

CONTINUUM PERCOLATION FOR GAUSSIAN ZEROES AND GINIBRE EIGENVALUES

BY SUBHROSHEKHAR GHOSH*,
MANJUNATH KRISHNAPUR† AND YUVAL PERES‡

*Princeton University**, *Indian Institute of Science†* and *Microsoft Research‡*

We study continuum percolation on certain negatively dependent point processes on \mathbb{R}^2 . Specifically, we study the Ginibre ensemble and the planar Gaussian zero process, which are the two main natural models of translation invariant point processes on the plane exhibiting local repulsion. For the Ginibre ensemble, we establish the uniqueness of infinite cluster in the supercritical phase. For the Gaussian zero process, we establish that a non-trivial critical radius exists, and we prove the uniqueness of infinite cluster in the supercritical regime.

1. Introduction. Let Π be a simple point process in Euclidean plane. We place open disks of the same radius r around each point of Π , and say that two points are neighbours if the corresponding disks overlap. Two points in Π are connected if there is a sequence of neighbouring points of Π that include these two points. We can then study the statistical properties of the maximal connected components (referred to as “clusters”) of the points of Π . Of particular interest are the infinite cluster(s). This is the basic setting of the continuum percolation model, also referred to as the Boolean model.

Using an easy coupling argument, we will show (in Section 1.1) that the probability that an infinite cluster exists is an increasing function of the radius of the disks. We say that there is a non-trivial critical radius if there exists an $0 < r_c < \infty$ such that the probability of having an infinite cluster is zero when $0 < r < r_c$ and the same probability is strictly positive when $r_c < r < \infty$. For $r_c < r < \infty$, one can ask whether the infinite cluster is unique. For point processes which are ergodic under the action of translations, the event that there is an infinite cluster is translation-invariant and, therefore, its probability is either 0 or 1. Similarly, the number of infinite clusters is a translation-invariant random variable and, therefore, a.s. a constant.

In this paper, we focus on the two main natural examples of repelling point processes on the plane: the Ginibre ensemble, arising as weak limits of certain random matrix eigenvalues, and the Gaussian zero process arising as weak limits

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of zeroes of certain random polynomials. The latter process will be abbreviated as the GAF zero process. For details on these models, see Section 2.

In [1] (see Corollary 3.7 and the discussion thereafter), it has been shown that there exists a non-zero and finite critical radius for the Ginibre ensemble.

In this paper, we prove the following theorems.

THEOREM 1.1. *In the Boolean percolation model on the Ginibre ensemble, a.s. there is exactly one infinite cluster in the supercritical regime.*

THEOREM 1.2. *In the Boolean percolation model on the GAF zero process, there exists a non-zero and finite critical radius. Moreover, in the supercritical regime, a.s. there is exactly one infinite cluster.*

Continuum percolation is well studied in theoretical and applied probability, as a model of communication networks, disease-spreading through a forest and many other phenomena. This model, also referred to as the Gilbert disk model or the Boolean model, is almost as old as the more popular discrete bond percolation theory. It was introduced by Gilbert in 1961 [10]. In the subsequent years, it has been studied extensively by different authors, such as [12, 17, 18] and [24], among others. Closely related models such as random geometric graphs, random connection models, face percolation in random Voronoi tessellations have also been studied. For a detailed discussion of continuum percolation and related models, we refer the reader to [17] and [3]. For further details on point processes, we refer to [5].

Much of the literature so far has focused on studying continuum percolation where the underlying point process Π is either a Poisson process or a variant thereof. Most of these models exhibit some kind of spatial independence. This property is extremely useful in the study of continuum percolation on these models, for example, the spatial independence enables us to carry over Peierls type argument from discrete percolation theory for establishing phase transitions in the existence of infinite clusters, or Burton and Keane-type arguments in order to prove uniqueness of infinite clusters.

While the Poisson process is the most extensively studied point process, the spatial independence built into it makes it less effective as a model for many natural phenomena. This makes it of interest to study point processes with non-trivial spatial correlation, particularly those where the points exhibit repulsive behaviour. On the complex plane, the main natural examples of translation-invariant point processes exhibiting repulsion are the Ginibre ensemble and the Gaussian zero process. The latter process is also known as the Gaussian analytic function (GAF) zero process. The former arises as weak limits of eigenvalues of (non-Hermitian) random matrices, while the latter arises as weak limits of zeros of Gaussian polynomials. For precise definitions of these processes, we refer the reader to Section 2.

The Ginibre ensemble was introduced by the physicist Ginibre [11] as a physical model based on non-Hermitian random matrices. In the mathematics literature,

it has been studied by [25] and [14] among others. The Gaussian zero process also has been studied in either field; see, for example, [2, 7, 21, 26–28]. We refer the reader to [19] for a survey. These models are distinguished elements in broader classes of repulsive point processes. For example, the Ginibre ensemble is essentially the unique determinantal process on the plane whose kernel $K(z, w)$ is holomorphic in the first variable, and conjugate holomorphic in the second [14]. The Gaussian zero process is essentially unique (up to scaling) among the zero sets of Gaussian power series in that its distribution is invariant under translations [26]. For an exposition on both the processes, we refer the reader to [13].

The strong spatial correlation present in the above models severely limits the effectiveness of standard independence-based arguments from the Poisson setting while studying continuum percolation. Our aim in this paper is to study continuum percolation on the two natural models of repulsive point processes mentioned above, and establish the basic results. Namely, there is indeed a non-trivial critical radius, and the infinite cluster is unique when we are in the supercritical regime.

While the spatial independence of the Poisson process is not available in these models, we observe that this obstacle can be largely overcome if we can obtain detailed understanding of spatial conditioning in these point processes. Recently, such understanding has been obtained in [9], where it has been shown that for a given domain \mathcal{D} , the point configuration outside \mathcal{D} determines a.s. the number of points in \mathcal{D} (in the Ginibre ensemble) and their number and the centre of mass (in the Gaussian zeroes ensemble), and “nothing further”. For a precise statement of the results, we refer the reader to the Theorems 5.3, 5.4, 5.5 and 5.6 quoted in this paper. In the present work, we demonstrate that along with certain estimates on the strength of spatial dependence, this understanding is sufficient to overcome the problem of lack of independence, and answer the basic questions in continuum percolation on these two processes.

For determinantal point processes in Euclidean space, it is known that a non-trivial critical radius exists; see, for example, [1]. This covers the Ginibre ensemble. The uniqueness of the infinite cluster (in the supercritical regime), however, was not known, and this is proved in Section 5.2. For the Gaussian zero process, both the existence of a non-trivial critical radius and the uniqueness of the infinite cluster (when one exists) are new results, and are established in Sections 4 and 5.3, respectively.

In the case of the GAF zero process, while proving our main results we derive new estimates for hole probabilities. Let $B(0; R)$ be the disk with centre at the origin and radius R . The hole probability for $B(0; R)$ is the probability $p(R)$ that $B(0; R)$ has no GAF zeroes. It has been studied in detail in [27], and culminated in the work of Nishry [23] where he obtained the precise asymptotics as $R \rightarrow \infty$. It turns out that as $R \rightarrow \infty$ we have $-\log p(R)/R^4 \rightarrow c$ where $c > 0$ is a constant. In the setting of our problem, however, we need to understand hole probabilities for much more general sets than disks.

Let us divide the plane into $\theta \times \theta$ squares given by the grid $\theta\mathbb{Z}^2$, each of them being called a standard square. Then we have the following.

THEOREM 1.3. *Let Γ be a connected set composed of L standard squares of side length θ . Let E and F_k denote the events that each standard square in Γ has no zeroes and has $\geq k$ zeroes, respectively. Then, for θ bigger than some universal constant, we have:*

$$(i) \quad \mathbb{P}[E] \leq \exp(-c_1(\theta)L), \quad (ii) \quad \mathbb{P}[F_k] \leq \exp(-c_2(\theta, k)L),$$

where $c_1(\theta) \rightarrow \infty$ as $\theta \rightarrow \infty$ and $\lim_{\theta \rightarrow \infty} \lim_{k \rightarrow \infty} c_2(\theta, k) = \infty$.

The techniques generally used in the literature to study hole probabilities do not readily apply to this situation. Instead, we exploit a certain ‘‘almost independence’’ property of GAF, and combine it with a Cantor set-type construction to obtain the desired result.

In fact, it seems that the upper bound (i) on the hole probability is not far from a lower bound which is only slightly worse. In [21] Section 9.1, the authors lower bound the hole probability of a tubular region (see Figure 6 in [21]) of length R and height 1 by $\exp(-cR(\log R)^{3/2})$. Our method of obtaining (i) can also be applied to such a domain, and would imply an upper bound of $\exp(-cR)$.

1.1. The Boolean model. Let Π be a point process in \mathbb{R}^2 whose one-point and two-point intensity measures are absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^2 and $\mathbb{R}^2 \times \mathbb{R}^2$, respectively, and fix a real number $r > 0$. We say two points x, y of Π are *neighbours* of each other if $\|x - y\|_2 < 2r$. Equivalently, we can place open disks of radius r around each point; then two points are neighbours if and only the corresponding disks intersect. Two points x, y of Π are *connected* if \exists a finite sequence of points $x_0, x_1, \dots, x_n \in \Pi$ such that $x_0 = x, x_n = y$ and x_{j+1} is the neighbour of x_j for $0 \leq j \leq n - 1$.

This is the Boolean percolation model on the point process Π with radius r , denoted by $X(\Pi, r)$.

Connectivity as defined above is an equivalence relation, and the maximal connected components are called *clusters*. The size of a cluster is the number of points of Π in that cluster. We say that the model percolates if there is at least one infinite cluster. We say that $x_0 \in \mathbb{R}^2$ is connected to the infinity if there is a point $x \in \Pi$ such that $\|x - x_0\|_2 < r$ and x belongs to an infinite cluster. The probability of having an infinite cluster and that of the origin being connected to infinity both depend on the parameter r . For $r' < r$, a disk of radius r' is contained inside a disk of radius r with the same centre. This gives us a coupling between the (random) graphs obtained in the Boolean percolation models $X(\Pi, r')$ and $X(\Pi, r)$, with the graph obtained in $X(\Pi, r')$ being a subgraph of the one obtained in $X(\Pi, r)$. This coupling shows that both the probability that the model percolates and the probability that a given point x_0 is connected to the infinity are non-decreasing in r .

Notation. Let $\Lambda(r)$ denote the number of infinite clusters when the disks are of radius r .

