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The Cramér condition for the Curie–Weiss model of SOC

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Abstract. We pursue the study of the Curie–Weiss model of self-organized criticality we designed in (*Ann. Probab.* **44** (2016) 444–478). We extend our results to more general interaction functions and we prove that, for a class of symmetric distributions satisfying a Cramér condition (C) and some integrability hypothesis, the sum S_n of the random variables behaves as in the typical critical generalized Ising Curie–Weiss model. The fluctuations are of order $n^{3/4}$ and the limiting law is $k\exp(-\lambda x^4)\,dx$ where k and λ are suitable positive constants. In (*Ann. Probab.* **44** (2016) 444–478), we obtained these results only for distributions having an even density.

1 Introduction

In their famous article Per Bak, Chao Tang and Kurt Wiesenfeld (1987) showed that certain complex systems are naturally attracted by critical points, without any external intervention. The amplification of small internal fluctuations can lead to a critical state and cause a chain reaction leading to a radical change of the system behavior. These systems exhibit the phenomenon of self-organized criticality (SOC). In general, SOC can be observed empirically or simulated on a computer in various models. However, the mathematical analysis of these models turns out to be extremely difficult, even for the sandpile model (the archetype of SOC, presented in Bak, Tang and Wiesenfeld (1987)) whose definition is yet simple.

In Cerf and Gorny (2016) and Gorny (2014), we introduced a Curie–Weiss model of self-organized criticality (SOC): we transformed the distribution associated to the generalized Ising Curie–Weiss model by implementing an automatic control of the inverse temperature which forces the model to evolve toward a critical state. This is the model given by $(X_n^k)_{1\leq k\leq n}$ such that, for all $n\geq 1$, $(X_n^1,\dots,X_n^n)$ has the distribution

$$\frac{1}{Z_n}\exp\biggl(\frac{1}{2}\frac{(x_1+\cdots+x_n)^2}{x_1^2+\cdots+x_n^2}\biggr)\mathbb{1}_{\{x_1^2+\cdots+x_n^2>0\}}\prod_{i=1}^n d\rho(x_i),$$

where ρ is a non-degenerate distribution on $\mathbb{R}$ and Z_n is a normalization constant. We proved that, if ρ has an even density which satisfies some integrability conditions, then the fluctuations of $S_n=X_n^1+\cdots+X_n^n$ are of order $n^{3/4}$ and the limiting

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law is

$$\left(\frac{4}{3}\right)^{1/4}\Gamma\left(\frac{1}{4}\right)^{-1}\exp\left(-\frac{s^4}{12}\right)ds.$$

This fluctuation result shows that the sum S_n behaves asymptotically as in the typical critical generalized Ising Curie–Weiss model. Moreover, by construction, it does not depend on any external parameter. In this sense, we can conclude that this is a Curie–Weiss model of self-organized criticality. Our result presents an unexpected universal feature. Indeed, this is in contrast to the situation in the critical generalized Ising Curie–Weiss model: at the critical point, the fluctuations are of order $n^{1-1/2k}$, where k depends on the distribution ρ. Moreover, our integrability conditions on ρ are weaker than those required to define the generalized Ising Curie–Weiss model, studied by Richard S. Ellis and Charles M. Newman (1978). For instance, our result holds for any centered Gaussian measure on $\mathbb{R}$. The Gaussian case of our model can be handled with the help of an explicit computation (Gorny (2014)).

In this paper, we extend the main results of Cerf and Gorny (2016) in three directions:

- We solve a problem about the mass at 0 of ρ that we met in Cerf and Gorny (2016) by using a conditioning argument. This allows us to extend the law of large numbers associated to our model.
- The hypothesis that the law ρ has a density is essential in the proof of the fluctuations result in Cerf and Gorny (2016). Here, we use arguments coming from the work of Anders Martin-Löf (1982) to extend this result to any symmetric probability measure which satisfies some integrability hypothesis and a Cramér condition:

$$\forall \alpha > 0 \qquad \sup_{\|(s,t)\|\geq\alpha}\left|\int_{\mathbb{R}} e^{isz+itz^2}\,d\rho(z)\right| < 1. \tag{C}$$

  This includes a much larger class of probability measures. However, the proof is much more technical.
- We extend our model to more general interaction functions. This extension is similar in spirit to the work of Richard S. Ellis and Theodor Eisele (1988) in the context of the generalized Ising Curie–Weiss model.

The model. Let g be a measurable real-valued function defined on $\mathbb{R}$ such that $g(u) \sim u^2/2$ in the neighbourhood of 0 and

$$\forall u \in \mathbb{R} \qquad g(u) \leq \frac{u^2}{2}.$$

Let ρ be a probability measure on $\mathbb{R}$, which is not the Dirac mass at 0. We consider an infinite triangular array of real-valued random variables $(X_n^k)_{1\leq k\leq n}$ such that, for all $n \geq 1$, $(X_n^1, \dots, X_n^n)$ has the distribution $\widetilde{\mu}_{n,\rho,g}$, whose density with respect to $\rho^{\otimes n}$ is

$$(x_1, \dots, x_n) \longmapsto \frac{1}{Z_{n,g}} \exp\biggl(ng\biggl(\frac{x_1 + \cdots + x_n}{\sqrt{n(x_1^2 + \cdots + x_n^2)}}\biggr)\biggr)\mathbb{1}_{\{x_1^2+\cdots+x_n^2>0\}},$$

where

$$Z_{n,g} = \int_{\mathbb{R}^n} \exp\biggl(ng\biggl(\frac{x_1 + \cdots + x_n}{\sqrt{n(x_1^2 + \cdots + x_n^2)}}\biggr)\biggr)\mathbb{1}_{\{x_1^2+\cdots+x_n^2>0\}} \prod_{i=1}^{n} d\rho(x_i).$$

We define $S_n = X_n^1 + \cdots + X_n^n$ and $T_n = (X_n^1)^2 + \cdots + (X_n^n)^2$.

We state next our main result, which is a strengthening of Theorems 1 and 2 of Cerf and Gorny (2016).

Theorem 1. *Let ρ be a symmetric probability measure on $\mathbb{R}$ with positive variance σ^2 and such that*

$$\exists v_0 > 0 \qquad \int_{\mathbb{R}} e^{v_0 z^2}\, d\rho(z) < +\infty.$$

Law of large numbers: *Under $\widetilde{\mu}_{n,\rho,g}$, $(S_n/n, T_n/n)$ converges in probability towards $(0, \sigma^2)$.*

We suppose in addition that g has a fourth derivative at 0 and that the following Cramér condition holds:

$$\forall \alpha > 0 \qquad \sup_{\|(s,t)\| \geq \alpha} \biggl| \int_{\mathbb{R}} e^{isz+itz^2}\, d\rho(z) \biggr| < 1. \tag{C}$$

Let μ_4 be the fourth moment of ρ. We denote $m_4 = -g^{(4)}(0)/2 \geq 0$.

Fluctuations result: *Under $\widetilde{\mu}_{n,\rho,g}$,*

$$\bigl(\mu_4 + m_4\sigma^4\bigr)^{1/4} \frac{S_n}{\sigma^2 n^{3/4}} \xrightarrow[n\to\infty]{\mathscr{L}} \left(\frac{4}{3}\right)^{1/4} \Gamma\left(\frac{1}{4}\right)^{-1} \exp\left(-\frac{s^4}{12}\right) ds.$$

The condition (C) is called the Cramér condition for the law of (Z, Z^2), where Z is a random variable with distribution ρ. The class of probability measures satisfying (C) is much larger than the class of probability measures having a density. Indeed, by the Lebesgue decomposition theorem (see Rudin (1987)), there exist three non-negative real numbers a, b, c such that $a + b + c = 1$ and

$$\rho = a\rho_{ac} + b\rho_d + c\rho_s,$$

where ρ_{ac} is a probability measure with density f, ρ_d is a discrete probability measure and ρ_s is a singular probability measure having no atoms. If $a > 0$, we say that ρ has an absolutely continuous component.

Proposition 2. *If ρ has an absolutely continuous component, then*

$$\forall\alpha>0\qquad\sup_{\|(s,t)\|\geq\alpha}\left|\int_{\mathbb{R}}e^{isz+itz^2}\,d\rho(z)\right|<1.$$

For example, the law

$$\rho_0=\frac{1}{16}\delta_{-1}+\frac{3}{4}\delta_0+\frac{1}{16}\delta_1+\exp\biggl(-\frac{x^2}{2}\biggr)\frac{dx}{8\sqrt{2\pi}}$$

satisfies the hypothesis of Theorem 1.

In Cerf and Gorny (2016), we treated the case where $g(u)=u^2/2$ for any $u\in\mathbb{R}$. We obtained a law of large numbers under $\widetilde{\mu}_{n,\rho,g}$, for symmetric probability measures ρ such that $\rho(\{0\})<e^{-1/2}$ or such that $\rho(]0,c[)=0$ for some $c>0$. The above distribution ρ_0 does not satisfy this hypothesis. Moreover, in the fluctuations theorem of Cerf and Gorny (2016), we only deal with a distribution ρ having an even density f which satisfies

$$\int_{\mathbb{R}^2}f^p(x+y)f^p(y)|x|^{1-p}\,dx\,dy<+\infty,$$

for some $p\in\,]1,2]$: once again this is not the case for ρ_0. Hence, Theorem 1 improves the main results of Cerf and Gorny (2016). Yet its proof is much more complicated: we have to use an approximation of the identity to obtain an asymptotic relation between ν_ρ^{*n} and its Cramér transform. The final Laplace's method is also much more technical than in Cerf and Gorny (2016).

Remark. If we start with the model studied in Eisele and Ellis (1988) and we follow the same road as in Cerf and Gorny (2016), then we end up with the distribution $\widetilde{\mu}^{\star}_{n,\rho,g}$ whose density with respect to $\rho^{\otimes n}$ is

$$(x_1,\dots,x_n)\longmapsto\frac{1}{Z^{\star}_{n,g}}\exp\biggl(n^2\frac{g((x_1+\cdots+x_n)/n)}{x_1^2+\cdots+x_n^2}\biggr)\mathbb{1}_{\{x_1^2+\cdots+x_n^2>0\}},$$

where $Z^{\star}_{n,g}$ is the renormalization constant. In this case, the result stated in Theorem 1 holds as well, but with $(\mu_4+m_4\sigma^6)^{1/4}$ instead of $(\mu_4+m_4\sigma^4)^{1/4}$.

This paper is organized as follows. In Section 2, we give some preliminaries containing a list of all the results derived from Cerf and Gorny (2016) which are essential for the proof of our main theorem. In Section 3, we extend the results of Cerf and Gorny (2016) around Varadhan's lemma with a conditioning argument. Next, in Section 4, we give some generalities on the Cramér condition, we prove Proposition 2 and a key theorem: an asymptotic relation between the n-fold tensor product of a probability measure and its Cramér transform (Theorem 8). Finally, in Section 5, we use Laplace's method in order to prove Theorem 1, with the help of the results from Sections 3 and 4. We end the paper by the Appendix presenting the proof of Theorem 8.

2 Preliminaries

Here, we give some notation and we list all the results derived from the Sections 3 and 5 of Cerf and Gorny (2016) which are essential for the proof of Theorem 1.

