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WHEN DOES A DISCRETE-TIME RANDOM WALK IN $\mathbb{R}^n$ ABSORB THE ORIGIN INTO ITS CONVEX HULL?

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We connect this question to a problem of estimating the probability that the image of certain random matrices does not intersect with a subset of the unit sphere $\mathbb{S}^{n-1}$. In this way, the case of a discretized Brownian motion is related to Gordon's escape theorem dealing with standard Gaussian matrices. We show that for the random walk $\mathrm{BM}_n(i)$, $i \in \mathbb{N}$, the convex hull of the first C^n steps (for a sufficiently large universal constant C) contains the origin with probability close to one. Moreover, the approach allows us to prove that with high probability the $\pi/2$-covering time of certain random walks on $\mathbb{S}^{n-1}$ is of order n. For certain spherical simplices on $\mathbb{S}^{n-1}$, we prove an extension of Gordon's theorem dealing with a broad class of random matrices; as an application, we show that C^n steps are sufficient for the standard walk on $\mathbb{Z}^n$ to absorb the origin into its convex hull with a high probability. Finally, we prove that the aforementioned bound is sharp in the following sense: for some universal constant $c > 1$, the convex hull of the n-dimensional Brownian motion $\mathrm{conv}\{\mathrm{BM}_n(t) : t \in [1, c^n]\}$ does not contain the origin with probability close to one.

1. Introduction. The goal of this paper is to study certain convexity aspects of high-dimensional random walks. Given a discrete-time random walk $W(i)$ with values in $\mathbb{R}^n$, we are interested in estimating the number of steps N when the origin becomes an interior point of the convex hull of $\{W(i)\}_{i \le N}$. This question was raised by I. Benjamini and considered by R. Eldan in [4]. Three models of random walks are treated in our paper: a walk given by a discretization of the standard Brownian motion in $\mathbb{R}^n$, the standard random walk on $\mathbb{Z}^n$ and a random walk on the unit sphere $\mathbb{S}^{n-1}$. We employ a novel approach that reduces the problem to a question about certain geometric properties of random matrices. Random matrix theory has strong connections with asymptotic geometric analysis (see, e.g., [2] and [22]); in particular, random matrices appear in Gordon's escape theorem [8] and in various estimates of diameters of random sections of convex sets [15, 18]. The interconnection between random walks, random matrix theory and high-dimensional convex geometry is at the heart of our paper.

The *standard Brownian motion* with values in $\mathbb{R}$ is defined as a centered Gaussian process $\mathrm{BM}_1(t)$, $t \in [0, \infty)$, such that the covariance $\mathrm{cov}(\mathrm{BM}_1(t), \mathrm{BM}_1(s)) = \min(t, s)$ for all $t, s \in [0, \infty)$. The Brownian motion in $\mathbb{R}^n$, denoted by BM_n, is a

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vector of n independent one-dimensional Brownian motions. We refer the reader to [17] for extensive information on the process BM_n. Various properties of the convex hull of the Brownian motion in high dimensions were studied recently in [4, 5] and [10]; in particular, results on interior and extremal points of the convex hulls were obtained. It is easy to see that the interior of $\mathrm{conv}\{\mathrm{BM}_n(t) : 0 < t < 1\}$ (with "conv" denoting the convex hull) contains the origin almost surely. In the case when the domain $t \in (0, 1)$ is replaced by a finite subset of the unit interval, the origin is outside of the convex hull with a nonzero probability. Our paper is motivated by the following problem which in a more specific form was considered by Eldan in [4].

Let $t_1 < t_2 < \cdots < t_N$ be points in $[0, 1]$. How is the probability that the origin belongs to the interior of $\mathrm{conv}\{\mathrm{BM}_n(t_i) : i \leq N\}$ related to the structure of the set $\{t_i\}_{i \leq N}$?

In [4], the numbers N and $t_1, t_2, \dots, t_N$ were generated by a homogeneous Poisson point process in $[0, 1]$. It was shown that when the expected number of generated points N is greater than $e^{Cn \log(n)}$, the origin belongs to the interior of $\mathrm{conv}\{\mathrm{BM}_n(t_i) : i \leq N\}$ with high probability [4], Theorem 3.1. A related result of [4] dealing with the standard walk on $\mathbb{Z}^n$ states that, with probability close to one, $e^{Cn \log(n)}$ steps are sufficient for the convex hull of the walk to absorb the origin. It was not clear, however, whether the bound $e^{Cn \log(n)}$ was sharp. This question is addressed in the first main theorem of our paper.

THEOREM A. *There exists a constant $C > 0$ such that for any $n \in \mathbb{N}$ and $N \geq \exp(Cn)$ the following holds*

- *Setting $t_i := i/N$, $i = 1, 2, \dots, N$, the set $\mathrm{conv}\{\mathrm{BM}_n(t_i), i \leq N\}$ contains the origin in its interior with probability at least $1 - \exp(-n)$.*
- *The convex hull of the first N steps of the standard random walk on $\mathbb{Z}^n$ starting at 0, contains the origin in its interior with probability at least $1 - \exp(-n)$.*

The first part of this theorem also holds when $\{t_i\}$ is a homogeneous Poisson process in $[0, 1]$ of intensity at least $\exp(Cn)$. Therefore, our result is strictly stronger than the bound proved in [4].

Let us discuss optimality of the estimates in Theorem A. Regarding the second assertion, it was proved in [4] that if the number of steps N is less than $\exp(cn/\log n)$ then with probability close to one the origin does not belong to the interior of the convex hull of the standard walk on $\mathbb{Z}^n$.

For the first assertion of Theorem A, we prove that it is optimal in the sense that the number of points N must be exponential in n in order to have, say, $\mathbb{P}\{0 \in \mathrm{conv}\{\mathrm{BM}_n(t_i), i \leq N\}\} \geq 1/2$. More precisely, we prove the following.

THEOREM B. *There exist universal constants $c > 0$ and $n_0 \in \mathbb{N}$ with the following property: let $n \geq n_0$ and $\mathrm{BM}_n(t)$ $(0 \leq t < \infty)$ be the standard Brownian* *motion in* $\mathbb{R}^n$. *Then*

$$\mathbb{P}\big\{0 \in \operatorname{conv}\big\{\mathrm{BM}_n(t) : t \in \big[1, 2^{cn}\big]\big\}\big\} \le \frac{1}{n}.$$

REMARK 1. The bound $\frac{1}{n}$ in the above theorem can be replaced with $\frac{1}{n^L}$ for any constant $L > 0$ at expense of decreasing c and increasing n_0.

The statement of Theorem B is equivalent to the estimate

$$\mathbb{P}\Big\{\min_{u \in S^{n-1}} \max_{t \in [1, 2^{cn}]} \langle u, \mathrm{BM}_n(t)\rangle < 0\Big\} \ge 1 - \frac{1}{n}, \tag{1}$$

where the quantity in the brackets is the "minimax" of the one dimensional Gaussian process $\langle u, \mathrm{BM}_n(t)\rangle$ indexed over $S^{n-1} \times [1, 2^{cn}]$. We note that a comparison theorem for the minimax of doubly indexed Gaussian processes was obtained in [7] (see also [11], Corollary 3.13 and Theorem 3.16), and was the central ingredient in proving the escape theorem of [8].

The second main result of our paper deals with discrete-time random walks on the sphere. For any $\theta \in (0, \pi/2)$, we consider a Markov chain W_θ with values in $\mathbb{S}^{n-1}$ such that the angle between two consecutive steps is θ (i.e., $\langle W_\theta(j), W_\theta(j+1)\rangle = \cos\theta$, $j \in \mathbb{N}$) and the direction from $W(j)$ to $W(j+1)$ is chosen *uniformly at random* in the sense that for any $u \in \mathbb{S}^{n-1}$, the distribution of $W_\theta(j+1)$ conditioned on $W_\theta(j) = u$ is uniform on the $(n-2)$-sphere $\mathbb{S}^{n-1} \cap \{x \in \mathbb{R}^n : \langle x, u\rangle = \cos\theta\}$.

THEOREM C. *For any* $\theta \in (0, \pi/2)$, *there exist* $L = L(\theta)$ *and* $n_0 = n_0(\theta)$ *depending only on* θ *such that the following holds*: *Let* $n \ge n_0$ *and* W_θ *be the process with values in* $\mathbb{S}^{n-1}$ *described above. Then for all* $N \ge Ln$ *we have*

$$\mathbb{P}\big\{0 \text{ belongs to } \operatorname{conv}\big\{W_\theta(i) : i \le N\big\}\big\} \ge 1 - \exp(-n).$$

It follows from dimension considerations that the estimate of the number of steps is optimal up to a factor depending only on θ. We note here that a related problem for the standard spherical Brownian motion was studied in [4].

Let us outline the main ideas behind the proofs of Theorems A and C. The following simple observation relates the question about random walks to a problem dealing with random matrices.

Let $X(t)$ ($t \in [0, \infty)$ or $t \in \mathbb{N} \cup \{0\}$) be a random process with values in $\mathbb{R}^n$, with $X(0) = 0$; let $0 = t_0 < t_1 < \cdots < t_N$ be a collection of nonrandom points and assume that the increments $X(t_i) - X(t_{i-1})$ are independent. Define A as the $N \times n$ random matrix with independent rows obtained by appropriately rescaling the increments $X(t_i) - X(t_{i-1})$, $i = 1, 2, \dots, N$. Then there exists a nonrandom $N \times N$ lower-triangular matrix F such that the rows of FA are precisely $X(t_i)$, $i = 1, 2, \dots, N$. Thus, we can restate our problem about the convex hull of $X(t_i)$'s in terms of certain properties of the matrix FA. Namely, the convex hull of $X(t_i)$'s contains the origin in its interior if and only if for any unit vector y in $\mathbb{R}^n$, the vector FAy has at least one negative coordinate. Geometrically, this problem is reduced to estimating the probability that the image of A *escapes* (i.e., does not intersect) the set $F^{-1}(\mathbb{R}_+^N) \cap \mathbb{S}^{N-1}$, where $\mathbb{R}_+^N$ denotes the cone of positive vectors. For the standard Brownian motion, A is the $N \times n$ standard Gaussian matrix. In this case, we apply Gordon's escape theorem [8] which estimates the probability that a random subspace uniformly distributed on the Grassmannian does not intersect with a given subset of $\mathbb{S}^{N-1}$. In a more general case, when the image of A is not uniformly distributed, Gordon's theorem cannot be applied. To treat that scenario, we prove a statement which can be seen as an extension of Gordon's theorem to a broad class of random matrices, however, with considerable restrictions on the subsets of $\mathbb{S}^{N-1}$.

THEOREM D. *For any $\tau, \delta \in (0, 1]$ and any $K > 1$, there exist L and $\eta > 0$ depending only on τ, δ and K with the following property: Let $N \geq Ln$ and let A be an $N \times n$ random matrix with independent rows $(R_i)_{i \leq N}$ satisfying*

$$\mathbb{P}\{\langle R_i, y\rangle < -\tau\} \geq \delta, \qquad \text{for any } y \in \mathbb{S}^{n-1} \text{ and any } i \leq N.$$

Then for any $N \times N$ random matrix F, matrix FA satisfies

$$\mathbb{P}\{\exists y \in \mathbb{S}^{n-1}, FAy \in \mathbb{R}_+^N\} \leq \exp(-\delta^2 N/4) + \mathbb{P}\{\|A\| > K\sqrt{N}\} + \mathbb{P}\{\|F - \mathbf{I}\| > \eta\}.$$

We use this result to deal with the random walk on $\mathbb{Z}^n$. For the random walks W_θ on the sphere we follow, with some modifications, the same scheme as for processes in $\mathbb{R}^n$ with independent increments.

The paper is organized as follows. Section 2 contains preliminaries and notation. Results about random matrices are given in Section 3, while corollaries for the Brownian motion and the standard random walk on $\mathbb{Z}^n$ are stated in Section 4. Section 5 is devoted to random walks on the sphere. Finally, we prove Theorem B in Section 6.

2. Preliminaries.

In this section, we introduce notation and discuss some classical or elementary facts.

For a finite set I, let $|I|$ be its cardinality. Let $\mathbb{R}_+$ and $\mathbb{R}_-$ be the closed positive and negative semi-axes, respectively. By $\{e_i\}_{i=1}^N$ we denote the standard unit basis in $\mathbb{R}^N$, by $\|\cdot\|$—the canonical Euclidean norm and by $\langle\cdot,\cdot\rangle$—the corresponding inner product. Let B_2^N and $\mathbb{S}^{N-1}$ be the Euclidean ball of radius 1 in $\mathbb{R}^N$ and the unit sphere, respectively.

For $N \geq n$ and an $N \times n$ matrix A, let $s_{\max}(A)$ and $s_{\min}(A)$ be its largest and smallest singular values, respectively, that is, $s_{\max}(A) = \|A\|$ (the operator norm of A) and $s_{\min}(A) = \inf_{y \in \mathbb{S}^{n-1}} \|Ay\|$. When A is an $N \times N$ invertible matrix, the condition number of A is $\|A\| \cdot \|A^{-1}\|$. Note that the condition number is equal to the ratio of the largest and the smallest singular values of A.

Throughout this paper, g denotes a standard Gaussian variable. The following estimate is well known (see, e.g., [6], Lemma VII.1.2):

$$\mathbb{P}\{g \geq t\} = \frac{1}{\sqrt{2\pi}} \int_t^\infty \exp(-r^2/2)\, dr < \frac{1}{\sqrt{2\pi} t} \exp(-t^2/2), \qquad t > 0. \tag{2}$$

A random vector X in $\mathbb{R}^n$ is *isotropic* if $\mathbb{E}X = 0$ and the covariance matrix of X is the identity, that is, $\mathbb{E}XX^t = \mathbf{I}$. The *standard Gaussian vector Y* in $\mathbb{R}^n$ is a random vector with i.i.d. coordinates having the same law as g. As a corollary of a concentration inequality for Gaussian variables (see [19], Theorem 4.7, or [16], Theorem V.1), we have for any $\varepsilon > 0$:

$$\mathbb{P}\{(1-\varepsilon)\sqrt{n} \leq \|Y\| \leq (1+\varepsilon)\sqrt{n}\} \geq 1 - 2\exp(-\tilde{c}\varepsilon^2 n) \tag{3}$$

for a universal constant $\tilde{c} > 0$. An $N \times n$ matrix is called the *standard Gaussian matrix* if its entries are i.i.d. having the same law as g. We denote this matrix by G (and recall that $N \geq n$). Then for any $t \geq 0$ we have

$$\begin{aligned} &\mathbb{P}\{\sqrt{N} - \sqrt{n} - t \leq s_{\min}(G) \leq s_{\max}(G) \leq \sqrt{N} + \sqrt{n} + t\} \\ &\qquad \geq 1 - 2\exp(-t^2/2) \end{aligned} \tag{4}$$

(see, e.g., [23], Corollary 5.35).

Given a vector $x \in \mathbb{R}^N$, we denote by x_+ and x_- its positive and negative part, respectively, that is,

$$x_+ = \sum_{i=1}^N \max(0, \langle x, e_i\rangle) e_i \quad \text{and} \quad x_- = \sum_{i=1}^N \max(0, -\langle x, e_i\rangle) e_i.$$

The following simple observation will be useful in the proof of the main theorems.

