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On the stability theorem of L^p solutions for multidimensional BSDEs with uniform continuity generators in z

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Abstract. In this paper, we first establish an existence and uniqueness result of L^p $(p > 1)$ solutions for multidimensional backward stochastic differential equations (BSDEs) whose generator g satisfies a certain one-sided Osgood condition with a general growth in y as well as a uniform continuity condition in z, and the ith component ${}^i g$ of g depends only on the ith row ${}^i z$ of matrix z besides (ω, t, y). Then we put forward and prove a stability theorem for L^p solutions of this kind of multidimensional BSDEs. This generalizes some known results.

1 Introduction

Nonlinear backward stochastic differential equations (BSDEs in short) were initially introduced by Pardoux and Peng (1990), which established an existence and uniqueness result of L^2 solutions for multidimensional BSDEs under a Lipschitz assumption of the generator g. Since then, the theory of BSDEs has been intensively developed thanks to its connections with many other fields of research, such as mathematical finance, stochastic control and partial differential equation theory; see Bahlali, Essaky and Hassani (2010), Briand et al. (2003), Delbaen, Hu and Bao (2011), El Karoui, Peng and Quenez (1997), Jia (2010), Kobylanski (2000), Pardoux (1999), Peng (1997), Richou (2012) and Xing (2012). The classical Lipschitz assumption on the generator g has also been relaxed by some researchers in recent years, in the case of multidimensional BSDEs; see Bahlali, Essaky and Hassani (2010), Briand et al. (2003), Fan (2014), Fan and Jiang (2013a, 2013b), Fan, Jiang and Davison (2010, 2013), Hamadène (2003), Hu and Tang (2014), Mao (1995) and Pardoux (1999), and in the one-dimensional case, see Briand and Hu (2008), Briand, Lepeltier and San Martin (2007), Delbaen, Hu and Bao (2011), Hamadène (1996), Jia (2010), Kobylanski (2000), Lepeltier and San Martin (1997), Ma, Fan and Song (2013), Richou (2012) and Xing (2012). At the same time, many works were done for properties of solutions for BSDEs, such as the comparison theorem and the stability theorem; see Briand and Hu (2008),

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Briand, Lepeltier and San Martin (2007), Delbaen, Hu and Bao (2011), El Karoui, Peng and Quenez (1997), Fan and Jiang (2013b), Hamadène (1996), Jia (2010), Kobylanski (2000), Lepeltier and San Martin (1997) and Pardoux (1999).

For the first time, El Karoui, Peng and Quenez (1997) established a stability theorem of L^2 solutions for multidimensional BSDEs whose generator g is Lipschitz continuous in (y, z). From then on, many researchers have been devoted to establishing existence and uniqueness results and the stability theorem of solutions for BSDEs under weaker conditions. For example, Pardoux (1999) proved an existence and uniqueness result as well as a stability theorem of L^2 solutions for multidimensional BSDEs when the generator g satisfies a monotonicity condition with a general growth in y. Recently, Fan, Jiang and Davison (2013), Fan and Jiang (2013b) and Fan (2014) further extended it step by step to the case of L^p $(p > 1)$ solutions for multidimensional BSDEs where the generator g satisfies a weak monotonicity condition with a general growth in y. Note that in these results for multidimensional BSDEs, the generators are all Lipschitz continuous in z, and then by establishing a priori estimates of solutions the authors proved the stability theorems.

On the other hand, in the one-dimensional case, Jia (2010) obtained an existence and uniqueness result of L^2 solutions of BSDEs where the generator g satisfies a Lipschitz condition in y and a uniform continuity condition in z. A stability theorem of solutions for this kind of BSDEs was also established by virtue of a comparison theorem for solutions of one-dimensional BSDEs. As for the multidimensional BSDEs, Hamadène (2003) and Fan, Jiang and Davison (2010), respectively, proved the existence and uniqueness result for L^2 solutions when the generator g satisfies the Osgood condition in y (which yields that g is of linear growth in y), a uniform continuity condition in z, and the ith component ${}^i g$ of g depends only on the ith row ${}^i z$ of matrix z besides (ω, t, y). Recently, they were further generalized by Fan and Jiang (2013a) to the case of L^p $(p > 1)$ solutions. Then a natural question is asked: Can we establish a stability theorem of the L^p $(p > 1)$ solutions for this kind of BSDEs?

Answering the question is the main objective of this paper. In the first part, under a uniform continuity condition of the generator g in z, we first extend the existence and uniqueness result obtained in Fan and Jiang (2013a) to the case when the generator g of multidimensional BSDEs satisfies a certain one-sided Osgood condition with a general growth in y and the ith component ${}^i g$ of g depends only on the ith row ${}^i z$ of matrix z besides (ω, t, y) (see Theorem 5 in Section 3). This is done by virtue of some updated results obtained in Fan (2014) (see Lemmas 2–3 in Section 2 and Lemma 9 in Section 3). The second part of this paper is devoted to establishing a stability theorem of L^p $(p > 1)$ solutions for the multidimensional BSDEs. It should be mentioned that it is very hard to establish a priori estimates of solutions for multidimensional BSDEs when the generator g is only uniformly continuous in z, and that the comparison theorem cannot be applied as in the one-dimensional case. By making use of Girsanov's theorem, the evolution-approaching technique, a priori estimations under the Lipschitz conditions of g in z and Bihari's inequality, we establish the stability theorem of L^p $(p > 1)$ solutions for the BSDEs (see Theorem 11 in Section 4), which yields the uniqueness result in Theorem 5 as a direct consequence. Note that Theorem 5 and Theorem 11 can be considered as a generalization of the corresponding results obtained in El Karoui, Peng and Quenez (1997), Hamadène (2003), Fan, Jiang and Davison (2010), Jia (2010) and Fan and Jiang (2013a).

This paper is organized as follows. Section 2 introduces some usual notations and lemmas. Section 3 is devoted to proving the existence and uniqueness result for L^p $(p > 1)$ solutions, and Section 4 aims to establish the stability theorem.

2 Notations and lemmas

First of all, let us fix a number $T > 0$, and two positive integers k and d. Let ${}^i x$ is the ith component of a vector x or the ith row of a matrix x. Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space carrying a standard d-dimensional Brownian motion $(B_t)_{t\geq 0}$. Let $(\mathcal{F}_t)_{t\geq 0}$ be the natural σ-algebra generated by $(B_t)_{t\geq 0}$. We assume that $\mathcal{F} = \mathcal{F}_T$ and $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous and complete. For each subset $A \subset \Omega \times [0, T]$, let $\mathbf{1}_A = 1$ in case of $(t, \omega) \in A$, otherwise, let $\mathbf{1}_A = 0$. In this paper, the Euclidean norm of a vector $y \in \mathbf{R}^k$ will be defined by $|y|$, and for an $k \times d$ matrix z, we define $|z| = \sqrt{\mathrm{Tr}(zz^*)}$, where z^* is the transpose of z. Let $\langle x, y\rangle$ represent the inner product of $x, y \in \mathbf{R}^k$. For each $p > 1$, we denote by $L^p(\Omega, \mathcal{F}_T, \mathrm{P}; \mathbf{R}^k)$ the set of all $\mathbf{R}^k$-valued and $\mathcal{F}_T$-measurable random variables ξ such that $\mathbf{E}[|\xi|^p] < +\infty$. Let $\mathcal{S}^p(0, T; \mathbf{R}^k)$ denote the set of $\mathbf{R}^k$-valued, adapted and continuous processes $(\phi_t)_{t\in[0,T]}$ such that

$$\|\phi\|_{\mathcal{S}^p} := \Big(\mathbf{E}\Big[\sup_{t\in[0,T]} |\phi_t|^p\Big]\Big)^{1/p} < +\infty.$$

Moreover, let $\mathrm{M}^p(0, T; \mathbf{R}^{k\times d})$ denote the set of $(\mathcal{F}_t)$-progressively measurable $\mathbf{R}^{k\times d}$-valued processes $(\varphi_t)_{t\in[0,T]}$ such that

$$\|\varphi\|_{\mathrm{M}^p} := \Big\{\mathbf{E}\Big[\Big(\int_0^T |\varphi_t|^2\,\mathrm{d}t\Big)^{p/2}\Big]\Big\}^{1/p} < +\infty.$$

Obviously, $\mathcal{S}^p$ and M^p are Banach spaces for each $p > 1$.

Finally, let $\mathbf{S}$ be the set of all nondecreasing continuous functions $\rho(\cdot) : \mathbf{R}^+ \mapsto \mathbf{R}^+$ with $\rho(0) = 0$ and $\rho(x) > 0$ for all $x > 0$, where and hereafter $\mathbf{R}^+ := [0, +\infty)$.

In this paper, we will deal only with the following multidimensional BSDE:

$$y_t = \xi + \int_t^T g(s, y_s, z_s)\,\mathrm{d}s - \int_t^T z_s\,\mathrm{d}B_s, \qquad t \in [0, T], \tag{2.1}$$

where $T > 0$ is called the time horizon; $\xi \in L^p(\Omega, \mathcal{F}_T, \mathrm{P}; \mathbf{R}^k)$ $(p > 1)$ called the terminal condition; the random function $g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbf{R}^k \times \mathbf{R}^{k\times d} \to \mathbf{R}^k$ is $(\mathcal{F}_t)$-progressively measurable for each (y, z), called the generator of BSDE (2.1). The triple (ξ, T, g) is called the parameters of BSDE (2.1). This BSDE is usually denoted by BSDE (ξ, T, g).

Definition 1. A pair of processes $(y_t, z_t)_{t\in[0,T]}$ is called an L^p $(p > 1)$ solution of BSDE (2.1), if $(y_t, z_t)_{t\in[0,T]} \in \mathcal{S}^p(0, T; \mathbf{R}^k) \times \mathrm{M}^p(0, T; \mathbf{R}^{k\times d})$ and satisfies BSDE (2.1).