Let F and F_g be the functions defined on $\mathbb{R}\times]0,+\infty[$ by

$$\forall (x,y)\in\mathbb{R}\times]0,+\infty[\qquad F(x,y)=\frac{x^2}{2y} \quad \text{and} \quad F_g(x,y)=g\Big(\frac{x}{\sqrt{y}}\Big).$$

We define the sets

$$\Delta=\big\{(x,y)\in\mathbb{R}^2 : x^2\leq y\big\} \quad \text{and} \quad \Delta^*=\Delta\setminus\big\{(0,0)\big\}.$$

We denote by ν_ρ the law of (Z,Z^2), where Z is a random variable with distribution ρ, and by $\widetilde{\nu}_{n,\rho}$ the law of $(S_n/n, T_n/n)$ under ρ^{*n}. Under $\widetilde{\mu}_{n,\rho,g}$, the law of $(S_n/n, T_n/n)$ is

$$\frac{\exp(nF_g(x,y))\mathbb{1}_{\Delta^*}(x,y)\,d\widetilde{\nu}_{n,\rho}(x,y)}{\int_{\Delta^*}\exp(nF_g(s,t))\,d\widetilde{\nu}_{n,\rho}(s,t)}.$$

Let ρ be a symmetric probability measure on $\mathbb{R}$ with variance σ^2. We define the Laplace transform Λ of ν_ρ by

$$\forall (u,v)\in\mathbb{R}^2 \qquad \Lambda(u,v)=\ln\int_{\mathbb{R}} e^{uz+vz^2}\,d\rho(z)$$

and by D_Λ the set of the points $(u,v)\in\mathbb{R}^2$ such that $\Lambda(u,v)<+\infty$. We define next the Cramér transform I of ν_ρ by

$$\forall (x,y)\in\mathbb{R}^2 \qquad I(x,y)=\sup_{(u,v)\in\mathbb{R}^2}\big(ux+vy-\Lambda(u,v)\big)$$

and by D_I the set of the points $(x,y)\in\mathbb{R}^2$ such that $I(x,y)<+\infty$.

We suppose that $(0,0)\in\mathring{D}_\Lambda$. Then I is a good rate function, that is, it is non-negative and for any $\alpha>0$, the set $\{(x,y)\in\mathbb{R}^2 : I(x,y)\leq\alpha\}$ is compact. Moreover Cramér's theorem states that $(\widetilde{\nu}_{n,\rho})_{n\geq 1}$ satisfies a large deviations principle, with speed n, governed by I. Next,

$$I(0,0)=\begin{cases} -\ln\rho\big(\{0\}\big) & \text{if } \rho\big(\{0\}\big)>0,\\ +\infty & \text{if } \rho\big(\{0\}\big)=0,\end{cases}$$

and the $I-F$ has a unique minimum on Δ^* at $(0,\sigma^2)$, with $(I-F)(0,\sigma^2)=0$. Moreover, if the support of ρ contains at least three points and if μ_4 denotes the fourth moment of ρ, then when (x,y) goes to $(0,\sigma^2)$,

$$I(x,y)-F(x,y)\sim\frac{\mu_4 x^4}{12\sigma^8}+\frac{(y-\sigma^2)^2}{2(\mu_4-\sigma^4)}.$$

Finally, since g has a fourth derivative at 0, the Taylor–Young formula implies that

$$g(u) = \frac{u^2}{2} + g^{(3)}(0)\frac{u^3}{6} - m_4\frac{u^4}{12} + o(u^4).$$

We have $g(u) \leq u^2/2$ for any $u \in \mathbb{R}$. Therefore, $g^{(3)}(0) = 0$, $m_4 \geq 0$ and thus, when (x, y) goes to $(0, \sigma^2)$,

$$F(x, y) - F_g(x, y) = \frac{m_4 x^4}{12y^2}(1 + o(1)) = \frac{m_4 x^4}{12\sigma^4} + o(\|(x, y)\|^4).$$

As a consequence,

$$I(x, y) - F_g(x, y) \sim \frac{(\mu_4 + m_4\sigma^4)x^4}{12\sigma^8} + \frac{(y - \sigma^2)^2}{2(\mu_4 - \sigma^4)}.$$

Remark. In the case of the model given by the distribution $\widetilde{\mu}^{\star}_{n,\rho,g}$, defined in the Remark at the end of the Introduction, we replace F_g by the function $(x, y) \in \mathbb{R}\times]0, +\infty[\longmapsto g(x)/y$ in the Sections 2–5. The only difference is that when (x, y) goes to $(0, \sigma^2)$,

$$I(x, y) - F_g(x, y) \sim \frac{(\mu_4 + m_4\sigma^6)x^4}{12\sigma^8} + \frac{(y - \sigma^2)^2}{2(\mu_4 - \sigma^4)}.$$

3 Around Varadhan's lemma

In Section 6 of Cerf and Gorny (2016), we proved the following result.

Lemma 3. *Let ρ be a symmetric probability measure on $\mathbb{R}$ such that $(0, 0)$ belongs to $\mathring{D}_\Lambda$ and $\rho(\{0\}) = 0$. Let σ^2 denote the variance of ρ. If A is a closed subset of $\mathbb{R}^2$ which does not contain $(0, \sigma^2)$, then*

$$\limsup_{n\to+\infty} \frac{1}{n} \ln \int_{\Delta^*\cap A} \exp\left(\frac{nx^2}{2y}\right) d\widetilde{\nu}_{n,\rho}(x, y) < 0.$$

Actually we obtained in Cerf and Gorny (2016) this same conclusion for symmetric measures ρ such that $\rho(\{0\}) < e^{-1/2}$ or such that $\rho(]0, c[) = 0$ for some $c > 0$. This restriction is due to the behaviour of $I - F$ near the point $(0, 0)$, which is a singularity of F.

In this section, we will extend this result to any non-degenerate symmetric probability measure on $\mathbb{R}$ such that $(0, 0) \in \mathring{D}_\Lambda$. To this end, we will rely on a conditioning argument in order to reduce the problem to the case of measures which have no point mass at 0, and to apply Lemma 3. We focus first on what happens in the neighbourhood of $(0, 0)$.

Proposition 4. *Suppose that ρ is a symmetric probability measure on $\mathbb{R}$ with positive variance σ^2 and such that $(0,0) \in \mathring{D}_\Lambda$. There exists $\gamma > 0$ such that, for $\delta \in]0, \sigma^2[$ small enough and for n large enough,*

$$\int_{\Delta^*} e^{nx^2/(2y)} \mathbb{1}_{0<y\leq\delta}\, d\widetilde{\nu}_{n,\rho}(x,y) \leq e^{-n\gamma}.$$

We notice that the constant γ only depends on ρ (and not δ).

Proof of Proposition 4. If $\rho(\{0\}) = 0$ then Lemma 3 implies that the constant

$$\gamma = -\frac{1}{2} \limsup_{n\to+\infty} \frac{1}{n} \ln \int_{\Delta^*} e^{nx^2/(2y)} \mathbb{1}_{0<y\leq\sigma^2/2}\, d\widetilde{\nu}_{n,\rho}(x,y)$$

is positive since $\{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq \sigma^2/2\}$ is a closed set which does not contain $(0,\sigma^2)$. For $\delta \in]0, \sigma^2/2[$, we have then

$$\limsup_{n\to+\infty} \frac{1}{n} \ln \int_{\Delta^*} e^{nx^2/(2y)} \mathbb{1}_{0<y\leq\delta}\, d\widetilde{\nu}_{n,\rho}(x,y) \leq -2\gamma < -\gamma.$$

Hence, the result holds for probability measures which have no point mass at 0.

We suppose now that $\rho(\{0\}) > 0$. Let $n \geq 1$ and $X_1, \ldots, X_n$ be independent random variables with common distribution ρ. We put

$$S_n = \sum_{i=1}^n X_i \quad \text{and} \quad T_n = \sum_{i=1}^n X_i^2.$$

For $\delta > 0$ small enough, we denote

$$E_{n,\delta} = \int_{\Delta^*} e^{nx^2/(2y)} \mathbb{1}_{0<y\leq\delta}\, d\widetilde{\nu}_{n,\rho}(x,y).$$

Since $\widetilde{\nu}_{n,\rho}(\Delta) = 1$, we have

$$E_{n,\delta} = \mathbb{E}\big(e^{S_n^2/(2T_n)} \mathbb{1}_{0<T_n\leq n\delta}\big).$$

For any $c > 0$, we have

$$E_{n,\delta} \leq \mathbb{E}\big(e^{S_n^2/(2T_n)} \mathbb{1}_{T_n>0} \mathbb{1}_{T_n/n\leq c|S_n/n|}\big) + \mathbb{E}\big(e^{S_n^2/(2T_n)} \mathbb{1}_{c|S_n/n|<T_n/n\leq\delta}\big)$$

and we write this sum $I_{n,1} + I_{n,2}$.

In Figure 1, $I_{n,1}$ is an integral on the vertically hatched area and $I_{n,2}$ is an integral on the horizontally hatched area.

We notice that, if $c|S_n/n| < T_n/n \leq \delta$, then

$$\frac{S_n^2}{2T_n} \leq \frac{T_n^2}{2c^2 T_n} \leq \frac{T_n}{2c^2} \leq \frac{n\delta}{2c^2}.$$

We have thus

$$I_{n,2} \leq \exp\left(\frac{n\delta}{2c^2}\right) \mathbb{P}\left(c\left|\frac{S_n}{n}\right| < \frac{T_n}{n} \leq \delta\right).$$

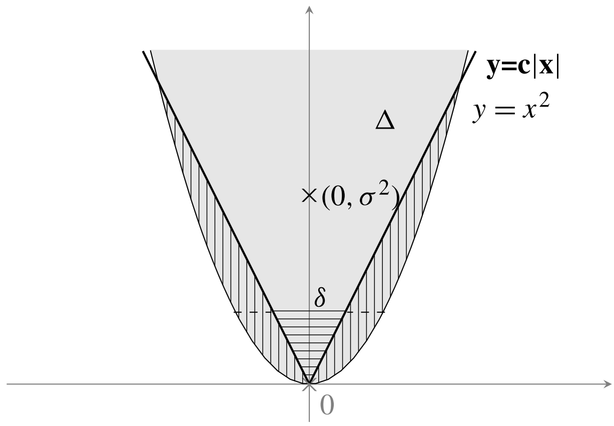

Figure 1 *The set $\{(x,y)\in\Delta : 0<y\le c|x|\}$ is vertically hatched. The set $\{(x,y)\in\mathbb{R}^2 : c|x|<y\le\delta\}$ is horizontally hatched.*

We denote $\alpha=-\ln\rho(\{0\})/2>0$. The function I is lower semi-continuous, thus there exists a neighbourhood $\mathcal{U}$ of $(0,0)$ such that

$$\forall (x,y)\in\overline{\mathcal{U}}\qquad I(x,y)\ge I(0,0)-\frac{\alpha}{2}=-\left(\ln\rho\big(\{0\}\big)+\frac{\alpha}{2}\right).$$

We can take δ small enough so that $\{(x,y)\in\mathbb{R}^2 : c|x|<y\le\delta\}\subset\overline{\mathcal{U}}$. We choose $c=\sigma/\sqrt{\alpha}$ (which only depends on ρ). Cramér's theorem (see Dembo and Zeitouni (2010)) implies that

$$\begin{aligned}\limsup_{n\to+\infty}\frac{1}{n}\ln I_{n,2}&\le\frac{\delta}{2c^2}-\inf_{\overline{\mathcal{U}}} I\\&\le\frac{\delta}{2c^2}+\ln\rho\big(\{0\}\big)+\frac{\alpha}{2}=\ln\rho\big(\{0\}\big)+\frac{\alpha}{2}\left(1+\frac{\delta}{\sigma^2}\right).\end{aligned}$$

If $\delta<\sigma^2$, then this last expression is smaller than

$$\ln\rho\big(\{0\}\big)+\alpha=-2\alpha+\alpha=-\alpha.$$

Hence, for n large enough,

$$I_{n,2}\le\exp\left(-\frac{n\alpha}{2}\right).$$

Let us focus now on $I_{n,1}$. We define the random variable N_n by

$$N_n=\big\{k\in\{0,\dots,n\} : X_k=0\big\}.$$

We have

$$\begin{aligned}I_{n,1}&=\mathbb{E}\big(e^{S_n^2/(2T_n)}\mathbb{1}_{T_n>0}\mathbb{1}_{T_n/n\le c|S_n/n|}\big)=\mathbb{E}\big(e^{S_n^2/(2T_n)}\mathbb{1}_{T_n>0}\mathbb{1}_{T_n\le c|S_n|}\big)\\&=\sum_{k=0}^{n-1}\mathbb{E}\big(e^{S_n^2/(2T_n)}\mathbb{1}_{T_n\le c|S_n|}\mathbb{1}_{N_n=k}\big)\end{aligned}$$

and, for any $k \in \{0, \dots, n-1\}$,

$$\begin{aligned}
&\mathbb{E}\big(e^{S_n^2/(2T_n)}\mathbb{1}_{T_n\le c|S_n|}\mathbb{1}_{N_n=k}\big)\\
&\quad=\mathbb{E}\Bigg(e^{S_n^2/(2T_n)}\mathbb{1}_{T_n\le c|S_n|}\sum_{1\le i_1<i_2<\cdots<i_k\le n}\mathbb{1}_{X_{i_1}=0}\cdots\mathbb{1}_{X_{i_k}=0}\mathbb{1}_{\forall j\notin\{i_1,\dots,i_k\}X_j\neq 0}\Bigg)\\
&\quad=\sum_{1\le i_1<i_2<\cdots<i_k\le n}\mathbb{E}\big(e^{S_n^2/(2T_n)}\mathbb{1}_{T_n\le c|S_n|}\mathbb{1}_{X_{i_1}=0}\cdots\mathbb{1}_{X_{i_k}=0}\mathbb{1}_{\forall j\notin\{i_1,\dots,i_k\}X_j\neq 0}\big).
\end{aligned}$$

