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Fractional absolute moments of heavy tailed distributions

Muneya Matsui[a] and Zbyněk Pawlas[b]

[a]*Nanzan University*
[b]*Charles University in Prague*

Abstract. Several convenient methods for calculation of fractional absolute moments are given with application to heavy tailed distributions. Our main focus is on an infinite variance case with finite mean, that is, we are interested in formulae for $\mathbb{E}[|X-\mu|^{\gamma}]$ with $1<\gamma<2$ and $\mu\in\mathbb{R}$. We review techniques of fractional differentiation of Laplace transforms and characteristic functions. Several examples are given with analytical expressions of $\mathbb{E}[|X-\mu|^{\gamma}]$. We also evaluate the fractional moment errors for both prediction and parameter estimation problems.

1 Fractional moments

The purpose of this paper is to study the evaluation tools for goodness of predictors and estimators which are of infinite variance. This is done by investigation of the fractional absolute moments. Although there exist several methods for their calculation, they are not always convenient to use. Among the methods we pursue the possibility of techniques based on the fractional differentiation of the Laplace (LP) transform or the characteristic function (ch.f.), which are found to supply attractive calculation tools when applied to heavy tailed distributions. We try to suggest an unified approach for fractional absolute moments, namely, to present possible ready-made expressions in terms of numerical calculations, though for some of them we succeed to obtain analytical expressions.

Heavy tailed distributions and stochastic processes with infinite variance have found applications in many diverse areas [see, e.g., Adler et al. (1998) and references therein]. Various statistical methods for these models have been investigated so far. Among them, the prediction problems have occupied an important place. To name a few contributions, Hardin et al. (1991) and Samorodnitsky and Taqqu (1991) studied the conditional expectation for stable random vectors, that is, the best predictor in the sense of minimizing mean squared error if it exists. In a recent paper, Matsui and Mikosch (2010) obtained the conditional expectations for Poisson cluster models with possibly infinite variance. The linear predictors for time series models have been considered in, for example, Cline and Brockwell (1985) and Kokoszka (1996). The regression type estimators have also been studied in,

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for example, Blattberg and Sargent (1971) or (Samorodnitsky and Taqqu, 1994, Section 4). See also Kozubowski (2001) for a parameter estimation directly by the fractional moment.

Although there have been multiple papers dealing with predictors, we find that little attention has been given to the measures of prediction errors and the methods of their calculation. In Cline and Brockwell (1985) and Kokoszka (1996), a certain dispersion measure has been proposed, but it is specially intended for the time series, and thus not quite general. The problem is that when concerning random elements with no finite second moments, we cannot apply the L^2 loss function, which is the most popular measure because it is easily tractable and intuitively clear. Therefore, the alternative measures are required.

In this paper, we adopt the L^p loss function with $0 < p < 2$ since we think it is a natural plausible candidate to evaluate the goodness of prediction or parameter estimation. Thus, we study the fractional absolute moments $m_p := \mathbb{E}[|X|^p]$ of order $0 < p < 2$. We also consider μ-centered moments $m_{\mu,p} := \mathbb{E}[|X - \mu|^p]$ with $\mu \in \mathbb{R}$. As far as we know, there have not been enough researches of the fractional absolute moments. Exceptions are the special case of first-order absolute moment m_1 and fractional moments for particular types of models [see, e.g., Mikosch et al. (2013)]. The reason is that the existing methods are unfamiliar or these methods seem to require a lot of numerical work. In other words, though there exist applicable mathematical theories, few attempts have been made on applications in the context.

Taking this into consideration, first we summarize existing methods for obtaining the fractional absolute moments. In particular, we focus on the methods exploiting LP and ch.f. of the corresponding distribution. It is well known that the moments of integer orders are related to the derivatives of ch.f. or LP transform at zero. More generally, the theory of fractional calculus can be utilized in order to obtain the non-integer real moments. Then we apply these methods to present some useful ready-made expressions for the fractional absolute moments of heavy tailed distributions. Our intention is to notice the methods and to process convenient derivative tools which are numerically tractable. The work would make unfamiliar tools more accessible for applied people.

There are several researches giving the relation between the fractional moments and the corresponding ch.f or LP transform. We refer to Hsu (1951), von Bahr (1965), Ramachandran (1969), Brown (1970, 1972), (Kawata, 1972, Section 11.4), Wolfe (1973, 1975a, 1975b, 1978), Laue (1980, 1986), (Zolotarev, 1986, Section 2.1), (Paolella, 2007, Section 8.3) and Pinelis (2011). The methods using moment generating functions have also been studied, for example, by Cressie et al. (1981) and Cressie and Borkent (1986).

Our main tool is the fractional calculus which generalizes ordinary differentiation and integration to arbitrary order; for details, we refer to monographs Podlubny (1999) and Samko et al. (1993). Though there exist different definitions of fractional derivatives, we will use the Marchaud fractional derivative. For a complex-valued function f, its fractional derivative of order $\gamma = k + \lambda$ with $k \in \mathbb{N}$, $0 < \lambda < 1$, is given by, for example, [Laue, 1980, equation (2.1)] or (Samko et al., 1993, Section 5),

$$\frac{\mathrm{d}^\gamma}{\mathrm{d}t^\gamma} f(t) = \frac{\mathrm{d}^\lambda}{\mathrm{d}t^\lambda} f^{(k)}(t) = \frac{\lambda}{\Gamma(1-\lambda)} \int_{-\infty}^{t} \frac{f^{(k)}(t) - f^{(k)}(u)}{(t-u)^{1+\lambda}} \,\mathrm{d}u, \qquad t \in \mathbb{R},$$

where $f^{(k)}$ is the kth derivative of f and Γ is the Gamma function. We are mostly interested in the fractional absolute moments $m_{1+\lambda} = \mathbb{E}[|X|^{1+\lambda}]$ with $0 < \lambda < 1$. For this reason, we will need the fractional derivative of order $1 + \lambda$ at zero,

$$\frac{\mathrm{d}^{1+\lambda}}{\mathrm{d}t^{1+\lambda}} f(t)\bigg|_{t=0} = \frac{\mathrm{d}^\lambda}{\mathrm{d}t^\lambda} f'(t)\bigg|_{t=0} = \frac{\lambda}{\Gamma(1-\lambda)} \int_0^\infty \frac{f'(0) - f'(-u)}{u^{1+\lambda}} \,\mathrm{d}u. \qquad (1)$$

Other popular definitions in fractional calculus literature involve the Caputo and the Riemann–Liouville derivatives. The Caputo derivative is suitable to the moment generating functions; see Cressie and Borkent (1986). The Riemann–Liouville derivative provides an alternative method to the computation of the fractional absolute moments.

The construction of our paper is as follows. In the remainder of Section 1, we make a brief survey on the relation between the fractional absolute moments and Marchaud fractional derivatives using references cited above. Several convenient formulae are also derived. In Section 2, we apply the mentioned methods to the infinitely divisible distributions and examine their fractional absolute moments. Heavy tailed distributions, such as stable, Pareto, geometric stable and Linnik distributions, are considered. Especially, in Section 3 we pay attention to the compound Poisson distribution that is popular in applications. In the final section, several applications are presented. The fractional errors of predictions with infinite variance, such as stable distributions, are explicitly calculated. In addition, the estimation errors in regression models are evaluated by the fractional absolute moments in heavy tailed cases.

1.1 Fractional derivatives of Laplace transforms

Let F be a distribution function (d.f.) of a non-negative random variable X. Its LP transform is defined as

$$\phi(t) := \int_0^\infty \mathrm{e}^{-tx} \,\mathrm{d}F(x), \qquad t \geq 0.$$

In (Wolfe, 1975a, Theorem 1), the relation between moments of X and the fractional derivative of ϕ at zero is given. We state this result in a slightly modified version. The proof is given in Appendix A.1.

Lemma 1.1. *Let $0 < \lambda < 1$ and let ϕ be the LP transform of the d.f. $F(x)$ such that $F(x) = 0$ for $x < 0$. Then $m_{1+\lambda}$ exists if and only if $\phi'(0+)$ exists and*

$$\int_0^\infty \frac{\phi'(u) - \phi'(0+)}{u^{1+\lambda}} \,\mathrm{d}u$$

exists, in which case

$$m_{1+\lambda} = \frac{\lambda}{\Gamma(1-\lambda)} \int_0^\infty \frac{\phi'(u) - \phi'(0+)}{u^{1+\lambda}} \, \mathrm{d}u.$$

Remark 1.2. Theorem 1 in Cressie and Borkent (1986) shows that under certain conditions an arbitrary moment of a positive random variable is equal to the Caputo fractional derivative (Podlubny, 1999, Section 2.4.1) of the corresponding moment generating function at zero.

1.2 Fractional derivatives of characteristic functions

We denote the ch.f. of a random variable X with d.f. F by

$$\varphi(t) := \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx} \, \mathrm{d}F(x), \qquad t \in \mathbb{R},$$

and denote that for $X - \mu$ with $\mu \in \mathbb{R}$ by

$$\varphi_\mu(t) := \mathrm{e}^{-\mathrm{i}t\mu} \varphi(t), \qquad t \in \mathbb{R}.$$

There are several papers dealing with the relation between the fractional derivative of φ and the fractional absolute moment. We will work mainly with the result of Laue (1980) who proved that

$$m_{n+\lambda} = \frac{1}{\cos((n+\lambda)\pi/2)} \Re\left[\frac{\mathrm{d}^{n+\lambda}}{\mathrm{d}t^{n+\lambda}} \varphi(t)\Big|_{t=0} \right] \tag{2}$$

for any integer $n \geq 0$ and $0 < \lambda < 1$. Here, $\Re z$ denotes the real part of the complex number z. In the following lemma, we state the consequences of results from Laue (1980) and Kawata (1972).

Lemma 1.3. *Let* $0 < \lambda < 1$ *and let* φ *be the ch.f. of an arbitrary d.f.* F.