$\Lambda(r)$ is a measurable function of the point configuration. To see this, fix $a > 0$ and look at the disk $B(0; a)$ of radius a and centre at the origin. Let $\Lambda(r, a)$ denote the number of infinite clusters in $X(\pi, r)$ which intersect this disk. Then $\Lambda(r) = \lim_{a \rightarrow \infty} \Lambda(r, a)$, so it suffices to prove that each $\Lambda(r, a)$ is measurable. To this end, consider $b \geq a$ and look at the graph obtained from $X(\Pi, r)$ restricted to the points in $B(0; b+r)$. Based on this, construct a new graph $G(a, b)$ as follows. The vertices of $G(a, b)$ are the those points of Π in $B(0; a)$ which are not part of any finite cluster in the Boolean model contained inside $B(0; b)$. Two vertices in $G(a, b)$ have an edge between them if the corresponding points are connected in the Boolean model by a path that only involves points in $B(0; b)$. Let $g(a, b)$ denote the number of connected components of $G(a, b)$. Since $G(a, b)$ is defined with respect to the finitely many points of Π in $B(0; b+r)$, therefore, $g(a, b)$ is easily seen to be measurable with respect to Π . However, $\Lambda(r, a) = \lim_{b \rightarrow \infty} g(a, b)$, which shows that $\Lambda(r, a)$ is measurable, as desired.

DEFINITION 1. The point process Π is said to have a critical radius $0 < r_c < \infty$ if $\Lambda(r) = 0$ a.s. when $0 < r < r_c$ and $\mathbb{P}(\Lambda(r) > 0) > 0$ when $r_c < r < \infty$.

For any point process Π in \mathbb{R}^2 , the group of translations of \mathbb{R}^2 acts in a natural way on Π : a translation T takes the point $x \in \Pi$ to $T(x)$, the resulting point process being denoted $T_*\Pi$. The process Π is said to be translation invariant if $T_*\Pi$ has the same distribution as Π for all translations T . The process is said to be ergodic under translations if this action is ergodic.

For any translation invariant point process, the probability of the origin being connected to infinity is the same as that for any $x \in \mathbb{R}^2$, so by a simple union bound over $x \in \mathbb{R}^2$ with rational co-ordinates, the probability of having an infinite cluster is positive if and only if the probability of the origin being connected to infinity is positive.

Clearly, $\Lambda(r)$ is a translation-invariant random variable. If the distribution of Π is ergodic under translations, $\Lambda(r)$ is a.s. a non-negative integer constant. In particular, the probability of having at least one infinite cluster is either 0 or 1.

1.2. The underlying graph. Consider the Boolean model with radius r on a point process Π in \mathbb{R}^2 . By the *underlying graph* \mathfrak{g} of this model we mean the graph whose vertices are the points of Π and two vertices x, y are neighbours iff $\|x - y\|_2 < 2r$. By $\Phi(\mathfrak{g})$ we denote the subset of \mathbb{R}^2 formed by the union of the points of Π and straight line segments drawn between two such points whenever their mutual distance is less than $2r$. Since the two point intensity measure of Π is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2 \times \mathbb{R}^2$, therefore, the probability that there are two points of Π at a mutual distance $2r$ is 0. Hence, if a take a large open disk D , then there exists an $\varepsilon > 0$ such that for each point

x of Π in D we have $\mathcal{B}(x; \varepsilon) \subset D$ and x can be moved to any new position in the open disk $\mathcal{B}(x; \varepsilon)$ without changing the connectivity properties of \mathfrak{g} . In other words, let Φ map each point of the configuration inside D to any point in its ε neighbourhood, and let it map every other point of the configuration to itself. Then $\Phi(x)$ and $\Phi(y)$ are neighbours if and only if x and y are neighbours.

2. The models.

2.1. *The Ginibre ensemble.* Let us consider an $n \times n$ matrix $X_n, n \geq 1$ whose entries are i.i.d. standard complex Gaussians. The vector of its eigenvalues, in uniform random order, has the joint density (with respect to the Lebesgue measure on \mathbb{C}^n) given by

$$p(z_1, \dots, z_n) = \frac{1}{\pi^n \prod_{k=1}^n k!} e^{-\sum_{k=0}^n |z_k|^2} \prod_{i < j} |z_i - z_j|^2.$$

Recall that a determinantal point process on the Euclidean space \mathbb{R}^d with kernel K and background measure μ is a point process on \mathbb{R}^d whose k -point intensity functions with respect to the measure $\mu^{\otimes k}$ are given by

$$\rho_k(x_1, \dots, x_k) = \det[(K(x_i, x_j))_{i,j=1}^k].$$

Typically, K has to be such that the integral operator defined by K is a non-negative trace class contraction mapping $L^2(\mu) \rightarrow L^2(\mu)$. For a detailed study of determinantal point processes, we refer the reader to [13, 15] or [29]. A simple calculation involving Vandermonde determinants shows that the eigenvalues of X_n (considered as a random point configuration) form a determinantal point process in \mathbb{R}^2 . Its kernel is given by $K_n(z, w) = \sum_{k=0}^{n-1} \frac{(z\bar{w})^k}{k!}$ with respect to the background measure $d\gamma(z) = \frac{1}{\pi} e^{-|z|^2} d\mathcal{L}(z)$ where \mathcal{L} denotes the Lebesgue measure on \mathbb{C} . This point process is the Ginibre ensemble (of dimension n), which we will denote by \mathcal{G}_n . As $n \rightarrow \infty$, these point processes converge, in distribution, to a determinantal point process given by the kernel $K(z, w) = e^{z\bar{w}} = \sum_{k=0}^{\infty} \frac{(z\bar{w})^k}{k!}$ with respect to the same background measure γ . This limiting point process is the infinite Ginibre ensemble \mathcal{G} . It is known that \mathcal{G} is ergodic under the natural action of the translations of \mathbb{R}^2 .

2.2. *The GAF zero process.* Let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence of i.i.d. standard complex Gaussians. Define, for $n \geq 0$,

$$f_n(z) = \sum_{k=0}^n \xi_k \frac{z^k}{\sqrt{k!}}, \quad f(z) = \sum_{k=0}^{\infty} \xi_k \frac{z^k}{\sqrt{k!}}.$$

These are complex Gaussian processes on \mathbb{C} with covariance kernels given by

$$K_n(z, w) = \sum_{k=0}^n \frac{(z\bar{w})^k}{k!} \quad \text{and} \quad K(z, w) = \sum_{k=0}^{\infty} \frac{(z\bar{w})^k}{k!},$$

respectively. A.s. f is an entire function and the f_n 's converge to f (in the sense of the uniform convergence of functions on compact sets). It can be shown, using Rouché's theorem, that this implies that the corresponding point processes of zeroes, denoted by \mathcal{F}_n , converge a.s. to the zero process \mathcal{F} of the GAF (in the sense of locally finite point configurations converging on compact sets). It is known that \mathcal{F} is ergodic under the natural action of the translations of the plane. For a detailed study of the GAF zero process (including a proof of translation invariance and ergodicity), we refer the reader to [13].

3. Discrete approximation and the critical radius. The first step in our study of continuum percolation will be to relate our events of interest to events defined with respect to a grid, so that the problem becomes amenable to techniques similar to the ones that are effective in studying percolation in discrete settings.

DEFINITION 2. Let $\theta > 0$ be a parameter, to be called base length, and consider the grid formed by $\theta\mathbb{Z}^2$ (which includes the horizontal and vertical edges connecting the points of $\theta\mathbb{Z}^2$). Each $\theta \times \theta$ closed square (including the interior) whose vertices are the points of $\theta\mathbb{Z}^2$ will be referred to as a *standard square*. Two (distinct) standard squares are said to be neighbours if their boundaries intersect. So, each standard square has 8 neighbours.

NOTATION 1. For $x \in \mathbb{R}^2$ and $R > 0$, we will denote by $\mathcal{B}(x; R)$ the open disk with centre x and radius R .

We define W_R , the box of size R , to be the set $W_R := \{x \in \mathbb{R}^2 : \|x\|_\infty = R\}$.

For a subset $K \subset \mathbb{R}^2$, we will denote by \overline{K} the topological closure of K .

DEFINITION 3. Fix a radius $r > 0$ and a base length $\theta > 0$.

A continuum path γ of length n is defined to be a piecewise linear curve whose vertices are given by the sequence of points $x_j \in \Pi$, $1 \leq j \leq n$ such that x_{i+1} is a neighbour of x_i for $1 \leq i \leq n-1$.

For a continuum path γ with vertices $\{x_1, \dots, x_n\}$, we denote by $S(\gamma)$ the set $\bigcup_{i=1}^n \mathcal{B}(x_i; r)$.

A lattice path Γ of length n is defined to be a sequence of standard squares $\{X_j\}_{j=1}^n$ such that X_{i+1} is a neighbour of X_i for $1 \leq i \leq n-1$. A lattice path $\{X_i\}_{i=1}^n$ is said to be non-repeating if $X_i \neq X_j$ for $i \neq j$.

For a lattice path $\Gamma = \{X_1, \dots, X_n\}$, we denote by $V(\Gamma)$ the set $\bigcup_{i=1}^n X_i$.

We say that a continuum path γ *connects* the origin to W_R if $0 \in S(\gamma)$ and $S(\gamma) \cap W_R \neq \emptyset$. For $R \in \mathbb{Z}_+$, we say that a lattice path Γ connects the origin to $W_{R\theta}$ if $0 \in V(\Gamma)$ and $V(\Gamma) \cap W_{R\theta} \neq \emptyset$.

With these notions in hand, we are ready to state the following.

PROPOSITION 3.1. *Consider the Boolean percolation model $X(\Pi, r)$. Let the base length $\theta = r/\sqrt{8}$. Suppose, for some $L \in \mathbb{Z}_+$, there exists a non-repeating lattice path $\Gamma = \{X_1, \dots, X_n\}$ that connects 0 to $W_{L\theta}$ with each X_i containing at least one point in Π . Then there exists a continuum path γ that connects 0 to $W_{L\theta}$.*

PROOF. The result follows from the fact that with $\theta = r/\sqrt{8}$, disks of radius r centred at any two points in adjacent standard squares intersect with each other. This is true because the maximum possible distance between two points in adjacent standard squares is $\theta\sqrt{8}$. \square

PROPOSITION 3.2. *Fix a base length θ and an integer $k \geq 0$. For any $0 < r < \theta/18k$ the following happens: Suppose in $X(\Pi, r)$ there exists a continuum path γ connecting 0 to $W_{L\theta}$ (where $L \in \mathbb{Z}_+$). Then there exists a non-repeating lattice path Γ connecting 0 to $W_{L\theta}$ such that each standard square in Γ contains $\geq k$ points $\in \Pi$.*

PROOF. Let r be a radius such that $k < \theta/18r$, and let γ be a continuum path with vertices $\{x_i\}_{i=1}^n$ connecting 0 to $W_{L\theta}$. A finite lattice path Γ_1 is said to be contained in another finite lattice path Γ_2 , denoted by $\Gamma_1 \subset \Gamma_2$, if $V(\Gamma_1) \subset V(\Gamma_2)$.

Now, consider the set Ξ of all finite lattice paths Γ (non-repeating or otherwise), $0 \in V(\Gamma)$, such that each standard square in Γ contains $\geq k$ points. Clearly, \subset is a partial order on Ξ . Moreover, Ξ is non-empty, because γ must reach L^∞ distance θ from the origin, and in doing so must have at least $\theta/2r$ points $\in \Pi$. The 4 standard squares whose closures contain the origin contain these $\theta/2r$ points, so at least one of them must have at least $\theta/8r \geq k$ points in Π .