The random variables $X_1, \dots, X_n$ are exchangeable, hence the expectations in the above sum are equal:

$$\begin{aligned}
&\mathbb{E}\big(e^{S_n^2/(2T_n)}\mathbb{1}_{T_n\le c|S_n|}\mathbb{1}_{N_n=k}\big)\\
&\quad=\binom{n}{k}\mathbb{E}\big(e^{S_n^2/(2T_n)}\mathbb{1}_{T_n\le c|S_n|}\mathbb{1}_{X_1\neq 0}\cdots\mathbb{1}_{X_{n-k}\neq 0}\mathbb{1}_{X_{n-k+1}=0}\cdots\mathbb{1}_{X_n=0}\big)\\
&\quad=\binom{n}{k}\mathbb{E}\big(e^{S_{n-k}^2/(2T_{n-k})}\mathbb{1}_{T_{n-k}\le c|S_{n-k}|}\\
&\qquad\qquad\times\mathbb{1}_{X_1\neq 0}\cdots\mathbb{1}_{X_{n-k}\neq 0}\mathbb{1}_{X_{n-k+1}=0}\cdots\mathbb{1}_{X_n=0}\big).
\end{aligned}$$

By the independence of $X_1, \dots, X_n$, we have

$$\begin{aligned}
&\mathbb{E}\big(e^{S_n^2/(2T_n)}\mathbb{1}_{T_n\le c|S_n|}\mathbb{1}_{N_n=k}\big)\\
&\quad=\binom{n}{k}\prod_{j=n-k+1}^{n}\mathbb{P}(X_j=0)\mathbb{E}\big(e^{S_{n-k}^2/(2T_{n-k})}\mathbb{1}_{T_{n-k}\le c|S_{n-k}|}\mathbb{1}_{X_1\neq 0}\cdots\mathbb{1}_{X_{n-k}\neq 0}\big)\\
&\quad=\binom{n}{k}\rho(\{0\})^k\big(1-\rho(\{0\})\big)^{n-k}\\
&\qquad\times\mathbb{E}\Bigg(e^{S_{n-k}^2/(2T_{n-k})}\mathbb{1}_{T_{n-k}\le c|S_{n-k}|}\prod_{j=1}^{n-k}\frac{\mathbb{1}_{X_j\neq 0}}{\mathbb{P}(X_j\neq 0)}\Bigg).
\end{aligned}$$

For any $k \in \{1, \dots, n\}$, we set

$$\overline{u}_k=\mathbb{E}\Bigg(e^{S_k^2/(2T_k)}\mathbb{1}_{T_k\le c|S_k|}\prod_{j=1}^{k}\frac{\mathbb{1}_{X_j\neq 0}}{\mathbb{P}(X_j\neq 0)}\Bigg)$$

so that we have

$$\begin{aligned}
I_{n,1}&=\sum_{k=0}^{n-1}\overline{u}_{n-k}\binom{n}{k}\rho(\{0\})^k\big(1-\rho(\{0\})\big)^{n-k}\\
&=\sum_{k=1}^{n}\overline{u}_k\binom{n}{k}\rho(\{0\})^{n-k}\big(1-\rho(\{0\})\big)^{k}.
\end{aligned}$$

We denote by $\overline{\rho}$ the probability measure ρ conditioned to $\mathbb{R}\setminus\{0\}$, that is,

$$\overline{\rho}=\rho\big(\cdot|\mathbb{R}\setminus\{0\}\big)=\frac{\rho(\cdot\cap\mathbb{R}\setminus\{0\})}{1-\rho(\{0\})},$$

so that

$$\forall k\in\{1,\dots,n\}\qquad \overline{u}_k=\int_{\Delta^*}e^{kx^2/(2y)}\mathbb{1}_{y\leq c|x|}\,d\widetilde{\nu}_{k,\overline{\rho}}(x,y).$$

The measure $\overline{\rho}$ is symmetric, $\overline{\rho}(\{0\})=0$ and

$$\forall(u,v)\in\mathbb{R}^2\qquad \overline{\Lambda}(u,v)=\ln\int_{\mathbb{R}}e^{uz+vz^2}\,d\overline{\rho}(z)\leq\Lambda(u,v)-\ln\big(1-\rho\big(\{0\}\big)\big),$$

thus $(0,0)\in\mathring{D}_{\overline{\Lambda}}$. Moreover, the variance of $\overline{\rho}$ is $\overline{\sigma}^2=\sigma^2(1-\rho(\{0\}))^{-1}$ and the closed set $\{(x,y)\in\mathbb{R}^2 : y\leq c|x|\}$ does not contain $(0,\overline{\sigma}^2)$. Applying Lemma 3, we get

$$\limsup_{k\to+\infty}\frac{1}{k}\ln\int_{\Delta^*}e^{kx^2/(2y)}\mathbb{1}_{y\leq c|x|}\,d\widetilde{\nu}_{k,\overline{\rho}}(x,y)<0.$$

Thus, there exist $\varepsilon_0>0$ and $n_0\geq 1$ such that

$$\forall k\geq n_0\qquad \overline{u}_k\leq\exp(-k\varepsilon_0).$$

For $n>n_0$, we write $I_{n,1}=A_n+B_n$ with

$$A_n=\sum_{k=1}^{n_0}\overline{u}_k\binom{n}{k}\rho\big(\{0\}\big)^{n-k}\big(1-\rho\big(\{0\}\big)\big)^k$$

and

$$B_n=\sum_{k=n_0+1}^{n}\overline{u}_k\binom{n}{k}\rho\big(\{0\}\big)^{n-k}\big(1-\rho\big(\{0\}\big)\big)^k.$$

For all $k\geq 1$, we have $\widetilde{\nu}_{k,\overline{\rho}}(\Delta)=1$, thus $\overline{u}_k\leq\exp(k/2)$ and then

$$\begin{aligned}A_n&\leq\rho\big(\{0\}\big)^n\sum_{k=1}^{n_0}e^{k/2}n^k\big(\rho\big(\{0\}\big)^{-1}-1\big)^k\\&\leq\rho\big(\{0\}\big)^n n_0e^{n_0/2}n^{n_0}\max\big(1,\big(\rho\big(\{0\}\big)^{-1}-1\big)^{n_0}\big).\end{aligned}$$

Moreover,

$$\begin{aligned}B_n&\leq\sum_{k=n_0+1}^{n}e^{-k\varepsilon_0}\binom{n}{k}\rho\big(\{0\}\big)^{n-k}\big(1-\rho\big(\{0\}\big)\big)^k\\&\leq\big(\rho\big(\{0\}\big)+e^{-\varepsilon_0}\big(1-\rho\big(\{0\}\big)\big)\big)^n.\end{aligned}$$

Therefore, setting

$$\beta = -\ln\big[\rho(\{0\}) + e^{-\varepsilon_0}(1-\rho(\{0\}))\big] > 0,$$

we have that, for n large enough,

$$I_{n,1} = A_n + B_n \leq \exp(-n\alpha) + \exp(-n\beta).$$

We notice that ε_0, α and β only depend on ρ.

Finally, we set $\gamma = \min(\alpha/4, \beta/2)$ (which only depends on ρ). For n large enough and $\delta \in]0, \sigma^2[$ small enough, we have

$$E_{n,\delta} \leq I_{n,1} + I_{n,2} \leq \exp(-n\gamma).$$

This proves the proposition. □

Now we can state the main result of this section, which is the announced refinement of Lemma 3 and which is essential to the proof of Theorem 1.

Proposition 5. *Let ρ be a symmetric probability measure on $\mathbb{R}$ with a positive variance σ^2 and such that $(0,0) \in \mathring{D}_\Lambda$. If A is a closed subset of $\mathbb{R}^2$ which does not contain $(0,\sigma^2)$ then*

$$\limsup_{n\to+\infty} \frac{1}{n} \ln \int_{\Delta^*\cap A} \exp\left(\frac{nx^2}{2y}\right) d\widetilde{\nu}_{n,\rho}(x,y) < 0.$$

Proof. By Proposition 4, there exist $\gamma > 0$ and $\delta > 0$ such that

$$\limsup_{n\to+\infty} \frac{1}{n} \ln \int_{\Delta^*} e^{nx^2/(2y)} \mathbb{1}_{0<y\leq\delta}\, d\widetilde{\nu}_{n,\rho}(x,y) \leq -\gamma.$$

We set $A_\delta = \{(x,y) \in \Delta \cap A : y \geq \delta\}$. We have

$$\Delta^* \cap A \subset \big\{(x,y) \in \Delta^* : 0 < y \leq \delta\big\} \cup A_\delta.$$

The set A_δ is closed, it does not contain $(0,\sigma^2)$ and F is continuous on it. The usual Varadhan's lemma (see Dembo and Zeitouni (2010)) implies that

$$\limsup_{n\to+\infty} \frac{1}{n} \ln \int_{A_\delta} e^{nx^2/(2y)}\, d\widetilde{\nu}_{n,\rho}(x,y) < -\inf_{A_\delta}(I-F).$$

As a consequence,

$$\limsup_{n\to+\infty} \frac{1}{n} \ln \int_{\Delta^*\cap A} \exp\left(\frac{nx^2}{2y}\right) d\widetilde{\nu}_{n,\rho}(x,y) \leq \max\Big(-\gamma, -\inf_{A_\delta}(I-F)\Big).$$

Since $(0,0) \in \mathring{D}_\Lambda$, I is a good rate function and $I-F$ attains its minimum on the closed set A_δ. Since A_δ does not contain $(0,\sigma^2)$, we have

$$\max\Big(-\gamma, -\inf_{A_\delta}(I-F)\Big) < 0$$

and the proposition is proved. □

4 The Cramér condition

Let $d \geq 1$. For any $z = (a_1 + ib_1, \dots, a_d + ib_d) \in \mathbb{C}^d$ and $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we denote

$$\langle z, x \rangle = \sum_{k=1}^{d} a_k x_k + i \sum_{k=1}^{d} b_k x_k.$$

If $z \in \mathbb{R}^d$ then $\langle z, x \rangle$ is the Euclidean inner product of z and x.

Let ν be a non-degenerate probability measure on $\mathbb{R}^d$. We denote by L its Log-Laplace and by J its Cramér transform. Let D_L and D_J be the domains of $\mathbb{R}^d$ where the functions L and J are respectively finite. We put

$$D_M = \{ z = a + ib \in \mathbb{C}^d : a \in D_L \}$$

and we define the function M by

$$\forall z \in D_M \qquad M(z) = \int_{\mathbb{R}^d} e^{\langle z, x \rangle} \, d\nu(x).$$

We notice that the function $s \in \mathbb{R}^d \longmapsto \ln M(s)$ is the Log-Laplace L of ν and that $s \in \mathbb{R}^d \longmapsto M(is)$ is the Fourier transform of ν.

One of the key ingredients for proving the main theorem of Cerf and Gorny (2016) is the Theorem 11 of Cerf and Gorny (2016) (which is extracted from Andriani and Baldi (1997)). This theorem allows us to express the density of ν^{*n} as a function of J and, under the condition

$$\forall \alpha > 0 \qquad \sup_{\|s\| \geq \alpha} \big| M(is) \big| < 1, \tag{C}$$

we can then obtain an asymptotic expansion. The condition (C) is called the Cramér condition. Anders Martin-Löf (1982) uses an approximation of the identity to obtain a similar expression for more general measures on $\mathbb{R}$ satisfying the condition (C), without requiring the existence of a density.

