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# AGGREGATION OF PREDICTORS FOR NONSTATIONARY SUB-LINEAR PROCESSES AND ONLINE ADAPTIVE FORECASTING OF TIME VARYING AUTOREGRESSIVE PROCESSES[1]

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In this work, we study the problem of aggregating a finite number of predictors for nonstationary sub-linear processes. We provide oracle inequalities relying essentially on three ingredients: (1) a uniform bound of the $\ell^1$ norm of the time varying sub-linear coefficients, (2) a Lipschitz assumption on the predictors and (3) moment conditions on the noise appearing in the linear representation. Two kinds of aggregations are considered giving rise to different moment conditions on the noise and more or less sharp oracle inequalities. We apply this approach for deriving an adaptive predictor for locally stationary time varying autoregressive (TVAR) processes. It is obtained by aggregating a finite number of well chosen predictors, each of them enjoying an optimal minimax convergence rate under specific smoothness conditions on the TVAR coefficients. We show that the obtained aggregated predictor achieves a minimax rate while adapting to the unknown smoothness. To prove this result, a lower bound is established for the minimax rate of the prediction risk for the TVAR process. Numerical experiments complete this study. An important feature of this approach is that the aggregated predictor can be computed recursively and is thus applicable in an online prediction context.

**1. Introduction.** In many applications where high frequency data are observed, we wish to predict the next values of this time series through an online prediction learning algorithm able to process a large amount of data. The classical stationarity assumption on the distribution of the observations has to be weakened to take into account some smooth evolution of the environment. From a statistical modelling point of view, this is described by some time varying parameters. In order to sequentially track them from high-frequency data, the algorithms must require few operations and a low storage capacity to update the parameters estimation and the prediction after each new observation. The most common online

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methods are least mean squares (LMS), normalized least mean squares (NLMS), regularized least squares (RLS) or Kalman. All of them rely on the choice of a gradient step, a forgetting factor, or more generally on a tuning parameter corresponding to some a priori knowledge on how smoothly the local statistical distribution of the data evolves along the time. To adapt automatically to this smoothness, usually unknown in practice, we propose to use an exponentially weighted aggregation of several such predictors, with various tuning parameters. We emphasize that to meet the online constraint, we cannot use methods that require a large amount of computations (such as cross validation).

The exponential weighting technique in aggregation have been developed in parallel in the machine learning community [see the seminal paper Vovk (1990)], in the statistical community [see Catoni (1997), Yang (2000a, 2004), Leung and Barron (2006), or more recently Audibert (2009), Dalalyan and Tsybakov (2008), Rigollet and Tsybakov (2012)] and in the game theory community for individual sequences prediction [see Cesa-Bianchi and Lugosi (2006) and Stoltz (2011) for recent surveys]. In contrast to the classical statistical setting, in the individual sequence setting the observations are not assumed to be generated by an underlying stochastic process. The link between both settings has been analyzed in Gerchinovitz (2011) for the regression model with fixed and random designs.

Exponential weighting has also been investigated in the case of weakly dependent stationary data in Alquier and Wintenberger (2012). More recently, an approach inspired from individual sequences prediction has been studied in Anava et al. (2013) for bounded ARMA processes under some specific conditions on the (constant) ARMA coefficients.

In this contribution, we consider two possible aggregation schemes based on exponential weights which can be computed recursively. We provide oracle inequalities applying to the aggregated predictor under the following main assumptions that (1) the observations are sub-linearly with respect to a sequence of random variables with possibly time varying linear coefficients and (2) the predictors to be aggregated are Lipschitz functions of the past. An important feature of our observation model is that it embeds the well-known class of *locally stationary* processes. We refer to Dahlhaus (2009) and the references therein for a recent general view about statistical inference for locally stationary processes. As an application, we focus on a particular locally stationary model, that of the time varying autoregressive (TVAR) process. The minimax rate of certain recursive estimators of the TVAR coefficients is studied in Moulines, Priouret and Roueff (2005). To our knowledge, there is not a well-established method on the automatic choice of the gradient step when the smoothness index is unknown. Here, we are interested in the prediction problem which is closely related to the estimation problem. We show that the proposed aggregation methods provide a solution to this question, in the sense that they give rise to recursive adaptive minimax predictors.

The paper is organized as follows. In Section 2, we provide oracle inequalities for the aggregated predictors under general conditions applying to nonstationary

sub-linear processes. TVAR processes are introduced in Section 3 in a nonparametric setting based on Hölder smoothness assumptions on the TVAR coefficients. A lower bound of the prediction risk is given in this setting and this result is used to show that the proposed aggregation methods achieve the minimax adaptive rate. Section 4 contains the proofs of the oracle inequalities. The proof of the lower bound of the minimax prediction risk is presented in Section 5. Numerical experiments illustrating these results are then described in Section 6. One Appendix and one supplementary material [Giraud, Roueff and Sanchez-Perez (2015)] complete this paper. Appendix and [Giraud, Roueff and Sanchez-Perez (2015), Section A] contain some postponed proofs and useful lemmas, [Giraud, Roueff and Sanchez-Perez (2015), Section B] explains how to build nonadaptive minimax predictors which can be used in the aggregation step and [Giraud, Roueff and Sanchez-Perez (2015), Section C] provides additional results with improved aggregation rates.

## 2. Online aggregation of predictors for nonstationary processes.

2.1. *General model.* In this section, we consider a time series $(X_t)_{t\in\mathbb{Z}}$ admitting the following *nonstationary* sub-linear property with respect to the nonnegative process $(Z_t)_{t\in\mathbb{Z}}$.

(M-1) The process $(X_t)_{t\in\mathbb{Z}}$ satisfies

$$|X_t| \le \sum_{j\in\mathbb{Z}} A_t(j) Z_{t-j}, \tag{2.1}$$

where $(A_t(j))_{t,j\in\mathbb{Z}}$ are nonnegative coefficients such that

$$A_* := \sup_{t\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} A_t(j) < \infty. \tag{2.2}$$

Additional assumptions will be required on $(Z_t)_{t\in\mathbb{Z}}$ to deduce useful properties for $(X_t)_{t\in\mathbb{Z}}$. Note, for instance, that the condition on $A_*$ in (2.2) guarantees that, if $(Z_t)_{t\in\mathbb{Z}}$ has a uniformly bounded $L^p$-norm, the convergence of the infinite sum in (2.1) holds almost surely and in the $L^p$-sense, with both convergences defining the same limit. It follows that $(X_t)_{t\in\mathbb{Z}}$ also has uniformly bounded $L^p$ moments. Let us give some particular contexts where the representation (M-1) can be used.

EXAMPLE 1 (Time varying linear processes). Standard weakly stationary processes such as ARMA processes [see Brockwell and Davis (2006)] admit a Wold decomposition of the form

$$X_t = \sum_{j\ge 0} a(j)\xi_{t-j},$$

where $(\xi_t)_{t\in\mathbb{Z}}$ is a weak white noise with, says, unit variance. This model, sometimes referred to as an MA($\infty$) representation, is often extended to a two-sided

sum representation

$$X_t = \sum_{j\in\mathbb{Z}} a(j)\xi_{t-j},$$

and additional assumptions on the existence of higher moments for $(\xi_t)_{t\in\mathbb{Z}}$ or on the independence of the $\xi_t$'s are often used for statistical inference or prediction; see Brockwell and Davis (2006), Chapters 7 and 8. Because the sequence $(A_t(j))_{j\in\mathbb{Z}}$ may vary with $t$ in (M-1), we may extend this standard stationary setting and also consider linear processes with time varying coefficients. In this case, we have

$$X_t = \sum_{j\in\mathbb{Z}} a_t(j)\xi_{t-j}, \tag{2.3}$$

where $(\xi_t)$ is a sequence of centered independent random variables with unit variance and $(a_t(j))_{t,j\in\mathbb{Z}}$ is supposed to satisfy (2.2) with $A_t(j) = |a_t(j)|$, so that (M-1) holds with $Z_t = |\xi_t|$. For this general class of processes, statistical inference is not easily carried out: each new observation $X_t$ comes with a new unknown sequence $(a_t(j))_{j\in\mathbb{Z}}$. However, additional assumptions on this set of sequences allow to derive and study appropriate statistical inference procedures. A sensible approach in this direction is to consider a *locally stationary* model as introduced in Dahlhaus (1996). In this framework, the set of sequences $\{(a_t(j))_{j\in\mathbb{Z}}, 1 \le t \le T\}$ is controlled as $T \to \infty$ by artificially (but meaningfully) introducing a dependence in $T$, hence is written as $(a_{t,T}(j))_{j\in\mathbb{Z},1\le t\le T}$, and by approximating it with a set of sequences rescaled on the time interval $[0, 1]$, $a(u, j)$, $u \in [0, 1]$, $j \in \mathbb{Z}$, for example, in the following way:

$$\sup_{T\ge 1}\sup_{j\in\mathbb{Z}}\sum_{t=1}^{T}\left|a_{t,T}(j) - a\left(\frac{t}{T}, j\right)\right| < \infty.$$

Then various interesting statistical inference problems based on $X_1, \ldots, X_T$ can be tackled by assuming some smoothness on the mapping $u \mapsto a(u, j)$ and, possibly, additional assumptions on the structure of the sequence $(a(u, j))_{j\in\mathbb{Z}}$ for each $u \in [0, 1]$ [see Dahlhaus (2009) and the references therein].

EXAMPLE 2 (TVAR model). A particular instance of Example 1 is the *time varying autoregressive* (TVAR) process, which is assumed to satisfy the recursive equation

$$X_t = \sum_{j=1}^{d} \theta_{j,t} X_{t-j} + \sigma_t \xi_t,$$

where $(\xi_t)_{t\in\mathbb{Z}}$ is a white noise process; see Grenier (1983). It turns out that, in the framework introduced by Dahlhaus (1996), under suitable assumptions, such processes admit a time varying linear representation of the form (2.3); see Dahlhaus (1996), Künsch (1995). In Section 3, we focus on such a class of processes and use

the aggregation of predictors to derive adaptive minimax predictors under specific smoothness assumptions on the time varying coefficients.

EXAMPLE 3 (A nonlinear extension). It can also be interesting to consider nonlinear extensions of Example 2. A simple example is obtained by setting

$$X_t = g_t(X_{t-1}) + \xi_t,$$

where $(\xi_t)_{t\in\mathbb{Z}}$ is an i.i.d. sequence and $g_t$ is a time varying sub-linear sequence of functions satisfying, for all $t$ that

$$\big|g_t(x)\big| \le \alpha|x|,$$

for some $\alpha \in (0, 1)$. Since $g_t$ is no longer linear but sub-linear, such a model does not enjoy an exact linear representation of the form (2.3). Nevertheless, since we have

$$|X_t| \le \alpha|X_{t-1}| + |\xi_t|,$$

and iterating this equation backwards yields assumption (M-1) with $Z_t = |\xi_t|$ and $A_t(j) = \alpha^j$. In the stationary case, where $g = g_t$ does not depend on $t$, a well-known nonlinear extension is the threshold autoregressive model where $g$ is piecewise linear; see Tong and Lim (1980).

Our goal in this section is to derive oracle bounds for the aggregation of predictors that hold for the general model (M-1) with one of the two following additional assumptions on $(Z_t)_{t\in\mathbb{Z}}$.

(N-1) The nonnegative process $(Z_t)_{t\in\mathbb{Z}}$ satisfies

$$m_p := \sup_{t\in\mathbb{Z}} \mathbb{E}\big[Z_t^p\big] < \infty.$$

(N-2) The nonnegative process $(Z_t)_{t\in\mathbb{Z}}$ is a sequence of independent random variables fulfilling

$$\phi(\zeta) := \sup_{t\in\mathbb{Z}} \mathbb{E}\big[\mathrm{e}^{\zeta Z_t}\big] < \infty.$$

Assumptions (N-1) and (N-2) appear to be quite mild. As mentioned in Example 1, basic assumptions in stationary time series usually include moments of sufficiently high order for the innovations and their independence, or rely on the Gaussian assumption, which is contained in (N-2). We also note that, in the context of locally stationary time series, our assumptions on the innovations are weaker than those used in the recent works Dahlhaus (2009), Dahlhaus and Polonik (2006, 2009). Precise comparisons between our assumptions and usual ones in the aggregation literature will be given after Corollary 1.

2.2. *Aggregation of predictors.* Let $(x_t)_{t\in\mathbb{Z}}$ be a real valued sequence. We say that $\widehat{x}_t$ is a predictor of $x_t$ if it is a measurable function of $(x_s)_{s\le t-1}$. Throughout this paper, the quality of a sequence of predictors $(\widehat{x}_t)_{1\le t\le T}$ is evaluated for some

$T \geq 1$ using the $\ell^2$ loss averaged over the time period $\{1, \ldots, T\}$

$$\frac{1}{T}\sum_{t=1}^{T}(\widehat{x}_t - x_t)^2.$$

Now, given a collection of $N$ sequences of predictors $\{(\widehat{x}_t^{(i)})_{1\leq t\leq T}, 1 \leq i \leq N\}$, we wish to sequentially derive a new predictor which predicts almost as accurately as or more accurately than the best of them.

In the present paper and for our purposes, aggregating the predictors amounts to compute a convex combination of them at each time $t$. This corresponds to choosing at each time $t$ an element $\alpha_t$ of the simplex

$$\mathcal{S}_N = \left\{\mathbf{s} = (s_1, \ldots, s_N) \in \mathbb{R}_+^N : \sum_{i=1}^{N} s_i = 1\right\} \tag{2.4}$$

and compute

$$\widehat{x}_t^{[\alpha_t]} = \sum_{i=1}^{N} \alpha_{i,t}\widehat{x}_t^{(i)}.$$

We consider two strategies of aggregation, which are studied in the context of bounded sequences in Catoni (2004), Cesa-Bianchi and Lugosi (2006). More recent contributions and extensions can be found in Gerchinovitz (2011). See also Stoltz (2011) for a pedagogical Introduction. These strategies are sequential and online, meaning that:

(i) to compute the aggregation weights $\alpha_t$ at time $t$, only the values of $\{\widehat{x}_s^{(i)}, 1 \leq i \leq N\}$ and $x_s$ up to time $s = t-1$ are used,

(ii) the computation can be done recursively by updating a set of quantities, the number of which does not depend on $t$.

These two properties are met in the Algorithm 1 detailed below.

We consider in the remaining of the paper a convex aggregation of predictors

$$\widehat{x}_t = \widehat{x}_t^{[\widehat{\alpha}_t]} = \sum_{i=1}^{N} \widehat{\alpha}_{i,t}\widehat{x}_t^{(i)}, \qquad 1 \leq t \leq T,$$

with some specific weights $\widehat{\alpha}_{i,t}$ defined as follows.

*Strategy* 1: *Building weights from the gradient of the quadratic loss.* The first strategy is to define for all $i = 1, \ldots, N$ and $t = 1, \ldots, T$, the weights $\widehat{\alpha}_{i,t}$ by

$$\widehat{\alpha}_{i,t} = \frac{\exp(-2\eta\sum_{s=1}^{t-1}(\sum_{j=1}^{N}\widehat{\alpha}_{j,s}\widehat{x}_s^{(j)} - x_s)\widehat{x}_s^{(i)})}{\sum_{k=1}^{N}\exp(-2\eta\sum_{s=1}^{t-1}(\sum_{j=1}^{N}\widehat{\alpha}_{j,s}\widehat{x}_s^{(j)} - x_s)\widehat{x}_s^{(k)})}, \tag{2.5}$$

with the convention that a sum over no element is zero, so $\widehat{\alpha}_{i,1} = 1/N$ for all $i$.

The parameter $\eta > 0$, usually called the *learning rate*, will be specified later.