Next, let us first introduce two lemmas, which will play an important role in the proof of our main results. First, the following Lemma 2 comes from Proposition 3 in Fan (2014). In stating it, the following assumption on the generator g will be used:

(A) $\mathrm{dP} \times \mathrm{d}t$-a.e., for each $(y, z) \in \mathbf{R}^k \times \mathbf{R}^{k\times d}$,

$$\left\langle \frac{y}{|y|}\mathbf{1}_{|y|\neq 0}, g(\omega, t, y, z)\right\rangle \leq \frac{\psi(|y|^p)}{|y|^{p-1}}\mathbf{1}_{|y|\neq 0} + \lambda|z| + f_t,$$

where λ is a nonnegative constant, f_t is a nonnegative and $(\mathcal{F}_t)$-progressively measurable process with $\mathbf{E}[(\int_0^T f_t\,\mathrm{d}t)^p] < +\infty$ and $\psi(\cdot) \in \mathbf{S}$ is a concave function.

Lemma 2. *Assume that $p > 1$ and* (A) *holds. Let $(y_t, z_t)_{t\in[0,T]}$ be an L^p solution of BSDE* (2.1). *Then there exists a nonnegative constant $C_{\lambda,p,T}$ depending only on λ, p and T such that for each $0 \leq u \leq t \leq T$,*

$$\begin{aligned}\mathbf{E}\Big[&\sup_{s\in[t,T]} |y_s|^p \big|\mathcal{F}_u\Big]\\ &\leq C_{\lambda,p,T}\left\{\mathbf{E}\big[|\xi|^p|\mathcal{F}_u\big] + \int_t^T \psi\big(\mathbf{E}[|y_s|^p|\mathcal{F}_u]\big)\,\mathrm{d}s + \mathbf{E}\left[\left(\int_t^T f_s\,\mathrm{d}s\right)^p\Big|\mathcal{F}_u\right]\right\}.\end{aligned}$$

Then the following Lemma 3 comes from Proposition 2 in Fan (2014). In stating it, the following assumption on the generator g will be used.

(B) $\mathrm{dP} \times \mathrm{d}t$-a.e., for each $(y, z) \in \mathbf{R}^k \times \mathbf{R}^{k\times d}$,

$$\langle y, g(\omega, t, y, z)\rangle \leq \mu|y|^2 + \lambda|y||z| + |y|f_t + \theta_t,$$

where μ and λ are two nonnegative constants, f_t and θ_t are two nonnegative and $(\mathcal{F}_t)$-progressively measurable processes with

$$\mathbf{E}\left[\left(\int_0^T f_t\,\mathrm{d}t\right)^p\right] < +\infty \qquad \text{and} \qquad \mathbf{E}\left[\left(\int_0^T \theta_t\,\mathrm{d}t\right)^{p/2}\right] < +\infty.$$

Lemma 3. *Assume that $p > 1$ and* (B) *holds. Let $(y_t, z_t)_{t\in[0,T]}$ be an L^p solution of BSDE* (2.1). *Then for each $0 \le u \le t \le T$, we have*

$$\mathbf{E}\Big[\Big(\int_t^T |z_s|^2\,\mathrm{d}s\Big)^{p/2}\Big|\mathcal{F}_u\Big] \le C_{\mu,\lambda,p,T}\mathbf{E}\Big[\sup_{s\in[t,T]}|y_s|^p\big|\mathcal{F}_u\Big] + C_p\mathbf{E}\Big[\Big(\int_t^T f_s\,\mathrm{d}s\Big)^p\Big|\mathcal{F}_u\Big] + C_p\mathbf{E}\Big[\Big(\int_t^T \theta_s\,\mathrm{d}s\Big)^{p/2}\Big|\mathcal{F}_u\Big],$$

where $C_{\mu,\lambda,p,T}$ is a nonnegative constant depending on (μ, λ, p, T), and C_p is a nonnegative constant depending only on p.

3 An existence and uniqueness result

In this section, we will establish the existence and uniqueness for L^p $(p > 1)$ solutions of multidimensional BSDEs under the condition that the generator g satisfies a certain one-sided Osgood condition with a general growth in y as well as a uniform continuity condition in z, and the ith component ${}^ig$ of g depends only on the ith row ${}^iz$ of matrix z besides (ω, t, y). In the remainder of this paper, we put an i at a left upper index of $y \in \mathbf{R}^k$, $z \in \mathbf{R}^{k\times d}$ and the generator g to represent the ith component of y and g, and the ith row of z, like ${}^iy$, ${}^iz$ and ${}^ig$. Let us start with introducing the following assumptions:

(H1) g satisfies a certain one-sided Osgood condition in y, that is, there exists a concave function $\rho(\cdot) \in \mathbf{S}$ with $\int_{0^+}\rho^{-1}(u)\,\mathrm{d}u = +\infty$ such that $\mathrm{dP}\times\mathrm{d}t$-a.e., for each $y_1, y_2 \in \mathbf{R}^k$, $z \in \mathbf{R}^{k\times d}$ and $i = 1, \ldots, k$,

$$\mathrm{sgn}\big({}^iy_1 - {}^iy_2\big)\big({}^ig(\omega, t, y_1, z) - {}^ig(\omega, t, y_2, z)\big) \le \rho\big(|y_1 - y_2|\big);$$

(H2) g satisfies a general growth bound in y, that is, for each $\alpha \ge 0$,

$$\mathbf{E}\Big[\int_0^T \varphi_\alpha(\omega, t)\,\mathrm{d}t\Big] < +\infty,$$

where

$$\varphi_\alpha(\omega, t) := \sup_{|y|\le\alpha}\big|g(\omega, t, y, 0) - g(\omega, t, 0, 0)\big|;$$

Furthermore, we assume that $\mathrm{dP}\times\mathrm{d}t$-a.e., for each $z \in \mathbf{R}^{k\times d}$, $y \mapsto g(\omega, t, y, z)$ is continuous;

(H3) g is uniformly continuous in z uniformly with respect to (ω, t, y), that is, there exists a linear growth function $\phi(\cdot) \in \mathbf{S}$ such that $\mathrm{dP}\times\mathrm{d}t$-a.e., for each $y \in \mathbf{R}^k$, $z_1, z_2 \in \mathbf{R}^{k\times d}$,

$$\big|g(\omega, t, y, z_1) - g(\omega, t, y, z_2)\big| \le \phi\big(|z_1 - z_2|\big);$$

(H4) For any $i = 1, \dots, k$, ${}^i g(\omega, t, y, z)$ depends only on $(\omega, t, y, {}^i z)$;
(H5) $\mathbf{E}[(\int_0^T |g(\omega, t, 0, 0)| \, \mathrm{d}t)^p] < +\infty$.

Remark 4. $\rho(\cdot)$ in assumption (H1) is a at most linear growth function since $\rho(\cdot)$ is nondecreasing and concave with $\rho(0) = 0$. In the sequel, we always denote the linear growth constant for $\rho(\cdot)$ in (H1) and $\phi(\cdot)$ in (H3) by a fixed constant $A > 0$, that is, $\rho(x) \le A(1 + x)$ and $\phi(x) \le A(1 + x)$ for all $x \in \mathbf{R}^+$.

The main result of this section is as follows.

Theorem 5. *Assume that $p > 1$ and the generator g satisfies assumptions* (H1)–(H5). *Then, for each $\xi \in L^p(\Omega, \mathcal{F}_T, \mathrm{P}; \mathbf{R}^k)$, BSDE (ξ, T, g) has a unique L^p solution.*

In order to compare Theorem 5 with some corresponding results in Fan (2014) and Fan and Jiang (2013a), let us further introduce the following assumption on g:

(H1′) g satisfies the one-sided Osgood condition in y, that is, there exists a concave function $\bar{\psi}(\cdot) \in \mathbf{S}$ with $\int_{0^+} \bar{\psi}^{-1}(u) \, \mathrm{d}u = +\infty$ such that $\mathrm{dP} \times \mathrm{d}t$-a.e., for each $y_1, y_2 \in \mathbf{R}^k$, $z \in \mathbf{R}^{k \times d}$,

$$\left\langle \frac{y_1 - y_2}{|y_1 - y_2|} \mathbf{1}_{|y_1 - y_2| \neq 0}, g(\omega, t, y_1, z) - g(\omega, t, y_2, z) \right\rangle \le \bar{\psi}\big(|y_1 - y_2|\big);$$

(H1″) g satisfies the Osgood condition in y, that is, there exists a concave function $\tilde{\psi}(\cdot) \in \mathbf{S}$ with $\int_{0^+} \tilde{\psi}^{-1}(u) \, \mathrm{d}u = +\infty$ such that $\mathrm{dP} \times \mathrm{d}t$-a.e., for each $y_1, y_2 \in \mathbf{R}^k$, $z \in \mathbf{R}^{k \times d}$,

$$\big|g(\omega, t, y_1, z) - g(\omega, t, y_2, z)\big| \le \tilde{\psi}\big(|y_1 - y_2|\big);$$

(H3′) g is Lipschitz continuous in z uniformly with respect to (ω, t, y), that is, there exists a constant $\mu \ge 0$ such that $\mathrm{dP} \times \mathrm{d}t$-a.e., for each $y \in \mathbf{R}^k$, $z_1, z_2 \in \mathbf{R}^{k \times d}$,

$$\big|g(\omega, t, y, z_1) - g(\omega, t, y, z_2)\big| \le \mu |z_1 - z_2|.$$

Remark 6. It is not hard to verify that (H3′) $\Longrightarrow$ (H3) and (H1″) $\Longrightarrow$ (H1) + (H2) $\Longrightarrow$ (H1′) + (H2). Furthermore, none of the inverse versions of these relations holds true.

In Fan and Jiang (2013a), the authors established the existence and uniqueness of L^p solutions for BSDE (2.1) under assumptions (H1″), (H3), (H4) and (H5). It then follows from Remark 6 that Theorem 5 generalizes this result. In Fan (2014), the author established the existence and uniqueness of L^p solutions for BSDE (2.1) (see Lemma 9 later) under assumptions (H1′), (H2), (H3′) and (H5), which is also closely related to our Theorem 5. Here, we would like to especially mention that to our knowledge, the issue of the existence and uniqueness of L^2 solutions for BSDE (2.1) under assumption (H1′), (H2), (H3), (H4) and (H5) is still an interesting open problem.