In this section, we will prove d-dimensional analogs of the results of Martin-Löf (1982).

4.1 Around the Cramér condition

We give here a sufficient condition for a measure ν on $\mathbb{R}^d$ to satisfy the Cramér condition (C).

Lemma 6. *If there exists $s_0 \neq 0$ such that $|M(is_0)| = 1$ then ν is an arithmetic measure, that is, there exists $(a, b) \in \mathbb{R}^2$ such that*

$$\nu\big(\{ x \in \mathbb{R}^d : \langle s_0, x \rangle \in a + b\mathbb{Z} \}\big) = 1.$$

Proof. Suppose that $|M(is_0)| = 1$ for some $s_0 \neq 0$. Thus,

$$1 = \left| \int_{\mathbb{R}^d} e^{i\langle s_0, x\rangle}\, d\nu(x) \right| \leq \int_{\mathbb{R}^d} d\nu(x) = 1.$$

We are in the equality case of this classical inequality, that is, there exists $b_0 \in \mathbb{R}$ such that

$$e^{i\langle s_0, x\rangle} = e^{ib_0} \qquad \nu \text{ a.s.},$$

whence

$$\nu\big(\{x \in \mathbb{R}^d : \langle s_0, x\rangle \in b_0 + 2\pi\mathbb{Z}\}\big) = 1$$

and the lemma is proved. □

Suppose that ν has a density with respect to the Lebesgue measure. By the Riemann–Lebesgue lemma,

$$\big|M(is)\big| = \left| \int_{\mathbb{R}^d} e^{i\langle s, x\rangle}\, d\nu(x) \right| \underset{\|s\| \to +\infty}{\longrightarrow} 0.$$

As a consequence, if ν does not satisfy (C), then there exists $s_0 \neq 0$, such that $|M(is_0)| = 1$. By the previous lemma, ν is arithmetic. This is absurd. Therefore, any probability measure having a density with respect to the Lebesgue measure satisfies (C). Moreover, by the Lebesgue decomposition theorem (see Rudin (1987)), a probability measure ν can be represented as the sum of three components:

$$\nu = a\nu_{ac} + b\nu_d + c\nu_s,$$

where ν_{ac} is an absolutely continuous probability measure, ν_d is a discrete probability measure, ν_s is a singular probability measure with no atoms and a, b, c are three non-negative real numbers such that $a + b + c = 1$. If $a > 0$, we say that ν has an absolutely continuous component. An absolutely continuous probability measure admits a density, thus we have the following proposition.

Proposition 7. *If ν has an absolutely continuous component, then it satisfies the Cramér condition* (C).

We end this section by giving the proof of Proposition 2: we suppose that $\rho = a\rho_{ac} + b\rho_d + c\rho_s$, where $a > 0$ and ρ_{ac} is a probability measure on $\mathbb{R}$ having a density f. We cannot use Proposition 7 directly because ν_ρ does not have a density. However, we saw in Lemma 16 of Cerf and Gorny (2016) that, if $\nu_{\rho_{ac}}$ denotes the law of (Z, Z^2) where Z is a random variable with distribution ρ_{ac}, then $\nu_{\rho_{ac}}^{*2}$ has the density

$$f_2 : (x, y) \longmapsto \frac{1}{\sqrt{2y - x^2}} f\left(\frac{x + \sqrt{2y - x^2}}{2}\right) f\left(\frac{x - \sqrt{2y - x^2}}{2}\right) \mathbb{1}_{x^2 < 2y}.$$

We can write $\rho^{*2} = a^2\rho_{ac}^{*2} + (1-a^2)\eta$, where η is the probability measure on $\mathbb{R}^2$ defined by

$$\eta = \frac{1}{1-a^2}\big(b^2\rho_d^{*2} + c^2\rho_s^{*2} + 2ab\rho_{ac} * \rho_d + 2ac\rho_{ac} * \rho_s + 2bc\rho_d * \rho_s\big).$$

We have then

$$\begin{aligned}
&\left|\int_{\mathbb{R}} e^{isz+itz^2}\, d\rho(z)\right|^2 \\
&\qquad = \left|\int_{\mathbb{R}^2} e^{is(x+y)+it(x^2+y^2)}\, d\rho(x)\, d\rho(y)\right| \\
&\qquad \le a^2\left|\int_{\mathbb{R}^2} e^{is(x+y)+it(x^2+y^2)}\, d\rho_{ac}^{*2}(x,y)\right| + \big(1-a^2\big)\left|\int_{\mathbb{R}^2} d\eta(x,y)\right| \\
&\qquad \le a^2\left|\int_{\mathbb{R}^2} e^{isu+itv}\, d\nu_{\rho_{ac}}^{*2}(u,v)\right| + 1 - a^2.
\end{aligned}$$

Hence,

$$\sup_{\|(s,t)\|\ge\alpha}\left|\int_{\mathbb{R}} e^{isz+itz^2}\, d\rho(z)\right|^2 \le a^2 \sup_{\|(s,t)\|\ge\alpha}\left|\int_{\mathbb{R}^2} e^{isu+itv} f_2(u,v)\, du\, dv\right| + 1 - a^2.$$

Proposition 7 implies that the supremum in the right-hand side of the previous inequality is strictly smaller that 1. This completes the proof of Proposition 2.

4.2 An asymptotic relation with the Cramér transform

We define the function k by

$$\forall x = (x_1,\dots,x_d) \in \mathbb{R}^d \qquad k(x) = \prod_{j=1}^{d} \max\big(1-|x_j|, 0\big)$$

and, for $c > 0$, the function k_c by

$$\forall x \in \mathbb{R}^d \qquad k_c(c) = \frac{1}{c^d} k\left(\frac{x}{c}\right).$$

It is an approximation of the identity on $\mathbb{R}^d$ since the integral of k is equal to 1. Finally, for any $n \ge 1$ and $c > 0$, we introduce

$$\varphi_{n,c} : x \in \mathbb{R}^d \longmapsto \int_{\mathbb{R}^d} k_c(s - nx)\, d\nu^{*n}(s).$$

We notice that $\varphi_{n,c}(x) = (k_c * \nu^{*n})(nx)$ for any $x \in \mathbb{R}^d$. A standard result on the approximations of the identity says that, if ν^{*n} has a density f_n, then

$$\lim_{c\to 0}\int_{\mathbb{R}^d}\big|\varphi_{n,c}(x) - f_n(nx)\big|\, dx = 0.$$

This suggests that the asymptotic behaviour of $\varphi_{n,c}$ and ν^{*n} are related, even in the general case when ν^{*n} does not have a density. The following theorem is the key result for the proof of Theorem 1.

Theorem 8. *Let ν be a non-degenerate probability measure on $\mathbb{R}^d$ such that the interior of D_L is not empty. Let K_J be a compact subset of A_J, the admissible domain of J. If ν satisfies the Cramér condition*

$$\forall\alpha > 0 \qquad \sup_{\|s\|\geq\alpha} \big|M(is)\big| < 1, \tag{C}$$

then there exists $\gamma > 0$ such that, when n goes to $+\infty$ and c goes to 0, uniformly over $x \in K_J$,

$$\varphi_{n,c}(x) = (2\pi n)^{-d/2}\big(\det \mathrm{D}_x^2 J\big)^{1/2} e^{-nJ(x)}\big(1 + o(1) + O\big(n^{d/2}e^{-\gamma n}c^{-d}\big)\big).$$

We postpone the proof of this theorem in the Appendix.

5 Proof of Theorem 1

In this section, we use first Proposition 5 to prove the law of large numbers under $\widetilde{\mu}_{n,\rho,g}$. Next, in order to prove the fluctuations theorem, we use Laplace's method: to this end, we introduce an integral with the approximation of the identity of Section 4. Then Proposition 5 gives the expansion of this integral. The technical part of the proof is to show that the remaining terms are negligible.

Suppose that ρ is a symmetric probability measure on $\mathbb{R}$ with positive variance σ^2 and such that

$$\exists v_0 > 0 \qquad \int_{\mathbb{R}} e^{v_0 z^2}\, d\rho(z) < +\infty.$$

5.1 Proof of the law of large numbers

The fact that $g(u) \sim u^2/2$ in the neighbourhood of 0 implies that F_g is positive on some open neighbourhood $\mathcal{V}$ of $(0,\sigma^2)$, which is included in Δ^*. We have then

$$Z_{n,g} = \int_{\Delta^*} \exp\big(nF_g(x,y)\big)\, d\widetilde{\nu}_{n,\rho}(x,y) \geq \widetilde{\nu}_{n,\rho}(\mathcal{V}).$$

The large deviations principle satisfied by $(\widetilde{\nu}_{n,\rho})_{n\geq 1}$ implies that

$$\liminf_{n\to+\infty} \frac{1}{n} \ln Z_{n,g} \geq \liminf_{n\to+\infty} \frac{1}{n} \ln \widetilde{\nu}_{n,\rho}(\mathcal{V}) \geq -\inf_{(x,y)\in\mathcal{V}} I(x,y) = 0.$$

We denote by $\theta_{n,\rho,g}$ the distribution of $(S_n/n, T_n/n)$ under $\widetilde{\mu}_{n,\rho,g}$. Let $\mathcal{U}$ be an open neighbourhood of $(0,\sigma^2)$ in $\mathbb{R}^2$. Since $F_g \leq F$, the results of Section 2 and

Proposition 5 imply that

$$\limsup_{n\to+\infty}\frac{1}{n}\ln\theta_{n,\rho,g}(\mathcal{U}^c)$$
$$\leq \limsup_{n\to+\infty}\frac{1}{n}\ln\int_{\Delta^*\cap\mathcal{U}^c}\exp\bigl(nF_g(x,y)\bigr)\,d\widetilde{\nu}_{n,\rho}(x,y)-\liminf_{n\to+\infty}\frac{1}{n}\ln Z_{n,g}<0.$$

Hence, there exist $\varepsilon>0$ and $n_0\in\mathbb{N}$ such that

$$\forall n>n_0\qquad \theta_{n,\rho}(\mathcal{U}^c)\leq\exp(-n\varepsilon).$$

Thus, for each open neighbourhood $\mathcal{U}$ of $(0,\sigma^2)$,

$$\lim_{n\to+\infty}\widetilde{\mu}_{n,\rho,g}\biggl(\biggl(\frac{S_n}{n},\frac{T_n}{n}\biggr)\in\mathcal{U}^c\biggr)=0.$$

This means that, under $\widetilde{\mu}_{n,\rho,g}$, $(S_n/n,T_n/n)$ converges in probability to $(0,\sigma^2)$.

