P}$ is a Poisson point process. We can construct the following example.

Example 1. The perturbation or extension of $X^{\mathrm{poi},\nu}$ with any other ergodic dynamical system is again an ergodic dynamical system under the product measure.

In fact, with (Daley and Vere-Jones, 2008, 12.3.II) considering bounded Borel measurable subsets of $\mathbb{R}^d$ we know that the Poisson point process satisfies (Daley and Vere-Jones, 2008, 12.3.I(iii)). In particular, according to (Rudolph, 1990, Definition 4.1) any transformation τ_x, $x \neq 0$, is weakly mixing under $\mathbb{P}$. Therefore, with help of Lemma 5 the statement follows.

We need some additional notation: With $\mathbb{E}_1$ we denote the expectation operator with respect to $\mathbb{P}_1$. We write $\Gamma^{\mathbb{P}_1}$, $\Gamma^{\mathbb{P},e}$ and $\Gamma^{\mathbb{P},p}$ for the variational functional in order to indicate the underlying dynamical system. $\pi_1 : \Omega_1 \times \Omega_2 \to \Omega_1$ denotes the projection onto Ω_1. For $y \in \mathbb{R}^{d_1}$ we set

$$\hat{y} := (y_1, \dots, y_{d_1}, 0, \dots, 0) \in \mathbb{R}^{d_1+d_2}.$$

The following result studies the effect of external input.

Theorem 6. *Let V be a potential on Ω. For any joining $\mathbb{P} \in J_e(X, Y)$, for any $y \in \mathbb{R}^{d_1}$,*

$$\Gamma_V^{\mathbb{P},e}(\hat{y}) \leq \Gamma^{\mathbb{P}_1}_{\mathbb{E}[V|\pi_1=\cdot]}(y).$$

If $d_1 = d_2$, the analogous inequality is valid for any joining $\mathbb{P} \in J_p(X, Y)$.

Note that in fact, $\Gamma^{\mathbb{P}_1}$ does not depend on the realisation of $\mathbb{E}[V|\pi_1 = \cdot]$.

Proof of Theorem 6. We prove the statement for τ^e. The same argument works for τ^p. Introduce $\mathbb{F}_1$ as the set $\mathbb{F}$ for the dynamical system X as defined on page 436. By $\mathbb{F}_e$, we denote the set $\mathbb{F}$ on Ω. Introduce Φ^1_y as the set Φ_y for X and $\Phi^e_{\hat{y}}$ as the set $\Phi_{\hat{y}}$ on Ω, see page 436. We define

$$\mathbb{F}^X := \{f \in \mathbb{F}_e : \forall \omega \in \Omega_1\ \exists c_\omega > 0 \text{ s.t. } f(\omega, \cdot) \equiv c_\omega\},$$

$$\Phi^X_{\hat{y}} := \{\phi \in \Phi^e_{\hat{y}} : \forall \omega \in \Omega_1\ \exists y_\omega \in \mathbb{R}^{d_1} \text{ s.t. } (\phi_i(\omega, \cdot))_{i=1,\dots d_1} \equiv y_\omega,$$
$$\forall d_1 < i \le d_2 : \phi_i \equiv 0\}.$$

Considering only the first component any $f \in \mathbb{F}^X$ can be identified uniquely with $\tilde{f} \in \mathbb{F}_1$ such that $f = \tilde{f} \circ \pi_1$. Then $|\nabla^{\tau_e} f|^2 = |\nabla^{\tau^{(1)}} \tilde{f}|^2 \circ \pi_1$ with obvious notation. Analogously any $\phi \in \Phi^X_{\hat{y}}$ can be identified uniquely with $\tilde{\phi} \in \Phi^1_y$ after a projection of ϕ onto its first d_1 components such that $(\phi_i)_{i=1,\dots,d_1} = (\tilde{\phi} \circ \pi_1)_{i=1,\dots,d_1}$. Then $|\phi|^2 = |\tilde{\phi} \circ \pi_1|^2$ and we get

$$\begin{aligned}
(\Gamma^{\mathbb{P},e}_V)^2(\hat{y}) &\le 4 \inf_{f \in \mathbb{F}^X} \inf_{\phi \in \Phi^X_{\hat{y}}} \mathbb{E}\Big[\frac{|\nabla f|^2}{8f} + Vf\Big] \mathbb{E}\Big[\frac{|\phi|^2}{2f}\Big] \\
&= 4 \inf_{\tilde{f} \in \mathbb{F}_1} \inf_{\tilde{\phi} \in \Phi^1_y} \mathbb{E}_1\Big[\frac{|\nabla \tilde{f}|^2}{8\tilde{f}} + \mathbb{E}[V|\pi_1 = \cdot]\tilde{f}\Big] \mathbb{E}_1\Big[\frac{|\tilde{\phi}|^2}{2\tilde{f}}\Big] \\
&= \big(\Gamma^{\mathbb{P}_1}_{\mathbb{E}[V|\pi_1 = \cdot]}\big)^2(y).
\end{aligned}$$

This shows the statement. □

We use this result to study sums and products of independent potentials.

Corollary 7. *Let $\mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2$. Assume $V_1, V_2 \in L^1(\mathbb{P})$ with V_1 constant in the second component and V_2 constant in the first component. Then for $y \in \mathbb{R}^{d_1}$,*

$$\Gamma^{\mathbb{P},e}_{V_1+V_2}(\hat{y}) \le \Gamma^{\mathbb{P}_1}_{V_1+\mathbb{E}V_2}(y), \qquad \Gamma^{\mathbb{P},e}_{V_1V_2}(\hat{y}) \le \Gamma^{\mathbb{P}_1}_{V_1\mathbb{E}V_2}(y),$$

where for the first inequality $V_1 + V_2$ and for the second V_1V_2 is required to be a potential. Analogous results hold for the action τ^p.

4 Continuity

In this section, we study continuity properties of the Lyapunov exponent. We consider continuity with respect to the underlying probability measure, continuity with respect to the potential and we are also interested in the exact rate of convergence of the Lyapunov exponent for scaled potentials. In Section 5, we give examples which show that the continuity results we obtain here are essentially all one can expect in general. Additional assumptions however should allow to derive stronger results. Possible enforcements of the prerequisites are, for example, mixing properties of the underlying probability measure, finite range dependence properties of the potential, or compactness of the space Ω. We show in Section 4.4 that compactness allows to deduce exact results. Both, compactness assumptions as well as additional mixing or independence properties are studied in literature in comparable situations:

For example, for the time constant in i.i.d. first-passage percolation continuity has been investigated in (Cox and Kesten, 1981, Theorem 3), see also (Smythe and Wierman, 1978, Chapter X.4). Recently, continuity of the Lyapunov exponent of random walk in i.i.d. random potential with respect to convergence in distribution of the underlying potential has been shown by Le (2013). Models with long range dependencies are considered, for example, by Scholler (2014). We also want to refer to (Mourrat, 2012, Section 11) where similar questions are addressed. In (Rassoul-Agha and Seppäläinen, 2014, Lemma 3.1) continuity of the quenched free energy of random walk in i.i.d. potential with respect to L^p convergence, $p > d$, of the potential is established. Continuity of quantities similar to the Lyapunov exponent is studied, for example, by Bourgain and Jitomirskaya (2002), Bourgain (2005), Jitomirskaya and Marx (2011), Duarte and Klein (2014) and You and Zhang (2014). There, compactness is a central feature in order to obtain continuity properties.

It is immediate to show continuity of the Lyapunov exponent with respect to uniform convergence of the potential:

Proposition 8. *Let V and V' be potentials. Assume $V \geq v_{\min} > 0$ and $\|V' - V\|_\infty < v_{\min}$. Then for $y \in \mathbb{R}^d$,*

$$\left|\Gamma_V^2(y) - \Gamma_{V'}^2(y)\right| \leq \|V' - V\|_\infty \Gamma_V^2(y)/v_{\min}.$$

Proof. Let $\varepsilon := \|V' - V\|_\infty$. Then $V' \leq V + \varepsilon \leq V(1 + \varepsilon/v_{\min})$. Now use (2.7) and (2.2) in order to get the lower bound. The upper bound follows analogously from $V' \geq V - \varepsilon \geq V(1 - \varepsilon/v_{\min})$, (2.7) and the corresponding inequality after (2.2). □

Consideration of continuity with respect to weak convergence of the potential as well as continuity with respect to the underlying measure turn out to be more delicate. While we are able to show upper semi-continuity, see Section 4.2, lower semi-continuity does not hold in general as indicated by examples given in Section 5. This resembles the situation in the articles of Cox and Kesten (1981) and Le (2013) where the proof of the lower bound was more involved than the proof of the upper bound.

4.1 Denseness

In Section 4.2, we study continuity of Γ_V with respect to weak convergence of the underlying probability measure $\mathbb{P}$ on Ω, and we therefore need to introduce function spaces of continuous functions. Assume Ω is a topological space and $\mathcal{F}$ is the Borel σ-algebra. We set

$$\mathbb{D}^c := \{ f \in \mathbb{D} : \forall n \in \mathbb{N}_0 \; D^n f \text{ is continuous w.r.t. the topology on } \Omega \},$$
$$\mathbb{F}^c := \mathbb{F} \cap \mathbb{D}^c,$$
$$\Phi_y^c := \Phi_y \cap (\mathbb{D}^c)^d.$$

We need the following condition on $(\Omega, \mathcal{F}, \mathbb{P}, \tau)$:

(T) Ω is a completely regular, first countable Hausdorff space s.t. $\mathcal{F}$ is the Borel σ-algebra, $\mathbb{P}$ is a Radon measure, the mapping $\omega \mapsto \tau_x \omega$ is continuous for all x.