Example 7. Let $k=2$ and for each $(y,z)\in\mathbf{R}^2\times\mathbf{R}^{2\times d}$, define the generator g by

$$g(\omega,t,y,z)=\begin{pmatrix} {}^1g(\omega,t,y,z)\\ {}^2g(\omega,t,y,z)\end{pmatrix}=\begin{pmatrix} e^{-{}^1y}+h(|{}^2y|)+f(|{}^1z|)\\ |y|+f(|{}^2z|)\end{pmatrix},$$

where

$$h(x)=\begin{cases} -x\ln x, & 0<x<\delta,\\ h'(\delta-)(x-\delta)+h(\delta), & x>\delta,\\ 0, & \text{otherwise}\end{cases}$$

with $\delta>0$ small enough, and $f(x):=\sqrt{x^2+2x}$ for all $x\in\mathbf{R}^+$.

It is not hard to verify that this generator g satisfies (H2), (H3) with $\phi(x)=f(x)$, (H4) and (H5). Furthermore, we can also prove that g satisfies (H1) by verify that e^{-x} is a decreasing function in x, $\int_{0^+}h^{-1}(u)\,\mathrm{d}u=+\infty$ and $h(\cdot)$ is concave and subadditive on $\mathbf{R}^+$, then the following two inequalities hold true:

$$\operatorname{sgn}({}^1y_1-{}^1y_2)\big({}^1g(\omega,t,y_1,z)-{}^1g(\omega,t,y_2,z)\big)\le h(|y_1-y_2|),$$
$$\operatorname{sgn}({}^2y_1-{}^2y_2)\big({}^2g(\omega,t,y_1,z)-{}^2g(\omega,t,y_2,z)\big)\le |y_1-y_2|.$$

Then, for each $\xi\in L^p(\Omega,\mathcal{F}_T,\mathrm{P};\mathbf{R}^k)$, BSDE (ξ,T,g) has a unique L^p solution by Theorem 5.

It is clear that this generator g does not satisfy (H3′). In addition, it should be noted that this generator g does not satisfy (H1″). In fact, if there exists a nondecreasing function $\tilde{\psi}(\cdot)$ such that (H1″) holds for g, then we have, $\mathrm{dP}\times\mathrm{d}t$-a.e., for each $y_1=({}^1y_1,{}^2y_1)$, $y_2=({}^1y_2,{}^2y_2)$,

$$\big|e^{-{}^1y_1}+h(|{}^2y_1|)-e^{-{}^1y_2}-h(|{}^2y_2|)\big|\le\tilde{\psi}(|y_1-y_2|).$$

Thus, letting ${}^1y_1={}^2y_1=x$, ${}^1y_2={}^2y_2=0$ with $x<0$ in the previous inequality and noticing that $\tilde{\psi}(x)$ is of linear growth, we can get that there exists a constant $K>0$ such that

$$\big|e^{-x}-1+h(|x|)\big|\le\tilde{\psi}(|x|)\le K|x|+K,$$

which leads to the contradiction that e^{-x} is of linear growth on $\mathbf{R}$. As a result, this generator g does not satisfy (H1″). Thus, the previous existence and uniqueness result can be obtained by the results in neither Fan (2014) nor Fan and Jiang (2013a).

As the first step to prove Theorem 5, we establish the following Proposition 8, whose proof is enlightened by Theorem 1 in Ma, Fan and Song (2013).

Proposition 8. *Assume that the generator g satisfies* (H1) *with* $\rho(\cdot)$, (H2), (H3) *with* $\phi(\cdot)$, (H4) *and* (H5). *Then there exists a generator sequence* $\{g^n\}_{n=1}^{+\infty}$ *such that*

(i) $\mathrm{dP} \times \mathrm{d}t$*-a.e., for each* $y \in \mathbf{R}^k$, $z \in \mathbf{R}^{k\times d}$ *and* $n \ge 1$,

$$\left|g^n(\omega, t, y, z) - g(\omega, t, y, z)\right| \le k\phi\left(\frac{2A}{n}\right); \tag{3.1}$$

(ii) *For each* $n \ge 1$, g^n *satisfies* (H1) *with* $\rho(\cdot)$, (H2), (H3) *with* $k\phi(\cdot)$, (H4), (H5) *and* (H1′) *with* $\bar{\psi}(\cdot) = k\rho(\cdot)$. *Furthermore,* g^n *is Lipschitz continuous in* z *uniformly with respect to* (ω, t, y), *that is, there exists a constant* $k_n > 0$ *such that* $\mathrm{dP} \times \mathrm{d}t$*-a.e., for each* $y \in \mathbf{R}^k$, $z_1, z_2 \in \mathbf{R}^{k\times d}$,

$$\left|g^n(\omega, t, y, z_1) - g^n(\omega, t, y, z_2)\right| \le k_n|z_1 - z_2|. \tag{3.2}$$

Proof. By a similar argument to that in Ma, Fan and Song (2013), we know that the sequence of generators $g^n(\omega, t, y, z) := ({}^1g^n, {}^2g^n, \ldots, {}^kg^n)$, where

$$\begin{aligned} {}^ig^n(\omega, t, y, z) = \inf_{v\in\mathbf{R}^{k\times d}} \bigl\{{}^ig\bigl(\omega, t, y, {}^iv\bigr) + (n + A)\bigl|{}^iv - {}^iz\bigr|\bigr\}, \\ i = 1, 2, \ldots, k, \end{aligned} \tag{3.3}$$

is well defined, and for each $n \ge 1$, g^n is $(\mathcal{F}_t)$-progressively measurable for each (y, z), and satisfies (i), (H2), (H4), (H5) and (3.2). Furthermore, we also know that $\mathrm{dP} \times \mathrm{d}t$-a.e., for each $y \in \mathbf{R}^k$, $z_1, z_2 \in \mathbf{R}^{k\times d}$, $n \ge 1$ and $i = 1, 2, \ldots, k$,

$$\bigl|{}^ig^n(\omega, t, y, z_1) - {}^ig^n(\omega, t, y, z_2)\bigr| \le \phi\bigl(\bigl|{}^iz_1 - {}^iz_2\bigr|\bigr) \le \phi\bigl(|z_1 - z_2|\bigr),$$

which yields that

$$\begin{aligned} \left|g^n(\omega, t, y, z_1) - g^n(\omega, t, y, z_2)\right| &= \sqrt{\sum_{i=1}^k \bigl|{}^ig^n(\omega, t, y, z_1) - {}^ig^n(\omega, t, y, z_2)\bigr|^2} \\ &\le \sqrt{\sum_{i=1}^k \phi^2\bigl(|z_1 - z_2|\bigr)} \\ &\le k\phi\bigl(|z_1 - z_2|\bigr). \end{aligned}$$

Hence, g^n also satisfies (H3) with $k\phi(\cdot)$.

Now, we show that g^n satisfies (H1) with $\rho(\cdot)$. In fact, by (H1) with $\rho(\cdot)$ for g, (3.3) and the inequality

$$\inf_{x\in D} f_1(x) - \inf_{x\in D} f_2(x) \le \sup_{x\in D}\bigl(f_1(x) - f_2(x)\bigr),$$

we can get that, $\mathrm{dP} \times \mathrm{d}t$-a.e., for each $n \geq 1$, $i = 1, 2, \ldots, k$, $y_1, y_2 \in \mathbf{R}^k$ with ${}^i y_1 \geq {}^i y_2$ and $z \in \mathbf{R}^{k \times d}$,

$$
\begin{aligned}
&\big({}^i y_1 - {}^i y_2\big)\big({}^i g^n(\omega, t, y_1, z) - {}^i g^n(\omega, t, y_2, z)\big) \\
&\quad \leq \sup_{v \in \mathbf{R}^{k \times d}} \big\{\big({}^i y_1 - {}^i y_2\big)\big({}^i g\big(\omega, t, y_1, {}^i v\big) - {}^i g\big(\omega, t, y_2, {}^i v\big)\big)\big\} \\
&\quad \leq \big|{}^i y_1 - {}^i y_2\big| \rho\big(|y_1 - y_2|\big).
\end{aligned}
$$

Note that the above inequality holds still true when ${}^i y_1 \leq {}^i y_2$. Then g^n also satisfies (H1) with $\rho(\cdot)$. Furthermore, by Remark 6 we know that (H1$'$) is also true for each g^n. Indeed, since g^n satisfies (H1) with $\rho(\cdot)$, we can deduce that $\mathrm{dP} \times \mathrm{d}t$-a.e., for each $n \geq 1$, $y_1, y_2 \in \mathbf{R}^k$ and $z \in \mathbf{R}^{k \times d}$,

$$
\begin{aligned}
&\big\langle y_1 - y_2, g^n(\omega, t, y_1, z) - g^n(\omega, t, y_2, z)\big\rangle \\
&\quad = \sum_{i=1}^{k} \big({}^i y_1 - {}^i y_2\big)\big({}^i g^n(\omega, t, y_1, z) - {}^i g^n(\omega, t, y_2, z)\big) \\
&\quad \leq \sum_{i=1}^{k} \big|{}^i y_1 - {}^i y_2\big| \rho\big(|y_1 - y_2|\big) \\
&\quad \leq k \left(\sum_{i=1}^{k} \big|{}^i y_1 - {}^i y_2\big|^2\right)^{1/2} \rho\big(|y_1 - y_2|\big) \\
&\quad = k|y_1 - y_2| \rho\big(|y_1 - y_2|\big).
\end{aligned}
$$

Hence, (H1$'$) holds true for g^n with the concave function $\bar{\psi}(\cdot) = k\rho(\cdot)$.

In conclusion, the sequence of generators $g^n = ({}^1 g^n, {}^2 g^n, \ldots, {}^k g^n)$ is just the one we desire. The proof is completed. □

The following Lemma 9 will be used in the proof of the Theorem 5, which comes from Corollary 2 in Fan (2014).

Lemma 9. *Assume that g satisfies* (H1$'$), (H2), (H3$'$) *and* (H5). *Then, for each $\xi \in L^p(\Omega, \mathcal{F}_T, \mathrm{P}; \mathbf{R}^k)$, BSDE (ξ, T, g) has a unique L^p solution.*

We are now in a position to prove Theorem 5.

Proof of Theorem 5. Assume $p > 1$ and the generator g satisfies assumptions (H1) with $\rho(\cdot)$, (H2), (H3) with $\phi(\cdot)$, (H4) and (H5). Let us first prove the existence part.