LEMMA 1. *Let $x, y \in \mathbb{R}^N$. Then $\|x_-\| \geq \|y_-\| - \|x - y\|$.*

PROOF. Writing $x = x_+ - x_-$ and $y = y_+ - y_-$, we obtain

$$\begin{aligned} \|x - y\|^2 &= \|(x_+ - y_+) - (x_- - y_-)\|^2 \\ &= \|x_- - y_-\|^2 + \|x_+ - y_+\|^2 - 2\langle x_+ - y_+, x_- - y_-\rangle \\ &\geq \|x_- - y_-\|^2 \\ &\geq (\|y_-\| - \|x_-\|)^2, \end{aligned}$$

where the first inequality in the above formula holds since $\langle x_+ - y_+, x_- - y_-\rangle$ is nonpositive. $\square$

Given a compact set $S \subset \mathbb{R}^N$, the *Gaussian width* of S is defined by

$$w(S) := \mathbb{E} \sup_{x \in S} \langle Y, x \rangle,$$

where Y is the standard Gaussian vector in $\mathbb{R}^N$ (see [1, 3, 22]). The following is a consequence of Urysohn's inequality (see, e.g., Corollary 1.4 in [19]) and the relation between the Gaussian and the mean width:

$$\sqrt{N-1}\left(\frac{\mathrm{Vol}_N(S)}{\mathrm{Vol}_N(B_2^N)}\right)^{1/N} \leq w(S). \tag{5}$$

Given a convex cone C in $\mathbb{R}^N$, the *polar cone* C^* of C is defined by

$$C^* := \{x \in \mathbb{R}^N, \langle x, y \rangle \leq 0 \text{ for any } y \in C\}.$$

The next lemma provides a useful relation between the Gaussian widths of the parts of a convex cone and its polar enclosed in the unit Euclidean ball. The lemma is proved in [3] for intersections of cones with the unit sphere (see [3], Lemma 3.7); we put it here in a version more convenient for us.

LEMMA 2. *Let $C \subset \mathbb{R}^N$ be a nonempty closed convex cone. Then*

$$w(C \cap B_2^N)^2 + w(C^* \cap B_2^N)^2 \leq N.$$

PROOF. For any $x \in \mathbb{R}^N$, let $P_C x := \arg\inf_{y \in C} \|x - y\|$ be the projection of x onto C. It can be checked that each vector $x \in \mathbb{R}^N$ can be decomposed as

$$x = P_C x + P_{C^*} x, \tag{6}$$

with $\langle P_C x, P_{C^*} x \rangle = 0$. As before, let Y be the standard Gaussian vector in $\mathbb{R}^N$. Having decomposition (6) in mind, we can write

$$w(C \cap B_2^N) = \mathbb{E} \sup_{x \in C \cap B_2^N} \langle Y, x \rangle \leq \mathbb{E} \sup_{x \in C \cap B_2^N} \langle P_C Y, x \rangle,$$

where the last inequality holds since $\langle P_{C^*} Y, x \rangle \leq 0$ for all $x \in C$. We deduce that

$$w(C \cap B_2^N) \leq \mathbb{E}\|P_C Y\|. \tag{7}$$

Now using the decomposition (6) and the above inequality, we obtain

$$w(C \cap B_2^N)^2 \leq \mathbb{E}\|P_C Y\|^2 = \mathbb{E}\|Y\|^2 - \mathbb{E}\|P_{C^*} Y\|^2 = N - \mathbb{E}\|P_{C^*} Y\|^2. \tag{8}$$

Note that (7) applied to the cone C^* yields $w(C^* \cap B_2^N)^2 \leq \mathbb{E}\|P_{C^*} Y\|^2$. Plugging it into (8), we complete the proof. $\square$

3. Escape theorems for random matrices. In this section, we estimate the probability that the image of a random $N \times n$ matrix A *escapes* the intersection of a given cone with the unit sphere $\mathbb{S}^{N-1}$ (we shall restrict ourselves to considering a special family of convex cones in $\mathbb{R}^N$). Similar questions have attracted considerable attention recently in connection with the theory of compressed sensing [1].

Given a closed subset $S \subset \mathbb{S}^{N-1}$, the problem of estimating the probability $\mathbb{P}\{\mathrm{Im}(A) \cap S = \varnothing\}$ can be treated in different ways. One may look at it as the question of bounding the diameter of the random section $\mathrm{conv}(S, -S) \cap \mathrm{Im}(A)$ of the convex set $\mathrm{conv}(S, -S)$: clearly, $\mathrm{Im}(A) \cap S = \varnothing$ if and only if $\mathrm{diam}(\mathrm{conv}(S, -S) \cap \mathrm{Im}(A)) < 2$. The study of random sections of convex sets is a central theme in the area of asymptotic geometric analysis and its importance has been highlighted in Milman's proof of Dvoretzky's theorem [16, 19]. The question of estimating diameters of random sections of proportional dimension was originally considered in [15] and [18] in the case when the corresponding random subspace is uniformly distributed on the Grassmannian (i.e., the randomness is given by a standard Gaussian matrix). More recently, results for much more general distributions of sections given by kernels and images of random matrices were obtained, among others, in papers [12] and [14]. In our setting, however, these papers do not seem directly applicable as they provide estimates for diameters up to a constant multiple: in particular, if a convex set K, say, satisfies $K \subset B_2^N \subset 2K$, and E is a random subspace given by a kernel or an image of a random matrix, those results only give a trivial bound $\mathrm{diam}(K \cap E) < C$ for a large constant C. At the same time, if $S = \mathbb{S}^{N-1} \cap \mathbb{R}^N_+$ then it is easy to show that $\mathrm{conv}(S, -S) \subset B_2^N \subset \sqrt{2}\,\mathrm{conv}(S, -S)$.

When the matrix A is Gaussian, a way of estimating the probability $\mathbb{P}\{\mathrm{Im}(A) \cap S = \varnothing\}$ which is more suitable in our setting is to apply the following result of Gordon (see Corollary 3.4 in [8]):

THEOREM 3 (Gordon's escape theorem). *Let S be a subset of the unit Euclidean sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$. Let E be a random n-dimensional subspace of $\mathbb{R}^N$, distributed uniformly on the Grassmannian with respect to the associated Haar measure. Assume that $w(S) < \sqrt{N-n}$. Then $E \cap S = \varnothing$ with probability at least*

$$1 - 3.5 \exp\left(-\frac{1}{18}\left(\frac{N-n}{\sqrt{N-n+1}} - w(S)\right)^2\right).$$

For the standard Gaussian matrix G, its image is uniformly distributed on the Grassmannian, and Gordon's result provides an efficient estimate of probability $\mathbb{P}\{\mathrm{Im}\, G \cap S = \varnothing\}$, as long as we have control over the Gaussian width of the set S. In our setting, the choice of S is determined by the applications to random walks; in fact, S shall always be a spherical simplex satisfying certain additional assumptions. A standard approach would be to bound $w(S)$ in terms of the covering numbers of S using the classical Dudley's inequality (see, e.g., [11], Theorem 11.17).

However, in our case the set S is relatively large, so the upper bound given by Dudley's inequality is trivial (greater than $\sqrt{N}$). Instead, we will estimate the Gaussian width of S using the following proposition which is a direct consequence of Lemma 2 and inequality (5).

PROPOSITION 4. *Let C be a convex cone in $\mathbb{R}^N$ and denote by C^* its polar cone. Then*

$$w\big(C\cap B_2^N\big)^2 \le N-(N-1)\left(\frac{\mathrm{Vol}_N(C^*\cap B_2^N)}{\mathrm{Vol}_N(B_2^N)}\right)^{2/N}.$$

The next theorem will be applied in Sections 4 and 5 to the discretized Brownian motion and to random walks on the sphere.

THEOREM 5. *For any $\gamma\in(0,1]$, there exist positive L, κ and η depending on γ such that the following is true: For $N\ge Ln$, let F be an $N\times N$ random matrix and $\tilde F$ be a deterministic invertible $N\times N$ matrix with the condition number satisfying $\|\tilde F\|\cdot\|\tilde F^{-1}\|\le\gamma^{-1}$. If G is the $N\times n$ standard Gaussian matrix, then*

$$\mathbb{P}\big\{\exists y\in\mathbb{S}^{n-1},\,FGy\in\mathbb{R}_+^N\big\}\le 5.5\exp(-\kappa N)+\mathbb{P}\big\{\|F-\tilde F\|>\eta\|\tilde F\|\big\}.$$

The statement holds with $L=64/\gamma^2$, $\kappa=2L^{-2}/9$ and $\eta=\gamma/4L$.

PROOF. Let $\gamma\in(0,1)$ and take L,κ, and η as stated above. In view of Lemma 1, we have

$$\begin{aligned}
&\mathbb{P}\big\{\exists y\in\mathbb{S}^{n-1},\,(FGy)_-=0\big\}\\
&\quad\le\mathbb{P}\big\{\exists y\in\mathbb{S}^{n-1},\,\big\|(\tilde FGy)_-\big\|\le\big\|(F-\tilde F)Gy\big\|\big\}\\
&\quad\le\mathbb{P}\big\{\exists y\in\mathbb{S}^{n-1},\,\big\|(\tilde FGy)_-\big\|\le\eta\|\tilde F\|\cdot\|G\|\big\}+\mathbb{P}\big\{\big\|F-\tilde F\big\|>\eta\|\tilde F\|\big\}.
\end{aligned}$$

Further,

$$\begin{aligned}
&\mathbb{P}\big\{\exists y\in\mathbb{S}^{n-1},\,\big\|(\tilde FGy)_-\big\|\le\eta\|\tilde F\|\cdot\|G\|\big\}\\
&\quad\le\mathbb{P}\big\{\exists y\in\mathbb{S}^{n-1},\,\tilde FGy\in\mathbb{R}_+^N+\eta\|\tilde F\|\cdot\|G\|B_2^N\big\}\\
&\quad\le\mathbb{P}\left\{\exists y\in\mathbb{S}^{n-1},\,\frac{Gy}{\|Gy\|}\in\tilde F^{-1}\big(\mathbb{R}_+^N\big)+\eta\|\tilde F\|\frac{\|G\|}{s_{\min}(G)}\tilde F^{-1}\big(B_2^N\big)\right\}\\
&\quad\le\mathbb{P}\left\{\exists y\in\mathbb{S}^{n-1},\,\frac{Gy}{\|Gy\|}\in\tilde F^{-1}\big(\mathbb{R}_+^N\big)+2\eta\cdot\gamma^{-1}B_2^N\right\}\\
&\qquad+\mathbb{P}\big\{\|G\|>2s_{\min}(G)\big\}\\
&\quad\le\mathbb{P}\big\{\mathrm{Im}(G)\cap\big(\tilde F^{-1}\big(\mathbb{R}_+^N\big)+2\eta\cdot\gamma^{-1}B_2^N\big)\cap\mathbb{S}^{N-1}\ne\varnothing\big\}+2e^{-N/128},
\end{aligned}\tag{9}$$

where the last estimate follows from (4).

To control the probability of escaping in (9) with help of Theorem 3, we have to estimate the Gaussian width of the set

$$\Gamma := \big(\tilde{F}^{-1}(\mathbb{R}_+^N) + 2\eta \cdot \gamma^{-1} B_2^N\big) \cap \mathbb{S}^{N-1}.$$

Note that $\Gamma \subset (1 + 2\eta \cdot \gamma^{-1})\tilde{F}^{-1}(\mathbb{R}_+^N) \cap B_2^N + 2\eta \cdot \gamma^{-1} B_2^N$. Therefore,

$$w(\Gamma) \le \big(1 + 2\eta \cdot \gamma^{-1}\big) \cdot w\big(\tilde{F}^{-1}(\mathbb{R}_+^N) \cap B_2^N\big) + 2\eta \cdot \gamma^{-1}\sqrt{N}. \tag{10}$$

It remains to bound the Gaussian width of $\tilde{F}^{-1}(\mathbb{R}_+^N) \cap B_2^N$. Denote by C the cone $\tilde{F}^{-1}(\mathbb{R}_+^N)$ and note that $C^* = \tilde{F}^t(\mathbb{R}_-^N)$. Then we have

$$\begin{aligned}\mathrm{Vol}_N\big(\tilde{F}^t(\mathbb{R}_-^N) \cap B_2^N\big) &= \big|\det(\tilde{F})\big| \cdot \mathrm{Vol}_N\big(\mathbb{R}_-^N \cap (\tilde{F}^t)^{-1}(B_2^N)\big) \\ &\ge \big|\det(\tilde{F})\big| \cdot \|\tilde{F}\|^{-N} \cdot \mathrm{Vol}_N\big(\mathbb{R}_-^N \cap B_2^N\big).\end{aligned}$$

Since $|\det(\tilde{F})| \ge \|\tilde{F}^{-1}\|^{-N}$, we get $\mathrm{Vol}_N(C^* \cap B_2^N) \ge (\gamma/2)^N \cdot \mathrm{Vol}_N(B_2^N)$. Now, applying Proposition 4, we deduce that

$$w\big(C \cap B_2^N\big) \le \sqrt{(1 - \gamma^2/8)N}. \tag{11}$$

Putting (10) and (11) together, we get that

$$w(\Gamma) \le \big(1 + 4\eta \cdot \gamma^{-1} - \gamma^2/16\big)\sqrt{N}.$$

The proof is completed by a straightforward application of Theorem 3. $\square$

As we will see in the next sections, Theorem 5 provides a way to deal with the standard Brownian motion in $\mathbb{R}^n$ and random walks W_θ on the sphere. To treat the standard walk on $\mathbb{Z}^n$, we shall derive a statement covering a rather broad class of random matrices. Let us introduce the following.

DEFINITION 6. A random variable ξ is said to have property $\mathcal{P}(\tau, \delta)$ (or safisfy condition $\mathcal{P}(\tau, \delta)$) for some $\tau, \delta \in (0, 1]$ if

$$\mathbb{P}\{\xi < -\tau\} \ge \delta.$$

A random vector X in $\mathbb{R}^n$ is said to have property $\mathcal{P}(\tau, \delta)$ for $\tau, \delta \in (0, 1]$ if for any $y \in \mathbb{S}^{n-1}$, the random variable $\langle X, y\rangle$ satisfies $\mathcal{P}(\tau, \delta)$.

Obviously, the above property holds (for some τ and δ) for any nonzero r.v. ξ with $\mathbb{E}\xi = 0$. As the next elementary lemma shows, with some additional assumptions on moments of ξ, the numbers τ and δ can be chosen as certain functions of the moments.

LEMMA 7. *Any random variable ξ such that $\mathbb{E}\xi = 0$, $\mathbb{E}\xi^2 = 1$ and $\mathbb{E}|\xi|^{2+\varepsilon} \le B < \infty$ for some $\varepsilon > 0$, has the property $\mathcal{P}(\tau, \delta)$, with τ and δ depending only on ε and B.*

PROOF. Indeed, an easy calculation shows that such ξ satisfies

$$\int_{L_\xi{}^2}^{\infty} \mathbb{P}\{\xi^2 \geq t\}\, dt \leq \frac{1}{2}$$

for some $L_\xi > 0$ depending only on B and ε. Then

$$\mathbb{E}|\xi| \geq \int_0^{L_\xi} \mathbb{P}\{|\xi| \geq t\}\, dt \geq \frac{1}{2L_\xi} \int_0^{L_\xi{}^2} \mathbb{P}\{\xi^2 \geq t\}\, dt \geq \frac{1}{4L_\xi},$$

implying, as $\mathbb{E}\max(0, -\xi) = \frac{1}{2}\mathbb{E}|\xi|$,

$$\begin{aligned}
\frac{1}{8L_\xi} &\leq \int_0^{\infty} \mathbb{P}\{\xi \leq -t\}\, dt \\
&\leq \int_0^{8L_\xi} \mathbb{P}\{\xi \leq -t\}\, dt + \int_{64L_\xi{}^2}^{\infty} \frac{1}{2\sqrt{t}} \mathbb{P}\{\xi^2 \geq t\}\, dt \\
&\leq \int_0^{8L_\xi} \mathbb{P}\{\xi \leq -t\}\, dt + \frac{1}{16L_\xi}.
\end{aligned}$$

Hence, $\mathbb{P}\{\xi < -2^{-5}L_\xi{}^{-1}\} \geq 2^{-8}L_\xi{}^{-2}$. $\square$

The following theorem will be used to treat the standard walk on $\mathbb{Z}^n$.

THEOREM 8. *For any $\tau, \delta \in (0, 1]$ and any $K > 1$, there exist L and $\eta > 0$ depending only on τ, δ and K with the following property: Let $N \geq Ln$ and let A be an $N \times n$ random matrix with independent rows having property $\mathcal{P}(\tau, \delta)$. Then for any $N \times N$ random matrix F, matrix FA satisfies*

$$\begin{aligned}
\mathbb{P}\{\exists y \in \mathbb{S}^{n-1}, FAy \in \mathbb{R}_+^N\} &\leq \exp(-\delta^2 N/4) \\
&\quad + \mathbb{P}\{\|A\| > K\sqrt{N}\} + \mathbb{P}\{\|F - \mathbf{I}\| > \eta\}.
\end{aligned}$$

PROOF. Define L as the smallest positive number satisfying

$$\left(\frac{3}{\eta}\right)^{1/L} \leq \exp(\delta^2/4),$$

where $\eta := \frac{\sqrt{\delta}\tau}{2\sqrt{2}K}$. Now, take any admissible $N \geq Ln$ and let A and F be as stated above.