Let Γ be a maximal element in Ξ under \subset . If Γ connects 0 to $W_{L\theta}$ then we are done. Otherwise, we define the *surround* $\Sigma(\Gamma)$ of Γ as the union of all standard squares which are neighbours of the standard squares in Γ and are contained in the unbounded component of the complement of Γ . Since γ connects 0 to $W_{L\theta}$, therefore γ intersects $\partial\Sigma(\Gamma) \setminus V(\Gamma)$. Let j be the least index $\in [n]$ such that the line segment $(x_{j-1}, x_j]$ intersects $\partial\Sigma(\Gamma) \setminus V(\Gamma)$. Since $r < \theta$, we must have $x_{j-1} \in \text{Int}(\Sigma(\Gamma))$, where $\text{Int}(H)$ denotes the interior of a set H . Let σ be a standard square in $\Sigma(\Gamma)$ such that σ contains x_{j-1} . Consider the continuum path γ' with vertices $\{x_j, x_{j-1}, \dots, x_i\}$ where i is the largest index $\leq j - 1$ such that $x_i \in \Gamma$. In other words, we trace the vertices of γ backward from x_j until we are in Γ . Now the part of γ' contained in σ and its neighbouring standard squares [that are in $\Sigma(\Gamma)$] is of length at least θ , therefore, it has at least $\theta/2r$ points $\in \Pi$ contained in these squares. But

Total number of such squares (including σ)

$$\leq 1 + \text{number of standard squares neighbouring } \sigma = 9.$$

Therefore, we have $\theta/2r$ points $\in \Pi$ contained in ≤ 9 squares in $\Sigma(\Gamma)$. Therefore, at least one square σ' in $\Sigma(\Gamma)$ has at least $\theta/18r \geq k$ points of γ . Let $\Gamma = \{X_i\}_{i=1}^N$ and let σ' be a neighbour of $X_j \in \Gamma$. We define a new lattice path $\Gamma' \in \Xi$ by

$$\Gamma' = \{X_1, X_2 \cdots, X_N, X_{N-1}, X_{N-2} \cdots, X_j, \sigma\},$$

that is, by backtracking along Γ until we reach X_j and then appending σ at the end. Clearly, $S(\Gamma') \supset S(\Gamma)$ as a proper subset, contradicting the maximality of Γ .

Since the procedure described above must terminate after finitely many steps because $W_{L\theta}$ is a compact set, a maximal element Γ of Ξ must connect 0 to $W_{L\theta}$. Such a lattice path may not be non-repeating. However, we can erase the loops in Γ in the chronological order to obtain a non-repeating lattice path of the desired kind that connects 0 to $W_{L\theta}$. \square

In the next theorem, we provide some general conditions under which there exists a non-trivial critical radius for the Boolean percolation model.

THEOREM 3.3. *Let Π be a translation invariant and ergodic point process with the property that for any connected set Γ of L standard squares (with base length θ) the following are true:*

(i) *For large enough θ , we have*

$$\mathbb{P}[\Gamma \text{ contains no points } \in \Pi] \leq \exp(-c_1(\theta)L),$$

with $c_1(\theta) \rightarrow \infty$ as $\theta \rightarrow \infty$.

(ii) *For large enough θ , we have*

$$\mathbb{P}[\text{Each standard square in } \Gamma \text{ has at least } k \text{ points } \in \Pi] \leq \exp(-c_2(\theta, k)L)$$

with $\lim_{\theta \rightarrow \infty} \lim_{k \rightarrow \infty} c_2(\theta, k) = \infty$.

In the Boolean percolation model $X(\Pi, r)$ on such a Π , let r denote the radius of each disk and let $\Lambda(r)$ denote the number of infinite clusters. Then there exists $0 < r_c < \infty$ such that for $0 < r < r_c$, we have $\Lambda(r) = 0$ a.s. and for $r_c < r < \infty$ we have $\Lambda(r) > 0$ a.s.

PROOF. The proof follows a Peierl's type argument from the classical bond percolation theory, after appropriate discretization using Propositions 3.1 and 3.2. We first note that by translation invariance, it suffices to show that $\mathbb{P}[0 \text{ is connected to } \infty \text{ with radius } r] > 0$ or $= 0$, respectively, in order to show that $\Lambda(r) > 0$ or $= 0$ a.s.

We want to show that for small enough r , there is no continuum path connecting 0 to ∞ . Consider possible base lengths θ so large that our hypothesis (ii) is valid. Fix base length θ and k a positive integer large enough such that $2 \log 3 - c_2(\theta, k) < 0$ where c_2 is as in (ii). We call a non-repeating lattice path

Γ to be k -full if each standard square in Γ contains $\geq k$ points $\in \Pi$. By condition (ii), if there are L distinct standard squares in Γ , then the probability of Γ being k -full $\leq \exp(-c_2(\theta, k)L)$. Since each standard square has ≤ 9 neighbours, therefore, the number of non-repeating lattice paths Γ containing 0 and having L standard squares $\leq 9^L$. So,

$$\begin{aligned} &\mathbb{P}[\text{There is a } k\text{-full lattice path of length } L \text{ containing the origin}] \\ &\leq \exp((2 \log 3 - c_2(\theta, k))L). \end{aligned}$$

The right-hand side is summable in positive integers L , hence by the Borell–Cantelli lemma,

$$\mathbb{P}[\text{There exists a } k\text{-full lattice path connecting the origin to } W_{L\theta} \text{ all } L \in \mathbb{Z}_+] = 0.$$

If there was a continuum path γ connecting 0 to ∞ , then for any integer $t > 0$ there will be a continuum path connecting 0 to $W_{t\theta}$. We now appeal to Proposition 3.2 for this k and find an r small enough such that for any continuum path γ connecting 0 to the box $W_{t\theta}$ we can find a k -full lattice path Γ connecting 0 to $W_{t\theta}$. But we have already seen that a.s. there are only finitely many k -full lattice paths, which gives us a contradiction, and proves that there is no continuum path connecting 0 to ∞ , with probability 1.

By translation invariance, this proves that for small enough r , we have $\Lambda(r) = 0$ a.s.

Next, we want to show that for large enough r , with positive probability there exists a continuum path connecting 0 to ∞ . Fix a radius r in the Boolean model. The event that there exists no continuum path from 0 to ∞ , implies by Proposition 3.1 that (choosing the base length to be θ as in Proposition 3.1 with $\theta = r/\sqrt{8}$) there exists $L \in \mathbb{Z}_+$ such that there is no lattice path connecting the origin to $W_{L\theta}$. The last statement implies that there exists a circuit of standard squares surrounding the origin such that the interiors of the standard squares in this circuit do not contain any point from Π . Therefore, it suffices to prove that the probability of this event can be made < 1 by choosing r sufficiently large.

To this end, we recall that the number of circuits of standard squares containing the origin and consisting of L distinct standard squares is $\exp(cL)$ for some constant $c > 0$. For details on this, we refer the reader to [3], Chapter 1, proof of Lemma 2.

The probability that a specific circuit of standard squares surrounding the origin and containing L standard squares is empty $\leq \exp(-c_1(\theta)L)$ when base length is θ , which follows from condition (i) in the present theorem. Therefore,

$$\begin{aligned} &\mathbb{P}[\text{There exists an empty circuit surrounding the origin}] \\ (1) \quad &\leq \sum_{L=1}^{\infty} e^{c(d)L} e^{-c_1(\theta)L}. \end{aligned}$$

Now, by choosing r large enough, we can make θ large enough (by Proposition 3.1), so that condition (i) would imply that the right-hand side of (1) is less than 1. This completes the proof that when r is large enough, 0 is connected to ∞ with positive probability. \square

REMARK 3.1. Theorem 3.3 carries over verbatim to d dimensions instead of 2, with standard squares replaced by d dimensional standard cubes, whose definition is analogous. The proof works on similar lines.

4. Critical radius for Gaussian zeros. In this section, we aim to study the Boolean model on the planar GAF zero process. First of all, we will prove an estimate on hole probabilities and overcrowding probabilities in the Gaussian zero ensemble, which is taken up in Section 4.1. It will subsequently be used to prove the existence of critical radius for the Boolean percolation model on Gaussian zeroes in Section 4.2.

4.1. *Exponential decay of hole and overcrowding probabilities.* The main goal of this section is to prove Theorem 1.3, which is an estimate on the hole and overcrowding probabilities of connected sets composed of standard squares.

We will perform a certain Cantor-type construction which will be used in proving Theorem 1.3. For the rest of this section, the symbol “log” denotes logarithm to the base 2.

We consider the normalised GAF $f^*(z) = e^{-1/2|z|^2} f(z)$. We will make use of the following almost independence theorem from [20].

THEOREM 4.1. *Let F be a GAF. There exists numerical constant $A > 1$ with the following property. Given a family of compact sets K_j in \mathbb{C} with diameters $d(K_j)$, let $\rho_j \geq \sqrt{\log(3 + d(K_j))}$. Suppose that $A\rho_j$ -neighbourhoods of the sets K_j are pairwise disjoint. Then*

$$F^* = F_j^* + G_j^* \quad \text{on } K_j,$$

where F_j are independent GAFs and for a positive numerical constant C we have

$$\mathbb{P}\left\{\max_{K_j} |G_j^*| \geq e^{-\rho_j^2}\right\} \leq C \exp[-e^{\rho_j^2}].$$

Our construction will be parameterised by two parameters: $\theta > 0$ and $0 < \lambda < 1$. We will think of θ to be large enough and λ to be small enough; the exact conditions demanded of θ and λ will be described as we proceed along the construction. It turns out that the resulting choice of θ and λ can be made to be uniform in all the other variables in the construction (like the length L), and it suffices to take λ smaller than some universal constant and θ large enough, depending on λ . To

begin with, we demand that θ be so large that $C \exp[-e^{(\log \theta)^2}] < 1$, for C as in Theorem 4.1, and

$$(2) \quad \sqrt{\log(3 + x\theta\sqrt{2})} < \theta \log x \quad \text{for all } x \geq 1.$$

4.1.1. *A Cantor-type construction.* Let B be a square in \mathbb{R}^2 of dimension $L\theta \times L\theta$. Let $0 < \lambda < 1/2$ be a fixed number, to be specified later. To begin with, we demand λ to be so small that $2A \log(1/4\lambda) > 3$. Observe that this guarantees the following: inside any square of side length $2A\theta \log(1/4\lambda)$ in \mathbb{R}^2 , there exists at least one standard square of side length θ . This fact will be used later, in the proof of Theorem 1.2. Let $N = \lceil \log \lambda L \rceil$. In this section, we will construct a subset $B_N \subset B$ which satisfies the following conditions:

- (a) B_N consists of 4^N connected components that are translates of each other.
- (b) Each connected component is a square of side length $\geq 2A\theta \log(1/4\lambda)$.
- (c) Euclidean distance between any two such components is

$$\geq 2A\theta \log(L/2^N) \geq 2A\theta \log(1/2\lambda).$$

Notice that the connected components of B_N are squares of side length $\geq 2A\theta \log(1/4\lambda)$, and since λ is so small that $2A \log(1/4\lambda) > 3$, each such component contains at least one standard square of side θ . We arbitrarily select one standard square from each connected component in B_N , and denote their union by Υ_N .