According to Proposition 8, we can construct a sequence of generators $\{g^n\}_{n=1}^{+\infty}$ satisfying (i) and (ii) in Proposition 8. Then it follows from Lemma 9 that for each

$n \geq 1$ and $\xi \in L^p(\Omega, \mathcal{F}_T, \mathrm{P}; \mathbf{R}^k)$, the following BSDE (ξ, T, g^n):

$$y_t^n = \xi + \int_t^T g^n(s, y_s^n, z_s^n)\,\mathrm{d}s - \int_t^T z_s^n\,\mathrm{d}B_s, \qquad t \in [0, T], \tag{3.4}$$

has a unique L^p solution $(y_\cdot^n, z_\cdot^n)$. The following proof will be split into five steps.

First step. In this step, we desire to prove that the process $(y_t^{n+r} - y_t^n)_{t\in[0,T]}$ is uniformly bounded, that is, there exists a positive constant $C_{k,A,p,T}$ depending on k, A, p and T such that $\mathrm{dP} \times \mathrm{d}t$-a.e., for each $n, r \geq 1$ and $t \in [0, T]$,

$$|y_t^{n+r} - y_t^n| \leq C_{k,A,p,T}. \tag{3.5}$$

First, for each $n, r \geq 1$, we set

$$\hat{y}_\cdot^{n,r} = y_\cdot^{n+r} - y_\cdot^n, \qquad \hat{z}_\cdot^{n,r} = z_\cdot^{n+r} - z_\cdot^n.$$

Thus, $(\hat{y}_t^{n,r}, \hat{z}_t^{n,r})_{t\in[0,T]}$ is the unique L^p solution of the following BSDE:

$$\hat{y}_t^{n,r} = \int_t^T \hat{g}^{n,r}(s, \hat{y}_s^{n,r}, \hat{z}_s^{n,r})\,\mathrm{d}s - \int_t^T \hat{z}_s^{n,r}\,\mathrm{d}B_s, \qquad t \in [0, T], \tag{3.6}$$

where for each $(y, z) \in \mathbf{R}^k \times \mathbf{R}^{k\times d}$,

$$\hat{g}^{n,r}(t, y, z) := g^{n+r}(t, y + y_t^n, z + z_t^n) - g^n(t, y_t^n, z_t^n).$$

It follows from (i) and (ii) of Proposition 8 that g^n satisfies (H1′) with $k\rho(\cdot)$, (H3) with $k\phi(\cdot)$, and (3.1). Then, in view of Remark 4, $\mathrm{dP} \times \mathrm{d}t$-a.e., for each $n, r \geq 1$ and $(y, z) \in \mathbf{R}^k \times \mathbf{R}^{k\times d}$, we have

$$\begin{aligned}
\left\langle \frac{y}{|y|}\mathbf{1}_{|y|\neq 0}, \hat{g}^{n,r}(s, y, z)\right\rangle &\leq |g^{n+r}(s, y + y_s^n, z + z_s^n) - g^n(s, y + y_s^n, z + z_s^n)| \\
&\quad + \left\langle \frac{y}{|y|}\mathbf{1}_{|y|\neq 0}, g^n(s, y + y_s^n, z + z_s^n) - g^n(s, y_s^n, z_s^n)\right\rangle \\
&\leq k\rho(|y|) + k\phi(|z|) + 2k\phi\left(\frac{2A}{n}\right) \\
&\leq kA|y| + kA|z| + 2kA + 2k\phi(2A).
\end{aligned}$$

Then, in the same way as in the proof of Proposition 3.2 of Briand et al. (2003) we can obtain that there exists a positive constant $C_{k,A,p,T}$ depending only on k, A, p and T such that for each $n, r \geq 1$ and $0 \leq u \leq t \leq T$,

$$\mathbf{E}\left[|\hat{y}_t^{n,r}|^p|\mathcal{F}_u\right] \leq \mathbf{E}\left[\sup_{s\in[t,T]}|\hat{y}_s^{n,r}|^p + \left(\int_t^T |\hat{z}_s^{n,r}|^2\,\mathrm{d}s\right)^{p/2}\Big|\mathcal{F}_u\right] \leq (C_{k,A,p,T})^p,$$

which yields (3.5) after taking $u = t$.

Second step. In this step, we desire to prove that for each $n, r \geq 1$, $i = 1, \ldots, k$ and $0 \leq u \leq t \leq T$,

$$\mathbf{E}^{n,r,i}\left[|{}^i y_t^{n+r} - {}^i y_t^n||\mathcal{F}_u\right] \leq b_n + \int_t^T \mathbf{E}^{n,r,i}\left[\rho(|y_s^{n+r} - y_s^n|)|\mathcal{F}_u\right]\mathrm{d}s, \tag{3.7}$$

where

$$b_n := Tk\Big(\phi\Big(\frac{2A}{n+2A}\Big)+2\phi\Big(\frac{2A}{n}\Big)\Big)$$

and $\mathbf{E}^{n,r,i}[X|\mathcal{F}_u]$ represents the conditional expectation of the random variable X with respect to $\mathcal{F}_u$ under a probability measure $\mathrm{P}^{n,r,i}$ on $(\Omega,\mathcal{F})$, which depends on n, r and i, and which is absolutely continuous with respect to P.

In fact, using Tanaka's formula for BSDE (3.6), we can obtain that for each $n, r\geq 1$ and $i=1,\dots,k$,

$$\begin{aligned}|{}^i y_t^{n+r}-{}^i y_t^n| \leq{}& \int_t^T \operatorname{sgn}({}^i y_s^{n+r}-{}^i y_s^n)({}^i g^{n+r}(s, y_s^{n+r}, {}^i z_s^{n+r})-{}^i g^n(s, y_s^n, {}^i z_s^n))\,\mathrm{d}s \\ &-\int_t^T \operatorname{sgn}({}^i y_s^{n+r}-{}^i y_s^n)({}^i z_s^{n+r}-{}^i z_s^n)\,\mathrm{d}B_s, \qquad t\in[0,T].\end{aligned} \tag{3.8}$$

It follows from (i) and (ii) of Proposition 8 that g^n satisfies (H1) with $\rho(\cdot)$, (H3) with $k\phi(\cdot)$, and (3.1). Then, for each $n,r\geq 1$, $i=1,\dots,k$ and $s\in[t,T]$, we have

$$\begin{aligned}&\operatorname{sgn}({}^i y_s^{n+r}-{}^i y_s^n)({}^i g^{n+r}(s, y_s^{n+r}, {}^i z_s^{n+r})-{}^i g^n(s, y_s^n, {}^i z_s^n)) \\ &\quad\leq \operatorname{sgn}({}^i y_s^{n+r}-{}^i y_s^n)({}^i g^{n+r}(s, y_s^{n+r}, {}^i z_s^{n+r})-{}^i g^{n+r}(s, y_s^n, {}^i z_s^n)) \\ &\qquad+|{}^i g^{n+r}(s, y_s^n, {}^i z_s^n)-{}^i g^n(s, y_s^n, {}^i z_s^n)| \\ &\quad\leq \rho(|y_s^{n+r}-y_s^n|)+k\phi(|{}^i z_s^{n+r}-{}^i z_s^n|)+2k\phi\Big(\frac{2A}{n}\Big).\end{aligned} \tag{3.9}$$

Furthermore, recalling that $\phi(\cdot)$ is a nondecreasing function from $\mathbf{R}^+$ to itself with linear growth, we can prove that for each $n\in\mathbf{N}$,

$$\phi(x)\leq (n+2A)x+\phi\Big(\frac{2A}{n+2A}\Big) \tag{3.10}$$

holds true for each $x\in\mathbf{R}^+$. In fact, if $0\leq x\leq\frac{2A}{n+2A}$, the conclusion is obvious considering $\phi(\cdot)$ is nondecreasing function. And if $\frac{2A}{n+2A}<x<1$, we have $(n+2A)x>2A=A+A>Ax+A\geq\phi(x)$. Finally, in the case of $x\geq 1$, we also have $(n+2A)x>2Ax=Ax+Ax>Ax+A\geq\phi(x)$. Thus, combining (3.8), (3.9) and (3.10), we get that for each $n,r\geq 1$ and $i=1,\dots,k$,

$$\begin{aligned}|{}^i y_t^{n+r}-{}^i y_t^n| \leq{}& Tk\Big(\phi\Big(\frac{2A}{n+2A}\Big)+2\phi\Big(\frac{2A}{n}\Big)\Big)+\int_t^T \rho(|y_s^{n+r}-y_s^n|)\,\mathrm{d}s \\ &-\int_t^T \operatorname{sgn}({}^i y_s^{n+r}-{}^i y_s^n)({}^i z_s^{n+r}-{}^i z_s^n)[-a_s^{n,r,i}\,\mathrm{d}s+\mathrm{d}B_s], \end{aligned} \tag{3.11}$$

$t\in[0,T]$,

where for each $t\in[0,T]$,

$$a_t^{n,r,i} := k(n+2A)\frac{\operatorname{sgn}({}^i y_t^{n+r}-{}^i y_t^n)({}^i z_t^{n+r}-{}^i z_t^n)^*}{|{}^i z_t^{n+r}-{}^i z_t^n|}\mathbf{1}_{|{}^i z_t^{n+r}-{}^i z_t^n|\neq 0}.$$

Note that $(a_t^{n,r,i})_{t\in[0,T]}$ is a $\mathbf{R}^d$-valued, bounded and $(\mathcal{F}_t)$-adapted process with $|a_t^{n,r,i}| \le k(n+2A)$. By Girsanov's theorem, we know that

$$B_t^{n,r,i} := B_t - \int_0^t a_s^{n,r,i}\,\mathrm{d}s, \qquad t\in[0,T]$$

is a d-dimensional Brownian motion under the probability measure $\mathrm{P}^{n,r,i}$ on $(\Omega,\mathcal{F})$ defined by

$$\frac{\mathrm{dP}^{n,r,i}}{\mathrm{dP}} = \exp\left\{\int_0^T (a_s^{n,r,i})^*\,\mathrm{d}B_s - \frac{1}{2}\int_0^T |a_s^{n,r,i}|^2\,\mathrm{d}s\right\}.$$

Furthermore, by the Burkholder–Davis–Gundy (BDG) inequality and Hölder's inequality we know that the process

$$\left(\int_0^t \operatorname{sgn}\left({}^i y_s^{n+r} - {}^i y_s^n\right)\left({}^i z_s^{n+r} - {}^i z_s^n\right)\mathrm{d}B_s^{n,r,i}\right)_{t\in[0,T]}$$

is an $(\mathcal{F}_t,\mathrm{P}^{n,r,i})$-martingale. Thus, for each $n,r\ge 1$, $i=1,\dots,k$ and $0\le u\le t\le T$, by taking the conditional expectation with respect to $\mathcal{F}_u$ under $\mathrm{P}^{n,r,i}$ in both sides of (3.11), we can get the desired result (3.7).