(a) $m_{1+\lambda}$ *exists if and only if*

$$\Re \int_0^\infty \frac{\varphi'(-u)}{u^{1+\lambda}} \, \mathrm{d}u \quad \textit{exists} \quad \textit{and} \quad \lim_{t \to 0+} \frac{1 - \Re\varphi(t)}{t^{1+\lambda}} \quad \textit{exists}. \tag{3}$$

In such a case,

$$m_{1+\lambda} = \frac{\lambda}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)} \Re \int_0^\infty \frac{\varphi'(-u)}{u^{1+\lambda}} \, \mathrm{d}u. \tag{4}$$

(b) *A necessary and sufficient condition for the existence of* $m_{1+\lambda}$, $0 < \lambda < 1$, *is that*

$$\Re \int_0^\infty \frac{1 - \varphi(u)}{u^{2+\lambda}} \, \mathrm{d}u < \infty. \tag{5}$$

In this case,

$$m_{1+\lambda} = \frac{\lambda(1+\lambda)}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)} \Re \int_0^\infty \frac{1 - \varphi(u)}{u^{2+\lambda}} \, \mathrm{d}u. \tag{6}$$

If $\varphi_\mu(t) = \mathrm{e}^{-\mathrm{i}t\mu}\varphi(t)$ *satisfies conditions* (3) *in* (a) *or condition* (5) *of* (b), *then the fractional absolute moment with center* μ ($m_{\mu,1+\lambda} = \mathbb{E}[|X-\mu|^{1+\lambda}]$) *is given by*

$$m_{\mu,1+\lambda} = \frac{\lambda}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)} \times \left[\mu\Im\int_0^\infty \frac{\mathrm{e}^{\mathrm{i}\mu u}\varphi(-u)}{u^{1+\lambda}}\,\mathrm{d}u + \Re\int_0^\infty \frac{\mathrm{e}^{\mathrm{i}\mu u}\varphi'(-u)}{u^{1+\lambda}}\,\mathrm{d}u\right], \tag{7}$$

where $\Im z$ *denotes the imaginary part of the complex number* z.

Part (a) is a special case of [Laue, 1980, Theorem 2.2(b)] and part (b) is contained in (Kawata, 1972, Theorem 11.4.3). However, for the consistency of the paper and reader's better understanding, we give the proof which is specific for our parameter ranges $0 < \lambda < 1$. It can be found in Appendix A.2.

Remark 1.4.

(i) Equation (4) follows from (1) and (2) with $n = 1$ by noticing that $\Re\varphi'(0) = 0$.
(ii) Equation (6) can also be found as (2.1.9) in Zolotarev (1986) or (8.30) in Paolella (2007), in both cases with differently written constant in front of the integral and with a typo contained.
(iii) Although we will mainly use expressions (4) and (7), expression (6) may be also useful in some purposes.

Moreover, (Kawata, 1972, Theorem 11.4.4) has obtained expressions for m_γ, $\gamma > 2$, in the form of

$$m_\gamma = C_\ell \int_0^\infty u^{-(1+\gamma)}\left[1 - \Re\varphi(u) + \sum_{k=1}^{\ell}\frac{u^{2k}}{(2k)!}\varphi^{(2k)}(0)\right]\mathrm{d}u,$$

where $\ell \in \mathbb{N}$ is such that $2\ell < \gamma < 2\ell + 2$ and C_ℓ is a positive constant depending on ℓ. In other context, Wolfe (1975a) has derived different formula for calculating moments m_γ of any real order $\gamma \in \mathbb{R}$ from the fractional derivatives of the ch.f. Recently, Pinelis (2011) has obtained integral expressions of positive-part moments $\mathbb{E}[X_+^p]$ with $p > 0$ in terms of the ch.f. His method is to apply the Fourier–Laplace transform and the Cauchy integral theorem, which is different from the fractional derivative approach.

2 Infinitely divisible distributions

In this section, we examine the class of infinitely divisible (ID for short) distributions, whose general definitions and many distributional properties are given by their ch.f. Many well-known distributions belong to this class and there are many applications in different areas (finance, insurance, physics, astronomy, etc.). Here, we work on the distribution without Gaussian part, its ch.f. is

$$\varphi(t) = \exp\left\{ \mathrm{i}\delta t + \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx\mathbf{1}_{\{x\le 1\}})\nu(\mathrm{d}x) \right\}, \qquad t \in \mathbb{R}, \tag{8}$$

where $\delta \in \mathbb{R}$ is a centering constant and ν is the Lévy measure satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(|x|^2 \wedge 1)\nu(\mathrm{d}x) < \infty$. For more details on the definition and properties, we refer to Sato (1999).

Although we cannot calculate $m_{1+\lambda}$ from density functions, because they are not available for most ID distributions, we can directly apply the fractional derivative of the ch.f. and obtain fractional absolute moments. An advantage is that we can check the existence of fractional moments by the Lévy measure of ID distributions and we do not need to check conditions of Lemma 1.3. The following result is a well-known criterion for moments [see, e.g., (Sato, 1999, Corollary 25.8) or (Wolfe, 1971, Theorem 2)]. In our case of interest $\mathbb{E}[|X|^{1+\lambda}]$, $0 < \lambda < 1$, we find a simple proof and give it in Appendix A.3.

Lemma 2.1. *Let X be an ID distribution with Lévy measure* ν. *Then for* $0 < \lambda < 1$, $m_{1+\lambda} < \infty$ *if and only if*

$$\int_{|x|>1} |x|^{1+\lambda}\nu(\mathrm{d}x) < \infty.$$

In what follows, we present the examples.

2.1 Stable distributions

As a representative of heavy tailed distributions we first consider stable distributions. A random variable X has a stable distribution with parameters $0 < \alpha \le 2$, $\sigma \ge 0$, $-1 \le \beta \le 1$ and $\delta \in \mathbb{R}$ if its ch.f. has the form, cf. (Samorodnitsky and Taqqu, 1994, Definition 1.1.6),

$$\varphi(t) = \exp\{\mathrm{i}\delta t - \sigma^\alpha |t|^\alpha \omega(t)\}, \qquad t \in \mathbb{R}, \tag{9}$$

where

$$\omega(t) = \begin{cases} 1 - \mathrm{i}\beta \tan\dfrac{\pi\alpha}{2}\,\mathrm{sign}(t), & \text{if } \alpha \ne 1 \\ 1 + \mathrm{i}\beta\dfrac{2}{\pi}\,\mathrm{sign}(t)\log|t|, & \text{if } \alpha = 1. \end{cases} \tag{10}$$

It is well known that if $\gamma < \alpha < 2$, the moment of order γ exists, while for $\gamma \ge \alpha$ it does not exist; see, for example, (Ramachandran, 1969, Section 4) or (Samorodnitsky and Taqqu, 1994, Property 1.2.16). We briefly review the existing results on the moments. If $0 < \alpha < 1$ and X is a stable subordinator with the LP transform given by $\mathbb{E}[\mathrm{e}^{-tX}] = \exp\{-\sigma^\alpha t^\alpha\}$, then for $-\infty < \gamma < \alpha$,

$$\mathbb{E}[X^\gamma] = \frac{\Gamma(1-\gamma/\alpha)}{\Gamma(1-\gamma)}\sigma^\gamma,$$

which is shown by (Wolfe, 1975a, Section 4) or Shanbhag and Sreehari (1977). In symmetric case ($\beta = 0$) with $\delta = 0$, it is shown in (Shanbhag and Sreehari, 1977, Theorem 3) that

$$m_\gamma = \frac{2^\gamma \Gamma((1+\gamma)/2)\Gamma(1-\gamma/\alpha)}{\Gamma(1-\gamma/2)\Gamma(1/2)}\sigma^\gamma, \qquad -1 < \gamma < \alpha, \tag{11}$$

where the authors rely on the decomposition of the symmetric stable distribution [see also Section 25 in (Sato, 1999)]. For general β and $\delta = 0$, the following relation is proved by two different methods in Section 8.3 of Paolella (2007), see also p. 18 in Samorodnitsky and Taqqu (1994),

$$m_\gamma = \kappa^{-1}\Gamma\left(1 - \frac{\gamma}{\alpha}\right)(1+\theta^2)^{\gamma/(2\alpha)} \cos\left(\frac{\gamma}{\alpha}\arctan\theta\right)\sigma^\gamma, \qquad -1 < \gamma < \alpha, \tag{12}$$

where $\theta = \beta \tan\frac{\pi\alpha}{2}$ and

$$\kappa = \begin{cases} \Gamma(1-\gamma)\cos\dfrac{\gamma\pi}{2}, & \text{if } \gamma \neq 1, \\ \dfrac{\pi}{2}, & \text{if } \gamma = 1. \end{cases}$$

Using the fractional derivative, we obtain from Lemma 1.3 not only another proof of (12), but also formulae for fractional absolute μ-centered moments which seem to be new.

Proposition 2.2. *Let X have a stable distribution with real parameters $\alpha > 1$, $|\beta| \leq 1$, $\delta = 0$ and $\sigma > 0$. Then, for $0 < \lambda < \alpha - 1$, we have*

$$\begin{aligned} m_{1+\lambda} = {} & \frac{\lambda\Gamma(1-(1+\lambda)/\alpha)}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)}\sigma^{1+\lambda}(1+\theta^2)^{(1+\lambda)/(2\alpha)-1/2} \\ & \times \left\{\cos\left[\left(1-\frac{1+\lambda}{\alpha}\right)\arctan\theta\right] + \theta\sin\left[\left(1-\frac{1+\lambda}{\alpha}\right)\arctan\theta\right]\right\}, \end{aligned} \tag{13}$$

and for $\mu \in \mathbb{R}$,

$$\begin{aligned} m_{\mu,1+\lambda} = {} & \frac{\lambda}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)} \\ & \times \left\{\mu\int_0^\infty u^{-(1+\lambda)} \mathrm{e}^{-\sigma^\alpha u^\alpha}\sin(\mu u - \theta\sigma^\alpha u^\alpha)\,\mathrm{d}u \right. \\ & + \alpha\sigma^\alpha \int_0^\infty u^{\alpha-\lambda-2}\mathrm{e}^{-\sigma^\alpha u^\alpha}\left[\cos(\mu u - \theta\sigma^\alpha u^\alpha)\right. \\ & \left.\left. - \theta\sin(\mu u - \theta\sigma^\alpha u^\alpha)\right]\mathrm{d}u\right\}, \end{aligned} \tag{14}$$

where $\theta = \beta\tan\frac{\pi\alpha}{2}$. If X is symmetric ($\beta = 0$), it follows that

$$m_{1+\lambda} = \frac{\lambda\Gamma(1-(1+\lambda)/\alpha)}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)}\sigma^{1+\lambda} \tag{15}$$

and

$$m_{\mu,1+\lambda} = \frac{\lambda\sigma^{1+\lambda}}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)}\left[\frac{\mu}{\sigma}\int_0^\infty u^{-(1+\lambda)}\mathrm{e}^{-u^\alpha}\sin\left(\frac{\mu u}{\sigma}\right)\mathrm{d}u + \alpha\int_0^\infty u^{\alpha-\lambda-2}\mathrm{e}^{-u^\alpha}\cos\left(\frac{\mu u}{\sigma}\right)\mathrm{d}u\right]. \tag{16}$$