In (Rueß, 2014, Proposition 2.2) it is shown that if $V \in L^2$ we may replace the function spaces in the definition of $\Gamma_V(y)$ by any of the sets in $\mathfrak{F}$ and $\mathfrak{P}_y$ without changing $\Gamma_V(y)$.

Proposition 9. *Assume that* $(\Omega, \mathcal{F}, \mathbb{P}, \tau)$ *satisfies* (T). *Then* $\mathbb{D}^c$ *is dense in* L^2. *Moreover,* $\mathbb{D}^c \in \mathfrak{D}$, $\mathbb{F}^c \in \mathfrak{F}$ *and for any* $y \in \mathbb{R}^d$ *one has* $\Phi_y^c \in \mathfrak{P}_y$.

Proof. This proof uses the concept of convolution on X, see, for example, (Jikov, Kozlov and Oleinik, 1994, p. 232) or (Rueß, 2014, Lemma 4.4). We need a "smoothing kernel" $\kappa \in C_c^\infty$ which is assumed to be an even function, $\kappa \geq 0$, and $\int_{\mathbb{R}^d} \kappa(x)\, dx = 1$. We rescale $\kappa_\varepsilon(x) := \varepsilon^{-d} \kappa(x/\varepsilon)$ for $\varepsilon > 0$.

We start by proving $\mathbb{F}^c \in \mathfrak{F}$: Let $f \in \mathbb{F}_w$. Without restriction, assume $f \leq \|f\|_\infty$ and $\inf_\Omega f > 0$. Choose δ s.t. $\inf_\Omega f > \delta > 0$. With (Rueß, 2014, (4.12), (4.14)) choose $\varepsilon > 0$ s.t.

$$\|f * \kappa_\varepsilon - f\|_\nabla \leq \delta/3. \tag{4.1}$$

Define $d_\varepsilon := \sup_i \int |\partial_i \kappa_\varepsilon| \, d\mathscr{L}$ and set $\delta_\varepsilon := \delta/(1 \vee d_\varepsilon) \leq \delta$. By Lusin's theorem, there exists a sequence of compact sets $K_n \subset \Omega$, $n \in \mathbb{N}$, s.t. f is continuous on K_n for $n \in \mathbb{N}$ and $\mathbb{P}[K_n] \nearrow 1$ for $n \to \infty$, see, for example, (Bogachev, 2007, Theorem 7.1.13). The function $f|_{K_n}$ can be extended from the compact set K_n to a continuous function g_n on whole Ω s.t. $g_n|_{K_n} = f|_{K_n}$ and $\inf_\Omega f \leq g_n \leq \|f\|_\infty$ as it is stated for completely regular Hausdorff spaces in (Bogachev, 2007, Exercise 6.10.22). Choose $n_0 \in \mathbb{N}$ s.t. for $n \geq n_0$,

$$\mathbb{P}[K_n^c] \leq \delta_\varepsilon (3\|f\|_\infty)^{-1},$$

where $K_n^c := \Omega \setminus K_n$. Let $a_n := 1 - \mathbb{E} g_n = \mathbb{E}[f - g_n]$. For $n \geq n_0$,

$$|a_n| \leq \mathbb{E}[|f - g_n|, K_n^c] \leq \|f\|_\infty \mathbb{P}[K_n^c] \leq \delta_\varepsilon/3.$$

We set

$$f_n := g_n + a_n.$$

Then $\mathbb{E}[f_n] = 1$. Moreover, since $\delta < \inf_\Omega f$ one has $\inf_{n \geq n_0} f_n \geq \inf_\Omega f + a_n \geq \delta/2 > 0$. And also $f_n * \kappa_\varepsilon \in \mathbb{F}$, use, for example, (Rueß, 2014, Lemma 4.4).

Moreover, $\omega \mapsto f_{n,\omega}(x)\kappa_\varepsilon(x)$ is continuous and bounded by $\|f_n\|_\infty \|\kappa_\varepsilon\|_\infty$ for any x. Since Ω is first countable, continuity is equivalent to sequential continuity, see (Willard, 1970, Corollary 10.5). Hence Lebesgue's dominated convergence theorem may be applied in order to show that $f_n * \kappa_\varepsilon$ is continuous in ω. ("continuity of integrals with respect to a parameter", see, for example, (Bauer, 2001, Lemma 16.1)). A similar argument together with equality $\partial_i(f_n * \kappa_\varepsilon) = -f_n * (\partial_i \kappa_\varepsilon)$ shows that $D^m f_{n,\varepsilon}$ is continuous and bounded for any $m \in \mathbb{N}_0$. In particular, $f_n * \kappa_\varepsilon \in \mathbb{F}^c$.

$f_n * \kappa_\varepsilon$ approximates f: Indeed, for $n \geq n_0$,

$$\begin{aligned} \|f - f_n\|_2 &= \|f - g_n - a_n\|_2 \leq \|f - g_n\|_2 + |a_n| \\ &\leq \|f\|_\infty \mathbb{P}\big[K_n^c\big] + \delta_\varepsilon/3 \leq 2\delta_\varepsilon/3. \end{aligned} \tag{4.2}$$

Further, Young's inequality, see, for example, (Rueß, 2014, (4.11)), gives

$$\|f * \kappa_\varepsilon - f_n * \kappa_\varepsilon\|_2 \leq \|f - f_n\|_2. \tag{4.3}$$

By (4.1), (4.2), (4.3) we get

$$\|f - f_n * \kappa_\varepsilon\|_2 \leq \|f - f * \kappa_\varepsilon\|_2 + \|f * \kappa_\varepsilon - f_n * \kappa_\varepsilon\|_2 \leq \delta.$$

We consider derivatives in an analogous manner: Again with Young's inequality,

$$\begin{aligned} \big\|\partial_i(f * \kappa_\varepsilon) - \partial_i(f_n * \kappa_\varepsilon)\big\|_2 &= \big\|f * \partial_i(\kappa_\varepsilon) - f_n * \partial_i(\kappa_\varepsilon)\big\|_2 \\ &\leq d_\varepsilon \|f - f_n\|_2. \end{aligned} \tag{4.4}$$

Hence, (4.1), (4.2), (4.4) imply

$$\begin{aligned} \big\|\partial_i f - \partial_i(f_n * \kappa_\varepsilon)\big\|_2 &= \big\|\partial_i f - \partial_i(f * \kappa_\varepsilon)\big\|_2 + \big\|\partial_i(f * \kappa_\varepsilon) - \partial_i(f_n * \kappa_\varepsilon)\big\|_2 \\ &\leq \delta/3 + d_\varepsilon 2\delta_\varepsilon/3 \leq \delta. \end{aligned}$$

This proves $\mathbb{F}^c \in \mathfrak{F}$.

In order to show $\mathbb{D}^c \in \mathfrak{D}$ note first, that it is sufficient to show $\mathbb{D}^c$ dense in $\mathbb{D}$ since $\mathbb{D} \subset \mathbb{D}_w$ in the desired way by Rueß (2014, Lemma 2.1). Let $w \in \mathbb{D}$, $w \neq 0$ and consider $\psi := (w - \mathbb{E}w)/(2\|w - \mathbb{E}w\|_\infty) + 1$. $\psi \in \mathbb{F}$ and we can apply the previous and get a sequence $\psi_n \to \psi$ in $\|\cdot\|_\nabla$, $(\psi_n)_n \subset \mathbb{F}^c$. Then $w_n := (\psi_n - 1)2\|w - \mathbb{E}w\|_\infty + \mathbb{E}w \to w$ in the desired way and $(w_n)_n \subset \mathbb{D}^c$. Thus, $\mathbb{D}^c \in \mathfrak{D}$.

In order to examine Φ_y^c, since $\mathbb{D}^c$ is dense in L^2 s.t. $\partial_i \mathbb{D}^c \subset \mathbb{D}^c$ and $\tau_x \mathbb{D}^c \subset \mathbb{D}^c$, we may apply (Rueß, 2014, Lemma 4.7). Using the fact that the space of weak divergence-free vector fields with expectation y equals Φ_y^w, use (Rueß, 2014, (4.18)), we get $(\mathbb{D}^c)^d \cap \Phi_y^w$ is dense in Φ_y^w with respect to $\|\cdot\|_2$. Any $\phi \in (\mathbb{D}^c)^d \cap \Phi_y^w$ equals up to an exceptional set some $\tilde{\phi} \in (\mathbb{D}^c)^d \cap \Phi_y$. This shows $\Phi_y^c \in \mathfrak{P}_y$. □

4.2 Semi-continuity

Our first continuity result considers also weak L^1 convergence.

Proposition 10. *Let V, V_n, $n \in \mathbb{N}$, be potentials on Ω. Assume that for all $f \in \mathbb{F}$ one has $\limsup_{n\to\infty} \mathbb{E}[V_n f] \le \mathbb{E}[V f]$, then for any $y \in \mathbb{R}^d$,*

$$\limsup_{n\to\infty} \Gamma_{V_n}(y) \le \Gamma_V(y). \tag{4.5}$$

Note that as soon as V_n, $n \in \mathbb{N}$, and V are potentials in L^2, in order to obtain (4.5) it suffices to know that there exists a set $\tilde{\mathbb{F}} \in \mathfrak{F}$ such that for all $f \in \tilde{\mathbb{F}}$ the condition $\limsup_{n\to\infty} \mathbb{E}[V_n f] \le \mathbb{E}[V f]$ is satisfied, use, for example, (Rueß, 2014, Proposition 2.2).