Third step. In this step, we will show that $\{(y_t^n)_{t\in[0,T]}\}_{n=1}^{+\infty}$ is a Cauchy sequence in $\mathcal{S}^p(0,T;\mathbf{R}^k)$. The proof is relatively classical (see, e.g., Hamadène (2003) and Fan, Jiang and Davison (2010)), which are likely to be initially enlightened by Yamada (1981). However, for readers' convenience we list it as follows.

First, for each $n\ge 1$, we define $\rho^n(\cdot):\mathbf{R}^+\to\mathbf{R}^+$ by

$$\rho^n(x) := \sup_{y\in\mathbf{R}^+}\{\rho(y) - (n+A)|x-y|\}.$$

Then $\rho^n(x)$ is well defined for each $n\ge 1$, Lipschitz continuous in x, nonincreasing in n and converges to $\rho(x)$. For each $n\ge 1$, let $h^n(\cdot)$ be the unique solution of the following deterministic backward differential equations (DBDE):

$$h^n(t) = b_n + \int_t^T \rho^n\big(k\cdot h^n(s)\big)\,\mathrm{d}s, \qquad t\in[0,T], \tag{3.12}$$

where b_n is defined in second step. Noticing that $\{\rho^n\}_n$ and $\{b_n\}_n$ are both decreasing in n, we have that $0\le h^{n+1}(t)\le h^n(t)$ for each $n\ge 1$, which implies that, in view of the fact that $b_n\to 0$ as $n\to+\infty$, the sequence $\{h^n(t)\}_{n=1}^{+\infty}$ converges pointwisely to a function $h(t)$, where $h(t)$ is a solution of the following DBDE:

$$h(t) = \int_t^T \rho\big(k\cdot h(s)\big)\,\mathrm{d}s.$$

Note that $\rho(\cdot)\in\mathbf{S}$ and $\int_{0^+}\rho^{-1}(u)\,\mathrm{d}u = +\infty$, Bihari's inequality yields that for each $t\in[0,T]$, $h(t)\equiv 0$.

Now, for each $n, j \geq 1$, let $h^{n,j}(t)$ be the function defined recursively as follows:

$$h^{n,1}(t) \equiv C_{k,A,p,T}; \qquad h^{n,j+1}(t) = b_n + \int_t^T \rho^n\big(k \cdot h^{n,j}(s)\big)\,\mathrm{d}s, \qquad t \in [0,T], \tag{3.13}$$

where $C_{k,A,p,T}$ is defined in (3.5). Noticing that ρ^n is Lipschitz continuous, we know that $h^{n,j}(t)$ converges pointwisely to $h^n(t)$ as $j \to +\infty$ for each $t \in [0,T]$. On the other hand, it is not hard to check by induction that for each $n, r, j \geq 1$ and $i = 1, \ldots, k$,

$$\big|{}^i y_t^{n+r} - {}^i y_t^n\big| \leq h^{n,j}(t) \leq h^{n,j}(0), \qquad t \in [0,T]. \tag{3.14}$$

Indeed, (3.14) holds true for $j = 1$ due to (3.5). Suppose (3.14) holds for some $j \geq 1$. Then, for each $n, r \geq 1$,

$$\rho\big(|y_s^{n+r} - y_s^n|\big) \leq \rho\big(k \cdot h^{n,j}(s)\big) \leq \rho^n\big(k \cdot h^{n,j}(s)\big), \qquad s \in [0,T]. \tag{3.15}$$

Then, letting $u = t$ in (3.7), from (3.13) and (3.15) we can deduce that for each $n, r \geq 1$ and $i = 1, 2, \ldots, k$,

$$\big|{}^i y_t^{n+r} - {}^i y_t^n\big| \leq h^{n,j+1}(t) \leq h^{n,j+1}(0), \qquad t \in [0,T].$$

Thus, (3.14) holds also true for $j + 1$.

Finally, taking a supremum with respect to t and r, sending first $j \to +\infty$ and then $n \to +\infty$ in (3.14), and using Lebesgue's dominated convergence theorem we obtain that for each $i = 1, 2, \ldots, k$,

$$\lim_{n\to\infty} \sup_{r\geq 1} \mathbf{E}\Big[\sup_{t\in[0,T]} \big|{}^i y_t^{n+r} - {}^i y_t^n\big|^p\Big] = 0,$$

which means that $\{(y_t^n)_{t\in[0,T]}\}_{n=1}^{+\infty}$ is a Cauchy sequence in the space of processes $\mathcal{S}^p(0,T;\mathbf{R}^k)$. We denote the limit by $(y_t)_{t\in[0,T]}$.

Fourth step. In this step, we show that $\{(z_t^n)_{t\in[0,T]}\}_{n=1}^{+\infty}$ is a Cauchy sequence in $\mathrm{M}^p(0,T;\mathbf{R}^{k\times d})$.

In fact, it follows from (i) and (ii) of Proposition 8 that g^n satisfies (H1′) with $k\rho(\cdot)$, (H3) with $k\phi(\cdot)$, and (3.1). Then, in view of (3.10), we can deduce that $\mathrm{dP} \times \mathrm{d}t$-a.e., for each $n, r, q \geq 1$ and each $(y,z) \in \mathbf{R}^k \times \mathbf{R}^{k\times d}$,

$$\begin{aligned}
\big\langle y, \hat{g}^{n,r}(s,y,z)\big\rangle &\leq |y|\big|g^{n+r}\big(s, y + y_s^n, z + z_s^n\big) - g^n\big(s, y + y_s^n, z + z_s^n\big)\big| \\
&\quad + \big\langle y, g^n\big(s, y + y_s^n, z + z_s^n\big) - g^n\big(s, y_s^n, z_s^n\big)\big\rangle \\
&\leq k|y|\rho\big(|y|\big) + k|y|\phi\big(|z|\big) + 2k\phi\left(\frac{2A}{n}\right)|y| \\
&\leq \left[k(q+2A) + \frac{k^2}{4}\right]|y|^2 + k(q+2A)|y||z| + 2k\phi\left(\frac{2A}{n}\right)|y| \\
&\quad + \left(\rho\left(\frac{2A}{q+2A}\right) + \phi\left(\frac{2A}{q+2A}\right)\right)^2,
\end{aligned}$$

which means that the assumption (B) is satisfied for the generator $\hat{g}^{n,r}(s,y,z)$ of BSDE (3.6) with $\mu = k(q+2A)+\frac{k^2}{4}$, $\lambda = k(q+2A)$, $f_t = 2k\phi(\frac{2A}{n})$ and $\theta_t = (\rho(\frac{2A}{q+2A})+\phi(\frac{2A}{q+2A}))^2$. Thus, it follows from Lemma 3 with $t=u=0$ that there exists a constant $K_{k,q,A,p,T}>0$ depending only on k, q, A, p and T, and a constant K_p depending only on p such that for each $n,r,q\geq 1$,

$$\mathbf{E}\Big[\Big(\int_0^T |\hat{z}_s^{n,r}|^2\,\mathrm{d}s\Big)^{p/2}\Big] \leq K_{k,q,A,p,T}\mathbf{E}\Big[\sup_{t\in[0,T]}|\hat{y}_t^{n,r}|^p\Big] + K_p\Big(2kT\phi\Big(\frac{2A}{n}\Big)\Big)^p$$
$$+ K_pT^{p/2}\Big(\rho\Big(\frac{2A}{q+2A}\Big)+\phi\Big(\frac{2A}{q+2A}\Big)\Big)^p.$$

In view of the fact that $\{(y_t^n)_{t\in[0,T]}\}_{n=1}^{+\infty}$ is a Cauchy sequence in $\mathcal{S}^p(0,T;\mathbf{R}^k)$, by taking a supremum in r and then taking limsup with respect to n in the previous inequality we deduce that for each $q\geq 1$,

$$\overline{\lim_{n\to\infty}}\sup_{r\geq 1}\mathbf{E}\Big[\Big(\int_0^T |\hat{z}_s^{n,r}|^2\,\mathrm{d}s\Big)^{p/2}\Big] \leq K_pT^{p/2}\Big(\rho\Big(\frac{2A}{q+2A}\Big)+\phi\Big(\frac{2A}{q+2A}\Big)\Big)^p.$$

Then, letting $q\to\infty$ in above inequality and recalling that $\rho(\cdot),\phi(\cdot)\in\mathbf{S}$ yields that

$$\lim_{n\to\infty}\sup_{r\geq 1}\mathbf{E}\Big[\Big(\int_0^T |z_t^{n+r}-z_t^n|^2\,\mathrm{d}t\Big)^{p/2}\Big]=0.$$

That is to say, $\{(z_t^n)_{t\in[0,T]}\}_{n=1}^{+\infty}$ is a Cauchy sequence in $\mathrm{M}^p(0,T;\mathbf{R}^{k\times d})$. We denote the limit by $(z_t)_{t\in[0,T]}$.

Fifth step. This step shows that the process $(y_\cdot, z_\cdot)$ is an L^p solution of BSDE (2.1).