Proof. We begin with the expression of $m_{\mu,1+\lambda}$. Let φ be the ch.f. of a stable distribution with $\delta = 0$ and $\alpha > 1$. Since we have, for $u > 0$,

$$\Im \mathrm{e}^{\mathrm{i}\mu u}\varphi(-u) = \exp\{-\sigma^\alpha u^\alpha\}\sin(\mu u - \theta\sigma^\alpha u^\alpha),$$
$$\Re \mathrm{e}^{\mathrm{i}\mu u}\varphi'(-u) = \alpha\sigma^\alpha u^{\alpha-1}\exp\{-\sigma^\alpha u^\alpha\}\cos(\mu u - \theta\sigma^\alpha u^\alpha) - \alpha\theta\sigma^\alpha u^{\alpha-1}\exp\{-\sigma^\alpha u^\alpha\}\sin(\mu u - \theta\sigma^\alpha u^\alpha),$$

inserting these into (7) of Lemma 1.3, we get (14). For $m_{1+\lambda}$, we let $\mu = 0$ in (14) and use change of variables theorem to obtain

$$m_{1+\lambda} = \frac{\lambda\sigma^{1+\lambda}}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)} \times \left(\int_0^\infty u^{-(1+\lambda)/\alpha}\mathrm{e}^{-u}\cos\theta u\,\mathrm{d}u + \theta\int_0^\infty u^{-(1+\lambda)/\alpha}\mathrm{e}^{-u}\sin\theta u\,\mathrm{d}u\right).$$

Now we get (13) by applying the formulae (3.944-5) and (3.944-6) from (Gradshteyn and Ryzhik, 2007, p. 498). Finally, letting $\beta = 0$ and applying change of variables, the symmetric case is obtained. □

After some manipulation, one can show that (15) coincides with (11) and (13) coincides with (12) for $\gamma = 1 + \lambda$. Figure 1 shows the fractional absolute moments with center μ, computed numerically from the representation (14). We remark that

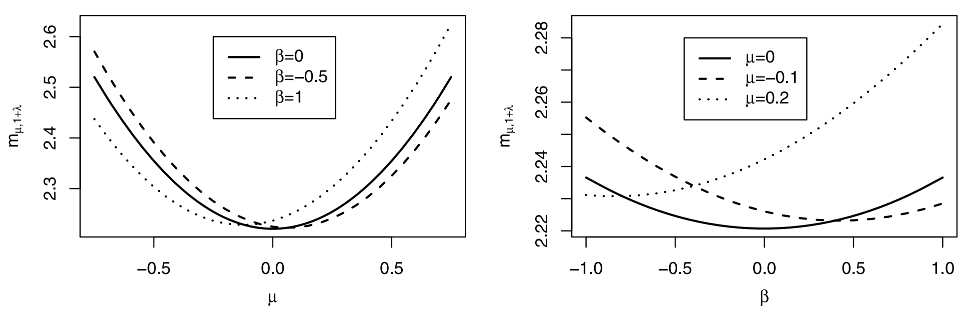

Figure 1 *The moments $m_{\mu,1+\lambda}$ of stable distribution with parameters $\alpha = 1.8$, $\beta \in [-1, 1]$, $\delta = 0$ and $\sigma = 1$. We choose $\lambda = 0.5$ and depict the dependence on μ for three choices of β (left) and the dependence on β for three choices of μ (right).*

even if this representation includes some integral expressions, it would be useful since most stable distributions have no explicit density functions. We appreciate that a use of infinite series representations of stable densities is another possibility to calculate $m_{1+\lambda}$ [see, e.g., Sections 2.4 and 2.5 in Zolotarev (1986) for these series]. However, these series are given with powers of x or x^{-1} and are different depending on both parameters and supports of densities. Besides, in some parameter ranges they only serve as asymptotic expansions for small and large values and do not cover the whole supports; see, for example, (Matsui and Takemura, 2006, Section 2.2). Therefore, the alternative approach yields other complexities.

2.2 Pareto law

Another heavy tailed distribution is the Pareto distribution which has density and ch.f. given by

$$f(x) = \alpha(1+x)^{-\alpha-1}, \qquad x > 0,$$
$$\varphi(t) = \alpha \int_0^\infty \mathrm{e}^{ity}(1+y)^{-\alpha-1}\,\mathrm{d}y, \qquad t \in \mathbb{R},$$

respectively, with real positive parameter $\alpha > 0$. This distribution belongs to ID distributions [see Remark 8.12 in Sato (1999)]. The fractional absolute moment $m_{1+\lambda}$ exists if and only if $1+\lambda < \alpha$. Though the density function is explicit, we obtain $m_{\mu,1+\lambda}$ from the fractional derivative of ch.f. Using (7) of Lemma 1.3, we have, for $1 < 1+\lambda < \alpha$,

$$m_{\mu,1+\lambda} = \alpha\bigg[(\mu+1)^{1+\lambda-\alpha} B(\alpha-1-\lambda, 2+\lambda) + \frac{\mu^{2+\lambda}}{2+\lambda} {}_2F_1(1, \alpha+1, 3+\lambda; -\mu)\bigg],$$

where B is the beta function and ${}_2F_1$ is the Gauss hypergeometric function.

Although the following examples are not always in ID distributions, they are closely related and could be heavy tailed.

2.3 Geometric stable law

A geometric stable distribution has similar properties to the stable distribution. The ch.f. is given as

$$\varphi(t) = \big[1 + \sigma^\alpha |t|^\alpha \omega(t) - \mathrm{i}\delta t\big]^{-1}, \qquad t \in \mathbb{R},$$

where $0 < \alpha < 2$, $\delta \in \mathbb{R}$ and $\omega(t)$ is defined by (10). However, its density function has no analytical expression. The tail behavior is the same as that of stable distribution; see, for example, Kozubowski et al. (1999). We apply Lemma 1.3 to a geometric stable distribution with $\delta = 0$. Then, for $1 < 1+\lambda < \alpha$ and $\mu \in \mathbb{R}$,

$$m_{\mu,1+\lambda} = \frac{\lambda}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)} \times \left\{ \mu \int_0^\infty u^{-(1+\lambda)} \frac{(1+\sigma^\alpha u^\alpha)\sin\mu u - \theta\sigma^\alpha u^\alpha \cos\mu u}{(1+\sigma^\alpha u^\alpha)^2 + (\theta\sigma^\alpha u^\alpha)^2}\, \mathrm{d}u \right. - \alpha\sigma^\alpha \int_0^\infty u^{\alpha-\lambda-2} \frac{\cos\mu u + \theta\sin\mu u}{(1+\sigma^\alpha u^\alpha)^2 + (\theta\sigma^\alpha u^\alpha)^2}\, \mathrm{d}u + 2\alpha\sigma^\alpha \int_0^\infty u^{\alpha-\lambda-2} \frac{(1+\sigma^\alpha u^\alpha)\cos\mu u + \theta\sigma^\alpha u^\alpha \sin\mu u}{[(1+\sigma^\alpha u^\alpha)^2 + (\theta\sigma^\alpha u^\alpha)^2]^2} \left. \times \left(1+\sigma^\alpha u^\alpha + \theta^2\sigma^\alpha u^\alpha\right) \mathrm{d}u \right\}$$

and

$$m_{1+\lambda} = \frac{\lambda\sigma^{1+\lambda}}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)} \int_0^\infty v^{-(1+\lambda)/\alpha} \frac{(1+v)^2 + (\theta v)^2 + 2\theta^2 v}{[(1+v)^2 + (\theta v)^2]^2}\, \mathrm{d}v,$$

where $\theta = \beta \tan\frac{\pi\alpha}{2}$.

If we put $\theta = 0$, the results coincide with the standard Linnik law case.

2.4 Linnik law

We consider a version of Linnik distribution given by Linnik (1953). Its density function is not explicit, while its ch.f. has the form

$$\varphi(t) = \left(1+\sigma^\alpha |t|^\alpha\right)^{-\beta}, \qquad t \in \mathbb{R},$$

where $0 < \alpha \le 2$ is the stability parameter, $\sigma > 0$ is the scale parameter and $\beta > 0$. By the method of fractional derivative, we recover the result of Lin (1998) as

$$m_{1+\lambda} = \frac{\lambda\beta\sigma^{1+\lambda}}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)} B\left(1 - \frac{1+\lambda}{\alpha}, \beta + \frac{1+\lambda}{\alpha}\right),$$

where $1 < 1+\lambda < \alpha$. The fractional absolute moment of order $1 < 1+\lambda < \alpha$ with center $\mu \in \mathbb{R}$ is

$$m_{\mu,1+\lambda} = \frac{\lambda\sigma^{1+\lambda}}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)} \times \left[\frac{\mu}{\sigma} \int_0^\infty \frac{u^{-(1+\lambda)}\sin((\mu u)/\sigma)}{(1+u^\alpha)^\beta}\, \mathrm{d}u + \alpha\beta \int_0^\infty \frac{u^{\alpha-\lambda-2}\cos((\mu u)/\sigma)}{(1+u^\alpha)^{\beta+1}}\, \mathrm{d}u \right]. \tag{17}$$

For $\beta = 1$, these equations coincide with those of geometric stable distributions for $\theta = 0$.