To this end, we proceed inductively as follows. For $0 \leq j < N$, we will start with a set B_j and describe how to obtain a set B_{j+1} from there. Set $B_0 = B$ and $l_0 = L\theta$. In our construction, B_j will satisfy the following inductive hypothesis:

- (i) B_j consists of 4^j connected components, denoted $\{B_j^i\}_{i=1}^{4^j}$ that are translates of each other.
- (ii) Each connected component is a square of side length $l_j \geq 2A\theta \log(L/2^{j+1})$. Furthermore, $l_j \leq \theta L/2^j$.
- (iii) Euclidean distance between any two such components is $\geq 2A\theta \log(L/2^j)$.

These conditions are trivially true for B_0 .

We will now describe an operation on an $l_j \times l_j$ square, that will give rise to 4 smaller squares of side length l_{j+1} . This operation will be performed on each of the connected components of B_j , and the union of the resulting $4 \cdot 4^j = 4^{j+1}$ smaller squares will be B_{j+1} . Of course, we would need to verify that the inductive hypotheses remain valid.

We consider S_j to be the $l_j \times l_j$ square formed by the points in \mathbb{R}^2 whose Cartesian co-ordinates are $(0, 0), (l_j, 0), (0, l_j), (l_j, l_j)$. Set $l_{j+1} = (l_j - 2A\theta \times \log(L/2^{j+1}))/2$. Consider the region

$$R_j := \{(x, y) \in \mathbb{R}^2 : l_{j+1} \leq x \leq l_j - l_{j+1} \text{ or } l_{j+1} \leq y \leq l_j - l_{j+1}\}.$$

A moment's thought would convince the reader that $S_j \setminus R_j$ is a disjoint union of four squares in \mathbb{R}^2 , each of side length l_{j+1} . In what follows, we will refer to these as "surviving squares". A general $l_j \times l_j$ square in \mathbb{R}^2 is a translate of S_j , so the construction on S_j under the same translation defines a similar construction on the general $l_j \times l_j$ square.

We perform this construction on each of the connected components of B_j . The union of the surviving squares formed as a result is B_{j+1} .

We need to verify that the inductive hypotheses remain valid for B_{j+1} .

This is the content of the following proposition.

PROPOSITION 4.2. *There is a number $\lambda_0 \in (0, 1/2)$ such that for any positive $\lambda < \lambda_0$ and $N = \lceil \log \lambda L \rceil$, the sets B_j in the above construction, for $0 \leq j < N$ satisfy conditions (i), (ii) and (iii) in the inductive hypothesis. With $\rho_j = \theta \log(L/2^j)$, the $A\rho_j$ -neighbourhoods of the connected components $\{B_j^i\}_{i=1}^{4^j}$ of B_j are disjoint. Furthermore, we have*

$$\rho_j \geq \sqrt{\log(3 + \text{Diam}(B_j^i))}.$$

PROOF. The fact that B_j has 4^j connected components, each of which are squares, is clear from the construction. This deals with condition (i) in the inductive hypothesis.

To check that $l_j \geq 2A\theta \log(L/2^{j+1})$, it suffices to check that for $0 \leq j < N$, we have $l_{j+1} = (l_j - 2A\theta \log(L/2^{j+1}))/2 \geq 0$. Substituting $l_k = (l_{k-1} - 2A\theta \log(L/2^k))/2$ for $k = j, j - 1, \dots, 0$, we deduce that we need to prove, for each $0 \leq j < N$, the inequality

$$\frac{1}{2^{j+1}} \left[l_0 - \sum_{k=0}^j 2^k \cdot 2A\theta \log \frac{L}{2^{k+1}} \right] \geq 0.$$

Recalling that $l_0 = L\theta$, it suffices to show that

$$\sum_{k=0}^{N-1} 2^k \cdot 2A\theta \log \frac{L}{2^{k+1}} \leq L\theta.$$

But the left-hand side above is equal to

$$\begin{aligned} & 2A(2^N - 1)\theta \log L - 2A\theta(N - 1)2^N - 2A\theta \\ &= 2^{N+1}A\theta(\log L - N) - 2A\theta \log L + 2^{N+1}A\theta - 2A\theta \\ &\leq 4A \left(\lambda \log \frac{1}{\lambda} \right) L\theta + 4A\lambda L\theta, \end{aligned}$$

where in the last step we have used

$$(3) \quad \lambda L \leq 2^N \leq 2\lambda L.$$

By choosing λ small enough (less than some universal constant λ_0), we can ensure that $4A(\lambda \log \frac{1}{\lambda})L\theta + 4A\lambda L\theta \leq \frac{1}{2}L\theta$. Finally,

$$l_j = (l_{j-1} - 2A\theta \log(L/2^j))/2 \leq l_{j-1}/2 \leq \dots \leq l_0/2^j = \theta L/2^j.$$

This completes the proof of condition (ii) in the inductive hypothesis.

Condition (iii) in the inductive hypothesis follows from the definition. It suffices to verify that the distance between two components in B_j , which have been created by subdividing the same component of B_{j-1} , is $\geq 2A\theta \log(L/2^j)$. This is because, the distance between any two of the other components in B_j is greater than or equal to the least distance between components of B_{j-1} , which is at least $2A\theta \log(L/2^{j-1}) \geq 2A\theta \log(L/2^j)$. By our procedure for constructing the components of B_j (recall the definition of R_j), the minimal distance between components of B_j obtained from the same component of B_{j-1} is indeed at least $2A\theta \log(L/2^j)$.

The fact that the $A\rho_j$ -neighbourhoods of the distinct B_j^i 's (for the same j) follows from the lower bound on the Euclidean distance between the components of B_j . The inequality

$$\rho_j \geq \sqrt{\log(3 + \text{Diam}(B_j^i))}$$

follows from the fact that $\text{Diam}(B_j^i) \leq l_j\sqrt{2} = \theta L\sqrt{2}/2^j$ and applying equation (2) with $x = L/2^j$. In doing so, we use the fact that for all $j \leq N$, we have $2^j \leq 2\lambda L$ [see (3)] and $1 \leq 1/2\lambda$ (recall that λ was chosen to be less than $1/2$), in order to ensure that $x \geq 1$, as required in (2). \square

4.1.2. *Functional decomposition in the Cantor construction.* In this section, we prove that the Cantor-type decomposition in the previous section leads to a decomposition of f^* (restricted to B_N) into a part that is essentially independent across the components of B_N and a residual part whose size, for a positive fraction of the components, is very small with high probability. In what follows, C will be a numerical quantity whose value might change from one line to another, but is not dependent on any of the parameters of our model.

PROPOSITION 4.3. *There are functions $\{f_i^*, G_i^*\}_{i=1}^{4^N}$ such that $f^* = f_i^* + G_i^*$ on B_N^i for each i , f_i^* are i.i.d. copies of f^* and with probability $\geq 1 - \exp(-c(\theta)L^2)$, the functions G_i^* satisfy*

$$\sup_{B_N^i} |G_i^*| \leq e^{-5\theta^2}$$

for at least $\frac{1}{2}\lambda^2 L^2$ of the sets B_N^i .

PROOF. We can consider the sets B_j^i to be the vertices of a tree \mathcal{T} of depth N where each vertex has 4 children (except at depth N). The children of the vertex

B_j^i are the vertices $B_{j+1}^{i'}$ where $B_{j+1}^{i'}$ are obtained by applying the $(j + 1)$ th level of the construction in Section 4.1.1 to B_j^i .

Corresponding to the tree \mathcal{T} , we can perform a decomposition of the normalised GAF f^* using Theorem 4.1. We start with f^* , which we also call f_0^* . We apply Theorem 4.1 to the compact sets $B_1^i, 1 \leq i \leq 4$ to obtain i.i.d. normalised GAFs $f_{1,i}^*$ and corresponding errors $g_{1,i}^*$. These are the functions corresponding to the first level of the tree. At the next level, we perform a similar decomposition on each $f_{1,i}^*$ to obtain $f_{2,j}^*$ and $g_{2,j}^*, 1 \leq j \leq 4^2$. So, on B_2^i we have $f^* = f_{2,i}^* + g_{2,i}^* + g_{1,i'}^*$ where $B_2^i \subset B_1^{i'}$. We continue this decomposition recursively until we reach level N in \mathcal{T} . At level N , we have $f^* = f_{N,i}^* + G_i^*$ on $B_N^i, 1 \leq i \leq 4^N$ where the $f_{N,i}^*$ i.i.d. normalised GAFs. The G_i^* are the cumulative errors given by $G_i^* = \sum_{k=1}^N g_{k,n(k,i)}^*$ where $n(k, i)$ are such that $B_N^i \subset B_k^{n(k,i)}$. The $g_{j,i}^*$'s are not independent. However, for any two distinct vertices $B_j^i, B_j^{i'}$ at the same level j in \mathcal{T} , the errors corresponding to the descendants of B_j^i and those of $B_j^{i'}$ are independent. Thus, at level j , the 4^j functions $g_{j,i}$ can be grouped into 4^{j-1} groups $\mathfrak{J}_{j,i''}, 1 \leq i'' \leq 4^{j-1}$ (each group consisting of 4 functions whose vertices have the same parent at level $j - 1$ in \mathcal{T}). Thus, the index i'' in $\mathfrak{J}_{j,i''}$ can be thought to be varying over the vertices in \mathcal{T} at level $j - 1$. Clearly, $\mathfrak{J}_{j,i}$ and $\mathfrak{J}_{j,i'}$ are independent sets of functions for $i \neq i'$. We call $\mathfrak{J}_{j,i}$ to be ‘‘good’’ if each $g_{j,k} \in \mathfrak{J}_{j,i}$ satisfies $\max_{B_j^k} |g_{j,k}^*| \leq e^{-\rho_j^2}$, otherwise we call it ‘‘bad’’. Recall from Theorem 4.1 (and a simple union bound) that $\mathbb{P}\{\mathfrak{J}_{j,i} \text{ is bad}\} \leq C \exp[-e^{\rho_j^2}]$.

Set $p_j = C \exp[-e^{\rho_j^2}]$ as in the last line. Denote by b_j the number of $\mathfrak{J}_{j,i}$ at level j which are not good. By a simple large deviation bound, we have, for any $0 < x_j < 1$,

$$(4) \quad \mathbb{P}(b_j > x_j \cdot 4^{j-1}) \leq \exp(-4^{j-1} I_j),$$

where $I_j = x_j \ln \frac{x_j}{p_j} + (1 - x_j) \ln \frac{1-x_j}{1-p_j}$ (for reference, see [6] Theorem 2.1.10).