**Algorithm 1:** Online computation of the aggregation algorithms

**parameters** the learning rate $\eta$ (in $(0, \infty)$) and the strategy (1 or 2);
**initialization** $t = 1$, $\widehat{\alpha}_t = (1/N)_{i=1,\dots,N}$;
**while input** *the predictions* $\widehat{x}_t^{(i)}$ *for* $i = 1, \dots, N$*;*
**do**
  $\widehat{x}_t = \widehat{x}_t^{[\widehat{\alpha}_t]} = \sum_{i=1}^N \widehat{\alpha}_{i,t} \widehat{x}_t^{(i)}$;
  **return** $\widehat{x}_t$*;*
  **and when input** *a new* $x_t$*;*
  **do**
    $t = t + 1$;
    **for** $i = 1$ **to** $N$ **do**
      **switch** *strategy* **do**
        **case** 1: $v_{i,t} = \widehat{\alpha}_{i,t-1} \exp(-2\eta(\widehat{x}_{t-1}^{[\widehat{\alpha}_{t-1}]} - x_{t-1})\widehat{x}_{t-1}^{(i)})$
        **case** 2: $v_{i,t} = \widehat{\alpha}_{i,t-1} \exp(-\eta(\widehat{x}_{t-1}^{(i)} - x_{t-1})^2)$
      $\widehat{\alpha}_t = (v_{i,t} / \sum_{k=1}^N v_{k,t})_{i=1,\dots,N}$;

*Strategy* 2: *Building weights from the quadratic loss*. The second strategy is to define for all $i = 1, \dots, N$ and $t = 1, \dots, T$, the weights $\widehat{\alpha}_{i,t}$ by

$$\widehat{\alpha}_{i,t} = \frac{\exp(-\eta \sum_{s=1}^{t-1} (\widehat{x}_s^{(i)} - x_s)^2)}{\sum_{k=1}^N \exp(-\eta \sum_{s=1}^{t-1} (\widehat{x}_s^{(k)} - x_s)^2)}, \tag{2.6}$$

with again the convention that a sum over no element is zero.

Both strategies yield the same algorithm up to the line where $v_{i,t}$ is computed. For sake of brevity, we write only one algorithm (see Algorithm 1) and use a switch/case statement to distinguish between the two strategies. Note, however, that the choice of the strategy (1 or 2) holds for the whole sequence of predictions.

2.3. *Oracle bounds*. We establish oracle bounds on the average prediction error of the aggregated predictors. These bounds ensure that the error is equal to that associated with the best convex combination of the predictors or with the best predictor (depending on the aggregation strategy), up to two remaining terms. One remaining term depends on the number $N$ of predictors to aggregate and the other one on the *variability* of the original process. The learning rate $\eta$ can then be chosen to achieve the best trade-off between these two terms.

The second remaining term indirectly depends on the variability of the predictors. We control below this variability in terms of the variability of the original process by using the following Lipschitz property.

DEFINITION 1. Let $L = (L_s)_{s\geq 1}$ be a sequence of nonnegative numbers. A predictor $\widehat{x}_t$ of $x_t$ from $(x_s)_{s\leq t-1}$ is said to be $L$-Lipschitz if

$$|\widehat{x}_t| \leq \sum_{s\geq 1} L_s |x_{t-s}|.$$

We more specifically consider a sequence $L$ satisfying the following assumption.

(L-1) The sequence $L = (L_s)_{s\geq 1}$ satisfies

$$L_* = \sum_{j\geq 1} L_j < \infty. \tag{2.7}$$

This condition is trivially satisfied by constant linear predictors depending only on a finite number of previous observations, that is, $\widehat{x}_t = \sum_{s=1}^{d} L_s x_{t-s}$. In Giraud, Roueff and Sanchez-Perez [(2015), Section B.1], we extend this case in the context of the TVAR process where the coefficients $L_s$ are replaced by estimates of the time varying autoregressive coefficients. More generally, assumption (L-1) appears to be quite natural in the general context where $\mathbb{E}[X_t|(X_{t-s})_{s\geq 1}] = f_t((X_{t-s})_{s\geq 1})$, where $f_t$ is a Lipschitz function from $\mathbb{R}^{\mathbb{N}^*}$ to $\mathbb{R}$, with Lipschitz coefficients satisfying a condition similar to (2.7); see, for instance, Doukhan and Wintenberger (2008) in the case of stationary time series.

We now state two upper-bounds on the mean quadratic prediction error of the aggregated predictors defined in the previous section, when the process $X$ fulfills the sub-linear property (M-1).

THEOREM 2.1. *Assume that assumption* (M-1) *holds. Let* $\{(\widehat{X}_t^{(i)})_{1\leq t\leq T}, 1 \leq i \leq N\}$ *be a collection of sequences of* $L$*-Lipschitz predictors with* $L$ *satisfying* (L-1).

(i) *Assume that the noise* $Z$ *fulfills* (N-1) *with* $p = 4$ *and let* $\widehat{X} = (\widehat{X}_t)_{1\leq t\leq T}$ *denote the aggregated predictor obtained using the weights* (2.5) *with any* $\eta > 0$. *Then we have*

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\big[(\widehat{X}_t - X_t)^2\big] \leq \inf_{\nu\in\mathcal{S}_N} \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\big[\big(\widehat{X}_t^{[\nu]} - X_t\big)^2\big] \\ + \frac{\log N}{T\eta} + 2\eta(1+L_*)^4 A_*^4 m_4.\end{aligned} \tag{2.8}$$

(ii) *Assume that the noise* $Z$ *satisfies* (N-1) *with a given* $p > 2$ *and let* $\widehat{X} = (\widehat{X}_t)_{1\leq t\leq T}$ *denote the aggregated predictor obtained using the weights* (2.6) *with any* $\eta > 0$. *Then we have*

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\big[(\widehat{X}_t - X_t)^2\big] \leq \min_{1\leq i\leq N} \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\big[\big(\widehat{X}_t^{(i)} - X_t\big)^2\big] \\ + \frac{\log N}{T\eta} + (2\eta)^{p/2-1} A_*^p (1+L_*)^p m_p.\end{aligned} \tag{2.9}$$

(iii) *Assume that the noise $Z$ fulfills* (N-2) *for some positive $\zeta$ and let $\widehat{X} = (\widehat{X}_t)_{1\le t\le T}$ denote the aggregated predictor obtained using the weights* (2.6) *with $\eta > 0$. Then, for any*

$$\lambda \in \left(0, \frac{\zeta}{a^*(L_*+1)}\right] \qquad \text{with } a^* := \sup_{j\in\mathbb{Z}}\sup_{t\in\mathbb{Z}} A_t(j) \le A_*, \tag{2.10}$$

*we have*

$$\begin{aligned}&\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\big[(\widehat{X}_t - X_t)^2\big]\\ &\le \min_{1\le i\le N}\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\big[\big(\widehat{X}_t^{(i)} - X_t\big)^2\big]\\ &\quad + \frac{\log N}{T\eta} + \frac{2}{\mathrm{e}}\lambda^{-2}\big(2+\lambda(2\eta)^{-1/2}\big)\mathrm{e}^{-\lambda(2\eta)^{-1/2}}\big(\phi(\zeta)\big)^{\lambda A_*(1+L_*)/\zeta}.\end{aligned} \tag{2.11}$$

The proof can be found in Section 4.2.

REMARK 1. The bounds (2.8), (2.9) and (2.11) are explicit in the sense that all the constants appearing in them are directly derived from those appearing in assumptions (M-1), (L-1), (N-1) and (N-2).

The following corollary is obtained by choosing $\eta$ [and $\lambda$ in the case (iii)] adequately in the three cases of Theorem 2.1.

COROLLARY 1. *Assume that assumption* (M-1) *holds. Let $\{(\widehat{X}_t^{(i)})_{1\le t\le T}, 1 \le i \le N\}$ be a collection of sequences of $L$-Lipschitz predictors with $L$ satisfying* (L-1).

(i) *Assume that the noise $Z$ fulfills* (N-1) *with $p = 4$ and let $\widehat{X} = (\widehat{X}_t)_{1\le t\le T}$ denote the aggregated predictor obtained using the weights* (2.5) *with*

$$\eta = \frac{1}{(2m_4)^{1/2}(1+L_*)^2A_*^2}\left(\frac{\log N}{T}\right)^{1/2}. \tag{2.12}$$

*This gives*

$$\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\big[(\widehat{X}_t - X_t)^2\big] \le \inf_{\nu\in\mathcal{S}_N}\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\big[\big(\widehat{X}_t^{[\nu]} - X_t\big)^2\big] + C_1\left(\frac{\log N}{T}\right)^{1/2}, \tag{2.13}$$

*with $C_1 = 2(2m_4)^{1/2}(1+L_*)^2A_*^2$.*

(ii) *Assume that the noise $Z$ satisfies* (N-1) *with a given $p > 2$ and let $\widehat{X} = (\widehat{X}_t)_{1\le t\le T}$ denote the aggregated predictor obtained using the weights* (2.6) *with*

$$\eta = \frac{1}{2m_p^{2/p}(1+L_*)^2A_*^2}\left(\frac{\log N}{T}\right)^{2/p}. \tag{2.14}$$

*We then have*

$$\begin{aligned}&\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\big[(\widehat{X}_t - X_t)^2\big]\\ &\quad\le \min_{1\le i\le N}\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\big[\big(\widehat{X}_t^{(i)} - X_t\big)^2\big] + C_2\left(\frac{\log N}{T}\right)^{1-2/p},\end{aligned} \tag{2.15}$$

*with $C_2 = 3m_p^{2/p}(1+L_*)^2A_*^2$.*

(iii) *Assume that the noise $Z$ fulfills* (N-2) *for some positive $\zeta$ and let $\widehat{X} = (\widehat{X}_t)_{1\le t\le T}$ denote the aggregated predictor obtained using the weights* (2.6) *with*

$$\eta = \frac{\zeta^2}{2(1+L_*)^2A_*^2}\left(\log\left(\frac{T}{\log N}\right)\right)^{-2}. \tag{2.16}$$

*Then we have*

$$\begin{aligned}&\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\big[(\widehat{X}_t - X_t)^2\big]\\ &\quad\le \min_{1\le i\le N}\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\big[\big(\widehat{X}_t^{(i)} - X_t\big)^2\big] + \frac{2A_*^2(L_*+1)^2}{\zeta^2}\frac{\log N}{T}\\ &\qquad\times\left\{\left(\log\left(\frac{T}{\log N}\right)\right)^2 + \frac{\phi(\zeta)}{\mathrm{e}}\left(2+\log\left(\frac{T}{\log N}\right)\right)\right\}.\end{aligned} \tag{2.17}$$

[*Note that when* $(\log N)/T \to 0$, *the term between curly brackets is equivalent to* $(\log(T/\log N))^2$.]

Cases (i) and (ii) in Corollary 1 follow directly from Theorem 2.1. Case (iii) is more delicate since it requires optimizing $\lambda$ as well as $\eta$ in the second line of (2.11). The details are postponed to Section 4.3.

REMARK 2. We observe that the bound in (2.17) improves that in (2.15) for any $p > 2$. For $p > 4$, the remaining term $(\log N/T)^{1-2/p}$ in (2.15) is smaller than the remaining term $(\log N/T)^{1/2}$ in (2.13). Similarly, the remaining term $\log N(\log T)^2/T$ in (2.17) is smaller than $(\log N/T)^{1/2}$ in (2.13). Yet, we emphasize that the oracle inequalities (2.15) and (2.17) compare the prediction risk of $\widehat{X}$ to the prediction risk of the *best predictor* $\widehat{X}^{(i)}$, while the oracle inequality (2.13) compare the prediction risk of $\widehat{X}$ to the prediction risk of the *best convex combination of the predictors* $\widehat{X}^{(i)}$, so they cannot be directly compared.

REMARK 3. As explained in Giraud, Roueff and Sanchez-Perez [(2015), Section C], under the hypotheses of cases (ii) and (iii) and for certain values of $T$ and $N$, using a more involved aggregation step, we can get a new predictor satisfying an oracle inequality better than that in (2.13). For example, under the hypotheses of case (iii), for $T > N^2(\log T)^6$, the remaining term $(\log N/T)^{1/2}$ in (2.13) can be replaced by $N(\log T)^3/T$ which is smaller; see Giraud, Roueff and Sanchez-Perez [(2015), inequality (C.7), page 8]. Yet, this aggregation has a prohibitive computational cost and seems difficult to implement in practice.

REMARK 4. In cases (ii) and (iii), which correspond to the weights (2.6), the choice of the optimal $\eta$ depends on the assumptions on the noise, namely (N-1) or (N-2). Under a moment condition of order $p$, the optimal $\eta$ is of order $(\log N/T)^{2/p}$ and under an exponential condition, it is of order $(\log T)^{-2}$. It is known from Catoni [(2004), Proposition 2.2.1] and Yang [(2004), Theorem 5] that $\eta$ can be chosen as a constant (provided that it is small enough) under a bounded noise condition, or under an exponential moment condition on the noise for predictors at a bounded distance from the conditional mean. Hence, coarsely speaking, the heavier the tail of the noise, the smallest $\eta$ should be chosen. Observing that $\eta$ allows us to tune the influence of the empirical risk on the weights from no influence at all ($\eta = 0$ yielding uniform weights) to the selection of the empirical risk minimizer ($\eta \to \infty$), the specific choices of $\eta$ can be interpreted as follows: the heavier the tail of the noise, the less we can trust the empirical risk.

*Comparison with previous works*. In the literature, prediction risk bounds of the form (2.13) [case (i) of Corollary 1] are sometimes called *convex regret bounds*, and prediction risk bounds of the form (2.15) and (2.17) [cases (ii) and (iii) of Corollary 1] are sometimes called *best predictor regret bounds*.

Sancetta (2010) exhibits convex regret bounds in a setting close to ours, namely for an online aggregation of predictors for a sequence of possibly dependent random variables. Under our moment condition (N-1) with $p = 4$, Sancetta [(2010), Theorem 2] provides an upper bound similar to (2.13) but with our remaining term $(\log N/T)^{1/2}$ replaced by $(N\log(N)/T)^{1/2}$. Under the exponential condition (N-2), Sancetta [(2010), Theorem 1] provides an upper bound similar to (2.13) but with a remaining term $(\log N/T)^{1/2} \times (\log(NT))^2$, which is still larger than our remaining term under moment conditions.

Best predictor regret bounds can be found in Yang (2004) for some sequences of possibly dependent random variables. The predictors are assumed to remain at a bounded distance to the conditional means and the scaled innovation noise is assumed to have either a known distribution (satisfying a certain technical condition) or an exponential moment. The regret bounds are presented in a slightly different fashion from ours but it is easy to see that a similar result as our bound (2.17) is obtained in this setting. However, we do not require bounded prediction errors and our conditions on the noise are milder.

The i.i.d. setting has received much more attention and, even if the setting is quite different, it is interesting to briefly compare our results to previous works in this case. Let us start with the convex regret bound in case (i) of Corollary 1. Most of the existing results [see, e.g., Juditsky and Nemirovski (2000), Yang (2000a), Tsybakov (2003) or Wang et al. (2014) for recent extensions to $\ell^q$ aggregation] assume the predictors to be bounded and various conditions on the noise are considered (very often the noise is assumed to be Gaussian). In such settings, the best possible remaining term typically takes the form $(\log N/T)^{1/2}$ when $N$ is much larger than $T^{1/2}$ and of the form $N/T$ if $N$ is smaller than $T^{1/2}$; see Juditsky and Nemirovski [(2000), Theorem 3.1], Yang [(2004), Theorem 6] and Tsybakov [(2003), Theorem 2]. Hence, our bound (2.13) is similar only in the case where $N$ is much larger than $T^{1/2}$. However, as explained in Remark 3 and [Giraud, Roueff and Sanchez-Perez (2015), Section C], when $T$ is larger than $N^2$ and under the moment condition (N-2), we can get via a more involved aggregation procedure, a convex regret bound with a remaining term of the same order $N/T$ up to a $(\log T)^3$ factor [see Giraud, Roueff and Sanchez-Perez (2015), inequality (C.7), page 8]. Let us now compare our bound (2.15) in case (ii) to optimal bounds in the i.i.d. setting under moment conditions on the noise. Corollary 7.2 and Theorem 8.6 in Audibert (2009) shows that the optimal aggregation rate is $(\log N/T)^{1-2/(p+2)}$ in the i.i.d. setting with bounded predictors and moment conditions of order $p$ on the noise. Our remaining term $(\log N/T)^{1-2/p}$ in (2.15) is slightly larger, yet an inspection of the proof of Audibert [(2009), Corollary 7.2] shows that the aggregation rate would also be $(\log N/T)^{1-2/p}$ in this corollary, if the predictors were assumed to have a moment condition of order $p$ instead of being uniformly bounded (we are not aware of any lower bound in this setting matching this rate). Finally, when the data and the predictors are bounded, the best aggregation rate is known to be $(\log N)/T$ in the i.i.d. setting; see, for example, Audibert (2009), Theorem 8.4. Our bound (2.17) in case (iii) achieves the same rate up to a $(\log T)^2$ factor.