2.5 Combination of stable law and Linnik law

Since

$$\lim_{\beta\to\infty}\left(1+\sigma^\alpha|t|^\alpha/\beta\right)^{-\beta}=\mathrm{e}^{-\sigma^\alpha|t|^\alpha},\qquad t\in\mathbb{R}, 0<\alpha\le 2,$$

a symmetric stable distribution is a limit of Linnik-type distributions. We consider their combination keeping both exponents α to be identical. Let X be a symmetric stable random variable with ch.f. $\varphi(t)=\mathrm{e}^{-|t|^\alpha}$ and let Y be a random variable with Linnik-type distribution and ch.f. $\varphi(t)=(1+|t|^\alpha/\beta)^{-\beta}$. Then we may express $\mathbb{E}[|X-Y|^{1+\lambda}]$ by taking expectation of (17) with μ replaced by X and $\sigma=\beta^{-1/\alpha}$. As the result, we obtain

$$\begin{aligned}\mathbb{E}[|X-Y|^{1+\lambda}]&=\frac{\lambda\beta^{1-(1+\lambda)/\alpha}}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)}\\&\quad\times\left[\int_0^\infty u^{-(1+\lambda)/\alpha}(1+u)^{-\beta}\mathrm{e}^{-\beta u}\,\mathrm{d}u\right.\\&\qquad\left.+\int_0^\infty u^{-(1+\lambda)/\alpha}(1+u)^{-\beta-1}\mathrm{e}^{-\beta u}\,\mathrm{d}u\right]\\&=\frac{\lambda\beta^{1-(1+\lambda)/\alpha}\Gamma(1-(1+\lambda)/\alpha)}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)}\\&\quad\times\left[U\left(1-\frac{1+\lambda}{\alpha},2-\beta-\frac{1+\lambda}{\alpha};\beta\right)\right.\\&\qquad\left.+U\left(1-\frac{1+\lambda}{\alpha},1-\beta-\frac{1+\lambda}{\alpha};\beta\right)\right],\end{aligned}$$

where U is the confluent hypergeometric function [Gradshteyn and Ryzhik, 2007, (9.210-2)].

2.6 Subordinator

For practical reasons, it is desirable to express the moments $m_{1+\lambda}$ through Lévy measure ν since ID distributions without Gaussian part are completely characterized by centering parameter δ and Lévy measure. However, in the light of (8), such expressions seem to be too formal and too complicated, thus they seem to be not very useful. Here, we confine our interest to some well-known distributions. However, for small classes of ID distributions general expressions of $m_{1+\lambda}$ by ν are worth considering. We pick out the class of subordinator (non-negative valued ID distributions) and that of compound Poisson distributions; the latter is treated in Section 3.

For subordinator, we apply Lemma 1.1 and obtain a relatively simple expression. The LP transform of a subordinator can be found in (Sato, 1999, Theorem 30.1).

Lemma 2.3. *Let X be a non-negative valued ID random variable with shift parameter $\delta \geq 0$ and Lévy measure ν such that $\int_{(0,\infty)}(1 \wedge |s|)\nu(\mathrm{d}s) < \infty$. The LP transform is given by*

$$\phi(t) = \mathrm{e}^{\Psi(-t)}, \qquad t \geq 0,$$

where

$$\Psi(t) = \delta t + \int_{(0,\infty)} (\mathrm{e}^{st} - 1)\nu(\mathrm{d}s).$$

Then it follows that

$$m_{1+\lambda} = \frac{\lambda}{\Gamma(1-\lambda)}\left[\delta \int_0^\infty \frac{1-\mathrm{e}^{\Psi(-u)}}{u^{1+\lambda}}\,\mathrm{d}u + \int_0^\infty s\nu(\mathrm{d}s)\int_0^\infty \frac{1-\mathrm{e}^{-us}\mathrm{e}^{\Psi(-u)}}{u^{1+\lambda}}\,\mathrm{d}u\right].$$

3 Compound Poisson distribution

Among ID distributions, we focus on the compound Poisson (CP for short) distribution which can easily manage the tail behavior by assuming a heavy tailed jump distribution. However, since most distributions do not have explicit representations, we rely on the ch.f. or the LP transform for calculating fractional moments. Let c be the intensity parameter of underlying Poisson distribution and ν jump measure. The CP distribution has the following ch.f.:

$$\varphi(t) = \exp\left\{c\int (\mathrm{e}^{\mathrm{i}tx} - 1)\nu(\mathrm{d}x)\right\} = \exp\{c(\varphi_J(t) - 1)\}, \qquad t \in \mathbb{R}, \tag{18}$$

where $\varphi_J(t) := \int \mathrm{e}^{\mathrm{i}tx}\nu(\mathrm{d}x)$ is the ch.f. of the jump distribution. If the jump distribution has positive support, we obtain the LP transform

$$\phi(t) = \exp\{c(\phi_J(t) - 1)\}, \qquad t \geq 0,$$

where $\phi_J(t) := \int \mathrm{e}^{-tx}\nu(\mathrm{d}x)$. The fractional absolute moments are expressed in the following lemma. The proof is just an application of Lemma 1.1 and Lemma 1.3.

Lemma 3.1. *Let $\varphi(t)$ be the ch.f. of CP given by (18), then we have the following form for fractional μ-centered moments of order $1 < 1+\lambda < 2$:*

$$\begin{aligned} m_{\mu,1+\lambda} = {}& \frac{\lambda}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)} \\ &\times \left\{\mu \int_0^\infty h_\lambda(u)\sin[\mu u + c\Im(\varphi_J(-u))]\,\mathrm{d}u \right. \\ &\quad + c\int_0^\infty h_\lambda(u)\Re(\varphi_J'(-u))\cos[\mu u + c\Im(\varphi_J(-u))]\,\mathrm{d}u \\ &\quad \left. - c\int_0^\infty h_\lambda(u)\Im(\varphi_J'(-u))\sin[\mu u + c\Im(\varphi_J(-u))]\,\mathrm{d}u\right\}, \end{aligned}$$

where $h_\lambda(u) = u^{-(1+\lambda)}\exp\{c[\Re(\varphi_J(-u)) - 1]\}$.

If the jump distribution is symmetric, that is, $\Im\varphi_J(u) = 0$, *we have*

$$m_{\mu,1+\lambda} = \frac{\lambda}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)} \times \left\{\mu \int_0^\infty u^{-(1+\lambda)} \sin(\mu u) \exp\{c(\varphi_J(-u)-1)\}\,\mathrm{d}u + c\int_0^\infty u^{-(1+\lambda)} \varphi_J'(-u)\cos(\mu u)\exp\{c(\varphi_J(-u)-1)\}\,\mathrm{d}u\right\}$$

and moreover

$$m_{1+\lambda} = \frac{\lambda c}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)} \int_0^\infty u^{-(1+\lambda)} \varphi_J'(-u)\exp\{c(\varphi_J(-u)-1)\}\,\mathrm{d}u.$$

If the jump distribution has positive support, we have

$$m_{1+\lambda} = \frac{\lambda c}{\Gamma(1-\lambda)} \int_0^\infty u^{-(1+\lambda)}\left[\phi_J'(u)\exp\{c(\phi_J(u)-1)\} - \phi_J'(0)\right]\mathrm{d}u.$$

In what follows, we will examine jumps given by well-known distributions, which are not always heavy tailed, and try to obtain analytical expressions. Since they require a lot of numerical integrals and special functions, we just mention the key steps of derivation.

3.1 Exponential jump

The LP transform of the exponential distribution with parameter β, that is, with density function $f(x) = \frac{1}{\beta}\mathrm{e}^{x/\beta}$, $x \ge 0$, is $\phi_J(t) = 1/(1+\beta t)$, $t \ge 0$. Then due to Lemma 3.1, fractional absolute moments for $0 < \lambda < 1$ are given by

$$\begin{aligned} m_{1+\lambda} &= \frac{\lambda c\beta}{\Gamma(1-\lambda)} \int_0^\infty \frac{(1+\beta u)^2 - \exp\{c(1/(1+\beta u)-1)\}}{u^{1+\lambda}(1+\beta u)^2}\,\mathrm{d}u \\ &= c\beta^{1+\lambda}\Gamma(2+\lambda)\left[{}_1F_1(1-\lambda;2;-c) + \frac{c}{2}\,{}_1F_1(1-\lambda;3;-c)\right], \end{aligned}$$

where ${}_1F_1$ is the confluent hypergeometric function [Gradshteyn and Ryzhik, 2007, (9.210-1)] and we use [Gradshteyn and Ryzhik, 2007, (3.383-1) and (3.191-3)].

3.2 Symmetric stable jump

Recall that the ch.f. is $\varphi_J(t) = \mathrm{e}^{-|t|^\alpha}$ with $1 < \alpha < 2$, and thus we apply Lemma 3.1

with $1 < 1+\lambda < \alpha$ to obtain the following series representation:

$$\begin{aligned} m_{1+\lambda} &= \frac{\lambda\alpha c}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)} \int_0^\infty u^{\alpha-\lambda-2} \mathrm{e}^{-u^\alpha} \exp\{c(\mathrm{e}^{-u^\alpha}-1)\}\,\mathrm{d}u \\ &= \frac{\lambda c}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)} \int_0^\infty v^{-(1+\lambda)/\alpha} \mathrm{e}^{-v} \exp\{c(\mathrm{e}^{-v}-1)\}\,\mathrm{d}v \\ &= \frac{\lambda c}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)} \mathrm{e}^{-c} \sum_{n=0}^\infty c^n \frac{\Gamma(1-(1+\lambda)/\alpha)}{n!(n+1)^{1-(1+\lambda)/\alpha}}. \end{aligned}$$

Again by Lemma 3.1, shifted fractional moments $\mathbb{E}[|X-\mu|^{1+\lambda}]$ with $1 < 1+\lambda < \alpha$ are obtained as

$$\begin{aligned} m_{\mu,1+\lambda} &= \frac{\lambda}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)} \\ &\quad \times \Big[\mu \int_0^\infty u^{-(1+\lambda)} \sin(\mu u) \exp\{c(\mathrm{e}^{-u^\alpha}-1)\}\,\mathrm{d}u \\ &\quad + c\alpha \int_0^\infty u^{\alpha-\lambda-2} \cos(\mu u) \mathrm{e}^{-u^\alpha} \exp\{c(\mathrm{e}^{-u^\alpha}-1)\}\,\mathrm{d}u\Big]. \end{aligned}$$

3.3 Linnik distribution jump

Let φ_J be the ch.f. of Linnik distribution with parameters $\alpha > 1$, $\beta > 0$ and $\sigma = 1$. Since

$$\varphi_J'(-u) = \alpha\beta(1+u^\alpha)^{-\beta-1} u^{\alpha-1}, \qquad u > 0,$$

from Lemma 3.1 and change of variables formula ($v = (1+u^\alpha)^{-\beta}$) it follows that

$$m_{1+\lambda} = \frac{\lambda c \mathrm{e}^{-c}}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)} \int_0^1 v^{(1+\lambda)/(\alpha\beta)} (1-v^{1/\beta})^{-(1+\lambda)/\alpha} \mathrm{e}^{cv}\,\mathrm{d}v$$

for $1 < 1+\lambda < \alpha$. If $\beta = 1$, the jump distribution is the symmetric geometric stable distribution and we have

$$m_{1+\lambda} = \frac{\lambda c \mathrm{e}^{-c}}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)} B\left(1-\frac{1+\lambda}{\alpha}, 1+\frac{1+\lambda}{\alpha}\right) {}_1F_1\left(1+\frac{1+\lambda}{\alpha}; 2; c\right),$$

where we use (3.383-1) in Gradshteyn and Ryzhik (2007).