Let $\mathcal{N}$ be an η-net on $\mathbb{S}^{n-1}$ of cardinality at most $(\frac{3}{\eta})^n$. In view of Lemma 1, we have

$$\begin{aligned}
&\mathbb{P}\{\exists y \in \mathbb{S}^{n-1}, FAy \in \mathbb{R}_+^N\} \\
&\quad \leq \mathbb{P}\{\exists y \in \mathbb{S}^{n-1}, \|(Ay)_-\| \leq \|(F - \mathbf{I})Ay\|\} \\
&\quad \leq \mathbb{P}\{\exists y \in \mathbb{S}^{n-1}, \|(Ay)_-\| \leq \eta\|A\|\} + \mathbb{P}\{\|F - \mathbf{I}\| > \eta\} \\
&\quad \leq \mathbb{P}\{\exists y' \in \mathcal{N}, \|(Ay')_-\| \leq 2\eta\|A\|\} + \mathbb{P}\{\|F - \mathbf{I}\| > \eta\}.
\end{aligned}$$

Further,

$$\begin{aligned}(12)\qquad &\mathbb{P}\{\exists y' \in \mathcal{N}, \|(Ay')_-\| \leq 2\eta\|A\|\}\\ &\quad \leq \mathbb{P}\{\exists y' \in \mathcal{N}, \|(Ay')_-\| \leq 2K\eta\sqrt{N}\} + \mathbb{P}\{\|A\| > K\sqrt{N}\}.\end{aligned}$$

Fix any $y' \in \mathcal{N}$. For all $i = 1, 2, \ldots, N$, the random variable $\langle Ay', e_i\rangle$ satisfies the property $\mathcal{P}(\tau, \delta)$. For any $i \leq N$, denote by χ_i the indicator function of the event $\{\langle Ay', e_i\rangle < -\tau\}$. Then $(\chi_i)_{i\leq N}$ are independent and $\mathbb{E}\chi_i \geq \delta$. Applying Hoeffding's inequality (see [9], Theorem 1), we get

$$\begin{aligned}\mathbb{P}\left\{\left|\{i \leq N : \langle Ay', e_i\rangle < -\tau\}\right| \leq \frac{\delta N}{2}\right\} &\leq \mathbb{P}\left\{\frac{1}{N}\sum_{i\leq N}(\chi_i - \mathbb{E}\chi_i) \leq -\frac{\delta}{2}\right\}\\ &\leq \exp(-\delta^2 N/2).\end{aligned}$$

Therefore, for any fixed $y' \in \mathcal{N}$, we have

$$\begin{aligned}\mathbb{P}\{\|(Ay')_-\| \leq 2K\eta\sqrt{N}\} &\leq \mathbb{P}\{\left|\{i \leq N : \langle Ay', e_i\rangle \leq -\tau\}\right| \leq 4K^2\eta^2 N/\tau^2\}\\ &\leq \exp(-\delta^2 N/2).\end{aligned}$$

Combining the last estimate with (12) and the upper estimate for $|\mathcal{N}|$, we get

$$\begin{aligned}&\mathbb{P}\{\exists y \in \mathbb{S}^{n-1}, FAy \in \mathbb{R}^N_+\}\\ &\quad \leq \left(\frac{3}{\eta}\right)^n \exp(-\delta^2 N/2) + \mathbb{P}\{\|A\| > K\sqrt{N}\} + \mathbb{P}\{\|F - \mathbf{I}\| > \eta\}.\end{aligned}$$

The result follows by the choice of L. □

REMARK 2. Theorem 8, applied to the Gaussian matrix G, gives a weaker form of Theorem 5 (with more restrictions on the choice of F). Let us emphasize that the theorems do not require F to be independent from G. This will be important in Section 5.

4. Applications to random walks in $\mathbb{R}^n$.

In this section, we will apply the statements about random matrices to the Brownian motion and the standard walk on $\mathbb{Z}^n$.

COROLLARY 9. *For any $K > 1$, there are constants L and κ depending only on K such that the following holds. Let $N \geq Ln$ and $t_1, \ldots, t_N$ be such that $t_i \geq K \cdot t_{i-1}$ for any $i = 2 \cdots N$ and $t_1 > 0$. Then*

$$\mathbb{P}\{0 \text{ belongs to the interior of } \operatorname{conv}\{\mathrm{BM}_n(t_i) : i \leq N\}\} \geq 1 - 5.5\exp(-\kappa N).$$

PROOF. Let $c_K := 1 + (K-1)^{-1/2}\sum_{j\geq 0} K^{-j/2}$ and $\gamma := c_K^{-1}\cdot(1+(K-1)^{-1/2})^{-1}$ be two constants depending only on K and take $L = 64/\gamma^2$ and $\kappa := 2L^{-2}/9$.

Denote $\delta_1 := \sqrt{t_1}$ and $\delta_i := \sqrt{t_i - t_{i-1}}$ for any $i = 2\cdots N$. Observe that for any $j < i$, we have $\delta_i \geq K^{(i-j-1)/2}\sqrt{K-1}\cdot\delta_j$.

Define F as the $N\times N$ lower triangular matrix whose entries are given by $f_{ii} = 1$ for any $i \leq N$ and $f_{ij} = \frac{\delta_j}{\delta_i}$ for any $i > j$. One can easily check that $\|F\| \leq c_K$. Moreover, the inverse of F is a lower bi-diagonal matrix with 1 on the main diagonal and $(\delta_i/\delta_{i+1})_{i<N}$ on the diagonal below. Hence, $\|F^{-1}\| \leq 1 + (K-1)^{-1/2}$, and the condition number of F satisfies

$$\|F\|\cdot\|F^{-1}\| \leq \gamma^{-1}.$$

Let $(R_i)_{i\leq N}$ be the rows of FG. One can check that $R_i = \mathrm{BM}_n(t_i)/\delta_i$ and, therefore,

$$0 \in \operatorname{conv}\{\mathrm{BM}_n(t_i) : i \leq N\} \quad\Longleftrightarrow\quad 0 \in \operatorname{conv}\{R_i : i \leq N\}.$$

Note that, by a standard separation argument, 0 does not belong to the interior of $\operatorname{conv}\{R_i : i \leq N\}$ if and only if $\operatorname{rank}(FG) < n$ or there is a vector $y \in \mathbb{S}^{n-1}$ such that $\langle FGy, e_i\rangle = \langle y, R_i\rangle \geq 0$ for any $i \leq N$, where $(e_i)_{i\leq N}$ denotes the canonical basis of $\mathbb{R}^N$. Since with probability one we have $\operatorname{rank}(FG) = n$, the result follows by applying Theorem 5 with $\tilde{F} := F$. $\square$

Suppose (t_i) is a finite increasing sequence of points in [0, 1]. The above statement tells us that if (t_i) contains a geometrically growing subsequence of length Ln for an appropriate $L > 0$ then with high probability the origin of $\mathbb{R}^n$ is contained in the interior of $\mathrm{BM}_n(t_i)$'s. We shall apply this result to the case when the t_i's are generated by the Poisson point process independent from BM_n.

Recall that *the homogeneous Poisson point process in* [0, 1] of intensity $s > 0$ is a random discrete measure N_s on [0, 1] such that (1) for each Borel subset $B \subset [0,1]$, the random variable $N_s(B)$ has the Poisson distribution with parameter $s\mu(B)$, where μ is the usual Lebesgue measure on $\mathbb{R}$, and (2) for any $j \in \mathbb{N}$ and pairwise disjoint Borel sets $B_1, B_2, \dots, B_j \subset [0,1]$, the random variables $N_s(B_1)$, $N_s(B_2), \dots, N_s(B_j)$ are jointly independent. The measure N_s admits a representation of the form

$$N_s = \sum_{i=1}^{\tau}\delta_{\xi_i},$$

where $\xi_1, \xi_2, \dots$ are i.i.d. random variables uniformly distributed on [0, 1], δ_{ξ_i} is the Dirac measure with the mass at ξ_i and τ is the random nonnegative integer with the Poisson distribution with parameter s.

Theorem 3.1 of [4] states that if τ and the points $\xi_1, \xi_2, \dots, \xi_\tau$ are generated by the homogeneous PPP in [0, 1] of intensity $s \geq n^{Cn}$ then the convex hull of

$\mathrm{BM}_n(\xi_i)$'s contains the origin in its interior with probability at least $1 - n^{-n}$. In our next statement, we weaken the assumptions on s at expense of decreasing the probability to $1 - \exp(-n)$:

COROLLARY 10. *There is a universal constant $\tilde{C} > 0$ with the following property: Let $n \in \mathbb{N}$ and let $\mathrm{BM}_n(t)$, $t \in [0, \infty)$, be the standard Brownian motion in $\mathbb{R}^n$. Further, let τ and the points $\xi_1, \xi_2, \dots, \xi_\tau$ be given by the homogeneous Poisson process on $[0, 1]$ of intensity $s \geq \exp(\tilde{C}n)$, which is independent from $\mathrm{BM}_n(t)$. Then*

$$\mathbb{P}\{0 \textit{ belongs to the interior of } \operatorname{conv}\{\mathrm{BM}_n(\xi_i) : i \leq \tau\}\} \geq 1 - \exp(-n).$$

PROOF. Let $K := 2$ and κ, L be as in Corollary 9. Then we define the constant $\tilde{C} := \max(\frac{32}{\kappa}, 8L)$. Let $n \in \mathbb{N}$ and let N_s be as stated above. Take $m := \lfloor \tilde{C}n \rfloor$ and

$$I_1 := [0, K^{-m+1}]; \qquad I_j := (K^{j-m-1}, K^{j-m}], \qquad j = 2, 3, \dots, m.$$

From the definition of N_s, we have

$$\begin{aligned}\mathbb{P}\{N_s(I_j) > 0 \text{ for all } j = 1, 2, \dots, m\} &\geq 1 - \sum_{j=1}^{m} \exp(-s\mu(I_j)) \\ &\geq 1 - m\exp(-sK^{-m}).\end{aligned}$$

In particular, with probability at least $1 - m\exp(-sK^{-m})$ the set $\{\xi_i\}_{i=1}^{\tau}$ contains a subset $\{\xi_{i_1}, \xi_{i_2}, \dots, \xi_{i_m}\}$ such that $\xi_{i_j} \in I_j$ for every admissible j, hence $\xi_{i_{j+2}} \geq K\xi_{i_j}$ for any $j \leq m - 2$. Conditioning on the realization of N_s, we obtain by Corollary 9:

$$\begin{aligned}&\mathbb{P}\{0 \text{ belongs to the interior of } \operatorname{conv}\{\mathrm{BM}_n(\xi_i) : i \leq \tau\}\} \\ &\quad\geq 1 - m\exp(-sK^{-m}) - 5.5\exp(-\kappa\lfloor m/2 \rfloor) \\ &\quad\geq 1 - \exp(-n),\end{aligned}$$

and the proof is complete. $\square$

The last result of this section concerns the standard random walk $W(j)$ on $\mathbb{Z}^n$, which is defined as a walk with independent increments such that each increment $W(j+1) - W(j)$ is uniformly distributed on the set $\{\pm e_j\}_{j \leq n}$. We note that the random variables $\langle \sqrt{n/m}W(m), y \rangle$ ($m \in \mathbb{N}$, $y \in \mathbb{S}^{n-1}$) are *not* uniformly sub-Gaussian; to be more precise, their sub-Gaussian moment depends on the dimension n. At the same time, the vectors $W(m)$ still have very strong concentration properties as the next lemma shows.

LEMMA 11. *Let $W(j)$ $(j \geq 0)$ be the standard walk on $\mathbb{Z}^n$ starting at the origin, and $m \geq n^4$ be any fixed integer. Then the vector $X := \sqrt{n/m}W(m)$ is isotropic and satisfies for any $y \in \mathbb{S}^{n-1}$:*

$$\mathbb{P}\{|\langle X, y\rangle| \geq t\} \leq \exp(-2(mn)^{1/4}) + 2\exp(-t^2/4), \qquad t > 0.$$

In particular, $\mathbb{E}|\langle X, y\rangle|^3 \leq 100$ for all $y \in \mathbb{S}^{n-1}$, and X has the property $\mathcal{P}(\tau, \delta)$ for some universal constants τ, δ.

PROOF. The isotropicity of X can be easily checked. Fix for a moment any vector $y \in \mathbb{S}^{n-1}$. The random variable $\langle X, y\rangle$ can be represented as

$$\langle X, y\rangle = \sqrt{n/m}\sum_{k=1}^{m} s_k,$$

where the variables $s_1, s_2, \dots, s_m$ are i.i.d. and each

$$s_k := \langle W(k) - W(k-1), y\rangle$$

is symmetrically distributed, has variance $\mathbb{E}{s_k}^2 = \frac{1}{n}$ and takes values in the interval $[-1, 1]$. Applying Hoeffding's inequality to the sum $\sum_{k=1}^m {s_k}^2$, we get

$$\mathbb{P}\left\{\sum_{k=1}^{m} {s_k}^2 \geq \frac{2m}{n}\right\} \leq \exp(-2m/n^2). \tag{13}$$

Further, since s_k is symmetric, the distribution of the sum $\sum_{k=1}^m s_k$ is the same as the distribution of $\sum_{k=1}^m r_k s_k$, where $r_1, r_2, \dots, r_m$ are Rademacher variables jointly independent with $s_1, s_2, \dots, s_m$. Conditioning on the values of s_k and using (13) and the Khintchine inequality, we obtain for every $t > 0$:

$$\begin{aligned}
&\mathbb{P}\left\{\left|\sum_{k=1}^{m} s_k\right| \geq mt\right\}\\
&\quad = \mathbb{P}\left\{\left|\sum_{k=1}^{m} r_k s_k\right| \geq mt\right\}\\
&\quad \leq \mathbb{P}\left\{\sum_{k=1}^{m} {s_k}^2 \geq \frac{2m}{n}\right\} + \mathbb{P}\left\{\sum_{k=1}^{m} {s_k}^2 \leq \frac{2m}{n} \text{ and } \left|\sum_{k=1}^{m} r_k s_k\right| \geq mt\right\}\\
&\quad \leq \exp(-2m/n^2) + 2\exp(-mnt^2/4).
\end{aligned}$$

Whence, in view of the bound $m \geq n^2(mn)^{1/4}$, we get

$$\mathbb{P}\{|\langle X, y\rangle| \geq t\} \leq \exp(-2(mn)^{1/4}) + 2\exp(-t^2/4), \qquad t > 0. \tag{14}$$

The condition (14), together with the bound $\|X\| \leq \sqrt{mn}$, gives $\mathbb{E}|\langle X, y\rangle|^3 \leq 100$. It remains to apply Lemma 7. $\square$

The next lemma follows from well-known concentration inequalities for sub-exponential random variables (see, e.g., [23], Corollary 5.17).

LEMMA 12. *There is a universal constant $\tilde{C} > 0$ such that for any $N \in \mathbb{N}$ and independent centered random variables $\tilde{\xi}_1, \tilde{\xi}_2, \dots, \tilde{\xi}_N$, each satisfying*

$$\mathbb{P}\{\tilde{\xi}_i \geq t\} \leq 3\exp(-t/4), \qquad t > 0, \tag{15}$$

we have

$$\mathbb{P}\left\{\sum_{i=1}^{N} \tilde{\xi}_i \geq \tilde{C}N\right\} \leq 40^{-N}. \tag{16}$$

In the next result, compared to Theorem 1.2 of [4], we decrease the bound on the number of steps N of the walk on $\mathbb{Z}^n$ sufficient to absorb the origin with high probability.