Proof of Proposition 10. By definition,

$$\limsup_{n\to\infty} \Gamma_{V_n}(y) = \inf_{n\ge 0} \sup_{m\ge n} \Gamma_{V_m}(y).$$

After an interchange of $\inf_{n\ge0} \sup_{m\ge n}$ and $\inf_{f\in\mathbb{F}} \inf_{\phi\in\Phi_y}$ in the variational expression, the statement follows. □

In order to study continuity with respect to weak convergence of the underlying probability measure, assume Ω is a topological space: Recall the condition (T) introduced on page 447.

Theorem 11. *Assume $(\Omega, \mathcal{F}, \mathbb{P}, \tau)$ satisfies* (T) *and V is a potential, which is bounded and continuous with respect to the topology on Ω. Let $(\mathbb{P}_n)_n$ be a sequence of Radon probability measures on $(\Omega, \mathcal{F})$ such that $(\Omega, \mathcal{F}, \mathbb{P}_n, \tau)$ is a metric dynamical system for all $n \in \mathbb{N}$. If $\mathbb{P}_n \to \mathbb{P}$ weakly, then for any $y \in \mathbb{R}^d$, with obvious notation,*

$$\limsup_{n\to\infty} \Gamma_V^{\mathbb{P}_n}(y) \le \Gamma_V^{\mathbb{P}}(y).$$

Proof. Let $\mathbb{F}_n^c$ and $\Phi_{y,n}^c$ denote the function spaces with respect to $\mathbb{P}_n$. We denote with $\mathbb{E}_n$ the expectation operator with respect to $\mathbb{P}_n$. Then one has bijective mappings

$$\mathbb{F}^c \to \mathbb{F}_n^c : f \mapsto \tilde{f} := f/\mathbb{E}_n[f], \quad \text{and} \quad \Phi_y^c \to \Phi_{y,n}^c : \phi \mapsto \tilde{\phi} := \phi - \mathbb{E}_n\phi + y.$$

Therefore,

$$\begin{aligned}
&\limsup_{n\to\infty} \Gamma_V^{\mathbb{P}_n}(y) \\
&\quad \le 2 \limsup_{n\to\infty} \inf_{f\in\mathbb{F}_n^c} \inf_{\phi\in\Phi_{n,y}^c} \left(\mathbb{E}_n\left[\frac{|\nabla f|^2}{8f} + Vf\right] \mathbb{E}_n\left[\frac{|\phi|^2}{2f}\right]\right)^{1/2} \\
&\quad = 2 \limsup_{n\to\infty} \inf_{f\in\mathbb{F}^c} \inf_{\phi\in\Phi_y^c} \left(\mathbb{E}_n\left[\frac{|\nabla f|^2}{8f} + Vf\right] \mathbb{E}_n\left[\frac{|\phi - \mathbb{E}_n\phi + y|^2}{2f}\right]\right)^{1/2}.
\end{aligned}$$

As in the proof of Proposition 10, the latter is less or equal to

$$2 \inf_{f\in\mathbb{F}^c} \inf_{\phi\in\Phi_y^c} \left(\inf_{n\geq 0} \sup_{m\geq n} \mathbb{E}_m\left[\frac{|\nabla f|^2}{8f} + Vf\right]\mathbb{E}_m\left[\frac{|\phi - \mathbb{E}_m\phi + y|^2}{2f}\right]\right)^{1/2}. \tag{4.6}$$

∇f is continuous and bounded for $f \in \mathbb{F}^c$. So is Vf by assumptions on V. Thus, weak convergence of $\mathbb{P}_n$ to $\mathbb{P}$ implies for $f \in \mathbb{F}^c$,

$$\mathbb{E}_n\left[\frac{|\nabla f|^2}{8f} + Vf\right] \to \mathbb{E}\left[\frac{|\nabla f|^2}{8f} + Vf\right] \qquad \text{as } n \to \infty. \tag{4.7}$$

Again weak convergence shows for $\phi \in \Phi_y^c$ that $\mathbb{E}_n\phi \to \mathbb{E}\phi$ for $n \to \infty$. Therefore, $\mathbb{E}_n[|y - \mathbb{E}_n\phi|^2/(2f)] \leq (2\min_\Omega f)^{-1}|y - \mathbb{E}_n\phi|^2 \to 0$, and we get for $n \to \infty$,

$$\begin{aligned}
&\mathbb{E}_n\left[\frac{|\phi - \mathbb{E}_n\phi + y|^2}{2f}\right] \\
&\quad = \mathbb{E}_n\left[\frac{|\phi|^2}{2f} + \frac{|y - \mathbb{E}_n\phi|^2}{2f} + \frac{2\phi\cdot(y - \mathbb{E}_n\phi)}{2f}\right] \to \mathbb{E}\left[\frac{|\phi|^2}{2f}\right].
\end{aligned} \tag{4.8}$$

By Lemma 9 and (Rueß, 2014, Proposition 2.2), we can substitute the spaces $\mathbb{F}^c$ and Φ_y^c with the spaces $\mathbb{F}$, Φ_y in the definition of Γ_V, and we get with (4.6), (4.7), (4.8),

$$\limsup_{n\to\infty} \Gamma_V^{\mathbb{P}_n}(y) \leq 2 \inf_{f\in\mathbb{F}^c} \inf_{\phi\in\Phi_y^c} \left(\mathbb{E}\left[\frac{|\nabla f|^2}{8f} + Vf\right]\mathbb{E}\left[\frac{|\phi|^2}{2f}\right]\right)^{1/2} = \Gamma_V^{\mathbb{P}}(y),$$

which was to be shown. □

4.3 Scaling

The variational formula also enables to determine convergence rates if scaled potentials are considered.

Proposition 12. *Let $c \geq 0$ and V be a potential. Let $V_n := V/n$. Then for all $y \in \mathbb{R}^d$,*

$$\Gamma_V^2(y) \leq n\big(\Gamma_{c+V_n}^2(y) - \Gamma_c^2(y)\big) \leq 2\mathbb{E}[V]|y|^2. \tag{4.9}$$

The rate of convergence as in Proposition 12 for scaled potentials has been investigated previously in the discrete space setting of random walk in i.i.d. integrable potential by Wang (2002) and Kosygina, Mountford and Zerner (2011). If V is not necessarily integrable the asymptotic behaviour has been recently established in the discrete setting by Mountford and Mourrat (2013, 2015). For Brownian motion in Poissonian potential speed of convergence is established by Rueß (2012). In the works of Kosygina, Mountford and Zerner (2011), Mountford and

Mourrat (2013) and Rueß (2012) c is assumed to equal zero. Kosygina, Mountford and Zerner (2011) and Rueß (2012) determined the speed of convergence to zero of α_{V_n} to equal $n^{-1/2}\sqrt{2\mathbb{E}[V]}|y|$. This coincides with the convergence speed $n^{-1/2}$ obtained from (4.9) for $c = 0$.

Additional assumptions allow to improve these results. For periodic potentials in Theorem 14, we get exact rates of convergence for more general scalings of the potential. Proposition 12 is essentially all one might expect in general. This is illustrated by an example given in Section 5.4.

Proof of Proposition 12. One has $n^{-1}\Gamma^2_{nV_n}(y) \le \Gamma^2_{c+V_n}(y) - \Gamma^2_c(y) \le 2\mathbb{E}[V_n] \times |y|^2$, where the upper bound follows from (3.1) and (2.3), the lower bound from (2.5) and (2.2). Since $nV_n = V$ this shows the statement. □

4.4 Continuity on the torus

The results obtained in Section 4.2 can be improved considerably if the underlying space Ω is assumed to be compact: In the case that $X = X^{\mathbb{T},1}$ where Ω is the one dimensional torus, see page 438, we get the following. We abbreviate for $f \in \mathbb{F}_w$,

$$B(f) := \inf_{\phi\in\Phi_y} \int \frac{|\phi|^2}{2f}\, d\mathbb{P}.$$

Theorem 13. *Let $X = X^{\mathbb{T},1}$. Let V_n, $n \in \mathbb{N}$, and V be potentials such that $V_n \to V$ in L^1 and $V \ge v_{\min} > 0$. Then there is a constant $C > 0$, depending only on $\mathbb{E}[V]$ and $v_{\min}$, and there is $n_0 \in \mathbb{N}$ such that for $n \ge n_0$, for $y \in \mathbb{R}^d$,*

$$\left|\Gamma^2_{V_n}(y) - \Gamma^2_V(y)\right| \le C\|V_n - V\|_1 |y|^2.$$

Proof. Without restriction, we may assume $|y| = 1$, see (2.1). Let $\varepsilon_n := \|V_n - V\|_1$ and choose n_0 such that for $n \ge n_0$,

$$\varepsilon_n \le (v_{\min}/2)\left(\sqrt{32\mathbb{E}[V]} + 1\right)^{-1}.$$

Note that in particular, $\varepsilon_n \le \mathbb{E}[V]$.