5.2 Proof of the fluctuations result

We suppose in addition that g has a fourth derivative at 0 and that ρ satisfies

$$\forall\alpha>0\qquad \sup_{\|(s,t)\|\geq\alpha}\biggl|\int_{\mathbb{R}}e^{isz+itz^2}\,d\rho(z)\biggr|<1. \tag{C}$$

This is the Cramér condition for ν_ρ. Let us prove that, under $\widetilde{\mu}_{n,\rho,g}$,

$$\frac{S_n}{n^{3/4}}\xrightarrow[n\to\infty]{\mathscr{L}}\biggl(\frac{4(\mu_4+m_4\sigma^4)}{3\sigma^4}\biggr)^{1/4}\Gamma\biggl(\frac{1}{4}\biggr)^{-1}\exp\biggl(-\frac{\mu_4+m_4\sigma^4}{12\sigma^8}s^4\biggr)\,ds.$$

This is equivalent to the convergence announced in Theorem 1. For $u\in\mathbb{R}$, we define

$$E_n(u)=\int_{\mathbb{R}^n}\exp\biggl(iu\frac{x_1+\cdots+x_n}{n^{3/4}}+ng\biggl(\frac{x_1+\cdots+x_n}{\sqrt{n(x_1^2+\cdots+x_n^2)}}\biggr)\biggr)$$
$$\times\mathbb{1}_{\{x_1^2+\cdots+x_n^2>0\}}\prod_{j=1}^n d\rho(x_j).$$

Let us notice that $Z_{n,g}=E_n(0)$ and that

$$\mathbb{E}_{\tilde{\mu}_{n,\rho}}\biggl[\exp\biggl(iu\frac{S_n}{n^{3/4}}\biggr)\biggr]=\frac{E_n(u)}{E_n(0)}.$$

By Paul Levy's theorem, in order to obtain the convergence in law stated in Theorem 1, it is necessary and sufficient to prove that, for any $u\in\mathbb{R}$, the sequence $(E_n(u)/E_n(0))_{n\geq1}$ converges toward

$$\frac{\int_{\mathbb{R}}\exp(iux-((\mu_4+m_4\sigma^4)x^4)/(12\sigma^8))\,dx}{\int_{\mathbb{R}}\exp(-((\mu_4+m_4\sigma^4)x^4)/(12\sigma^8))\,dx}.$$

To this end, we will compute the expansion of $E_n(u)$, $n \geq 1$, $u \in \mathbb{R}$. We denote by $\widetilde{\nu}_{n,\rho}$ the law of $(S_n/n, T_n/n)$ under $\rho^{\otimes n}$. We have

$$\forall u \in \mathbb{R} \qquad E_n(u) = \int_{\Delta^*} \exp\bigl(iuxn^{1/4} + nF_g(x,y)\bigr)\, d\widetilde{\nu}_{n,\rho}(x,y).$$

Let $u \in \mathbb{R}$ and $\delta > 0$. We denote by B_δ the open ball in $\mathbb{R}^2$ of radius δ centered at $(0, \sigma^2)$. We choose δ small enough so that B_δ is included in K_I, a compact subset of $A_I \subset \Delta^*$. We define

$$f_n : (x,y) \in \mathbb{R}^2 \longmapsto \exp\bigl(iuxn^{1/4}\bigr).$$

For all $n \geq 1$, we write $E_n(u) = A_n + B_n$ with

$$A_n = \int_{\mathrm{B}_\delta} f_n e^{nF_g}\, d\widetilde{\nu}_{n,\rho} \quad \text{and} \quad B_n = \int_{(\mathrm{B}_\delta)^c \cap \Delta^*} f_n e^{nF_g}\, d\widetilde{\nu}_{n,\rho}.$$

First, since $F_g \leq F$, Proposition 5 implies that there exists $\varepsilon_0 > 0$ such that, for n large enough,

$$|B_n| \leq \exp(-n\varepsilon_0).$$

We next compute the expansion of A_n, using the results of the last section. We define the function k by

$$\forall (x,y) \in \mathbb{R}^2 \qquad k(x,y) = \max\bigl(1 - |x|, 0\bigr) \times \max\bigl(1 - |y|, 0\bigr)$$

and, for $c > 0$, we define k_c by

$$\forall (x,y) \in \mathbb{R}^2 \qquad k_c(x,y) = \frac{1}{c^2} k\left(\frac{x}{c}, \frac{y}{c}\right).$$

We put

$$A_{n,c,1} = \int_{\mathbb{R}^2} k_{c/n} * \bigl(f_n e^{nF_g} \mathbb{1}_{\mathrm{B}_\delta}\bigr)(s,t)\, d\widetilde{\nu}_{n,\rho}(s,t)$$

and $A_{n,c,2} = A_n - A_{n,c,1}$. Fubini's theorem implies that

$$\begin{aligned}
A_{n,c,1} &= \int_{\mathbb{R}^2} k_{c/n} * \bigl(f_n e^{nF_g} \mathbb{1}_{\mathrm{B}_\delta}\bigr)\left(\frac{s}{n}, \frac{t}{n}\right) d\nu_\rho^{*n}(s,t) \\
&= \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} k_{c/n}\left(\frac{s}{n} - x, \frac{t}{n} - y\right) \right. \\
&\qquad\qquad \left. \times f_n(x,y) e^{nF_g(x,y)} \mathbb{1}_{\mathrm{B}_\delta}(x,y)\, dx\, dy \right) d\nu_\rho^{*n}(s,t) \\
&= \int_{\mathbb{R}^2} f_n(x,y) e^{nF_g(x,y)} \mathbb{1}_{\mathrm{B}_\delta}(x,y) \\
&\qquad \times \left(\int_{\mathbb{R}^2} n^2 k_c(s - nx, t - ny)\, d\nu_\rho^{*n}(s,t) \right) dx\, dy \\
&= n^2 \int_{\mathrm{B}_\delta} f_n(x,y) e^{nF_g(x,y)} \varphi_{n,c}(x,y)\, dx\, dy,
\end{aligned}$$

where

$$\forall (x,y)\in\mathbb{R}^2 \qquad \varphi_{n,c}(x,y)=\int_{\mathbb{R}^2} k_c(s-nx,t-ny)\,d\nu_\rho^{*n}(s,t).$$

We denote

$$H_{n,c}:(x,y)\in\mathbb{R}^2\longmapsto ne^{nI(x,y)}\varphi_{n,c}(x,y).$$

Hence,

$$A_{n,c,1}=n\int_{\mathrm{B}_\delta} f_n(x,y)e^{-n(I-F_g)(x,y)}H_{n,c}(x,y)\,dx\,dy.$$

The measure ν_ρ satisfies the Cramér condition, thus by Theorem 8, there exists $\gamma>0$ such that, when n goes to $+\infty$ and c goes to 0, uniformly over $(x,y)\in K_I$,

$$H_{n,c}(x,y)=\frac{1}{2\pi}\big(\det \mathrm{D}^2_{(x,y)}I\big)^{1/2}\big(1+o(1)+O\big(ne^{-\gamma n}c^{-2}\big)\big).$$

We suppose that

$$\varepsilon_{n,c}=ne^{-\gamma n}c^{-2}\underset{\substack{n\to\infty\\ c\to 0}}{\longrightarrow}0.$$

Then, uniformly over $(x,y)\in K_I$,

$$H_{n,c}(x,y)\underset{\substack{n\to\infty\\ c\to 0}}{\longrightarrow}\frac{1}{2\pi}\big(\det \mathrm{D}^2_{(x,y)}I\big)^{1/2}.$$

We denote

$$\mathrm{B}_{\delta,n}=\big\{(x,y)\in\mathbb{R}^2:\big\|\big(xn^{-1/4},yn^{-1/2}\big)\big\|\le\delta\big\},$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^2$. Let us make the change of variable given by $(x,y)\longmapsto(xn^{-1/4},yn^{-1/2}+\sigma^2)$ with Jacobian $n^{-3/4}$:

$$\begin{aligned}A_{n,c,1}=n^{1/4}\int_{\mathrm{B}_{\delta,n}}&\exp\big(iux-n(I-F_g)\big(xn^{-1/4},yn^{-1/2}+\sigma^2\big)\big)\\ &\times H_{n,c}\big(xn^{-1/4},yn^{-1/2}+\sigma^2\big)\,dx\,dy.\end{aligned}$$

We check now that we can apply the dominated convergence theorem to this integral. The uniform expansion of $H_{n,c}$ means that for any $\alpha>0$, there exist $n_0\ge 1$ and $c_0>0$ such that

$$(x,y)\in K_I, n\ge n_0, c\le c_0 \quad\Longrightarrow\quad \big|H_{n,c}(x,y)2\pi\big(\det \mathrm{D}^2_{(x,y)}I\big)^{-1/2}-1\big|\le\alpha.$$

If $(x,y)\in\mathrm{B}_{\delta,n}$, then $(x_n,y_n)=(xn^{-1/4},yn^{-1/2}+\sigma^2)\in\mathrm{B}_\delta\subset K_I$, thus for all $n\ge n_0$, $c\le c_0$ and $(x,y)\in\mathrm{B}_{\delta,n}$,

$$\left|H_{n,c}\left(\frac{x}{n^{1/4}},\frac{y}{\sqrt{n}}+\sigma^2\right)2\pi\big(\det \mathrm{D}^2_{(x_n,y_n)}I\big)^{-1/2}-1\right|\le\alpha.$$

Moreover, $(x_n, y_n) \to (0, \sigma^2)$ thus, by continuity,

$$\big(\mathrm{D}^2_{(x_n,y_n)} I\big)^{-1/2} \underset{n\to+\infty}{\longrightarrow} \big(\mathrm{D}^2_{(0,\sigma^2)} I\big)^{-1/2} = \big(\mathrm{D}^2_{(0,0)} \Lambda\big)^{1/2},$$

whose determinant is equal to $\sqrt{\sigma^2(\mu_4-\sigma^4)}$. Therefore,

$$\mathbb{1}_{\mathrm{B}_{\delta,n}}(x,y) H_{n,c}\Big(\frac{x}{n^{1/4}}, \frac{y}{\sqrt{n}} + \sigma^2\Big) \underset{\substack{n\to\infty \\ c\to 0}}{\longrightarrow} \big(4\pi^2\sigma^2(\mu_4-\sigma^4)\big)^{-1/2}.$$

We proved in Section 2 that, when (x, y) goes to $(0, \sigma^2)$,

$$I(x,y) - F_g(x,y) \sim \frac{(\mu_4 + m_4\sigma^4)x^4}{12\sigma^8} + \frac{(y-\sigma^2)^2}{2(\mu_4-\sigma^4)}.$$

It follows that

$$n(I - F_g)\Big(\frac{x}{n^{1/4}}, \frac{y}{\sqrt{n}} + \sigma^2\Big) \underset{n\to+\infty}{\longrightarrow} \frac{(\mu_4 + m_4\sigma^4)x^4}{12\sigma^8} + \frac{y^2}{2(\mu_4-\sigma^4)}.$$

Let us check that the integrand is dominated by an integrable function, which is independent of n. The function

$$(x,y) \longmapsto \big(\mathrm{D}^2_{(x,y)} I\big)^{-1/2}$$

is bounded on B_δ by some $M_\delta > 0$. The uniform expansion of $H_{n,c}$ implies that for all $(x, y) \in \mathrm{B}_\delta$, $H_{n,c}(x, y) \le C_\delta$ for some constant $C_\delta > 0$. Finally, it follows from the above expansion of the proposition that, for $\delta > 0$ small enough,

$$\forall (x,y) \in \mathrm{B}_\delta \qquad G(x,y) = I(x,y) - F_g(x,y) \ge \frac{(\mu_4 + m_4\sigma^4)x^4}{24\sigma^8} + \frac{(y-\sigma^2)^2}{4(\mu_4-\sigma^4)}$$

and thus, for δ small enough, for any $(x, y) \in \mathbb{R}^2$, $n \ge n_0$ and $c \le c_0$,

$$\begin{aligned} &\mathbb{1}_{\mathrm{B}_{\delta,n}}(x,y) \exp\Big(-n(I - F_g)\Big(\frac{x}{n^{1/4}}, \frac{y}{\sqrt{n}} + \sigma^2\Big)\Big) H_{n,c}\Big(\frac{x}{n^{1/4}}, \frac{y}{\sqrt{n}} + \sigma^2\Big) \\ &\qquad \le C_\delta \exp\Big(-\frac{(\mu_4 + m_4\sigma^4)x^4}{24\sigma^8} - \frac{y^2}{4(\mu_4-\sigma^4)}\Big) \end{aligned}$$

and the right term is an integrable function on $\mathbb{R}^2$. It follows from the dominated convergence theorem that, when n goes to $+\infty$ and c goes to 0, then $n^{-1/4} A_{n,c,1}$ converges to

$$\int_{\mathbb{R}^2} \frac{\exp(iux)}{\sqrt{2\pi\sigma^2}\sqrt{2\pi(\mu_4-\sigma^4)}} \exp\Big(-\frac{(\mu_4 + m_4\sigma^4)x^4}{12\sigma^8} - \frac{y^2}{2(\mu_4-\sigma^4)}\Big)\, dx\, dy.$$

By Fubini's theorem, we get

$$A_{n,c,1} \underset{\substack{n\to\infty \\ c\to 0}}{\sim} \frac{n^{1/4}}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} \exp\Big(iux - \frac{(\mu_4 + m_4\sigma^4)x^4}{12\sigma^8}\Big)\, dx.$$

Now we deal with $A_{n,c,2}$. We will introduce an indicator function in order to simplify the expression of $A_{n,c,2}$. We put $\alpha = \delta/(2\sqrt{2})$ and

$$A_{n,c,3} = \int_{\mathrm{B}_\alpha} \big[f_n(s,t)e^{nF_g(s,t)}\mathbb{1}_{\mathrm{B}_\delta}(s,t) - k_{c/n} * \big(f_n e^{nF_g}\mathbb{1}_{\mathrm{B}_\delta}\big)(s,t)\big]\, d\widetilde{\nu}_{n,\rho}(s,t),$$

$$A_{n,c,4} = \int_{(\mathrm{B}_\alpha)^c} f_n(s,t)e^{nF_g(s,t)}\mathbb{1}_{\mathrm{B}_\delta}(s,t)\, d\widetilde{\nu}_{n,\rho}(s,t),$$

$$A_{n,c,5} = \int_{(\mathrm{B}_\alpha)^c} k_{c/n} * \big(f_n e^{nF_g}\mathbb{1}_{\mathrm{B}_\delta}\big)(s,t)\, d\widetilde{\nu}_{n,\rho}(s,t),$$

so that $A_{n,c,2} = A_{n,c,3} + A_{n,c,4} - A_{n,c,5}$. Since $\mathrm{B}_\delta \subset \Delta^*$ and $F_g \le F$, we have

$$|A_{n,c,4}| \le \int_{(\mathrm{B}_\alpha)^c \cap \Delta^*} e^{nF}\, d\widetilde{\nu}_{n,\rho}$$

and Proposition 5 ensures that there exists $\varepsilon_1 > 0$ such that, for n large enough,

$$A_{n,c,4} \underset{\substack{n\to\infty \\ c\to 0}}{=} O\big(\exp(-n\varepsilon_1)\big).$$

Until now we used the standard techniques of Laplace's method (cf. the proof of the main result of Cerf and Gorny (2016)) together with an approximation of the identity. The computation of the expansion of $A_{n,c,3}$ and $A_{n,c,5}$ is the technical part of this proof.