COROLLARY 13. *There is a universal constant $C > 0$ with the following property: Let $n, R \in \mathbb{N}$, $R \geq \exp(Cn)$ and let $W(j)$, $j \geq 0$, be the standard random walk on $\mathbb{Z}^n$ starting at the origin. Then*

$$\mathbb{P}\{0 \textit{ belongs to the interior of } \operatorname{conv}\{W(j) : j = 1, \dots, R\}\} \geq 1 - 2\exp(-n).$$

PROOF. *Definition of constants and the matrix A.* Let $\tau, \delta > 0$ be taken from Lemma 11 and $\tilde{C}$—from Lemma 12. Now, we define $K := 2\sqrt{\tilde{C}}$ and let L and η be taken from Theorem 8. Finally, we define $C > 0$ as the smallest positive number satisfying

$$\exp(Cn) \geq (28N)^4 \left\lceil \frac{4}{\eta^2} + 1 \right\rceil^N$$

for any $n \in \mathbb{N}$ and $N = n\lceil\max(L, 4/\delta^2)\rceil$.

Fix any numbers $n > 0$ and $R \geq \exp(Cn)$, and let $N := n\lceil\max(L, 4/\delta^2)\rceil$. Further, let t_i $(i = 0, 1, \dots, N)$ be numbers from $\{0, 1, \dots, R\}$, with $t_0 = 0$, $t_1 = (28N)^4$ and $t_i = \lceil\frac{4}{\eta^2} + 1\rceil t_{i-1}$, $i = 2, 3, \dots, N$. Denote

$$X_i := \sqrt{n}(t_i - t_{i-1})^{-1/2}\big(W(t_i) - W(t_{i-1})\big), \qquad i = 1, 2, \dots, N.$$

Then the vectors are isotropic, jointly independent and, in view of Lemma 11, satisfy

$$\mathbb{P}\{|\langle X_i, y\rangle| \geq t\} \leq \exp\big(-2(nt_i - nt_{i-1})^{1/4}\big) + 2\exp\big(-t^2/4\big), \qquad t > 0 \tag{17}$$

for all $y \in \mathbb{S}^{n-1}$. We let A to be the $N \times n$ random matrix with rows X_i.

Estimating the norm of A. Let $\mathcal{N}$ be a 1/2-net on $\mathbb{S}^{n-1}$ of cardinality at most 5^n. Fix any $y' \in \mathcal{N}$. For each $i = 1, 2, \dots, N$, let $\xi_i := \langle X_i, y'\rangle^2$, and let $\tilde{\xi}_i$ be its truncation at level $(nt_i - nt_{i-1})^{1/4}$, that is,

$$\tilde{\xi}_i(\omega) = \begin{cases} \xi_i(\omega), & \text{if } \xi_i(\omega) \leq (nt_i - nt_{i-1})^{1/4}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that, in view of (17), the variables $\tilde{\xi}_i$ satisfy (15), and

$$\mathbb{P}\{\xi_i \neq \tilde{\xi}_i\} \le 3\exp\bigl(-(nt_i - nt_{i-1})^{1/4}/4\bigr).$$

Hence, by (16) and the above estimate, we have

$$\begin{aligned}\mathbb{P}\{\|Ay'\| \ge \sqrt{\tilde{C}N}\} &= \mathbb{P}\Biggl\{\sum_{i=1}^{N} \xi_i \ge \tilde{C}N\Biggr\} \\ &\le 40^{-n} + \mathbb{P}\{\xi_i \neq \tilde{\xi}_i \text{ for some } i \in \{1,2,\dots,N\}\} \\ &\le 40^{-n} + 3\sum_{i=1}^{N} \exp\bigl(-(nt_i - nt_{i-1})^{1/4}/4\bigr) \\ &\le 40^{-n} + 3N\exp\bigl(-7Nn^{1/4}\bigr) \\ &\le 20^{-n}.\end{aligned}$$

Taking the union bound for all $y' \in \mathcal{N}$ and applying the standard approximation argument, we obtain $\|A\| \le 2\sqrt{CN} = K\sqrt{N}$ with probability at least $1 - \exp(-n)$.

Construction of the matrix F and application of Theorem 8. Let F be the $N \times N$ nonrandom lower-triangular matrix, with the entries

$$f_{ij} = \sqrt{\frac{t_j - t_{j-1}}{t_i - t_{i-1}}}, \qquad i \ge j.$$

Obviously, FA is the matrix whose ith row ($i = 1, \dots, N$) is precisely the vector

$$\sqrt{\frac{n}{t_i - t_{i-1}}}W(t_i).$$

Then, in view of the definition of t_i's, we have

$$\|F - \mathbf{I}\| \le \frac{\eta/2}{1 - \eta/2} \le \eta.$$

Finally, applying Theorem 8, we obtain

$$\begin{aligned}&\mathbb{P}\bigl\{0 \text{ belongs to the interior of } \operatorname{conv}\bigl\{W(j) : j = 1,2,\dots,R\bigr\}\bigr\} \\ &\quad \ge \mathbb{P}\bigl\{0 \text{ belongs to the interior of } \operatorname{conv}\bigl\{W(t_i) : i = 1,2,\dots,N\bigr\}\bigr\} \\ &\quad = \mathbb{P}\bigl\{\operatorname{rank} A = n \text{ and } \operatorname{Im}(FA) \cap \mathbb{R}^n_+ = \{0\}\bigr\} \\ &\quad \ge 1 - 2\exp(-n).\end{aligned}$$

□

5. Random walks on the sphere. Let $n > 1$ and $\theta \in (0, \pi/2)$. Here, we consider the Markov chain W_θ taking values on $\mathbb{S}^{n-1}$ such that the angle between two consecutive steps is θ, that is, for any $i \ge 1$ we have $\langle W_\theta(i), W_\theta(i+1)\rangle = \cos\theta$ a.s., and the direction from $W_\theta(i)$ to $W_\theta(i+1)$ is chosen uniformly at random. The latter condition means that for any $u \in \mathbb{S}^{n-1}$, the distribution of $W_\theta(i+1)$ conditioned on $W_\theta(i) = u$, is uniform on the $(n-2)$-sphere $\mathbb{S}^{n-1} \cap \{x \in \mathbb{R}^n : \langle x, u\rangle = \cos\theta\}$. See [20] for a study of these walks and some of their generalizations.

The question addressed in this section is how many steps it takes for W_θ to absorb the origin into its convex hull. Note that the answer does not depend on the distribution of the first vector $W_\theta(1)$, and we shall further assume that $W_\theta(1)$ is uniformly distributed on the sphere. The question can be equivalently reformulated as a problem of estimating $\pi/2$-covering time of W_θ. For $\phi \in (0, \pi/2]$, a *ϕ-covering* of $\mathbb{S}^{n-1}$ is any subset S of the sphere such that the geodesic distance from any point of the sphere to S is at most ϕ. Then the ϕ-covering time for W_θ is the random variable

$$T = \min\{N : \text{the set } \{W_\theta(i), i \le N\} \text{ is a } \phi\text{-covering of } \mathbb{S}^{n-1}\}.$$

A related problem of estimating ϕ-covering time of the *spherical Brownian motion* was considered in [13] and [4], for $\phi \to 0$ and $\phi = \pi/2$, respectively. It is not clear whether the argument developed in [4] can be adopted to the walks W_θ. Our approach to the above problem is based on the results of Section 3 and is completely different from the argument in [4].

The walk W_θ can be constructively described as follows: Let $Y_1, Y_2, \dots$ be a sequence of independent standard Gaussian vectors in $\mathbb{R}^n$. Let $\beta_1 := \|Y_1\|$ and define

$$W_\theta(1) := \frac{Y_1}{\|Y_1\|} = \frac{Y_1}{\beta_1}.$$

Further, for any $i \ge 1$ let

$$W_\theta(i+1) := \frac{\alpha_{i+1} W_\theta(i) + Y_{i+1}}{\beta_{i+1}}, \tag{18}$$

where

$$\begin{aligned} &\beta_{i+1} := \|\alpha_{i+1} W_\theta(i) + Y_{i+1}\| \quad \text{and} \\ &\alpha_{i+1} := \cot\theta \|P_i Y_{i+1}\| - \langle Y_{i+1}, W_\theta(i)\rangle, \qquad i \ge 1, \end{aligned} \tag{19}$$

with P_i denoting the (random) orthogonal projection onto the hyperplane orthogonal to $W_\theta(i)$. It can be easily checked that

$$\beta_i = \frac{\|P_{i-1} Y_i\|}{\sin\theta}, \qquad i \ge 2,$$

and that W_θ is the Markov process described at the beginning of the section. For any $i = 2, 3, \dots$ the coefficients α_i and β_i are random variables depending on Y_i and $W_\theta(i-1)$. Using (2) and (3), one can deduce the following concentration inequalities.

LEMMA 14. *There exist a universal constant $c > 0$ such that for $\delta_\theta := c\min(1, \cot\theta)$ and for any $i = 2, 3, \ldots$ and $\varepsilon > 0$ we have*

$$\mathbb{P}\{(1-\varepsilon)\sqrt{n}\cot\theta \le \alpha_i \le (1+\varepsilon)\sqrt{n}\cot\theta\} \ge 1 - 2\exp(-{\delta_\theta}^2\varepsilon^2 n)$$

and

$$\mathbb{P}\{(1-\varepsilon)\sin\theta/\sqrt{n} \le {\beta_i}^{-1} \le (1+\varepsilon)\sin\theta/\sqrt{n}\} \ge 1 - 2\exp(-{\delta_\theta}^2\varepsilon^2 n).$$

Moreover, (3) immediately implies

$$\mathbb{P}\{(1-\varepsilon)/\sqrt{n} \le {\beta_1}^{-1} \le (1+\varepsilon)/\sqrt{n}\} \ge 1 - 2\exp(-c\varepsilon^2 n), \qquad \varepsilon > 0, \tag{20}$$

provided that the constant c is sufficiently small. Before we state the main result of the section, let us consider the following elementary lemma.

LEMMA 15. *For any $q \in (0,1)$ and $0 < \varepsilon \le \frac{1-q}{8}$, we have*

$$\sum_{k=0}^{\infty}((1+\varepsilon)^{2k+1} - 1)q^k \le \frac{4\varepsilon}{(1-q)^2}.$$

PROOF. First, note that the conditions on ε and q imply

$$q(1+\varepsilon)^2 \le \frac{81q}{64} - \frac{9q^2}{32} + \frac{q^3}{64} \le q + \frac{17q}{64} - \frac{17q^2}{64} \le \frac{1+q}{2},$$

whence

$$1 - q(1+\varepsilon)^2 \ge \frac{1-q}{2}.$$

Using the last inequality, we obtain

$$\begin{aligned}\sum_{k=0}^{\infty}((1+\varepsilon)^{2k+1} - 1)q^k &= (1+\varepsilon)\sum_{k=0}^{\infty}(q(1+\varepsilon)^2)^k - \sum_{k=0}^{\infty} q^k \\ &= \frac{(1+\varepsilon)}{1 - q(1+\varepsilon)^2} - \frac{1}{1-q} \\ &= \frac{\varepsilon + \varepsilon q + \varepsilon^2 q}{(1-q)(1-q(1+\varepsilon)^2)} \\ &\le \frac{4\varepsilon}{(1-q)^2}.\end{aligned}$$

□

THEOREM 16. *For any $\theta \in (0, \pi/2)$ there exist $n_0 = n_0(\theta)$ and $K = K(\theta)$ depending only on θ such that the following holds: Let $n \ge n_0$ and let W_θ be the random walk on $\mathbb{S}^{n-1}$ defined above. Then for all $N \ge Kn$ we have*

$$\mathbb{P}\{0 \textit{ belongs to } \operatorname{conv}\{W_\theta(i) : i \le N\}\} \ge 1 - \exp(-n).$$

PROOF. Fix an angle $\theta \in (0, \pi/2)$. Let $\gamma := \frac{\sin\theta(1-\cos\theta)}{1+\cos\theta}$ and let η, L and κ be as in Theorem 5. Define $\varepsilon := \eta \sin\theta(1 - \cos\theta)^2/4$ and let n_0 be the smallest integer such that for all $n \geq n_0$ we have

$$5.5 \exp\bigl(-\kappa \lceil Ln \rceil\bigr) + 4\lceil Ln \rceil \exp\bigl(-\delta_\theta{}^2 \varepsilon^2 n\bigr) \leq \exp(-\mu n),$$

where $\mu = \frac{1}{2} \min(\kappa, \delta_\theta{}^2 \varepsilon^2)$ and δ_θ is taken from Lemma 14.

Fix $n \geq n_0$. First, we show that $\tilde{N} := \lceil Ln \rceil$ steps is sufficient to get the origin in the convex hull of $W_\theta(i)$ $(i \leq \tilde{N})$ with probability $1 - \exp(-\mu n)$. This shall be done by using the representation (18) for the walk W_θ and by applying Theorem 5. Then we will augment the probability estimate to $1 - \exp(-n)$ by increasing the number of steps.

Let G be the standard $\tilde{N} \times n$ Gaussian matrix with rows Y_i $(i \leq \tilde{N})$. We shall construct a random lower-triangular $\tilde{N} \times \tilde{N}$ matrix F such that the ith row of FG is $W_\theta(i)$. Define $F := (f_{ij})$ with

$$f_{ij} := \frac{\prod_{k=j+1}^{i} \alpha_k}{\prod_{k=j}^{i} \beta_k} \qquad \text{for } j < i \leq \tilde{N} \quad \text{and} \quad f_{ii} := \frac{1}{\beta_i} \qquad \text{for } i \leq \tilde{N},$$

where α_k and β_k are given by (19). Since $FG = (W_\theta(1), W_\theta(2), \dots, W_\theta(\tilde{N}))^t$, the origin does not belong to $\operatorname{conv}\{W_\theta(i) : i \leq \tilde{N}\}$ only if there exists $y \in \mathbb{S}^{n-1}$ such that $FGy \in \mathbb{R}_+^{\tilde{N}}$. Now define $\tilde{F}$ as the $\tilde{N} \times \tilde{N}$ lower triangular matrix whose entries are given by

$$\tilde{f}_{i1} = \frac{(\cos\theta)^{i-1}}{\sqrt{n}} \qquad \text{for any } i \leq \tilde{N} \quad \text{and}$$

$$\tilde{f}_{ij} := \sin\theta \frac{(\cos\theta)^{i-j}}{\sqrt{n}} \qquad \text{for } 2 \leq j \leq i.$$

It is not difficult to see that

$$\frac{\sin\theta}{\sqrt{n}} \leq \|\tilde{F}\| \leq \frac{1}{(1-\cos\theta)\sqrt{n}}. \tag{21}$$

Further, let Q be the matrix obtained from $\tilde{F}$ by multiplying the first column of $\tilde{F}$ by $\sin\theta$ and leaving the other columns unchanged. Then, clearly, $s_{\min}(Q) \leq s_{\min}(\tilde{F})$ implying $\|\tilde{F}^{-1}\| \leq \|Q^{-1}\|$. On the other hand, the inverse of Q is a lower bi-diagonal matrix with $\frac{\sqrt{n}}{\sin\theta}$ on the main diagonal and $-\cos\theta \frac{\sqrt{n}}{\sin\theta}$ on the diagonal below. Hence, $\|\tilde{F}^{-1}\| \leq \|Q^{-1}\| \leq (1+\cos\theta)\frac{\sqrt{n}}{\sin\theta}$, and the condition number of $\tilde{F}$ satisfies

$$\|\tilde{F}\| \cdot \|\tilde{F}^{-1}\| \leq \frac{1+\cos\theta}{\sin\theta(1-\cos\theta)} = \gamma^{-1}.$$

Applying Theorem 5, we get

$$\mathbb{P}\{\exists y \in \mathbb{S}^{n-1}, FGy \in \mathbb{R}_+^{\tilde{N}}\} \le 5.5\exp(-\kappa\tilde{N}) + \mathbb{P}\{\|F - \tilde{F}\| > \eta\|\tilde{F}\|\}.$$

It remains to bound the probability $\mathbb{P}\{\|F - \tilde{F}\| > \eta\|\tilde{F}\|\}$. In view of Lemma 14 and (20), with probability at least $1 - 4\tilde{N}\exp(-\delta_\theta{}^2\varepsilon^2 n)$ we have

$$|f_{ij} - \tilde{f}_{ij}| \le \bigl((1+\varepsilon)^{2(i-j)+1} - 1\bigr)\tilde{f}_{ij} \qquad \text{for any } j \le i.$$

This, together with Lemma 15 and (21), implies that

$$\|F - \tilde{F}\| \le \frac{1}{\sqrt{n}}\sum_{k=0}^{\infty}\bigl((1+\varepsilon)^{2k+1} - 1\bigr)(\cos\theta)^k \le \frac{4\varepsilon}{(1-\cos\theta)^2\sqrt{n}} \le \eta\|\tilde{F}\|$$

with probability at least $1 - 4\tilde{N}\exp(-\delta_\theta{}^2\varepsilon^2 n)$. Hence, by the restriction on n_0,

$$\mathbb{P}\{\exists y \in \mathbb{S}^{n-1}, FGy \in \mathbb{R}_+^{\tilde{N}}\} \le 5.5\exp(-\kappa\tilde{N}) + 4\tilde{N}\exp\bigl(-\delta_\theta{}^2\varepsilon^2 n\bigr) \le \exp(-\mu n),$$

where $\mu = \frac{1}{2}\min(\kappa, \delta_\theta{}^2\varepsilon^2)$. Finally, if $N \ge \lceil\mu^{-1}\rceil\tilde{N}$ then the above estimate implies

$$\begin{aligned}
&\mathbb{P}\bigl\{0 \text{ does not belong to } \operatorname{conv}\{W_\theta(i) : i \le N\}\bigr\}\\
&\quad\le \mathbb{P}\bigl\{0 \text{ does not belong to } \operatorname{conv}\{W_\theta(i) : i \le \tilde{N}\}\bigr\}^{\lceil\mu^{-1}\rceil}\\
&\quad\le \exp(-n).
\end{aligned}$$

□

6. Minimax of the *n*-dimensional Brownian motion. In this section, we will prove Theorem B which, as noted in the Introduction, is equivalent to estimate (1).