## 3. Time-varying autoregressive (TVAR) model.

### 3.1. *Nonparametric TVAR model.*

3.1.1. *Vector norms and Hölder smoothness norms.* We introduce some preliminary notation before defining the model. In the remainder of this article, vectors are denoted using boldface symbols and $|\mathbf{x}|$ denotes the Euclidean norm of $\mathbf{x}$, $|\mathbf{x}| = (\sum_i |x_i|^2)^{1/2}$.

For $\beta \in (0, 1]$ and an interval $I \subseteq \mathbb{R}$, the $\beta$-Hölder semi-norm of a function $\mathbf{f} : I \to \mathbb{R}^d$ is defined by

$$|\mathbf{f}|_\beta = \sup_{0<|s-s'|<1} \frac{|\mathbf{f}(s) - \mathbf{f}(s')|}{|s - s'|^\beta}.$$

This semi-norm is extended to any $\beta > 0$ as follows. Let $k \in \mathbb{N}$ and $\alpha \in (0, 1]$ be such that $\beta = k + \alpha$. If $\mathbf{f}$ is $k$ times differentiable on $I$, we define

$$|\mathbf{f}|_\beta = |\mathbf{f}^{(k)}|_\alpha,$$

and $|\mathbf{f}|_\beta = \infty$ otherwise. We consider the case $I = (-\infty, 1]$. For $R > 0$ and $\beta > 0$, the $(\beta, R)$-Hölder ball is denoted by

$$\Lambda_d(\beta, R) = \{\mathbf{f} : (-\infty, 1] \to \mathbb{R}^d, \text{ such that } |\mathbf{f}|_\beta \le R\}.$$

3.1.2. *TVAR parameters in rescaled time.* The idea of using a rescaled time with the sample size $T$ for the TVAR parameters goes back to Dahlhaus (1996). Since then, it has always been a central example of locally stationary linear processes. In this setting, the time varying autoregressive coefficients and variance which generate the observations $X_{t,T}$ for $1 \le t \le T$ are represented by functions from $[0, 1]$ to $\mathbb{R}^d$ and from $[0, 1]$ to $\mathbb{R}_+$, respectively. The definition sets of these functions are extended to $(-\infty, 1]$ in the following definition.

DEFINITION 2 (TVAR model). Let $d \ge 1$. Let $\theta_1, \dots, \theta_d$ and $\sigma$ be functions defined on $(-\infty, 1]$ and $(\xi_t)_{t\in\mathbb{Z}}$ be a sequence of i.i.d. random variables with zero mean and unit variance. For any $T \ge 1$, we say that $(X_{t,T})_{t\le T}$ is a TVAR process with time varying parameters $\theta_1, \dots, \theta_d, \sigma^2$ sampled at frequency $T^{-1}$ and normalized innovations $(\xi_t)$ if the two following assertions hold:

(i) The process $X$ fulfills the time varying autoregressive equation

$$X_{t,T} = \sum_{j=1}^{d} \theta_j\left(\frac{t-1}{T}\right) X_{t-j,T} + \sigma\left(\frac{t}{T}\right)\xi_t \qquad \text{for } -\infty < t \le T. \tag{3.1}$$

(ii) The sequence $(X_{t,T})_{t\le T}$ is bounded in probability,

$$\lim_{M\to\infty} \sup_{-\infty<t\le T} \mathbb{P}(|X_{t,T}| > M) = 0.$$

This definition extends the usual definition of TVAR processes, where the time varying parameters $\theta_1, \dots, \theta_d$ and $\sigma^2$ are assumed to be constant on $\mathbb{R}_-$; see, for example, Dahlhaus [(1996), page 144]. The TVAR model is generally used for the sample $(X_{t,T})_{1\le t\le T}$. The definition of the process for negative times $t$ can be seen as a way to define initial conditions for $X_{1-d,T}, \dots, X_{0,T}$, which are then sufficient to compute $(X_{t,T})_{1\le t\le T}$ by iterating (3.1). However, in the context of prediction, it can be useful to consider predictors $\widehat{X}_{t,T}$ which may rely on historical data $X_{s,T}$ arbitrarily far away in the past, that is, with $s$ tending to $-\infty$. To cope with this situation, our definition of the TVAR process $(X_{t,T})$ holds for all time indices $-\infty < t \le T$ and we use the following definition for predictors.

DEFINITION 3 (Predictor). For all $1 \le t \le T$, we say that $\widehat{X}_{t,T}$ is a predictor of $X_{t,T}$ if it is $\mathcal{F}_{t-1,T}$-measurable, where

$$\mathcal{F}_{t,T} = \sigma(X_{s,T}, s = t, t-1, t-2, \ldots) \tag{3.2}$$

is the $\sigma$-field generated by $(X_{s,T})_{s \le t}$. For any $T \ge 1$, we denote by $\mathcal{P}_T$ the set of sequences $\widehat{X}_T = (\widehat{X}_{t,T})_{1 \le t \le T}$ of predictors for $(X_{t,T})_{1 \le t \le T}$, that is, the set of all processes $\widehat{X}_T = (\widehat{X}_{t,T})_{1 \le t \le T}$ adapted to the filtration $(\mathcal{F}_{t-1,T})_{1 \le t \le T}$.

In this general framework, the time $t = 1$ corresponds to the beginning of the aggregation procedure. Such a framework applies in two practical situations. In the first one, we start collecting data $X_{t,T}$ at $t \ge 1$ and compute several predictors $\widehat{X}^{(j)}_{t,T}$, $j = 1, \ldots, N$ from them. Thus, the resulting aggregated predictor only depends on $(X_{s,T})_{1 \le s \le t-1}$. A somewhat different situation is when historical data is available beforehand the aggregation step, so that a given predictor $\widehat{X}^{(j)}_{t,T}$ is allowed to depend also on data $X_{s,T}$ with $s \le 0$, while the aggregation step only starts at $t \ge 1$, and thus depends on the data $(X_{s,T})_{s \le 0}$ only through the predictors. It is important to note that, in contrast to the usual stationary situation, having observed the process $X_{s,T}$ for infinitely many $s$'s in the past (for all $s \le t-1$) is not so decisive for deriving a predictor of $X_{t,T}$, since observations far away in the past may have a completely different statistical behavior.

3.1.3. *Stability conditions*. The next proposition proves that under standard stability conditions on the time varying parameters $\theta_1, \ldots, \theta_d$ and $\sigma^2$, condition (ii) in Definition 2 ensures the existence and uniqueness of the solution of equation (3.1) for $t \le 0$ (and thus for all $t \le T$). We define the time varying autoregressive polynomial by

$$\boldsymbol{\theta}(z; u) = 1 - \sum_{j=1}^{d} \theta_j(u) z^j.$$

Let us denote, for any $\delta > 0$,

$$s_d(\delta) = \big\{\boldsymbol{\theta} : (-\infty, 1] \to \mathbb{R}^d, \boldsymbol{\theta}(z; u) \ne 0, \forall |z| < \delta^{-1}, u \in [0, 1]\big\}. \tag{3.3}$$

Define, for $\beta > 0$, $R > 0$, $\delta \in (0, 1)$, $\rho \in [0, 1]$ and $\sigma_+ > 0$, the class of parameters

$$\mathcal{C}(\beta, R, \delta, \rho, \sigma_+) = \big\{(\boldsymbol{\theta}, \sigma) : (-\infty, 1] \to \mathbb{R}^d \times [\rho\sigma_+, \sigma_+] : \boldsymbol{\theta} \in \Lambda_d(\beta, R) \cap s_d(\delta)\big\}.$$

The definition of the class $\mathcal{C}$ is very similar to that of Moulines, Priouret and Roueff (2005). The domain of definition in their case is $[0, 1]$ whereas it is $(-\infty, 1]$ in ours. We have the following stability result.

PROPOSITION 1. *Assume that the time varying AR coefficients $\theta_1, \dots, \theta_d$ are uniformly continuous on $(-\infty, 1]$ and the time varying variance $\sigma^2$ is bounded on $(-\infty, 1]$. Assume moreover that there exists $\delta \in (0,1)$ such that $\boldsymbol{\theta} \in s_d(\delta)$. Then there exists $T_0 \geq 1$ such that, for all $T \geq T_0$, there exists a unique process $(X_{t,T})_{t \leq T}$ which satisfies* (i) *and* (ii) *in Definition 2. This solution admits the linear representation*

$$X_{t,T} = \sum_{j=0}^{\infty} a_{t,T}(j)\sigma\left(\frac{t-j}{T}\right)\xi_{t-j}, \qquad -\infty < t \leq T, \tag{3.4}$$

*where the coefficients $(a_{t,T}(j))_{t \leq T, j \geq 0}$ satisfy that for any $\delta_1 \in (\delta, 1)$,*

$$\bar{K} = \sup_{T \geq T_0} \sup_{-\infty < t \leq T} \sup_{j \geq 0} \delta_1^{-j} |a_{t,T}(j)| < \infty.$$

*Moreover, if $(\boldsymbol{\theta}, \sigma) \in \mathcal{C}(\beta, R, \delta, 0, \sigma_+)$ for some positive constants $\beta$, $R$ and $\sigma_+$, then the constants $T_0$ and $\bar{K}$ can be chosen only depending on $\delta_1$, $\delta$, $\beta$ and $R$.*

A proof of Proposition 1 is provided in Appendix. This kind of result is classical under various smoothness assumptions on the parameters and initial conditions for $X_{1-k,T}$, $k = 1, \dots, d$. For instance, in Dahlhaus and Polonik (2009), bounded variations and a constant $\boldsymbol{\theta}$ for negative times are used for the smoothness assumption on $\boldsymbol{\theta}$ and for defining the initial conditions. The linear representation (3.4) of TVAR processes was first obtained in the seminal papers Dahlhaus (1996), Künsch (1995). We note that an important consequence of Proposition 1 is that for any $T \geq T_0$, the process $(X_{t,T})_{t \leq T}$ satisfies assumption (M-1) with $Z_t = |\xi_t|$ and $A_t(j) = |a_{t,T}(j)\sigma((t-j)/T)|$ for $j \geq 0$. Moreover, the constant $A_*$ in (2.2) is bounded independently of $T$, and we have, for all $(\boldsymbol{\theta}, \sigma) \in \mathcal{C}(\beta, R, \delta, 0, \sigma_+)$,

$$A_* \leq \frac{\bar{K}\sigma_+}{1-\delta_1}, \tag{3.5}$$

where $\bar{K} > 0$ and $\delta_1 \in (0,1)$ can be chosen only depending on $\delta$, $\beta$ and $R$.

3.1.4. *Main assumptions.* Based on Proposition 1, given an i.i.d. sequence $(\xi_t)_{t \in \mathbb{Z}}$ and constants $\delta \in (0,1)$, $\rho \in [0,1]$, $\sigma_+ > 0$, $\beta > 0$ and $R > 0$, we consider the following assumption.

(M-2) The sequence $(X_{t,T})_{t \leq T}$ is a TVAR process with time varying standard deviation $\sigma$, time varying AR coefficients $\theta_1, \dots, \theta_d$ and innovations $(\xi_t)_{t \in \mathbb{Z}}$, and $(\boldsymbol{\theta}, \sigma) \in \mathcal{C}(\beta, R, \delta, \rho, \sigma_+)$.

Let $\xi$ denote a generic random variable with the same distribution as the $\xi_t$'s. Under assumption (M-2), the distribution of $(X_{t,T})_{1-d \leq t \leq T}$ only depends on that of $\xi$ and on the functions $\boldsymbol{\theta}$ and $\sigma$. For a given distribution $\psi$ on $\mathbb{R}$ for $\xi$, we

denote by $\mathbb{P}^{\psi}_{(\boldsymbol{\theta},\sigma)}$ the probability distribution of the whole sequence $(X_{t,T})_{t\le T}$ and by $\mathbb{E}^{\psi}_{(\boldsymbol{\theta},\sigma)}$ its corresponding expectation.

The next two assumptions on the innovations are useful to prove upper bounds of the prediction error.

(I-1) The innovations $(\xi_t)_{t\in\mathbb{Z}}$ satisfy $m_p := \mathbb{E}[|\xi|^p] < \infty$.
(I-2) The innovations $(\xi_t)_{t\in\mathbb{Z}}$ satisfy $\phi(\zeta) := \mathbb{E}[\mathrm{e}^{\zeta|\xi|}] < \infty$.

The following one will be used to obtain a lower bound.

(I-3) The innovations $(\xi_t)_{t\in\mathbb{Z}}$ admit a density $f$ such that

$$\kappa = \sup_{v\neq 0} v^{-2} \int f(u) \log \frac{f(u)}{f(u+v)}\,\mathrm{d}u < \infty.$$

Assumption (I-3) is standard for proving lower bounds in nonparametric regression estimation, see Tsybakov (2009), Chapter 2. It is satisfied by Gaussian density with $\kappa = 1$.

3.1.5. *Nonparametric setting.* The setting of Definition 2 and of assumptions derived thereafter is essentially nonparametric, since for given initial distribution $\psi$, the distribution of the observations $X_{1,T}, \dots, X_{T,T}$ are determined by the unknown parameter function $(\boldsymbol{\theta}, \sigma)$. The doubly indexed $X_{t,T}$ refers to the fact that this distribution cannot be seen as a distribution on $\mathbb{R}^{\mathbb{Z}}$ marginalized on $\mathbb{R}^T$ as the usual time series setting but rather as a sequence of distributions on $\mathbb{R}^T$ indexed by $T$. It corresponds to the usual nonparametric approach for studying statistical inference based on this model. In this contribution, we focus on the prediction problem, which is to answer the question: for given smoothness conditions on $(\boldsymbol{\theta}, \sigma)$, what is the mean prediction error for predicting $X_{t,T}$ from its past? The standard nonparametric approach is to answer this question in a minimax sense by determining, for a given sequence of predictors $\widehat{X}_T = (\widehat{X}_{t,T})_{1\le t\le T}$, the maximal risk

$$\begin{aligned} &S_T(\widehat{X}_T; \psi, \beta, R, \delta, \rho, \sigma_+) \\ &\quad = \sup_{(\boldsymbol{\theta},\sigma)} \frac{1}{T} \sum_{t=1}^{T} \left( \mathbb{E}^{\psi}_{(\boldsymbol{\theta},\sigma)}[(\widehat{X}_{t,T} - X_{t,T})^2] - \sigma^2\left(\frac{t}{T}\right)\right), \end{aligned} \tag{3.6}$$

where:

(a) $\widehat{X}_T$ is assumed to belong to $\mathcal{P}_T$ as in Definition 3,
(b) the sup is taken over $(\boldsymbol{\theta}, \sigma) \in \mathcal{C}(\beta, R, \delta, \rho, \sigma_+)$ within a smoothness class of functions,
(c) the expectation $\mathbb{E}^{\psi}_{(\boldsymbol{\theta},\sigma)}$ is that associated to assumption (M-2).