First, since $y_\cdot^n\to y_\cdot$ in $\mathcal{S}^p(0,T;\mathbf{R}^k)$, passing to a subsequence if needed, still denoted by $y_\cdot^n$, we know that dP-a.s.,

$$\lim_{n\to\infty}\sup_{t\in[0,T]}|y_t^n-y_t|=0.$$

Then, for almost all ω, there exists a constant $M_1(\omega)$ depending only on ω such that

$$\sup_{n\geq 1}\sup_{t\in[0,T]}|y_t^n-y_t|\leq M_1(\omega),$$

which means the existence of a constant $M_2(\omega)$ depending only on ω such that

$$\sup_{n\geq 1}\sup_{t\in[0,T]}|y_t^n|\leq M_2(\omega). \tag{3.16}$$

In the sequel, since $z_\cdot^n$ converges in $\mathrm{M}^p(0,T;\mathbf{R}^{k\times d})$ to $z_\cdot$, we can assume, choosing a subsequence if necessary, still denoted by $z_\cdot^n$, that

$$\Big\{\mathbf{E}\Big[\Big(\int_0^T |z_s^n-z_s|^2\,\mathrm{d}s\Big)^{p/2}\Big]\Big\}^{1/p}\leq\frac{1}{2^n}.$$

Note that

$$\begin{aligned}\mathbf{E}\bigg[\bigg(\int_0^T \sup_n |z_s^n - z_s|^2\,\mathrm{d}s\bigg)^{p/2}\bigg] &\le \mathbf{E}\bigg[\bigg(\int_0^T \sum_{n=1}^{+\infty} |z_s^n - z_s|^2\,\mathrm{d}s\bigg)^{p/2}\bigg] \\ &\le \mathbf{E}\bigg[\bigg(\int_0^T \bigg(\sum_{n=1}^{+\infty} |z_s^n - z_s|\bigg)^2\,\mathrm{d}s\bigg)^{p/2}\bigg] \\ &= \bigg\|\sum_{n=1}^{+\infty} |z_s^n - z_s|\bigg\|_{\mathrm{M}^p}^p \le \bigg(\sum_{n=1}^{+\infty} \|z_s^n - z_s\|_{\mathrm{M}^p}\bigg)^p \\ &\le \bigg(\sum_{n=1}^{+\infty} \frac{1}{2^n}\bigg)^p < +\infty.\end{aligned}$$

We have

$$\begin{aligned}\mathbf{E}\bigg[\bigg(\int_0^T \sup_n |z_s^n|^2\,\mathrm{d}s\bigg)^{p/2}\bigg] &\le 2^p\mathbf{E}\bigg[\bigg(\int_0^T \sup_n |z_s^n - z_s|^2\,\mathrm{d}s\bigg)^{p/2}\bigg] \\ &\quad + 2^p\mathbf{E}\bigg[\bigg(\int_0^T |z_s|^2\,\mathrm{d}s\bigg)^{p/2}\bigg] \\ &< +\infty.\end{aligned}\tag{3.17}$$

Furthermore, it follows from (i) and (ii) of Proposition 8, (H2) and the definitions of $y_\cdot$ and $z_\cdot$ that for each $t \in [0, T]$, dP-a.s.,

$$g^n\big(t, y_t^n, z_t^n\big) \to g(t, y_t, z_t), \qquad n \to +\infty. \tag{3.18}$$

On the other hand, it follows from (i) of Proposition 8, (H2), (H3), (H5), (3.16), (3.17) and Remark 4 that for each $t \in [0, T]$, dP-a.s.,

$$\begin{aligned}&\big|g^n\big(t, y_t^n, z_t^n\big)\big| \\ &\quad \le \big|g\big(t, y_t^n, z_t^n\big)\big| + k\phi(2A) \\ &\quad \le \big|g\big(t, y_t^n, z_t^n\big) - g\big(t, y_t^n, 0\big)\big| + \big|g\big(t, y_t^n, 0\big)\big| + k\phi(2A) \\ &\quad \le \phi\big(\big|z_t^n\big|\big) + \big|g\big(t, y_t^n, 0\big) - g(t, 0, 0)\big| + \big|g(t, 0, 0)\big| + k\phi(2A) \\ &\quad \le A \sup_n \big|z_t^n\big| + A + \sup_{|y| \le M_2} \big|g(t, y, 0) - g(t, 0, 0)\big| \\ &\qquad + \big|g(t, 0, 0)\big| + k\phi(2A) \in L^1\big([0, T], \mathrm{d}t\big).\end{aligned}\tag{3.19}$$

Thus, in view of (3.18) and (3.19), it follows from Lebesgue's dominated convergence theorem that dP-a.s., for each $t \in [0, T]$,

$$\int_t^T g^n\big(s, y_s^n, z_s^n\big)\,\mathrm{d}s \to \int_t^T g(s, y_s, z_s)\,\mathrm{d}s, \qquad n \to +\infty.$$

Finally, we pass to the limit in uniform convergence in probability for BSDE (3.4), to see that $(y_t, z_t)_{t\in[0,T]}$ is an L^p solution of BSDE (2.1). Thus, we prove the existence part of Theorem 5.

The uniqueness part of Theorem 5 is an immediate corollary of the following Theorem 11 in Section 4. The proof of Theorem 5 is then completed. □

4 A stability theorem

In this section, we shall put forward and prove a stability theorem of L^p solutions for BSDEs with generators satisfying assumptions (H1)–(H5).

In the sequel, for each $m \in \mathbf{N}$, let $\xi^m \in L^p(\Omega, \mathcal{F}_T, \mathrm{P}; \mathbf{R}^k)$ and let $(y_t^m, z_t^m)_{t\in[0,T]}$ be an L^p solution of the following BSDEs depending on parameter m:

$$y_t^m = \xi^m + \int_t^T g^m(s, y_s^m, z_s^m)\,\mathrm{d}s - \int_t^T z_s^m\,\mathrm{d}B_s, \qquad t \in [0, T].$$

Furthermore, we introduce the following assumptions:

(A1) All g^m satisfy assumptions (H1)–(H5) with the same functions $\rho(\cdot)$ and $\phi(\cdot)$.
(A2) $\lim_{m\to\infty} \mathbf{E}[|\xi^m - \xi^0|^p] = 0$. And, there exists a sequence $a_m > 0$ depending only on m and satisfying $\lim_{m\to\infty} a_m = 0$ such that $\mathrm{dP} \times \mathrm{d}t$-a.e., for each $(y, z) \in \mathbf{R}^k \times \mathbf{R}^{k\times d}$ and $m \geq 1$, we have

$$\left|g^m(\omega, t, y, z) - g^0(\omega, t, y, z)\right| \leq a_m.$$

Remark 10. By Proposition 1 of Fan (2014), we know that if $\rho(\cdot) : \mathbf{R}^+ \to \mathbf{R}^+$ is a concave function such that $\rho(\cdot) \in \mathbf{S}$ and $\int_{0^+} \rho^{-1}(u)\,\mathrm{d}u = +\infty$, then there exists another nondecreasing and concave function $\kappa(\cdot) \in \mathbf{S}$ with $\int_{0^+} \kappa^{-1}(u)\,\mathrm{d}u = +\infty$ such that $|x|^{p-1}\rho(|x|) \leq \kappa(|x|^p)$ for each $x \in \mathbf{R}$.

The following Theorem 11 is the main result of this section.

Theorem 11. *Under assumptions* (A1)–(A2), *we have*

$$\lim_{m\to\infty} \mathbf{E}\left[\sup_{t\in[0,T]} |y_t^m - y_t^0|^p + \left(\int_0^T |z_s^m - z_s^0|^2\,\mathrm{d}s\right)^{p/2}\right] = 0.$$

Proof. Assume that assumptions (A1) and (A2) hold. Let $g^{m,n}$ be the function defined as follows:

$${}^i g^{m,n}(\omega, t, y, z) = \inf_{v\in\mathbf{R}^{k\times d}} \left\{{}^i g^m(\omega, t, y, {}^i v) + (n + A)\left|{}^i v - {}^i z\right|\right\}, \qquad i = 1, 2, \ldots, k.$$

According to Proposition 8, the sequence of generators $g^{m,n} := ({}^1 g^{m,n}, {}^2 g^{m,n}, \ldots, {}^k g^{m,n})$ is well defined for $m \in \mathbf{N}$, it is $(\mathcal{F}_t)$-progressively measurable for each (y, z), and the following two statements hold:

(a) $\mathrm{dP} \times \mathrm{d}t$-a.e., for each $m \in \mathbf{N}$, $n \geq 1$ and $y \in \mathbf{R}^k$, $z \in \mathbf{R}^{k\times d}$,

$$\left|g^{m,n}(\omega, t, y, z) - g^m(\omega, t, y, z)\right| \leq k\phi\left(\frac{2A}{n}\right); \tag{4.1}$$

(b) For each $m \in \mathbf{N}$ and $n \geq 1$, $g^{m,n}$ satisfies (H1) with $\rho(\cdot)$, (H2), (H3) with $k\phi(\cdot)$, (H4), (H5) and (H1′) with $\bar{\psi}(\cdot) = k\rho(\cdot)$. Furthermore, $g^{m,n}$ is Lipschitz continuous in z uniformly with respect to (ω, t, y), that is, there exists a constant $K_n > 0$ depending only on n such that $\mathrm{dP} \times \mathrm{d}t$-a.e., for each $y \in \mathbf{R}^k$, $z_1, z_2 \in \mathbf{R}^{k\times d}$,

$$\left|g^{m,n}(\omega, t, y, z_1) - g^{m,n}(\omega, t, y, z_2)\right| \leq K_n|z_1 - z_2|. \tag{4.2}$$

As a result, it follows from Lemma 9 that for each $m \in \mathbf{N}$ and $n \geq 1$, the following BSDE:

$$y_t^{m,n} = \xi^m + \int_t^T g^{m,n}\left(s, y_s^{m,n}, z_s^{m,n}\right) \mathrm{d}s - \int_t^T z_s^{m,n}\, \mathrm{d}B_s, \qquad t \in [0, T],$$

has a unique L^p solution $(y_t^{m,n}, z_t^{m,n})_{t\in[0,T]}$.

In the sequel, note that

$$\left|y_t^m - y_t^0\right| \leq \left|y_t^m - y_t^{m,n}\right| + \left|y_t^{m,n} - y_t^{0,n}\right| + \left|y_t^{0,n} - y_t^0\right|.$$

It follows from the basic inequality $(a + b)^p \leq 2^p(a^p + b^p)$ that there exists a constant $c_p > 0$ depending only on p such that

$$\begin{aligned}\mathbf{E}\Big[\sup_{t\in[0,T]}\left|y_t^m - y_t^0\right|^p\Big] &\leq c_p\mathbf{E}\Big[\sup_{t\in[0,T]}\left|y_t^m - y_t^{m,n}\right|^p\Big] + c_p\mathbf{E}\Big[\sup_{t\in[0,T]}\left|y_t^{m,n} - y_t^{0,n}\right|^p\Big] \\ &\quad + c_p\mathbf{E}\Big[\sup_{t\in[0,T]}\left|y_t^{0,n} - y_t^0\right|^p\Big].\end{aligned} \tag{4.3}$$

In the following, we will estimate, respectively, every term of the right-hand side in (4.3). First, the following Proposition 12 gives an estimate with respect to the second term.