We set $x_j = 1/4^{N-j+1}$, whereas recall that $p_j = C \exp(-e^{\rho_j^2})$, and

$$\rho_j = \theta \log(L/2^j) = \rho_N + (N - j)\theta.$$

Further, $\theta \log \frac{1}{2\lambda} \leq \rho_N \leq \theta \log \frac{1}{\lambda}$ (recall that $N = \lceil \log \lambda L \rceil$). Combining all these facts, we have

$$-x_j \ln p_j = \left[\exp\left(\theta^2 \left(\frac{1}{\theta} \rho_N + (N - j)\right)^2\right) - \ln C \right] / 4^{N-j+1}.$$

By choosing θ larger than and λ smaller than certain absolute constants, we can make the numerator of the above expression $\geq 2\theta 4^{2(N-j+1)}$ for all $N \geq 1$ and $1 \leq j \leq N$. Since $|x_j| \leq 1/4$ for each $1 \leq j \leq N$, we have $|x_j \ln x_j| \leq \frac{1}{4} \ln 4$. Also,

for θ bigger than and λ smaller than some absolute constants, we have $c_1 \leq |(1 - x_j) \ln \frac{1-x_j}{1-p_j}| \leq c_2 \forall j \leq N$ where c_1 and c_2 are two positive constants. The upshot of all this is that by choosing θ larger than a constant we can make $I_j \geq \theta 4^{N-j+1}$ for all j , where we recall that $I_j = x_j \ln \frac{x_j}{p_j} + (1 - x_j) \ln \frac{1-x_j}{1-p_j}$.

Hence, we have

$$\mathbb{P}(b_j > x_j \cdot 4^{j-1}) \leq \exp(-\theta 4^{j-1} 4^{N-j+1}) = \exp(-\theta 4^N) \leq \exp(-\theta \lambda^2 L^2).$$

We denote by Ω the event $\{b_j \geq x_j \cdot 4^j \text{ for some } j \leq N\}$. By a union bound over $1 \leq j \leq N$, we have $\mathbb{P}(\Omega) \leq N \exp(-\theta \lambda^2 L^2) \leq \exp(-c_2(\theta)L^2)$ when θ is large enough, depending only on λ . Here, we recall again that $N = \lceil \log \lambda L \rceil$.

We call G_N^i to be “good” if each summand $g_{k,n(k,i)}^*$ in $G_i^* = \sum_{k=1}^N g_{k,n(k,i)}^*$ belongs to good \mathfrak{J} ’s. If G_N^i is not good then we call it “bad”. Now, each bad \mathfrak{J} at level j gives rise to 4^{N-j+1} bad G_N^i ’s at level N . Outside the event Ω , there are at most $x_j 4^{j-1}$ bad \mathfrak{J} ’s at level j , leading to $x_j 4^N$ bad G_i^* ’s. But $\sum_{j=1}^N x_j < 1/2$, hence except with probability $\leq \exp(-c_2(\theta)L^2)$, we have $\geq \frac{1}{2} 4^N \geq \frac{1}{2} \lambda^2 L^2$ good G_i^* ’s. For any good G_i^* , we have, for θ larger than and λ smaller than absolute constants,

$$\sup_{B_N^i} |G_i^*| \leq \sum_{k=1}^N \sup_{B_k^{n(k,i)}} |g_{k,n(k,i)}^*| \leq \sum_{k=1}^N e^{-\rho_k^2} \leq 2e^{-\rho_N^2} \leq e^{-5\theta^2}. \quad \square$$

For the specific scenario under our consideration, we would require a result similar to Proposition 4.3, which states that with high probability, a positive fraction of the constituent squares are good for a general set (which is not necessarily a box). We state this formally as the following.

PROPOSITION 4.4. *Let γ be a fixed positive number and $\Gamma \subset B_N$ be a set of the form $\bigcup_{i \in \Lambda} B_N^i$, where $\Lambda \subset \{1, \dots, 4^N\}$ with $|\Lambda| \geq \gamma L$ (here $N = \log \lceil \lambda L \rceil$ as before). Then the functions f_i^* and G_i^* as constructed in Proposition 4.3 are such that, with probability $\geq 1 - \exp(-c(\theta, \gamma)L)$ for some positive quantity $c(\theta, \gamma)$, the functions G_i^* satisfy*

$$\sup_{B_N^i} |G_i^*| \leq e^{-5\theta^2}$$

for at least $\frac{1}{2} \gamma L$ of the sets $\{B_N^i : i \in \Lambda\}$. The quantity $c(\theta, \gamma) \rightarrow \infty$ as $\theta \rightarrow \infty$ for fixed γ .

Remark. Note that the statement of the proposition holds no matter where the starting set B_0 (and hence the final set B_N) is located in \mathbb{R}^2 . In particular, it works for any translate of B_N . The statement also holds when we restrict to subsets of B_N^i ’s, for example, any collection of standard squares that are in Υ_N .

PROOF. We refer to the proof of Proposition 4.3 for the construction of the tree \mathcal{T} . The connected components of Γ correspond to some subset of the leaves of \mathcal{T} . We say that a leaf in \mathcal{T} is in Γ if it corresponds to one such component. Such leaves, after being connected to the root, define a subtree of \mathcal{T} . At each level j of the tree, we would like to consider the $\mathfrak{J}_{j,i}$'s which pertain to this subtree (for a definition of $\mathfrak{J}_{j,i}$ look at the proof of Proposition 4.3). Define

$$\mathcal{N}_j := \{ \mathfrak{J}_{j,i} : \exists k \text{ with } g_{j,k} \in \mathfrak{J}_{j,i} \text{ such that the vertex } B_j^k \in \mathcal{T} \text{ is connected to a leaf in } \Gamma \}.$$

Let $N_j = |\mathcal{N}_j|$.

As in the proof of Proposition 4.3, recall that $p_j = C \exp[-e^{\rho_j^2}]$, where $\rho_j = \theta \log L/2^j$ and C is as in Theorem 4.1. Set $x_j = \theta 4^{N-j+1}/(-\ln p_j)$ and $I_j = x_j \ln \frac{x_j}{p_j} + (1-x_j) \ln \frac{1-x_j}{1-p_j}$. As before, observe that the dominant term in I_j is $-x_j \ln p_j$, and hence $I_j \geq C' \theta 4^{N-j+1}$, for some positive number C' . We call $\mathfrak{J}_{j,i}$ to be “good” if each $g_{j,k} \in \mathfrak{J}_{j,i}$ satisfies $\max_{B_j^k} |g_{j,k}^*| \leq e^{-\rho_j^2}$, otherwise we call it “bad”. Recall from Theorem 4.1 (and a simple union bound) that $\mathbb{P}\{\mathfrak{J}_{j,i} \text{ is bad}\} \leq C \exp[-e^{\rho_j^2}]$. Denote by b_j the number of $\mathfrak{J}_{j,i}$ at level j which are not good. By a large deviation bound, we have

$$(5) \quad \mathbb{P}(b_j > x_j N_j) \leq \exp(-N_j I_j).$$

Clearly, $N_j \geq 1$ for each j , and N_j is non-decreasing in j . Note that N_N is the number of leaves of \mathcal{T} which are in Γ , and hence $N_N \geq \gamma L \geq c(\gamma) 4^{(1/2)N}$. Notice that $N_j \geq N_{j+1}/4$. Hence, we have $N_j \geq N_N/4^{N-j} \geq c(\gamma) 4^{j-(1/2)N}$. So, by summing equation (5), we see that

$$\begin{aligned} & \mathbb{P}(b_j > x_j N_j \text{ for some } j \in \{1, \dots, N\}) \\ & \leq \sum_{j=1}^N \exp(-N_j I_j) \\ & \leq \sum_{j=1}^{\lfloor (1/2)N \rfloor} \exp(-I_j) + \sum_{j=\lfloor (1/2)N \rfloor}^N \exp(-c(\gamma) I_j 4^{j-(1/2)N}), \end{aligned}$$

where we have used the inequality $N_j \geq 1$ for $j \leq \lfloor \frac{1}{2}N \rfloor$ and $N_j \geq c(\gamma) 4^{j-(1/2)N}$ for $j \geq \lfloor \frac{1}{2}N \rfloor$. But $\exp(-c(\gamma) I_j 4^{j-(1/2)N}) \leq \exp(-c(\gamma) \theta 4^{(1/2)N+1}) \leq \exp(-c(\gamma, \theta)L)$. For $j \leq \lfloor N/2 \rfloor$, we have $I_j \geq C \theta 4^{N-j+1} \geq C' \theta 4^{N/2} \geq C'' \theta L$. Therefore,

$$\mathbb{P}(b_j > x_j N_j \text{ for some } j \in \{1, \dots, N\}) \leq \exp(-c_1(\theta, \gamma)L).$$

Therefore, except on an event Ω with probability $\leq \exp(-c_1(\theta, \gamma)L)$, we have $b_j \leq x_j N_j \forall j$. As before, we call G_N^i to be “good” if each summand $g_{k,n(k,i)}^*$ in

$G_i^* = \sum_{k=1}^N g_{k,n(k,i)}^*$ belongs to good \mathfrak{J} 's. If G_N^i is not good, then we call it “bad”. Now, each bad \mathfrak{J} at level j gives rise to 4^{N-j+1} bad G_N^i 's at level N . Outside the event Ω , there are at most $x_j N_j$ bad \mathfrak{J} 's at level j , leading to $x_j N_j 4^{N-j+1}$ bad G_N^i 's. Observe that, as $N - j$ increases, x_j decreases super-exponentially in $N - j$. Moreover, $x_N = -(\ln p_N)^{-1}$ can be made arbitrarily small by choose λ to be appropriately small. Hence, $\sum_{j=1}^N x_j 4^{N-j+1} = \sum_{j=1}^N -(\ln p_j)^{-1} < 1/2$ (for small enough λ). Also recall that $N_j \leq N_N$ for each j . Thus, outside the event Ω , the number of bad G_N^i 's is at most $\sum_{j=1}^N x_j N_j 4^{N-j+1} \leq \frac{1}{2} N_N$. Hence, except with probability $\leq \exp(-c(\theta, \gamma)L)$, we have $\geq \frac{1}{2} N_N \geq \frac{1}{2} \gamma L$ good G_i^* 's. \square

4.1.3. *Proof of Theorem 1.3.* Suppose we have a connected set Γ of standard squares of base length θ and consisting of L standard squares. Then there is a square B of side length $L\theta$, consisting of L^2 standard squares of base length θ , such that $\Gamma \subset B$. Let B_0 denote the square with vertices $(0, 0), (0, L\theta), (L\theta, 0), (L\theta, L\theta)$. Denote by $T[m, n]$ the translation map on \mathbb{R}^2 by the vector $(m\theta, n\theta)$. Because of the translation invariance of the GAF zero process, without loss of generality we can consider B to be the square $T[L, L](B_0)$.

Consider the set of translations $\mathcal{I} := \{T[m, n] : 1 \leq m, n \leq 2L\}$. Observe that each $T[m, n]$ maps standard squares to standard squares (for base length θ). For standard squares $\sigma_1 \in B_0$ and $\sigma_2 \in B$, it is an easy check that there exists an element T_{σ_1, σ_2} of \mathcal{I} such that $T_{\sigma_1, \sigma_2}(\sigma_1) = \sigma_2$. Moreover, since this equation uniquely determines T_{σ_1, σ_2} , there is only one such translation in \mathcal{I} .