The reason for subtracting the average $\sigma^2(t/T)$ over all $1 \le t \le T$ in this prediction risk is that it corresponds to the best prediction risk, would the parameters $(\boldsymbol{\theta}, \sigma)$ be exactly known. We observe that dividing $X_{t,T}$ by the class parameter $\sigma_+$ amounts to take $\sigma_+ = 1$. In addition, we have

$$S_T(\widehat{X}_T; \psi, \beta, R, \delta, \rho, \sigma_+) = \sigma_+^2 S_T\big(\widehat{X}_T \sigma_+^{-1}; \psi, \beta, R, \delta, \rho, 1\big),$$

so the prediction problem in the class $\mathcal{C}(\beta, R, \delta, \rho, \sigma_+)$ can be reduced to the prediction problem in the class $\mathcal{C}(\beta, R, \delta, \rho, 1)$. Accordingly, we define the reduced minimax risk by

$$\begin{aligned} &\overline{M}_T(\psi, \beta, R, \delta, \rho) \\ &\quad = \inf_{\widehat{X}_T \in \mathcal{P}_T} S_T(\widehat{X}_T; \psi, \beta, R, \delta, \rho, 1) \\ &\quad = \inf_{\widehat{X}_T \in \mathcal{P}_T} \sigma_+^{-2} S_T(\widehat{X}_T; \psi, \beta, R, \delta, \rho, \sigma_+) \qquad \text{for all } \sigma_+ > 0. \end{aligned} \tag{3.7}$$

In Section 3.2, we provide a lower bound of the minimax rate in the case where the smoothness class is of the form $\mathcal{C}(\beta, R, \delta, \rho, \sigma_+)$. Then, in Section 3.3, relying on the aggregation oracle bounds of Section 2.3, we derive an upper bound with the same rate as the lower bound using the same smoothness class of the parameters. Moreover, we exhibit an online predictor which does not require any knowledge about the smoothness class and which is thus minimax adaptive. In other words, it is able to adapt to the unknown smoothness of the parameters from the data. To our knowledge, such theoretical results are new for locally stationary models.

3.2. *Lower bound.* A lower bound on the minimax rate for the estimation error of $\boldsymbol{\theta}$ is given by Moulines, Priouret and Roueff [(2005), Theorem 4]. Clearly, a predictor

$$\widehat{X}_{t,T} = \sum_{k=1}^{d} \widehat{\boldsymbol{\theta}}_{t,T}(k) X_{t-k,T}$$

can be defined from an estimator $\widehat{\boldsymbol{\theta}}_{t,T}$, and the resulting prediction rate can be controlled using the estimation rate (see Giraud, Roueff and Sanchez-Perez [(2015), Section B.1] for the details). The next theorem provides a lower bound of the minimax rate of the risk of *any* predictor of the process $(X_{t,T})_{1 \le t \le T}$. Combining this result with [Giraud, Roueff and Sanchez-Perez (2015), Lemma 9], we show that a predictor obtained by [Giraud, Roueff and Sanchez-Perez (2015), equation (B.1)] from a minimax rate estimator of $\boldsymbol{\theta}$ automatically achieves the minimax prediction rate.

THEOREM 3.1. *Let* $\delta \in (0, 1)$, $\beta > 0$, $R > 0$ *and* $\rho \in [0, 1]$. *Suppose that assumption* (M-2) *holds and assume* (I-3) *on the distribution* $\psi$ *of the innovations. Then we have*

$$\liminf_{T \to \infty} T^{2\beta/(1+2\beta)} \overline{M}_T(\psi, \beta, R, \delta, \rho) > 0, \tag{3.8}$$

*where* $\overline{M}_T$ *is defined in* (3.7).

The proof is postponed to Section 5.

3.3. *Minimax adaptive forecasting of the TVAR process.* In Arkoun (2011), an adaptive estimator of the autoregressive function of a Gaussian TVAR process of order 1 is studied. It relies on the Lepskiĭ's procedure [see Lepskiĭ (1990)], which seems difficult to implement in an online context.

Our minimax adaptive predictor is based on the aggregation of sufficiently many predictors, assuming that at least one of them converges at the minimax rate. The oracle bounds found in Section 2.3 imply that the aggregated predictor is minimax rate adaptive under appropriate assumptions. Seminal works using the aggregation to adapt to the minimax convergence rate are Yang (2000a) (nonparametric regression) and Yang (2000b) (density estimation); see also Catoni (2004) for a more general presentation.

In the TVAR model (M-2), it is natural to consider $L$-Lipschitz predictors $(\widehat{X}_{t,T})_{1\le t\le T}$ of $(X_{t,T})_{1\le t\le T}$ with a sequence $L$ supported on $\{1,\dots,d\}$. Then $L^*$ in (2.7) corresponds to the maximal $\ell^1$-norm of the TVAR parameters. Since for the process itself to be stable, this norm has to be bounded independently of $T$, condition (L-1) is a quite natural assumption for the TVAR model; see Giraud, Roueff and Sanchez-Perez [(2015), Section B.1] for the details.

A practical advantage of the proposed procedures is that, given a set of predictors that behaves well under specific smoothness assumptions, we obtain an aggregated predictor which performs almost as well as or better than the best of these predictors, hence which behaves well without any prior knowledge on the smoothness of the unknown parameter. Such an adaptive property can be formally demonstrated by exhibiting an adaptive minimax rate for the aggregated predictor which coincides with the lower bound given in Theorem 3.1.

The first ingredient that we need is the following.

DEFINITION 4 [$(\psi,\beta)$-minimax-rate predictor]. Let $\psi$ be a distribution on $\mathbb{R}$ and $\beta>0$. We say that $\widehat{X}=(\widehat{X}_T)_{T\ge1}$ is a $(\psi,\beta)$-minimax-rate sequence of predictors if, for all $T\ge1$, $\widehat{X}_T\in\mathcal{P}_T$ and, for all $\delta\in(0,1)$, $R>0$, $\rho\in(0,1]$ and $\sigma_+>0$,

$$\limsup_{T\to\infty} T^{2\beta/(1+2\beta)} S_T(\widehat{X}_T;\psi,\beta,R,\delta,\rho,\sigma_+)<\infty, \tag{3.9}$$

where $S_T$ is defined by (3.6).

The term *minimax-rate* in this definition refers to the fact that the maximal rate in (3.9) is equal to the minimax lower bound (3.8) for the class $\mathcal{C}(\beta,R,\delta,\rho,\sigma_+)$. We explain in Giraud, Roueff and Sanchez-Perez [(2015), Section B] how to build such predictors which are moreover $L$-Lipschitz for some $L$ only depending on $d$.

To adapt to an unknown smoothness, we rely on a collection of $(\psi, \beta)$-minimax-rate predictors with $\beta$ within $(0, \beta_0)$, where $\beta_0$ is the (possibly infinite) maximal smoothness index.

DEFINITION 5 (Locally bounded set of $\psi$-minimax-rate predictors). Let $\psi$ be a distribution on $\mathbb{R}$ and $\beta_0 \in (0, \infty]$. We say that $\{\widehat{X}^{(\beta)}, \beta \in (0, \beta_0)\}$ is a locally bounded set of $\psi$-minimax-rate predictors if for each $\beta$, $\widehat{X}^{(\beta)}$ is a $(\psi, \beta)$-minimax-rate predictor and if moreover, for all $\delta \in (0, 1)$, $R > 0$, $\rho \in (0, 1]$, $\sigma_+ > 0$ and for each closed interval $J \subset (0, \beta_0)$,

$$\limsup_{T\to\infty} \sup_{\beta\in J} T^{2\beta/(1+2\beta)} S_T\big(\widehat{X}_T^{(\beta)}; \psi, \beta, R, \delta, \rho, \sigma_+\big) < \infty,$$

where $S_T$ is defined by (3.6).

The following lemma shows that, given a locally bounded set of minimax-rate predictors, we can always pick a finite subset of at most $N = \lceil (\log T)^2 \rceil$ predictors among which the best one achieves the minimax rate of any unknown smoothness index.

LEMMA 1. *Let $\psi$ be a distribution on $\mathbb{R}$. Let $\beta_0 \in (0, \infty]$ and $\{\widehat{X}^{(\beta)}, \beta \in (0, \beta_0)\}$ be a corresponding locally bounded set of $\psi$-minimax-rate predictors. Set, for any $N \geq 1$,*

$$\beta_i = \begin{cases} (i-1)\beta_0/N, & \text{if } \beta_0 < \infty, \\ (i-1)/N^{1/2}, & \text{otherwise}, \end{cases} \qquad 1 \leq i \leq N. \tag{3.10}$$

*Suppose moreover, in the case where $\beta_0 < \infty$, that $N \geq \lceil \log T \rceil$, and, in the case where $\beta_0 = \infty$, that $N \geq \lceil (\log T)^2 \rceil$. Then we have, for all $\beta \in (0, \beta_0)$, $\delta \in (0, 1)$, $R > 0$, $\rho > 0$ and $\sigma_+ > 0$,*

$$\limsup_{T\to\infty} T^{2\beta/(1+2\beta)} \min_{i=1,\dots,N} S_T\big(\widehat{X}_T^{(\beta_i)}; \psi, \beta, R, \delta, \rho, \sigma_+\big) < \infty.$$

The proof of this lemma is postponed to Giraud, Roueff and Sanchez-Perez [(2015), Section A.8]. Lemma 1 says that to obtain a minimax-rate predictor which adapts to an unknown smoothness index $\beta$, it is sufficient to select it judiciously among $\log T$ or $(\log T)^2$ well chosen nonadaptive minimax-rate predictors.

As a consequence of Theorem 2.1 and Lemma 1, we obtain an adaptive predictor by aggregating them (instead of selecting one of them), as stated in the following result.

THEOREM 3.2. *Let $\psi$ be a distribution on $\mathbb{R}$. Let $\beta_0 \in (0, \infty]$ and $\{\widehat{X}^{(\beta)}, \beta \in (0, \beta_0)\}$ be a locally bounded set of $\psi$-minimax-rate and $L$-Lipschitz predictors with $L$ satisfying* (L-1). *Define $(\widehat{X}_{t,T})_{1\leq t\leq T}$ as the predictor aggregated from*

*$\{\widehat{X}^{(\beta_i)}, 1 \le i \le N\}$ with $N$ defined by*

$$N = \begin{cases} \lceil \log T \rceil, & \text{if } \beta_0 < \infty, \\ \lceil (\log T)^2 \rceil, & \text{otherwise}, \end{cases} \tag{3.11}$$

*$\beta_i$ defined by* (3.10), *and with weights defined according to one of the following setting depending on the assumption on $\psi$ and $\beta_0$*:

(i) *If $\psi$ satisfies* (I-1) *with $p \ge 4$ and $\beta_0 \le 1/2$, use the weights* (2.5) *with $\eta = \sigma_+^{-2}(\log(\lceil \log T \rceil)/T)^{1/2}$.*

(ii) *If $\psi$ satisfies* (I-1) *with $p > 2$ and $\beta_0 \le (p-2)/4$, use the weights* (2.6) *with $\eta = \sigma_+^{-2}(\log(\lceil \log T \rceil)/T)^{2/p}$.*

(iii) *If $\psi$ satisfies* (I-2), *use the weights* (2.6) *with $\eta = \sigma_+^{-2}(\log T)^{-3}$.*

*Then we have, for any $\beta \in (0, \beta_0)$, $\delta \in (0,1)$, $R > 0$, $\rho \in (0,1]$ and $\sigma_+ > 0$,*

$$\limsup_{T \to \infty} T^{2\beta/(1+2\beta)} S_T(\widehat{X}_T; \psi, \beta, R, \delta, \rho, \sigma_+) < \infty. \tag{3.12}$$

The proof of this theorem is postponed to Giraud, Roueff and Sanchez-Perez [(2015), Section A.9].

REMARK 5. The limitation to $\beta_0 \le 1/2$ in (i) under assumption (I-1) for $\psi$ follows from the factor $(\log N/T)^{1/2}$ obtained in the oracle inequality (2.8) of Theorem 2.1 after optimizing in $\eta$ [see (2.13)]. If $p > 4$ this restriction is weakened to $\beta_0 \le (p-2)/4$ in (ii) taking into account the factor $(\log N/T)^{1-2/p}$ obtained in the oracle inequality (2.9) of Theorem 2.1 after optimizing in $\eta$ [see (2.15)]. In the last case, the limitation of $\beta_0$ drops when applying the oracle inequality (2.11) of the same theorem. However, a stronger condition on $\psi$ is then required.

REMARK 6. It may happen that the locally bounded set of $\psi$-minimax-rate predictors is limited to some $\beta_0 < \infty$ [see the example of the NLMS predictors in Giraud, Roueff and Sanchez-Perez (2015), Section B.2]. In this case, the result roughly needs $\log T$ predictors and the computation of the aggregated one requires less operations than if $\beta_0$ were infinite. For these reasons, we do not consider in general that $\beta_0 = \infty$. On the one hand, a finite $\beta_0$ yields a restriction on the set of (unknown) smoothness indices $\beta$ for which the aggregated predictors are minimax rate adaptive. On the other hand, if $\beta_0 = \infty$, Theorem 3.2 then requires the stronger assumption (I-2) on the process.

REMARK 7. The constant $\sigma_+^{-2}$ present in the definitions of $\eta$ in the three cases (i), (ii) and (iii) corresponds to the homogenization of the remaining terms appearing in Theorem 2.1 [the second lines of (2.8), (2.9) and (2.11)]. Indeed with the proposed choices and in the three cases, the constant $\sigma_+^2$ factors out in front of the remaining terms [see the last three displayed equations in Giraud, Roueff and

Sanchez-Perez (2015), Section A.9]. However, the $\sigma_+^{-2}$ in the definitions of $\eta$ does not impact the convergence rate in the sense that Theorem 3.2 is still valid using any other constant (1, e.g.) in these definitions.

## 4. Proofs of the upper bounds.

4.1. *Preliminary results.* We start with a lemma which gathers useful adaptations of well-known inequalities applying to the aggregation of deterministic predicting sequences.

LEMMA 2. *Let $(x_t)_{1\le t\le T}$ be a real valued sequence and $\{(\widehat{x}_t^{(i)})_{1\le t\le T}, 1\le i\le N\}$ be a collection of predicting sequences. Define $(\widehat{x}_t)_{1\le t\le T}$ as the sequence of aggregated predictors obtained from this collection with the weights (2.5). Then, for any $\eta>0$, we have*

$$\frac{1}{T}\sum_{t=1}^{T}(\widehat{x}_t - x_t)^2 \le \inf_{\nu\in\mathcal{S}_N}\frac{1}{T}\sum_{t=1}^{T}\big(\widehat{x}_t^{[\nu]} - x_t\big)^2 + \frac{\log N}{T\eta} + \frac{2\eta}{T}\sum_{t=1}^{T} y_t^4, \tag{4.1}$$

*where $y_t = |x_t| + \max_{1\le i\le N}|\widehat{x}_t^{(i)}|$.*

*Define now $(\widehat{x}_t)_{1\le t\le T}$ as the sequence of aggregated predictors obtained with the weights (2.6). Then, for any $\eta>0$, we have*

$$\begin{aligned}\frac{1}{T}&\sum_{t=1}^{T}(\widehat{x}_t - x_t)^2 \\ &\le \min_{i=1,\dots,N}\frac{1}{T}\sum_{t=1}^{T}\big(\widehat{x}_t^{(i)} - x_t\big)^2 + \frac{\log N}{T\eta} + \frac{1}{T}\sum_{t=1}^{T}\Big(y_t^2 - \frac{1}{2\eta}\Big)_+,\end{aligned} \tag{4.2}$$

*where $y_t = |x_t| + \max_{1\le i\le N}|\widehat{x}_t^{(i)}|$.*

PROOF. With the weights defined by (2.5), by slightly adapting [Stoltz (2011), Theorem 1.7], we have that

$$\frac{1}{T}\sum_{t=1}^{T}(\widehat{x}_t - x_t)^2 - \inf_{\nu\in\mathcal{S}_N}\frac{1}{T}\sum_{t=1}^{T}\big(\widehat{x}_t^{[\nu]} - x_t\big)^2 \le \frac{\log N}{T\eta} + \frac{\eta}{8T}s_T^*,$$

where $s_T^* = \sum_{t=1}^{T} s_t^2$ and $s_t = 2\max_{1\le i\le N}|2(\sum_{j=1}^{N}\widehat{\alpha}_{j,t}\widehat{x}_t^{(j)} - x_t)\widehat{x}_t^{(i)}|$. The bound (4.1) follows by using that $\{\widehat{\alpha}_{i,t}\}_{1\le i\le N}$ is in the simplex $\mathcal{S}_N$ defined in (2.4).