Proposition 12. *For each $n \geq 1$,*

$$\lim_{m\to\infty} \mathbf{E}\Big[\sup_{t\in[0,T]}\left|y_t^{m,n} - y_t^{0,n}\right|^p\Big] = 0.$$

Proof. For each $m, n \geq 1$, let

$$\hat{y}_\cdot^{m,n} := y_\cdot^{m,n} - y_\cdot^{0,n}, \qquad \hat{z}_\cdot^{m,n} := z_\cdot^{m,n} - z_\cdot^{0,n}.$$

Then $(\hat{y}_\cdot^{m,n}, \hat{z}_\cdot^{m,n})$ is an L^p solution of the following BSDE:

$$\begin{aligned}\hat{y}_t^{m,n} &= \xi^m - \xi^0 + \int_t^T \hat{g}^{m,n}\left(s, \hat{y}_s^{m,n}, \hat{z}_s^{m,n}\right) \mathrm{d}s - \int_t^T \hat{z}_s^{m,n}\, \mathrm{d}B_s, \\ t &\in [0, T],\end{aligned} \tag{4.4}$$

where for each $(y,z)\in\mathbf{R}^k\times\mathbf{R}^{k\times d}$,

$$\hat{g}^{m,n}(t,y,z):=g^{m,n}\big(t,y+y_t^{0,n},z+z_t^{0,n}\big)-g^{0,n}\big(t,y_t^{0,n},z_t^{0,n}\big).$$

In the sequel, it follows from (b) that $g^{m,n}$ satisfies (H1′) with $k\rho(\cdot)$, and (4.2). Then, by Remark 10, the following basic inequality:

$$\Big|\inf_{x\in D}f_1(x)-\inf_{x\in D}f_2(x)\Big|\le\sup_{x\in D}\big|f_1(x)-f_2(x)\big|,$$

and (A2), we can get the existence of a nondecreasing and concave function $\kappa(\cdot)\in\mathbf{S}$ with $\int_{0^+}\kappa^{-1}(u)\,\mathrm{d}u=+\infty$ such that $\mathrm{dP}\times\mathrm{d}t$-a.e., for each $m,n\ge 1$ and $(y,z)\in\mathbf{R}^k\times\mathbf{R}^{k\times d}$,

$$\begin{aligned}
&\left\langle\frac{y}{|y|}\mathbf{1}_{|y|\neq 0},\hat{g}^{m,n}(s,y,z)\right\rangle\\
&\quad\le\left\langle\frac{y}{|y|}\mathbf{1}_{|y|\neq 0},g^{m,n}\big(s,y+y_s^{0,n},z+z_s^{0,n}\big)-g^{m,n}\big(s,y_s^{0,n},z_s^{0,n}\big)\right\rangle\\
&\qquad+\big|g^{m,n}\big(s,y_s^{0,n},z_s^{0,n}\big)-g^{0,n}\big(s,y_s^{0,n},z_s^{0,n}\big)\big|\\
&\quad\le k\rho\big(|y|\big)+K_n|z|+\big|g^{m,n}\big(s,y_s^{0,n},z_s^{0,n}\big)-g^{0,n}\big(s,y_s^{0,n},z_s^{0,n}\big)\big|\\
&\quad\le k\frac{\kappa(|y|^p)}{|y|^{p-1}}\mathbf{1}_{|y|\neq 0}+K_n|z|\\
&\qquad+\sqrt{\sum_{i=1}^{k}\sup_{v\in\mathbf{R}^{k\times d}}\big|{}^i g^m\big(s,y_s^{0,n},{}^i v\big)-{}^i g^0\big(s,y_s^{0,n},{}^i v\big)\big|^2}\\
&\quad\le k\frac{\kappa(|y|^p)}{|y|^{p-1}}\mathbf{1}_{|y|\neq 0}+K_n|z|+ka_m,
\end{aligned}$$

which means that the assumption (A) is satisfied for the generator $\hat{g}^{m,n}(s,y,z)$ of BSDE (4.4) with $\psi(u)=k\kappa(u)$, $\lambda=K_n$ and $f_t=ka_m$. Then it follows from Lemma 2 with $u=0$ that there exists a constant $K_{n,p,k,T}>0$ depending only on n, p, k and T such that for each $m,n\ge 1$ and $t\in[0,T]$,

$$\begin{aligned}
&\mathbf{E}\Big[\sup_{r\in[t,T]}\big|\hat{y}_r^{m,n}\big|^p\Big]\\
&\quad\le K_{n,p,k,T}\left\{\mathbf{E}\big[\big|\xi^m-\xi^0\big|^p\big]+k\int_t^T\kappa\Big(\mathbf{E}\Big[\sup_{r\in[s,T]}\big|\hat{y}_r^{m,n}\big|^p\Big]\Big)\,\mathrm{d}s\right\} \qquad (4.5)\\
&\qquad+K_{n,p,k,T}\mathbf{E}\left[\left(\int_0^T ka_m\,\mathrm{d}s\right)^p\right].
\end{aligned}$$

Furthermore, in view of the facts that $\kappa(\cdot)$ is of linear growth by Remark 10, and $\mathbf{E}[|\xi^m-\xi^0|^p]$ and a_m are bounded by (A2), Gronwall's inequality yields that there exists a positive constant $M(n) > 0$ which depends on n but is independent of m such that

$$\mathbf{E}\Big[\sup_{r\in[0,T]} |\hat{y}_r^{m,n}|^p\Big] \le M(n).$$

Finally, by taking limsup in (4.5) with respect to m and then using Fatou's lemma, the monotonicity and continuity of $\kappa(\cdot)$ and Bihari's inequality, in view of (A2), we can conclude that for each $n \ge 1$ and $t \in [0, T]$,

$$\lim_{m\to\infty} \mathbf{E}\Big[\sup_{s\in[t,T]} |\hat{y}_s^{m,n}|^p\Big] = 0.$$

The proof of Proposition 12 is completed. □

In the sequel, we will estimate the first term and the third term of the right-hand side in (4.3). In fact, we have the following Proposition 13. It will play an important role in the proof of Theorem 11.

Proposition 13. *There exists a real sequence h_n depending only on n and satisfying* $\lim_{n\to\infty} h_n = 0$ *such that* dP-*a.s., for each $m \in \mathbf{N}$ and $n \ge 1$,*

$$\sup_{t\in[0,T]} |y_t^{m,n} - y_t^m| \le k h_n. \tag{4.6}$$

Proof. For each $m \in \mathbf{N}$ and $n \ge 1$, let

$$\tilde{y}_\cdot^{m,n} := y_\cdot^{m,n} - y_\cdot^m, \qquad \tilde{z}_\cdot^{m,n} := z_\cdot^{m,n} - z_\cdot^m.$$

Then $(\tilde{y}_\cdot^{m,n}, \tilde{z}_\cdot^{m,n})$ is an L^p solution of the following BSDE:

$$\tilde{y}_t^{m,n} = \int_t^T \tilde{g}^{m,n}(s, \tilde{y}_s^{m,n}, \tilde{z}_s^{m,n})\,\mathrm{d}s - \int_t^T \tilde{z}_s^{m,n}\,\mathrm{d}B_s, \qquad t \in [0, T],$$

where for each $(y, z) \in \mathbf{R}^k \times \mathbf{R}^{k\times d}$,

$$\tilde{g}^{m,n}(t, y, z) := g^{m,n}(t, y + y_t^m, z + z_t^m) - g^m(t, y_t^m, z_t^m).$$

It follows from (a) and (b) that $g^{m,n}$ satisfies (H1′) with $k\rho(\cdot)$, (H3) with $k\phi(\cdot)$ and (4.1). Then, in view of Remark 4, we deduce that dP × dt-a.e., for each $m \in \mathbf{N}$, $n \ge 1$ and $(y, z) \in \mathbf{R}^k \times \mathbf{R}^{k\times d}$,

$$\begin{aligned}
&\Big\langle \frac{y}{|y|}\mathbf{1}_{|y|\ne 0}, \tilde{g}^{m,n}(s, y, z)\Big\rangle \\
&\quad\le \Big\langle \frac{y}{|y|}\mathbf{1}_{|y|\ne 0}, g^{m,n}(s, y + y_s^m, z + z_s^m) - g^{m,n}(s, y_s^m, z_s^m)\Big\rangle \\
&\qquad + |g^{m,n}(s, y_s^m, z_s^m) - g^m(s, y_s^m, z_s^m)| \\
&\quad\le k\rho(|y|) + k\phi(|z|) + k\phi\Big(\frac{2A}{n}\Big) \\
&\quad\le kA|y| + kA|z| + 2kA + k\phi(2A).
\end{aligned}$$

Then, in the same way as in the proof of Proposition 3.2 of Briand et al. (2003) we can prove that there exists a positive constant $M_{k,A,p,T}$ depending only on k, A, p and T such that for each $m \in \mathbf{N}$, $n \geq 1$ and $0 \leq u \leq t \leq T$,

$$\begin{aligned}\mathbf{E}\big[|\tilde{y}_t^{m,n}|^p|\mathcal{F}_u\big] &\leq \mathbf{E}\bigg[\sup_{s\in[t,T]}|\tilde{y}_s^{m,n}|^p + \bigg(\int_t^T |\tilde{z}_s^{m,n}|^2\,\mathrm{d}s\bigg)^{p/2}\bigg|\mathcal{F}_u\bigg] \\ &\leq (M_{k,A,p,T})^p,\end{aligned} \tag{4.7}$$

which yields, by taking $u = t$ in (4.7), that for each $m \in \mathbf{N}$ and $n \geq 1$, $\mathrm{dP} \times \mathrm{d}t$-a.e., $|y_t^{m,n} - y_t^m| \leq M_{k,A,p,T}$. That is to say, the process $(y_t^{m,n} - y_t^m)_{t\in[0,T]}$ is uniformly bounded.