Let us give an informal description of the proof. We construct a random unit vector $\bar{v}$ in $\mathbb{R}^n$ such that with probability close to one

$$\langle\bar{v}, \mathrm{BM}_n(t)\rangle > 0 \qquad \text{for any } t \in [1, 2^{cn}]. \tag{22}$$

The construction procedure shall be divided into a series of steps. At the initial step, we produce a random vector $\bar{v}_0$ such that

$$\langle\bar{v}_0, \mathrm{BM}_n(2^i)\rangle > 0 \qquad \text{for any } i = 0, 1, \dots, cn$$

(in fact, $\bar{v}_0$ will satisfy a stronger condition). At a step k, $k \ge 1$, we update the vector $\bar{v}_{k-1}$ by adding a small perturbation in such a way that

$$\langle\bar{v}_k, \mathrm{BM}_n(2^{j2^{-k}})\rangle > 0 \qquad \text{for any } j = 0, 1, \dots, 2^k cn.$$

(Again $\bar{v}_k$ will in fact satisfy a stronger condition.) Finally, using some standard properties of the Brownian bridge, we verify that $\bar{v} := \bar{v}_{\log_2 \ln n}$ satisfies (22) with a large probability.

6.1. *Auxiliary facts.* In this subsection, we introduce several auxiliary results that will be used within the proof. The proof of the next lemma is straightforward, so we omit it.

LEMMA 17. *Let* $\mathrm{BM}_n(t)$ $(0 \le t < \infty)$ *be the standard Brownian motion in* $\mathbb{R}^n$ *and let* $0 < a < b$. *Fix any* $s \in (a, b)$ *and set*

$$w(s) := \frac{b-s}{b-a}\mathrm{BM}_n(a) + \frac{s-a}{b-a}\mathrm{BM}_n(b); \qquad u(s) := \mathrm{BM}_n(s) - w(s).$$

Then the process $u(s)$, $s \in (a, b)$, *is a Brownian bridge, and*

1. $u(s)$ *is a centered Gaussian vector with the covariance matrix*

$$\frac{(b-s)(s-a)}{b-a}\mathbf{I}_n.$$

2. *The random vector* $u(s)$ *is independent from the process* $\mathrm{BM}_n(t)$ *indexed over* $t \in (0, a] \cup [b, \infty)$.

LEMMA 18. *Let* $d, m \in \mathbb{N}$ *be such that* $m \le d/2$. *Let* $X_1, X_2, \dots, X_m$ *be independent standard Gaussian vectors in* $\mathbb{R}^d$. *Then for any nonrandom vector* $b \in S^{m-1}$, *there exists a random unit vector* $\bar{u}_b \in \mathbb{R}^d$ *such that*

$$\mathbb{P}\bigl\{\langle \bar{u}_b, X_i\rangle \ge c_{18}\sqrt{d}|b_i| \text{ for all } i = 1, 2, \dots, m\bigr\} \ge 1 - \exp(-c_{18}d),$$

where c_{18} *is a universal constant and* b_i*'s are the coordinates of* b. *Moreover,* $\bar{u}_b$ *can be defined as a Borel function of* X_i*'s and* b.

PROOF. Without loss of generality, we can assume that $b_i \neq 0$ for any $i \le m$ and that X_i's are linearly independent on the entire probability space. Denote by E the random affine subspace of $\mathbb{R}^d$ spanned by $\{|b_i|^{-1}X_i\}_{i\le m}$. Define $\bar{u}_b$ as the unique unit vector in $\mathrm{span}\{X_1, \dots, X_m\}$ such that $\bar{u}_b$ is orthogonal to E and for any $i \le m$ we have

$$\bigl\langle \bar{u}_b, |b_i|^{-1}X_i\bigr\rangle = \mathrm{d}(0, E),$$

where $\mathrm{d}(0, E)$ stands for the distance from the origin to E. Then we have

$$\sum_{i\le m}\langle \bar{u}_b, X_i\rangle^2 = \sum_{i\le m}\biggl\langle \bar{u}_b, \frac{X_i}{|b_i|}\biggr\rangle^2 |b_i|^2 = \sum_{i\le m}\mathrm{d}(0, E)^2 \cdot |b_i|^2 = \mathrm{d}(0, E)^2. \tag{23}$$

Let G be the $d \times m$ standard Gaussian matrix with columns X_i, $i = 1, 2, \dots, m$. Using the definition of $\bar{u}_b$ together with (23), we obtain for any $\tau > 0$:

$$\begin{aligned}\mathbb{P}\bigl\{\langle \bar{u}_b, X_i\rangle \ge \tau\sqrt{d}|b_i| \text{ for all } i = 1, 2, \dots, m\bigr\} &= \mathbb{P}\bigl\{\mathrm{d}(0, E) \ge \tau\sqrt{d}\bigr\}\\ &= \mathbb{P}\biggl\{\sqrt{\sum_{i\le m}\langle \bar{u}_b, X_i\rangle^2} \ge \tau\sqrt{d}\biggr\}\\ &= \mathbb{P}\bigl\{\|G^t\bar{u}_b\| \ge \tau\sqrt{d}\bigr\}\\ &\ge \mathbb{P}\bigl\{s_{\min}(G) \ge \tau\sqrt{d}\bigr\},\end{aligned}$$

where the last inequality holds since $\bar{u}_b \in \operatorname{Im} G$. The proof is completed by choosing a sufficiently small $c_{18} := \tau$ and applying (4). $\square$

LEMMA 19. *Let $q \in \mathbb{N}$ and $r \in \mathbb{R}$ with $e \le r \le \sqrt{\ln q}$, and let $g_1, g_2, \dots, g_q$ be independent standard Gaussian variables. Define a random vector $b = (b_1, b_2, \dots, b_q) \in \mathbb{R}^q$ by $b_i := \max(0, g_i - r)$, $i \le q$. Then*

$$\mathbb{P}\{\|b\| \le 4\sqrt{q}\exp(-r^2/8)\} \ge 1 - \exp(-2\sqrt{q}).$$

PROOF. Let $\lambda \in (0, 1/2)$. We have

$$\mathbb{E}e^{\lambda\|b\|^2} = \prod_{i=1}^{q} \mathbb{E}e^{\lambda {b_i}^2} = \left(1 + \int_1^\infty \mathbb{P}\{e^{\lambda {b_1}^2} \ge \tau\}\, d\tau\right)^q.$$

Next, using (2), we get

$$\begin{aligned}
\int_1^\infty \mathbb{P}\{e^{\lambda b_1^2} \ge \tau\}\, d\tau &\le (r-1)\mathbb{P}\{g_1 > r\} + \int_r^\infty \mathbb{P}\{e^{\lambda b_1^2} \ge \tau\}\, d\tau \\
&\le e^{-r^2/2} + \int_r^\infty \mathbb{P}\left\{g_1 \ge \sqrt{\frac{\ln\tau}{\lambda}}\right\} d\tau \\
&\le e^{-r^2/2} + \int_r^\infty \tau^{-1/(2\lambda)}\, d\tau \\
&= e^{-r^2/2} + \frac{r^{1-1/(2\lambda)}}{1/(2\lambda) - 1}.
\end{aligned}$$

Now, take $\lambda = (2 + \frac{r^2}{\ln r})^{-1}$ so that $\frac{1}{2\lambda} - 1 = \frac{r^2}{2\ln r}$. After replacing λ with its value, we deduce that

$$\mathbb{E}e^{\lambda\|b\|^2} \le (1 + 2e^{-r^2/2})^q \le \exp(2qe^{-r^2/2}). \tag{24}$$

Using Markov's inequality together with (24), we obtain

$$\mathbb{P}\{\lambda\|b\|^2 \ge 4qe^{-r^2/2}\} \le \exp(-2qe^{-r^2/2}) \le \exp(-2\sqrt{q}),$$

where the last inequality holds since $r \le \sqrt{\ln q}$. To complete the proof, it remains to note that

$$\frac{4qe^{-r^2/2}}{\lambda} \le 8qr^2e^{-r^2/2} \le 16qe^{-r^2/4}.$$ $\square$

6.2. *Proof of Theorem* B. Throughout this part, we assume that $c > 0$ and $n_0 \in \mathbb{N}$ are appropriately chosen constants (with c sufficiently small and n_0 sufficiently large) and $n \ge n_0$ is fixed. The admissible values for c and n_0 can be recovered from the proof, however, we prefer to avoid these technical details. Further, in order not to overload the presentation, from now on we treat certain real-valued parameters are integers. In particular, this concerns the product cn, as well as several other quantities depending on n (we will point them out later). To prove relation (1), we will construct a random unit vector $\bar{v} \in \mathbb{R}^n$ such that

$$\langle \bar{v}, \mathrm{BM}_n(t) \rangle > 0 \qquad \text{for any } t \in [1, 2^{cn}] \tag{25}$$

with probability close to one.

Let $N := cn$ and define

$$a_0 := 0 \quad \text{and} \quad a_i := 2^{i-1}, \qquad i = 1, 2, \dots, N+1.$$

The starting point of the proof is to define a random vector $\bar{v}_0$ such that $\langle \bar{v}_0, \mathrm{BM}_n(a_i) \rangle$ is large for all $i \le N+1$. For this, we will use Lemma 18 taking all coordinates of the vector b equal. It will be more convenient to state the next lemma (which is a direct consequence of Lemma 18) with generic parameters m and d instead of N, n.

LEMMA 20. *Let $d, m \in \mathbb{N}$ with $m \le d/2$ and $\mathrm{BM}_d(t)$ be the standard Brownian motion in $\mathbb{R}^d$. Then there exists a random unit vector $\bar{v}_0 \in \mathbb{R}^d$ such that*

$$\mathbb{P}\left\{ \langle \bar{v}_0, \mathrm{BM}_d(a_{i+1}) - \mathrm{BM}_d(a_i) \rangle \ge \frac{c_{18}}{2} \sqrt{\frac{d a_{i+1}}{m}}, i = 0, \dots, m \right\}$$
$$\ge 1 - \exp(-c_{18} d).$$

We note that, conditioned on a realization of $\mathrm{BM}_d(a_1), \dots, \mathrm{BM}_d(a_{m+1})$ (hence, $\bar{v}_0$), for each admissible $i \ge 1$ the process

$$\langle \bar{v}_0, \mathrm{BM}_d(a_i + t(a_{i+1} - a_i)) \rangle, \qquad t \in [0, 1],$$

is a (noncentered) Brownian bridge, and standard estimates (see, e.g., [21], page 34) together with above lemma imply that given i, we have $\langle \bar{v}_0, \mathrm{BM}_d(a_i + t(a_{i+1} - a_i)) \rangle > 0$ for all $t \in [0, 1]$ with probability at least $1 - 2\exp(-c'' d/m)$ for a universal constant c''. If $m \ll d/\ln d$ then applying the union bound we get $\langle \bar{v}_0, \mathrm{BM}_d(t) \rangle > 0$ for all $1 \le t \le a_{m+1}$ with high probability.

The argument described above is given in [4]. Note that for $m \gg d/\ln d$ the probability that the ith Brownian bridge is not positive becomes too large to apply the union bound over all i. For this reason, we significantly modified the approach of [4]. Let $M := \log_2 \ln n$ (we will further treat the quantity as an integer, omitting a truncation operation). Our construction will be iterative: after defining vector $\bar{v}_0$ as described above, we will produce a sequence of random vectors $\bar{v}_k$, $k = 1, \dots, M$, where each $\bar{v}_k$ with a high probability satisfies $\langle \bar{v}_k, \mathrm{BM}_n(t) \rangle > 0$ for all t in a certain discrete subset of $[1, 2^{cn}]$. The subset for $\bar{v}_k$ is obtained by zooming in and adding mid-points between every two neighboring points of the subset generated for $\bar{v}_{k-1}$. The size of those discrete subsets grows with k exponentially, so that the vector $\bar{v} := \bar{v}_M$ will possess the required property (25) with probability close to one. The definition of the subsets is made more precise below.

We split the interval $[0, a_{N+1}]$ into blocks. For each admissible $i \geq 0$, *the ith block* is the interval $[a_i, a_{i+1}]$. With the ith block, we associate a sequence of sets I_k^i, $k = 0, 1, \dots, M$, in the following way: for $i = 0$ we have $I_k^i = \varnothing$ for all $k \geq 0$; for $i \geq 1$, we set $I_0^i = \varnothing$ and

$$I_k^i := \{2^{1/2^k} a_i, 2^{2/2^k} a_i, 2^{3/2^k} a_i, \dots, 2^{(2^k-1)/2^k} a_i\}, \qquad k = 1, 2, \dots, M.$$

Given any $0 < k \leq M$, the vector $\bar{v}_k$ will be a small perturbation of the vector $\bar{v}_{k-1}$. The operation of constructing $\bar{v}_k$ will be referred to as *the kth step* of the construction. We must admit that the construction is rather technical. In fact, each step itself is divided into a sequence of *substeps*. To make the exposition of the proof as clear as possible, we will not provide all the details at once but instead introduce them sequentially.

At each step, to avoid issues connected with probabilistic dependencies, the already constructed vector $\bar{v}_{k-1}$ and the perturbation added to it will be defined on disjoint coordinate subspaces of $\mathbb{R}^n$. Namely, we split $\mathbb{R}^n$ into $M + 1$ coordinate subspaces as follows:

$$\mathbb{R}^n := \prod_{k=0}^{M} \mathbb{R}^{J^k},$$

where J^k are pairwise disjoint subsets of $\{1, \dots, n\}$ with $|J^k| = \tilde{c}n2^{-k/8}$ for an appropriate constant $\tilde{c}$ (chosen so that $\sum_{k \leq M} |J^k| = n$) and $\mathbb{R}^{J^k} = \operatorname{span}\{e_i\}_{i \in J^k}$. Again, for a lighter exposition we treat the quantities $\tilde{c}n2^{-k/8}$ as integers. For every $k \leq M$, define $\mathrm{P}^k : \mathbb{R}^n \to \mathbb{R}^n$ as the orthogonal projection onto $\mathbb{R}^{J^k}$.

Let $F, H : \mathbb{N} \to \mathbb{R}_+$ be a decreasing and an increasing function, respectively, satisfying the relations

$$8cF(1)^2 = \tilde{c}{c_{18}}^2 \quad \text{and} \quad \forall k \leq M, \qquad F(k) \geq C_f \geq 2H(k), \tag{26}$$

where $C_f > 0$ is a constant which will be determined later.