We now prove (4.2). We adapt the proof of Catoni [(2004), Proposition 2.2.1.] to unbounded sequences by replacing the convexity argument by the following lemma.

LEMMA 3. *Let $a > 0$ and $\mathbb{P}$ a probability distribution supported on $[-a, a]$. Then we have*

$$\int \exp(-x^2)\,\mathrm{d}\mathbb{P}(x) \le \exp\biggl(-\biggl(\int x\,\mathrm{d}\mathbb{P}(x)\biggr)^2 + \biggl(a^2 - \frac{1}{2}\biggr)_+\biggr).$$

The proof of Lemma 3 is postponed to Section A.3 in Appendix. Now, let $\eta > 0$ and $t = 1, \dots, T$. Using Lemma 3 with the probability distribution $\mathbb{P}$ defined by $\mathbb{P}(A) = \sum_{i=1}^N \widehat{\alpha}_{i,t} \mathbb{1}_A(\eta^{1/2}(\widehat{x}_t^{(i)} - x_t))$ and $a = \eta^{1/2} y_t$, we get that

$$\sum_{i=1}^N \widehat{\alpha}_{i,t} \exp\bigl(-\eta(\widehat{x}_t^{(i)} - x_t)^2\bigr) \le \exp\biggl(-\eta(\widehat{x}_t - x_t)^2 + \eta\biggl(y_t^2 - \frac{1}{2\eta}\biggr)_+\biggr).$$

Taking the log, multiplying by $-\eta^{-1}$ and re-ordering the terms, we obtain that

$$\begin{aligned}(\widehat{x}_t - x_t)^2 &\le -\frac{1}{\eta}\log\biggl(\sum_{j=1}^N \widehat{\alpha}_{i,t} \exp\bigl(-\eta(\widehat{x}_t^{(i)} - x_t)^2\bigr)\biggr) \\ &\quad + \biggl(y_t^2 - \frac{1}{2\eta}\biggr)_+.\end{aligned}$$

Taking the average over $t = 1, \dots, T$ and developing the expression of $\widehat{\alpha}_{i,t}$, we obtain

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^T (x_t - \widehat{x}_t)^2 &\le -\frac{1}{\eta T}\log\biggl(\frac{1}{N}\sum_{i=1}^N \exp\biggl(-\eta\sum_{t=1}^T (\widehat{x}_t^{(i)} - x_t)^2\biggr)\biggr) \\ &\quad + \frac{1}{T}\sum_{t=1}^T \biggl(y_t^2 - \frac{1}{2\eta}\biggr)_+.\end{aligned} \tag{4.3}$$

Using that $\sum_{i=1}^N \exp(-\eta\sum_{t=1}^T (\widehat{x}_t^{(i)} - x_t)^2) \ge \exp(-\eta \min_{i=1,\dots,N} \sum_{t=1}^T (\widehat{x}_t^{(i)} - x_t)^2)$, we get the bound (4.2). □

4.2. *Proof of Theorem 2.1.* We prove the cases (i), (ii) and (iii) successively. We denote $Y_t = |X_t| + \max_{1 \le i \le N} |\widehat{X}_t^{(i)}|$.

*Case* (i). Applying (4.1) in Lemma 2 with $\mathbb{E}[\inf \cdots] \le \inf \mathbb{E}[\cdots]$, we obtain

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^T \mathbb{E}\bigl[(\widehat{X}_t - X_t)^2\bigr] &\le \inf_{\boldsymbol{\nu} \in \mathcal{S}_N} \frac{1}{T}\sum_{t=1}^T \mathbb{E}\bigl[\bigl(\widehat{X}_t^{[\boldsymbol{\nu}]} - X_t\bigr)^2\bigr] \\ &\quad + \frac{\log N}{T\eta} + \frac{2\eta}{T}\sum_{t=1}^T \mathbb{E}\bigl[Y_t^4\bigr].\end{aligned} \tag{4.4}$$

Using that the predictors are $L$-Lipschitz and the process $(X_t)_{t\in\mathbb{Z}}$ satisfies (M-1), we have, for all $1 \le t \le T$,

$$\begin{aligned} Y_t &= |X_t| + \max_{1\le i\le N} |\widehat{X}_t^{(i)}| \\ &\le \sum_{j\in\mathbb{Z}} A_t(j) Z_{t-j} + \sum_{s\ge 1}\sum_{j\in\mathbb{Z}} L_s A_{t-s}(j) Z_{t-s-j} \\ &\le \sum_{j\in\mathbb{Z}} B_t(j) Z_{t-j}, \end{aligned} \tag{4.5}$$

where

$$B_t(j) = A_t(j) + \sum_{s\ge 1} L_s A_{t-s}(j-s).$$

Applying the Minkowski inequality together with (4.5), (2.2) and (2.7), we obtain, for all $1 \le t \le T$,

$$\mathbb{E}\big[Y_t^4\big] \le \mathbb{E}\bigg[\bigg(\sum_{j\in\mathbb{Z}} B_t(j) Z_{t-j}\bigg)^4\bigg] \le A_*^4 (1+L_*)^4 \sup_{t\in\mathbb{Z}} \mathbb{E}\big[Z_t^4\big].$$

Since the process $Z$ fulfills (N-1) with $p = 4$, plugging this bound in (4.4) we obtain (2.8).

*Case* (ii). We use (4.2) in Lemma 2 and the inequality $(x^2 - 1/(2\eta))_+ \le (2\eta)^{p/2-1} x^p$ which holds for $x \ge 0$ and $p \ge 2$. We get, taking the expectation,

$$\begin{aligned} \frac{1}{T}\sum_{t=1}^T \mathbb{E}\big[(\widehat{X}_{t,T} - X_{t,T})^2\big] \le \min_{i=1,\dots,N} \frac{1}{T}\sum_{t=1}^T \mathbb{E}\big[\big(\widehat{X}_{t,T}^{(i)} - X_{t,T}\big)^2\big] + \frac{\log N}{T\eta} \\ + (2\eta)^{p/2-1} \max_{t=1,\dots,T} \mathbb{E}\big[Y_t^p\big]. \end{aligned} \tag{4.6}$$

Applying the Minkowski inequality, (4.5) and assumption (N-2),

$$\mathbb{E}\big[Y_t^p\big] \le \bigg(\sum_{j\in Z} B_t(j) \big(\mathbb{E}\big[Z_{t-j}^p\big]\big)^{1/p}\bigg)^p \le A_*^p (1+L_*)^p \sup_{t\in Z} \mathbb{E}\big[Z_t^p\big].$$

Using this bound which is independent of $t$, with (N-1) and (4.6), the inequality (2.9) follows.

*Case* (iii). To obtain (2.11), we again use (4.2) in Lemma 2 but now with an exponential bound for $(Y_t^2 - 1/(2\eta))_+$. We note that, or all $u > 0$,

$$\sup_{x\ge 1} (x^2-1)\mathrm{e}^{-ux} = (x_0^2-1)\mathrm{e}^{-ux_0} \qquad \text{with } x_0 = u^{-1}\big(1 + \big(1+u^2\big)^{1/2}\big).$$

It follows that, for all $x \in \mathbb{R}$ and $u > 0$,

$$\big(x^2-1\big)_+ \le \mathrm{e}^{ux}\big(x_0^2-1\big)\mathrm{e}^{-ux_0} \le \mathrm{e}^{ux} 2u^{-2}(2+u)\mathrm{e}^{-1-u}.$$

Applying this bound with $x = (2\eta)^{1/2} Y_t$ and $u = \lambda(2\eta)^{-1/2}$ we get

$$\left(Y_t^2 - \frac{1}{2\eta}\right)_+ = (2\eta)^{-1}(x^2-1)_+ \le 2\lambda^{-2}\bigl(2+\lambda(2\eta)^{-1/2}\bigr)\mathrm{e}^{-1-\lambda(2\eta)^{-1/2}}\mathrm{e}^{\lambda Y_t}.$$

Plugging this into (4.2) and taking the expectation, we obtain that

$$\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\bigl[(\widehat{X}_{t,T} - X_{t,T})^2\bigr] \\
&\le \min_{i=1,\dots,N}\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\bigl[\bigl(\widehat{X}_{t,T}^{(i)} - X_{t,T}\bigr)^2\bigr] + \frac{\log N}{T\eta} \\
&\quad + 2\lambda^{-2}\bigl(2+\lambda(2\eta)^{-1/2}\bigr)\mathrm{e}^{-1-\lambda(2\eta)^{-1/2}}\max_{t=1,\dots,T}\mathbb{E}\bigl[\mathrm{e}^{\lambda Y_t}\bigr].
\end{aligned} \tag{4.7}$$

We now use assumption (N-2). Since $B_t(j) \le a^*(1+L_*)$ for all $j, t \in \mathbb{Z}$ and

$$\sum_{j\in\mathbb{Z}} B_t(j) \le A_*(1+L_*),$$

Jensen's inequality and (4.5) gives that, for any $\lambda \le \zeta/(a^*(1+L_*))$,

$$\begin{aligned}
\mathbb{E}\bigl[\mathrm{e}^{\lambda Y_t}\bigr] &\le \mathbb{E}\bigl[\mathrm{e}^{\lambda(|X_t| + \max_{1\le i\le N}|\widehat{X}_t^{(i)}|)}\bigr] \\
&\le \prod_{j\in\mathbb{Z}}\mathbb{E}\bigl[\mathrm{e}^{\lambda B_t(j) Z_{t-j}}\bigr] \\
&\le \prod_{j\in\mathbb{Z}}\bigl(\phi(\zeta)\bigr)^{\lambda B_t(j)/\zeta} \le \bigl(\phi(\zeta)\bigr)^{\lambda A_*(1+L_*)/\zeta}.
\end{aligned}$$

The combination of this bound with (4.7) gives (2.11). The proof of Theorem 2.1 is complete.

4.3. *Proof of case* (iii) *in Corollary* 1. Minimizing the sum of the two terms appearing in the second line of (2.11) is a bit more involved, since it depends both on $\eta$ and $\lambda$. Under condition (2.10), the quantity $(\phi(\zeta))^{\lambda A_*(1+L_*)/\zeta}$ remains between two positive constants while, for any $\eta > 0$, $\lambda^{-2}(2+\lambda(2\eta)^{-1/2})$ is decreasing as $\lambda$ increases. To simplify $(\phi(\zeta))^{\lambda A_*(1+L_*)/\zeta}$ into $\phi(\zeta)$, we simply take

$$\lambda = \frac{\zeta}{A_*(1+L_*)},$$

which satisfies (2.10). Now that $\lambda$ is set, it remains to choose a value of $\eta$ which (almost) minimizes

$$\frac{\log N}{T\eta} + \frac{2\phi(\zeta)}{\mathrm{e}}\lambda^{-2}\bigl(2+\lambda(2\eta)^{-1/2}\bigr)\mathrm{e}^{-\lambda(2\eta)^{-1/2}}.$$

The $\eta$ defined as in (2.16) is chosen so that $(\log N)/T = \mathrm{e}^{-\lambda(2\eta)^{-1/2}}$, and we get (2.17).

**5. Proof of the lower bound.** We now provide a proof of Theorem 3.1. We consider an autoregressive equation of order one

$$X_{t,T} = \theta\Big(\frac{t-1}{T}\Big) X_{t-1,T} + \xi_t, \tag{5.1}$$

where $(\xi_t)_{t\in\mathbb{Z}}$ is i.i.d. with density $f$ as in (I-3). In this case, if $\sup_{u\le 1} |\theta(u)| < 1$, the representation (3.4) of the stationary solution reads, for all $t \le T$ as

$$X_{t,T} = \sum_{j=0}^{\infty} \prod_{s=1}^{j} \theta\Big(\frac{t-s}{T}\Big) \xi_{t-j}, \tag{5.2}$$

with the convention $\prod_{s=1}^{0} \theta((t-s)/T) = 1$. The class of models so defined with $\theta \in \Lambda_1(\beta, R) \cap s_1(\delta)$ corresponds to assumption (M-2) with $(\boldsymbol{\theta}, \sigma)$ in $\mathcal{C}(\beta, R, \delta, \rho, 1)$ such that only the first component of $\boldsymbol{\theta}$ is nonzero and $\sigma$ is constant and equal to one.

We write henceforth in this proof $\mathbb{P}_\theta$ for the law of the process $X = (X_{t,T})_{t\le T, T\ge 1}$ and $\mathbb{E}_\theta$ for the corresponding expectation.

Let $\widehat{X} = (\widehat{X}_{t,T})_{1\le t\le T}$ be any predictor of $(X_{t,T})_{1\le t\le T}$ in the sense of Definition 3. Define $\widehat{\theta} = (\widehat{\theta}_{t,T})_{0\le t\le T-1} \in \mathbb{R}^T$ by

$$\widehat{\theta}_{t,T} = \begin{cases} \widehat{X}_{t+1,T}/X_{t,T}, & \text{if } X_{t,T} \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

For any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^T$, we define

$$d_X(\mathbf{u}, \mathbf{v}) = \left( \frac{1}{T} \sum_{t=0}^{T-1} X_{t,T}^2 (u_t - v_t)^2 \right)^{1/2}. \tag{5.3}$$

By (5.1), since $X_{t,T}$ and $\widehat{\theta}_{t,T}$ are $\mathcal{F}_{t,T}$-measurable, they are independent of $\xi_{t+1}$ and we have

$$\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_\theta\big[(\widehat{X}_{t,T} - X_{t,T})^2\big] - 1 = \mathbb{E}_\theta\big[d_X^2\big(\widehat{\theta}, \mathrm{v}_T\{\theta\}\big)\big],$$

where, for any $\theta : (-\infty, 1] \to \mathbb{R}$, $\mathrm{v}_T\{\theta\} \in \mathbb{R}^T$ denotes the $T$-sample of $\theta$ on the regular grid $0, 1/T, \dots, (T-1)/T$,

$$\mathrm{v}_T\{\theta\} = \Big(\theta\Big(\frac{t}{T}\Big)\Big)_{0\le t\le T-1}.$$

Hence, to prove the lower bound of Theorem 3.1, it is sufficient to show that there exist $\theta_0, \dots, \theta_M \in \Lambda_1(\beta, R) \cap s_1(\delta)$, $c > 0$ and $T_0 \ge 1$ both depending only on $\delta$, $\beta$, $R$ and the density $f$, such that for any $\widehat{\theta} = (\widehat{\theta}_{t,T})_{0\le t\le T-1}$ adapted to $(\mathcal{F}_{t,T})_{0\le t\le T-1}$ and $T \ge T_0$, we have

$$\max_{j=0,\dots,M} \mathbb{E}_{\theta_j}\big[d_X^2\big(\widehat{\theta}, \mathrm{v}_T\{\theta_j\}\big)\big] \ge c T^{-2\beta/(2\beta+1)}. \tag{5.4}$$

We now face the more standard problem of providing a lower bound for the minimax rate of an estimation error, since $\widehat{\theta}$ is an estimator of $\mathrm{v}_T\{\theta\}$. The path for deriving such a lower bound is explained in [Tsybakov (2009), Chapter 2]. However, we have to deal with a loss function $d_X$ which depends on the observed process $X$. Not only the loss function is random, but it is also not independent of the estimator $\widehat{\theta}$. The proof of the lower bound (5.4) thus requires nontrivial adaptations. It relies on some intermediate lemmas.