Let τ be a translation chosen uniformly at random from \mathcal{I} . Then $\mathbb{P}[\tau(\sigma_1) = \sigma_2] = 1/4L^2$. Now suppose $K_1 \subset B_0$ such that K_1 is a union of $|K_1|$ standard squares in B_0 . Then $\mathbb{P}[\sigma_2 \subset \tau(K_1)] = |K_1|/4L^2$. Now suppose $K_2 \subset B$ such that K_2 is a union of $|K_2|$ standard squares. For a set H that is a union of standard squares (of base length θ), let $\mathcal{S}(H)$ denote the set of constituent standard squares of H . Then

$$\begin{aligned} \mathbb{E}[|\tau(K_1) \cap K_2|] &= \mathbb{E}\left[\sum_{i \in \mathcal{S}(K_2)} 1_{i \subset \tau(K_1)}\right] \\ &= \sum_{i \in \mathcal{S}(K_2)} \mathbb{P}(i \subset \tau(K_1)) = |K_1||K_2|/4L^2. \end{aligned}$$

Let, as in Section 4.1.1, Υ_N be the set of standard squares obtained by performing the Cantor-type construction on B_0 . Then, applying the above argument with $K_1 = \Upsilon_N$ and $K_2 = \Gamma$, we get

$$\mathbb{E}[|\tau(\Upsilon_N) \cap \Gamma|] = |\Upsilon_N||\Gamma| \geq (\lambda L)^2 \cdot L/4L^2 = \lambda^2 L/4.$$

This implies that there exists $T \in \mathcal{I}$ such that $|T(\Upsilon_N) \cap \Gamma| \geq \lambda^2 L/4$. Fix such a T . The set $T(\Upsilon_N) \cap \Gamma$ is a union of at least $\lambda^2 L/4$ standard squares, the distance between any two of which is at least $\geq 2A\theta \log(1/2\lambda)$, and which satisfy a functional decomposition on the lines of Proposition 4.4.

Recall from Proposition 4.4 that on $T(\Upsilon_N) \cap \Gamma$, we have that except on an event Ω of probability $< e^{-c(\theta, \lambda^2/4)L}$, we have at least $\frac{1}{2}$ of the constituent standard squares of $T(\Upsilon_N) \cap \Gamma$ to be good (where, in the Cantor-type construction, a standard square $S \subset B_N^i$ is good if $\sup_{B_N^i} |G_i^*| \leq e^{-5\theta^2}$). Let $\Gamma_i, 1 \leq i \leq L$ denote the standard squares in Γ . Call a standard square to be “empty” or “full” according as it contains respectively 0 or $\geq k$ points in \mathcal{F} (recall from Section 2 that \mathcal{F} denotes the GAF zero process). Call Γ “empty” or “full” if all standard squares in Γ are empty or full, respectively. In what follows, we treat the state “empty”, but in all steps it can be replaced by the state “full”.

We observe that

$$\{\Gamma \text{ is empty}\} \subset \Omega \cup \{\text{Some subset of } \lfloor \lambda^2 L/8 \rfloor \text{ standard squares in } \Gamma, \\ \text{are all good and empty}\}.$$

We have, via a union bound,

$$\mathbb{P}(\Gamma \text{ is empty}) \leq \mathbb{P}(\Omega) + \sum_{\mathfrak{F}} \mathbb{P}\left(\bigcap_{\{\Gamma_{i_k}\} \in \mathfrak{F}} \{\Gamma_{i_k} \text{ is empty and good}\}\right),$$

where the last summation is over \mathfrak{F} which is the collection of all possible subsets $\{\Gamma_{i_k}\}$ of $\lfloor \lambda^2 L/8 \rfloor$ standard squares in $T(\Upsilon_N) \cap \Gamma$. Since there are at most 2^L subsets of standard squares in Γ , it suffices to show that for any fixed $\{\Gamma_{i_k}\} \in \mathfrak{F}$, we have for large enough θ

$$(6) \quad \mathbb{P}\left(\bigcap_{\{\Gamma_{i_k}\} \in \mathfrak{F}} \{\Gamma_{i_k} \text{ is empty and good}\}\right) \leq \exp(-c_1(\theta)L),$$

$$(7) \quad \mathbb{P}\left(\bigcap_{\{\Gamma_{i_k}\} \in \mathfrak{F}} \{\Gamma_{i_k} \text{ is full and good}\}\right) \leq \exp(-c_2(\theta, k)L),$$

where $c_1(\theta) \rightarrow \infty$ as $\theta \rightarrow \infty$ and $\lim_{\theta \rightarrow \infty} \lim_{k \rightarrow \infty} c_2(\theta, k) = \infty$.

Let A_{i_k} denote the event that $\{\Gamma_{i_k} \text{ is empty and good}\}$. Recall that Γ_{i_k} being empty implies that $f^*|_{\Gamma_{i_k}}$ does not have any zeros, and Γ_{i_k} being good implies that $\max_{\Gamma_{i_k}} |G_{i_k}^*| \leq e^{-5\theta^2}$, where $G_{i_k}^*$ are the cumulative errors in the cantor set construction, as estimated in Section 4.1.2.

Define A'_{i_k} to be the event that $f_{i_k}^*|_{\Gamma_{i_k}}$ does not have any zeros. Here, $f_{i_k}^*$ are the final independent normalised GAFs obtained in Proposition 4.4. Clearly, the events A'_{i_k} are independent.

We will show that $A_{i_k} \subset A'_{i_k} \cup \Omega_{i_k}$, where the Ω_{i_k} 's are independent events with $\mathbb{P}(\Omega_{i_k}) < e^{-c\theta}$. To this end, we note that on Γ_{i_k} , we have $f^* = f_{i_k}^* + G_{i_k}^*$, and also $\max_{\Gamma_{i_k}} |G_{i_k}^*| \leq e^{-5\theta^2}$. Applying Corollary 4.6 to the square Γ_{i_k} , we deduce that except for a bad event Ω_{i_k} of probability $\leq e^{-c\theta}$, we have $|f_{i_k}^*| > e^{-5\theta^2}$ on $\partial\Gamma_{i_k}$.

Hence, the equation $f^* = f_{i_k}^* + G_{i_k}^*$ on Γ_{i_k} along with Rouché’s theorem implies that f^* and $f_{i_k}^*$ have the same number of zeros in Γ_{i_k} . So, on $A_{i_k} \cap \Omega_{i_k}^c$ we have that A'_{i_k} holds, in other words $A_{i_k} \subset A'_{i_k} \cup \Omega_{i_k}$, as desired. The Ω_{i_k} ’s are independent since Ω_{i_k} is defined in terms of $f_{i_k}^*$ which are independent normalised GAFs.

Therefore, we can write, for a fixed $\{\Gamma_{i_k}\} \in \mathcal{S}$

$$\mathbb{P}\left(\bigcap_k A_{i_k}\right) \leq \mathbb{P}\left(\bigcap_k (A'_{i_k} \cup \Omega_{i_k})\right) = \prod_k \mathbb{P}(A'_{i_k} \cup \Omega_{i_k}).$$

But it is not hard to see that for the state “empty” we have $\mathbb{P}(A'_{i_k} \cup \Omega_{i_k}) \leq \mathbb{P}(A'_{i_k}) + \mathbb{P}(\Omega_{i_k}) \leq e^{-c(\theta)}$ where $c(\theta) \rightarrow \infty$ as $\theta \rightarrow \infty$. This is because, $\mathbb{P}(\Omega_{i_k}) \leq e^{-c\theta}$ by definition of Ω_{i_k} as in the previous paragraph. On the other hand, by the translation invariance of the zero set of f^* , we have

$$\mathbb{P}[A'_{i_k}] = \mathbb{P}[f^* \text{ has no zeroes in a box of size } \theta \text{ with centre at the origin}].$$

The probability on the right-hand side of the above equation clearly decreases with increasing θ and converges to 0 as $\theta \rightarrow \infty$. Combining these two observations, we deduce the decay of $\mathbb{P}(A'_{i_k} \cup \Omega_{i_k})$ to 0 as $\theta \rightarrow \infty$.

It can also be seen that if we consider the state “full” instead of “empty” (and define A_{i_k} and A'_{i_k} accordingly), then we have $\mathbb{P}(A'_{i_k}) \leq e^{-c(\theta,k)}$ where $c(\theta, k) \rightarrow \infty$ as $k \rightarrow \infty$ for fixed θ , and $\mathbb{P}(\Omega_{i_k}) \leq e^{-c\theta}$. To see the upper bound on $\mathbb{P}(A'_{i_k})$, notice that by the translation invariance of the zero set of f^* , we have

$$\mathbb{P}[A'_{i_k}] = \mathbb{P}[f^* \text{ has no zeroes in a box of size } \theta \text{ with centre at the origin}].$$

For fixed θ , this probability clearly decreases to zero as $k \rightarrow \infty$ because a.s. f is an entire function whose zero set does not have any accumulation point. Therefore, we have $\mathbb{P}(A'_{i_k} \cup \Omega_{i_k}) \leq \exp(-c_2(\theta, k))$ where $\lim_{\theta \rightarrow \infty} \lim_{k \rightarrow \infty} c_2(\theta, k) = \infty$.

This proves equations (6) and (7), and hence completes the proof of the theorem.

4.1.4. *Lower bound on the size of f^* .* Our goal in this section is to establish that with large probability, the size of a normalised GAF on the perimeter of a circle (or a square) cannot be too small. Of course, there is a trade-off between the “largeness” of the probability and “smallness” of the GAF, depending on the radius of the circle or the side length of the square. Such estimates, along with Rouché’s theorem, would be useful in replacing $f^*|_{\Gamma_{i_k}}$ with the independent $f_{i_k}^*$ on “good” Γ_{i_k} ’s in Section 4.1.3.

To this end, we will use Lemma 8 from [22], which we quote here the following.

PROPOSITION 4.5 (Lemma 8, [22]). *Let γ be a curve of length at most $r \geq 1$. Then, for any positive $\varepsilon \leq 1/4$,*

$$\mathbb{P}\left\{\min_{z \in \gamma} |f^*(z)| < \varepsilon\right\} < 100r\varepsilon \sqrt{\log \frac{1}{\varepsilon}}.$$

Here, $f^*(z) = e^{-1/2|z|^2} f(z)$ where f is the standard planar GAF.

For our purposes, the precise form in which we will use this result is given by the following.

COROLLARY 4.6. *Let us consider a square B of side length $S > 1$, and let $\nu > 1$. Then we have*

$$(8) \quad \mathbb{P}(|f^*(z)| \leq e^{-\nu S^2} \text{ for some } z \in \partial B) \leq e^{-C(\nu)S}$$

for some constant $C(\nu) > 0$. Here, $f^*(z) = e^{-(1/2)|z|^2} f(z)$ where f is the standard planar GAF.