Now, we can state more precisely what we mean by the kth step of the construction ($k = 0, 1, \dots, M$). *The goal of the kth step is to produce a random unit vector $\bar{v}_k$ with the following properties*:

$$\text{1. } \bar{v}_k \text{ is supported on } \prod_{p=0}^{k} \mathbb{R}^{J^p}; \tag{27}$$

$$\begin{aligned} &\text{2. } \bar{v}_k \text{ is measurable with respect to the } \sigma\text{-algebra generated by} \\ &\mathrm{P}^p\big(\mathrm{BM}_n(t)\big) \text{ for all } 0 \leq p \leq k,\ t \in \bigcup_{i=0}^{N} \big(\{a_{i+1}\} \cup I_k^i\big); \end{aligned} \tag{28}$$

3. The event

$$\begin{aligned}\mathcal{E}_k := \{&\langle \bar{v}_k, \mathrm{BM}_n(t) - \mathrm{BM}_n(a_i)\rangle \geq -H(k+1)\sqrt{a_i} \text{ and} \\ &\langle \bar{v}_k, \mathrm{BM}_n(a_{i+1}) - \mathrm{BM}_n(a_i)\rangle \geq F(k+1)\sqrt{a_{i+1}} \\ &\text{for all } t \in I_k^i \text{ and } i = 0, 1, \dots, N\}\end{aligned}$$

has probability close to one.

Quantitative estimates of $\mathbb{P}(\mathcal{E}_k)$ are provided by the following lemma which will be proved in the next section.

LEMMA 21 (kth Step). *For a small enough constant $c > 0$ and a large enough $C_f > 0$, there exist F and H satisfying* (26) *such that the following holds. Let $1 \leq k \leq M$ and assume that a random unit vector $\bar{v}_{k-1}$ satisfying properties* (27), (28) *has been constructed. Then there exists a random unit vector $\bar{v}_k$ satisfying* (27)–(28) *and such that*

$$\mathbb{P}(\mathcal{E}_k) \geq \mathbb{P}(\mathcal{E}_{k-1}) - \frac{1}{n^2}.$$

PROOF OF THEOREM B. In view of the relation (26), we have

$$2F(1) = c_{18}\sqrt{\frac{\tilde{c}}{2c}} \leq c_{18}\sqrt{\frac{|J^0|}{N}}.$$

Hence, in view of Lemma 20 (applied with $m = N$ and $d = |J^0|$), there exists a random unit vector $\bar{v}_0 \in \mathbb{R}^{J^0}$ measurable with respect to the σ-algebra generated by vectors $\mathrm{P}^0(\mathrm{BM}_n(a_{i+1}) - \mathrm{BM}_n(a_i))$, $i = 0, 1, \dots, N$, and such that

$$\begin{aligned}\mathbb{P}(\mathcal{E}_0) &= \mathbb{P}\{\langle \bar{v}_0, \mathrm{BM}_n(a_{i+1}) - \mathrm{BM}_n(a_i)\rangle \geq F(1)\sqrt{a_{i+1}} \text{ for } i = 0, 1, \dots, N\} \\ &\geq 1 - \exp(-c_{18}|J^0|) \\ &\geq 1 - \frac{1}{n^2}.\end{aligned}$$

Applying Lemma 21 M times, we obtain a random unit vector $\bar{v}_M$ satisfying (27)–(28) such that

$$\mathbb{P}(\mathcal{E}_M) \geq 1 - \frac{M+1}{n^2}.$$

Note that everywhere on $\mathcal{E}_M$, we have

$$\langle \bar{v}_M, \mathrm{BM}_n(a_{i+1})\rangle \geq \langle \bar{v}_M, \mathrm{BM}_n(a_{i+1}) - \mathrm{BM}_n(a_i)\rangle \geq C_f\sqrt{a_{i+1}}$$

and

$$\langle \bar{v}_M, \mathrm{BM}_n(t)\rangle \geq \langle \bar{v}_M, \mathrm{BM}_n(a_i)\rangle - \frac{C_f}{2}\sqrt{a_i} \geq \frac{C_f}{2}\sqrt{a_i}, \qquad t \in I_k^i$$

for all $i = 0, 1, \dots, N$. Hence, denoting $Q := \{a_1, a_2, \dots, a_{N+1}\} \cup \bigcup_{i=1}^N I_M^i$, we get

$$\mathcal{E}_M \subset \left\{ \left\langle \bar{v}_M, \frac{\mathrm{BM}_n(t)}{\sqrt{t}} \right\rangle \geq \frac{C_f}{4}, t \in Q \right\}. \tag{29}$$

Now, take any two neighboring points $t_1 < t_2$ from Q. Note that, conditioned on a realization of vectors $\mathrm{BM}_n(t)$, $t \in Q$, the random process

$$X(s) = \left\langle \bar{v}_M, \frac{s\mathrm{BM}_n(t_2) + (1-s)\mathrm{BM}_n(t_1)}{\sqrt{t_2 - t_1}} \right\rangle - \left\langle \bar{v}_M, \frac{\mathrm{BM}_n(t_1 + s(t_2 - t_1))}{\sqrt{t_2 - t_1}} \right\rangle,$$

defined for $s \in [0, 1]$, is a standard Brownian bridge. Hence (see, e.g., [21], page 34), we have for any $\tau > 0$

$$\mathbb{P}\{X(s) \geq \tau \text{ for some } s \in [0, 1]\} = \exp(-2\tau^2).$$

Taking $\tau := 2\sqrt{\ln n}$, we obtain

$$\begin{aligned} &\mathbb{P}\{\langle \bar{v}_M, \mathrm{BM}_n(t)\rangle \leq \min(\langle \bar{v}_M, \mathrm{BM}_n(t_1)\rangle, \langle \bar{v}_M, \mathrm{BM}_n(t_2)\rangle) \\ &\quad - 2\sqrt{t_2 - t_1}\sqrt{\ln n} \text{ for some } t \in [t_1, t_2]\} \\ &\quad \leq \frac{1}{n^8}. \end{aligned}$$

Finally, note that, in view of (29), everywhere on $\mathcal{E}_M$ we have

$$\begin{aligned} &(t_2 - t_1)^{-1/2} \min(\langle \bar{v}_M, \mathrm{BM}_n(t_1)\rangle, \langle \bar{v}_M, \mathrm{BM}_n(t_2)\rangle) - 2\sqrt{\ln n} \\ &\quad \geq \frac{C_f}{4}\sqrt{\frac{t_1}{t_2 - t_1}} - 2\sqrt{\ln n} \\ &\quad \geq 2^{M/2-3} C_f - 2\sqrt{\ln n} \\ &\quad > 0. \end{aligned}$$

Taking the union bound over all adjacent pairs in Q (clearly, $|Q| \leq n^2$), we come to the relation

$$\mathbb{P}\{\langle \bar{v}_M, \mathrm{BM}_n(t)\rangle > 0 \text{ for all } t \in [1, 2^{cn}]\} \geq \mathbb{P}(\mathcal{E}_M) - \frac{|Q|}{n^8} \geq 1 - \frac{1}{n}. \qquad \square$$

6.3. *Proof of Lemma* 21. Let $M' = \frac{1}{4}\log_2 \ln n$. For every $k \leq M$, we split J^k into pairwise disjoint subsets J_ℓ^k, $\ell \leq M'$, with $|J_\ell^k| = c' n 2^{-(k+\ell)/8}$ for an appropriate constant c', chosen so that $\sum_{\ell \leq M'} |J_\ell^k| = |J^k|$ (to make computations lighter, we will treat the quantities $c' n 2^{-(k+\ell)/8}$, $k \leq M$, $\ell \leq M'$, as integers). For every $k \leq M$, $\ell \leq M'$, define $\mathrm{P}_\ell^k : \mathbb{R}^n \to \mathbb{R}^n$ as the orthogonal projection onto $\mathbb{R}^{J_\ell^k}$.

Further, we define two functions $f, h : \mathbb{N} \times \mathbb{N}_0 \to \mathbb{R}_+$ as follows:

1. f is decreasing in both arguments; $f(1,0) = C_f + 2^{-1/2}(1-2^{-1/4})^{-2}C_f$; for each $k > 0$ and $\ell > 0$ we have $f(k,\ell-1) - f(k,\ell) = C_f 2^{-(k+\ell)/4}$; finally, $f(k,0) = \lim_{\ell\to\infty} f(k-1,\ell)$ for all $k > 1$. The constant $C_f > 0$ is defined via the relation $8cf(1,0)^2 = \tilde{c}{c_{18}}^2$, where $\tilde{c}$ is taken from the definition of sets J^k and c_{18} comes from Lemma 18.
2. h is increasing in both arguments; $h(1,0) = 0$; for each $k > 0$ and $\ell > 0$ we have $h(k,\ell) - h(k,\ell-1) = C_h 2^{-(k+\ell)/4}$; moreover, $h(k,0) = \lim_{\ell\to\infty} h(k-1,\ell)$ for all $k > 1$. The constant C_h is defined by $C_h = 2^{-1/2}(1-2^{-1/4})^2 C_f$.

Now define $F : \mathbb{N} \to \mathbb{R}$ and $H : \mathbb{N} \to \mathbb{R}$ by $F(k) := f(k,0)$ and $H(k) := h(k,0)$ for any $k \in \mathbb{N}$. Note that F and H satisfy (26).

Fix $k \geq 1$. Assuming that the vector $\bar{v}_{k-1}$ is already constructed, the aim is to construct $\bar{v}_k$ such that the event $\mathcal{E}_k$ has large probability. The vector $\bar{v}_k$ is obtained via an embedded iteration procedure realized as a sequence of substeps. Namely, we set $\bar{v}_{k,0} := \bar{v}_{k-1}$ and inductively construct random vectors $\bar{v}_{k,\ell}$, $1 \leq \ell \leq M'$ and take $\bar{v}_k = \bar{v}_{k,M'}$. Let us give a partial description of the procedure, omitting some details.

For each $\ell = 1, 2, \dots, M'+1$ and every block $i = 0, 1, 2, \dots, N$ *the ith block statistic* is

$$\begin{aligned}\mathcal{B}_i(k,\ell) := \max\biggl(0, \max_{t\in I_k^i}\biggl\langle \bar{v}_{k,\ell-1}, \frac{\mathrm{BM}_n(a_i) - \mathrm{BM}_n(t)}{\sqrt{a_i}}\biggr\rangle - h(k,\ell),\\ \biggl\langle \bar{v}_{k,\ell-1}, \frac{\mathrm{BM}_n(a_i) - \mathrm{BM}_n(a_{i+1})}{\sqrt{a_{i+1}}}\biggr\rangle + f(k,\ell)\biggr).\end{aligned} \tag{30}$$

Note that the statistic for the zero block is simply

$$\max\bigl(0, -\langle \bar{v}_{k,\ell-1}, \mathrm{BM}_n(a_1)\rangle + f(k,\ell)\bigr).$$

The $(N+1)$-dimensional vector $(\mathcal{B}_0(k,\ell), \dots, \mathcal{B}_N(k,\ell))$ will be denoted by $\mathcal{B}(k,\ell)$. Let us also denote

$$\mathcal{I}(k,\ell) := \{i : \mathcal{B}_i(k,\ell) \neq 0\}.$$

Note that the event $\{\mathcal{I}(k, M'+1) = \varnothing\}$ is contained inside $\mathcal{E}_k$. At each substep, using information about the statistics $\mathcal{B}(k,\ell)$ and choosing an appropriate perturbation of $\bar{v}_{k,\ell-1}$ to obtain $\bar{v}_{k,\ell}$, we will control the measure of the event $\{\mathcal{I}(k,\ell+1) = \varnothing\}$, and in this way will be able to estimate the probability of $\mathcal{E}_k$ from below.

Given $\bar{v}_{k,\ell-1}$, the goal of the ℓth substep is to construct a random unit vector $\bar{v}_{k,\ell}$ such that

$$\begin{aligned}&1.\ \bar{v}_{k,\ell} \text{ is supported on } \prod_{(p,q)\precsim(k,\ell)} \mathbb{R}^{J_q^p}, \text{ where the notation}\\ &(p,q) \precsim (k,\ell) \text{ means ``} p<k \text{ or } p=k, q\leq\ell\text{''};\end{aligned} \tag{31}$$

$$\text{2. } \bar{v}_{k,\ell} \text{ is measurable with respect to the } \sigma\text{-algebra generated by} \\ \mathrm{P}_q^p\big(\mathrm{BM}_n(t)\big) \text{ for all } (p,q) \precsim (k,\ell) \text{ and } t \in \bigcup_{i=0}^{N}\big(\{a_{i+1}\} \cup I_k^i\big); \tag{32}$$

3. $\|\mathcal{B}(k,\ell+1)\|$ is typically smaller than $\|\mathcal{B}(k,\ell)\|$.

The third property will be made more precise later. For now, we note that the typical value of $\|\mathcal{B}(k,\ell)\|$ will decrease with ℓ in such a way that, after the M'th substep, the vector $\mathcal{B}(k, M'+1)$ will be zero with probability close to one.

The vector $\bar{v}_{k,\ell}$ will be defined as

$$\bar{v}_{k,\ell} = \frac{\bar{v}_{k,\ell-1} + \alpha_{k,\ell}\bar{\Delta}_{k,\ell}}{\sqrt{1+{\alpha_{k,\ell}}^2}}, \tag{33}$$

where $\bar{\Delta}_{k,\ell}$ is a random unit vector (perturbation) and $\alpha_{k,\ell} := 16^{-k-\ell}$.

The vector $\bar{\Delta}_{k,\ell}$ *will satisfy the following properties*:

$$\text{1. } \bar{\Delta}_{k,\ell} \text{ is supported on } \mathbb{R}^{J_\ell^k}; \tag{34}$$

$$\text{2. } \bar{\Delta}_{k,\ell} \text{ is measurable with respect to the } \sigma\text{-algebra generated by} \\ \mathrm{P}_q^p\big(\mathrm{BM}_n(t)\big) \text{ for all admissible } (p,q) \precsim (k,\ell),\ t \in \bigcup_{i=0}^{N}\big(\{a_{i+1}\} \cup I_k^i\big); \tag{35}$$

$$\text{3. For any subset } I \subset \{0,1,\dots,N\} \text{ such that } \mathbb{P}\big\{\mathcal{I}(k,\ell) = I\big\} > 0, \\ \bar{\Delta}_{k,\ell} \text{ is } \textit{conditionally} \text{ independent from the collection of vectors} \\ \big\{\mathrm{P}_\ell^k\big(\mathrm{BM}_n(t) - \mathrm{BM}_n(a_i)\big), t \in I_k^i \cup \{a_{i+1}\}, i \notin I\big\} \\ \text{given the event } \big\{\mathcal{I}(k,\ell) = I\big\}. \tag{36}$$

4. The event

$\mathcal{E}_{k,\ell} := \big\{\mathcal{B}_i(k,\ell+1) = 0 \text{ for all } i \in \mathcal{I}(k,\ell)\big\}$

has probability close to one.

Again, we will make the last property more precise later.

Let us sum up the construction procedure. We sequentially produce random unit vectors $\bar{v}_0 = \bar{v}_{1,0},\ \bar{v}_{1,1},\ \bar{v}_{1,2},\dots,\bar{v}_{1,M'} = \bar{v}_1 = \bar{v}_{2,0},\ \bar{v}_{2,1},\ \bar{v}_{2,2},\dots,\bar{v}_{2,M'} = \bar{v}_2 = \bar{v}_{3,0},\dots,\dots,\bar{v}_{M,M'} = \bar{v}_M$ (in the given order). Each next vector is a random perturbation of the previous one. In a certain sense [quantified with help of order statistics $\mathcal{B}(k,\ell)$], each newly produced vector is a refinement of the previous one in such a way that $\bar{v}_M = \bar{v}$ will possess the required characteristics.

In the next two lemmas, we establish certain important properties of the block statistics.