LEMMA 4. *We write $\mathcal{K}(\mathbb{P}, \mathbb{P}')$ for the Kullback–Leibler divergence between $\mathbb{P}$ and $\mathbb{P}'$. For any functions $\theta_0, \dots, \theta_M$ from $[0, 1]$ to $\mathbb{R}$ such that*

$$\max_{j=0,\dots,M} \mathcal{K}(\mathbb{P}_{\theta_j}, \mathbb{P}_{\theta_0}) \le \frac{2\mathrm{e}}{2\mathrm{e}+1} \log(1+M) \tag{5.5}$$

*and any $r > 0$ we have*

$$\max_{j=0,\dots,M} \mathbb{E}_{\theta_j}\big[d_X^2\big(\widehat{\theta}, \mathrm{v}_T\{\theta_1\}\big)\big] \ge \frac{r^2}{4}\bigg(\frac{1}{2\mathrm{e}+1} - \max_{j=0,\dots,M} \mathbb{P}_{\theta_j}\Big(\min_{i:i\neq j} d_{X,T}(\theta_i, \theta_j) \le r\Big)\bigg),$$

*where we denote, for any two functions $\theta, \theta'$ from $(-\infty, 1]$ to $\mathbb{R}$,*

$$d_{X,T}(\theta, \theta') = d_X\big(\mathrm{v}_T\{\theta\}, \mathrm{v}_T\{\theta'\}\big).$$

The proof is postponed to Section A.4 in Appendix.

We next construct certain functions $\theta_0, \dots, \theta_M \in \Lambda_1(\beta, R) \cap s_1(\delta)$ fulfilling (5.5) and well spread in terms of the pseudo-distance $d_{X,T}$. Consider the infinitely differentiable kernel $K$ defined by

$$K(u) = \exp\bigg(-\frac{1}{1-4u^2}\bigg)\mathbb{1}_{|u|<1/2}.$$

Given any $m \ge 8$, Vershamov–Gilbert's lemma [Tsybakov (2009), Lemma 2.9] ensures the existence of $M+1$ points $w^{(0)}, \dots, w^{(M)}$ in the hypercube $\{0,1\}^m$ such that

$$M \ge 2^{m/8}, \qquad w^{(0)} = 0 \quad \text{and} \quad \mathrm{card}\big\{\ell : w_\ell^{(j)} \neq w_\ell^{(i)}\big\} \ge m/8 \quad \text{for all } j \neq i. \tag{5.6}$$

We then define $\theta_0, \dots, \theta_M$ by setting, for all $x \le 1$,

$$\theta_j(x) = \frac{R_0}{m^\beta} \sum_{\ell=1}^{m} w_l^{(j)} K\bigg(mx - \ell + \frac{1}{2}\bigg) \qquad \text{for } j = 0, \dots, M, \tag{5.7}$$

where

$$R_0 = \min\bigg(\delta, \frac{R}{(2|K|_\beta)}\bigg). \tag{5.8}$$

Since $K = 0$ out of $(-1/2, 1/2)$, we observe that

$$\theta_j(x) = 0 \qquad \text{for all } x \le 0, \tag{5.9}$$

and

$$\theta_j(x) = \frac{R_0}{m^\beta} w^{(j)}_{\lfloor mx \rfloor + 1} K\left(\{mx\} - \frac{1}{2}\right) \qquad \text{for all } x \in [0, 1], \tag{5.10}$$

where $\{mx\} = mx - \lfloor mx \rfloor$ denotes the fractional part of $mx$. Thus, we have

$$\theta^* := \max_{0 \le j \le M} \sup_{x \in [0,1]} |\theta_j(x)| \le \frac{R_0 \mathrm{e}^{-1}}{m^\beta} \le \delta < 1. \tag{5.11}$$

We first check that the definition of $R_0$ ensures that the $\theta_j$'s are in the expected set of parameters.

LEMMA 5. *For all $j = 0, \dots, M$, we have $\theta_j \in \Lambda_1(\beta, R) \cap s_1(\delta)$.*

The proof can be found in Section A.5 of Appendix.

Next, we provide a bound to check the required condition (5.5) on the chosen $\theta_j$'s.

LEMMA 6. *For all $j = 1, \dots, M$, we have*

$$\mathcal{K}(\mathbb{P}_{\theta_j}, \mathbb{P}_{\theta_0}) \le \frac{8\mathrm{e}^{-2}\kappa R_0^2}{(1-\delta^2)\log 2} \frac{T}{m^{1+2\beta}} \log(1+M),$$

*where $\kappa$ is the constant appearing in* (I-3).

We prove it in Section A.6 of Appendix.

Finally, we need a control on the distances $d^2_{X,T}(\theta_i, \theta_j)$.

LEMMA 7. *For any $\varepsilon > 0$, there exists a constant $A$ depending only on $\varepsilon$ and the density $f$ of $\xi$ such that for all $m \ge 16$, $T \ge 4m$ and $j = 0, \dots, M$,*

$$\mathbb{P}_{\theta_j}\left(\min_{i : i \ne j} d^2_{X,T}(\theta_i, \theta_j) \le A \frac{R_0^2}{m^{2\beta}}\right) \le \varepsilon + \frac{2R_0 \mathrm{e}^{-3}}{A(1-\delta)m^\beta}. \tag{5.12}$$

The proof is postponed to Section A.7 of Appendix.

We can now complete the proof of Theorem 3.1.

PROOF OF THEOREM 3.1. Recall that $\theta_0, \dots, \theta_M$ in (5.7) are some parameters only depending on $\beta$ and $\delta$ and a certain integer $m \ge 8$ and that, whatever the value of $m$, Lemma 5 insures that $\theta_0, \dots, \theta_M$ belongs to $\Lambda_1(\beta, R) \cap s_1(\delta)$.

Hence, it is now sufficient to show that (5.4) holds for a correct choice of $m$, relying on Lemmas 4, 6 and 7. Let us set

$$m = \max\{\lceil c_0 T^{1/(2\beta+1)} \rceil, 16\}, \tag{5.13}$$

where $c_0$ is a constant to be chosen. Then $Tm^{-1-2\beta} \le c_0^{-1-2\beta}$ and, by Lemma 6, we can choose $c_0$ only depending on $\beta$, $R$, $\kappa$ and $\delta$ so that condition (5.5) of Lemma 4 is met. We thus get that, for any $r > 0$,

$$\max_{j=0,\dots,M} \mathbb{E}_{\theta_j}\big[d_X^2\big(\widehat{\theta}, \mathrm{v}_T\{\theta_j\}\big)\big] \ge \frac{r^2}{4}\bigg(\frac{1}{2\mathrm{e}+1} - \max_{j=0,\dots,M} \mathbb{P}_{\theta_j}\Big(\min_{i:i\neq j} d_{X,T}(\theta_i,\theta_j) \le r\Big)\bigg).$$

Applying Lemma 7 with $\varepsilon = 1/(4\mathrm{e}+2)$ and the previous bound with $r^2 = AR_0^2 m^{-2\beta}$, we get, as soon as $T \ge 4m$,

$$\max_{j=0,\dots,M} \mathbb{E}_{\theta_j}\big[d_X^2\big(\widehat{\theta}, \mathrm{v}_T\{\theta_j\}\big)\big] \ge \frac{r^2}{4}\bigg(\frac{1}{4\mathrm{e}+2} - \frac{2R_0\mathrm{e}^{-1}}{A(1-\delta)m^\beta}\bigg).$$

The proof is concluded by observing that, as a consequence of (5.13), we can choose a constant $T_0$ only depending on $\beta$, $R$, $\kappa$ and $\delta$ such that $T \ge T_0$ implies that $T \ge 4m$ and that the term between parentheses is bounded by $1/(8\mathrm{e}+4)$ from below. □

**6. Numerical experiments.** In this section, we test the proposed aggregation methods on data simulated according to a TVAR process with $d = 3$. The choice of a smooth parameter function $t \mapsto \boldsymbol{\theta}(t)$ within $s_d(\delta)$ for some $\delta \in (0,1)$ is done by first picking randomly some smoothly time varying partial autocorrelation functions up to the order $d$ that are bounded between $-1$ and 1 and then by relying on the Levinson–Durbin algorithm. We show the three components of the obtained $\boldsymbol{\theta}(t)$ on $t \in [0,1]$ in the top parts of Figure 1. Realizations of the TVAR process are then obtained from an innovation sequence $(\xi_t)_{t\in\mathbb{Z}}$ of i.i.d. centered Gaussian process with unit variance as in Definition 2 by sampling $\boldsymbol{\theta}$ at a given rate $T \ge 1$. Figure 1 displays one realization of such a TVAR process for $T = 2^{10}$.

The NLMS algorithm [see Giraud, Roueff and Sanchez-Perez (2015), Algorithm 1] studied in Moulines, Priouret and Roueff (2005) provides an online estimator of $\boldsymbol{\theta}$ depending on a gradient step size $\mu$. For any $\beta \in (0,1]$, choosing $\mu \propto T^{-2\beta/(2\beta+1)}$ yields a $\mathcal{C}(\beta, R, \delta, \rho, 1)$-minimax-rate online $L$-Lipschitz predictor as explained in Giraud, Roueff and Sanchez-Perez (2015), Section B.1. Hence, proceeding as in Lemma 1 to define $N$ and $\beta_i$, $i = 1,\dots,N$, with $\beta_0 = 0.5$, we obtain a finite set of NLMS predictors corresponding to gradient step sizes $\mu_1 > \cdots > \mu_N$. This set of predictors is aggregated in two possible ways according to the online Algorithm 1 with the specifications on $\eta$ and $N$ given in Theorem 3.2. The overall running time of $T$ iterates of the algorithm leading to the aggregated predictors from the data $X_1,\dots,X_T$ is then $O(dNT)$. Since the algorithm is recursive, the corresponding required storage capacity is $O(dN)$.

We evaluate the obtained NLMS predictors and their aggregated predictors by running 1000 simulations based on equally distributed realizations of the above

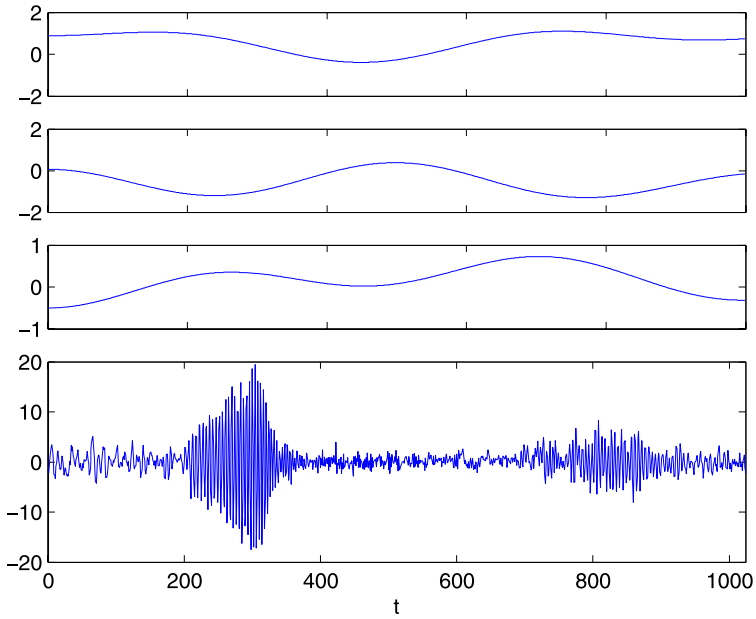


FIG. 1. *The first three plots represent* $\theta_1$, $\theta_2$ *and* $\theta_3$ *on the interval* $[0, 1]$. *The last plot displays* $T = 2^{10}$ *samples of the corresponding TVAR process with Gaussian innovations*.

Gaussian TVAR process in the case $T = 2^{10}$ which yields $N = 7$. In Figure 2, we compare the averaged downward shifted empirical losses defined for any predictor $(\widehat{X}_{t,T})_{1\le t\le T}$ by

$$L_T = \frac{1}{T}\sum_{t=1}^{T}\left((\widehat{X}_{t,T} - X_{t,T})^2 - \sigma^2\left(\frac{t}{T}\right)\right).$$

This empirical averaged loss mimics the risk considered in (3.6).

We observe that the best NLMS predictor is the third one while the aggregated predictor of strategy 1 enjoys a smaller loss and that of strategy 2 a slightly larger one. This is in accordance with Theorem 2.1(i) and (iii) where it is shown that the aggregated predictor of the first strategy may outperform the best predictor as it nearly achieves the loss of the best possible convex combination of the original predictors while the aggregated predictor of the second strategy nearly achieves the loss of the best original predictor.

## APPENDIX: POSTPONED PROOFS

**A.1. A useful lemma.** The following lemma provides a uniform bound on the norm of a product of matrices sampled from a continuous function defined on an interval $I$ and valued in a set of $d \times d$ matrices with bounded spectral radius and norm.

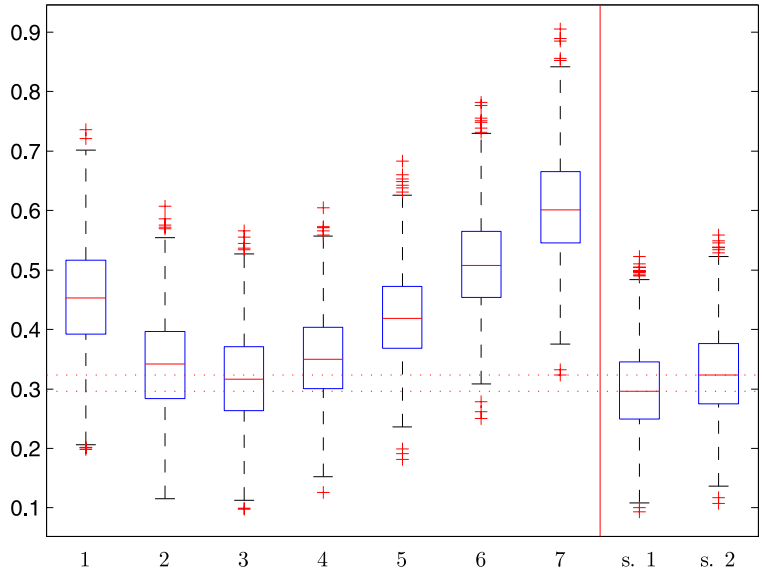


FIG. 2. *The seven boxplots on the left of the vertical red line correspond to the averaged downward shifted empirical losses $L_T$ of the NLMS predictors $\widehat{X}^{(1)}, \ldots, \widehat{X}^{(7)}$. The ones on the right of the same line are those associated to the aggregated predictors using the weights* (2.5) *and* (2.6).

LEMMA 8. *Let $d \geq 1$ and $I$ an interval of $\mathbb{R}$. Let $A$ be a function defined on $I$ taking values in the set of $d \times d$ matrices with eigenvalues moduli at most equal to $\delta$. Let $|\cdot|$ be any matrix norm. Denote by $A^*$ the corresponding uniform norm of $A$,*

$$A^* = \sup_{t \in I} |A(t)|,$$

*and, for any $h > 0$, $\omega_h(A, I)$ the modulus of continuity of $A$ over $I$,*

$$\omega_h(A; I) = \sup\{|A(t) - A(s)| : s, t \in I, |s - t| \leq h\}.$$

*Let $\delta_1 > \delta$ and assume that $A^* < \infty$. Then there exist some positive constants $\varepsilon$, $\ell$ and $K$ only depending on $A^*$, $\delta$ and $\delta_1$ such that, for any $h \in (0, 1)$ fulfilling $\omega_h(A; I) \leq \varepsilon$, we have, for all $s < t$ in $I$ and all integer $p \geq \ell(t - s)/h$,*

$$\underbrace{\left| A(t) A\left(t - \frac{t-s}{p}\right) A\left(t - \frac{2(t-s)}{p}\right) \cdots A(s) \right|}_{p+1 \text{ terms}} \leq K \delta_1^{p+1}. \tag{A.1}$$

PROOF. Denote by $\Pi(s, t; p)$ the product of matrices appearing in the left-hand side of (A.1). The proof goes along the same lines as [Moulines, Priouret and Roueff (2005), Proposition 13] but we use the modulus of continuity instead of the $\beta$-Lipschitz norm to control the local oscillation of matrices.