Lemma 9. *If δ, c/n and $cn^{1/4}$ are small enough, then*

$$A_{n,c,3} \underset{\substack{n\to\infty \\ c\to 0}}{=} o\big(E_n(0)\big),$$

$$A_{n,c,5} \underset{\substack{n\to\infty \\ c\to 0}}{=} O\bigg(\int_{(\mathrm{B}_\alpha)^c} e^{nF(s,t)}\, d\widetilde{\nu}_{n,\rho}(s,t)\bigg).$$

Suppose that Lemma 9 has been proved. Then Proposition 5 ensures that there exists $\varepsilon_2 > 0$ such that, for n large enough,

$$A_{n,c,5} \underset{\substack{n\to\infty \\ c\to 0}}{=} O\big(\exp(-n\varepsilon_2)\big).$$

We put now together the previous estimates in order to conclude. We take $c = 1/n$ so that c, $ne^{-\gamma n}c^{-2}$ and $cn^{1/4}$ go to 0 when $n \to +\infty$. For δ small enough, when n goes to $+\infty$, we have

$$A_n = \frac{n^{1/4}}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} \exp\bigg(iux - \frac{(\mu_4 + m_4\sigma^4)x^4}{12\sigma^8}\bigg)\, dx\big(1 + o(1)\big)$$
$$+ o\big(E_n(0)\big) + O\big(e^{-n\varepsilon_1} + e^{-n\varepsilon_2}\big).$$

Finally,

$$e^{-n\varepsilon_0}+e^{-n\varepsilon_1}+e^{-n\varepsilon_2}\underset{n\to\infty}{=}o\biggl(\frac{n^{1/4}}{\sqrt{2\pi\sigma^2}}\int_{\mathbb{R}}\exp\biggl(iux-\frac{(\mu_4+m_4\sigma^4)x^4}{12\sigma^8}\biggr)\,dx\biggr)$$

thus $E_n(u)=A_n+B_n$ is equal to

$$\frac{n^{1/4}}{\sqrt{2\pi\sigma^2}}\int_{\mathbb{R}}\exp\biggl(iux-\frac{(\mu_4+m_4\sigma^4)x^4}{12\sigma^8}\biggr)\,dx\bigl(1+o(1)\bigr)+o\bigl(E_n(0)\bigr).$$

Hence,

$$E_n(0)\sim\frac{n^{1/4}}{\sqrt{2\pi\sigma^2}}\int_{\mathbb{R}}\exp\biggl(iux-\frac{(\mu_4+m_4\sigma^4)x^4}{12\sigma^8}\biggr)\,dx.$$

Therefore,

$$\frac{E_n(0)}{E_n(0)}\underset{n\to+\infty}{\longrightarrow}\frac{\int_{\mathbb{R}}\exp(iux-((\mu_4+m_4\sigma^4)x^4)/(12\sigma^8))\,dx}{\int_{\mathbb{R}}\exp(-((\mu_4+m_4\sigma^4)x^4)/(12\sigma^8))\,dx}.$$

This completes the proof of Theorem 1.

We still have to prove the expansions of $A_{n,c,3}$ and $A_{n,c,5}$ which are stated in Lemma 9.

Proof of Lemma 9. For $(s,t)\in \mathrm{B}_\alpha$, if we have $k_{c/n}(x-s,y-t)\neq 0$, then

$$1-\bigl|n(x-s)/c\bigr|>0\quad\text{and}\quad 1-\bigl|n(y-t)/c\bigr|>0$$

and thus, for $c/n<\alpha$,

$$|x|\le|x-s|+|s|<\frac{c}{n}+\frac{\delta}{2\sqrt{2}}<\frac{\delta}{\sqrt{2}},$$

$$\bigl|y-\sigma^2\bigr|\le|y-t|+\bigl|t-\sigma^2\bigr|<\frac{c}{n}+\frac{\delta}{2\sqrt{2}}<\frac{\delta}{\sqrt{2}}.$$

Hence, $(x,y)\in \mathrm{B}_\delta$ and

$$\forall (s,t)\in \mathrm{B}_\alpha\qquad k_{c/n}(x-s,y-t)=k_{c/n}(x-s,y-t)\mathbb{1}_{\mathrm{B}_\delta}(x,y).$$

This implies that

$$\mathbb{1}_{\mathrm{B}_\alpha}\times\bigl(k_{c/n}*\bigl(f_ne^{nF_g}\mathbb{1}_{\mathrm{B}_\delta}\bigr)\bigr)=\mathbb{1}_{\mathrm{B}_\alpha}\times\bigl(k_{c/n}*\bigl(f_ne^{nF_g}\bigr)\bigr).$$

We have shown that, for $c/n<\alpha$,

$$A_{n,c,3}=\int_{\mathbb{R}^2}\mathbb{1}_{\mathrm{B}_\alpha}(s,t)\bigl[f_n(s,t)e^{nF_g(s,t)}-k_{c/n}*\bigl(f_ne^{nF_g}\bigr)(s,t)\bigr]\,d\widetilde{\nu}_{n,\rho}(s,t).$$

Let $(s,t)\in \mathrm{B}_\alpha$. We have

$$
\begin{aligned}
&\big[f_n e^{nF_g} - k_{c/n} * \big(f_n e^{nF_g}\big)\big](s,t)\\
&\quad = \int_{\mathbb{R}^2} \big(f_n(s,t) e^{nF_g(s,t)} - f_n(s-x,t-y) e^{nF_g(s-x,t-y)}\big) k_{c/n}(x,y)\,dx\,dy\\
&\quad = e^{nF_g(s,t)} f_n(s,t) \int_{\mathbb{R}^2} \big(1 - e^{n\Psi_{s,t,n}(cx/n,cy/n)}\big) k(x,y)\,dx\,dy\\
&\quad = e^{nF_g(s,t)} f_n(s,t) \int_{[-1,1]^2} \big(1 - e^{n\Psi_{s,t,n}(cx/n,cy/n)}\big) k(x,y)\,dx\,dy,
\end{aligned}
$$

with, for each $(x,y)\in\mathbb{R}^2$,

$$
\Psi_{s,t,n}(x,y) = F_g(s-x,t-y) - F_g(s,t) - iuxn^{1/4}.
$$

By hypothesis, the function g has a fourth derivative at 0, thus g is C^1 in a neighbourhood of 0. As a consequence, F_g is C^1 in a neighbourhood of $(0,\sigma^2)$. Hence, the mean value inequality implies that there exist $r>0$ and $M>0$ such that, for any $(s,t)\in\mathrm{B}_r$ and $(x,y)\in[-1,1]^2$,

$$
|x|<r \quad \text{and} \quad |y|<r \quad \Longrightarrow \quad \big|F_g(s-x,t-y) - F_g(s,t)\big| \le M\big\|(x,y)\big\|.
$$

If δ is small enough (so that $\alpha\le r$) and $c\le rn$ then, for any $(s,t)\in\mathrm{B}_\alpha$ and $(x,y)\in[-1,1]^2$,

$$
\begin{aligned}
\left|n\Psi_{s,t,n}\left(\frac{cx}{n},\frac{cy}{n}\right)\right| &\le Mn\left\|\left(\frac{cx}{n},\frac{cy}{n}\right)\right\| + n\left|u\frac{cx}{n}\right|n^{1/4}\\
&\le M\sqrt{2}c + |u|cn^{1/4}.
\end{aligned}
$$

By applying the mean value inequality to the function $(x,y)\in\mathbb{R}^2\longmapsto e^{x+iy}$, we prove that, if $z\in\mathbb{C}$ has a small enough real part, then $|1-e^z|\le 2|z|$. Therefore, if $cn^{1/4}$ goes to 0, then, for any $(s,t)\in\mathrm{B}_\alpha$, uniformly over $(x,y)\in[-1,1]^2$,

$$
\big|1 - e^{n\Psi_{s,t,n}(cx/n,cy/n)}\big| \le 2M\sqrt{2}c + 2|u|cn^{1/4} = o(1).
$$

Hence, if δ, c/n and $cn^{1/4}$ are small enough, then $A_{n,c,3} = o(E_n(0))$ when n goes to $+\infty$ and c goes to 0. Next, for $(s,t)\in\mathbb{R}^2$, we have

$$
\begin{aligned}
&k_{c/n} * \big(f_n e^{nF_g} \mathbb{1}_{\mathrm{B}_\delta}\big)(s,t)\\
&\quad = \int_{[-c/n,c/n]^2} k_{c/n}(x,y) \big(f_n e^{nF_g} \mathbb{1}_{\mathrm{B}_\delta}\big)(s-x,t-y)\,dx\,dy.
\end{aligned}
$$

We suppose that $\|s,t-\sigma^2\| > \delta + \sqrt{2}c/n$. For $|x|\le c/n$ and $|y|\le c/n$, we have then

$$
\begin{aligned}
\big\|(s-x,t-y) - \big(0,\sigma^2\big)\big\| &\ge \|s,t-\sigma^2\| - \|x,y\|\\
&> \delta + \sqrt{2}c/n - \sqrt{(c/n)^2 + (c/n)^2} > \delta
\end{aligned}
$$

so that $\mathbb{1}_{\mathrm{B}_\delta}(s-x, t-y) = 0$ and then

$$k_{c/n} * \big(f_n e^{nF_g} \mathbb{1}_{\mathrm{B}_\delta}\big)(s,t) = 0.$$

If c/n is small enough so that $\delta + \sqrt{2}c/n \le 2\delta$, then

$$k_{c/n} * \big(f_n e^{nF_g} \mathbb{1}_{\mathrm{B}_\delta}\big) = \big(k_{c/n} * \big(f_n e^{nF_g} \mathbb{1}_{\mathrm{B}_\delta}\big)\big) \times \mathbb{1}_{\mathrm{B}_{2\delta}}.$$

Hence,

$$\begin{aligned}
&|A_{n,c,5}| \\
&\quad \le \int_{(\mathrm{B}_\alpha)^c \cap \mathrm{B}_{2\delta}} \bigg(\int_{\mathbb{R}^2} \big| k_{c/n}(s-x, t-y) \big(f_n e^{nF_g} \mathbb{1}_{\mathrm{B}_\delta}\big)(x,y) \big| \, dx \, dy \bigg) \, d\widetilde{\nu}_{n,\rho}(s,t) \\
&\quad \le \int_{(\mathrm{B}_\alpha)^c \cap \mathrm{B}_{2\delta}} \big(k_{c/n} * e^{nF_g}\big)(s,t) \, d\widetilde{\nu}_{n,\rho}(s,t).
\end{aligned}$$

We note that, for δ small enough, we have on $\mathrm{B}_{2\delta}$,

$$\big| k_{c/n} * e^{nF_g} \big| \le e^{nF_g} + \big| e^{nF_g} - k_{c/n} * e^{nF_g} \big| \le e^{nF} (1 + 2M\sqrt{2}c),$$

if c/n is small enough (we use here the same argument as in the control of $A_{n,c,3}$, with $u = 0$). Finally,

$$A_{n,c,5} \underset{\substack{n\to\infty \\ c\to 0}}{=} O\bigg(\int_{(\mathrm{B}_\alpha)^c} e^{nF(s,t)} \, d\widetilde{\nu}_{n,\rho}(s,t) \bigg).$$

This completes the proof of the lemma. □

Appendix: Proof of Theorem 8

The ideas of the proof of Theorem 8 come from the article of Anders Martin-Löf (1982). It relies also on the following proposition.