PROOF. We apply Proposition 4.5 with $r = S$ and $\varepsilon = e^{-\nu S^2}$, and obtain an upper bound of $100\sqrt{\nu}S^2e^{-\nu S^2}$. We then find $C(\nu) > 0$ such that for all $\nu > 1$ and $S > 1$ we have

$$100\sqrt{\nu}S^2e^{-\nu S^2} \leq e^{-C(\nu)S}. \quad \square$$

4.2. Proof of Theorem 1.2: Existence of critical radius. We simply observe that Theorem 1.3 proves that the criteria outlined in Proposition 3.3 are valid for \mathcal{F} , thereby establishing that a critical radius exists for \mathcal{F} .

5. Uniqueness of infinite cluster. In this section, we will prove that in the supercritical regime for the Boolean percolation models (\mathcal{G}, r) and (\mathcal{F}, r) , a.s. there is exactly one infinite cluster.

5.1. An approach to uniqueness. We will first describe a proposition which has important implications regarding such uniqueness for a translation invariant point process Π .

PROPOSITION 5.1. *Let $r > r_c$ for the Boolean percolation model $X(\Pi, r)$, where Π is a translation invariant point process on \mathbb{R}^2 , and $0 < r_c < \infty$ is the critical radius. For $R > 0$, let B_R denote the set $\{x \in \mathbb{R}^2 : \|x\|_\infty \leq R\}$. Define the event*

$$E(R) = \{\text{There is an infinite cluster } C' \text{ with the property that } C' \cap (B_R)^c \text{ contains at least three infinite clusters, any two of which are connected via a path lying inside } B_R, \text{ and such that there is at least one point from } \Pi \text{ in } C' \cap B_R\}.$$

Then $\mathbb{P}(E(R)) = 0$.

The proof of the above proposition is on the lines of the proof of Theorem 3.6 in [17]. The event $E(R)$ from Proposition 5.1 corresponds to the event $E^0(N)$ there. For the sake of completeness, we provide an outline of the proof (from [17])

here. The central ideas of the proof can be traced back to [4] and [8] in the discrete setting and [16] in the continuum.

PROOF. Let, if possible, the event $E(R)$ occur with positive probability η . On the event $E(R)$ we call the components of $C' \cap (B_R)^c$ as “branches”. Let K be a large constant, whose value is to be decided later. Given K , for every $M > 1$ we define the event

$$E(R, M) := E(R) \cap \{\text{Each branch of } C' \cap (B_R)^c \text{ contain at least } K \text{ points of } \Pi \text{ in } B_{MR} \setminus B_R\}.$$

We choose M so large that the event $E(R, M)$ has probability at least $\frac{1}{2}\eta$. The events $E^z(R)$ and $E^z(R, M)$ are defined by translating $E(R)$ and $E(R, M)$, respectively, by the vector $z \in \mathbb{R}^2$. For $x \in \mathbb{R}^2$, let B_R^x denote the box $x + B_R$. For $L > 0$, define J_L to be the (random) subset of \mathbb{Z}^2 given by

$$J_L := \{z \in \mathbb{Z}^2 : B_{MR}^{2Rz} \subset B_{LR}, E^{2Rz}(R, M) \text{ occurs}\}.$$

Then, for large enough L (depending on R and M), we have $\mathbb{E}[|J_L|] \geq \frac{1}{4}\eta L^2$ (due to translation invariance). Also observe that if $z \neq z'$ are elements of \mathbb{Z}^2 , then the interiors of B_{MR}^{2Rz} and $B_{MR}^{2Rz'}$ are disjoint.

For $z \in J_L$, pick any infinite cluster C' such that $C' \cap (B_R^{2Rz})^c$ has at least three infinite clusters, any two of which are connected via a path lying inside B_{MR} . Group these clusters arbitrarily into three disjoint collections. Let $C_z^{(1)}, C_z^{(2)}$ and $C_z^{(3)}$ to be the points of $\Pi \cap B_{MR}^{2Rz}$ in these three collections, respectively. It is clear that each $C_z^{(1)}$ is contained in B_{MR}^{2Rz} , and no two $C_z^{(i)}$'s intersect, and each one of them contains at least K points. Identify each $z \in J_L$ with an arbitrarily chosen point of Π lying in B_R^{2Rz} [there exists one such point by the definition of $E^{2Rz}(R, M)$]. Such identification enables us to invoke the combinatorial Lemma 3.2 from [17] (quoted below as Lemma 5.2). It can be checked that if the points of Π in B_R^{2Rz} and those in $B_R^{2Rz'}$ are in different components of $C' \cap B_{LR}$ (for some infinite cluster C'), then (i) of part (b) in Lemma 5.2 holds. Otherwise, (ii) of part (b) in the same lemma is satisfied. The conclusion of Lemma 3.2, combined with the inequality $\mathbb{E}[|J_L|] \geq \frac{1}{4}\eta L^2$, then gives us

$$\mathbb{E}[|\Pi \cap B_{LR}|] \geq K \left(\frac{\eta}{4} L^2 + 2 \right).$$

But if μ is the first intensity of the translation invariant point process Π , then $\mathbb{E}[|\Pi \cap B_{LR}|] = \mu(2LR)^2$. This gives us the inequality

$$K \left(\frac{\eta}{4} L^2 + 2 \right) \leq \mu(2LR)^2.$$

But for K large enough, this gives us a contradiction. \square

We now complete this discussion by quoting Lemma 3.2 from [17] (for a more general version see Lemma 2 in [8]).

LEMMA 5.2. *Let S be a set and R be a non-empty finite subset of S . Suppose that:*

(a) *For all $r \in R$, we have a family $(C_r^{(1)}, C_r^{(2)}, C_r^{(3)})$ of disjoint non-empty subsets (which we shall call branches of) S , not containing r , and $|C_r^{(i)}| \geq K$ for all i and r .*

(b) *For all $r, r' \in R$, one of the following events occurs, writing C_r for $\bigcup_{i=1}^3 C_r^{(i)}$:*

(i) $(\{r\} \cup C_r) \cap (\{r'\} \cup C_{r'}) = \emptyset$,

(ii) *there exist i, j such that $C_r^{(i)} \supset \{r'\} \cup (C_{r'} \setminus C_{r'}^{(j)})$ and $C_{r'}^{(j)} \supset \{r\} \cup (C_r \setminus C_r^{(i)})$.*

Then $|S| \geq K(|R| + 2)$.

A general approach to a proof that a.s. there cannot be infinitely many infinite clusters is to show that such an event would imply $E(R)$ would occur for some R .

5.2. *Uniqueness of infinite clusters: Ginibre ensemble.* In this section, we prove that in $X(\mathcal{G}, r)$ with $r > r_c$, we have $\Lambda(r) = 1$ a.s.

To this end, we would need to have an understanding of the conditional distribution of the points of \mathcal{G} inside a domain given the points outside. This has been obtained in [9] Theorems 1.1 and 1.2. We state these results below.

Let \mathcal{D} be a bounded open set in \mathbb{C} whose boundary has zero Lebesgue measure, and let \mathcal{S}_{in} and \mathcal{S}_{out} denote the Polish spaces of locally finite point configurations on \mathcal{D} and \mathcal{D}^c , respectively. \mathcal{G}_{in} and \mathcal{G}_{out} , respectively, denote the point processes obtained by restricting \mathcal{G} to \mathcal{D} and \mathcal{D}^c .

THEOREM 5.3. *For the Ginibre ensemble, there is a measurable function $N : \mathcal{S}_{\text{out}} \rightarrow \mathbb{N} \cup \{0\}$ such that a.s.*

$$\text{Number of points in } \mathcal{G}_{\text{in}} = N(\mathcal{G}_{\text{out}}).$$

Let the points of \mathcal{G}_{in} , taken in uniform random order, be denoted by the vector $\underline{\zeta}$. Let $\rho(\Upsilon_{\text{out}}, \cdot)$ denote the conditional measure of $\underline{\zeta}$ given $\mathcal{G}_{\text{out}} = \Upsilon_{\text{out}}$. Since a.s. the length of $\underline{\zeta}$ equals $N(\mathcal{G}_{\text{out}})$, we can as well assume that each measure $\rho(\Upsilon_{\text{out}}, \cdot)$ is supported on $\mathcal{D}^{N(\Upsilon_{\text{out}})}$.

THEOREM 5.4. *For the Ginibre ensemble, $\mathbb{P}[\mathcal{G}_{\text{out}}]$ -a.s. $\rho(\mathcal{G}_{\text{out}}, \cdot)$ and the Lebesgue measure \mathcal{L} on $\mathcal{D}^{N(\mathcal{G}_{\text{out}})}$ are mutually absolutely continuous.*

We are now ready to prove Theorem 1.1.

PROOF OF THEOREM 1.1. Let r be such that $\Lambda(r) > 0$ a.s. In what follows, we will repeatedly use the fact that if there are two points $x, y \in \mathbb{R}^2$ at Euclidean distance d , then there can be connected to each other by $(1 + \lceil d/2r \rceil)$ overlapping open disks of radius r , such that the centres of no two disks are exactly at a distance $2r$.

We will first deal with the case where a.s. $\Lambda(r) > 1$ but finite. A similar argument will show that if $3 \leq \Lambda(r) \leq \infty$ then the event $E(R)$ as in Proposition 5.1 occurs, with a suitable choice of R . This would rule out the possibility $\Lambda(r) = \infty$, and complete the proof.

We argue by contradiction, and let if possible $1 < \Lambda(r) < \infty$ a.s. Let $\mathcal{D}_1 \subset \mathcal{D}_2$ be two concentric open disks centred at the origin and respectively having radii $R_1 < R_2$. Recall the definition of the underlying graph \mathfrak{g} from Section 1.2. Let \mathcal{E} be the event that:

- (i) There are two infinite clusters \mathcal{C}_1 and \mathcal{C}_2 in the underlying graph \mathfrak{g} such that $\mathcal{C}_1 \cap \mathcal{D}_1 \neq \emptyset \neq \mathcal{C}_2 \cap \mathcal{D}_1$ (in the sense that there is at least one vertex from each \mathcal{C}_i in \mathcal{D}_1).
- (ii) There exists a finite cluster \mathcal{C}_3 of vertices of \mathfrak{g} which has $\geq 1 + \lceil 2R_1/r \rceil$ vertices such that $\mathcal{C}_3 \subset \text{Int}(\mathcal{D}_2 \setminus \mathcal{D}_1)$, where $\text{Int}(A)$ is the interior of the set A .

It is not hard to see that the event \mathcal{E} depends on the parameters R_1 and R_2 , and when R_1 and R_2 are large enough, we have $\mathbb{P}(\mathcal{E}) > 0$. Fix such disks \mathcal{D}_1 and \mathcal{D}_2 . We denote the configuration of points outside \mathcal{D}_2 by ω and those inside \mathcal{D}_2 by ζ . Let the number of points in \mathcal{D}_2 be denoted by $N(\omega)$. Any two points of Π inside \mathcal{D}_1 are at most at a Euclidean distance of $2R_1$, and hence can be connected by at most $(1 + \lceil R_1/r \rceil)$ open disks of radius r such that the centres of no two disks are exactly at a distance $2r$. We define an event \mathcal{E}' as follows: corresponding to every configuration (ζ, ω) in \mathcal{E} , we define a new configuration (ζ', ω) where ζ' is obtained by moving $(1 + \lceil R_1/r \rceil)$ points of \mathcal{C}_3 to the interior of \mathcal{D}_1 and placing them such that in the new underlying graph \mathfrak{g}' (for definition see Section 1.2) the clusters \mathcal{C}_1 and \mathcal{C}_2 become connected with each other.