For $\ell_1 \geq 1$ and any square matrices $A_1, \ldots, A_{\ell_1}$, adopting the convention $\prod_{i=i_1}^{i_2} A_i = A_{i_1} \cdots A_{i_2}$ if $i_1 \leq i_2$ and $\prod_{i=i_1}^{i_2} A_i$ is the identity matrix if $i_1 > i_2$, we have

$$\begin{aligned}\prod_{k=1}^{\ell_1} A_k &= A_1^{\ell_1} + \sum_{k=1}^{\ell_1-1} \Bigg( A_1^{\ell_1-k} \prod_{i=\ell_1-k+1}^{\ell_1} A_i - A_1^{\ell_1-(k-1)} \prod_{i=\ell_1-k+2}^{\ell_1} A_i \Bigg) \\ &= A_1^{\ell_1} + \sum_{k=1}^{\ell_1-1} A_1^{\ell_1-k} (A_{\ell_1-k+1} - A_1) \prod_{i=\ell_1-k+2}^{\ell_1} A_i.\end{aligned} \tag{A.2}$$

Given a positive integer $\ell$, using the Euclidean division of $p+1$ by $\ell$, $p+1 = \ell q + r$, we decompose the product $\Pi(s,t;p)$ as

$$\begin{aligned}\Pi(s,t;p) = \prod_{j=0}^{q-1} \Bigg( \prod_{k=1}^{\ell} A\Bigg( t - \frac{(j\ell+k-1)(t-s)}{p} \Bigg) \Bigg) \\ \times \prod_{k=1}^{r} A\Bigg( t - \frac{(q\ell+k-1)(t-s)}{p} \Bigg).\end{aligned} \tag{A.3}$$

Using (A.2), we have for any $h \geq \ell(t-s)/p$, $0 \leq j \leq q$ and $0 \leq \ell_1 \leq \ell$,

$$\begin{aligned}&\Bigg| \prod_{k=1}^{\ell_1} A\Bigg( t - \frac{(j\ell+k-1)(t-s)}{p} \Bigg) \Bigg| \\ &\quad \leq \Bigg| \Bigg( A\Bigg( t - \frac{j\ell(t-s)}{p} \Bigg) \Bigg)^{\ell_1} \Bigg| + (\ell_1 - 1)\big(A^*\big)^{\ell_1-1} \omega_h(A; I).\end{aligned} \tag{A.4}$$

Take an arbitrary $\delta_2 \in (\delta, \delta_1)$ (say the middle point). The eigenvalues of $A$ are at most $\delta$ on $I$ and $A^* < \infty$. Applying [Moulines, Priouret and Roueff (2005), Lemma 12] we obtain that there is a constant $K_1 \geq 1$ only depending on $\delta$, $\delta_2$ and $A^*$ such that $|(A(t - j\ell(t-s)/p))^{\ell_1}| \leq K_1 \delta_2^{\ell_1}$.

From (A.3) and (A.4), we derive the following inequality:

$$\big|\Pi(s,t;p)\big| \leq \big(K_1\delta_2^{\ell} + K_2\omega_h(A;I)\big)^q \big(K_1\delta_2^{r} + K_2\omega_h(A;I)\big),$$

where $K_2 = (\ell-1)(\max\{A^*, 1\})^{\ell-1}$.

We can choose a positive integer $\ell$ and a positive number $\varepsilon_0$ only depending on $\delta_2$, $\delta_1$ and $K_1$ such that

$$K_1\delta_2^{\ell} \leq \delta_1^{\ell} - \varepsilon_0.$$

In the following, we set $\varepsilon = \varepsilon_0/K_2$. The previous bound gives that for any $h \in (0,1)$ such that $\omega_h(A;I) \leq \varepsilon$ and $\ell(t-s)/p \leq h$,

$$\begin{aligned}\big|\Pi(s,t;p)\big| &\leq \delta_1^{\ell q}\big(K_1\delta_2^{r} + \varepsilon_0\big) \leq K_1\delta_1^{p+1} + \varepsilon_0\delta_1^{\ell q} \\ &\leq \big(K_1 + \varepsilon_0 \max\{1, \delta_1^{1-\ell}\}\big)\delta_1^{p+1}.\end{aligned}$$

Hence, we have the result. □

**A.2. Proof of Proposition 1.** We can now provide a proof of Proposition 1. Equation (3.1) can be more compactly written as

$$X_{t,T} = \boldsymbol{\theta}'\left(\frac{t-1}{T}\right)\mathbf{X}_{t-1,T} + \sigma\left(\frac{t}{T}\right)\xi_{t,T}. \tag{A.5}$$

For all $k \geq 0$, iterating this recursive equation $k$ times, we have

$$\begin{aligned} X_{t,T} &= \mathbf{e}_1'\left[\prod_{i=1}^{k+1} A\left(\frac{t-i}{T}\right)\right]\mathbf{X}_{t-k-1,T} \\ &\quad + \sum_{j=0}^{k} \sigma\left(\frac{t-j}{T}\right)\mathbf{e}_1'\left[\prod_{i=1}^{j} A\left(\frac{t-i}{T}\right)\right]\mathbf{e}_1\xi_{t-j}, \end{aligned} \tag{A.6}$$

where $\mathbf{e}_1 = [1 \;\; 0 \cdots 0]'$ and

$$A(u) = \begin{bmatrix} \theta_1(u) & \theta_2(u) & \cdots & \cdots & \theta_d(u) \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

Note that the eigenvalues of $A(u)$ are the reciprocals of the roots of the local time varying autoregressive polynomial $z \mapsto \boldsymbol{\theta}(z; u)$, and thus are at most $\delta < 1$. Moreover, since $\boldsymbol{\theta}$ is bounded by a constant only depending on $d$ and is uniformly continuous on $I = (-\infty, 1]$, so is $A$ as a function defined on $I$ and we can find $h \in (0, 1)$ such that $\omega_h(A, I) \leq \varepsilon$ for any positive $\varepsilon$. If $\boldsymbol{\theta} \in \Lambda_d(\beta, R)$, this $h$ can be chosen depending only on $\varepsilon$, $\beta$ and $R$ (and also on the matrix norm $|\cdot|$).

Consider $\delta_1 \in (\delta, 1)$. Lemma 8 gives that there exist some positive constant $\varepsilon$, $\ell$ and $K$ only depending on $A^*$, $\delta$ and $\delta_1$ such that, for any $h \in (0, 1)$ fulfilling $\omega_h(A; I) \leq \varepsilon$, we have, for all $T \geq 1$, $t \leq T$ and $j \geq 1$ so that $T \geq \ell/h$,

$$\left|\prod_{i=1}^{j} A\left(\frac{t-i}{T}\right)\right| \leq K\delta_1^j.$$

We here consider the $\ell^\infty$ operator norm which is the maximum absolute row sum of the matrix, in which case $A^* = \max\{1, \sup_{u\in I}(|\theta_1(u)| + \cdots + |\theta_d(u)|)\} \leq 2^d d^{1/2}$. Hence, by (A.6) we obtain that

$$\begin{aligned} X_{t,T} &= \sum_{i=1}^{d} b_{t,T}(k, i) X_{t-k-i,T} + \sum_{j=0}^{k} a_{t,T}(j)\sigma\left(\frac{t-j}{T}\right)\xi_{t-j,T}, \\ &\qquad 1 \leq t \leq T, \end{aligned} \tag{A.7}$$

with, provided that $T > \ell/h$, for all $t \le T$, $k, j \ge 1$ and $i = 1, \dots, d$,

$$|b_{t,T}(k,i)| \le K\delta_1^{k+1},$$
$$|a_{t,T}(j)| \le K\delta_1^{j}.$$

The result follows.

**A.3. Proof of Lemma 3.** Denote $\omega(x) = \min\{2^{-1/2}, \max\{x, -2^{-1/2}\}\}$, so that $\omega(x)^2 = \min(1/2, x^2) \le x^2$. The function $x \mapsto \exp(-x^2)$ is concave on $[-2^{-1/2}, 2^{-1/2}]$, so introducing $\omega(x)$ and then using Jensen's inequality, we get

$$\begin{aligned}
&\int \exp(-x^2)\,\mathrm{d}\mathbb{P}(x) \\
&\quad\le \int \exp(-\omega^2(x))\,\mathrm{d}\mathbb{P}(x) \le \exp\biggl(-\biggl(\int \omega(x)\,\mathrm{d}\mathbb{P}(x)\biggr)^2\biggr) \\
&\quad= \exp\biggl(-\biggl(\int x\,\mathrm{d}\mathbb{P}(x)\biggr)^2 + \biggl(\int x\,\mathrm{d}\mathbb{P}(x)\biggr)^2 - \biggl(\int \omega(x)\,\mathrm{d}\mathbb{P}(x)\biggr)^2\biggr).
\end{aligned}$$

It only remains to show that $(\int x\,\mathrm{d}\mathbb{P}(x))^2 - (\int \omega(x)\,\mathrm{d}\mathbb{P}(x))^2 \le (a^2 - 1/2)_+$, with the assumption that $\mathbb{P}$ has support on $[-a, a]$. This is verified if $a \le 2^{-1/2}$, so we now assume $a > 2^{-1/2}$. We write

$$\begin{aligned}
&\biggl(\int x\,\mathrm{d}\mathbb{P}(x)\biggr)^2 - \biggl(\int \omega(x)\,\mathrm{d}\mathbb{P}(x)\biggr)^2 \\
&\qquad= \int \bigl(x - \omega(x)\bigr)\bigl(y + \omega(y)\bigr)\,\mathrm{d}\mathbb{P}(x)\,\mathrm{d}\mathbb{P}(y).
\end{aligned}$$

We note that $|x - \omega(x)| = (|x| - 1/2)_+$ and $|y + \omega(y)| \in \{2|y|, |y| + 2^{-1/2}\}$. We deduce that the product $(x - \omega(x))(y + \omega(y))$ either take nonpositive values or positive values of the form

$$\begin{cases}
2|y|\bigl(|x| - 2^{-1/2}\bigr), & \text{with } |x| > 2^{-1/2}, |y| < 2^{-1/2}, \\
\bigl(|x| - 2^{-1/2}\bigr)\bigl(|y| + 2^{-1/2}\bigr), & \text{with } |x| > 2^{-1/2}, |y| > 2^{-1/2}.
\end{cases}$$

Now, for $x, y \in [-a, a]$ with $a > 2^{-1/2}$, in the first case, we have $2|y|(|x| - 2^{-1/2}) \le 2^{1/2}(a - 2^{-1/2}) \le a^2 - 1/2$ since $2^{1/2} \le a + 2^{-1/2}$, and, in the second case, $(|x| - 2^{-1/2})(|y| + 2^{-1/2}) \le (a - 2^{-1/2})(a + 2^{-1/2}) = a^2 - 1/2$. The lemma follows.

**A.4. Proof of Lemma 4.** We define $\hat{\mathrm{J}}$ as the (random) smallest index which minimizes $d_X(\widehat{\theta}, \mathrm{v}_T\{\theta_j\})$ over $j \in \{0, \dots, M\}$ so that $d_X(\widehat{\theta}, \mathrm{v}_T\{\theta_{\hat{\mathrm{J}}}\}) = \min_{\theta \in \{\theta_0, \dots, \theta_M\}} d_X(\widehat{\theta}, \mathrm{v}_T\{\theta\})$. Note that $d_{X,T}(\theta_{\hat{\mathrm{J}}}, \theta_j) \le d_X(\mathrm{v}_T\{\theta_{\hat{\mathrm{J}}}\}, \widehat{\theta}) + d_X(\widehat{\theta},$

$\mathrm{v}_T\{\theta_j\}) \le 2d_X(\widehat{\theta}, \mathrm{v}_T\{\theta_j\})$. Hence,

$$\begin{aligned}
&\max_{j=0,\dots,M} \mathbb{E}_{\theta_j}\big[d_X^2(\widehat{\theta}, \mathrm{v}_T\{\theta_j\})\big] \\
&\quad \ge \frac{1}{4} \max_{j=0,\dots,M} \mathbb{E}_{\theta_j}\big[d_{X,T}^2(\theta_{\hat{\mathrm{J}}}, \theta_j)\big] \\
&\quad \ge \frac{r^2}{4} \max_{j=0,\dots,M} \mathbb{P}_{\theta_j}\Big(\{\hat{\mathrm{J}} \ne j\} \cap \Big\{\min_{i:i\ne j} d_{X,T}(\theta_i, \theta_j) > r\Big\}\Big) \\
&\quad \ge \frac{r^2}{4}\Big(1 - \min_{j=0,\dots,M} \mathbb{P}_{\theta_j}(\hat{\mathrm{J}} = j) - \max_{j=0,\dots,M} \mathbb{P}_{\theta_j}\Big(\min_{i:i\ne j} d_{X,T}(\theta_i, \theta_j) \le r\Big)\Big).
\end{aligned}$$

Birgé's lemma [Massart (2007), Corollary 2.18] implies that

$$\min_{j=0,\dots,M} \mathbb{P}_{\theta_j}(\hat{\mathrm{J}} = j) \le \max\left\{\left(\frac{2\mathrm{e}}{2\mathrm{e}+1}\right), \left(\frac{\max_{j=0,\dots,M} \mathcal{K}(\mathbb{P}_{\theta_j}, \mathbb{P}_{\theta_0})}{\log(1+M)}\right)\right\},$$

so the lemma follows from condition (5.5).

**A.5. Proof of Lemma 5.** By (5.11), we have $\theta_j \in s_1(\delta)$ for all $j = 0, \dots, M$. Decompose the Hölder-exponent $\beta = k + \alpha$ where $k$ is an integer and $\alpha \in (0, 1]$. Differentiating (5.7) $k$ times, we have, as in (5.10),

$$\theta_j^{(k)}(x) = \frac{R_0}{m^\alpha} w_{\lfloor mx\rfloor+1}^{(j)} K^{(k)}\left(\{mx\} - \frac{1}{2}\right) \qquad \text{for all } x \in [0, 1].$$

Thus, for $s, s'$ in the same interval $[\ell/m, (\ell+1)/m]$ with $\ell = 0, \dots, m-1$, we get

$$\begin{aligned}
|\theta_j^{(k)}(s) - \theta_j^{(k)}(s')| &\le \frac{R_0}{m^\alpha}\left|K^{(k)}\left(ms - \ell - \frac{1}{2}\right) - K^{(k)}\left(ms' - \ell - \frac{1}{2}\right)\right| \\
&\le R_0 |K|_\beta |s - s'|^\alpha.
\end{aligned}$$

The same inequality then follows with $R_0$ replaced by $2R_0$ for $s, s'$ in two such consecutive intervals. Now, if $s, s'$ are separated by at least one such interval, we have $|s - s'| \ge m^{-1}$ and, using that $K$ has support in $(-1/2, 1/2)$, we have that $|K^{(k)}(x)|$ is bounded by $|K|_\beta$. We thus get in this case that

$$|\theta_j^{(k)}(s) - \theta_j^{(k)}(s')| \le \frac{2R_0}{m^\alpha} \sup_{-1/2\le x\le 1/2} |K^{(k)}(x)| \le 2R_0 |K|_\beta |s - s'|^\alpha.$$

The last two displays and (5.8) then yields $\theta_j \in \Lambda_1(\beta, R)$.