In the sequel, it follows from Tanaka's formula and (A2) that for each $m \in \mathbf{N}$, $n \geq 1$ and $i = 1, \ldots, k$,

$$\begin{aligned}\big|{}^i y_t^{m,n} - {}^i y_t^m\big| \leq &\int_t^T \mathrm{sgn}\big({}^i y_s^{m,n} - {}^i y_s^m\big)\big({}^i g^{m,n}\big(s, y_s^{m,n}, {}^i z_s^{m,n}\big) - {}^i g^m\big(s, y_s^m, {}^i z_s^m\big)\big)\,\mathrm{d}s \\ &- \int_t^T \mathrm{sgn}\big({}^i y_s^{m,n} - {}^i y_s^m\big)\big({}^i z_s^{m,n} - {}^i z_s^m\big)\,\mathrm{d}B_s, \qquad t \in [0, T].\end{aligned}$$

Furthermore, It follows from (a) and (b) that $g^{m,n}$ satisfies (H1) with $\rho(\cdot)$, (H3) with $k\phi(\cdot)$, and (4.1). Then, for each $m \in \mathbf{N}$, $n \geq 1$ and $s \in [t, T]$, we have

$$\begin{aligned}&\mathrm{sgn}\big({}^i y_s^{m,n} - {}^i y_s^m\big)\big({}^i g^{m,n}\big(s, y_s^{m,n}, {}^i z_s^{m,n}\big) - {}^i g^m\big(s, y_s^m, {}^i z_s^m\big)\big) \\ &\quad\leq \mathrm{sgn}\big({}^i y_s^{m,n} - {}^i y_s^m\big)\big({}^i g^{m,n}\big(s, y_s^{m,n}, {}^i z_s^{m,n}\big) - {}^i g^{m,n}\big(s, y_s^m, {}^i z_s^m\big)\big) \\ &\qquad + \big|{}^i g^{m,n}\big(s, y_s^m, {}^i z_s^m\big) - {}^i g^m\big(s, y_s^m, {}^i z_s^m\big)\big| \\ &\quad\leq \rho\big(\big|y_s^{m,n} - y_s^m\big|\big) + k\phi\big(\big|{}^i z_s^{m,n} - {}^i z_s^m\big|\big) + k\phi\bigg(\frac{2A}{n}\bigg).\end{aligned}$$

Combining the above two inequalities with (3.10), we get that for each $m \in \mathbf{N}$, $n \geq 1$ and $i = 1, \ldots, k$,

$$\begin{aligned}\big|{}^i y_t^{m,n} - {}^i y_t^m\big| \leq\; &Tk\bigg(\phi\bigg(\frac{2A}{n+2A}\bigg) + \phi\bigg(\frac{2A}{n}\bigg)\bigg) \\ &+ \int_t^T \big[\rho\big(\big|y_s^{m,n} - y_s^m\big|\big) + k(n+2A)\big|{}^i z_s^{m,n} - {}^i z_s^m\big|\big]\,\mathrm{d}s \\ &- \int_t^T \mathrm{sgn}\big({}^i y_s^{m,n} - {}^i y_s^m\big)\big({}^i z_s^{m,n} - {}^i z_s^m\big)\,\mathrm{d}B_s, \qquad t \in [0, T].\end{aligned}$$

Based on this inequality, in the same way as in the proof of second step of Theorem 5, by virtue of Girsanov's theorem, we can deduce that for each $m \in \mathbf{N}$, $n \geq 1$, $i = 1, \ldots, k$ and $0 \leq u \leq t \leq T$,

$$\mathbf{E}^{m,n,i}\big[\big|{}^i y_t^{m,n} - {}^i y_t^m\big|\big|\mathcal{F}_u\big] \leq \bar{b}_n + \int_t^T \mathbf{E}^{m,n,i}\big[\rho\big(\big|y_s^{m,n} - y_s^m\big|\big)\big|\mathcal{F}_u\big]\,\mathrm{d}s,$$

where $\bar{b}_n = Tk(\phi(\frac{2A}{n+2A}) + \phi(\frac{2A}{n}))$ and $\mathbf{E}^{m,n,i}[X|\mathcal{F}_u]$ represents the conditional expectation of the random variable X with respect to $\mathcal{F}_u$ under a probability measure $\mathrm{P}^{m,n,i}$ on $(\Omega, \mathcal{F})$, which depends on n, m and i, and which is absolutely continuous with respect to P.

Finally, note that $\bar{b}_n$ tends nonincreasingly to 0 as $n \to +\infty$. By the same argument as that in the proof of the third step of Theorem 5, we can find a real sequence h_n depending on n and satisfying $\lim_{n\to\infty} h_n = 0$ such that for each $m \in \mathbf{N}$, $n \geq 1$ and $i = 1, 2, \ldots, k$,

$$\sup_{t\in[0,T]} \left|{}^i y_t^{m,n} - {}^i y_t^m\right| \leq h_n.$$

The proof of Proposition 13 is then completed. □

Now, let us come back to the proof of Theorem 11. Combining (4.3) and (4.6), we get that

$$\mathbf{E}\Big[\sup_{t\in[0,T]} |y_t^m - y_t^0|^p\Big] \leq 2c_p k^p (h_n)^p + c_p \mathbf{E}\Big[\sup_{t\in[0,T]} |y_t^{m,n} - y_t^{0,n}|^p\Big]. \tag{4.8}$$

By taking limsup in (4.8) with respect to m and in view of Proposition 12, we deduce that for each $n \geq 1$,

$$\varlimsup_{m\to\infty} \mathbf{E}\Big[\sup_{t\in[0,T]} |y_t^m - y_t^0|^p\Big] \leq 2c_p k^p (h_n)^p. \tag{4.9}$$

Furthermore, note that h_n tends to 0 as $n \to +\infty$. Sending $n \to +\infty$ in (4.9) yields that

$$\lim_{m\to\infty} \mathbf{E}\Big[\sup_{t\in[0,T]} |y_t^m - y_t^0|^p\Big] = 0. \tag{4.10}$$

Finally, we show that

$$\lim_{m\to\infty} \mathbf{E}\Big[\Big(\int_0^T |z_s^m - z_s^0|^2 \,\mathrm{d}s\Big)^{p/2}\Big] = 0.$$

For each $m \geq 1$, let

$$\hat{\xi}^m := \xi^m - \xi^0, \qquad \hat{y}_\cdot^m := y_\cdot^m - y_\cdot^0, \qquad \hat{z}_\cdot^m := z_\cdot^m - z_\cdot^0.$$

Then $(\hat{y}_\cdot^m, \hat{z}_\cdot^m)$ is an L^p solution of the following BSDE:

$$\hat{y}_t^m = \hat{\xi}^m + \int_t^T \hat{g}^m(s, \hat{y}_s^m, \hat{z}_s^m)\,\mathrm{d}s - \int_t^T \hat{z}_s^m \,\mathrm{d}B_s, \qquad t \in [0, T], \tag{4.11}$$

where for each $(y, z) \in \mathbf{R}^k \times \mathbf{R}^{k\times d}$,

$$\hat{g}^m(t, y, z) := g^m(t, y + y_t^0, z + z_t^0) - g^0(t, y_t^0, z_t^0).$$

Furthermore, in view of (A1), (A2) and (3.10), we can deduce that $\mathrm{dP} \times \mathrm{d}t$-a.e., for each $m, n \geq 1$ and $(y, z) \in \mathbf{R}^k \times \mathbf{R}^{k\times d}$,

$$\begin{aligned}\langle y, \hat{g}^m(t, y, z)\rangle &\leq \langle y, g^m(t, y + y_t^0, z + z_t^0) - g^m(t, y_t^0, z_t^0)\rangle \\ &\quad + |y||g^m(t, y_t^0, z_t^0) - g^0(t, y_t^0, z_t^0)| \\ &\leq |y|\rho(|y|) + |y|\phi(|z|) + |y||g^m(t, y_t^0, z_t^0) - g^0(t, y_t^0, z_t^0)| \\ &\leq \left(n + 2A + \frac{1}{4}\right)|y|^2 + (n + 2A)|y||z| + a_m|y| \\ &\quad + \left(\rho\left(\frac{2A}{n+2A}\right) + \phi\left(\frac{2A}{n+2A}\right)\right)^2,\end{aligned}$$

which means that the assumption (B) is satisfied for the generator $\hat{g}^m(t, y, z)$ of BSDE (4.11) with $\mu = n + 2A + \frac{1}{4}$, $\lambda = n + 2A$, $f_t = a_m$ and $\theta_t = (\rho(\frac{2A}{n+2A}) + \phi(\frac{2A}{n+2A}))^2$. Thus, it follows from Lemma 3 with $t = u = 0$ that there exists a constant $M_{n,A,p,T} > 0$ depending only on n, A, p and T, and a constant $M_p > 0$ depending only on p such that for each $m, n \geq 1$,

$$\begin{aligned}\mathbf{E}\left[\left(\int_0^T |\hat{z}_s^m|^2\,\mathrm{d}s\right)^{p/2}\right] &\leq M_{n,A,p,T}\mathbf{E}\left[\sup_{t\in[0,T]} |\hat{y}_t^m|^p\right] + M_p\mathbf{E}\left[\left(\int_0^T a_m\,\mathrm{d}s\right)^p\right] \\ &\quad + M_pT^{p/2}\left(\rho\left(\frac{2A}{n+2A}\right) + \phi\left(\frac{2A}{n+2A}\right)\right)^p.\end{aligned}$$

In view of (4.10) and $\lim_{m\to\infty} a_m = 0$, by taking limsup with respect to m in the previous inequality yields that for each $n \geq 1$,

$$\overline{\lim_{m\to\infty}}\,\mathbf{E}\left[\left(\int_0^T |\hat{z}_s^m|^2\,\mathrm{d}s\right)^{p/2}\right] \leq M_pT^{p/2}\left(\rho\left(\frac{2A}{n+2A}\right) + \phi\left(\frac{2A}{n+2A}\right)\right)^p.$$

Then, letting $n \to \infty$ in above inequality and recalling that $\rho(\cdot), \phi(\cdot) \in \mathbf{S}$ yields that

$$\lim_{m\to\infty} \mathbf{E}\left[\left(\int_0^T |z_t^m - z_t^0|^2\,\mathrm{d}t\right)^{p/2}\right] = 0.$$

The proof of Theorem 11 is then completed. □

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