Similar to the observations made in Section 1.2, we can move each point in ζ' in a sufficiently small disk around itself, resulting in new configurations (ζ'', ω) such that the connectivity properties of \mathfrak{g}' as well as the number of points in \mathcal{D}_2 remain unaltered. The event \mathcal{E}' consists of all such configurations (ζ'', ω) as (ζ, ω) varies over all configurations in \mathcal{E} . Observe that for each ω , the set of configurations $\{\zeta'' : (\zeta'', \omega) \in \mathcal{E}'\}$ constitutes an open subset of $\mathcal{D}^{N(\omega)}$, when considered as a vector in the usual way. Since $\mathbb{P}(\mathcal{E}) > 0$, by Theorem 5.4 applied to the domain \mathcal{D}_2 , we also have $\mathbb{P}(\mathcal{E}') > 0$. But on \mathcal{E}' , there is one less infinite cluster than on \mathcal{E} . But $\Lambda(r)$ being a translation invariant random variable, and \mathcal{G} being ergodic under the action of translations, $\Lambda(r)$ must be a.s. constant. This gives us the desired contradiction, and proves that $\mathbb{P}(1 < \Lambda(r) < \infty) = 0$.

Had it been the case $\Lambda(r) \geq 3$ a.s., observe that an argument analogous to the previous paragraph can be carried through with three instead of two infinite clusters (\mathcal{C}_1 and \mathcal{C}_2 above). The end result would be that with positive probability we can connect all the three clusters with each other. If $\Lambda(r) = \infty$ a.s. then we carry out the above argument with three of the infinite clusters, and observe that the event $E(R)$ as in Proposition 5.1 occurs with a set B_R where $R > R_2$, on the modified event analogous to \mathcal{E}' above. This proves that $\mathbb{P}(\Lambda(r) = \infty) = 0$. \square

Denote by $T[m, n]$ the translation map on \mathbb{R}^2 by the vector $(m\theta, n\theta)$.

5.3. *Uniqueness of infinite clusters: Gaussian zeroes.* In this section, we prove that in $X(\mathcal{F}, r)$ with $r > r_c$, we have $\Lambda(r) = 1$ a.s.

To this end, we would need to have an understanding of the conditional distribution of the points of \mathcal{F} inside a domain given the points outside. This has been obtained in [9] Theorems 1.3 and 1.4. \mathcal{F}_{in} and \mathcal{F}_{out} , respectively, denote the point processes obtained by restricting \mathcal{F} to \mathcal{D} and \mathcal{D}^c respectively. We state these results below. Some of the notation is from Section 5.2.

THEOREM 5.5. *For the GAF zero ensemble:*

(i) *There is a measurable function $N : \mathcal{S}_{\text{out}} \rightarrow \mathbb{N} \cup \{0\}$ such that a.s.*

$$\text{Number of points in } \mathcal{F}_{\text{in}} = N(\mathcal{F}_{\text{out}}).$$

(ii) *There is a measurable function $S : \mathcal{S}_{\text{out}} \rightarrow \mathbb{C}$ such that a.s.*

$$\text{Sum of the points in } \mathcal{F}_{\text{in}} = S(\mathcal{F}_{\text{out}}).$$

Define the set

$$\Sigma_{S(\mathcal{F}_{\text{out}})} := \left\{ \underline{\zeta} \in \mathcal{D}^{N(\mathcal{F}_{\text{out}})} : \sum_{j=1}^{N(\mathcal{F}_{\text{out}})} \zeta_j = S(\mathcal{F}_{\text{out}}) \right\},$$

where $\underline{\zeta} = (\zeta_1, \dots, \zeta_{N(\mathcal{F}_{\text{out}})})$.

Since a.s. the length of $\underline{\zeta}$ equals $N(\mathcal{F}_{\text{out}})$, we can as well assume that each measure $\rho(\Upsilon_{\text{out}}, \cdot)$ gives us the distribution of a random vector in $\mathcal{D}^{N(\Upsilon_{\text{out}})}$ supported on $\Sigma_{S(\Upsilon_{\text{out}})}$.

THEOREM 5.6. *For the GAF zero ensemble, $\mathbb{P}[\mathcal{F}_{\text{out}}]$ -a.s. $\rho(\mathcal{F}_{\text{out}}, \cdot)$ and the Lebesgue measure \mathcal{L}_Σ on $\Sigma_{S(\mathcal{F}_{\text{out}})}$ are mutually absolutely continuous.*

We are now ready to prove Theorem 1.2.

PROOF OF THEOREM 1.2: UNIQUENESS OF INFINITE CLUSTER. The proof follows the contour of Section 5.2, with extended arguments to take care of the

fact that for \mathcal{F} there are two conserved quantities for local perturbations of the zeros inside a disk: their number and their sum, unlike \mathcal{G} where only the number of points is conserved.

We first show that it cannot be true that a.s. $1 < \Lambda(r) < \infty$. We argue by contradiction, and let if possible $1 < \Lambda(r) < \infty$ a.s. We will define events \mathcal{E} and \mathcal{E}' in analogy to the proof of Theorem 1.1 such that on \mathcal{E}' there one less infinite cluster than on \mathcal{E} and $\mathbb{P}(\mathcal{E}) > 0$ and $\mathbb{P}(\mathcal{E}') > 0$.

Let $\mathcal{D}_1, \mathcal{D}_2$ and \mathcal{D}_3 be two concentric open disks centred at the origin and respectively having radii $R_1 < R_2 < R_3$.

Let \mathcal{E} be the event that:

- (i) $\mathcal{C}_1 \cap \mathcal{D}_1 \neq \emptyset \neq \mathcal{C}_2 \cap \mathcal{D}_1$ for two infinite clusters \mathcal{C}_1 and \mathcal{C}_2 (in the sense that there is at least one vertex from each \mathcal{C}_i inside \mathcal{D}_1),
- (ii) \exists a cluster \mathcal{C}_3 of vertices of the underlying graph \mathfrak{g} which has $\geq n = 1 + \lceil R_1/r \rceil$ vertices such that $\mathcal{C}_3 \subset \text{Int}(\mathcal{D}_2 \setminus \mathcal{D}_1)$,
- (iii) \exists a cluster $\mathcal{C}_4 \subset \text{Int}(\mathcal{D}_3 \setminus \mathcal{D}_2)$ with $\geq n' = \lceil 2R_2n \rceil$ vertices [where n is as in (ii) above] such that

$$\text{Euclidean dist}(\text{vertices in } \mathcal{C}_4, \text{vertices in } \mathfrak{g} \setminus \mathcal{C}_4) > 10$$

(recall that \mathfrak{g} denotes the underlying graph; for the definition see Section 1.2).

It is not hard to see that $\mathbb{P}(\mathcal{E}) > 0$ when $R_i, i = 1, 2, 3$ are large enough. Fix such disks $\mathcal{D}_i, i = 1, 2, 3$. We denote the configuration of points of \mathcal{F} outside \mathcal{D}_3 by ω and those inside \mathcal{D}_3 by ζ . Let the number of points in \mathcal{D}_3 be denoted by $N(\omega)$ and let their sum be $S(\omega)$.

We start with a configuration (ζ, ω) in \mathcal{E} . With the vertices inside \mathcal{D}_2 , we first perform the same operations as in the proof of Theorem 1.1. However, in \mathcal{F} , unlike in \mathcal{G} , we need to further ensure that the sum of the points inside \mathcal{D}_3 remain unchanged at $S(\omega)$ in order to stay absolutely continuous. We note that due to the operations already performed on the points inside \mathcal{D}_2 , the sum of the points inside \mathcal{D}_3 has changed by at most $2R_2n$, since $\leq n$ points have been moved and each of them can move by at most $2R_2$ which is the diameter of \mathcal{D}_2 . We observe that we can compensate for this by translating each point in \mathcal{C}_4 by an distance ≤ 1 in an appropriate direction. Due to the separation condition in (iii) in the definition of \mathcal{E} , this does not change the connectivity properties of any vertex in $\mathfrak{g} \setminus \mathcal{C}_4$.

By the observations made in Section 1.2, we can move each point in ζ' in a sufficiently small disk around itself, resulting in new configurations (ζ'', ω) such that the connectivity properties of \mathfrak{g}' as well as the number of the points in \mathcal{D}_3 remain unaltered. The event \mathcal{E}' consists of all such configurations (ζ'', ω) as (ζ, ω) varies over all configurations in \mathcal{E} . Observe that for each ω , the set of configurations $\{\zeta'' : (\zeta'', \omega) \in \mathcal{E}'\}$ constitutes an open subset of $\mathcal{D}_3^{N(\omega)}$, when considered as a vector in the usual way. Hence, its intersection with $\Sigma_{S(\omega)}$ is an open subset of $\Sigma_{S(\omega)}$. Since $\mathbb{P}(\mathcal{E}) > 0$, by Theorem 5.6 applied to the domain \mathcal{D}_3 , we also have

$\mathbb{P}(\mathcal{E}') > 0$. But in \mathcal{E}' , there is one less infinite cluster than in \mathcal{E} . But $\Lambda(r)$ being a translation invariant random variable, and \mathcal{F} being ergodic under the action of translations, $\Lambda(r)$ must be a.s. constant. This gives us the desired contradiction, and proves that $\mathbb{P}(1 < \Lambda(r) < \infty) = 0$.

We take care of the case $\Lambda(r) = \infty$ as we did in the proof of Theorem 1.1. Had it been the case $\Lambda(r) \geq 3$ a.s., an argument analogous to the previous paragraph can be carried through with three instead of two infinite clusters (\mathcal{C}_1 and \mathcal{C}_2 above), with the end result that with positive probability we can connect all the three infinite clusters with each other. If $\Lambda(r) = \infty$ a.s. then we carry out the above argument with three of the infinite clusters, and observe that the event $E(R)$ in Proposition 5.1 occurs on the modified event (analogous to \mathcal{E}' above) with a set B_R where $R > R_3$. This proves that $\mathbb{P}(\Lambda(r) = \infty) = 0$. \square

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S. GHOSH
DEPARTMENT OF ORFE
PRINCETON UNIVERSITY
PRINCETON, NEW JERSEY 08544
USA
E-MAIL: subhrowork@gmail.com

M. KRISHNAPUR
DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF SCIENCE
BANGALORE 560 012
INDIA
E-MAIL: manju@math.iisc.ernet.in

Y. PERES
MICROSOFT RESEARCH
ONE MICROSOFT WAY
REDMOND, WASHINGTON 98052
USA
E-MAIL: peres@microsoft.com