One has by (3.1) and (2.3) for $n \ge n_0$,

$$\Gamma^2_{V_n}(y) \le 4\mathbb{E}[V] =: C_0. \tag{4.10}$$

We choose a "minimising" sequence $(f_n)_n \subset \mathbb{F}$ such that for $n \in \mathbb{N}$,

$$\Gamma^2_{V_n}(y) \ge 4\mathbb{E}\left[\frac{|f_n'|^2}{8f_n} + V_n f_n\right] B(f_n) - \varepsilon_n. \tag{4.11}$$

An application of the "inverse" Hölder inequality (2.9) with $r = 2$ shows

$$\mathbb{E}\left[\frac{|f_n'|^2}{f_n}\right] \ge \mathbb{E}[|f_n'|]^2, \tag{4.12}$$

since $\mathbb{E}[f_n] = 1$. For $n \geq n_0$, by (4.10), (4.11), (4.12), (2.10), since $\mathbb{E}[V_n f_n] \geq 0$,

$$C_1 := (8C_0)^{1/2} \geq \mathbb{E}[|f_n'|]. \tag{4.13}$$

An application of the fundamental theorem of calculus shows for $n \geq n_0$, for all $x < y \in \mathbb{T}^1$,

$$|f_n(y) - f_n(x)| = \left| \int_x^y f_n'(t)\,dt \right| \leq \int_x^y |f_n'(t)|\,dt \leq C_1. \tag{4.14}$$

$\mathbb{E}[f_n] = 1$, thus, each f_n attains the value 1. We get for n large, $f_n(x) \leq C_1 + 1 =: C_2$ for all $x \in \mathbb{T}^1$. Therefore, for $n \geq n_0$,

$$\big|\mathbb{E}[V_n f_n] - \mathbb{E}[V f_n]\big| \leq C_2 \|V_n - V\|_1. \tag{4.15}$$

We need an upper bound on $B(f_n)$: By (4.10), (4.11) and (4.15), for $n \geq n_0$,

$$2C_0 \geq C_0 + \varepsilon_n \geq 4\mathbb{E}[V_n f_n]B(f_n) \geq 4\big(\mathbb{E}[V f_n] - C_2\varepsilon_n\big)B(f_n) \geq 2v_{\min}B(f_n).$$

This shows that for $n \geq n_0$,

$$B(f_n) \leq C_0/v_{\min} =: C_3. \tag{4.16}$$

Finally, by (4.11), (4.15), (4.16), for $n \geq n_0$

$$\begin{aligned}\Gamma_{V_n}^2(y) &\geq 4\mathbb{E}\left[\frac{|f_n'|^2}{8f_n} + V f_n\right]B(f_n) - 4C_2C_3\varepsilon_n - \varepsilon_n \\ &\geq \Gamma_V^2(y) - (1 + 4C_2C_3)\varepsilon_n.\end{aligned}$$

The proof of the upper bound is similar: Choose a minimising sequence $(g_n)_n \subset \mathbb{F}$ such that for $n \in \mathbb{N}$,

$$\Gamma_V^2(y) \geq 4\mathbb{E}\left[\frac{|g_n'|^2}{8g_n} + V g_n\right]B(g_n) - \varepsilon_n. \tag{4.17}$$

As in (4.13) by (3.1) and (2.3), "inverse" Hölder inequality, for $n \geq n_0$,

$$\mathbb{E}[|g_n'|] \leq C_1.$$

Thus, as in (4.14) for $n \geq n_0$, for $x \in \mathbb{T}^1$ one has $g_n(x) \leq C_2$. This shows

$$\big|\mathbb{E}[V_n g_n] - \mathbb{E}[V g_n]\big| \leq C_2 \|V_n - V\|_1. \tag{4.18}$$

We have similar to (4.16) $2C_0 \geq 4v_{\min}B(g_n)$, in particular,

$$B(g_n) \leq C_3. \tag{4.19}$$

Therefore, by (4.18), (4.19) and (4.17), for $n \geq n_0$,

$$\begin{aligned}\Gamma_{V_n}^2(y) &\leq 4\mathbb{E}\left[\frac{|g_n'|^2}{8g_n} + V_n g_n\right]B(g_n) \leq 4\mathbb{E}\left[\frac{|g_n'|^2}{8g_n} + V g_n\right]B(g_n) + 4C_2C_3\varepsilon_n \\ &\leq \Gamma_V^2(y) + (1 + 4C_2C_3)\varepsilon_n.\end{aligned}$$

This shows the statement. □

As we have an L^1-Poincaré inequality on the d-dimensional torus, we can calculate the convergence rate on the torus exactly:

Theorem 14. *Let $X = X^{\mathbb{T},d}$ and V_n, $n \in \mathbb{N}$, V be potentials. Assume $nV_n \to V$ for $n \to \infty$ in L^1 and V is bounded. Let $c \geq 0$, then for $y \in \mathbb{R}^d$,*

$$n\big(\Gamma^2_{V_n+c}(y) - \Gamma^2_c(y)\big) \to 2\mathbb{E}[V]|y|^2 \qquad \text{as } n \to \infty.$$

Proof. Let $y \neq 0$. The upper bound follows from (3.1), (2.3). For the lower, let $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ one has $\mathbb{E}[nV_n] \leq 2\mathbb{E}[V]$. By (3.1), (2.3) for $n \geq n_0$,

$$\psi_n := n\Gamma^2_{V_n}(y) \leq 2\mathbb{E}[nV_n]|y|^2 \leq 4\mathbb{E}[V]|y|^2 =: C_0. \tag{4.20}$$

Choose $(f_n)_n \subset \mathbb{F}$ such that

$$\begin{aligned}\psi_n &= 4n \inf_{f\in\mathbb{F}} \mathbb{E}\left[\frac{|\nabla f|^2}{8f} + Vf\right]B(f) \\ &\geq 4n\mathbb{E}\left[\frac{|\nabla f_n|^2}{8f_n} + Vf_n\right]B(f_n) - 1/n.\end{aligned} \tag{4.21}$$

Therefore, with (4.20) and (2.10), for $n \geq n_0$, $(C_0 + 1/n)/n \geq \psi_n/n \geq 2\mathbb{E}[|\nabla f_n|^2/(8f_n)]|y|^2$, which shows

$$\mathbb{E}\left[\frac{|\nabla f_n|^2}{8f_n}\right] \to 0 \qquad \text{as } n \to \infty. \tag{4.22}$$

Using "inverse" Hölder inequality (2.9) and Poincaré inequality, see (Gilbarg and Trudinger, 1983, (7.45)), we get for $n \in \mathbb{N}$,

$$\mathbb{E}\left[\frac{|\nabla f_n|^2}{8f_n}\right] \geq \mathbb{E}\big[|\nabla f_n|\big]^2/8 \geq c_p\mathbb{E}\big[|f_n - 1|\big]^2/8,$$

where the constant c_p comes from the Poincaré inequality. Thus, by (4.22)

$$\|f_n - 1\|_1 \to 0 \qquad \text{as } n \to \infty. \tag{4.23}$$

In particular, the set $\{f_n : n \in \mathbb{N}\}$ is uniformly integrable, see (Durrett, 1996, Theorem 4.5.2), and we get for $M_n := \|nV_n - V\|_1^{-1/2}$ that

$$\varepsilon_{1,n} := \mathbb{E}[f_n, f_n \geq M_n] \to 0 \qquad \text{as } n \to \infty. \tag{4.24}$$

We may estimate for $n \in \mathbb{N}$,

$$\begin{aligned}\mathbb{E}[nV_nf_n] &\geq \mathbb{E}\big[V(f_n \wedge M_n)\big] - \|nV_n - V\|_1 M_n \\ &\geq \mathbb{E}[Vf_n] - \|V\|_\infty \mathbb{E}\big[|f_n - f_n \wedge M_n|\big] - \|nV_n - V\|_1^{1/2} \\ &= \mathbb{E}[Vf_n] - \|V\|_\infty\varepsilon_{1,n} - \|nV_n - V\|_1^{1/2} \geq \mathbb{E}[V] - \varepsilon_{2,n},\end{aligned} \tag{4.25}$$

with $\varepsilon_{2,n} := \|V\|_\infty(\|f_n - 1\|_1 + \varepsilon_{1,n}) + \|nV_n - V\|_1^{1/2}$. Note that by (4.23), (4.24) and by assumptions on V one has $\varepsilon_{2,n} \to 0$ as $n \to \infty$. We need control of $B(f_n)$: Let $n_1 \geq n_0$ such that for $n \geq n_1$ one has $1/n \leq C_0$ and $\varepsilon_{2,n} \leq \mathbb{E}[V]/2$. Then with (4.20), (4.21), (4.25) for $n \geq n_1$

$$2C_0 \geq C_0 + 1/n \geq 4\mathbb{E}[nV_n f_n]B(f_n) \geq 4\big(\mathbb{E}[V] - \varepsilon_{2,n}\big)B(f_n) \geq 2\mathbb{E}[V]B(f_n).$$

This shows for $n \geq n_1$,

$$B(f_n) \leq C_0/\mathbb{E}[V] =: C_1. \tag{4.26}$$

Therefore, by (2.5), (4.21), (4.25), (4.26) and (2.10), for $n \geq n_1$,

$$\begin{aligned} n\big(\Gamma^2_{c+V_n}(y) - \Gamma^2_c(y)\big) &\geq n\Gamma^2_{V_n}(y) \\ &\geq 4n\mathbb{E}\Big[\frac{|\nabla f_n|^2}{8f_n} + V_n f_n\Big]B(f_n) - 1/n \\ &\geq 4\big(\mathbb{E}[V] - \varepsilon_{2,n}\big)B(f_n) - 1/n \\ &\geq 4\mathbb{E}[V]B(f_n) - 4C_1\varepsilon_{2,n} - 1/n \\ &\geq 2\mathbb{E}[V]|y|^2 - 4C_1\varepsilon_{2,n} - 1/n. \end{aligned}$$

This finishes the argument. □

5 Examples

The Lyapunov exponent is semi-continuous in many cases as outlined in Section 4. We provide examples which show that continuity of the Lyapunov exponent with respect to weak convergence of the underlying measure, continuity with respect to L^p convergence of the potential, $1 \leq p < \infty$, and also a speed of convergence as the one established in Proposition 12 are not valid in general. This should be compared to similar models such as random walks in random potential, and we refer to the discussion in Section 4.