**A.6. Proof of Lemma 6.** Let $j = 1, \dots, M$. Recall that $\theta_0 \equiv 0$ by (5.6) and (5.7). By (5.9) and (5.1), we have that $(X_{s,T})_{s\le 0}$ has the same distribution under $\mathbb{P}_{\theta_j}$ and $\mathbb{P}_{\theta_0}$ [which is the distribution of $(\xi_s)_{s\le 0}$]. Hence, the likelihood ratio $\mathrm{d}\mathbb{P}_{\theta_j}/\mathrm{d}\mathbb{P}_{\theta_0}$ of $(X_{s,T})_{s\le T}$ is given by the corresponding conditional likelihood

ratio of $(X_{s,T})_{1\le s\le T}$ given $(X_{s,T})_{s\le 0}$. Hence, under (I-3), we obtain that

$$\frac{\mathrm{d}\mathbb{P}_{\theta_j}}{\mathrm{d}\mathbb{P}_{\theta_0}} = \prod_{t=1}^{T} \frac{f(X_{t,T} - \theta_j((t-1)/T)X_{t-1,T})}{f(X_{t,T} - \theta_0((t-1)/T)X_{t-1,T})}$$
$$= \prod_{t=1}^{T} \frac{f(X_{t,T} - \theta_j((t-1)/T)X_{t-1,T})}{f(X_{t,T})},$$

where, in the second equality, we used again that $\theta_0 \equiv 0$. Now, under $\mathbb{P}_{\theta_j}$, we have $X_{t,T} = \theta_j((t-1)/T)X_{t-1,T} + \xi_t$. Thus, we get

$$\mathcal{K}(\mathbb{P}_{\theta_j}, \mathbb{P}_{\theta_0}) = \mathbb{E}_{\theta_j}\left[\log \frac{\mathrm{d}\mathbb{P}_{\theta_j}}{\mathrm{d}\mathbb{P}_{\theta_0}}\right]$$
$$= \sum_{t=1}^{T} \mathbb{E}_{\theta_j}\left[\log \frac{f(\xi_t)}{f(\theta_j((t-1)/T)X_{t-1,T} + \xi_t)}\right]$$
$$= \sum_{t=1}^{T} \mathbb{E}_{\theta_j} \int \log\left(\frac{f(u)}{f(\theta_j((t-1)/T)X_{t-1,T} + u)}\right) f(u)\,\mathrm{d}u.$$

Using assumption (I-3) yields

$$\mathcal{K}(\mathbb{P}_{\theta_j}, \mathbb{P}_{\theta_0}) \le \sum_{t=1}^{T} \mathbb{E}_{\theta_j}\left[\kappa\theta_j^2\left(\frac{t-1}{T}\right)X_{t-1,T}^2\right] \le \kappa\theta^{*2} \sum_{t=1}^{T} \mathbb{E}_{\theta_j}\big[X_{t-1,T}^2\big]. \tag{A.8}$$

The series representation (5.2), the fact that $\xi$ is centered with unit variance and (5.11) imply that for all $t = 0, \dots, T$

$$\mathbb{E}_{\theta_j}\big[X_{t,T}^2\big] \le \big(1 - \theta^{*2}\big)^{-1}.$$

Using this bound and (5.11) in (A.8), we obtain

$$\mathcal{K}(\mathbb{P}_{\theta_j}, \mathbb{P}_{\theta_0}) \le \frac{R_0^2 \mathrm{e}^{-2}\kappa T}{(1-\delta^2)m^{2\beta}}.$$

The proof of Lemma 6 now follows by applying the first bound in (5.6).

**A.7. Proof of Lemma 7.** The proof relies on an upper bound of $d_{X,T}^2(\theta_i, \theta_j)$ involving the noise $(\xi_t)$. By the expression of $\theta_j$ in (5.10), we have

$$d_{X,T}^2(\theta_i, \theta_j) = \frac{R_0^2}{Tm^{2\beta}} \sum_{t=0}^{T-1} X_{t,T}^2 \big(w_{k(t)}^{(i)} - w_{k(t)}^{(j)}\big)^2 K^2\big(\varphi(t)\big), \tag{A.9}$$

where we denoted $\varphi(t) = \{mt/T\} - 1/2$ and $k(t) = \lfloor mt/T \rfloor + 1$. Using (5.2) and (5.11), we have, for all $0 \le t \le T-1$,

$$|X_{t,T}| \ge |\xi_t| - \sum_{j=1}^{\infty} \theta^{*j} |\xi_{t-j}|,$$

which implies

$$X_{t,T}^2 \geq \xi_t^2 - 2|\xi_t| \sum_{j=1}^{\infty} \theta^{*j} |\xi_{t-j}|.$$

Inserting this bound in (A.9), we get

$$\frac{m^{2\beta}}{R_0^2} d_{X,T}^2(\theta_i, \theta_j) \geq \frac{1}{T} \sum_{t=0}^{T-1} \xi_t^2 \big(w_{k(t)}^{(i)} - w_{k(t)}^{(j)}\big)^2 K^2\big(\varphi(t)\big) - \mathcal{R}_T, \tag{A.10}$$

where

$$\mathcal{R}_T = \frac{2\mathrm{e}^{-2}}{T} \sum_{t=0}^{T-1} \sum_{j=1}^{\infty} \theta^{*j} |\xi_t| |\xi_{t-j}|.$$

Thus, with (A.10), the left-hand side of inequality (5.12) is upper bounded by

$$\mathbb{P}_{\theta_j}\left( \min_{i:i\neq j} \frac{1}{T} \sum_{t=0}^{T-1} \xi_t^2 \big(w_{k(t)}^{(i)} - w_{k(t)}^{(j)}\big)^2 K^2\big(\varphi(t)\big) < 2A \right) + \mathbb{P}(\mathcal{R}_T > A).$$

Using that $\xi$ is centered with unit variance and then (5.11), we easily get that

$$\mathbb{E}_{\theta_j}[\mathcal{R}_T] \leq \frac{2\mathrm{e}^{-2}}{T} \sum_{t=0}^{T-1} \sum_{j=1}^{\infty} \theta^{*j} \leq \frac{2\mathrm{e}^{-2}\theta^*}{1-\theta^*} \leq \frac{2R_0\mathrm{e}^{-3}}{(1-\delta)m^{\beta}}.$$

Hence, by Markov's inequality, to conclude the proof, it now suffices to show that, for $A$ well chosen,

$$\mathbb{P}_{\theta_j}\left( \min_{i:i\neq j} \frac{1}{T} \sum_{t=0}^{T-1} \xi_t^2 \big(w_{k(t)}^{(i)} - w_{k(t)}^{(j)}\big)^2 K^2\big(\varphi(t)\big) < 2A \right) \leq \varepsilon. \tag{A.11}$$

For $k \in \{1, \ldots, m\}$ we define $J_k = \{\lfloor (k-1)T/m \rfloor + i\lceil T/(4m) \rceil + 1 \leq i \leq \lfloor 3T/(4m) \rfloor\}$. We observe that the cardinality of $J_k$ is

$$\Gamma\left(\frac{T}{m}\right) = \left\lfloor \frac{3T}{4m} \right\rfloor - \left\lceil \frac{T}{4m} \right\rceil \geq 1,$$

where the lower bound is a consequence of the assumption $T \geq 4m$ in the lemma. Moreover, it is easy to check that we have $|\varphi(t)| \leq 1/4$ for all index $t \in J_k$ and that, for each $1 \leq k \leq m$, the set $J_k$ is included in the set $\{1 \leq t \leq T-1 : k(t) = k\}$ (so that, in particular, $J_k \cap J_{k'} = \varnothing$ for $k < k'$). It follows that random variables

$$S_k = \frac{1}{\Gamma(T/m)} \sum_{t \in J_k} \xi_{t-1}^2 \qquad \text{for } k = 1, \ldots, m$$

are i.i.d. By the monotonicity of $K$ in $\mathbb{R}_-$ and its symmetry, we have

$$\frac{1}{T}\sum_{t=0}^{T-1}\xi_t^2\big(w_{k(t)}^{(i)}-w_{k(t)}^{(j)}\big)^2K^2\big(\varphi(t)\big)\geq\frac{1}{T}\sum_{k=1}^{m}\big(w_k^{(i)}-w_k^{(j)}\big)^2\sum_{t\in J_k}\xi_t^2K^2\big(\varphi(t)\big)$$
$$\geq\frac{K^2(1/4)\Gamma(T/m)}{T}\sum_{k=1}^{m}\big(w_k^{(i)}-w_k^{(j)}\big)^2S_k.$$

From (5.6), for any $i,j\in\{1,\dots,M\}$ there exist at least $\lceil m/8\rceil$ values of $k$ for which $(w_k^{(i)}-w_k^{(j)})^2$ equals one in the above sum. Hence, using the order statistics $S_{(1,m)}\leq\cdots\leq S_{(m,m)}$, we thus obtain that

$$\min_{i:i\neq j}\frac{1}{T}\sum_{t=0}^{T-1}\xi_t^2\big(w_{k(t)}^{(i)}-w_{k(t)}^{(j)}\big)^2K^2\big(\varphi(t)\big)\geq\frac{K^2(1/4)\Gamma(T/m)}{T}\sum_{k=1}^{\lceil m/8\rceil}S_{(k,m)}$$
$$\geq\frac{K^2(1/4)m\Gamma(T/m)}{16T}S_{(\lfloor m/16\rfloor,m)}$$
$$\geq\frac{K^2(1/4)}{128}S_{(\lfloor m/16\rfloor,m)},$$

where we used $\Gamma(T/m)\geq T/(8m)$ for $T/m\geq 4$ in the last inequality. Let us denote by $F$ the cumulative distribution function of $S_1$, which only depends on $\Gamma(T/m)$ and on the distribution of $\xi_0$. For $x>0$, we have

$$\mathbb{P}(S_{(\lfloor m/16\rfloor,m)}\leq x)=\mathbb{P}\bigg(\mathrm{Bin}\big(m,F(x)\big)\geq\bigg\lfloor\frac{m}{16}\bigg\rfloor\bigg)$$
$$\leq\frac{m}{\lfloor m/16\rfloor}F(x)\leq 32F(x).$$

Gathering the last two bounds, we get that

$$\mathbb{P}_{\theta_j}\bigg(\min_{i:i\neq j}\frac{1}{T}\sum_{t=1}^{T-1}\xi_t^2\big(w_{k(t)}^{(i)}-w_{k(t)}^{(j)}\big)^2K^2\big(\varphi(t)\big)\leq 2A\bigg)$$
$$\leq\mathbb{P}\bigg(S_{(\lfloor m/16\rfloor,m)}\leq\frac{256A}{K^2(1/4)}\bigg)$$
$$\leq 32F\bigg(\frac{256A}{K^2(1/4)}\bigg).$$

Recall that $\Gamma(T/m)\geq 1$ and note that $S_1$ admits a density, since $\xi$ does. By the strong law of large numbers, we further have that the random variable $S_1$ converges to 1 almost surely when $\Gamma(T/m)$ goes to infinity, so there exists $x_0>0$ depending only on the density of $\xi$ such that $F(x_0)\leq\varepsilon/32$ whatever the value of $\Gamma(T/m)\geq 1$. Therefore, there exists some $A>0$, depending only on the distribution of $\xi$, such that (A.11) holds, which achieves the proof.

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## SUPPLEMENTARY MATERIAL

**Supplementary material for: Aggregation of predictors for nonstationary sub-linear processes and online adaptive forecasting of time varying autoregressive processes** (DOI: 10.1214/15-AOS1345SUPP; .pdf). We explain how to build nonadaptive minimax predictors which can be used in the aggregation step. The document also contains some technical proofs and provides additional results with improved aggregation rates.

## REFERENCES

ALQUIER, P. and WINTENBERGER, O. (2012). Model selection for weakly dependent time series forecasting. *Bernoulli* **18** 883–913. MR2948906

ANAVA, O., HAZAN, E., MANNOR, S. and SHAMIR, O. (2013). Online learning for time series prediction. Preprint. Available at arXiv:1302.6927.

ARKOUN, O. (2011). Sequential adaptive estimators in nonparametric autoregressive models. *Sequential Anal*. **30** 229–247. MR2801140

AUDIBERT, J.-Y. (2009). Fast learning rates in statistical inference through aggregation. *Ann. Statist*. **37** 1591–1646. MR2533466

BROCKWELL, P. J. and DAVIS, R. A. (2006). *Time Series*: *Theory and Methods*. Springer, New York. Reprint of the second (1991) edition. MR2839251

CATONI, O. (1997). A mixture approach to universal model selection. Technical report, École Normale Supérieure.

CATONI, O. (2004). *Statistical Learning Theory and Stochastic Optimization*. *Lecture Notes in Math*. **1851**. Springer, Berlin. MR2163920

CESA-BIANCHI, N. and LUGOSI, G. (2006). *Prediction*, *Learning*, *and Games*. Cambridge Univ. Press, Cambridge. MR2409394

DAHLHAUS, R. (1996). On the Kullback–Leibler information divergence of locally stationary processes. *Stochastic Process*. *Appl*. **62** 139–168. MR1388767

DAHLHAUS, R. (2009). Local inference for locally stationary time series based on the empirical spectral measure. *J. Econometrics* **151** 101–112. MR2559818

DAHLHAUS, R. and POLONIK, W. (2006). Nonparametric quasi-maximum likelihood estimation for Gaussian locally stationary processes. *Ann. Statist*. **34** 2790–2824. MR2329468

DAHLHAUS, R. and POLONIK, W. (2009). Empirical spectral processes for locally stationary time series. *Bernoulli* **15** 1–39. MR2546797

DALALYAN, A. S. and TSYBAKOV, A. B. (2008). Aggregation by exponential weighting, sharp PAC-Bayesian bounds and sparsity. *Mach. Learn*. **72** 39–61.

DOUKHAN, P. and WINTENBERGER, O. (2008). Weakly dependent chains with infinite memory. *Stochastic Process*. *Appl*. **118** 1997–2013. MR2462284

GERCHINOVITZ, S. (2011). Prediction of individual sequences and prediction in the statistical framework: Some links around sparse regression and aggregation techniques. Ph.D. thesis, Univ. Paris Sud-Paris XI.

GIRAUD, C., ROUEFF, F. and SANCHEZ-PEREZ, A. (2015). Supplement to "Aggregation of predictors for non stationary sub-linear processes and online adaptive forecasting of time varying autoregressive processes." DOI:10.1214/15-AOS1345SUPP.

GRENIER, Y. (1983). Time-dependent ARMA modeling of nonstationary signals. *IEEE Transactions on ASSP* **31** 899–911.

JUDITSKY, A. and NEMIROVSKI, A. (2000). Functional aggregation for nonparametric regression. *Ann. Statist.* **28** 681–712. MR1792783

KÜNSCH, H. R. (1995). A note on causal solutions for locally stationary AR-processes. Unpublished preprint, ETH Zürich.

LEPSKIĬ, O. V. (1990). A problem of adaptive estimation in Gaussian white noise. *Teor. Veroyatn. Primen.* **35** 459–470. MR1091202

LEUNG, G. and BARRON, A. R. (2006). Information theory and mixing least-squares regressions. *IEEE Trans. Inform. Theory* **52** 3396–3410. MR2242356

MASSART, P. (2007). *Concentration Inequalities and Model Selection. Lecture Notes in Math.* **1896**. Springer, Berlin. MR2319879

MOULINES, E., PRIOURET, P. and ROUEFF, F. (2005). On recursive estimation for time varying autoregressive processes. *Ann. Statist.* **33** 2610–2654. MR2253097

RIGOLLET, P. and TSYBAKOV, A. B. (2012). Sparse estimation by exponential weighting. *Statist. Sci.* **27** 558–575. MR3025134

SANCETTA, A. (2010). Recursive forecast combination for dependent heterogeneous data. *Econometric Theory* **26** 598–631. MR2600575

STOLTZ, G. (2011). Contributions to the sequential prediction of arbitrary sequences: Applications to the theory of repeated games and empirical studies of the performance of the aggregation of experts. Habilitation à diriger des recherches, Univ. Paris Sud-Paris XI.

TONG, H. and LIM, K. S. (1980). Threshold autoregression, limit cycles and cyclical data. *J. Roy. Statist. Soc. Ser. B* **42** 245–292.

TSYBAKOV, A. B. (2003). Optimal rates of aggregation. In *Learning Theory and Kernel Machines* (B. Schölkopf and M. K. Warmuth, eds.). *Lecture Notes in Computer Science* **2777** 303–313. Springer, Berlin.

TSYBAKOV, A. B. (2009). *Introduction to Nonparametric Estimation*. Springer, New York. MR2724359

VOVK, V. G. (1990). Aggregating strategies. In *Proc. Third Workshop on Computational Learning Theory* 371–383. Morgan Kaufmann, San Mateo, CA.

WANG, Z., PATERLINI, S., GAO, F. and YANG, Y. (2014). Adaptive minimax regression estimation over sparse $\ell_q$-hulls. *J. Mach. Learn. Res.* **15** 1675–1711. MR3225246

YANG, Y. (2000a). Combining different procedures for adaptive regression. *J. Multivariate Anal.* **74** 135–161. MR1790617

YANG, Y. (2000b). Mixing strategies for density estimation. *Ann. Statist.* **28** 75–87. MR1762904

YANG, Y. (2004). Combining forecasting procedures: Some theoretical results. *Econometric Theory* **20** 176–222. MR2028357

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