LEMMA 22 (Initial substep for block statistics). *Fix any* $1 \le k \le M$ *and assume that a random unit vector* $\bar{v}_{k,0} := \bar{v}_{k-1}$ *satisfying properties* (27) *and* (28) *has been constructed. Then*

$$\mathbb{P}\left\{\left|\mathcal{I}(k,1)\right| \le N\exp\bigl(-{C_h}^2 2^{k/2}/16\bigr) \text{ and } \|\mathcal{B}(k,1)\| \le \frac{8\sqrt{N}}{\exp({C_h}^2 2^{k/2}/32)}\right\}$$
$$\ge \mathbb{P}(\mathcal{E}_{k-1}) - 2\exp(-2\sqrt{N}).$$

PROOF. Let $i > 0$ so that $I_k^i \neq \varnothing$. For each $t \in I_k^i \setminus I_{k-1}^i$, let t_L be the maximal number in $\{a_i\} \cup I_{k-1}^i$ strictly less than t ("left neighbor") and, similarly, t_R be the minimal number in $I_{k-1}^i \cup \{a_{i+1}\}$ strictly greater than t ("right neighbor"). For every such t, let

$$w_t := \frac{t_R - t}{t_R - t_L}\mathrm{BM}_n(t_L) + \frac{t - t_L}{t_R - t_L}\mathrm{BM}_n(t_R); \qquad u_t := \mathrm{BM}_n(t) - w_t.$$

It is not difficult to see that

$$\begin{aligned}
&\left\langle \bar{v}_{k,0}, \frac{\mathrm{BM}_n(a_i) - w_t}{\sqrt{a_i}}\right\rangle \\
&\quad\le \max\left(\left\langle \bar{v}_{k,0}, \frac{\mathrm{BM}_n(a_i) - \mathrm{BM}_n(t_L)}{\sqrt{a_i}}\right\rangle, \left\langle \bar{v}_{k,0}, \frac{\mathrm{BM}_n(a_i) - \mathrm{BM}_n(t_R)}{\sqrt{a_i}}\right\rangle\right) \\
&\quad\le \max\left(0, \max_{\tau\in I_{k-1}^i}\left\langle \bar{v}_{k,0}, \frac{\mathrm{BM}_n(a_i) - \mathrm{BM}_n(\tau)}{\sqrt{a_i}}\right\rangle,\right. \\
&\qquad\left.\left\langle 2\bar{v}_{k,0}, \frac{\mathrm{BM}_n(a_i) - \mathrm{BM}_n(a_{i+1})}{\sqrt{a_{i+1}}}\right\rangle\right).
\end{aligned}$$

Hence, the ith block statistic (for $i = 0, 1, \ldots, N$) can be (deterministically) bounded as

$$\begin{aligned}
\mathcal{B}_i(k,1) &\le \max\left(0, \max_{t\in I_{k-1}^i}\left\langle \bar{v}_{k,0}, \frac{\mathrm{BM}_n(a_i) - \mathrm{BM}_n(t)}{\sqrt{a_i}}\right\rangle - h(k,1),\right. \\
&\qquad \max_{t\in I_k^i\setminus I_{k-1}^i}\left\langle \bar{v}_{k,0}, \frac{\mathrm{BM}_n(a_i) - w_t}{\sqrt{a_i}}\right\rangle - h(k,1) + \max_{t\in I_k^i\setminus I_{k-1}^i}\left\langle \bar{v}_{k,0}, \frac{-u_t}{\sqrt{a_i}}\right\rangle, \\
&\qquad \left.\left\langle \bar{v}_{k,0}, \frac{\mathrm{BM}_n(a_i) - \mathrm{BM}_n(a_{i+1})}{\sqrt{a_{i+1}}}\right\rangle + f(k,1)\right) \\
&\le \max\left(0, \max_{t\in I_{k-1}^i}\left\langle \bar{v}_{k,0}, \frac{\mathrm{BM}_n(a_i) - \mathrm{BM}_n(t)}{\sqrt{a_i}}\right\rangle - h(k,0),\right. \\
&\qquad \left.\left\langle 2\bar{v}_{k,0}, \frac{\mathrm{BM}_n(a_i) - \mathrm{BM}_n(a_{i+1})}{\sqrt{a_{i+1}}}\right\rangle + 2f(k,0)\right) \\
&\qquad + \max\left(0, \max_{t\in I_k^i\setminus I_{k-1}^i}\left\langle \bar{v}_{k,0}, \frac{-u_t}{\sqrt{a_i}}\right\rangle + h(k,0) - h(k,1)\right).
\end{aligned}$$

Let us denote the first summand in the last estimate by ξ_i, so that

$$\mathcal{B}_i(k,1) \le \xi_i + \max\Bigl(0, \max_{t\in I_k^i\setminus I_{k-1}^i} \Bigl\langle \bar{v}_{k,0}, \frac{-u_t}{\sqrt{a_i}}\Bigr\rangle + h(k,0) - h(k,1)\Bigr).$$

Note that

$$\mathcal{E}_{k-1} = \{\xi_i = 0 \text{ for all } i = 0, 1, \dots, N\}. \tag{37}$$

Further, the property (28) of the vector $\bar{v}_{k,0} = \bar{v}_{k-1}$, together with Lemma 17 and the independence of the Brownian motion on disjoint intervals, imply that the Gaussian variables $\langle \bar{v}_{k,0}, \frac{-u_t}{\sqrt{a_i}}\rangle$ are jointly independent for $t \in I_k^i \setminus I_{k-1}^i$, $i = 1, 2, \dots, N$, and the variance of each one can be estimated from above by 2^{1-k}. Thus, the vector $\mathcal{B}(k,1)$ can be majorized coordinate-wise by the vector

$$\Bigl(\xi_i + \max_{t\in I_k^i\setminus I_{k-1}^i} \bigl(0, 2^{(1-k)/2} g_t + h(k,0) - h(k,1)\bigr)\Bigr)_{i=0}^N,$$

where g_t ($t \in I_k^i \setminus I_{k-1}^i$, $i = 0, 1, \dots, N$) are i.i.d. standard Gaussians (in fact, appropriate scalar multiples of $\langle \bar{v}_{k,0}, \frac{-u_t}{\sqrt{a_i}}\rangle$). Denoting by g the standard Gaussian variable, we get from the definition of h:

$$\begin{aligned}\mathbb{P}\Bigl\{\max_{t\in I_k^i\setminus I_{k-1}^i} \bigl(0, 2^{(1-k)/2} g_t + h(k,0) - h(k,1)\bigr) > 0\Bigr\} &\le 2^k \mathbb{P}\{g > C_h 2^{k/4}/2\} \\ &\le 2^k \exp\bigl(-{C_h}^2 2^{k/2}/8\bigr) \\ &\le \frac{1}{2}\exp\bigl(-{C_h}^2 2^{k/2}/16\bigr).\end{aligned}$$

(In the last two inequalities, we assumed that C_h is sufficiently large.) Applying Hoeffding's inequality to corresponding indicators, we infer

$$\bigl|\mathcal{I}(k,1)\bigr| \le \bigl|\{i : \xi_i \neq 0\}\bigr| + N \exp\bigl(-{C_h}^2 2^{k/2}/16\bigr)$$

with probability at least $1 - \exp(-2\sqrt{N})$ [we note that, in view of the inequality $k \le M$, we have $\frac{1}{2}\exp(-{C_h}^2 2^{k/2}/16) \ge N^{-1/4}$]. Next, it is not hard to see that the Euclidean norm of $\mathcal{B}(k,1)$ is majorized (deterministically) by the sum

$$\bigl\|(\xi_i)_{i=0}^N\bigr\| + 2^{(1-k)/2}\bigl\|\bigl(\max\bigl(0, g_t - C_h 2^{k/4}/2\bigr)\bigr)_t\bigr\|,$$

with the second vector having $\sum_{i=0}^N |I_k^i \setminus I_{k-1}^i| \le 2^k N$ coordinates. Applying Lemma 19 to the second vector (note that for sufficiently large n we have $C_h 2^{k/4}/2 \le \sqrt{\ln N}$), we get

$$\bigl\|\mathcal{B}(k,1)\bigr\| \le \bigl\|(\xi_i)_{i=0}^N\bigr\| + \frac{8\sqrt{N}}{\exp({C_h}^2 2^{k/2}/32)}$$

with probability at least $1 - \exp(-2\sqrt{N})$. Combining the estimates with (37), we obtain the result. $\square$

LEMMA 23 (Subsequent substeps for block statistics). *Fix any* $1 \le k \le M$ *and* $1 < \ell \le M' + 1$ *and assume that the random unit vectors* $\bar{v}_{k,\ell-2}$ *and* $\bar{\Delta}_{k,\ell-1}$ *satisfying properties* (31)–(32) *and* (34)–(36), *respectively, are constructed, and* $\bar{v}_{k,\ell-1}$ *is defined according to formula* (33). *Then*

$$\mathbb{P}\left\{\left|\mathcal{I}(k,\ell)\right| \le N\exp\bigl(-{C_h}^2 2^{(k+\ell)/2}\bigr) \text{ and } \|\mathcal{B}(k,\ell)\| \le \frac{\sqrt{N}}{\exp({C_h}^2 2^{(k+\ell)/2})}\right\}$$
$$\ge \mathbb{P}(\mathcal{E}_{k,\ell-1}) - 2\exp(-2\sqrt{N}).$$

Moreover,

$$\mathbb{P}\{\mathcal{I}(k,\ell) \ne \varnothing\} \le N\exp\bigl(-{C_h}^2/\alpha_{k,\ell-1}\bigr) + 1 - \mathbb{P}(\mathcal{E}_{k,\ell-1}).$$

PROOF. To shorten the notation, we will use α in place of $\alpha_{k,\ell-1}$ within the proof. Using the definition of $\bar{v}_{k,\ell-1}$ in terms of $\bar{v}_{k,\ell-2}$ and $\bar{\Delta}_{k,\ell-1}$, we get for every $i = 0, 1, \dots, N$

$$\begin{aligned}
\mathcal{B}_i(k,\ell) &= \max\Biggl(0, \max_{t\in I_k^i}\left\langle\frac{\bar{v}_{k,\ell-2}+\alpha\bar{\Delta}_{k,\ell-1}}{\sqrt{1+\alpha^2}}, \frac{\mathrm{BM}_n(a_i)-\mathrm{BM}_n(t)}{\sqrt{a_i}}\right\rangle - h(k,\ell),\\
&\qquad \left\langle\frac{\bar{v}_{k,\ell-2}+\alpha\bar{\Delta}_{k,\ell-1}}{\sqrt{1+\alpha^2}}, \frac{\mathrm{BM}_n(a_i)-\mathrm{BM}_n(a_{i+1})}{\sqrt{a_{i+1}}}\right\rangle + f(k,\ell)\Biggr)\\
&\le \frac{\mathcal{B}_i(k,\ell-1)}{\sqrt{1+\alpha^2}}\\
&\quad + \max\Biggl(0, \max_{t\in I_k^i}\left\langle\alpha\bar{\Delta}_{k,\ell-1}, \frac{\mathrm{BM}_n(a_i)-\mathrm{BM}_n(t)}{\sqrt{a_i}}\right\rangle\\
&\quad + h(k,\ell-1) - h(k,\ell),\\
&\quad \left\langle\alpha\bar{\Delta}_{k,\ell-1}, \frac{\mathrm{BM}_n(a_i)-\mathrm{BM}_n(a_{i+1})}{\sqrt{a_{i+1}}}\right\rangle + \sqrt{1+\alpha^2}f(k,\ell) - f(k,\ell-1)\Biggr).
\end{aligned}$$

Let us denote the second summand by η_i so that

$$\mathcal{B}_i(k,\ell) \le \frac{\mathcal{B}_i(k,\ell-1)}{\sqrt{1+\alpha^2}} + \eta_i.$$

Fix for a moment any subset I of $\{0,1,\dots,N\}$ such that $\mathbb{P}\{\mathcal{I}(k,\ell-1) = I\} > 0$. A crucial observation is that, conditioned on the event $\mathcal{I}(k,\ell-1) = I$, the variables η_i, $i \notin I$, are jointly independent. This follows from properties (34), (36) of $\bar{\Delta}_{k,\ell-1}$ and from independence of the Brownian motion on disjoint intervals. Next, the same properties tell us that, conditioned on $\mathcal{I}(k,\ell-1) = I$, each variable $\langle\bar{\Delta}_{k,\ell-1}, \frac{\mathrm{BM}_n(a_i)-\mathrm{BM}_n(t)}{\sqrt{a_i}}\rangle$, $t \in I_k^i$, and $\langle\bar{\Delta}_{k,\ell-1}, \frac{\mathrm{BM}_n(a_i)-\mathrm{BM}_n(a_{i+1})}{\sqrt{a_{i+1}}}\rangle$ have Gaussian distributions with variances at most 1. Further, note that, by the choice of α

and the functions f and h, we have

$$\sqrt{1+\alpha^2} f(k,\ell) - f(k,\ell-1) \le h(k,\ell-1) - h(k,\ell) = -C_h 2^{(-k-\ell)/4}.$$

Thus, denoting by g the standard Gaussian variable, we get

$$\begin{aligned}\mathbb{P}\{\eta_i > 0 | \mathcal{I}(k,\ell-1) = I\} &\le 2^k \mathbb{P}\{g > \alpha^{-1} C_h 2^{(-k-\ell)/4}\} \\ &\le \tfrac{1}{2}\exp(-{C_h}^2 \alpha^{-1}), \qquad i \in \{0,1,\dots,N\} \setminus I.\end{aligned} \tag{38}$$

Hence, by Hoeffding's inequality [note that $\exp(-{C_h}^2 2^{(k+\ell)/2}) > 2N^{-1/4}$]:

$$\mathbb{P}\{|\{i \notin I : \eta_i > 0\}| \ge N \exp(-{C_h}^2 2^{(k+\ell)/2}) | \mathcal{I}(k,\ell-1) = I\} \le \exp(-2\sqrt{N}).$$

Next, it is not difficult to see that for any $\tau > 0$ and $i \notin I$

$$\begin{aligned}&\mathbb{P}\{\eta_i^2 \ge \tau | \mathcal{I}(k,\ell-1) = I\} \\ &\quad\le 2^k \mathbb{P}\{\max(0, \alpha g - C_h 2^{(-k-\ell)/4})^2 \ge \tau\} \\ &\quad\le 1 - \exp(-2^{k+1}\mathbb{P}\{\max(0, \alpha g - C_h 2^{(-k-\ell)/4})^2 \ge \tau\}) \\ &\quad\le 1 - \mathbb{P}\{\max(0, \alpha g - C_h 2^{(-k-\ell)/4})^2 < \tau\}^{2^{k+1}} \\ &\quad\le \mathbb{P}\Bigg\{\sum_{j=1}^{2^{k+1}} \max(0, \alpha g_j - C_h 2^{(-k-\ell)/4})^2 \ge \tau\Bigg\} \\ &\quad\le \mathbb{P}\Bigg\{\sum_{j=1}^{2^{k+1}} \max(0, \alpha g_j - 4\alpha C_h 2^{(k+\ell)/4})^2 \ge \tau\Bigg\},\end{aligned}$$

where g_j $(j = 1,2,\dots,2^{k+1})$ are i.i.d. copies of g. Hence, the conditional c.d.f. of $\|(\eta_i)_{i\notin I}\|$ given $\mathcal{I}(k,\ell-1) = I$ majorizes the c.d.f. of

$$\alpha \big\| \big(\max(0, g_j - 4C_h 2^{(k+\ell)/4})\big)_{j=1}^{2^{k+1}N} \big\| =: \alpha Z$$

for i.i.d. standard Gaussians g_j, $j = 1,2,\dots,2^{k+1}N$. Applying Lemma 19 (note that $4C_h 2^{(k+\ell)/4} \le \sqrt{\ln N}$), we obtain

$$\begin{aligned}&\mathbb{P}\Bigg\{\|(\eta_i)_{i\notin I}\| > \frac{\sqrt{N}}{\exp({C_h}^2 2^{(k+\ell)/2})} \Big| \mathcal{I}(k,\ell-1) = I\Bigg\} \\ &\quad\le \mathbb{P}\Bigg\{Z > \frac{\alpha^{-1}\sqrt{N}}{\exp({C_h}^2 2^{(k+\ell)/2})} \Big| \mathcal{I}(k,\ell-1) = I\Bigg\} \\ &\quad\le \mathbb{P}\Bigg\{Z > \frac{4\sqrt{2^{k+1}N}}{\exp(2{C_h}^2 2^{(k+\ell)/2})} \Big| \mathcal{I}(k,\ell-1) = I\Bigg\} \\ &\quad\le \exp(-2\sqrt{N}).\end{aligned}$$