The example we present is built on homogeneous Poisson line processes. In particular, the underlying probability measure is isotropic, whereas it does not satisfy a "finite range dependence property". We start by recalling Poisson line processes and refer to (Daley and Vere-Jones, 2008, Section 15.3), (Stoyan, Kendall and Mecke, 1987, Chapter 8) for more detailed descriptions.

5.1 The Poisson line process

Let e_1 and e_2 denote the unit vectors in $\mathbb{R}^2$. Any (undirected) line ℓ in $\mathbb{R}^2$ can be represented by its angle θ with a reference line and its (signed) distance r to a reference point. We choose as reference line the x_1-axis and as reference point the origin. The angle is measured starting from the x_1-axis counterclockwise. The distance r is chosen to be nonnegative if ℓ intersects $\{te_2 : t \geq 0\}$ or if ℓ is parallel to e_2 intersecting $\{te_1 : t > 0\}$. Else, r is chosen negative. This leads to a bijective correspondence $\rho : \mathcal{L} \to \mathbf{C}$ between the set $\mathcal{L}$ of lines in $\mathbb{R}^2$ and the "representation space" $\mathbf{C} := \mathbb{R} \times (0, \pi]$. If $\ell = \rho^{-1}(r, \theta)$ we also simply write $\ell = (r, \theta)$. Let $\mathcal{B}(\mathbf{C})$ denote the Borel σ-algebra on $\mathbf{C}$.

Let Ω be the set of locally finite measures on $(\mathbf{C}, \mathcal{B}(\mathbf{C}))$ equipped with the topology of vague convergence and let $\mathcal{F}$ be the Borel σ-algebra on Ω. We introduce an action of $\mathbb{R}^2$ on Ω in the following way: $(\mathbb{R}^2, +)$ is acting on $\mathcal{L}$ via $\tau_x^{\mathcal{L}}$: $\ell \mapsto \ell + x$, where $x \in \mathbb{R}^2$, $\ell \subset \mathcal{L}$. This induces an action of $(\mathbb{R}^2, +)$ on $\mathbf{C}$ given by $\tau_x^{\mathbf{C}} : (r, \theta) \mapsto \rho(\tau_x^{\mathcal{L}}(\rho^{-1}(r, \theta)))$, where $(r, \theta) \in \mathbf{C}$, $x \in \mathbb{R}^2$. Finally we introduce the action of $(\mathbb{R}^2, +)$ on Ω as $\tau_x : \Omega \to \Omega$, $\tau_x \omega[A] := \omega[\tau_x^{\mathbf{C}} A]$, where $A \in \mathcal{B}(\mathbf{C})$, $\omega \in \Omega$ and $x \in \mathbb{R}^2$. Note that the action $\tau^{\mathbf{C}}$ is no simple shift on the cylinder, but a shear, see (Stoyan, Kendall and Mecke, 1987, (8.2.1)) or (Daley and Vere-Jones, 2008, (15.3.1)) where formulae for directed lines are given. The continuity properties of $\tau_{\cdot}^{\mathbf{C}}\cdot$ obtained from such formulae ensure that τ is product measurable analogous to (Daley and Vere-Jones, 2008, Exercise 12.1.1(a)).

The (homogeneous) Poisson line process is given by the representation ρ and the distribution $\mathbb{P}_\kappa$ of a Poisson point process on $(\mathbf{C}, \mathcal{B}(\mathbf{C}))$ having intensity measure $\nu = \kappa \cdot \mathscr{L} \otimes \mu$ with μ the uniform distribution on $(0, \pi]$ and $\kappa > 0$. The tuple $(\Omega, \mathcal{F}, \mathbb{P}_\kappa, \tau)$ is an ergodic dynamical system, as outlined, for example, in (Cowan, 1980, p. 99) and (Miles, 1964, Theorem 1). Moreover, it is isotropic, see, for example, (Miles, 1964, p. 902).

5.2 Discontinuity with respect to the underlying measure

Some additional notation is needed: Let $R > 0$, $x \in \mathbb{R}^2$, and let ℓ be a line in $\mathbb{R}^2$. We denote by $B_R(x)$ the closed ball with centre x and radius R, and we introduce stripes $Q_R(\ell)$ given by

$$Q_R(\ell) := \{y \in \mathbb{R}^2 : d(y, \ell) < R\}.$$

By $H_R(x)$, we denote the entrance time of Z into $B_R(x)$, that is $H_R(x) := \inf\{t \geq 0 : Z_t \in B_R(x)\}$. We introduce the exit time of Z from $Q_R(\ell)$ by $\tau_R(\ell) := \inf\{t \geq 0 : Z_t \notin Q_R(\ell)\}$. Then $H_R(x)$ and $\tau_R(\ell)$ are stopping times with respect to the canonical filtration of $(Z_t)_t$, see, for example, (Karatzas and Shreve, 1991, Problem 1.2.7).

The potential we consider is defined as follows: For $\omega \in \Omega$, let $[\omega]$ be the support of ω. If F is a subset of $\mathbb{R}^2$ and $\omega \in \Omega$, we introduce the intersection of F with the lines of ω by $[\omega] \cap F := \bigcup_{z \in [\omega]} (F \cap \rho^{-1}(z))$. Let $c, M \geq 0$ and $R > 0$. We define the potential $V : \Omega \to [0, \infty)$,

$$V(\omega) := V_{c,R,M}(\omega) := c + M \cdot 1_{\{\tilde{\omega} \in \Omega : [\tilde{\omega}] \cap B_R(0) = \varnothing\}}(\omega), \tag{5.1}$$

which equals $M + c$ outside of stripes of radius R along the lines of ω, and which equals c inside these stripes.

We show the following: Let λ_2 be the principal Dirichlet eigenvalue of $-(1/2)\Delta$ in the unit ball in $\mathbb{R}^2$.

Theorem 15. *For any $D > 0$ there is $R_0 > 0$ such that for $\kappa > 0$ one has $\mathbb{P}_\kappa$-a.s. for all $c \geq 0$ and for all $y \in S^1$,*

$$\sup_{R\geq R_0}\sup_{M\geq 0}\limsup_{u\to\infty} -\frac{1}{u}\ln E_0\left[\exp\left\{-\int_0^{H_1(uy)}(V_{c,R,M})_\omega(Z_s)\,ds\right\}\right] \\ \leq \sqrt{2c}+D. \tag{5.2}$$

We may choose $R_0 = 4\sqrt{\lambda_2}/D + 1$.

Recall, that by (2.3) the right side of (5.2) equals $\alpha_c(y) + D$.

This result contradicts continuity of the Lyapunov exponent with respect to weak convergence of the underlying probability measure: The convolution with an even and smooth function $g : \mathbb{R}^2 \to [0, \infty)$ of support $\operatorname{supp} g \subset B_{R/2}(0)$ and $\int_{\mathbb{R}^2} g(x)\,dx = 1$, see, for example, (Rueß, 2014, Lemma 4.4), leads to a *regular potential* $W := V_{1,2R,1} * g \leq V_{1,R,1}$ for which the Lyapunov exponent exists and can be expressed as follows: $\mathbb{P}_\kappa$-a.s. the limit in the following exists and equals

$$\lim_{u\to\infty} -\frac{1}{u}\ln E_0\left[\exp\left\{-\int_0^{H_1(ue_1)} W_\omega(Z_s)\,ds\right\}\right] = \alpha_W^{\mathbb{P}_\kappa}(e_1),$$

see (Rueß, 2014, (1.9)). With Theorem 15 for $D = (\alpha_2(e_1) - \alpha_1(e_1))/2$, there is $R > 0$ such that

$$\sup_{\kappa>0}\alpha_W^{\mathbb{P}_\kappa}(e_1) \leq \sup_{\kappa>0}\alpha_{V_{1,R,1}}^{\mathbb{P}_\kappa}(e_1) \leq \alpha_1(e_1) + D < \alpha_2(e_1) = \alpha_W^{\delta_{\mathbf{0}}}(e_1), \tag{5.3}$$

where $\mathbf{0}$ is the zero measure on $\mathbf{C}$. On the other hand note that $\mathbb{P}_\kappa \to \delta_{\mathbf{0}}$ weakly as $\kappa \to 0$. Such convergence follows, for example, with help of Laplace transforms of Point processes, see (Daley and Vere-Jones, 2008, (9.4.17), Theorem 11.1.VIII). This together with (5.3) shows discontinuity as stated.