3.4 Deterministic jump of size 1 (simple Poisson)

Substituting its LP transform $\phi_J(t) = \mathrm{e}^{-t}$ into the expression in Lemma 3.1, we have

$$m_{1+\lambda} = \frac{\lambda c \mathrm{e}^{-c}}{\Gamma(1-\lambda)} \int_0^\infty \frac{\mathrm{e}^c - \mathrm{e}^{-u}\mathrm{e}^{c\mathrm{e}^{-u}}}{u^{1+\lambda}}\,\mathrm{d}u,$$

which is rewritten by the Taylor expansion as

$$m_{1+\lambda} = \frac{\lambda c \mathrm{e}^{-c}}{\Gamma(1-\lambda)} \sum_{k=0}^{\infty} \frac{c^k}{k!} \int_0^\infty \frac{1-\mathrm{e}^{-(k+1)u}}{u^{1+\lambda}}\,\mathrm{d}u = \mathrm{e}^{-c} \sum_{k=0}^{\infty} \frac{k^{1+\lambda} c^k}{k!},$$

where the final expression can be directly obtained from the probability mass function.

Remark 3.2. If the jump distribution has reproductive property, that is, it is convolution-closed, we have another method for determining the fractional absolute moments. Write the CP random variable as $S_N = \sum_{j=1}^{N} X_j$, where N has the Poisson distribution with parameter c and (X_j) is an i.i.d. sequence such that X_1 has reproduction property. Denote the ch.f. of kth convolution of X_1 by $\varphi_k(t)$, then under suitable conditions we have

$$\begin{aligned} m_{1+\lambda} &= \frac{\lambda}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)} \mathbb{E}\Big[\Re \int_0^\infty \frac{\varphi_N'(-u)}{u^{1+\lambda}}\,\mathrm{d}u\Big] \\ &= \frac{\lambda}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)} \sum_{k=0}^{\infty} \frac{c^k}{k!} \mathrm{e}^{-c} \Big[\Re \int_0^\infty \frac{\varphi_k'(-u)}{u^{1+\lambda}}\,\mathrm{d}u\Big]. \end{aligned}$$

In case of the LP transform, we denote that of kth convolution of X_1 by $\phi_k(t)$, $t \ge 0$, and from Lemma 1.1 we obtain

$$m_{1+\lambda} = \frac{\lambda \mathrm{e}^{-c}}{\Gamma(1-\lambda)} \sum_{k=0}^{\infty} \frac{c^k}{k!} \int_0^\infty \frac{\phi_k'(u) - \phi_k'(0+)}{u^{1+\lambda}}\,\mathrm{d}u.$$

4 Applications

4.1 Evaluation of conditional expectation for stable law

Conditional expectations of stable random vectors have been intensively investigated in Hardin et al. (1991) and Samorodnitsky and Taqqu (1991), since stable laws are often thought as natural generalization of the Gaussian random vector for which the minimizer of the mean squared error given some components of the vector is the conditional expectation. However, their evaluations have not been examined thoroughly. In what follows, we evaluate the goodness of several predictors given by conditional expectations through their fractional moments.

First, we consider general results for a bivariate stable random vector with ch.f.

$$\varphi(t_1, t_2) := \mathbb{E}\big[\mathrm{e}^{\mathrm{i}(t_1 X_1 + t_2 X_2)}\big], \qquad (t_1, t_2) \in \mathbb{R}^2,$$

which can be written as

$$\begin{aligned} \varphi(t_1, t_2) = \exp\Big\{ &- \int_{\mathbb{S}^1} |t_1 s_1 + t_2 s_2|^\alpha \Big[1 - \mathrm{i} \tan \frac{\pi\alpha}{2} \operatorname{sign}(t_1 s_1 + t_2 s_2)\Big] \Gamma(\mathrm{d}s) \\ &+ \mathrm{i}(t_1 \delta_1 + t_2 \delta_2) \Big\}, \end{aligned} \tag{19}$$

where Γ is a finite measure on the unit sphere $\mathbb{S}^1$, called spectral measure, and we let $\alpha > 1$; see (Samorodnitsky and Taqqu, 1994, Theorem 2.3.1). Our aim is to linearly approximate X_2 by X_1 and evaluate the fractional error of order $1 < \gamma < 2$. The situation includes various settings, for example, if stable random vectors are symmetric, that is,

$$\varphi(t_1, t_2) = \exp\left\{-\int_{\mathbb{S}^1} |t_1 s_1 + t_2 s_2|^\alpha \Gamma(\mathrm{d}s)\right\},$$

then it is proved that $\mathbb{E}[X_2 \mid X_1] = cX_1$ a.s. with some real constant c; see Theorem 4.1.2 in Samorodnitsky and Taqqu (1994) or Theorem 3.1 in Samorodnitsky and Taqqu (1991). For the general case, we refer to Theorem 3.1 in Hardin et al. (1991). For convenience, we assume $(\delta_1, \delta_2) = \mathbf{0}$, the general result for $(\delta_1, \delta_2) \neq \mathbf{0}$ can be obtained in the same manner.

Proposition 4.1. *Let (X_1, X_2) be a bivariate stable random vector defined by (19) such that $(\delta_1, \delta_2) = \mathbf{0}$. Then for any constant c and $1 < 1 + \lambda < \alpha$, it follows that*

$$\mathbb{E}\big[|X_2 - cX_1|^{1+\lambda}\big] = \frac{\lambda\Gamma(1 - (1+\lambda)/\alpha)}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)} \sigma_0^{1+\lambda}\big(1 + \theta_0^2\big)^{(1+\lambda)/(2\alpha)-1/2}(\cos\psi_\lambda + \theta_0 \sin\psi_\lambda),$$

where

$$\begin{aligned} \sigma_0 &= \left(\int_{\mathbb{S}^1} |s_2 - cs_1|^\alpha \Gamma(\mathrm{d}s)\right)^{1/\alpha}, \\ \beta_0 &= \frac{\int_{\mathbb{S}^1} \operatorname{sign}(s_2 - cs_1)|s_2 - cs_1|^\alpha \Gamma(\mathrm{d}s)}{\int_{\mathbb{S}^1} |s_2 - cs_1|^\alpha \Gamma(\mathrm{d}s)}, \end{aligned} \tag{20}$$

$\theta_0 = \beta_0 \tan\frac{\pi\alpha}{2}$ *and* $\psi_\lambda = (1 - \frac{1+\lambda}{\alpha}) \arctan\theta_0$. *In symmetric case, we have*

$$\mathbb{E}\big[|X_2 - cX_1|^{1+\lambda}\big] = \frac{\lambda\Gamma(1 - (1+\lambda)/\alpha)}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)} \left(\int_{\mathbb{S}^1} |s_2 - cs_1|^\alpha \Gamma(\mathrm{d}s)\right)^{(1+\lambda)/\alpha}.$$

Proof. The fractional derivative of the ch.f. of $X_2 - cX_1$ is calculated. We put $t_1 = -cu$ and $t_2 = u$ in (19), then we regard it as a function of u,

$$\mathbb{E}\big[\mathrm{e}^{\mathrm{i}u(X_2 - cX_1)}\big] = \exp\left\{-|u|^\alpha \int_{\mathbb{S}^1} |s_2 - cs_1|^\alpha \Gamma(\mathrm{d}s)\big(1 - \mathrm{i}\theta_0 \operatorname{sign}(u)\big)\right\}.$$

In view of (9), this is the ch.f. of one-dimensional α-stable distribution with parameters $(\beta, \sigma, \delta) = (\beta_0, \sigma_0, 0)$. Hence, we apply Proposition 2.2 to obtain the result. □

Examples. As examples, we consider predictions for two bivariate stable random vectors and one stable process. First we treat a bivariate stable random vector considered by (Nguyen, 1995, p. 183) such that ch.f. of (X_1, X_2) satisfies for $|a| < 1$, $\alpha \neq 1$,

$$\begin{aligned}\varphi(t_1, t_2) &= \mathbb{E}\big[e^{i(t_1 X_1 + t_2 X_2)}\big] \\ &= \exp\bigg\{-\sigma^\alpha |t_2|^\alpha \bigg[1 + i\beta \tan\frac{\pi\alpha}{2}\operatorname{sign}(t_2)\bigg] \\ &\qquad - \frac{\sigma^\alpha |t_1 + a t_2|^\alpha}{1-|a|^\alpha}\bigg[1 + i\beta\tan\frac{\pi\alpha}{2}\frac{1-|a|^\alpha}{1-\operatorname{sign}(a)|a|^\alpha}\operatorname{sign}(t_1 + a t_2)\bigg]\bigg\}.\end{aligned}$$

The conditional ch.f. is

$$\varphi_{X_1=x}(t) := \mathbb{E}\big[e^{itX_2} \mid X_1 = x\big] = \exp\bigg\{iaxt - \sigma^\alpha |t|^\alpha \bigg[1 + i\beta\tan\frac{\pi\alpha}{2}\operatorname{sign}(t)\bigg]\bigg\}$$

and hence for $1 < \alpha < 2$, $\mathbb{E}[X_2 \mid X_1 = x] = ax$. The support of spectral measure Γ consists of four points in $\mathbb{S}^1$,

$$\begin{aligned}\Gamma(0, \pm 1) &= \frac{1}{2}\sigma^\alpha (1 \pm \beta), \\ \Gamma\bigg(\pm\frac{1}{\sqrt{1+a^2}}, \pm\frac{a}{\sqrt{1+a^2}}\bigg) &= \frac{1}{2}\frac{\sigma^\alpha}{1-|a|^\alpha}(1+a^2)^{\alpha/2}\bigg(1 \pm \beta\frac{1-|a|^\alpha}{1-\operatorname{sign}(a)|a|^\alpha}\bigg).\end{aligned}$$

Hence,

$$\begin{aligned}\int_{\mathbb{S}^1} |s_2 - c s_1|^\alpha \Gamma(ds) &= \sigma^\alpha\bigg(1 + \frac{|a-c|^\alpha}{1-|a|^\alpha}\bigg), \\ \int_{\mathbb{S}^1} \operatorname{sign}(s_2 - c s_1)|s_2 - c s_1|^\alpha \Gamma(ds) &= \beta\sigma^\alpha\bigg(1 + \frac{\operatorname{sign}(a-c)|a-c|^\alpha}{1-\operatorname{sign}(a)|a|^\alpha}\bigg).\end{aligned}$$

Substitution of these relations into (20) yields σ_0 and β_0 that can be used for calculating the fractional absolute prediction error by Proposition 4.1.