Clearly, $\mathcal{B}_i(k, \ell - 1) = 0$ for all $i \notin I$ given $\mathcal{I}(k, \ell - 1) = I$. Hence, the above estimates give

$$\begin{aligned}
&\mathbb{P}\Big\{\big|\mathcal{I}(k,\ell)\big| \ge N\exp\big(-C_h{}^2 2^{(k+\ell)/2}\big) \\
&\qquad \text{or } \|\mathcal{B}(k,\ell)\| > \frac{\sqrt{N}}{\exp(C_h{}^2 2^{(k+\ell)/2})}\Big|\mathcal{I}(k,\ell-1) = I\Big\} \\
&\qquad\quad \le \mathbb{P}\big\{\mathcal{B}_i(k,\ell) > 0 \text{ for some } i \in I | \mathcal{I}(k,\ell-1) = I\big\} + 2\exp(-2\sqrt{N}).
\end{aligned}$$

Summing over all admissible subsets I, we get

$$\begin{aligned}
&\mathbb{P}\Big\{\big|\mathcal{I}(k,\ell)\big| \ge N\exp\big(-C_h{}^2 2^{(k+\ell)/2}\big) \text{ or } \|\mathcal{B}(k,\ell)\| > \frac{\sqrt{N}}{\exp(C_h{}^2 2^{(k+\ell)/2})}\Big\} \\
&\quad \le 2\exp(-2\sqrt{N}) \\
&\qquad + \sum_I \mathbb{P}\big\{\mathcal{B}_i(k,\ell) > 0 \text{ for some } i \in I | \mathcal{I}(k,\ell-1) = I\big\}\mathbb{P}\big\{\mathcal{I}(k,\ell-1) = I\big\} \\
&\quad = 2\exp(-2\sqrt{N}) + \mathbb{P}\big\{\mathcal{B}_i(k,\ell) > 0 \text{ for some } i \in \mathcal{I}(k,\ell-1)\big\} \\
&\quad = 2\exp(-2\sqrt{N}) + 1 - \mathbb{P}(\mathcal{E}_{k,\ell-1}).
\end{aligned}$$

By analogous argument, as a corollary of (38),

$$\mathbb{P}\big\{\mathcal{I}(k,\ell) \ne \varnothing\big\} \le N\exp\big(-C_h{}^2\alpha^{-1}\big) + 1 - \mathbb{P}(\mathcal{E}_{k,\ell-1}). \qquad \square$$

The next lemma, which is the heart of the proof, provides a construction procedure for the perturbation $\bar{\Delta}_{k,\ell}$. Given vector $\bar{v}_{k,\ell-1}$, we examine its block statistics $\mathcal{B}(k,\ell)$, and define the perturbation in such a way that its inner product with increments of the Brownian motion is large on bad blocks $\mathcal{I}(k,\ell)$ [in fact, it will be proportional to the values of corresponding $\mathcal{B}_i(k,\ell)$], and random on other blocks. This is achieved using Lemma 18.

LEMMA 24 (Construction of $\bar{\Delta}_{k,\ell}$). *Let $1 \le k \le M$ and $1 \le \ell \le M'$ and assume that the random unit vector $\bar{v}_{k,\ell-1}$ satisfying properties (31) and (32) has been constructed. Then one can construct a random unit vector $\bar{\Delta}_{k,\ell}$ satisfying properties (34)–(36) and such that*

$$\begin{aligned}
&\mathbb{P}(\mathcal{E}_{k,\ell}) \ge \mathbb{P}(\mathcal{E}_{k,\ell-1}) - 3\exp(-\sqrt{N}) \qquad \textit{if } \ell > 1, \quad \textit{or} \\
&\mathbb{P}(\mathcal{E}_{k,\ell}) \ge \mathbb{P}(\mathcal{E}_{k-1}) - 3\exp(-\sqrt{N}) \qquad \textit{if } \ell = 1.
\end{aligned}$$

PROOF. Fix for a moment any subset $I \subset \{0, 1, \dots, N\}$ such that the event

$$\Gamma_I = \big\{\mathcal{I}(k,\ell) = I\big\}$$

has a nonzero probability. If $|I| > N\exp(-C_h{}^2 2^{(k+\ell)/2}/32)$ then define a random vector $\bar{\Delta}^I_{k,\ell}$ on Γ_I by setting $\bar{\Delta}^I_{k,\ell} := u$ for an arbitrary fixed unit vector $u \in \mathbb{R}^{J^k_\ell}$. Otherwise, if $|I| \le N\exp(-C_h{}^2 2^{(k+\ell)/2}/32)$, we proceed as follows:

Define a set of double indices

$$T_I := \{(i,p) : i \in I \setminus \{0\}, p \in \{1,\dots,2^k - 1\}\} \cup \bigcup_{i\in I}\{(i,0)\}.$$

For each $(i,p) \in T_I$, define an increment $X_{i,p}$ on the probability space $(\Gamma_I, \mathbb{P}(\cdot|\Gamma_I))$ by

$$X_{i,p} := \frac{\mathrm{P}^k_\ell(\mathrm{BM}_n(t_{i,p+1}) - \mathrm{BM}_n(t_{i,p}))}{\sqrt{t_{i,p+1} - t_{i,p}}},$$

where $t_{i,p} = 2^{i-1+p2^{-k}}$ for $p = 0, 1, \dots, 2^k$ and $i \in I \setminus \{0\}$; additionally, if $0 \in I$, then $t_{0,1} = 1$ and $t_{0,0} = 0$.

Note that $\mathcal{B}(k,\ell)$ is measurable with respect to the σ-algebra generated by processes $\mathrm{P}^q_s\mathrm{BM}_n(t)$, $(q,s) \precsim (k,\ell-1)$, where the notation "$\precsim$" is taken from (31); see formula (30). It implies that $\mathrm{P}^k_\ell(\mathrm{BM}_n(t))$ (on Ω) is independent from the event Γ_I; moreover, considered on the space $(\Gamma_I, \mathbb{P}(\cdot|\Gamma_I))$, the set $\{X_{i,p}, (i,p) \in T_I\}$ is a collection of standard Gaussian vectors, such that all $X_{i,p}$ and the vector $\mathcal{B}(k,\ell)$ are *jointly independent*. Let us define a random vector $\tilde{b}^I \in \mathbb{R}^{T_I}$ on $(\Gamma_I, \mathbb{P}(\cdot|\Gamma_I))$ by

$$\tilde{b}^I_{i,p} = \begin{cases} 2^{-k/2}\mathcal{B}_i(k,\ell)/\|\mathcal{B}(k,\ell)\|, & \text{if } \mathcal{B}(k,\ell) \neq \mathbf{0}; \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $\|\tilde{b}^I\| \le 1$ (deterministically) and that

$$|T_I| \le 2^k|I| \le 2^k N\exp(-C_h{}^2 2^{(k+\ell)/2}/32) \le \tfrac{1}{2}|J^k_\ell|.$$

(In the last estimate, we used the assumption that C_h is a large constant.) Hence, in view of Lemma 18, there exists a random unit vector $\bar{\Delta}^I_{k,\ell}$ on the space $(\Gamma_I, \mathbb{P}(\cdot|\Gamma_I))$ with values in $\mathbb{R}^{J^k_\ell}$, which is a Borel function of $X_{i,p}$ and $\tilde{b}^I$, and such that

$$\begin{aligned}\mathbb{P}\Big\{\langle\bar{\Delta}^I_{k,\ell}, X_{i,p}\rangle \ge c_{18}\sqrt{|J^k_\ell|}\tilde{b}^I_{i,p} \text{ for all } (i,p) \in T_I|\Gamma_I\Big\} &\ge 1 - \exp(-c_{18}|J^k_\ell|) \\ &\ge 1 - \exp(-\sqrt{N}).\end{aligned}$$

It will be convenient for us to denote by $\tilde{\Gamma}_I$ the event

$$\Big\{\langle\bar{\Delta}^I_{k,\ell}, X_{i,p}\rangle \ge c_{18}\sqrt{|J^k_\ell|}\tilde{b}^I_{i,p} \text{ for all } (i,p) \in T_I\Big\} \subset \Gamma_I.$$

By gluing together $\bar{\Delta}^I_{k,\ell}$ for all I, we obtain a random vector $\bar{\Delta}_{k,\ell}$ defined on the entire probability space Ω.

Clearly, $\bar{\Delta}_{k,\ell}$ satisfies properties (34) and (35). Next, on each Γ_I with $\mathbb{P}(\Gamma_I) > 0$ the vector $\bar{\Delta}_{k,\ell}$ was defined as a Borel function of $\mathcal{B}(k,\ell)$ and $\mathrm{P}_\ell^k(\mathrm{BM}(t) - \mathrm{BM}(\tau))$, $t, \tau \in I_k^i \cup \{a_i, a_{i+1}\}$, $i \in I$, so, in view of independence of the Brownian motion on disjoint intervals, $\bar{\Delta}_{k,\ell}$ satisfies (36).

Finally, we shall estimate the probability of $\mathcal{E}_{k,\ell}$. Define

$$\mathcal{E} = \biggl\{ |\mathcal{I}(k,\ell)| \le N \exp(-{C_h}^2 2^{(k+\ell)/2}/32) \text{ and } \|\mathcal{B}(k,\ell)\| \le \frac{\sqrt{N}}{\exp({C_h}^2 2^{(k+\ell)/2}/64)} \biggr\}.$$

Note that, according to Lemmas 22 and 23, the probability of $\mathcal{E}$ can be estimated from below by $\mathbb{P}(\mathcal{E}_{k,\ell-1}) - 2\exp(-2\sqrt{N})$ for $\ell > 1$ and $\mathbb{P}(\mathcal{E}_{k-1}) - 2\exp(-2\sqrt{N})$ for $\ell = 1$.

Take any subset $I \subset \{0, 1, \dots, N\}$ with $|I| \le N \exp(-{C_h}^2 2^{(k+\ell)/2}/32)$ and such that $\tilde{\Gamma}_I \cap \mathcal{E} \ne \varnothing$, and let $\omega \in \tilde{\Gamma}_I \cap \mathcal{E}$. If $\mathcal{I}(k,\ell) = \varnothing$ at point ω then, obviously, $\omega \in \mathcal{E}_{k,\ell}$. Otherwise, we have

$$\begin{aligned} &\biggl\langle \bar{\Delta}_{k,\ell}(\omega), \frac{\mathrm{BM}_n(t_{i,p+1})(\omega) - \mathrm{BM}_n(t_{i,p})(\omega)}{\sqrt{t_{i,p+1} - t_{i,p}}} \biggr\rangle \\ &\qquad \ge \frac{c_{18} 2^{-k/2} \sqrt{|J_\ell^k|} \mathcal{B}_i(k,\ell)(\omega)}{\|\mathcal{B}(k,\ell)(\omega)\|} \qquad \text{for all } (i,p) \in T_I, \end{aligned}$$

whence, using the estimate $t_{i,p+1} - t_{i,p} \ge \frac{2^{i-k}}{4}$ $[(i,p) \in T_I]$, we obtain for any $i \in I$ and $t \in I_k^i \cup \{a_{i+1}\}$:

$$\begin{aligned} &\langle \bar{\Delta}_{k,\ell}(\omega), \mathrm{BM}_n(t)(\omega) - \mathrm{BM}_n(a_i)(\omega) \rangle \\ &\qquad = \sum_{p: t_{i,p} < t} \langle \bar{\Delta}_{k,\ell}(\omega), \mathrm{BM}_n(t_{i,p+1})(\omega) - \mathrm{BM}_n(t_{i,p})(\omega) \rangle \\ &\qquad \ge \frac{c_{18} 2^{-k-1} \sqrt{a_{i+1} |J_\ell^k|} \mathcal{B}_i(k,\ell)(\omega)}{\|\mathcal{B}(k,\ell)(\omega)\|}. \end{aligned}$$

Further,

$$\frac{c_{18} 2^{-k-1} \sqrt{|J_\ell^k|}}{\|\mathcal{B}(k,\ell)(\omega)\|} \ge \frac{c_{18} 2^{-k-1} \sqrt{c' n 2^{(-k-\ell)/8}} \exp({C_h}^2 2^{(k+\ell)/2}/64)}{\sqrt{N}} \ge \frac{1}{\alpha_{k,\ell}}.$$

Using the definition of $\bar{v}_{k,\ell}$ in terms of $\bar{v}_{k,\ell-1}$ and $\bar{\Delta}_{k,\ell}$ and the above estimates, we get

$$\begin{aligned} &\biggl\langle \bar{v}_{k,\ell}(\omega), \frac{\mathrm{BM}_n(t)(\omega) - \mathrm{BM}_n(a_i)(\omega)}{\sqrt{a_i}} \biggr\rangle \\ &\qquad \ge \frac{\alpha_{k,\ell}}{\sqrt{1 + {\alpha_{k,\ell}}^2}} \biggl\langle \bar{\Delta}_{k,\ell}(\omega), \frac{\mathrm{BM}_n(t)(\omega) - \mathrm{BM}_n(a_i)(\omega)}{\sqrt{a_i}} \biggr\rangle \end{aligned}$$

$$
\begin{aligned}
&- \frac{h(k,\ell) + \mathcal{B}_i(k,\ell)(\omega)}{\sqrt{1+\alpha_{k,\ell}{}^2}} \\
&\geq \frac{-h(k,\ell)}{\sqrt{1+\alpha_{k,\ell}{}^2}} \\
&\geq -h(k,\ell+1), \qquad t \in I_k^i, i \in I,
\end{aligned}
$$

and, similarly,

$$
\left\langle \bar{v}_{k,\ell}(\omega), \frac{\mathrm{BM}_n(a_{i+1})(\omega) - \mathrm{BM}_n(a_i)(\omega)}{\sqrt{a_{i+1}}} \right\rangle \geq \frac{f(k,\ell)}{\sqrt{1+\alpha_{k,\ell}{}^2}} \geq f(k,\ell+1), \qquad i \in I.
$$

Thus, by the definition of the event $\mathcal{E}_{k,\ell}$, we get $\omega \in \mathcal{E}_{k,\ell}$.

The above argument shows that

$$
\mathbb{P}(\mathcal{E}_{k,\ell}) \geq \sum_I \mathbb{P}(\tilde{\Gamma}_I \cap \mathcal{E}),
$$

where the sum is taken over all I with $|I| \leq N \exp(-C_h{}^2 2^{(k+\ell)/2}/32)$. Finally,

$$
\sum_I \mathbb{P}(\tilde{\Gamma}_I \cap \mathcal{E}) \geq \sum_I \mathbb{P}(\Gamma_I \cap \mathcal{E}) - \sum_I \mathbb{P}(\Gamma_I \setminus \tilde{\Gamma}_I) \geq \mathbb{P}(\mathcal{E}) - \exp(-\sqrt{N}),
$$

and we get the result. □

PROOF OF LEMMA 21. As before, we set $\bar{v}_{k,0} := \bar{v}_{k-1}$. Consecutively applying Lemma 24 and formula (33) M' times, we obtain a random unit vector $\bar{v}_{k,M'}$ satisfying (31) and (32). Moreover, the same lemma provides the estimate

$$
\mathbb{P}(\mathcal{E}_{k,\mathcal{M}'}) \geq \mathbb{P}(\mathcal{E}_{k-1}) - 3M' \exp(-\sqrt{N}).
$$

Then, in view of Lemma 23 and the definition of M', we have

$$
\mathbb{P}\{\mathcal{I}(k, M'+1) \neq \varnothing\} \leq N \exp(-C_h{}^2/\alpha_{k,M'}) + 1 - \mathbb{P}(\mathcal{E}_{k,M'}) \leq \frac{1}{n^2} + 1 - \mathbb{P}(\mathcal{E}_{k-1}).
$$

Combining the above estimate with the definition of $\mathcal{E}_k$, we get for $\bar{v}_k := \bar{v}_{k,M'}$ that

$$
\mathbb{P}(\mathcal{E}_k) \geq \mathbb{P}(\mathcal{E}_{k-1}) - \frac{1}{n^2}. \qquad \square
$$

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