We start with an estimate on the travel costs along stripes which is analogous to (Sznitman, 1998, (5.2.32)). Let ℓ_0 denote the x_1-axis.

Lemma 16. *Let $R > 0$, $c \geq 0$ and $u > R$. Then*

$$E_0\left[\exp\{-cH_R(ue_1)\}, \tau_R(\ell_0) > H_R(ue_1)\right] \geq C\exp\left\{-u\sqrt{2(c+\lambda_2/R^2)}\right\},$$

where $C > 0$ is a constant.

Proof. For any $t > 0$, with Girsanov's formula, see (Karatzas and Shreve, 1991, Theorem 3.5.1, Corollary 3.5.13),

$$\begin{aligned} &E_0\left[\exp\{-cH_R(ue_1)\}, \tau_R(\ell_0) > H_R(ue_1)\right] \\ &\quad\geq \exp\{-ct\}P_0\left[\sup_{0\leq s\leq t}\left|Z_s - \frac{s}{t}ue_1\right| < R\right] \\ &\quad= \exp\{-ct\}E_0\left[\exp\left\{-\frac{u}{t}e_1\cdot Z_t - \frac{u^2}{2t}\right\}, \sup_{0\leq s\leq t}|Z_s| < R\right]. \end{aligned} \tag{5.4}$$

We abbreviate $\mathcal{B} := \{\sup_{0\le s\le t} |Z_s| < R\}$. Note that $E_0[Z_t|\mathcal{B}] = 0$, since $-Z \stackrel{d}{=} Z$ and $Z \in \mathcal{B}$ if and only if $-Z \in \mathcal{B}$. An application of Jensen inequality shows, that (5.4) is greater or equal

$$\exp\{-ct\}\exp\left\{-\frac{u}{t}e_1 \cdot E_0[Z_t|\mathcal{B}] - \frac{u^2}{2t}\right\}P_0[\mathcal{B}]$$
$$= \exp\left\{-ct - \frac{u^2}{2t}\right\}P_0[\mathcal{B}] \ge C\exp\left\{-ct - \frac{u^2}{2t} - \lambda_2 t/R^2\right\},$$

where for the last estimate we used (Sznitman, 1998, (3.1.53)). The choice $t := u/\sqrt{2(c+\lambda_2/R^2)}$ shows the statement. □

In order to prove Theorem 15, we need to construct a path such that travelling along this path is relatively cheap for the Brownian motion. Therefore, a great part of this path should lie in regions of low potential. This forces the path to follow closely the lines of the Poisson line process. On the other hand, the path should not be too long. A path following mainly the lines and "exceeding" the Euclidean distance only logarithmically can be found in the articles of Aldous and Kendall (2008) and Kendall (2011). For our purposes, it suffices to find a path with linear "exceedance".

Proof of Theorem 15. For any direction $y \in S^1$ we need to have a "suitable" line leading into this direction: Let ι denote the complex number $(0, 1) \in \mathbb{C}$. For $y \in S^1$, for $\psi \in (0, \pi/2)$ we introduce the event $\mathcal{A}(y, \psi) \in \mathcal{F}$ consisting of those $\omega \in \Omega$ for which there is a line $\ell = (r, \theta) \in [\omega]$ such that the angle between ℓ and y, measured from y counterclockwise, is in $[\pi - \psi, \pi)$, and $\ell \cap \{e^{t\iota}, t > 0\} \ne \varnothing$. Set $\Omega_1 := \bigcap_{0<\psi<\pi/2}\bigcap_{y\in S^1}\mathcal{A}(y, \psi)$.

For all $\kappa > 0$ one has $\mathbb{P}_\kappa[\Omega_1] = 1$. In fact, for any $y \in S^1$ and $\psi \in (0, \pi/2)$ one has $\mathbb{P}_\kappa[\mathcal{A}(y, \psi)] = 1$. This can be verified by considering the distribution of the intersection points and angles of the lines $\ell \in \rho^{-1}[\omega]$ with a fixed line, see, for example, (Miles, 1964, Theorem 2). Recognise, that for $\psi \in (0, \pi/2)$ and for $0 \le t \le \psi/4$ one has $\mathcal{A}(ye^{t\iota}, \psi) \supset \mathcal{A}(y, \psi/4)$. Thus, if we divide the interval $[0, 2\pi)$ into a finite number of intervals $I_k = [x_k, x_{k+1})$, $k = 1, \dots, \bar{k}$ of same length $x_{k+1} - x_k \le \psi/4$, $x_1 = 0$, $x_{\bar{k}+1} = 2\pi$, we get $\mathbb{P}_\kappa[\bigcap_{y\in S^1}\mathcal{A}(y, \psi)] \ge \mathbb{P}_\kappa[\bigcap_{k=1,\dots,\bar{k}}\mathcal{A}(e^{x_k\iota}, \psi/4)] = 1$. For all $y \in S^1$ the events $\bigcap_{y\in S^1}\mathcal{A}(y, \psi)$ are monotone increasing as ψ increases. Therefore, looking only at a countable number of angles $(\psi_n)_n$, $\psi_n \to 0$ as $n \to \infty$, we even have $\mathbb{P}_\kappa[\Omega_1] = \mathbb{P}_\kappa[\bigcap_{n\in\mathbb{N}}\bigcap_{y\in S^1}\mathcal{A}(y, \psi_n)] = 1$.

Let $\omega \in \Omega_1$. Let $D > 0$ and define

$$R_0 := 4\sqrt{\lambda_2}/D + 1.$$

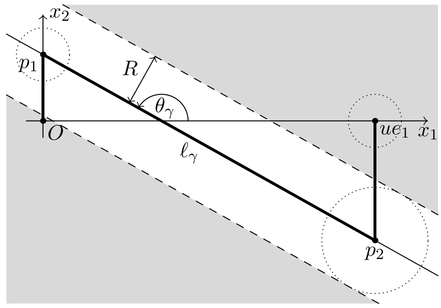

Figure 1 *We see a small sector of* $\mathbb{R}^2$. *The only line of* ω *passing this sector is* ℓ_γ. *The emphasised line segments illustrate the path* γ *from* 0 *to* ue_1 *through* p_1 *and* p_2. Z *is observed until hitting* $B_1(p_1)$, *then until reaching* $B_R(p_2)$ *"forced" to stay in the region* $Q_R(\ell_\gamma)$ *of low potential, and thereafter until hitting* $B_1(ue_1)$.

Let $R \geq R_0$ and $c, M \geq 0$, and let the potential $V = V_{c,R,M}$ be given as in (5.1). Define $\zeta_1 := \sqrt{2c + 2\lambda_2/(R_0-1)^2}$, $\zeta_2 := \alpha_c(e_1) + D/2$, and

$$\varphi := \min\{\arctan\bigl(D/\bigl(16\alpha_{c+M}(e_1)\bigr)\bigr), \arccos(\zeta_1/\zeta_2)\}.$$

Note that by (2.3) $0 < \zeta_1/\zeta_2 < 1$, and therefore $\varphi \in (0, \pi/2)$.

We restrict ourselves in the following to the case $y = e_1$. By rotation invariance of the law of Brownian motion, the same argument shows the statement for any $y \in S^1$.

Let $u \geq R+1$. We construct a path γ starting in 0 and leading to ue_1 as follows, see Figure 1. We start the path in 0 in direction of e_2 until hitting a line $\ell_\gamma = (r, \theta_\gamma) \in [\omega]$ of angle $\theta_\gamma \in [\pi - \varphi, \pi)$. Such a line exists by choice of $\omega \in \Omega_1 \subset \mathcal{A}(e_1, \varphi)$. We denote the intersection point by p_1. The path now follows the line ℓ_γ until the intersection of ℓ_γ with the line $\{ue_1 + se_2 : s \in \mathbb{R}\}$. We denote this intersection point by p_2. Then the path follows this vertical line until hitting ue_1.

We divide the journey of the Brownian motion into three different parts, see also Figure 1: Define stopping times

$$H^{(2)} := H_R(p_2) \circ \Theta_{H_1(p_1)} + H_1(p_1), \qquad H^{(3)} := H_1(ue_1) \circ \Theta_{H^{(2)}} + H^{(2)},$$

where Θ denotes the shift on the pathspace Σ, that is $\Theta_t((w_s)_{s\geq 0}) = (w_{s+t})_{s\geq 0}$ for $w \in \Sigma$, $t \geq 0$. Let $\mathcal{A} := \{\tau_R(\ell_\gamma) \circ \Theta_{H_1(p_1)} > H_R(p_2) \circ \Theta_{H_1(p_1)}\}$. We estimate and split the integral:

$$\begin{aligned} &E_0\Bigl[\exp\Bigl\{-\int_0^{H_1(ue_1)} V_\omega(Z_s)\,ds\Bigr\}\Bigr] \\ &\quad \geq E_0\Bigl[\exp\Bigl\{-\int_0^{H^{(3)}} V_\omega(Z_s)\,ds\Bigr\}, \mathcal{A}\Bigr] \end{aligned} \tag{5.5}$$

$$= E_0\Big[\exp\Big\{-\int_0^{H_1(p_1)} V_\omega(Z_s)\,ds - \int_{H_1(p_1)}^{H^{(2)}} V_\omega(Z_s)\,ds$$
$$- \int_{H^{(2)}}^{H^{(3)}} V_\omega(Z_s)\,ds\Big\}, \mathcal{A}\Big].$$