Proposition 10. *Let ν be a non-degenerate probability measure on $\mathbb{R}^d$ such that the interior of D_L is non-empty. Let A_J be the admissible domain of J.*

(a) *The function ∇L is a C^∞-diffeomorphism from $\mathring{D}_L$ to A_J. Moreover,*

$$A_J \subset D_J = \big\{ x \in \mathbb{R}^d : J(x) < +\infty \big\}.$$

(b) *Denote by λ the inverse C^∞-diffeomorphism of ∇L. Then the map J is C^∞ on A_J and for any $x \in A_J$,*

$$\begin{aligned}
J(x) &= \big\langle x, \lambda(x) \big\rangle - L\big(\lambda(x)\big), \\
\nabla J(x) &= (\nabla L)^{-1}(x) = \lambda(x) \quad \textit{and} \quad \mathrm{D}_x^2 J = \big(\mathrm{D}_{\lambda(x)}^2 L\big)^{-1}.
\end{aligned}$$

(c) *If D_L is an open subset of $\mathbb{R}^d$ then $A_J = \mathring{D}_J = \mathring{\mathcal{C}}$ where $\mathcal{C}$ denotes the convex hull of the support of ν.*

The points (a) and (b) of the above proposition are proved in Andriani and Baldi (1997) and Borovkov and Mogulskii (1992) and the point (c) in Cerf and Gorny (2016). We will also need the two following lemmas.

Lemma 11. *For any $c > 0$ and $z \in \mathbb{C}$,*

$$\int_{\mathbb{R}^d} e^{\langle x,z\rangle} k_c(x)\,dx = \prod_{j=1}^{d} \frac{2(\cosh(cz_j) - 1)}{(cz_j)^2}.$$

Moreover, for any compact K of $\mathbb{R}$, there exists $M > 0$ such that

$$\forall s \in \mathbb{R} \qquad \sup_{u \in K} \left| \frac{2(\cosh(u + is) - 1)}{(u + is)^2} \right| \le \frac{M}{1 + s^2}.$$

Proof. For any $\zeta \in \mathbb{C} \setminus \{0\}$,

$$\begin{aligned}
\int_{\mathbb{R}} e^{\zeta s} \max\big(1 - |s|, 0\big)\,ds &= \int_{-1}^{1} e^{\zeta s}\big(1 - |s|\big)\,ds \\
&= \int_{-1}^{1} e^{\zeta s}\,ds - 2\int_0^1 s\cosh(\zeta s)\,ds \\
&= \frac{2\sinh(\zeta)}{\zeta} - 2\left(\frac{\sinh(\zeta)}{\zeta} - \frac{\cosh(\zeta) - 1}{(\zeta)^2}\right) \\
&= \frac{2(\cosh(\zeta) - 1)}{\zeta^2}
\end{aligned}$$

and this last function can be extended to a continuous function at $\zeta = 0$. By Fubini's theorem, we have, for any $c > 0$ and $z \in \mathbb{C}^d$,

$$\begin{aligned}
\int_{\mathbb{R}^d} e^{\langle x,z\rangle} k_c(x)\,dx &= \prod_{j=1}^{d} \frac{1}{c} \int_{\mathbb{R}} e^{x_j z_j} \max\left(1 - \left|\frac{x_j}{c}\right|, 0\right) dx_j \\
&= \prod_{j=1}^{d} \int_{\mathbb{R}} e^{x_j c z_j} \max\big(1 - |x_j|, 0\big)\,dx_j \\
&= \prod_{j=1}^{d} \frac{2(\cosh(cz_j) - 1)}{(cz_j)^2}.
\end{aligned}$$

Next, we define

$$f : (s, u) \in \mathbb{R} \times K \longmapsto \frac{2(1 + s^2)(\cosh(u + is) - 1)}{(u + is)^2}.$$

This is a continuous function on $\mathbb{R} \times K$ (at $u = s = 0$ it can be extended to a continuous function by setting $f(0,0) = 1$). Thus, f is bounded over the compact set $[-1,1] \times K$. Moreover, if $|s| > 1$ and $u \in K$, we have

$$\begin{aligned}\big|f(s,u)\big| &= \frac{2(1+s^2)}{u^2+s^2}\big|\cosh(u+is)-1\big| \leq 2\bigg(\frac{1}{s^2}+1\bigg)\big(\cosh(u)+1\big)\\ &\leq 4\sup_{u\in K}\big(\cosh(u)+1\big) < +\infty.\end{aligned}$$

Hence, f is bounded over $\mathbb{R} \times K$ by some constant $M > 0$. This completes the proof of the lemma. □

Lemma 12 (Uniform dominated convergence theorem). *Let $\mathcal{X}$ be a separable metric space and let $(\Omega, \mathcal{F}, \mu)$ be a measurable space. Let f and f_n, $n \geq 1$, be real or complex-valued measurable functions defined on $\mathcal{X} \times \Omega$. Suppose that, for any $\omega \in \Omega$, the functions $x \longmapsto f(x,\omega)$ and $x \longmapsto f_n(x,\omega)$, $n \in \mathbb{N}$, are continuous on $\mathcal{X}$ and that*

$$\sup_{x\in\mathcal{X}}\big|f_n(x,\omega) - f(x,\omega)\big| \underset{n\to\infty}{\longrightarrow} 0.$$

Suppose also that there exists a non-negative and integrable function g on Ω such that

$$\forall n \in \mathbb{N}\ \forall x \in \mathcal{X}\ \forall \omega \in \Omega \qquad \big|f_n(x,\omega)\big| \leq g(\omega).$$

Then for any $x \in \mathcal{X}$, the function $\omega \longmapsto f(x,\omega)$ is integrable and

$$\sup_{x\in\mathcal{X}}\bigg|\int_\Omega f_n(x,\omega)\,d\mu(\omega) - \int_\Omega f(x,\omega)\,d\mu(\omega)\bigg| \underset{n\to\infty}{\longrightarrow} 0.$$

Proof. We adapt the proof of the classical dominated convergence theorem in Rudin (1987). Sending n to $+\infty$ in the domination inequality, we get

$$\forall (x,\omega) \in \mathcal{X} \times \Omega \qquad \big|f(x,\omega)\big| \leq g(\omega).$$

This shows that $\omega \longmapsto f(x,\omega)$ is integrable. For any $n \in \mathbb{N}$, we set

$$h_n : \omega \longmapsto \sup_{x\in\mathcal{X}}\big|f_n(x,\omega) - f(x,\omega)\big|.$$

For all $n \in \mathbb{N}$ and $\omega \in \Omega$, the function $x \in \mathcal{X} \longmapsto |f_n(x,\omega) - f(x,\omega)|$ is continuous and, since $\mathcal{X}$ is separable, its supremum is equal to its supremum on a countable dense subset of $\mathcal{X}$. Therefore, h_n is a measurable function. Moreover, $(2g - h_n)_{n\in\mathbb{N}}$ is a sequence of non-negative functions whose limit is the function $2g$. Fatou's lemma implies that

$$\begin{aligned}\int_\Omega 2g\,d\mu &= \int_\Omega \liminf_{n\to+\infty}(2g - h_n)\,d\mu \leq \liminf_{n\to+\infty}\int_\Omega (2g-h_n)\,d\mu\\ &= \int_\Omega 2g\,d\mu - \limsup_{n\to+\infty}\int_\Omega h_n\,d\mu.\end{aligned}$$

Since g is integrable, we get that

$$\limsup_{n\to+\infty}\int_\Omega h_n\,d\mu \le 0.$$

Hence, $\int_\Omega h_n\,d\mu \to 0$ since for any $n\in\mathbb{N}$, h_n is a non-negative function. Finally,

$$\begin{aligned}
&\sup_{x\in\mathcal{X}}\left|\int_\Omega f_n(x,\omega)\,d\mu(\omega)-\int_\Omega f(x,\omega)\,d\mu(\omega)\right|\\
&\qquad\le \sup_{x\in\mathcal{X}}\int_\Omega \big|f_n(x,\omega)-f(x,\omega)\big|\,d\mu(\omega)\le \int_\Omega h_n\,d\mu \underset{n\to\infty}{\longrightarrow} 0
\end{aligned}$$

and the lemma is proved. □

Proof of Theorem 8. Lemma 11 implies that

$$\forall s\in\mathbb{R}^d\qquad \widehat{k}_c(s)=\prod_{j=1}^d\frac{2(1-\cos(cs_j))}{(cs_j)^2}$$

and, for any $u\in\mathbb{R}^d$, the function $x\longmapsto e^{\langle u,x\rangle}k_c(x)$ has the Fourier transform

$$s\in\mathbb{R}^d\longmapsto \prod_{j=1}^d\frac{2(\cosh(c(u_j+is_j))-1)}{(c(u_j+is_j))^2},$$

which can be rewritten as

$$s\in\mathbb{R}^d\longmapsto \prod_{j=1}^d\frac{2(1-\cos(c(s_j-iu_j)))}{(c(s_j-iu_j))^2}=\widehat{k}_c(s-iu).$$

This is an integrable function, thus the Fourier inversion formula (see Rudin (1987)) implies that the Fourier transform of $s\longmapsto(2\pi)^{-d}\widehat{k}_c(s-iu)$ is the function $y\longmapsto e^{-\langle u,y\rangle}k_c(y)$. Let $x\in K_J$ and $u\in\mathbb{R}^d$. A straightforward computation yields us that the Fourier transform of

$$s\longmapsto\frac{1}{(2\pi)^d}e^{-n\langle x,u+is\rangle}\widehat{k}_c(s-iu)$$

is the function $y\longmapsto e^{-\langle u,y\rangle}k_c(y-nx)$. We have then

$$\begin{aligned}
\varphi_{n,c}(x)&=\int_{\mathbb{R}^d}e^{-\langle u,y\rangle}k_c(y-nx)e^{\langle u,y\rangle}\,d\nu^{*n}(y)\\
&=\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}e^{i\langle s,y\rangle}\frac{e^{-n\langle x,u+is\rangle}\widehat{k}_c(s-iu)}{(2\pi)^d}\,ds\right)e^{\langle u,y\rangle}\,d\nu^{*n}(y).
\end{aligned}$$

By Fubini's theorem,

$$\begin{aligned}
\varphi_{n,c}(x)&=\int_{\mathbb{R}^d}\frac{e^{-n\langle x,u+is\rangle}\widehat{k}_c(s-iu)}{(2\pi)^d}\left(\int_{\mathbb{R}^d}e^{i\langle s,y\rangle}e^{\langle u,y\rangle}\,d\nu^{*n}(y)\right)ds\\
&=\int_{\mathbb{R}^d}\frac{e^{-n\langle x,u+is\rangle}\widehat{k}_c(s-iu)}{(2\pi)^d}M(u+is)^n\,ds.
\end{aligned}$$

However, $x \in A_J$ thus, if λ denotes the inverse function of ∇L, then Theorem 8 states that

$$J(x) = \langle \lambda(x), x \rangle - \ln M\big(\lambda(x)\big).$$

Replacing u by $\lambda(x)$ in the previous integral, we get

$$\varphi_{n,c}(x) = e^{-nJ(x)} \int_{\mathbb{R}^d} e^{-in\langle x,s\rangle} \frac{M(\lambda(x)+is)^n}{M(\lambda(x))^n} \widehat{k}_c\big(s - i\lambda(x)\big) \frac{ds}{(2\pi)^d}.$$