Another example is the prediction for sub-Gaussian random vector. Let $0 < \alpha < 2$, $|\gamma| \le 1$, and let (G_1, G_2) be zero mean Gaussian random vector with covariance matrix

$$\Sigma = \begin{pmatrix} 1 & \gamma \\ \gamma & 1 \end{pmatrix}. \tag{21}$$

Let A be a positive $\alpha/2$-stable random variable, given by the LP transform

$$\mathbb{E}\big[e^{-tA}\big] = e^{-t^{\alpha/2}}, \qquad t > 0,$$

such that it is independent of (G_1, G_2). The random vector $(X_1, X_2) = (A^{1/2}G_1, A^{1/2}G_2)$ is called a sub-Gaussian symmetric α-stable random vector. In Samorodnitsky and Taqqu (1991), $\mathbb{E}[X_2 \mid X_1] = \gamma X_1$ is shown. Since we have the ch.f.

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{\mathrm{i}t(X_2-\gamma X_1)}] &= \mathbb{E}[\mathbb{E}[\mathrm{e}^{\mathrm{i}t(A^{1/2}G_2-\gamma A^{1/2}G_1)}] \mid A] \\ &= \mathbb{E}\left[\exp\left\{-\frac{t^2}{2}A(-\gamma,1)\Sigma(-\gamma,1)'\right\}\right] \\ &= \mathbb{E}\left[\exp\left\{-\frac{t^2(1-\gamma^2)}{2}A\right\}\right] = \exp\left\{-\left(\frac{1-\gamma^2}{2}\right)^{\alpha/2} t^\alpha\right\},\end{aligned}$$

due to the fractional moment (15), we get the fractional error

$$\mathbb{E}\left[\left|X_2 - \mathbb{E}[X_2 \mid X_1]\right|^{1+\lambda}\right] = \frac{\lambda\Gamma(1-(1+\lambda)/\alpha)}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)}\left(\frac{1-\gamma^2}{2}\right)^{(1+\lambda)/2},$$

where $1 < 1+\lambda < \alpha$. If X_2 is predicted by a linear function cX_1, in a similar manner, we obtain

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}t(X_2-cX_1)}] = \exp\left\{-\left(\frac{1-2\gamma c+c^2}{2}\right)^{\alpha/2} t^\alpha\right\}, \qquad t \in \mathbb{R},$$

which yields

$$\mathbb{E}\left[|X_2 - cX_1|^{1+\lambda}\right] = \frac{\lambda\Gamma(1-(1+\lambda)/\alpha)}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)}\left(\frac{1-2\gamma c+c^2}{2}\right)^{(1+\lambda)/2}.$$

Alternatively, we could use the spectral measure of the sub-Gaussian random vector given in (Samorodnitsky and Taqqu, 1994, Proposition 2.5.8). Since it is given in a closed form, we obtain the fractional error directly from ch.f. here.

Next, we examine the prediction of the α-stable Ornstein–Uhlenbeck (OU for short) process with $0 < \alpha < 2$ and $\gamma > 0$ given by

$$X_t = \mathrm{e}^{-\gamma t}X_0 + \int_0^t \mathrm{e}^{-\gamma(t-s)}\,\mathrm{d}Z_s, \qquad t > 0,$$

where $\{Z_t\}_{t\in\mathbb{R}}$ is the symmetric α-stable motion. We set $X_0 = \int_{-\infty}^0 \mathrm{e}^{\gamma s}\,\mathrm{d}Z_s$ to obtain the stationary version; see (Samorodnitsky and Taqqu, 1994, Example 3.6.3) for its definition. Then the conditional ch.f. of X_t given X_0 is

$$\begin{aligned}\varphi_{X_0}(u) &= \mathbb{E}[\mathrm{e}^{\mathrm{i}uX_t} \mid X_0] = \exp\{\mathrm{i}u\mathrm{e}^{-\gamma t}X_0\}\exp\left\{-\int_0^t \left|u\mathrm{e}^{-\gamma(t-s)}\right|^\alpha \mathrm{d}s\right\} \\ &= \exp\left\{\mathrm{i}u\mathrm{e}^{-\gamma t}X_0 - \frac{1-\mathrm{e}^{-\alpha\gamma t}}{\alpha\gamma}|u|^\alpha\right\},\end{aligned}$$

which yields $\mathbb{E}[X_t \mid X_0] = \mathrm{e}^{-\gamma t}X_0$ for $\alpha > 1$. Since the mean squared error of the prediction is not available, we use the fractional absolute moment of order $1 < 1+\lambda < \alpha$. More generally, we measure the error of a linear approximation cX_0 with $c \in \mathbb{R}$.

Lemma 4.2. *Let X_t be an α-stable OU process driven by the symmetric stable motion with the location parameter $\delta = 0$. Then for $1 < 1+\lambda < \alpha$,*

$$\mathbb{E}[|X_t - cX_0|^{1+\lambda}] = \frac{\lambda\Gamma(1-(1+\lambda)/\alpha)}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)}\left[\frac{1-\mathrm{e}^{-\alpha\gamma t} + (c-\mathrm{e}^{-\gamma t})^\alpha}{\alpha\gamma}\right]^{(1+\lambda)/\alpha}$$

and hence putting $c = \mathrm{e}^{-\gamma t}$, we obtain

$$\mathbb{E}\big[\big|X_t - \mathbb{E}[X_t \mid X_0]\big|^{1+\lambda}\big] = \frac{\lambda\Gamma(1-(1+\lambda)/\alpha)}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)}\left(\frac{1-\mathrm{e}^{-\alpha\gamma t}}{\alpha\gamma}\right)^{(1+\lambda)/\alpha}.$$

Proof. Since

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}u(X_t - cX_0)} \mid X_0] = \exp\{\mathrm{i}u(e^{-\gamma t} - c)X_0\}\exp\left\{-\frac{1}{\alpha\gamma}(1-\mathrm{e}^{-\alpha\gamma t})|u|^\alpha\right\},$$

$u \in \mathbb{R}$, we may use formula (16) in Proposition 2.2 with $\mu = (c - \mathrm{e}^{-\gamma t})X_0$ and $\sigma^\alpha = \frac{1}{\alpha\gamma}(1-\mathrm{e}^{-\alpha\gamma t})$. Consequently,

$$\begin{aligned}
&\mathbb{E}[|X_t - cX_0|^{1+\lambda} \mid X_0] \\
&\quad = \frac{\lambda\sigma^{1+\lambda}}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)}\left\{\frac{(c-\mathrm{e}^{-\gamma t})X_0}{\sigma}\int_0^\infty v^{-(1+\lambda)}\mathrm{e}^{-v^\alpha}\right. \\
&\qquad\qquad \times \sin\left[\frac{(c-\mathrm{e}^{-\gamma t})X_0}{\sigma}v\right]\mathrm{d}v \\
&\qquad\qquad \left. + \alpha\int_0^\infty v^{\alpha-\lambda-2}\mathrm{e}^{-v^\alpha}\cos\left[\frac{(c-e^{-\gamma t})X_0}{\sigma}v\right]\mathrm{d}v\right\}.
\end{aligned}$$

After taking expectation w.r.t. X_0 and applying Fubini's theorem, we get

$$\begin{aligned}
\mathbb{E}[|X_t - cX_0|^{1+\lambda}] &= \frac{\lambda\sigma^{1+\lambda}}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)}\left[1 + \frac{1}{\alpha\gamma}\left(\frac{c-\mathrm{e}^{-\gamma t}}{\sigma}\right)^\alpha\right] \\
&\quad \times \int_0^\infty u^{-(1+\lambda)/\alpha}\exp\left\{-\left[1+\frac{1}{\alpha\gamma}\left(\frac{c-\mathrm{e}^{-\gamma t}}{\sigma}\right)^\alpha\right]u\right\}\mathrm{d}u.
\end{aligned}$$

Then the result is implied by $\sigma^\alpha = \frac{1}{\alpha\gamma}(1-\mathrm{e}^{-\alpha\gamma t})$ and definition of the gamma function. □

Since the finite dimensional distribution of the α-stable OU process is a multivariate stable, we may use Proposition 4.1 similarly as before. The spectral measure is given in (Samorodnitsky and Taqqu, 1991, Example 3.6.4).

4.2 Evaluation of conditional expectation related with Linnik law

(1) Let (X_1, X_2) be a bivariate Linnik distribution with ch.f.

$$\varphi(t_1, t_2) = \left[1 + \left(\mathbf{t}'\Sigma\mathbf{t}\right)^{\alpha/2}\right]^{-\beta}, \qquad \mathbf{t}' = (t_1, t_2) \in \mathbb{R}^2,$$

where $0 < \alpha \le 2$, $\beta > 0$ and Σ is given by (21); see, for example, Lim and Teo (2010).

Lemma 4.3. *Let (X_1, X_2) be a bivariate Linnik random vector. Then $\mathbb{E}[X_2 \mid X_1] = \gamma X_1$ and for any $c \in \mathbb{R}$ and $1 < \alpha < 2$ it follows that*

$$\mathbb{E}\left[|X_2 - cX_1|^{1+\lambda}\right] = \frac{\lambda\beta|c^2 - 2\gamma c + 1|^{(1+\lambda)/2}}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)} B\left(1 - \frac{1+\lambda}{\alpha}, \beta + \frac{1+\lambda}{\alpha}\right)$$

and, therefore,

$$\mathbb{E}\left[\left|X_2 - \mathbb{E}[X_2 \mid X_1]\right|^{1+\lambda}\right] = \frac{\lambda\beta(1-\gamma^2)^{(1+\lambda)/2}}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)} B\left(1 - \frac{1+\lambda}{\alpha}, \beta + \frac{1+\lambda}{\alpha}\right),$$

where γ is the covariance term from Σ given by (21).