The potential V is bounded by $c + M$, and on $\mathcal{A}$ for $H_1(p_1) \le t \le H^{(2)}$ we have $V_\omega(Z_t) = c$. All considered stopping times are P_0-a.s. finite. Thus, an application of the strong Markov property, see (Karatzas and Shreve, 1991, Theorem 2.6.15), shows that we can bound (5.5) from below by

$$E_0\big[\exp\{-(c+M)H_1(p_1)\}\big] \inf_{x\in B_1(p_1)} E_x\big[\exp\{-cH_R(p_2)\}, \tau_R(\ell_\gamma) > H_R(p_2)\big]$$
$$\times \inf_{x\in B_R(p_2)} E_x\big[\exp\{-(c+M)H_1(ue_1)\}\big]. \tag{5.6}$$

We start estimating the middle term of (5.6): Since $u \ge R+1$ we have $|p_2 - p_1| = u/\cos(\pi - \theta_\gamma) \ge R+1$. For $x \in B_1(p_1)$, we set $\bar{x} := x - p_1$. Since $R > 1$ we have $B_R(p_2) \supset B_{R-1}(p_2 + \bar{x})$ and $Q_R(\ell_\gamma) \supset Q_{R-1}(\ell_\gamma + \bar{x})$. Therefore, for $x \in B_1(p_1)$,

$$E_x\big[\exp\{-cH_R(p_2)\}, \tau_R(\ell_\gamma) > H_R(p_2)\big]$$
$$\ge E_x\big[\exp\{-cH_{R-1}(p_2+\bar{x})\}, \tau_{R-1}(\ell_\gamma + \bar{x}) > H_{R-1}(p_2 + \bar{x})\big].$$

$\ell_\gamma + \bar{x}$ leads through x and through $x + p_2 - p_1$. The law of Brownian motion is invariant under translations and rotations. Thus, Lemma 16 applied with radius $R-1$ shows for $R \ge R_0$ that the middle term in (5.6) can be bounded from below by

$$C\exp\{-|p_2 - p_1|\sqrt{2c + 2\lambda_2/(R_0-1)^2}\}. \tag{5.7}$$

In order to get a bound on the last term of (5.6), let $a_u := u\tan(\pi - \theta_\gamma) - |p_1|$. Note that $0 < \tan(\pi - \theta_\gamma) \le \tan\varphi$, and $a_u \sim u\tan(\pi - \theta_\gamma)$ as $u \to \infty$. Note also that $a_u = |p_2 - ue_1|$ if $a_u \ge 0$. Let $u_0 \ge R+1$ such that for $u \ge u_0$ one has $d(B_R(p_2), ue_1) \le 2a_u$ and

$$-\frac{1}{u}\ln E_0\big[\exp\{-(c+M)H_1(2a_ue_1)\}\big] \le \frac{4a_u}{u}\alpha_{c+M}(e_1) \le 8(\tan\varphi)\alpha_{c+M}(e_1),$$

where the first inequality is a consequence of the existence of the Lyapunov exponent for constant potential. Then for $u \ge u_0$ the last term of (5.6) can be bounded from below by

$$E_0\big[\exp\{-(c+M)H_1(2a_ue_1)\}\big] \ge \exp\{-8u(\tan\varphi)\alpha_{c+M}(e_1)\}. \tag{5.8}$$

Since the first term in (5.6) only depends on ω, since $|p_2 - p_1| = u/\cos(\pi - \theta_\gamma) \le u/\cos\varphi$, we get with (5.6), (5.7) and (5.8) for $R \ge R_0$ and for $u \ge u_0$,

$$\limsup_{u\to\infty} -\frac{1}{u}\ln E_0\Big[\exp\Big\{-\int_0^{H_1(ue_1)} V_\omega(Z_s)\,ds\Big\}\Big]$$
$$\le \sqrt{2\big(c + \lambda_2/(R_0-1)^2\big)}/\cos\varphi + 8(\tan\varphi)\alpha_{c+M}(e_1)$$

which is lower or equal $\alpha_c(e_1) + D$ by the choice of φ. $\square$

5.3 Discontinuity with respect to the potential

A slight modification of the previous setting also shows, that the Lyapunov exponent cannot be continuous in general with respect to L^p-convergence of the potential, $1 \le p < \infty$. Extend $\mathbf{C}$ to $\hat{\mathbf{C}} := \mathbf{C} \times [0, \infty)$. Let $\hat{\Omega}$ be the space of locally finite discrete measures on $\hat{\mathbf{C}}$ provided with the topology of vague convergence and let $\hat{\mathcal{F}}$ be the Borel σ-algebra. Let $\hat{\mathbb{P}}$ be the law of a homogeneous Poisson point process on $\hat{\Omega}$ with intensity measure $\mathscr{L} \otimes \mu \otimes \mathscr{L}$. Then $\mathbb{R}^2$ is acting on $\hat{\mathbf{C}}$ via $\hat{\tau}_x^{\hat{\mathbf{C}}} : (r, \theta, s) \mapsto (\tau_x^{\mathbf{C}}(r, \theta), s)$, where $x \in \mathbb{R}^2$. This again leads to an action $\hat{\tau}$ of $\mathbb{R}^2$ on $\hat{\Omega}$ as before, under which $\hat{\mathbb{P}}$ is invariant and ergodic.

For $0 < \kappa$ let Φ_κ be the mapping from $\hat{\Omega}$ to Ω defined as follows: We may represent discrete $\omega \in \hat{\Omega}$ as sums of Dirac measures: $\omega = \sum_{i \in \mathbb{N}} \delta_{(r_i, \theta_i, s_i)}$, see, for example, (Daley and Vere-Jones, 2008, Proposition 9.1.III). We define for discrete $\omega \in \hat{\Omega}$ the mapping

$$\Phi_\kappa : \omega = \sum_{i \in \mathbb{N}} \delta_{(r_i, \theta_i, s_i)} \mapsto \sum_{i \in \mathbb{N} : s_i < \kappa} \delta_{(r_i, \theta_i)}.$$

Then $\Phi_\kappa(\hat{\mathbb{P}}) = \mathbb{P}_\kappa$.

We introduce for $\omega \in \hat{\Omega}$ and for $\kappa, R > 0$ the potential

$$\hat{V}_{\kappa, R}(\omega) := V_{1, R, 1} \circ \Phi_\kappa(\omega).$$

For all $1 \le p < \infty$ and $R > 0$, we have $\hat{V}_{\kappa, R} \to 2$ in $L^p(\hat{\mathbb{P}})$ as $\kappa \to 0$. Indeed,

$$\begin{aligned} \hat{\mathbb{E}}\big[|2 - \hat{V}_{\kappa, R}|^p\big] &= \hat{\mathbb{P}}\big[\{\omega \in \hat{\Omega} : [\Phi_\kappa(\omega)] \cap B_R(0) \neq \varnothing\}\big] \\ &= \mathbb{P}_\kappa\big[\{\omega \in \Omega : [\omega] \cap B_R(0) \neq \varnothing\}\big] \\ &= \mathbb{P}_\kappa\big[\{\omega \in \Omega : \omega[[-R, R] \times (0, \pi]] \neq 0\}\big] \\ &= 1 - e^{-2\kappa R} \to 0 \qquad \text{as } \kappa \to 0, \end{aligned} \tag{5.9}$$

since for discrete $\omega \in \Omega$ one has $[\omega] \cap B_R(0) \neq \varnothing$ if and only if $\omega[[-R, R] \times (0, \pi]] \neq 0$. On the other hand as after Theorem 15 let $\hat{W}_\kappa := (V_{1, 2R, 1} \circ \Phi_\kappa) * g$, then $\hat{W}_\kappa$ is a *regular potential* such that $\hat{W}_\kappa \le \hat{V}_{\kappa, R}$. In order to clarify dependence on ω we introduce

$$a(u, U, \omega) := -\ln E_0\bigg[\exp\bigg\{-\int_0^{H(ue_1)} U_\omega(Z_s)\, ds\bigg\}\bigg],$$

where $u > 0$, where U is some potential on Ω or $\hat{\Omega}$, and where $\omega \in \Omega$ or $\omega \in \hat{\Omega}$, respectively. Then with (Rueß, 2014, (1.9)) and by Theorem 15 applied to $D :=$

$(\alpha_2(e_1)-\alpha_1(e_1))/2$, there is $R>0$ such that we have for $\kappa>0$ $\hat{\mathbb{P}}$-a.s.,

$$\begin{aligned}\alpha_{\hat{W}_\kappa}(e_1) &= \lim_{u\to\infty}\frac{1}{u}a(u,\hat{W}_\kappa,\omega) \le \limsup_{u\to\infty}\frac{1}{u}a(u,\hat{V}_{\kappa,R},\omega)\\ &= \limsup_{u\to\infty}\frac{1}{u}a\big(u,V_{1,R,1},\Phi_\kappa(\omega)\big) \le \alpha_1(e_1)+D<\alpha_2(e_1).\end{aligned}$$

This, continuity of the convolution, see, for example, (Rueß, 2014, (4.11)), and (5.9) show discontinuity.

5.4 Untypical scaling

The previous example can also be used to show that in general convergence of a sequence of potentials $(V_n)_n$ to zero such that there is a potential V with $nV_n\to V$ in L^p for some $1\le p<\infty$ does not guarantee $\sqrt{n}\alpha_{V_n}(e_1)\to\sqrt{2\mathbb{E}V}$.