We denote by μ_x the measure on $\mathbb{R}^d$ such that

$$d\mu_x(y) = \frac{e^{\langle x+y, \lambda(x)\rangle}}{M(\lambda(x))}\, d\nu(y+x).$$

Its Fourier transform is the function

$$s \longmapsto e^{-i\langle x,s\rangle} \frac{M(\lambda(x)+is)}{M(\lambda(x))}$$

so that

$$\varphi_{n,c}(x) = e^{-nJ(x)} \int_{\mathbb{R}^d} \big(\widehat{\mu}_x(s)\big)^n \widehat{k}_c\big(s - i\lambda(x)\big) \frac{ds}{(2\pi)^d}.$$

For any $x \in K_J$, the mean of μ_x is

$$\int_{\mathbb{R}^d} y \frac{e^{\langle x+y,\lambda(x)\rangle}}{\exp M(\lambda(x))}\, d\nu(y+x) = \int_{\mathbb{R}^d} (z-x) \frac{e^{\langle z,\lambda(x)\rangle}}{M(\lambda(x))}\, d\nu(z) = \nabla L\big(\lambda(x)\big) - x = 0$$

and its covariance matrix is $\Gamma_x = \mathrm{D}^2_{\lambda(x)} L$ since for $1 \le i, j \le d$ and $s \in D_L$,

$$\begin{aligned}
(\Gamma_x)_{i,j} &= \frac{\int_{\mathbb{R}^d} y_i y_j e^{\langle \lambda(x), y+x\rangle}\, d\nu(y+x)}{M(\lambda(x))} = \frac{\int_{\mathbb{R}^d} (z_i - x_i)(z_j - x_j) e^{\langle \lambda(x), z\rangle}\, d\nu(z)}{M(\lambda(x))} \\
&= \frac{\int_{\mathbb{R}^d} z_i z_j e^{\langle \lambda(x), z\rangle}\, d\nu(z)}{M(\lambda(x))} - x_i x_j = \frac{\partial^2 L}{\partial s_i s_j}\big(\lambda(x)\big).
\end{aligned}$$

When $t \to 0$, uniformly over $x \in K_J$, we have the expansion

$$\widehat{\mu}_x(t) = 1 - \frac{1}{2}\langle \Gamma_x t, t\rangle + o\big(\|t\|^2\big).$$

Indeed the function $(x,t) \longmapsto \widehat{\mu}_x(t)$ is C^∞ on $A_J \times \mathbb{R}^d$ (by Proposition 10), thus the Taylor–Lagrange formula guarantees that the remainder term is uniformly controlled over $x \in K_J$. Therefore, for any $t \in \mathbb{R}^d$, uniformly over $x \in K_J$,

$$\widehat{\mu}_x\left(\frac{t}{\sqrt{n}}\right)^n \underset{n\to\infty}{\longrightarrow} \exp\left(-\frac{1}{2}\langle \Gamma_x t, t\rangle\right).$$

Moreover, for any $c > 0$, $n \ge 1$, $t \in \mathbb{R}^d$ and $x \in K_J$,

$$\widehat{k}_c\left(\frac{t}{\sqrt{n}} - i\lambda(x)\right) = \int_{\mathbb{R}^d} f_{c,n}(x,s)\, ds,$$

with

$$\forall s \in \mathbb{R}^d \qquad f_{c,n}(x,s) = \exp\Big(i\frac{c}{\sqrt{n}}\langle s,t\rangle + c\langle s,\lambda(x)\rangle\Big)k(s).$$

We have

$$\begin{aligned}&\sup_{x\in K_J}\big|f_{c,n}(x,s)-k(s)\big|\\&\quad = k(s)\sup_{x\in K_J}\left|\exp\Big(i\frac{c}{\sqrt{n}}\langle s,t\rangle + c\langle s,\lambda(x)\rangle\Big)-1\right| \underset{\substack{n\to+\infty\\ c\to 0}}{\longrightarrow} 0\end{aligned}$$

and, for all $s\in\mathbb{R}^d$, $x\in K_I$, $c\le 1$ and $n\ge 1$,

$$\big|f_{c,n}(x,s)\big| \le k(s) \sup_{\substack{x\in K_I\\ t\in[-1,1]^d}} \exp\langle t,\lambda(x)\rangle.$$

The term on the right defines an integrable function on $\mathbb{R}^d$ since $k(s)=0$ for any $s\notin[-1,1]^d$. Thus, the uniform dominated convergence theorem (Lemma 12) states that, for any $t\in\mathbb{R}^d$, uniformly over $x\in K_J$,

$$\widehat{k}_c\Big(\frac{t}{\sqrt{n}}-i\lambda(x)\Big) \underset{\substack{n\to+\infty\\ c\to 0}}{\longrightarrow} 1.$$

The functions $x\longmapsto\widehat{\mu}_x(t)$ and $x\longmapsto\exp(-\langle\Gamma_x t,t\rangle/2)$, $t\in\mathbb{R}^d$, are continuous on K_J. In order to apply the dominated convergence theorem (the uniform variant), we need to get a uniform domination of the sequence of functions. For $x\in A_J$, Γ_x is a positive definite symmetric matrix thus ε_x, its smallest eigenvalue, is positive. The largest eigenvalue of the inverse of Γ_x is ε_x^{-1}. Therefore, for any $x\in A_J$,

$$\varepsilon_x = \big(\max\{\alpha:\alpha \text{ eigenvalue of } \Gamma_x^{-1}\}\big)^{-1} = \Big(\sup_{y\neq 0}\frac{\langle\Gamma_x^{-1}y,\Gamma_x^{-1}y\rangle}{\langle y,y\rangle}\Big)^{-1/2}.$$

The term on the right is the inverse of the operator norm of the linear application associated to the matrix Γ_x^{-1}. Moreover, $x\longmapsto\Gamma_x=\mathrm{D}^2_{\lambda(x)}L$ is continuous on A_J thus the function $x\longmapsto\varepsilon_x$ is continuous. Let us denote by ε_0 its minimum on K_J. The compactness of K_J ensures that $\varepsilon_0>0$. The previous expansion implies that there exists $\delta>0$ such that

$$\forall (t,x)\in \mathrm{B}(0,\delta)\times K_J \qquad \big|\widehat{\mu}_x(t)\big| \le 1-\frac{1}{2}\Big\langle\Big(\Gamma_x-\frac{\varepsilon_0}{2}\mathrm{I}_d\Big)t,t\Big\rangle.$$

The spectral theorem for real symmetric matrices yields that, for any $x\in K_J$, the matrix $\Gamma_x-\varepsilon_0\mathrm{I}_d$ is positive symmetric. Thus,

$$\forall t\in\mathbb{R}^d \qquad \Big\langle\Big(\Gamma_x-\frac{\varepsilon_0}{2}\mathrm{I}_d\Big)t,t\Big\rangle-\frac{\varepsilon_0}{2}\|t\|^2 = \big\langle(\Gamma_x-\varepsilon_0\mathrm{I}_d)t,t\big\rangle \ge 0.$$

It follows that

$$\forall (t,x) \in \mathrm{B}(0,\delta) \times K_J \qquad \big|\widehat{\mu}_x(t)\big| \le 1 - \frac{\varepsilon_0}{4}\|t\|^2.$$

Since $1-y \le e^{-y}$ for all $y \ge 0$, we get

$$\forall n \ge 1\ \forall (t,x) \in \mathrm{B}(0,\delta\sqrt{n}) \times K_J \qquad \left|\widehat{\mu}_x\Big(\frac{t}{\sqrt{n}}\Big)\right|^n \le \exp\Big(-\frac{\varepsilon_0}{4}\|t\|^2\Big).$$

The right term is integrable and does not depend on $x \in K_J$ and n. Moreover, $\widehat{k}_c(t) = \widehat{k}(ct)$ for $t \in \mathbb{R}$, and by Lemma 11, the function $\widehat{k}_c(\cdot/\sqrt{n} - i\lambda(x))$ is bounded uniformly over $x \in K_J$, $c > 0$ and $n \ge 1$. The uniform dominated convergence theorem (Lemma 12) implies that, uniformly over $x \in K_J$,

$$\int_{\|t\|<\delta\sqrt{n}} \widehat{\mu}_x\Big(\frac{t}{\sqrt{n}}\Big)^n \widehat{k}_c\Big(\frac{t}{\sqrt{n}} - i\lambda(x)\Big)\,dt \underset{\substack{n\to+\infty\\ c\to 0}}{\longrightarrow} \int_{\mathbb{R}^d} \exp\Big(-\frac{1}{2}\big\langle\big(\mathrm{D}^2_{\lambda(x)}L\big)t,t\big\rangle\Big)\,dt.$$

Moreover, this second integral is equal to $(2\pi)^{d/2}(\det\Gamma_x)^{-1/2}$ and Proposition 10 guarantees that, for $x \in A_J$, $\mathrm{D}^2_{\lambda(x)}L$ is the inverse matrix of $\mathrm{D}^2_x J$. Therefore, when $n \to \infty$ and $c \to 0$, uniformly over $x \in K_J$,

$$\begin{aligned}
&\int_{\|t\|<\delta} \widehat{\mu}_x(t)^n \widehat{k}_c\big(s - i\lambda(x)\big)\,ds \\
&\quad = n^{-d/2} \int_{\|t\|<\delta\sqrt{n}} \widehat{\mu}_x\Big(\frac{t}{\sqrt{n}}\Big)^n \widehat{k}_c\Big(\frac{t}{\sqrt{n}} - i\lambda(x)\Big)\,dt \\
&\quad \sim \Big(\frac{2\pi}{n}\Big)^{d/2} \big(\det \mathrm{D}^2_x J\big)^{1/2}.
\end{aligned}$$

Let us consider now the remaining integral

$$\int_{\|t\|\ge\delta} \widehat{\mu}_x(t)^n \widehat{k}_c\big(s - i\lambda(x)\big)\,ds.$$

The measure ν satisfies the Cramér condition and ν is absolutely continuous with respect to μ_x. By Lemma 4 of Bahadur and Ranga Rao (1960), we get that μ_x also satisfies the Cramér condition:

$$\sup_{\|s\|\ge\delta} \big|\widehat{\mu}_x(s)\big| < 1.$$

Therefore, by the compactness of K_J,

$$\sup_{x\in K_J} \sup_{\|s\|\ge\delta} \big|\widehat{\mu}_x(s)\big| = e^{-\gamma} < 1,$$

for some $\gamma > 0$. As a consequence,

$$\sup_{x \in K_J} \left| \int_{\|s\| \geq \delta} \widehat{\mu}_x(s)^n \widehat{k}_c\big(s - i\lambda(x)\big)\, ds \right| \leq e^{-n\gamma} \int_{\mathbb{R}^d} \sup_{x \in K_J} \widehat{k}_c\big(s - i\lambda(x)\big)\, ds.$$

By Lemma 11, we have

$$\int_{\mathbb{R}^d} \sup_{x \in K_J} \widehat{k}_c\big(s - i\lambda(x)\big)\, ds = O\left(\prod_{j=1}^{d} \int_{\mathbb{R}^d} \frac{1}{1 + (cs_j)^2}\, ds_j \right) = O\left(\frac{1}{c^d} \right).$$

Finally, when $n \to +\infty$ and $c \to 0$,

$$\begin{aligned} \varphi_{n,c}(x) &= \frac{e^{-nJ(x)}}{(2\pi)^d} \left(\left(\frac{2\pi}{n} \right)^{d/2} \big(\det \mathrm{D}_x^2 J\big)^{1/2} \big(1 + o(1)\big) + O\big(e^{-n\gamma} c^{-d}\big) \right) \\ &= (2\pi n)^{-d/2} \big(\det \mathrm{D}_x^2 J\big)^{1/2} e^{-nJ(x)} \big(1 + o(1) + O\big(n^{d/2} e^{-\gamma n} c^{-d}\big)\big). \end{aligned}$$

The boundedness of the function $x \longmapsto (\det \mathrm{D}_x^2 J)^{1/2}$ on K_J and the previous study show us that this expansion is uniform over $x \in K_J$. This completes the proof of Theorem 8. $\square$

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