Proof. To obtain $\mathbb{E}[X_2 \mid X_1]$, we use the decomposition by Devroye (1990) of univariate Linnik law, which is also applicable in our bivariate case. Let (Y_1, Y_2) be a sub-Gaussian random vector with ch.f.

$$\varphi(t_1, t_2) = \mathrm{e}^{-(\mathbf{t}'\Sigma\mathbf{t})^{\alpha/2}}, \qquad \mathbf{t} = (t_1, t_2)'$$

and let Z be an independent random variable with density

$$f(x) = \frac{\mathrm{e}^{-x^{1/\beta}}}{\Gamma(1+\beta)}, \qquad x > 0.$$

Then we observe that $(X_1, X_2) \stackrel{d}{=} (Y_1 Z^{1/\alpha\beta}, Y_2 Z^{1/\alpha\beta})$, which leads to

$$\begin{aligned}\mathbb{E}[X_2 \mid X_1] &\stackrel{d}{=} \mathbb{E}\left[\mathbb{E}\left[Y_2 Z^{1/\alpha\beta} \mid Y_1, Z^{1/\alpha\beta}\right] \mid Y_1 Z^{1/\alpha\beta}\right] \\ &= \mathbb{E}\left[Z^{1/\alpha\beta}\mathbb{E}[Y_2 \mid Y_1] \mid Y_1 Z^{1/\alpha\beta}\right] = \gamma Y_1 Z^{1/\alpha\beta} \stackrel{d}{=} \gamma X_1,\end{aligned}$$

where the conditional expectation of the sub-Gaussian random vector is used. Now put $t_1 = -cu$ and $t_2 = u$ in $\varphi(t_1, t_2)$ to obtain

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}u(X_2 - cX_1)}\right] = \left[1 + \left(c^2 - 2\gamma c + 1\right)^{\alpha/2}|u|^\alpha\right]^{-\beta}$$

and we conclude our result from Section 2.4. □

(2) Let Z be a symmetric stable random variable with exponent $0 < \alpha < 2$ and E be the standard exponential random variable, that is,

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}tZ}\right] = \mathrm{e}^{\mathrm{i}t\delta - \sigma^\alpha|t|^\alpha} \quad \text{and} \quad \mathbb{E}\left[\mathrm{e}^{\mathrm{i}tE}\right] = (1 - \mathrm{i}t)^{-1}, \qquad t \in \mathbb{R}, \delta \in \mathbb{R}, \sigma > 0.$$

We consider a bivariate distribution

$$(X_1, X_2) \stackrel{d}{=} (E^{1/\alpha} Z, E)$$

as in Kozubowski and Meerschaert (2009), where Z is a stable subordinator, which yields a bivariate distribution with exponential and Mittag–Leffler marginals. Note that the marginal X_1 has no second moment, whereas X_2 has any power moments. Since the conditional ch.f. of X_1 given X_2 is

$$\varphi_{X_2}(t) = \mathbb{E}[\mathrm{e}^{\mathrm{i}tE^{1/\alpha}Z} \mid X_2] = \mathrm{e}^{\mathrm{i}t\delta X_2^{1/\alpha} - \sigma^\alpha X_2 |t|^\alpha},$$

the conditional expectation has the form $\mathbb{E}[X_1 \mid X_2] = \delta X_2^{1/\alpha}$. If we predict X_1 by $cX_2^{1/\alpha}$ with a constant c, the following result holds.

Lemma 4.4. *Let* $(X_1, X_2) \stackrel{d}{=} (E^{1/\alpha} Z, E)$ *be a bivariate random vector such that* Z *is a symmetric stable with location* $\delta \in \mathbb{R}$ *and scale* $\sigma > 0$ *and* E *is the standard exponential. Then* $\mathbb{E}[X_1 \mid X_2] = \delta X_2^{1/\alpha}$ *and for any* $c \in \mathbb{R}$ *and* $1 < \alpha < 2$ *it follows that*

$$\mathbb{E}\big[|X_1 - cX_2^{1/\alpha}|^{1+\lambda}\big] = \frac{\lambda\sigma^{1+\lambda}\Gamma(1+(1+\lambda)/\alpha)}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)} \times \Big[\frac{\delta - c}{\sigma}\int_0^\infty u^{-(1+\lambda)}\mathrm{e}^{-u^\alpha}\sin\Big(\frac{\delta-c}{\sigma}u\Big)\,\mathrm{d}u + \alpha\int_0^\infty u^{\alpha-\lambda-2}\mathrm{e}^{-u^\alpha}\cos\Big(\frac{\delta-c}{\sigma}u\Big)\,\mathrm{d}u\Big]$$

and, therefore,

$$\mathbb{E}\big[|X_1 - \mathbb{E}[X_1 \mid X_2]|^{1+\lambda}\big] = \frac{\lambda\sigma^{1+\lambda}\Gamma(1+(1+\lambda)/\alpha)\Gamma(1-(1+\lambda)/\alpha)}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)}.$$

Proof. Since $\mathbb{E}[\mathrm{e}^{\mathrm{i}t(X_1 - cX_2^{1/\alpha})} \mid X_2] = \mathrm{e}^{\mathrm{i}t(\delta-c)X_2^{1/\alpha} - \sigma^\alpha X_2|t|^\alpha}$ is ch.f. of a symmetric stable distribution, we apply (16) of Proposition 2.2 and then take expectation with respect to X_2, which is justified by Fubini's theorem. □

4.3 Estimation errors of regression model

We consider the basic regression model

$$Y_j = \theta_0 + x_j\theta_1 + \varepsilon_j, \qquad j = 1, 2, \ldots, n,$$

where (ε_j) is an i.i.d. sequence of symmetric random variables. It is well known that the least squares estimator $(\widehat{\theta}_0, \widehat{\theta}_1)$, which is the best linear unbiased estimator if the ε_j follow Gaussian distribution, has the form

$$\widehat{\theta}_0 = \overline{Y} - \widehat{\theta}_1\overline{x}, \qquad \widehat{\theta}_1 = \frac{\sum_{j=1}^n (x_j - \overline{x})(Y_j - \overline{Y})}{\sum_{j=1}^n (x_j - \overline{x})^2},$$

where $\overline{x} = \frac{1}{n}\sum_{j=1}^{n} x_j$ and $\overline{Y} = \frac{1}{n}\sum_{j=1}^{n} Y_j$. For our purpose, it will be convenient to rewrite this as

$$\widehat{\theta}_0 = \theta_0 - \sum_{i=1}^{n} \frac{(x_i - \overline{x})\overline{x} - \sum_{j=1}^{n}(x_j - \overline{x})^2/n}{\sum_{j=1}^{n}(x_j - \overline{x})^2}\varepsilon_i,$$

$$\widehat{\theta}_1 = \theta_1 + \sum_{i=1}^{n} \frac{x_i - \overline{x}}{\sum_{j=1}^{n}(x_j - \overline{x})^2}\varepsilon_i.$$

We express the fractional errors for the case of α-stable noise distributions. In a similar manner, it would be possible to calculate the fractional errors for other regression-type estimators [e.g., Blattberg and Sargent (1971) and Nolan (2013)], and compare the goodness of estimators.

If ε_1 is a standard symmetric stable random variable with parameters $\delta = 0$, $\sigma = 1$ and $\alpha > 1$, then the characteristic functions of estimation errors are

$$\mathbb{E}\big[\mathrm{e}^{\mathrm{i}t(\widehat{\theta}_k - \theta_k)}\big] = \mathrm{e}^{-\sigma_k^\alpha |t|^\alpha}, \qquad t \in \mathbb{R}, k = 0, 1,$$

where

$$\sigma_0^\alpha = \sum_{i=1}^{n}\left(\frac{|(x_i - \overline{x})\overline{x} - \sum_{j=1}^{n}(x_j - \overline{x})^2/n|}{\sum_{j=1}^{n}(x_i - \overline{x})^2}\right)^\alpha$$

and

$$\sigma_1^\alpha = \sum_{i=1}^{n}\left(\frac{|x_i - \overline{x}|}{\sum_{j=1}^{n}(x_j - \overline{x})^2}\right)^\alpha.$$

This together with (15) yields, for $1 < 1 + \lambda < \alpha$,

$$\mathbb{E}\big[|\widehat{\theta}_k - \theta_k|^{1+\lambda}\big] = \frac{\lambda\Gamma(1 - (1+\lambda)/\alpha)}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)}\sigma_k^{1+\lambda}, \qquad k = 0, 1.$$

Interestingly, if $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ is an elliptically contoured stable random vector with ch.f. $\mathbb{E}[\mathrm{e}^{\mathrm{i}\mathbf{t}'\boldsymbol{\varepsilon}}] = \mathrm{e}^{-|\mathbf{t}'\mathbf{I}\mathbf{t}|^{\alpha/2}}$ for $\mathbf{t}' \in \mathbb{R}^n$, where $\mathbf{I}$ is $n \times n$ identity matrix, then we obtain closer results to Gaussian case. Namely, for $1 < 1 + \lambda < \alpha$,

$$\mathbb{E}\big[|\widehat{\theta}_k - \theta_0|^{1+\lambda}\big] = \frac{\lambda\Gamma(1 - (1+\lambda)/\alpha)}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)}\overline{\sigma}_k^{1+\lambda}, \qquad k = 0, 1,$$

where

$$\overline{\sigma}_0 = \left(\frac{\overline{x}^2}{\sum_{j=1}^{n}(x_j - \overline{x})^2} + \frac{1}{n}\right)^{1/2} \quad \text{and} \quad \overline{\sigma}_1 = \frac{1}{[\sum_{j=1}^{n}(x_j - \overline{x})^2]^{1/2}}.$$