We consider $(\hat{\Omega},\hat{\mathcal{F}},\hat{\mathbb{P}},\hat{\tau})$ and define for $b,n\in\mathbb{N}$ the potential

$$\tilde{V}_{n,b} := \frac{1}{n}\hat{V}_{1/n^2,bn} = V_{1/n,bn,1/n}\circ\Phi_{1/n^2}.$$

Then for $1\le p<\infty$, for $b\in\mathbb{N}$, one has $n\tilde{V}_{n,b}\to 2$ in L^p as $n\to\infty$. Indeed, as in (5.9), $\hat{\mathbb{E}}[|n\tilde{V}_{n,b}-2|^p]=\mathbb{P}_{1/n^2}[\{\omega\in\Omega:\omega[[-bn,bn]\times(0,\pi]]\neq 0\}]$, thus,

$$\hat{\mathbb{E}}\big[|n\tilde{V}_{n,b}-2|^p\big] = 1-e^{-2bn/n^2}\to 0 \tag{5.10}$$

as $n\to\infty$. On the other hand, set $\tilde{W}_n:=\tilde{V}_{n,2}*g$ where g is given as after Theorem 15. Then $\tilde{W}_n$ is a *regular potential* and $\tilde{W}_n\le\tilde{V}_{n,1}$. Theorem 15 applied to $D_n:=4\sqrt{\lambda_2}/(n-1)$ shows for $R=n$, $\hat{\mathbb{P}}$-a.s.,

$$\begin{aligned}\sqrt{n}\alpha_{\tilde{W}_n}(e_1) &\le \sqrt{n}\limsup_{u\to\infty}\frac{1}{u}a(u,\tilde{V}_{n,1},\omega)\\ &= \sqrt{n}\limsup_{u\to\infty}\frac{1}{u}a\big(u,V_{1/n,n,1/n},\Phi_{1/n^2}(\omega)\big) \le \sqrt{2}+\sqrt{n}D_n<\alpha_2(e_1)\end{aligned}$$

for n large enough. This, continuity of the convolution, and (5.10) show untypical scaling.

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References

Aldous, D. J. and Kendall, W. S. (2008). Short-length routes in low-cost networks via Poisson line patterns. *Adv. in Appl. Probab.* **40**, 1–21. MR2411811

Armstrong, S. N. and Souganidis, P. E. (2012). Stochastic homogenization of Hamilton–Jacobi and degenerate Bellman equations in unbounded environments. *J. Math. Pures Appl. (9)* **97**, 460–504. MR2914944

Bauer, H. (2001). *Measure and Integration Theory*. Berlin: Walter de Gruyter & Co. MR1897176

Bogachev, V. I. (2007). *Measure Theory. Vols. I, II*. Berlin: Springer. MR2267655

Bourgain, J. (2005). Positivity and continuity of the Lyapounov exponent for shifts on $\mathbb{T}^d$ with arbitrary frequency vector and real analytic potential. *J. Anal. Math.* **96**, 313–355. MR2177191

Bourgain, J. and Jitomirskaya, S. (2002). Continuity of the Lyapunov exponent for quasiperiodic operators with analytic potential. *J. Stat. Phys.* **108**, 1203–1218. MR1933451

Cornfeld, I. P., Fomin, S. V. and Sinaĭ, Y. G. (1982). *Ergodic Theory*. New York: Springer. MR0832433

Cowan, R. (1980). Properties of ergodic random mosaic processes. *Math. Nachr.* **97**, 89–102. MR0600826

Cox, J. T. and Kesten, H. (1981). On the continuity of the time constant of first-passage percolation. *J. Appl. Probab.* **18**, 809–819. MR0633228

Dal Maso, G. and Modica, L. (1986). Nonlinear stochastic homogenization and ergodic theory. *J. Reine Angew. Math.* **368**, 28–42. MR0850613

Daley, D. J. and Vere-Jones, D. (2008). *An Introduction to the Theory of Point Processes, Vol. II*, 2nd ed. New York: Springer. MR2371524

Duarte, P. and Klein, S. (2014). Continuity of the Lyapunov exponents for quasiperiodic cocycles. *Comm. Math. Phys.* **332**, 1113–1166. MR3262622

Durrett, R. (1996). *Probability: Theory and Examples*, 2nd ed. Belmont, CA: Duxbury Press. MR1609153

Furstenberg, H. (1981). *Recurrence in Ergodic Theory and Combinatorial Number Theory*. Princeton, NJ: Princeton University Press. MR0603625

Gilbarg, D. and Trudinger, N. S. (1983). *Elliptic Partial Differential Equations of Second Order*, 2nd ed. Berlin: Springer. MR0737190

Jikov, V. V., Kozlov, S. M. and Oleinik, O. A. (1994). *Homogenization of Differential Operators and Integral Functionals*. Berlin: Springer. MR1329546

Jitomirskaya, S. and Marx, C. A. (2011). Continuity of the Lyapunov exponent for analytic quasi-periodic cocycles with singularities. *J. Fixed Point Theory Appl.* **10**, 129–146. MR2825743

Karatzas, I. and Shreve, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. New York: Springer. MR1121940

Kendall, W. S. (2011). Geodesics and flows in a Poissonian city. *Ann. Appl. Probab.* **21**, 801–842. MR2830605

Kosygina, E., Mountford, T. S. and Zerner, M. P. W. (2011). Lyapunov exponents of Green's functions for random potentials tending to zero. *Probab. Theory Related Fields* **150**, 43–59. MR2800903

Le, T. T. H. (2013). On the continuity of Lyapunov exponents of random walks in random potentials. ArXiv e-prints.

Miles, R. E. (1964). Random polygons determined by random lines in a plane. *Proc. Natl. Acad. Sci. USA* **52**, 901–907. MR0168000

Mountford, T. and Mourrat, J. C. (2013). Lyapunov exponents of random walks in small random potential: The lower bound. *Comm. Math. Phys.* **323**, 1071–1120. MR3106503

Mountford, T. and Mourrat, J. C. (2015). Lyapunov exponents of random walks in small random potential: the upper bound. *Electron. J. Probab.* **20**, 1–18. MR3339869

Mourrat, J. C. (2012). Lyapunov exponents, shape theorems and large deviations for the random walk in random potential. *ALEA Lat. Am. J. Probab. Math. Stat.* **9**, 165–209. MR2923190

Pinsky, R. G. (1995). *Positive Harmonic Functions and Diffusion*. Cambridge: Cambridge University Press. MR1326606

Rassoul-Agha, F. and Seppäläinen, T. (2014). Quenched point-to-point free energy for random walks in random potentials. *Probab. Theory Related Fields* **158**, 711–750. MR3176363

Rinott, Y. and Saks, M. (1992). On FKG-type and permanental inequalities. In *Stochastic Inequalities*. *IMS Lecture Notes Monogr. Ser.* **22**, 332–342. Hayward, CA: IMS. MR1228073

Rudolph, D. J. (1990). *Fundamentals of Measurable Dynamics—Ergodic Theory on Lebesgue Spaces*. New York: Oxford University Press. MR1086631

Rueß, J. (2012). Lyapunov exponents of Brownian motion: Decay rates for scaled Poissonian potentials and bounds. *Markov Process. Related Fields* **18**, 595–612. MR3051654

Rueß, J. (2014). A variational formula for the Lyapunov exponent of Brownian motion in stationary ergodic potential. *ALEA Lat. Am. J. Probab. Math. Stat.* **11**, 679–709. MR3317557

Ryzhikov, V. V. (1991). Joinings of dynamical systems. Approximations and mixing. *Uspekhi Mat. Nauk* **46**, 177–178. MR1160346

Scholler, J. (2014). On the time constant in a dependent first passage percolation model. *ESAIM Probab. Stat.* **18**, 171–184. MR3230873

Schroeder, C. (1988). Green's functions for the Schrödinger operator with periodic potential. *J. Funct. Anal.* **77**, 60–87. MR0930391

Smythe, R. T. and Wierman, J. C. (1978). *First-Passage Percolation on the Square Lattice*. Berlin: Springer. MR0513421

Stoyan, D., Kendall, W. S. and Mecke, J. (1987). *Stochastic Geometry and Its Applications*. Chichester: John Wiley & Sons Ltd. MR0895588

Sznitman, A. S. (1994). Shape theorem, Lyapounov exponents, and large deviations for Brownian motion in a Poissonian potential. *Comm. Pure Appl. Math.* **47**, 1655–1688. MR1303223

Sznitman, A. S. (1998). *Brownian Motion, Obstacles and Random Media*. Berlin: Springer. MR1717054

Wang, W. M. (2002). Mean field upper and lower bounds on Lyapunov exponents. *Amer. J. Math.* **124**, 851–878. MR1925337

Willard, S. (1970). *General Topology*. Reading, Mass.–London–Don Mills, Ont: Addison-Wesley Publishing Co. MR0264581

You, J. and Zhang, S. (2014). Hölder continuity of the Lyapunov exponent for analytic quasiperiodic Schrödinger cocycle with weak Liouville frequency. *Ergodic Theory Dynam. Systems* **34**, 1395–1408. MR3227161

Zerner, M. P. W. (1998). Directional decay of the Green's function for a random nonnegative potential on $\mathbf{Z}^d$. *Ann. Appl. Probab.* **8**, 246–280. MR1620370

Auf der Morgenstelle 10
Mathematisches Institut
72076 Tübingen
Germany