Moreover, if $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ is a multivariate Linnik random vector with ch.f. $\mathbb{E}[\mathrm{e}^{\mathrm{i}\mathbf{t}'\boldsymbol{\varepsilon}}] = \{1 + (\mathbf{t}'\mathbf{I}\mathbf{t})^{\alpha/2}\}^{-\beta}$ for $\mathbf{t}' \in \mathbb{R}^n$, we obtain in a similar manner that $(1 < 1 + \lambda < \alpha)$

$$\mathbb{E}\big[|\widehat{\theta}_k - \theta_k|^{1+\lambda}\big] = \frac{\lambda\beta B(1 - (1+\lambda)/\alpha, \beta + (1+\lambda)/\alpha)}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)}\overline{\sigma}_k^{1+\lambda}, \qquad k = 0, 1.$$

Appendix: Proofs

A.1 Proof of Lemma 1.1

Suppose that $m_{1+\lambda}$ exists, then $\phi'(u)$ exists for all $u > 0$ and is equal to $-\int_0^\infty x\mathrm{e}^{-xu}\,\mathrm{d}F(x)$ and $\phi'(0+)$ exists and is equal to $-\int_0^\infty x\,\mathrm{d}F(x)$. We use Fubini's theorem to see that

$$\begin{aligned}
&\frac{\lambda}{\Gamma(1-\lambda)}\int_0^\infty \frac{\phi'(u)-\phi'(0+)}{u^{1+\lambda}}\,\mathrm{d}u \\
&\quad= \frac{\lambda}{\Gamma(1-\lambda)}\int_0^\infty u^{-(1+\lambda)}\int_0^\infty x\big(1-\mathrm{e}^{-xu}\big)\,\mathrm{d}F(x)\,\mathrm{d}u \\
&\quad= \frac{\lambda}{\Gamma(1-\lambda)}\int_0^\infty x^{1+\lambda}\int_0^\infty \frac{1-\mathrm{e}^{-xu}}{(xu)^{1+\lambda}}x\,\mathrm{d}u\,\mathrm{d}F(x) \\
&\quad= \int_0^\infty x^{1+\lambda}\,\mathrm{d}F(x) < \infty.
\end{aligned}$$

Conversely, the existence of $\phi'(0+)$ implies

$$-\phi'(0+) \geq \int_0^\infty x\mathrm{e}^{-ux}\,\mathrm{d}F(x) = -\phi'(u)$$

for any $u > 0$. Hence, the reverse argument yields $m_{1+\lambda} < \infty$.

A.2 Proof of Lemma 1.3

A simple calculation yields

$$\begin{aligned}
\int_{-\infty}^\infty |x|^{1+\lambda}\,\mathrm{d}F(x) &= \frac{\lambda(1+\lambda)}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)}\int_{-\infty}^\infty |x|^{1+\lambda}\int_0^\infty \frac{1-\cos u}{u^{2+\lambda}}\,\mathrm{d}u\,\mathrm{d}F(x) \\
&= \frac{\lambda(1+\lambda)}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)}\int_{-\infty}^\infty\int_0^\infty \frac{1-\cos ux}{u^{2+\lambda}}\,\mathrm{d}u\,\mathrm{d}F(x) \qquad (22)\\
&= \frac{\lambda(1+\lambda)}{\sin((\lambda\pi)/2)\Gamma(1-\lambda)}\int_0^\infty \frac{1-\Re\varphi(u)}{u^{2+\lambda}}\,\mathrm{d}u,
\end{aligned}$$

where we use Fubini's theorem in the last step. Hence, we obtain the first part of (b). Moreover, due to the integration by parts, we have for $x \in \mathbb{R}$,

$$\begin{aligned}
\int_0^\infty \frac{1-\cos ux}{u^{2+\lambda}}\,\mathrm{d}u &= \left[-\frac{u^{-(1+\lambda)}}{1+\lambda}(1-\cos ux)\right]_0^\infty + \frac{1}{1+\lambda}\int_0^\infty \frac{x\sin ux}{u^{1+\lambda}}\,\mathrm{d}u \\
&= \lim_{u\to 0+}\frac{1}{1+\lambda}\frac{1-\cos ux}{u^{1+\lambda}} + \frac{1}{1+\lambda}\int_0^\infty \frac{x\sin ux}{u^{1+\lambda}}\,\mathrm{d}u.
\end{aligned} \qquad (23)$$

If condition (5) holds, then by the Lebesgue dominated convergence theorem,

$$\int_{-\infty}^\infty \lim_{u\to 0+}\frac{1}{1+\lambda}\frac{1-\cos ux}{u^{1+\lambda}}\,\mathrm{d}F(x) = \frac{1}{1+\lambda}\lim_{u\to 0+}\frac{1-\Re\varphi(u)}{u^{1+\lambda}} < \infty,$$

and by Fubini's theorem,

$$\frac{1}{1+\lambda}\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{x\sin ux}{u^{1+\lambda}}\,\mathrm{d}u\,\mathrm{d}F(x)=\frac{1}{1+\lambda}\int_{0}^{\infty}\frac{\Re\varphi'(-u)}{u^{1+\lambda}}\,\mathrm{d}u<\infty.$$

Thus, conditions (3) in (a) are satisfied. We can prove the converse in a similar manner, and hence we showed that conditions in (a) and (b) are equivalent. If $m_{1+\lambda}<\infty$, then

$$\lim_{t\to 0+}\frac{1-\Re\varphi(t)}{t^{1+\lambda}}=0$$

follows from (5). This, together with (22) and (23), yields the expression (4).

Finally, we substitute the first derivative of the ch.f. φ_{μ}, which is

$$\varphi_{\mu}'(t)=-\mathrm{i}\mu\mathrm{e}^{-\mathrm{i}t\mu}\varphi(t)+\mathrm{e}^{-\mathrm{i}t\mu}\varphi'(t),\qquad t\in\mathbb{R},$$

into (4) to obtain the desired result (7). We may decompose the integral into two parts as in (7) because the existence of the integral

$$\Im\int_{0}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\mu u}\varphi(-u)}{u^{1+\lambda}}\,\mathrm{d}u=\int_{0}^{\infty}\frac{\cos\mu u\Im\varphi(-u)+\sin\mu u\Re\varphi(-u)}{u^{1+\lambda}}\,\mathrm{d}u$$

follows from $|\Re\varphi(-u)|\le 1$ and

$$\left|\Im\varphi(-u)\right|\le\int_{-\infty}^{\infty}\left|\sin(-ux)\right|\mathrm{d}F(x)\wedge 1\le\int_{-\infty}^{\infty}|ux|\,\mathrm{d}F(x)\wedge 1.$$

A.3 Proof of Lemma 2.1

We express ch.f. $\varphi(t)$, given by (8), as the product $\varphi_1(t)\cdot\varphi_2(t)$, where

$$\varphi_1(t):=\exp\left\{\mathrm{i}\delta t+\int_{|x|\le 1}\left(\mathrm{e}^{\mathrm{i}tx}-1-\mathrm{i}tx\right)\nu(\mathrm{d}x)\right\},\qquad t\in\mathbb{R},$$

$$\varphi_2(t):=\exp\left\{\int_{|x|>1}\left(\mathrm{e}^{\mathrm{i}tx}-1\right)\nu(\mathrm{d}x)\right\},\qquad t\in\mathbb{R}.$$

Since the distribution with ch.f. $\varphi_1(t)$ has moments of any positive order, it suffices to consider $\varphi_2(t)$. We use the necessary and sufficient condition (5) for the existence of $m_{1+\lambda}$; see Lemma 1.3. The following inequalities

$$\frac{a}{1+a}\le 1-\mathrm{e}^{-a}\le a,\qquad a\ge 0,$$

$$1-\cos b\le\frac{b^2}{2}\le\frac{b}{2},\qquad 0\le b\le 1,$$

and the fact

$$\int_{|x|>1}(1-\cos tx)\nu(\mathrm{d}x)\le\int_{|x|>1}\left(\frac{(tx)^2}{2}\wedge 1\right)\nu(\mathrm{d}x)=:c_t<\infty$$

are used to obtain

$$\begin{aligned}1-\Re\varphi_2(t) &\geq 1-\exp\Big\{\int_{|x|>1}(\cos tx-1)\nu(\mathrm{d}x)\Big\}\\ &\geq \frac{1}{1+c_t}\int_{|x|>1}(1-\cos tx)\nu(\mathrm{d}x),\\ 1-\Re\varphi_2(t) &= 1-\exp\Big\{\int_{|x|>1}(\cos tx-1)\nu(\mathrm{d}x)\Big\}\\ &\quad+\exp\Big\{\int_{|x|>1}(\cos tx-1)\nu(\mathrm{d}x)\Big\}\\ &\quad-\cos\Big(\int_{|x|>1}\sin tx\,\nu(\mathrm{d}x)\Big)\exp\Big\{\int_{|x|>1}(\cos tx-1)\nu(\mathrm{d}x)\Big\}\\ &\leq \int_{|x|>1}(1-\cos tx)\nu(\mathrm{d}x)+\frac{1}{2}\int_{|x|>1}|\sin tx|\nu(\mathrm{d}x).\end{aligned}$$

Then we notice by Fubini's theorem that

$$\int_0^\infty t^{-(2+\lambda)}\int_{|x|>1}(1-\cos tx)\nu(\mathrm{d}x)\,\mathrm{d}t=\int_0^\infty\frac{1-\cos v}{v^{2+\lambda}}\,\mathrm{d}v\int_{|x|>1}|x|^{1+\lambda}\nu(\mathrm{d}x),$$

$$\int_0^\infty t^{-(2+\lambda)}\int_{|x|>1}|\sin tx|\nu(\mathrm{d}x)\,\mathrm{d}t=\int_0^\infty\frac{|\sin v|}{v^{2+\lambda}}\,\mathrm{d}v\int_{|x|>1}|x|^{1+\lambda}\nu(\mathrm{d}x).$$

Since double integrals on the left-hand sides exist if and only if the integrals on the right-hand sides exist, condition (5) is equivalent to the existence of $\int_{|x|>1}|x|^{1+\lambda}\nu(\mathrm{d}x)$.

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Department of Business Administration
Nanzan University
18 Yamazato-cho
Showa-ku
Nagoya 466-8673
Japan
E-mail: muneya@nanzan-u.ac.jp

Department of Probability and Mathematical Statistics
Faculty of Mathematics and Physics
Charles University in Prague
Sokolovská 83
186 75 Prague 8
Czech Republic
E-mail: zbynek.pawlas@mff.cuni.cz